Juliusz Schauder Center for Nonlinear Studies
Nicolaus Copernicus University

Algebraic Constructions of Solutions for the Translation Equation

Andrzej Mach

Institute of Mathematics
The Jan Kochanowski University
of Humanities and Sciences

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Recenzenci:

prof. dr hab. Roman Ger
prof. dr hab. Zenon Moszner

Centrum Badań Nieliniowych im. Juliusza Schaudera
Uniwersytet Mikołaja Kopernika
ul. Chopina 12/18, 87-100 Toruń

Redakcja: tel. +48 (56) 611 34 28, faks: +48 (56) 622 89 79

e-mail: tmna@ncu.pl
http://www.cbn.ncu.pl

Wydanie pierwsze. Nakład 250 egz.

Dedicated on the occasion
of Professor Zenon Moszner's
80th bithday anniversary,
who guided me
through translation equation theory

CONTENTS

INTRODUCTION

Let X be a nonempty set and let $(S, \cdot)$ be an algebraic structure, with a binary operation $\cdot(S \times S) \subseteq S$. By the *translation equation* we understand the functional equation

$$F(F(\alpha, x), y) = F(\alpha, x \cdot y), \tag{0.1}$$

where the unknown function F is defined on the subset of the set $X \times S$ and takes its values in X. When the operation "$\cdot$" is defined for all pairs $(x, y) \in S \times S$ and the function F is defined on the whole set $X \times S$, then the notion of fulfilment by F of the translation equation is clear, since both sides of the translation equation are defined and equal for any $\alpha \in X$ and $x, y \in S$. Otherwise, one can define the fulfilment by F of the translation equation in miscellaneous ways. Such problems were considered for example in the papers [12], [42], [60], [61]. In the present paper we assume that "$\cdot$" is defined for all pairs $(x, y) \in S \times S$ and the function F is defined on the whole set $X \times S$. If $(S, \cdot) = (G, \cdot)$ is a groupoid, then $\cdot(G \times G) \subseteq G$ and the solution of the translation equation (0.1) is a function $F: X \times G \to X$. If

$$F(\alpha, 1) = \alpha, \quad \alpha \in X, \tag{0.2}$$

where 1 denote, if there exists, the unit element of $(G, \cdot)$, then we say that the solution F satisfies the *identity condition*.

Every solution of the translation equation is a homomorphism from the algebraic structure S to the monoid $(X^X, \circ)$ where "$\circ$" denotes the operation of superpostion.

Conversely, every homorphism $h: S \to X$ where $(X, \#)$ is a semigroup, determines the solution of the translation equation $F(\alpha, x) := \alpha \# h(x)$, for $\alpha \in X$ and $x \in S$. For these reasons, it is natural to use algebraic methods to describe the translation equation.

The translation equation appears in several mathematical domains: abstract geometric and algebraic objects, abstract automata, groups of transformations, iterations, representations of groups, dynamical systems and others (see [36]) and its theory is applied extensively (see [41]).

We will describe some particular examples of the occurences or applications of the translation equation in various fields of mathematics.

In the widely developed theory of abstract geometric objects, the solutions of the translation equation are the so-called transformation formulae of the geometric object. In that theory, the translation equation is considered for the Euclidean space $X = \mathbb{R}^n$. The algebraic structure G can be the differential group L^n_r, or the Brandt groupoid (see Definition 2.3.4, below), or the pseudogroup of transformations in Gołąb's sense (see [17]). The references concerning the theory of geometric objects can be found for example in [36], [2], [17], [32]. The other papers on geometric objects are for example [58], [57], [53]. The algebraic objects (see [56]) are defined as generalization of geometric objects. The other papers on this subject are for example [52], [5].

In the theory of abstract machines (initiated by S. Ginsburg in [8]), the function F, called the next state function of abstract machine from the state α to the state $\beta = F(\alpha, x)$ by the signal x, is a solution of the translation equation (see [47]).

If F is a solution of (0.1) where $(S, \cdot\,) = (G, \cdot\,)$ is a group, then G acts on X by means of the mapping: $a \to F(\,\cdot\,, a)\colon X \to X$, see for example [7], [54].

The function $F(\alpha, n) := f^{(n)}(\alpha)$ where $f^{(n)}$ denotes the n-th iteration of the function f satisfies the translation equation and the identity condition. The generalization of the notion of iteration to the case where the index of iteration varies continuously leads to the theory of iteration group, continuous iteration group (or semigroup), and further to the theory of dynamical systems (see for example [18], [19], [6], [45], [33], [51], [3], [59], [9], [46], [48]).

As we can see, the translation equation plays an essential role in several areas. In conclusion, the theory of translation equation is the very important topic.

The aim of this paper is to provide an overview and comparative analysis of existing results concerning algebraic constructions of solutions of this equation.

The most general construction of solutions of the translation equation is the construction given in the paper [16] where the considered algebraic structure is a category (see Definition 2.3.1, below). In that construction the parameters are involved in the so-called compatibility condition (the concrete form of the compatibility condition for the case of semigroup, can be seen in (2.1)). Let us remark that there does not exist any construction of solutions of the translation equation without some kind of dependencies, however complicated, of parameters. Indeed, in the best known construction on a group G, given by Z. Moszner (see Construction 2.1.1, below), we take two parameters, for example a subgroup G_k and a subset (fibre) X_k, and if the cardinalities of the sets G/G_k and X_k are the same, the construction works. Otherwise, we are obliged to change the parameters. However, the compatibility condition in the form (2.1) determines the special dependencies of parameters as a system of functional equations. In the present paper, we choose only the constructions without the compatibility condition of the form of a system of functional equations. The last section of the paper includes the constructions of the generalized translation equation.

The paper has a character of a survey. Therefore, the majority of theorems, already published somewhere, is presented without proofs with only a few exceptions. We present the proof of Theorem 1.3.3 describing some invariant decompositions, which is very important and useful in applications. Using the form of invariant decompositions, we can construct the solutions of the translation equation and it can be applied to describe additional and interesting properties of solutions as in papers [40], [25], [26]. The proof of Theorem 2.2.2 is omitted in [30]; so, we present it here. We also present the modified formulation (with an altered and simpler proof) of Theorem 2.2.27, which is included in unpublished Ph. D. thesis by B. Nowak ([43]). The proof of Theorem 2.2.32 was published in Polish ([23]); so, it is included too.

Several years ago, J. Aczél, in a letter to Z. Moszner, asked about the solutions of the translation equation on monoid of natural numbers. The results of investigations initiated by this question are quoted from [30] as Theorems 2.2.2, 2.2.10, 2.2.11. I hope that the presented overview can be useful for those who are interested in the theory and applications of the translation equation.

CHAPTER 1

INVARIANT DECOMPOSITIONS

Invariant decompositions are one of the parameters occurring in every construction of solutions of the translation equation on an arbitrary algebraic structure.

1.1. Groupoids and groups

Let us suppose that $(G, \cdot)$ is a groupoid.

Definition 1.1.1 ([30])**.** A family $\{E_j\}_{j\in J}$ of nonempty pairwise disjoint subsets of G is called an *G-invariant decomposition* (or simply: *invariant decomposition*) of the groupoid $(G, \cdot)$, if $G = \bigcup_{j\in J} E_j$ and

$$\forall j \in J\ \forall k \in G\ \exists l \in J : (E_j \cdot k \subseteq E_l). \tag{1.1}$$

Remark 1.1.2. One can easily verify the following statement. If $F\colon X\times G \to X$ is a solution of the translation equation and we define the so-called *almost fibres* $F_\alpha := F(\{\alpha\} \times G)$, then the family $\{E_\alpha(\beta)\}_{\beta\in F_\alpha}$ where $E_\alpha(\beta) := \{x \in G : F(\alpha, x) = \beta\}$, forms an invariant decomposition of G. Indeed, the sets $\{E_\alpha(\beta)\}_{\beta\in F_\alpha}$ are evidently disjoint, nonempty and $G = \bigcup_{\beta\in F_\alpha} E_\alpha(\beta)$. Moreover, let $\beta \in F_\alpha$ and $y \in G$. We put $\gamma := F(\beta, y)$. Consequently, we have

$$\forall x \in E_\alpha(\beta) : F(\alpha, x \cdot y) = F(F(\alpha, x), y) = F(\beta, y) = \gamma,$$

therefore $E_\alpha(\beta) \cdot y \subseteq E_\alpha(\gamma)$.

Remark 1.1.3 ([30])**.** The decomposition $\{E_j\}_{j\in J}$ of G is invariant if and only if the relation

$$a \equiv b \Leftrightarrow \exists j \in J : a, b \in E_j$$

is right-compatible with the groupoid action, i.e.

$$\forall a, b, c \in G : [a \equiv b \Leftrightarrow a \cdot c \equiv b \cdot c].$$

If the groupoid G is abelian, then every equivalence relation $\equiv$ right-compatible with the groupoid action is a congruence relation, that is,

$$\forall a, b, c, d \in G : [(a \equiv b \wedge c \equiv d) \Leftrightarrow a \ \cdot c \equiv b \cdot d].$$

An equivalence relation $\equiv$ on a groupoid G is a congruence (right-compatible with the groupoid action, respectively) if and only if there exists a function $h\colon G \to G$ such that

$$a \equiv b \Leftrightarrow h(a) = h(b) \text{ and } h(a \cdot b) = h[h(a) \cdot h(b)] \tag{1.2}$$

$(h(a \cdot b) = h[h(a) \cdot b]$, respectively) for $a, b \in G$. In the case of a congruence relation, the function h is a homomorphism of G onto the groupoid $h(G)$ with the operation $c\#d = h(c \cdot d)$. This means that the equivalence relation $\equiv$ is a congruence in the groupoid $(G, \cdot\,)$ if and only if there exists a homomorphism H of G into a groupoid T such that $a \ \equiv b \Leftrightarrow H(a) = H(b)$ (see [10, pp. 35–37]). This yields a method of constructing invariant decompositions for the monoid of natural numbers (see Remark 1.2.1).

Remark 1.1.4 ([30])**.** If the groupoid G is a group, then its invariant decompositions are sets of right cosets of some subgroup (see [4, pp. 34–35]). Additionally, if the group G is abelian then invariant decompositions are determined by the quotient groups.

1.2. Monoid of natural numbers $(\mathbb{N}, \cdot\,)$

Remark 1.2.1 (by A. Schinzel, see [30])**.** By Remark 1.1.3 all congruences $\equiv$ in the monoid $(\mathbb{N}, \cdot\,)$ are obtained by the following construction.

Construction 1.2.2.

(1) Take an arbitrary abelian semigroup $(T, +)$ with neutral element 0.
(2) Take an arbitrary function $\phi\colon P \to T$ where P is the set of all prime numbers. Define a homomorphism $H\colon (\mathbb{N}, \cdot\,) \to (T, +)$ by setting, for $a = \prod_{p\in P} p^{\alpha(p)} \in \mathbb{N}$ where $\alpha(p)$ are non-negative integers, $H(a) := \sum_{p\in P} \alpha(p)\phi(p)$.
(3) For $a, b \in \mathbb{N}$ define: $a \ \equiv b \Leftrightarrow H(a) = H(b)$, that is

$$\prod_{p\in P} p^{\alpha(p)} \equiv \prod_{p\in P} p^{\beta(p)} \Leftrightarrow \sum_{p\in P} \alpha(p)\phi(p) = \sum_{p\in P} \beta(p)\phi(p)$$

where $\alpha(p)$, $\beta(p)$ are non-negative integers.

Describing all congruence relations means describing all semigroups and, in consequence, solving the associativity equation

$$F(F(a, b), c) = F(a, F(b, c)) \quad \text{where } F\colon G \times G \to G,$$

which is accounted for by a rather complex theory (see e.g. [1, pp. 297, 308]).

Example 1.2.3 (by A. Schinzel, see [30])**.** Let $T := 2^{\mathbb{N}}$ be the monoid with the union operation. If we define $\phi(p) := \{p\}$ we get the congruence relation

$$a \equiv b \Leftrightarrow a \text{ and } b \text{ have the same prime factors,}$$

which means that components of invariant decomposition of $\mathbb{N}$ are sets of natural numbers having the same prime factors.

Remark 1.2.4 ([30])**.** To obtain the same invariant decomposition as in Example 1.2.3, it is possible to take $\mathbb{N}$ with suitable operation within. The function $h\colon \mathbb{N} \to \mathbb{N}$ such that $h(a)$ equals the product of prime factors of a for $a > 1$ and $h(1) = 1$ satisfies (1.2), hence h is a homomorphism of $(\mathbb{N}, \cdot\,)$ into $\mathbb{N}$ with the operation $a \# b = h(a \cdot b)$.

Remark 1.2.5 ([30])**.** If we define $\phi\colon P \to T = 2^{\mathbb{N}}$ by $\phi(p) := \emptyset$ for $p \neq 2$ and $\phi(2) := \{1\}$ where $T = 2^{\mathbb{N}}$ denotes the monoid described in Example 1.2.3, then, by Construction 1.2.2 (3), we get the congruence equivalent to the following invariant decomposition: $E_1 := \{1, 3, 5, \dots\}$, $E_2 := \{2, 4, 6, \dots\}$. Similarly, if we define $\phi\colon P \to T = 2^{\mathbb{N}}$ by $\phi(p) := \mathbb{N}$ for all $p \in P$, then, by Construction 1.2.2 (3), we get the congruence equivalent to the invariant decomposition: $E_1 := \{1\}$, $E_2 := \mathbb{N} \setminus \{1\}$. To obtain the following invariant decomposition: $E_0 := \{2, 4, 6, \dots\}$ and $E_j := \{2j - 1\}$ for $j \in \mathbb{N}$, it is sufficient to consider the semigroup $(T, \cdot\,) := (2^{\mathbb{R}\setminus\{0\}}, \cdot\,)$ where the operation is defined by $A \cdot B := \{a \cdot b : a \in A,\ b \in B\}$ for $A, B \in 2^{\mathbb{R}\setminus\{0\}}$, and to define $\phi\colon P \to T = 2^{\mathbb{R}\setminus\{0\}}$ by $\phi(p) := \{p\}$ for $p \neq 2$ and $\phi(2) := \mathbb{R} \setminus \{0\}$.

Remark 1.2.6. The examples of invariant decompositions of the monoid of natural numbers $(\mathbb{N}, \cdot\,)$ can be also obtained in the following way. Let us consider a linearly ordered and archimedean group (see the next subsection) $(\mathbb{R}_+, \cdot, \leq)$ where $\mathbb{R}_+ =]0, +\infty[$. Using Theorem 1.3.11 below, take an arbitrary invariant decomposition of the semigroup of positive elements of the considered group, namely invariant decomposition of $[1, +\infty[$. One can easily observe that the family of all nonempty intersections of components of this decomposition with $\mathbb{N}$ forms an invariant decomposition of the monoid of natural numbers. If we take, for example, the invariant decomposition of $[1, +\infty[$ of the form: every element of the interval $[1, 4]$ is a component and the last component is the set $[1, +\infty[\setminus [1, 4]$, then we obtain the following invariant decomposition of the monoid of natural numbers: $E_1 = \{1\}$, $E_2 = \{2\}$, $E_3 = \{3\}$, $E_4 = \{4\}$ and $E_5 = \{5, 6, 7, \dots\}$.

1.3. Semigroup of positive elements of linearly ordered and abelian group

Let $(G, \cdot, 1)$ and P denote a group and its subsemigroup, respectively. It is well known ([20]) that the relation "$\leq$", defined by

$$\forall a, b \in G : a \leq b \Leftrightarrow a^{-1} \cdot b \in P, \tag{1.3}$$

determines a linear ordering of the group G, if and only if, the subsemigroup P satisfies the following conditions:

- $1 \in P$,
- $P \cap P^{-1} = \{1\}$, where $P^{-1} := \{x \in G : x^{-1} \in P\}$,
- $\forall x \in G\ \forall a \in P : x^{-1} \cdot a \cdot x \in P$,
- $P \cup P^{-1} = G$.

For the semigroup P we have $P = \{x \in G : x \geq 1\}$ and P is called the *semigroup of positive elements* of the group G. Moreover, linearly ordered, by relation (1.3), group G is said to be archimedean, i.e. there are no convex subgroups in G, apart from $\{1\}$ and G if and only if the semigroup P additionally satisfies the condition:

- $\forall x, y \in P \setminus \{1\}\ \exists n \in \mathbb{N} : y^{-1} \cdot x^n \in P$.

Let us also recall that, by Hölder's Theorem (see [20, p. 325]), the linearly ordered and archimedean group is abelian (the converse is not true).

From now onwards, we will use the additive notation for linearly ordered group, i.e. $(G, +, \leq)$ and 0 is the neutral element of G.

In this subsection, we will present (quoted from [22] and [24]) the general form of invariant decompositions of the subsemigroup G^+ of positive elements (or more general of final interval) of the group $(G, +, \leq)$ linearly ordered and abelian. Before the presentation of the construction of the decompositions, we will cite two definitions.

Definition 1.3.1 ([22])**.** A subset A of G is called *bounded* if

$$\exists z \in G^+\ \forall a \in A : (a < z \text{ and } a > -z),$$

and *unbounded* if it is not bounded.

Definition 1.3.2 ([22])**.** Let A, B be subsets of G and $A \subseteq B \subseteq G$. We say:

(a) A is an *initial interval* of B if $\{a \in B : a \leq a_0\} \subseteq A$, for all $a_0 \in A$,
(b) A is a *final interval* of B if $\{a \in B : a_0 \leq a\} \subseteq A$ for all $a_0 \in A$.

Theorem 1.3.3 ([22] and [24]). *All G^+-invariant decompositions of the semigroup G^+ of positive elements of a linearly ordered, abelian group G are obtained by the construction presented below. The same construction determines the invariant decomposition of a final interval Δ of G.*

Construction 1.3.4.

(1) Take a family $\{G_s\}_{s\in S}$ of distinct, bounded subgroups of G forming a chain, i.e. $G_s \subset G_t$ or $G_t \subset G_s$ for $s,t \in S$, and an unbounded subgroup G^* such that $G^* \supseteq \bigcup_{s\in S} G_s$.

(2) Let Φ be a function from the family $\{G_s\}_{s\in S}$ onto a family of initial intervals of G^+ (onto a family of initial intervals of Δ, respectively), such that
   (a) $\Phi(G_s)$ is an union of intersections with G^+ (with Δ, respectively) cosets of $C(G_s)$ in G where $C(G_s)$ denotes the smallest convex subgroup containing G_s (the convexity of $C(G_s)$ means that together with every positive element a the subgroup $C(G_s)$ contains all elements $x \in G^+$ with $x \le a$),
   (b) if $G_s \subset G_t$, then $\Phi(G_s) \subset \Phi(G_t)$.

(3) Every nonempty set

$$W \cap \left[\Phi(G_s) \setminus \bigcup_{G_t \subsetneq G_s} \Phi(G_t)\right], \quad W \in G_s,\ s \in S,$$

is a component of the decomposition.

(4) The sets

$$V \cap \left[G^+ \setminus \bigcup_{s\in S} \Phi(G_s)\right], \quad V \in G/G^*,$$

$$\left(V \cap \left[\Delta \setminus \bigcup_{s\in S} \Phi(G_s)\right], \quad V \in G/G^*,\ \text{respectively}\right)$$

are the remaining components.

Before demonstrating the proof of Theorem 1.3.3 we shall present some lemmas. The omitted proofs are exactly the same as those proved in the particular case $\Delta = G^+$ in [22]. Let $\{E_k\}_{k\in K}$ be an arbitrary G^+-invariant decomposition of a final interval Δ of the group G.

Lemma 1.3.5. *Every component E_k is a final interval of a coset of subgroup $G_k := \{x - y :\ x, y \in E_k\}$ in G.*

Lemma 1.3.6. *If $G_k \neq \{0\}$ is a bounded subgroup of G, then G_k is a subgroup of the convex subgroup $C(G_k) := \{c \in G : \exists c_1, c_2 \in G_k : c_1 \leq c \leq c_2\}$ such that $\{0\} \neq C(G_k) \neq G$.*

Lemma 1.3.7. *If a component E_k is a final interval of a coset of an unbounded subgroup G_k in G and E_l is a final interval of a coset of a bounded subgroup G_l in G, then*

$$\forall x \in E_k \ \forall y \in E_l : x > y.$$

Lemma 1.3.8. *If a components E_k, E_l are final intervals of cosets of the bounded subgroups G_k, G_l in G, respectively, then one of the following conditions is satisfied*

$$\forall x \in E_k \ \forall y \in E_l : x > y \quad \textit{or} \quad \forall x \in E_k \ \forall y \in E_l : \ x < y.$$

Lemma 1.3.9. *If E_k, E_l, G_k, G_l are such as in the previous lemma and $G_k \subsetneq G_l$, then*

$$\forall x \in E_k \ \forall y \in E_l : x < y.$$

Lemma 1.3.10. *If the components E_k, E_l are final intervals of cosets of the unbounded subgroups G_k, G_l in G, respectively, then $G_k = G_l$.*

Proof of Theorem 1.3.3. Since it is visible that in Construction 1.3.4 we obtain the G^+-invariant decomposition of the final interval Δ, we restrict ourselves to the proof that every G^+-invariant decomposition of the final interval Δ of group G can be obtained by this construction. Let $\{E_k\}_{k \in K}$ be an arbitrary G^+-invariant decomposition of a final interval Δ of the group G. Firstly, let us suppose that for every $k \in K$ the subgroup G_k defined in Lemma 1.3.5 is unbounded. Consequently, by Lemma 1.3.10 we have $G_k = G_l =: G^*$, for $k, l \in K$. Therefore, every component E_k is the final interval of a coset of unbounded subgroup G^* in G. Fix a $k \in K$. Let $V \in G/G^*$ and $E_k \subseteq V$. The equalities

$$\Delta = \bigcup_{l \in K} E_l \quad \text{and} \quad (E_l \cap V = \emptyset, \quad \text{for } l \in K, \ l \neq k)$$

imply

$$V \cap \Delta = V \cap \bigcup_{l \in K} E_l = \bigcup_{l \in K} (V \cap E_l) = V \cap E_k = E_k.$$

Assuming the empty chain of bounded subgroups as well as the function Φ in Construction 1.3.4, all components $\{E_k\}_{k \in K}$ are defined by point (4) of this construction as follows

$$V \cap \Delta \quad \text{for } V \in G/G^*.$$

Now, let us suppose the existence of $k \in K$ for which G_k is bounded subgroup of G. Let $\{G_s\}_{s \in S}$ be the family of all different and bounded subgroups from the set $\{G_k : k \in K\}$. Furthermore, let $\overline{E}_s$ denote the union of these components E_k of invariant decomposition $\{E_k\}_{k\in K}$ which are final intervals of cosets of G_s in G. We will show that the set $\overline{E}_s$, for $s \in S$, is convex and it is an union of final intervals of cosets of $C(G_s)$ in G where $C(G_s)$ is the convex subgroup defined in Lemma 1.3.6. Take an arbitrary $s \in S$, $x_1, x_2 \in \overline{E}_s$ and $x_1 < x_2$. Let $c \in \Delta$ and $x_1 \le c \le x_2$. Assume that $c \in E_l \in \{E_k\}_{k\in K}$ and take the component $E_k, E_m \subset \overline{E}_s$ containing x_1, x_2, respectively. We have $E_k + c - x_1 \subseteq E_l$ and $E_l + x_2 - c \subseteq E_m$, hence $G_s \subseteq G_l$ and $G_l \subseteq G_s$, so $G_l = G_s$. Therefore, $E_l \subset \overline{E}_s$ and $c \in \overline{E}_s$, was to have been proven for the convexity of $\overline{E}_s$. In order to prove that $\overline{E}_s$ is a union of final intervals of cosets of $C(G_s)$ in G, let us notice that by the convexity of cosets of $C(G_s)$ in G and by convexity of $\overline{E}_s$, it is sufficient to show the following formula

$$\forall x \in \overline{E}_s\ \forall u \in C(G_s) \cap G^+ : u + x \in \overline{E}_s. \tag{1.4}$$

So, let $u \in C(G_s) \cap G^+$, $x \in \overline{E}_s$ and $u + x \in E_l \in \{E_k\}_{k\in K}$. Since $C(G_s)$ is the smallest convex subgroup of G containing G_s, then there exists $v \in G_s$ such that $v \ge u$. Let E_k be the component of the decomposition such that $x+v \in E_k$. We have $x \in \overline{E}_s$, $v \in G_s$ and $x+v \ge x$, so $E_k \subset \overline{E}_s$. From the above, $E_l+v-u \subseteq E_k$ and $G_l \subseteq G_s$. Hence G_l is a bounded subgroup of G and by $u+x \ge x$, $u+x \in E_l$ and by Lemma 1.3.9, we obtain $G_l = G_s$, and so $E_l \subset \overline{E}_s$, whence $u + x \in \overline{E}_s$ and the formula (1.4) is proved.

For every $s \in S$, define the initial interval P_s of Δ in the following way

$$P_s := \{x \in \Delta : \exists y \in \overline{E}_s\ x \le y\}.$$

It follows from the above that every initial interval P_s is an union of intersections with Δ of cosets of $C(G_s)$ in G. Let us define the function Φ by the formula

$$\Phi(G_s) := P_s, \quad \text{for } s \in S.$$

We will show that the subgroups $\{G_s\}_{s\in S}$ and the function Φ satisfy the points (1), (2) of Construction 1.3.4. The subgroups G_s, for $s \in S$, are bounded. Take $s, t \in S$ and $s \ne t$. Let $x \in E_k \subset \overline{E}_s$ and $y \in E_l \subset \overline{E}_t$. Since G is a linearly ordered group then $x \le y$ or $y \le x$. Therefore, $E_k+y-x \subseteq E_l$ or $E_l+x-y \subseteq E_k$. Hence, $G_s \subsetneq G_t$ or $G_t \subsetneq G_s$, which means that the subgroups $\{G_s\}_{s\in S}$ form a chain such as in Construction 1.3.4.

We remarked earlier that every initial interval $P_s = \Phi(G_s)$, for $s \in S$, is an union of intersections with Δ of cosets of $C(G_s)$ in G. By lemma 1.3.9 the function Φ satisfies the condition (2)(b) of Construction 1.3.4.

Assume that $x \in E_l \subset \overline{E}_s$, for certain $s \in S$, so $x \in \Phi(G_s)$. Let us observe that $x \notin \bigcup_{G_t \subsetneq G_s} \Phi(G_t)$. If not, we have

$$\exists t \in S\ \exists y \in E_k \subset \overline{E}_t : x \le y \text{ and } G_t \subsetneq G_s.$$

Hence, $E_l + y - x \subseteq E_k$, whence a contradiction $G_s \subseteq G_t$. We have shown the inclusion

$$\overline{E}_s \subseteq \Phi(G_s) \setminus \bigcup_{G_t \subsetneq G_s} \Phi(G_t).$$

In order to prove the opposite inclusion, assume that $x \in \Phi(G_s)$ and $x \notin \Phi(G_t)$, for every $G_t \subsetneq G_s$. Hence, $x \leq y$, for a certain $y \in E_l \subset \overline{E}_s$. Suppose, by *reductio ad absurdum*, that $x \in E_k \subset \Phi(G_k)$ and $E_k \not\subset \overline{E}_s$. Hence, $E_k + y - x \subseteq E_l$, whence $G_k \subsetneq G_s$ and so a contradiction with $x \notin \bigcup_{G_t \subsetneq G_s} \Phi(G_t)$. Therefore we have

$$\overline{E}_s = \Phi(G_s) \setminus \bigcup_{G_t \subsetneq G_s} \Phi(G_t), \quad \text{for } s \in S.$$

Now, we will show that the bounded components of the invariant decomposition $\{E_k\}_{k \in K}$ are as in the point (3) of Construction 1.3.4, i.e.

$$\forall s \in S \ \forall E_k \subset \overline{E}_s \ \exists W \in G/G_s : E_k = W \cap \overline{E}_s.$$

Fix $s \in S$ and $E_k \subset \overline{E}_s$. By Lemma 1.3.5, the component E_k is a final interval of a coset of subgroup G_k ($G_k = G_s$) in G. Therefore, there exists $W \in G/G_s$ such that $E_k \subseteq W$. Evidently,

$$W \cap E_l = \emptyset, \quad \text{for all } E_l \subset \overline{E}_s,\ E_l \neq E_k.$$

Since $\overline{E}_s = \bigcup_{G_l = G_s} E_l$ then

$$W \cap \overline{E}_s = W \cap \bigcup_{G_l = G_s} E_l = \bigcup_{G_l = G_s} (W \cap E_l) = W \cap E_k = E_k.$$

By Lemma 1.3.10, in the set $\{G_k : k \in K\}$, there exists at most one unbounded subgroup G^*. Therefore, every unbounded component of the decomposition is a final interval of a coset of G^* in G. By Lemma 1.3.7, for an unbounded component E_k we have

$$\forall s \in S \ \forall x \in \overline{E}_s \ \forall y \in E_k : x < y$$

and hence,

$$\forall s \in S \ \forall x \in E_l \subset \overline{E}_s \ \forall y \in E_k : E_l + y - x \subseteq E_k,$$

and so $G_s \subseteq G^*$, for all $s \in S$, i.e. $G^* \supseteq \bigcup_{s \in S} G_s$.

Now, we will show that every unbounded component of decomposition is such as in point (4) of Construction 1.3.4. Let E_k be an arbitrary unbounded component, $V \in G/G^*$ and $E_k \subseteq V$. Since

$$\Delta \setminus \bigcup_{s \in S} \Phi(G_s) = \bigcup_{G_l = G^*} E_l,$$

then using the fact $E_l \cap V = \emptyset$, for every $E_l \neq E_k$, $G_l = G^*$, we obtain

$$V \cap \left[\Delta \setminus \bigcup_{s \in S} \Phi(G_s)\right] = V \cap \bigcup_{G_l = G^*} E_l = \bigcup_{G_l = G^*} (V \cap E_l) = V \cap E_k = E_k.$$

As for Theorem 1.3.3, that was to have been proven. □

If we assume additionally that $(G, +, \leq)$ is an archimedean group, then Construction 1.3.4 is reducible to the following result from the paper [39].

Theorem 1.3.11 ([39]). *Every G^+-invariant decomposition of the semigroup G^+ of positive elements of a linearly ordered, archimedean group G (of a final interval Δ of G, respectively), is of the following form:*

(a) *there exists a right-closed or right-open initial interval Δ_1 of G^+ (of Δ, respectively), such that every element belonging to Δ_1, is a component of the decomposition; it is possible that $\Delta_1 = \emptyset$ or $\Delta_1 = G^+$ (or $\Delta_1 = \Delta$, respectively),*

(b) *the remaining components are the intersections with $G^+ \setminus \Delta_1$ (with $\Delta \setminus \Delta_1$, respectively) of cosets of some subgroup $G^* \neq \{0\}$ in G.*

Example 1.3.12 ([30]). Using Construction 1.3.4 we can obtain examples of invariant decompositions for the semigroup G^+ of positive elements of a linearly ordered, abelian group $(G, +, \leq)$. Let $\mathbb{Z}$ denote the set of integers and $G := \{ax+b : a, b \in \mathbb{Z}\}$ be the group of linear polynomials with ordinary addition and with linear order defined as follows:

$$(ax + b \leq cx + d) \Leftrightarrow (a < c) \quad \text{or} \quad (a = c \text{ and } b \leq d).$$

The semigroup of positive elements is

$$G^+ := \{ax + b : a > 0,\ b \in \mathbb{Z}\} \cup \mathbb{Z}^+$$

where $\mathbb{Z}^+ := \{a \in \mathbb{Z} : a \geq 0\}$. According to Construction 1.3.4, take the chain of bounded subgroups $\{0\} \subset \mathbb{Z}$ and the unbounded subgroup $G^* := G$. Define $\Phi(\{0\}) := \mathbb{Z}^+$; $\Phi(\mathbb{Z}) := \mathbb{Z}^+ \cup (\mathbb{Z} + x)$ where $\mathbb{Z} + x \in G/\mathbb{Z}$. Every element of $\mathbb{Z}^+$ is a component of the decomposition. The sets $\mathbb{Z} + x$, $G^+ \setminus (\mathbb{Z}^+ \cup (\mathbb{Z} + x))$ are also components. The obtained invariant decomposition can be illustrated as in Figure 1.

Example 1.3.13 ([23]). Let the group $(G, +, \leq)$ and semigroup G^+ be the same as in Example 1.3.12. Let $\Delta := G^+ \setminus \{0, 1, 2\}$ be the final interval of G. According to Construction 1.3.4, let us take the chain of bounded subgroups of G, namely

$$\mathbb{Z}_4 \subset \mathbb{Z}_2 \subset \mathbb{Z} \tag{1.5}$$

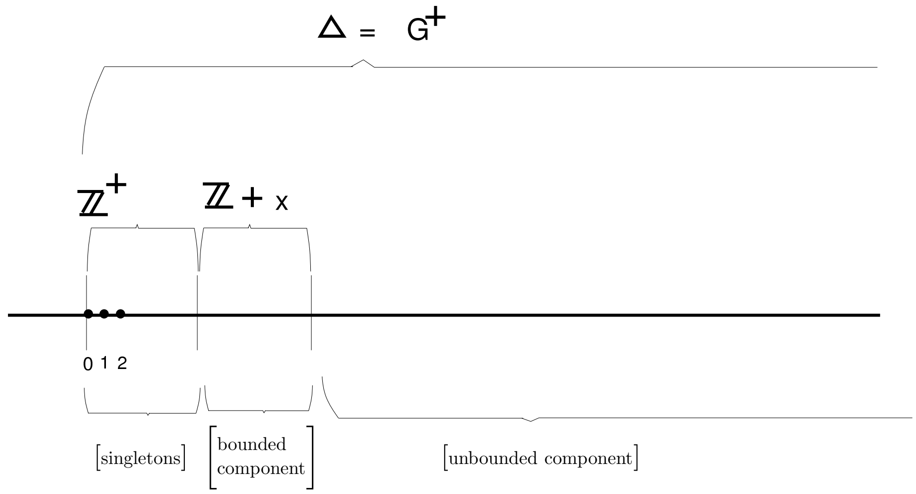

FIGURE 1

where $\mathbb{Z}_s := \{z \cdot s : z \in \mathbb{Z}\}$ for $s \in \mathbb{N}$. Let us take the unbounded subgroup $G^* := \mathbb{Z}_2 \cdot x + \mathbb{Z} = \{ax + b : a \in \mathbb{Z}_2 \text{ and } b \in \mathbb{Z}\}$. Continuing, let us define two functions Φ and Ψ from the chain (1.5) onto a family of initial intervals of Δ, which will determine two different G^+-invariant decompositions of the final interval Δ. Define

$$\begin{aligned} \Phi(\mathbb{Z}_4) &:= \Delta \cap \mathbb{Z}^+, \\ \Phi(\mathbb{Z}_2) &:= \Phi(\mathbb{Z}_4) \cup (\mathbb{Z} + x), \\ \Phi(\mathbb{Z}) &:= \Phi(\mathbb{Z}_2) \cup (\mathbb{Z} + 2x) \cup (\mathbb{Z} + 3x) \end{aligned}$$

and

$$\begin{aligned} \Psi(\mathbb{Z}_4) &:= (\Delta \cap \mathbb{Z}^+) \cup (\mathbb{Z} + x), \\ \Psi(\mathbb{Z}_2) &:= \Psi(\mathbb{Z}_4) \cup (\mathbb{Z} + 2x), \\ \Psi(\mathbb{Z}) &:= \Psi(\mathbb{Z}_2) \cup (\mathbb{Z} + 3x) \cup (\mathbb{Z} + 4x). \end{aligned}$$

We obtain the following G^+-invariant decompositions of the final interval Δ. For Φ:

$$\begin{aligned} E_1 &= \{4, 8, 12, \dots\}, & E_2 &= \{5, 9, 13, \dots\}, \\ E_3 &= \{6, 10, 14, \dots\}, & E_4 &= \{3, 7, 11, \dots\}, \\ E_5 &= \mathbb{Z}_2 + x, & E_6 &= (\mathbb{Z} + x) \setminus (\mathbb{Z}_2 + x), \\ E_7 &= \mathbb{Z} + 2x, & E_8 &= \mathbb{Z} + 3x, \\ E_9 &= (\mathbb{Z} + 4x) \cup (\mathbb{Z} + 6x) \cup \dots, & E_{10} &= (\mathbb{Z} + 5x) \cup (\mathbb{Z} + 7x) \cup \dots . \end{aligned}$$

For Ψ:

$$
\begin{aligned}
E_1 &= \{4, 8, 12, \dots\}, & E_2 &= \{5, 9, 13, \dots\}, \\
E_3 &= \{6, 10, 14, \dots\}, & E_4 &= \{3, 7, 11, \dots\}, \\
E_5 &= \mathbb{Z}_4 + x, & E_6 &= \{\dots, -3, 1, 5, \dots\} + x, \\
E_7 &= \{\dots, -2, 2, 6, \dots\} + x, & E_8 &= \{\dots, -1, 3, 7, \dots\} + x, \\
E_9 &= \mathbb{Z}_2 + 2x, & E_{10} &= (\mathbb{Z} + 2x) \setminus (\mathbb{Z}_2 + 2x), \\
E_{11} &= \mathbb{Z} + 3x, & E_{12} &= \mathbb{Z} + 4x, \\
E_{13} &= (\mathbb{Z} + 5x) \cup (\mathbb{Z} + 7x) \cup \dots, & E_{14} &= (\mathbb{Z} + 6x) \cup (\mathbb{Z} + 8x) \cup \dots .
\end{aligned}
$$

The obtained invariant decompositions can be illustrated in Figures 2 and 3.

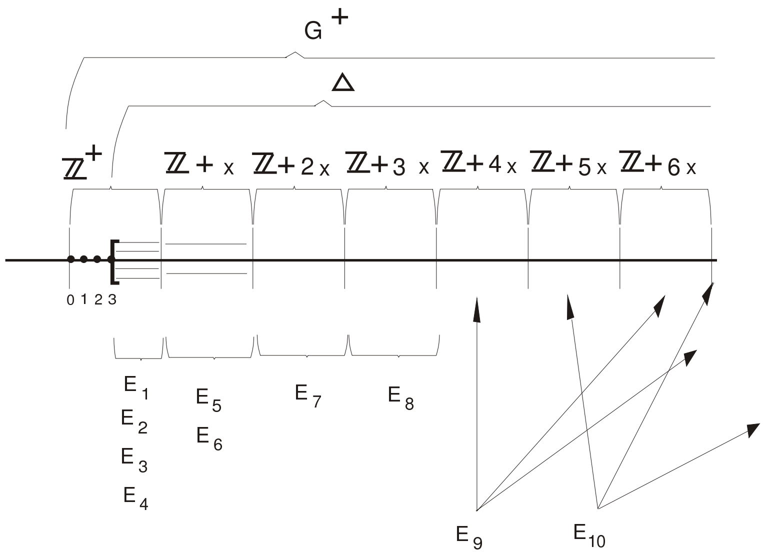

FIGURE 2. For Φ

Example 1.3.14 ([23]). Let the group $(G, +, \leq)$ and semigroup G^+ be the same as in Examples 1.3.12, 1.3.13. Let $\Delta := G$. According to Construction 1.3.4, take the chain of bounded subgroups

$$\mathbb{Z}_1 \supset \mathbb{Z}_2 \supset \mathbb{Z}_4 \supset \mathbb{Z}_8 \supset \mathbb{Z}_{16} \supset \dots \tag{1.6}$$

and the unbounded subgroup $G^* := G$. The function Φ is defined on the chain (1.6) as follows:

$$\text{if } s = 2^{k_s}, \text{ then } \Phi(\mathbb{Z}_s) := \bigcup_{k=k_s}^{\infty} [\mathbb{Z} - (k-1) \cdot x].$$

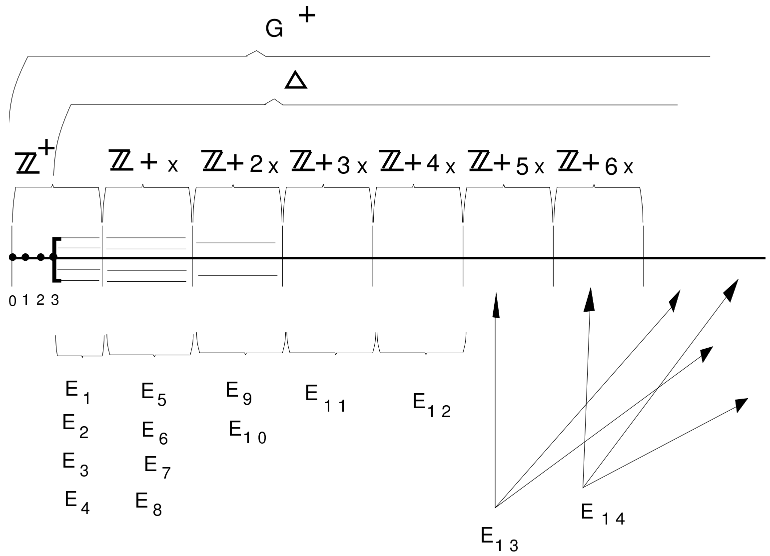

FIGURE 3. For Ψ

The only unbounded component is the set $G^+ \setminus (\mathbb{Z} \cup (\mathbb{Z} + x))$. The bounded components are cosets of the subgroup $\mathbb{Z}_s$ in G, included in the set $\mathbb{Z} - (k_s - 1) \cdot x$. The obtained invariant decomposition can be illustrated as in Figure 4.

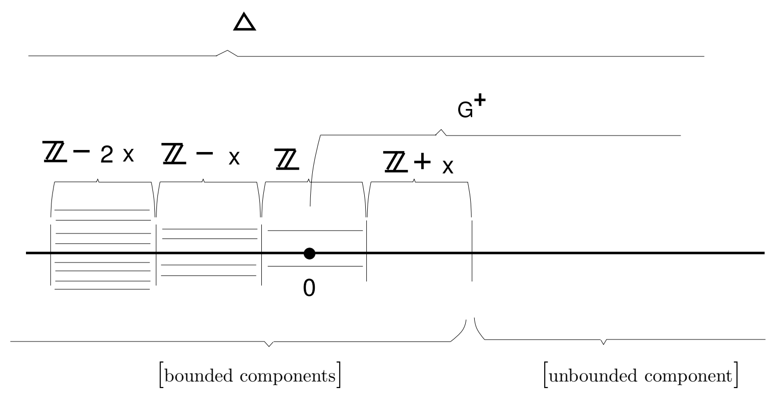

FIGURE 4

Definition 1.3.15 ([23]). An invariant decomposition of a final interval Δ of G, obtained by Construction 1.3.4, is called *simple* if the chain of bounded subgroups $\{G_s\}_{s \in S}$ in the point (1) of this construction is empty or

$$\exists s_0 \in S \ \forall s \in S, s \neq s_0 : G_{s_0} \subsetneq G_s.$$

Remark 1.3.16. Let us remark that the invariant decompositions given in Examples 1.3.12, 1.3.13 are simple, but the invariant decomposition, given in example 1.3.14, is not simple. Let us also notice that every invariant decomposition of a final interval Δ of archimedean group G, is simple.

CHAPTER 2

GENERAL CONSTRUCTIONS

2.1. Group

The general solution of the translation equation (0.1) satisfying the identity condition $F(\alpha, 1) = \alpha$ has been given by S. Łojasiewicz in [21] and independently by Z. Moszner in the paper [35]. Moreover, in the construction by Z. Moszner, the identity condition (0.2) is not supposed; namely, we have the following construction.

Construction 2.1.1.

(1) Let $f: X \to X$ be a function such that $f \circ f = f$.
(2) Let $f(X) = \bigcup_{k \in K} X_k$ be a union of nonempty disjoint sets (fibres) X_k such that for every $k \in K$ there exists a subgroup G_k of G and a bijection $g_k: G/G_k \to X_k$ where G/G_k is the set of right cosets of G_k in G.
(3) Then $F(\alpha, a) = g_k(g_k^{-1}(f(\alpha)) \cdot a)$, $f(\alpha) \in X_k$, $a \in G$.

2.2. Semigroups

The general solution $F: X \times P \to X$ of the translation equation on an arbitrary semigroup with unity $(P, \cdot, 1)$ is given in the paper [16] through the following construction.

Construction 2.2.1.

(1) Let K be an arbitrary set of indices. For every $k \in K$ we take an P-invariant decomposition $\{W_{ik}\}_{i \in I_k}$ of the semigroup P such that $\operatorname{card} I_k \leq \operatorname{card} X$.
(2) Let $h_k: \{W_{ik}\}_{i \in I_k} \to X$ be an injection and let $X_k := h_k(\{W_{ik}\}_{i \in I_k})$. Define the function $\overline{h}_k: P \to X_k$ as follows:

$$\overline{h}_k(x) := h_k(W_{ik}), \quad \text{for } x \in W_{ik}.$$

Let us assume that the functions $\overline{h}_k$ satisfy the so-called *compatibility condition*

$$\forall k_1, k_2 \in K \; \forall \alpha \in X_{k_1} \cap X_{k_2} \; \forall x \in P : \\ [\overline{h}_{k_1}(\overline{h}_{k_1}^{-1}(\{\alpha\}) \cdot x) = \overline{h}_{k_2}(\overline{h}_{k_2}^{-1}(\{\alpha\}) \cdot x)]. \tag{2.1}$$

Let X^* denote $X^* := \bigcup_{k \in K} X_k$.

(3) Let $f\colon X \to X^*$ be a surjection such that $f \circ f = f$.

(4) Let the function $g\colon X^* \to K$ satisfy the condition: $\forall \alpha \in X^* : \alpha \in X_{g(\alpha)}$.

(5) Define

$$F(\alpha, x) = \}\overline{h}_k(\overline{h}_k^{-1}(\{f(\alpha)\}) \cdot x)\{, \quad \text{for } (\alpha, x) \in X \times P \text{ where } k = g(f(\alpha)). \tag{2.2}$$

The symbol $\}A\{$ in (2.2) denotes the element of a set A when $\operatorname{card} A = 1$.

The compatibility condition (2.1) in Construction 2.2.1 has the form of a system of functional equations and involves the parameters. We quote below the constructions of some classes of solutions of the translation equation on the semigroups without the compatibility condition.

2.2.1. Monoid of natural numbers $(\mathbb{N}, \cdot)$**.** In what follows $(G, \cdot)$, $(\mathbb{N}, \cdot)$ and $(\mathbb{Q}_+, \cdot)$ denote an arbitrary monoid, the monoid of natural numbers and the group of rational positive numbers, respectively. The paper [30] includes the following theorem.

Theorem 2.2.2 ([30])**.** *Let X be a nonempty set. Let $X = \bigcup_{s \in S} X_s$ be a decomposition of X into a union of nonempty disjoint sets such that for every $s \in S$ there exists an invariant decomposition $\{E_{js}\}_{j \in J_s}$ of the monoid $(G, \cdot)$ with* $\operatorname{card} X_s = \operatorname{card} J_s$*. Let $\overline{g}_s\colon \{E_{js}\}_{j \in J_s} \to X_s$ be an arbitrary bijection and set $g_s(k) := \overline{g}_s(E_{js})$ for $k \in E_{js}$. Then the function $F\colon X \times G \to X$ defined by*

$$F(\alpha, k) = \}g_s(g_s^{-1}(\{\alpha\}) \cdot k)\{, \quad \alpha \in X_s, \; k \in G, \tag{2.3}$$

is a solution of the translation equation (0.1) *for which $F(\alpha, 1) = \alpha$.*

Proof. Let $\alpha \in X$ and $k, l \in G$. By the form of F in (2.3), by definition of g_s and by (1.1), we have the implication

$$\alpha \in X_s \Rightarrow F(\alpha, k) \in X_s, \quad \text{for } k \in G.$$

Therefore, we have

$$F(\alpha, k \cdot l) = \}g_s(g_s^{-1}(\{\alpha\}) \cdot k \cdot l)\{$$

and

$$\begin{aligned} F(F(\alpha, k), l) &= \}g_s(g_s^{-1}[\{F(\alpha, k)\}] \cdot l)\{ \\ &= \}g_s(g_s^{-1}[g_s(g_s^{-1}(\{\alpha\}) \cdot k)] \cdot l)\{ = \}g_s(g_s^{-1}(\{\alpha\}) \cdot k \cdot l)\{. \end{aligned}$$

Moreover, by definition of g_s

$$F(\alpha, 1) = \}g_s(g_s^{-1}(\{\alpha\}) \cdot 1)\{ = \}\{\alpha\}\{ = \alpha. \qquad \square$$

Remark 2.2.3. When the monoid $(G, \cdot\,)$ is the group, then Theorem 2.2.2 yields all solutions of the translation equation (0.1) satisfying $F(\alpha, 1) = \alpha$. In this case the invariant decompositions consist of right cosets of some subgroup G_s of G (see Remark 1.1.4), $\overline{g}_s$ is equal to g_s and $g_s\colon G/G_s \to X_s$. The general solution of the translation equation (0.1) satisfying $F(\alpha, 1) = \alpha$ has been given also in [37] in the Construction 2.1.1 with $f = \mathrm{id}_X$.

Using Theorem 2.2.2 we can obtain examples of solutions for $(G, \cdot\,) = (\mathbb{N}, \cdot\,)$.

Example 2.2.4 ([30])**.** Let $X := \,]1/4, 1]$ and take $S := \,]1/2, 1]$, $X_s := \{s/2, s\}$, $J_s := \{1, 2\}$ for $s \in S$. Moreover, $E_{1s} := \{1, 3, 5, \dots\}$, $E_{2s} := \{2, 4, \dots\}$ for $s \in S$. Define $\overline{g}_s(E_{1s}) := s/2$, $\overline{g}_s(E_{2s}) := s$ for $s \in S$. In that case we get the following solution:

$$F(\alpha, k) = \begin{cases} 2\alpha & \text{for } \alpha \in \,]1/4, 1/2],\ \ k \in \{2, 4, 6, \dots\}, \\ \alpha & \text{for } \alpha \in \,]1/2, 1],\ k \in \mathbb{N} \text{ or } \alpha \in]1/4, 1/2],\, k \in \{1, 3, 5, \dots\}. \end{cases}$$

Example 2.2.5 ([30])**.** Let X, S, $\{X_s\}$, J_s for $s \in S$ be the same as in Example 2.2.4. We take $E_{1s} := \{1\}$, $E_{2s} := \mathbb{N} \setminus \{1\}$ for $s \in S$. The functions $\overline{g}_s$ are defined the same as in Example 2.2.4. In that case we get the following solution:

$$F(\alpha, k) = \begin{cases} 2\alpha & \text{for } \alpha \in \,]1/4, 1/2],\, k \in \mathbb{N} \setminus \{1\}, \\ \alpha & \text{for } \alpha \in \,]1/2, 1],\ k \in \mathbb{N} \text{ or } \alpha \in \,]1/4, 1/2],\, k = 1. \end{cases}$$

Example 2.2.6 ([30])**.** Let $X := [0, +\infty[$, $S := [0, 1[$, $X_s := \{s + j : j = 0, 1, 2, \dots\}$, $J_s := \mathbb{N} \cup \{0\}$ for $s \in S$. Moreover, $E_{0s} := \{2, 4, 6, \dots\}$ and $E_{js} := \{2j - 1\}$ for $j \in \mathbb{N}$ and $s \in S$. Define $\overline{g}_s(E_{js}) := s + j$ for $j \in \mathbb{N} \cup \{0\}$ and $s \in S$. Now we get the solution:

$$F(\alpha, k) = \begin{cases} \alpha - E(\alpha) & \text{for } \alpha \in X \setminus [0, 1[\\ & \text{and } k \in \{2, 4, 6, \dots\} \\ & \text{or } \alpha \in [0, 1[\text{ and } k \in \mathbb{N}, \\ \alpha + E(\alpha)(k - 1) - (k - 1)/2 & \text{for } \alpha \in X \setminus [0, 1[\\ & \text{and } k \in \{1, 3, 5, \dots\}, \end{cases}$$

where $E(\alpha)$ denotes the integer part of α.

Example 2.2.7. Let $X := [0,5[$, $S := [0,1[$, $X_s := \{s+j : j = 0,1,2,3,4\}$, $J_s := \{0,1,2,3,4\}$ for $s \in S$. Moreover, $E_{0s} := \{1\}$, $E_{1s} := \{2\}$, $E_{2s} := \{3\}$, $E_{3s} := \{4\}$ and $E_{5s} := \{5,6,7,\dots\}$, for $s \in S$. Define $\overline{g}_s(E_{js}) := s + j$ for $j \in \{0,1,2,3,4\}$ and $s \in S$. We get the following solution:

$$F(\alpha,k) = \begin{cases} \alpha & \text{if } E(\alpha) \in \{0,1,2,3\} \text{ and } k = 1 \\ & \text{or } E(\alpha) = 4 \text{ and } k \in \mathbb{N}, \\ \alpha - E(\alpha) + 4 & \text{if } E(\alpha) \in \{0,1\} \text{ and } k \geq 5 - 2E(\alpha) \\ & \text{or } E(\alpha) \in \{2,3\} \text{ and } k \geq 2, \\ \alpha + E(\alpha) + k - 1 & \text{if } E(\alpha) \in \{0,1\} \text{ and } 1 < k < 5 - 2E(\alpha). \end{cases}$$

Remark 2.2.8 ([30])**.** If the solution of equation (0.1) is trivial, that is $F(\alpha,k) := \alpha$ for every $(\alpha,k) \in X \times \mathbb{N}$ where X denotes an arbitrary nonempty set, then the invariant decomposition of $\mathbb{N}$ has exactly one element $\{\mathbb{N}\}$, the set X is decomposed into singletons and $\overline{g}_s(\mathbb{N}) := s$.

Remark 2.2.9 ([30])**.** The function $F(\alpha,k) := k \cdot \alpha$ for $(\alpha,k) \in X \times \mathbb{N}$ and $X :=]0,\infty[$ is a solution of the translation equation (0.1). This solution is not of the form (2.3) (see remark 2.2.12).

Theorem 2.2.10 ([30])**.** *Let $X \subset \mathbb{R}$ be an arbitrary nonempty set. Suppose that a solution $F\colon X \times \mathbb{N} \to X$ of the translation equation* (0.1) *satisfying $F(\alpha,1) = \alpha$ for $\alpha \in X$ can be extended to a solution $\overline{F}\colon X \times \mathbb{Q}_+ \to X$ of this equation. Then there exists a family $\{X_s\}_{s\in S}$ of disjoint sets such that $\bigcup_{s\in S} X_s = X$ and for every $s \in S$ there exists a subgroup $\mathbb{Q}_s \leq \mathbb{Q}_+$ and a bijection $g_s\colon \mathbb{Q}_+/\mathbb{Q}_s \to X_s$ for which*

$$F(\alpha,k) = g_s(g_s^{-1}(\alpha)\cdot k), \quad \alpha \in X_s,\ k \in \mathbb{N}.$$

Theorem 2.2.11 ([30])**.** *Let $X \subset \mathbb{R}$ be an arbitrary interval. A function $F\colon X \times \mathbb{N} \to X$ is a solution of the translation equation* (0.1) *such that for every $\alpha \in X$ the function $F(\alpha,\cdot\,)$ is increasing and for every $k \in \mathbb{N}$ the function $F(\,\cdot\,,k)$ is increasing and surjective if and only if there exists a family $\{X_s\}_{s\in S}$ of disjoint sets such that $\bigcup_{s\in S} X_s = X$ and there exists a family of increasing bijections $g_s\colon \mathbb{Q}_+ \to X_s$, $s \in S$ such that*

$$F(\alpha,k) = g_s(g_s^{-1}(\alpha)\cdot k), \quad \alpha \in X_s,\ k \in \mathbb{N}.$$

Remark 2.2.12 ([30])**.** If $F\colon X \times \mathbb{N} \to X$ satisfies the assumptions of Theorem 2.2.11, then F cannot be obtained by means of Theorem 2.2.2. Indeed, otherwise let $g_s(1) =: \alpha_0$ for some $s \in S$. Then $X_s = \{F(\alpha_0,k) : k \in \mathbb{N}\} = g_s(\mathbb{N})$ and $\alpha_0 \in X_s$. Let $\overline{F}\colon X \times \mathbb{Q}_+ \to X$ be an extension of the solution F. Since

$\overline{F}(\alpha_0, 1/2) < F(\alpha_0, k)$ for $k \in \mathbb{N}$, we have $\overline{F}(\alpha_0, 1/2) \notin X_s$. Let $\overline{F}(\alpha_0, 1/2) \in X_t$, $t \neq s$. Hence,

$$F\left(\overline{F}\left(\alpha_0, \frac{1}{2}\right), 2\right) = F(\alpha_0, 1) = \alpha_0,$$

so $\alpha_0 \in X_t$, which contradicts the relation $X_t \cap X_s = \emptyset$.

Remark 2.2.13 ([30])**.** Let $X := [0, +\infty[$ and define $F\colon X \times \mathbb{N} \to X$ by

$$F(\alpha, k) = \begin{cases} \alpha & \text{for } \alpha \in X,\ k = 1, \\ 1 & \text{for } \alpha \in [0,1],\ k \in \mathbb{N} \setminus \{1\}, \\ k\alpha & \text{for } \alpha \in X \setminus [0,1],\ k \in \mathbb{N} \setminus \{1\}. \end{cases}$$

Then F is a solution of (0.1), which cannot be extended to a solution $\overline{F}\colon X \times \mathbb{Q}_+ \to X$ and is not of the form (2.3). Indeed, for every solution $\overline{F}\colon X \times \mathbb{Q}_+ \to X$ of (0.1) satisfying $\overline{F}(\alpha, 1) = \alpha$, all functions $\overline{F}(\,\cdot\,, k)$ ought to be bijections. But

$$F\left(\frac{1}{2}, 2\right) = 1 = F\left(\frac{3}{4}, 2\right),$$

therefore F cannot be extended to a solution $\overline{F}\colon X \times \mathbb{Q}_+ \to X$. Moreover, by Theorem 2.2.2, $\operatorname{card} X_s = \operatorname{card} J_s$ for $s \in S$. It is easy to see that for the solution F one of the elements of the family $\{X_s\}_{s \in S}$ is the set $X_n = [0,1]$ for some $n \in S$. This implies the following contradiction:

$$\mathfrak{c} = \operatorname{card}[0,1] = \operatorname{card} J_n \leq \operatorname{card}\mathbb{N} = \aleph_0.$$

2.2.2. Semigroup of positive elements of linearly ordered and archimedean group. Let X be an arbitrary set and G^+ a semigroup of positive elements of a linearly ordered and archimedean, so abelian, group $(G, +, 0, \leq)$. The Construction 2.2.16 presented below describing all solutions of the translation equation (0.1) where $F\colon X \times G^+ \to X$, for which the almost fibres $F_\alpha = F(\{\alpha\} \times G^+)$ satisfy the condition

$$\forall \alpha, \beta \in X : F_\alpha \cap F_\beta \neq \emptyset \Rightarrow F_\alpha \subset F_\beta \text{ or } F_\beta \subset F_\alpha, \tag{2.6}$$

is cited from the paper [31].

Definition 2.2.14 ([23])**.** Let us define the operation "$\oplus$" by the following way: if $\{E_k\}_{k \in K}$ is an invariant decomposition of the final interval Δ of the group G, we have

$$\forall k \in K\ \forall x \in G^+ : (E_k \oplus x = E_l \Leftrightarrow E_k + x \subseteq E_l).$$

Theorem 2.2.15 (see [31]). *The function $F\colon X \times G^+ \to X$ is a solution of the translation equation* (0.1) *satisfying the condition* (2.6) *if and only if F is obtained by the following construction.*

Construction 2.2.16.

(1) Let $f\colon X \to X$ be a function such that

$$\forall \alpha \in X : f(f(\alpha)) = f(\alpha).$$

(2) Let $f(X) = \bigcup_{k\in K} X_k$ be a union of nonempty disjoint sets X_k such that for every $k \in K$ there exists an invariant decomposition $\{E_{jk}\}_{j\in J_k}$ of a final interval Δ_k of G for which $G^+ \subseteq \Delta_k$ and $\operatorname{card} J_k = \operatorname{card} X_k$.

(3) Let $h_k\colon \{E_{jk}\}_{j\in J_k} \to X_k$ be a bijection for $k \in K$.

(4) Define

$$F(\alpha, x) = h_k(h_k^{-1}(f(\alpha)) \oplus x), \quad \text{for } f(\alpha) \in X_k,\ x \in G^+. \tag{2.7}$$

Remark 2.2.17. The Construction 2.2.16 is a special case of the more complex construction in [23] where it is not supposed that G is archimedean (see Theorem 2.2.32 and the Construction 2.2.33, below). The analogue of Theorem 2.2.15 is also given in [38] with the proof different and more complicated, than the proof given in [31].

Remark 2.2.18. Let us remark that if we define the functions $\overline{h}_k\colon \Delta_k :\to X_k$ by the formula

$$\overline{h}_k(x) := h_k(W), \quad \text{when } x \in W \in \{E_{jk}\}_{j\in J_k}, \tag{2.8}$$

we can replace (2.7) by

$$F(\alpha, x) := \}\overline{h}_k((\overline{h}_k)^{-1}(\{f(\alpha)\}) + x)\{, \quad \text{for } f(\alpha) \in X_k,\ x \in G^+. \tag{2.9}$$

and the formula (2.9) represents the form of solutions exactly the same as in the paper [38].

We will give a solution of the translation equation (0.1) by Construction 2.2.16.

Example 2.2.19 ([31]). Put $X = [0, +\infty[$, $(G, +, \leq) = (\mathbb{R}_+ =]0, +\infty[, \cdot, \leq)$, $f = \mathrm{id}_{\mathbb{R}_+}$, $K = \{1, 2\}$, $X_1 = [0, 1]$, $X_2 =]1, +\infty[$, $G^+ = [1, +\infty[$, $\Delta_1 = [1, +\infty[$, $\Delta_2 =]1/2, +\infty[$. Let us take the following invariant decompositions of the intervals Δ_1 and Δ_2: $\Delta_1 = \bigcup_{w\in[1,2[}\{w\} \cup [2, +\infty[$ at $\Delta_2 = \bigcup_{w\in\Delta_2}\{w\}$. Define $h_1(\{w\}) := w - 1$, for $w \in [1, 2[$, $h_1([2, +\infty[) := 1$ and $h_2(\{w\}) := 2w$. The function $H(\alpha, x) = h_k(h_k^{-1}(f(\alpha)) \odot x)$, for $\alpha \in X_k$, $x \in G^+$ is a solution of (0.1) given by Construction 2.2.16.

Remark 2.2.20 ([31])**.** If the solution $F: X \times G^+ \to X$ of (0.1) can be extended to the solution $F^*: X \times G \to X$ of (0.1), F can be obtained by Construction 2.2.16 because the condition (2.6) is fulfilled. The converse implication is not true. For the solution H in Example 2.2.19, we have: $H(0,1) = 0$ and $H(0,x) \neq 0$ for $x \neq 1$. Moreover, if an extension of H to the solution $H^*: X \times \mathbb{R}_+ \to X$ of (0.1) could exist, we would conclude by $H(0,2) = H(0,3)$ that $0 = H(0,1) = H^*(H(0,2), 1/2) = H^*(H(0,3), 1/2)$, whence $0 = H(0, 3/2)$, which leads to a contradiction.

Remark 2.2.21 ([31])**.** The solution $F: X \times \mathbb{N} \to X$ of the translation equation where $X := [0, \infty[$, given in Remark 2.2.13 by the formula

$$F(\alpha, k) = \begin{cases} \alpha & \text{for } \alpha \in X,\ k = 1, \\ 1 & \text{for } \alpha \in [0,1],\ k \in \mathbb{N} \setminus \{1\}, \\ k\alpha & \text{for } \alpha \in X \setminus [0,1],\ k \in \mathbb{N} \setminus \{1\} \end{cases}$$

cannot be obtained by theorems 2.2.2, 2.2.10, 2.2.11. For this solution the condition (2.6) is not fulfilled, but the solution is the restriction of the solution $H: X \times [1, +\infty[\to X$ obtained by Construction 2.2.16 in Example 2.2.19.

Remark 2.2.22 ([31])**.** Every function $F: X \times M \to X$ where M is a monoid, such that $F(\alpha, 1) = \alpha$, for $\alpha \in X$, which can be extended to a solution $F^*: X \times G \to X$ where $(G, \cdot)$ is a group being an extension of the monoid $(M, \cdot)$, is a restriction to the set $X \times M$ of the solution H received by Construction 2.2.16. It is true since the function F^* fulfills the condition (2.6).

Example 2.2.23 ([31])**.** Let $F: X \times \mathbb{N} \to X$ where $X := [0,1]$, is a solution of (0.1) given by the following formula

$$F(\alpha, k) = \begin{cases} \alpha & \text{for } k \in 2\mathbb{N} - 1, \\ 1 & \text{for } k \in 2\mathbb{N}. \end{cases}$$

For this solution the condition (2.6) is not fulfilled. This solution cannot be obtained as restriction of a solution $H: X \times \mathbb{Q}_+ \to X$ of (0.1) given by Construction 2.2.16, since there are no proper subgroup of the group $(\mathbb{Q}_+, \cdot)$ of positive rational numbers containing the set $2\mathbb{N} - 1$, for which an equivalence class includes the set $2\mathbb{N}$. Since every group G being an extension of the monoid $(\mathbb{N}, \cdot)$ ought to contain a group isomorphic to $\mathbb{Q}_+$, we have the same conclusion for a solution $H: X \times G \to X$.

In the next example we shall present a simple solution of equation (0.1) for which the condition (2.6) is not satisfied; therefore, this solution cannot be obtained by Construction 2.2.16.

Example 2.2.24 ([31])**.** Let $(\mathbb{N}_0, +)$ be a semigroup of positive integers of additive group of integers and let $X = \{\alpha_1, \alpha_2, \alpha_3\}$ where α_1, α_2, α_3 are different elements. The function $F\colon X \times \mathbb{N}_0 \to X$ defined by $F(\alpha_i, 0) = \alpha_i$ and $F(\alpha_i, n) = \alpha_3$ for $i = 1, 2, 3$ and $n \neq 0$ is a solution of (0.1) which does not fulfill (2.6). Indeed, $F(\{\alpha_1\} \times \mathbb{N}_0) = \{\alpha_1, \alpha_3\}$, $F(\{\alpha_2\} \times \mathbb{N}_0) = \{\alpha_2, \alpha_3\}$.

In the paper [43], some generalization of the Construction 2.2.16 is considered. The author investigates the condition for solutions $F\colon X \times G^+ \to X$ generalizing (2.6), namely

$$\begin{aligned}(2.10)\qquad &\forall \alpha, \beta \in X : \{F_\alpha \cap F_\beta \neq \emptyset \Rightarrow (F_\alpha \subset F_\beta \text{ or } F_\beta \subset F_\alpha) \quad \text{or} \\ &\forall \delta \in X : [F_\delta \cap F_\alpha \neq \emptyset \Rightarrow \ (F_\delta \subset F_\alpha \text{ or } F_\alpha \subset F_\delta \text{ or } F_\delta \subset F_\beta \text{ or } F_\beta \subset F_\delta)]\}.\end{aligned}$$

We will give examples of solutions satisfying the condition (2.10) and not satisfying the condition (2.6).

Example 2.2.5 ([43])**.** Define $H\colon [0, 2] \times \mathbb{R}^+ \to [0, 2]$ as follows:

$$H(\alpha, x) = \begin{cases} \alpha + x & \text{if } \alpha \in [0, 1] \text{ and } x \in [0, 1 - \alpha], \\ 1 & \text{if } (\alpha \in [0, 1] \text{ and } x \geq 1 - \alpha) \text{ or } (\alpha \in \,]1, 2] \text{ and } x \geq \alpha - 1), \\ \alpha - x & \text{if } \alpha \in \,]1, 2] \text{ and } x \in [0, \alpha - 1]. \end{cases}$$

We have $H_0 = [0, 1]$, $H_2 = [1, 2]$, so $H_0 \cap H_2 \neq \emptyset$ and $\neg H_0 \subset H_2$ and $\neg H_2 \subset H_0$. The fulfillment of the condition (2.10) is evident.

Example 2.2.26. The following more complex function $H\colon [0, 4] \times \mathbb{R}^+ \to [0, 4]$ is also an example of the solution which satisfies the condition (2.10) and does not satisfy the condition (2.6).

$$H(\alpha, x) = \begin{cases} \alpha + x & \text{if } \alpha \in \,]2, 3] \text{ and } x \in [0, 3 - \alpha], \\ 3 & \text{if } (\alpha \in \,]2, 3] \text{ and } x \geq 3 - \alpha) \text{ or } (\alpha \in \,]3, 4] \text{ and } x \geq \alpha - 3), \\ \alpha - x & \text{if } \alpha \in \,]3, 4] \text{ and } x \in [0, \alpha - 3], \\ 2 & \text{if } \alpha = 2 \text{ and } x \in \mathbb{R}^+, \\ \alpha + x & \text{if } \alpha \in [0, 1] \text{ and } x \in [0, 1 - \alpha], \\ 1 & \text{if } (\alpha \in [0, 1] \text{ and } x \geq 1 - \alpha) \text{ or } (\alpha \in]1, 2] \text{ and } x \geq \alpha - 1), \\ \alpha - x & \text{if } \alpha \in]1, 2[\text{ and } x \in [0, \alpha - 1]. \end{cases}$$

The Theorem 2.2.27 presented below is the reformulation of Theorem 2.7 given in [43].

Theorem 2.2.27. *The function $F\colon X \times G^+ \to X$ is a solution of the translation equation* (0.1) *satisfying the condition* (2.10) *if and only if F is obtained by the following construction.*

Construction 2.2.28.

(1) Let $f: X \to X$ be a function such that

$$\forall \alpha \in X : f(f(\alpha)) = f(\alpha).$$

(2) Let $f(X) = \bigcup_{k \in K} X_k$ be a union of nonempty disjoint sets X_k such that for every $k \in K$:

(a) there exist: an invariant decomposition $\{E_{jk}\}_{j \in J_k}$ of a final interval Δ_k^0 of G, for which $G^+ \subseteq \Delta_k^0$ and an injection $h_k^0: \{E_{jk}\}_{j \in J_k} \to X_k$ (let X_k^0 denote: $X_k^0 := h_k^0(\{E_{jk}\}_{j \in J_k})$),

(b) there exist: interval $A \subset G$ which is not final or $A = \emptyset$ and a bijection $\varphi_k: \mathcal{A} \to X_k \setminus X_k^0 =: X_k^1$ where $\mathcal{A} := \{\{x\} : x \in A\}$.

(3) Let Δ_k^1 denote:

$$\Delta_k^1 := \begin{cases} A \cup \{x \in G : \forall u \in A : x \geq u\} & \text{when } A \neq \emptyset, \\ \Delta_k^0 & \text{when } A = \emptyset. \end{cases}$$

Choose $t \in G^+$ such that $(\Delta_k^1 \setminus A) + t \subset \Delta_k^0$ and define

$$\mathcal{V} := \{W \in \{E_{jk}\}_{j \in J_k} : W \cap (\Delta_k^1 \setminus A) \neq \emptyset\},$$

(observe that we can state that the family of sets $\mathcal{A} \cup \mathcal{V}$ forms an invariant decomposition of Δ_k^1) and define the function $h_k^1: \mathcal{A} \cup \mathcal{V} \to X_k$

$$h_k^1(W) := \begin{cases} \varphi_k(W) & \text{if } W \in \mathcal{A}, \\ h_k^0(W \oplus t) & \text{if } W \in \mathcal{V}. \end{cases}$$

(4) Define

$$F(\alpha, x) := h_k^i((h_k^i)^{-1}(f(\alpha)) \oplus x), \quad \text{for } f(\alpha) \in X_k^i,\ i \in \{0, 1\}. \tag{2.11}$$

Remark 2.2.29. Since the paper [43] is not published, we will present below a proof of Theorem 2.2.27. The presented proof differs from the original one and it seems to be simpler. Let us remark firstly that if we define the functions $\overline{h}_k^i, \Delta_k^i : \to X_k$ for $i \in \{0, 1\}$ by the formula

$$\overline{h}_k^i(x) := h_k^i(W), \quad \text{when } x \in W, \tag{2.12}$$

then we can replace (2.11) by

$$F(\alpha, x) := \}\overline{h}_k^i((\overline{h}_k^i)^{-1}(\{f(\alpha)\}) + x)\{, \quad \text{for } f(\alpha) \in X_k^i,\ i \in \{0, 1\}. \tag{2.13}$$

and the formula (2.13) represents the form of solutions exactly the same as in the paper [43].

Proof. ($\Leftarrow$) Let us suppose that the function F has the form (2.11) with parameters described by Construction 2.2.28. We will verify the statement that

F satisfies the translation equation (0.1) and the condition (2.10). Let $\alpha \in X$, $x, y \in G^+$. Let us consider two cases:

(a) $f(\alpha) \in X_k^0$,
(b) $f(\alpha) \in X_k^1$.

Ad (a). In this case we have

$$F(\alpha, x+y) = h_k^0((h_k^0)^{-1}(f(\alpha)) \oplus (x+y))$$

and

$$F(F(\alpha, x), y) = h_k^0((h_k^0)^{-1}(f[h_k^0((h_k^0)^{-1}(f(\alpha)) \oplus x)]) \oplus y),$$

hence,

$$F(F(\alpha, x), y) = h_k^0((h_k^0)^{-1}(f(\alpha)) \oplus (x+y)),$$

so

$$F(\alpha, x+y) = F(F(\alpha, x), y).$$

Ad (b). Let us denote: $(h_k^1)^{-1}(f(\alpha)) =: \{a\ \} \in \mathcal{A}$. We will consider the following subcases.

(i) $\{a\ \} \oplus (x+y) = \{b\} \in \mathcal{A}$,
(ii) $\{a\ \} \oplus x = \{b\} \in \mathcal{A}$, and $\{a\ \} \oplus (x+y) = \{b\} \oplus y = W \in \mathcal{V}$,
(iii) $\{a\ \} \oplus x = W \in \mathcal{V}$.

Ad (i). We have

$$F(\alpha, x+y) = h_k^1((h_k^1)^{-1}(f(\alpha)) \oplus (x+y)) = h_k^1(\{a\ \} \oplus (x+y)) = h_k^1(\{b\}) = \varphi_1(\{b\})$$

and

$$\begin{aligned} F(F(\alpha, x), y) &= h_k^1((h_k^1)^{-1}(f[h_k^1((h_k^1)^{-1}(f(\alpha)) \oplus x)]) \oplus y) = h_k^1((\{a\ \} \oplus x) \oplus y) \\ &= h_k^1(\{a\ \} \oplus (x+y)) = h_k^1(\{b\}) = \varphi_1(\{b\}) = F(\alpha, x+y). \end{aligned}$$

Ad (ii). We have

$$\begin{aligned} F(\alpha, x+y) &= h_k^1((h_k^1)^{-1}(f(\alpha)) \oplus (x+y)) \\ &= h_k^1(\{a\ \} \oplus (x+y)) = h_k^1(W) = h_k^0(W \oplus t) \end{aligned}$$

and

$$\begin{aligned} F(F(\alpha, x), y) &= h_k^1((h_k^1)^{-1}(f[h_k^1((h_k^1)^{-1}(f(\alpha)) \oplus x)]) \oplus y) = h_k^1(\{b\} \oplus y) \\ &= h_k^1(W) = h_k^0(W \oplus t) = F(\alpha, x+y). \end{aligned}$$

Ad (iii). We have

$$\begin{aligned} F(\alpha, x+y) &= h_k^1(\{a\ \} \oplus (x+y)) = h_k^1(W \oplus y) = h_k^0(W \oplus y \oplus t), \\ F(F(\alpha, x), y) &= h_k^1(\{a\ \} \oplus x \oplus y) = h_k^1(W \oplus y) = h_k^0(W \oplus y \oplus t). \end{aligned}$$

Now, we will verify the statement that the solution (2.11) satisfies the condition (2.10). Let us suppose that $\alpha, \beta \in X$ and $F_\alpha \cap F_\beta \neq \emptyset$. By (2.11) we easily conclude that $f(\alpha), f(\beta) \in X_k$ for a certain $k \in K$. Let us consider the following cases:

(a) $f(\alpha), f(\beta) \in X_k^0$,
(b) $f(\alpha), f(\beta) \in X_k^1$,
(c) $f(\alpha) \in X_k^1, f(\beta) \in X_k^0$,
(d) $f(\alpha) \in X_k^0, f(\beta) \in X_k^1$.

Ad (a). There exist $E_{j_1k}, E_{j_2k} \in \{E_{jk}\}_{j\in J_k}$ such that $h_k^0(E_{j_1k}) = f(\alpha)$ and $h_k^0(E_{j_2k}) = f(\beta)$. We have, for a certain $x \in G^+$, two possible subcases:

(i) $E_{j_1k} \oplus x = E_{j_2k}$,
(ii) $E_{j_2k} \oplus x = E_{j_1k}$.

Ad (i). We have

$$f(\beta) = h_k^0(E_{j_2k}) = h_k^0(E_{j_1k} \oplus x) = h_k^0((h_k^0)^{-1}(f(\alpha)) \oplus x) = F(\alpha, x).$$

Hence,

$$\forall u \in G^+ : F(\beta, u) = F(f(\beta), u) = F(F(\alpha, x), u) = F(\alpha, x + u),$$

therefore $F_\beta \subset F_\alpha$.

Ad (ii). By the similar reasoning we conclude that $F_\alpha \subset F_\beta$.

Ad (b). In this case, there exist $\{a\ \} \in \mathcal{A}$, $\{b\} \in \mathcal{A}$ such that $h_k^1(\{a\ \}) = f(\alpha)$ and $h_k^1(\{b\}) = f(\beta)$. Furthermore, the reasoning is similar to one in the case (a). Therefore, $F_\alpha \subset F_\beta$ or $F_\beta \subset F_\alpha$.

Ad (c). Let δ be such that $F_\delta \cap F_\alpha \neq \emptyset$. Since, by (2.11), $\delta \in X_k$, then we have two possibilities

(i) $f(\delta) \in X_k^0$ or
(ii) $f(\delta) \in X_k^1$.

In the first case by the reasoning similar to one in (a), we get $F_\alpha \subset F_\delta$ or $F_\delta \subset F_\alpha$. In the second case similarly we get $F_\beta \subset F_\delta$ or $F_\delta \subset F_\beta$.

Ad (d). In this case the reasoning is similar.

The proof of the ($\Leftarrow$) part is finished.

($\Rightarrow$) Let F be a solution of of the equation (0.1) satisfying the condition (2.10). Let $f(\alpha) := F(\alpha, 0)$, for $\alpha \in X$. We have

$$f(f(\alpha)) = F(F(\alpha, 0), 0) = F(\alpha, 0) = f(\alpha).$$

Since

$$F(\alpha, x) = F(F(\alpha, x), 0) \in F(X \times \{0\}),$$

then

$$f(X) = F(X \times \{0\}) = F(X \times G^+).$$

Remark that $F(\alpha, 0) = \alpha$, for $\alpha \in f(X)$.

Let us distinguish for $\alpha \in F(X, G^+)$, the family of sets

$$\{E_\alpha(\beta)\}_{\beta \in F_\alpha} \quad \text{where } E_\alpha(\beta) := \{x \in G^+ : F(\alpha, x) = \beta\}$$

and define

$$G_\alpha := E_\alpha(\alpha) \cup E_\alpha^{-1}(\alpha) \quad \text{where } E_\alpha^{-1}(\alpha) := \{x \in G : -x \in E_\alpha(\alpha)\}.$$

It is evident that the sets $\{E_\alpha(\beta)\}_{\beta \in F_\alpha}$ are disjoint, nonempty and $G^+ = \bigcup_{\beta \in F_\alpha} E_\alpha(\beta)$. Let $\beta \in F_\alpha$ and $x \in G^+$. Put $\gamma = F(\beta, x)$. We have

$$\forall y \in E_\alpha(\beta) : F(\alpha, y + x) = F(F(\alpha, y), x) = F(\beta, x) = \gamma,$$

then $E_\alpha(\beta) + x \subseteq E_\alpha(\gamma)$. Therefore, the family $\{E_\alpha(\beta)\}_{\beta \in F_\alpha}$, for every $\alpha \in F(X, G^+)$, forms an invariant decomposition of semigroup G^+. Moreover, G_α is a subgroup of G. Indeed, since $\alpha \in F(X, G^+)$, $F(\alpha, 0) = \alpha$ and $0 \in G_\alpha$. Let $x, y \in G_\alpha$. We will verify the statement that $x - y \in G_\alpha$ considering all possibilities.

(1) If $x, y \in E_\alpha(\alpha)$, then:

- (i) if $x - y \in G^+ \cap E_\alpha(\beta)$, for a certain $\beta \in F_\alpha$, then, by virtue of $0 \in E_\alpha(\alpha)$, we have $E_\alpha(\alpha) + x - y \subseteq E_\alpha(\beta)$. Moreover, since $y \in E_\alpha(\alpha)$, then $x \in E_\alpha(\beta)$. Therefore, $E_\alpha(\alpha) = E_\alpha(\beta)$ and hence $x - y \in E_\alpha(\alpha) \subseteq G_\alpha$.
- (ii) if $y - x \in G^+ \cap E_\alpha(\beta)$, for a certain $\beta \in F_\alpha$, one can prove similarly that $E_\alpha(\alpha) = E_\alpha(\beta)$ and hence $x - y \in G_\alpha$.

(2) If $x \in E_\alpha(\alpha)$ and $y \in E_\alpha^{-1}(\alpha)$, then $E_\alpha(\alpha) + x \subseteq E_\alpha(\alpha)$ and $E_\alpha(\alpha) - y \subseteq E_\alpha(\alpha)$. Hence $E_\alpha(\alpha) + x - y \subseteq E_\alpha(\alpha)$, so $x - y \in E_\alpha(\alpha) \subseteq G_\alpha$.

(3) If $x \in E_\alpha^{-1}(\alpha)$ and $y \in E_\alpha(\alpha)$, then we have similarly $E_\alpha(\alpha) + y \subseteq E_\alpha(\alpha)$ and $E_\alpha(\alpha) - x \subseteq E_\alpha(\alpha)$. Hence $E_\alpha(\alpha) + y - x \subseteq E_\alpha(\alpha)$, so $y - x \in E_\alpha(\alpha)$ and $x - y \in G_\alpha$.

(4) If $x \in E_\alpha^{-1}(\alpha)$ and $y \in E_\alpha^{-1}(\alpha)$, then:

- (i) if $x - y \in G^+$, then $E_\alpha(\alpha) + x - y \subseteq E_\alpha(\alpha)$, hence $x - y \in E_\alpha(\alpha) \subseteq G_\alpha$.
- (ii) if $y - x \in G^+$, then similarly $E_\alpha(\alpha) + y - x \subseteq E_\alpha(\alpha)$, hence $y - x \in E_\alpha(\alpha)$ and $x - y \in G_\alpha$.

The relation "$\sim$" in $F(X \times G^+)$ defined as follows:

$$\alpha \sim \beta \Leftrightarrow F_\alpha \cap F_\beta \neq \emptyset \tag{2.14}$$

is an equivalence relation. Indeed, evidently this relation is reflexive and symmetrical. We will verify that the relation is transitive. Let us suppose that $\alpha, \beta, \gamma \in F(X \times G^+)$, $F_\alpha \cap F_\beta \neq \emptyset$ and $F_\beta \cap F_\gamma \neq \emptyset$. There exist $x_1, x_2, y_1, y_2 \in G^+$ such that $F(\alpha, x_1) = F(\beta, x_2)$ and $F(\beta, y_1) = F(\gamma, y_2)$. If $y_1 - x_2 \in G^+$, we have

$$\begin{aligned} F(\gamma, y_2) &= F(\beta, y_1) = F(\beta, x_2 + y_1 - x_2) = F(F(\beta, x_2), y_1 - x_2) \\ &= F(F(\alpha, x_1), y_1 - x_2) = F(\alpha, x_1 + y_1 - x_2), \end{aligned}$$

this means $F_\gamma \cap F_\alpha \neq \emptyset$. If $x_2 - y_1 \in G^+$, the reasoning is the same.

Let K denote a selection of the set $F(X \times G^+)/\sim$ and let X_k be the class for which $k \in K \cap X_k$. Therefore $F(X \times G^+)/\sim = \{X_k\}_{k \in K}$.

Let $k \in K$. Let us consider two cases:

(A) $\forall \alpha \in X_k : G_\alpha \neq \{0\}$,
(B) $\exists \alpha_0 \in X_k : G_{\alpha_0} = \{0\}$.

Ad (A). Fix $\alpha_0 \in X_k$. We will prove that $G_\alpha = G_{\alpha_0}$ for all $\alpha \in X_k$.

Indeed, firstly, let us observe that for an arbitrary $\alpha, \gamma \in X_k$, every component of the invariant decomposition $\{E_\alpha(\beta)\}_{\beta \in F_\alpha}$ is a singleton if and only if the invariant decomposition $\{E_\gamma(\beta)\}_{\beta \in F_\gamma}$ has the same property. Let us suppose that every component of the invariant decomposition $\{E_\alpha(\beta)\}_{\beta \in F_\alpha}$ is a singleton. By *reductio ad absurdum* let us suppose that there exists $\beta_1 \in F_\gamma$ such that $\operatorname{card} E_\gamma(\beta_1) > 1$. Therefore, $E_\gamma(\beta_1)$ is the intersection with semigroup G^+ of a coset of some subgroup $G_1^* \neq \{0\}$ in G. Since $\alpha, \gamma \in X_k$, then

$$\exists x_1, x_2 \in G^+ : F(\alpha, x_1) = F(\gamma, x_2).$$

Moreover, there exists $x \in E_\gamma(\beta_1)$, such that $x \geq x_2$. From the above

$$\begin{aligned} \beta_1 &= F(\gamma, x) = F(\gamma, x - x_2 + x_2) = F(F(\gamma, x_2), x - x_2) \\ &= F(F(\alpha, x_1), x - x_2) = F(\alpha, x_1 + x - x_2), \end{aligned}$$

so $\beta_1 \in F_\alpha$. Let $E_\alpha(\beta_1) := \{x_0\}$ and $u \in G_1^* \cap G^+$. We have

$$\begin{aligned} \beta_1 &= F(\gamma, x) = F(\gamma, x + u) = F(F(\gamma, x), u) = F(\beta_1, u) \\ &= F(F(\alpha, x_0), u) = F(\alpha, x_0 + u). \end{aligned}$$

Hence, $x_0 + u \in E_\alpha(\beta_1)$, for all $u \in G_1^* \cap G^+$ so we get a contradiction with $E_\alpha(\beta_1) := \{x_0\}$. Now, let us consider the invariant decompositions $\{E_{\alpha_0}(\beta)\}_{\beta \in F_{\alpha_0}}$ and $\{E_\alpha(\beta)\}_{\beta \in F_\alpha}$ where α is an arbitrary element of X_k. From the above, the invariant decomposition $\{E_{\alpha_0}(\beta)\}_{\beta \in F_{\alpha_0}}$ contains only the components being the intersections with semigroup G^+ of a coset of the subgroup $G_{\alpha_0} \neq \{0\}$ in G and the invariant decomposition $\{E_\alpha(\beta)\}_{\beta \in F_\alpha}$ contains the components being the intersections with semigroup G^+ of a coset of the subgroup $G_\alpha \neq \{0\}$ in G.

Let $x_1, y_1 \in G^+$ and $\beta_1 := F(\alpha_0, x_1) = F(\alpha, y_1)$. For every $u \in G_{\alpha_0} \cap G^+$ we have

$$\beta_1 = F(\alpha_0, x_1 + u) = F(F(\alpha_0, x_1), u) = F(\beta_1, u) = F(F(\alpha, y_1), u) = F(\alpha, y_1 + u)$$

and hence $G_{\alpha_0} \subseteq G_\alpha$.

We will verify the statement that $F_{\alpha_0} = F_\alpha = X_k$, for all $\alpha \in X_k$.

Let $\alpha \in X_k$. There exist $\overline{x}, \overline{y} \in G^+$ such that $F(\alpha_0, \overline{x}) = F(\alpha, \overline{y})$. Since $G_\alpha = G_{\alpha_0} \neq \{0\}$, t hen there exist $u, v \in G^+$ such that $v + \overline{y}, u + \overline{x} \in G_{\alpha_0} \cap G^+ = G_\alpha \cap G^+$. Therefore,

$$\begin{aligned} F(\alpha, \overline{y} + u) &= F(F(\alpha, \overline{y}), u) = F(F(\alpha_0, \overline{x}), u) = F(\alpha_0, \overline{x} + u) = \alpha_0, \\ F(\alpha_0, \overline{x} + v) &= F(F(\alpha_0, \overline{x}), v) = F(F(\alpha, \overline{y}), v) = F(\alpha, \overline{y} + v) = \alpha. \end{aligned}$$

From the above, for every x belonging to G^+, we have

$$\begin{aligned} F(\alpha_0, x) &= F(F(\alpha, \overline{y} + u), x) = F(\alpha, \overline{y} + u + x), \\ F(\alpha, x) &= F(F(\alpha_0, \overline{x} + v), x) = F(\alpha_0, \overline{x} + v + x), \end{aligned}$$

whence $F_{\alpha_0} \subseteq F_\alpha$ and $F_\alpha \subseteq F_{\alpha_0}$, so $F_\alpha = F_{\alpha_0} = X_k$.

Let us put $\Delta_k^0 := G^+$ and define the bijection $h_k^0 \colon \{E_{\alpha_0}(\beta)\}_{\beta \in F_{\alpha_0}} \to X_k$ as follows

$$\forall \beta \in F_{\alpha_0} : h_k^0(E_{\alpha_0}(\beta)) = \beta.$$

Let $f(\alpha) \in X_k$ and $x \in G^+$. Since for $y \in E_{\alpha_0}(F(\alpha, 0))$ we have

$$F(\alpha_0, y + x) = F(F(\alpha_0, y), x) = F(F(\alpha, 0), x) = F(\alpha, x),$$

then

$$E_{\alpha_0}(F(\alpha, 0)) \oplus x = E_{\alpha_0}(F(\alpha, x)).$$

Therefore,

$$h_k^0((h_k^0)^{-1}(f(\alpha)) \oplus x) = h_k^0(E_{\alpha_0}(F(\alpha, 0)) \oplus x) = h_k^0(E_{\alpha_0}(F(\alpha, x))) = F(\alpha, x).$$

Moreover, let us remark that in this case $A = \emptyset$, $X_k^1 = \emptyset$, $\Delta_k^1 = \Delta_k^0$ and $h_k^1 = h_k^0$.

Ad (B). Now, we need a reformulation of the same results from [43], especially results included in Lemma 2.2 and Corollary 2.1 of [43]. The relation

$$\alpha \preceq \beta \Leftrightarrow F_\beta \subset F_\alpha, \tag{2.15}$$

defined in the set $F(X \times G^+)$ is reflexive and transitive; so, it is *a quasi-ordering* relation (see [44, p. 123]). By *quasi-string* we understand a subset B of the set $F(X \times G^+)$ for which

$$\forall \alpha, \beta \in B : (\alpha \preceq \beta \text{ or } \beta \preceq \alpha).$$

We will prove, that the set X_k is a maximal quasi-string in $F(X \times G^+)$ (note X_k has the form QS – Quasi String) or X_k is a union of maximal quasi-strings in $F(X \times G^+)$ mutually flowing one in other (note X_k has the form UQS – Union of Quasi Strings). Let us consider two cases:

(1) $\forall \alpha, \beta \in X_k : (F_\alpha \subset F_\beta \text{ or } F_\beta \subset F_\alpha)$,
(2) $\exists \alpha_1, \beta_1 \in X_k : (F_{\alpha_1} \not\subset F_{\beta_1} \text{ and } F_{\beta_1} \not\subset F_{\alpha_1})$.

Ad (1). In this case X_k is evidently a quasi-string and for every $\alpha \in X_k$ we have

$$X_k = B_\alpha := \{\beta \in X_k : F_\alpha \subset F_\beta \text{ or } F_\beta \subset F_\alpha\}. \tag{2.16}$$

Ad (2). Let us choose $\delta \in X_k$. Then $F_\delta \cap F_{\alpha_1} \neq \emptyset$ and $F_{\alpha_1} \cap F_{\beta_1} \neq \emptyset$. By (2.10) and (2.16), we get $\delta \in B_{\alpha_1}$ or $\delta \in B_{\beta_1}$. We will verify the statement that the sets B_{α_1} and B_{β_1} are the quasi-strings. Let us suppose, by *reductio ad absurdum* that there exist $\tau_1, \tau_2 \in B_{\alpha_1}$ such that

$$F_{\tau_1} \not\subset F_{\tau_2} \quad \text{and} \quad F_{\tau_2} \not\subset F_{\tau_1}. \tag{2.17}$$

Since τ_1, $\tau_2 \in B_{\alpha_1}$, then we have

$$(F_{\tau_1} \subset F_{\alpha_1} \text{ or } F_{\alpha_1} \subset F_{\tau_1}) \quad \text{and} \quad (F_{\tau_2} \subset F_{\alpha_1} \text{ or } F_{\alpha_1} \subset F_{\tau_2}). \tag{2.18}$$

Therefore, we have the following possibilities:

(i) $F_{\tau_1} \subset F_{\alpha_1}$ and $F_{\tau_2} \subset F_{\alpha_1}$,
(ii) $F_{\tau_1} \subset F_{\alpha_1}$ and $F_{\alpha_1} \subset F_{\tau_2}$,
(iii) $F_{\alpha_1} \subset F_{\tau_1}$ and $F_{\tau_2} \subset F_{\alpha_1}$,
(iv) $F_{\alpha_1} \subset F_{\tau_1}$ and $F_{\alpha_1} \subset F_{\tau_2}$.

Considering the cases (ii) and (iii), by transitivity of inclusion, we easily get the contradiction with (2.17).

In the case (i), there exist $x, y \in G^+$ such that

$$F(\tau_1, 0) = \tau_1 = F(\alpha_1, x) \quad \text{and} \quad F(\tau_2, 0) = \tau_2 = F(\alpha_1, y).$$

If $x \leq y$, then we have

$$\tau_2 = F(\alpha_1, y) = F(\alpha_1, x + y - x) = F(F(\alpha_1, x), y - x) = F(\tau_1, y - x),$$

hence, for every $u \in G^+$ we get

$$F(\tau_2, u) = F(\tau_1, y - x), u) = F(\tau_1, y - x + u),$$

whence $F_{\tau_2} \subset F_{\tau_1}$, so we get a contradiction. The reasoning is similar if $x \geq y$; in this case we get $F_{\tau_1} \subset F_{\tau_2}$. Now, we will consider the case (iv). Since $F_{\alpha_1} \cap F_{\beta_1} \neq \emptyset$ and $F_{\alpha_1} \cap F_{\tau_1} \neq \emptyset$, then $F_{\beta_1} \cap F_{\tau_1} \neq \emptyset$. Moreover,

$$F_{\beta_1} \not\subset F_{\tau_1} \quad \text{and} \quad F_{\tau_1} \not\subset F_{\beta_1}. \tag{2.19}$$

Indeed, the inclusion $F_{\beta_1} \subset F_{\tau_1}$ and (iv), by reasoning similar to one in the case (i), imply $F_{\beta_1} \subset F_{\alpha_1}$ or $F_{\alpha_1} \subset F_{\beta_1}$, so we get a contradiction with (2). However, the inclusion $F_{\tau_1} \subset F_{\beta_1}$ and (iv) imply $F_{\alpha_1} \subset F_{\beta_1}$, and this also leads to the contradiction with (2). Similarly, we can prove

$$F_{\beta_1} \not\subset F_{\tau_2} \quad \text{and} \quad F_{\tau_2} \not\subset F_{\beta_1}. \tag{2.20}$$

Since $F_{\tau_2} \cap F_{\beta_1} \neq \emptyset$, then by (2.19), (2.20), (2.17), we have the following contradiction with (2.10):

$$\begin{aligned}\exists \beta_1, \tau_1 \in X : \{F_{\beta_1} \cap F_{\tau_1} \neq \emptyset \text{ and } F_{\beta_1} \not\subset F_{\tau_1} \text{ and } F_{\tau_1} \not\subset F_{\beta_1} \\ \text{and } \exists \tau_2 \in X : [F_{\tau_2} \cap F_{\beta_1} \neq \emptyset \text{ and } F_{\tau_2} \not\subset F_{\beta_1} \\ \text{and } F_{\beta_1} \not\subset F_{\tau_2} \text{ and } F_{\tau_2} \not\subset F_{\tau_1} \text{ and } F_{\tau_1} \not\subset F_{\tau_2}]\}.\end{aligned}$$

We have proved that B_{α_1} is the quasi-string. By the same token, B_{β_1} is the quasi-string. Therefore, in the case (2), the set X_k is the union of quasi-strings B_{α_1} and B_{β_1}. One can easily observe that the sets B_α, defined by (2.16), are maximal quasi-strings. Moreover, it is easy to observe if $\omega \in B_{\alpha_1} \cap B_{\beta_1}$, then for every σ such that $\omega \preceq \sigma$, we have $\sigma \in B_{\alpha_1} \cap B_{\beta_1}$. Therefore, the quasi-strings having a nonempty common part can be illustrated (see drawing below) as the union of mutually flowing one in other quasi-strings.

Continuing the proof in the case (B), we can suppose that α_0, defined in (B), belongs to X_k as in the Figure 5. The arrows indicate the direction of the relation "$\preceq$".

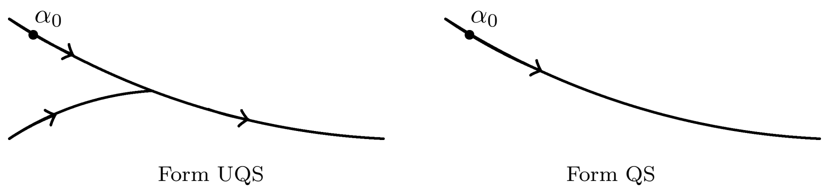

FIGURE 5

Let us define:

$$\begin{aligned} C_{\alpha_0}, &:= \{\alpha \in X_k : F_{\alpha_0} \subsetneq F_\alpha\}, \\ E_\alpha^{-1}(\beta) &:= \{-x : x \in E_\alpha(\beta)\}, \text{ pour } \alpha, \beta \in F(X \times G^+), \\ \mathcal{W} &:= \{E_{\alpha_0}(\beta)\}_{\beta \in F_{\alpha_0}} \cup \{E_\beta^{-1}(\alpha_0)\}_{\beta \in C_{\alpha_0}}, \\ \Delta_k^0 &:= \bigcup_{W \in \mathcal{W}} W. \end{aligned} \tag{2.21}$$

Firstly, we will prove that Δ_k^0 is a final interval of the group G. Let $a \in \Delta_k^0$ and $b \geq a$. If $b \geq 0$, we have $b \in \{E_{\alpha_0}(\beta)\}_{\beta \in F_{\alpha_0}}$ for $\beta = F(\alpha_0, b)$. Let us suppose that $b < 0$. Then $a < 0$, so $a \in E_\beta^{-1}(\alpha_0)$ for $\beta \in C_{\alpha_0}$, whence $F(\beta, -a) = \alpha_0$. Put $\delta := F(\beta, b-a)$. We have $F(\delta, -b) = F(\beta, -a) = \alpha_0$, therefore $b \in E_\delta^{-1}(\alpha_0)$. If $\delta \in C_{\alpha_0}$ then $b \in \Delta_k^0$. If $\delta \notin C_{\alpha_0}$ then $\delta \in F_\delta \subseteq F_{\alpha_0}$, therefore there exists $\overline{x} \in G^+$ such that $\delta = F(\alpha_0, \overline{x})$, whence

$$F(\alpha_0, \overline{x} - b) = F(F(\alpha_0, \overline{x}), -b) = F(\delta, -b) = \alpha_0,$$

so $b = 0$ by (B). This contradicts $b < 0$.

We have proved that Δ_k^0 is a final interval. Evidently $\Delta_k^0 \supseteq G^+$.

The sets $E_\beta^{-1}(\alpha_0)$, for $\beta \in C_{\alpha_0}$, are singletons. Indeed, if $x, y \in G^+$, $x \neq y$, $x \geq y$ and $-x, -y \in E_\beta^{-1}(\alpha_0)$, then

$$F(\alpha_0, x - y) = F(F(\beta, y), x - y) = F(\beta, x) = \alpha_0,$$

so $x = y$, whence $-x = -y$.

Therefore the family of sets $\{W\}_{W \in \mathcal{W}}$ forms an invariant decomposition of interval Δ_k^0.

Let the bijection $h_k^0 \colon \{W\}_{W \in \mathcal{W}} \to X_k^0 \subset X_k$ be defined in the following way:

$$h_k^0(W) = \beta \text{ such that } W = E_{\alpha_0}(\beta),\ \beta \in F_{\alpha_0} \text{ or } W = E_\beta^{-1}(\alpha_0),\ \beta \in C_{\alpha_0}.$$

Let us consider two cases.

(a) $X_k^1 := X_k \setminus X_k^0 = \emptyset$,
(b) $X_k^1 := X_k \setminus X_k^0 \neq \emptyset$.

Ad (a). In this case X_k has the form QS. Therefore, the function h_k^0 is the bijection: $\{W\}_{W \in \mathcal{W}} \to X_k$. Let $f(\alpha) \in X_k$ and $x \in G^+$. Let us consider two subcases:

(I) $f(\alpha) \in F_{\alpha_0}$,
(II) $f(\alpha) \in C_{\alpha_0}$.

Ad (I). If $W = (h_k^0)^{-1}(f(\alpha))$, then $W = E_{\alpha_0}(f(\alpha))$. For $W_1 := W \oplus x$ we have $W_1 = E_{\alpha_0}(F(f(\alpha), x))$. Therefore,

$$\begin{aligned} h_k^0((h_k^0)^{-1}(f(\alpha)) \oplus x) &= h_k^0(W \oplus x) = h_k^0(W_1) \\ &= F(f(\alpha), x) = F(F(\alpha, 0), x) = F(\alpha, x), \end{aligned}$$

so we have (2.11).

Ad (II). If $W = (h_k^0)^{-1}(f(\alpha))$, then $W = E_{f(\alpha)}^{-1}(\alpha_0)$. Let $W_1 := W \oplus x = E_{f(\alpha)}^{-1}(\alpha_0) \oplus x$.

(i) If $W_1 = W \oplus x = E_{f(\alpha)}^{-1}(\alpha_0) \oplus x = E_\beta^{-1}(\alpha_0)$, for a certain $\beta \in C_{\alpha_0}$, then

$$h_k^0((h_k^0)^{-1}(f(\alpha)) \oplus x) = h_k^0(W \oplus x) = h_k^0(W_1)$$

and for $E_{f(\alpha)}(\alpha_0) = \{y\}$, we have $E_\beta(\alpha_0) = \{y - x\}$.

From the above,

$$E_{f(\alpha)}(\alpha_0) = \{y\} \Rightarrow F(F(\alpha, x), y - x) = F(\alpha, y) = F(f(\alpha), y) = \alpha_0,$$

therefore, $E_{F(\alpha,x)}(\alpha_0) = \{y - x\}$, hence,

$$W_1 = E_\beta^{-1}(\alpha_0) = E_{F(\alpha,x)}^{-1}(\alpha_0).$$

This implies

$$F(\alpha,x)=h_k^0(W_1)=h_k^0((h_k^0)^{-1}(f(\alpha))\oplus x).$$

(ii) If $W_1=E_{\alpha_0}(\beta)$ and $\beta\in F_{\alpha_0}$, then, similarly,

$$h_k^0((h_k^0)^{-1}(f(\alpha))\oplus x)=h_k^0(W\oplus x)=h_k^0(W_1)=\beta$$

and for $E_{f(\alpha)}(\alpha_0)=\{y\}$, we have $x-y\in E_{\alpha_0}(\beta)$.

From the above,

$$\beta=F(\alpha_0,x-y)=F(F(f(\alpha),y),x-y)=F(f(\alpha),x)=F(\alpha,x),$$

so

$$F(\alpha,x)=\beta=h_k^0((h_k^0)^{-1}(f(\alpha))\oplus x).$$

In this case similarly to the case (A), we have: $A=\emptyset$, $X_k^1=\emptyset$, $\Delta_k^1=\Delta_k^0$ and $h_k^1=h_k^0$.

Ad (b). In this case X_k has the form UQS. Therefore the function h_k^0 is the injection: $\{W\}_{W\in\mathcal{W}}\to X_k$ and $X_k^0\subsetneq X_k$. Firstly, let us observe that

$$\forall\beta\in X_k^1: E_\beta(\beta)=\{0\}.$$

Fixing $\beta_0\in X_k^1$ and putting $A:=\{x\in G:\exists\beta\in X_k^1: x\in E_{\beta_0}(\beta)$ or $x\in E_\beta(\beta_0)\}$, one can verify easily the statement that A is an non-final interval and $\operatorname{card} A=\operatorname{card} X_k^1$. Define the set $\mathcal{A}:=\{\{x\}: x\in A\}$ and define the bijection $\varphi_k\colon\mathcal{A}\to X_k^1$ as follows:

$$\varphi_k(\{x\}):=\beta \text{ such that } x\in E_{\beta_0}(\beta) \text{ or } x\in E_\beta(\beta_0).$$

Let us denote $\Delta_k^1:=A\cup\{x\in G:\forall u\in A: x\geq u\}$. We can choose $t\in G^+$ such that

$$\forall u\in G^+: F(\beta_0,u)\in F_{\alpha_0}\Rightarrow F(\beta_0,u)=F(\alpha_0,u+t).$$

Indeed, let us put $a:=$min $\{x\in G^+: F(\alpha_0,x)\in F_{\beta_0}\}$. Evidently $a>0$. Let $b\in G^+$ and $F(\alpha_0,a)=F(\beta_0,b)$. For $a\geq b$ and $u\geq b\geq$max $\{a,b\}$, we have

$$F(\beta_0,u)=F(\beta_0,b+u-b)=F(\alpha_0,a+u-b)=F(\alpha_0,u+t)\quad \text{where} t:=a-b.$$

In the case $b\geq a$, we can apply the same reasoning by changing β_0 by β_1 defined as follows: $F(\beta_0,b-a)=:\beta_1\in X_k^1$, since $F(\beta_1,a)=F(F(\beta_0,b-a),a))=F(\beta_0,b)=F(\alpha_0,a)$.

Let $\mathcal{V}:=\{W\in\{E_{\alpha_0}(\beta)\}_{\beta\in F_{\alpha_0}\cap F_{\beta_0}}: W\cap(\Delta_k^1\setminus A)\neq\emptyset\}$ and define the function $h_k^1\colon\mathcal{A}\cup\mathcal{V}\to X_k$ as follows:

$$h_k^1(W)=\begin{cases}\varphi_k(W) & \text{for } W\in\mathcal{A},\\ h_k^0(W\oplus t) & \text{for } W\in\mathcal{V}.\end{cases}$$

We will verify the statement that

$$F(\alpha,x)=h_k^i((h_k^i)^{-1}(f(\alpha))\oplus x),\quad \text{for } f(\alpha)\in X_k^i,\ i\in\{0,1\}.$$

For this purpose, it is sufficient to consider the case $f(\alpha) \in X_k^1$ only. We have

$$h_k^1((h_k^1)^{-1}(f(\alpha)) \oplus x) = h_k^1(\varphi_k^{-1}(f(\alpha)) \oplus x).$$

If $\varphi_k^{-1}(f(\alpha)) = \{y\} \in \mathcal{A}$ and $\{y\} \oplus x = \{y + x\} \in \mathcal{A}$ then

$$\begin{aligned} h_k^1((h_k^1)^{-1}(f(\alpha)) \oplus x) &= \varphi_k(\varphi_k^{-1}(f(\alpha)) \oplus x) \\ &= \varphi_k(\{y + x\}) = F(\beta_0, y + x) = F(f(\alpha), x) = F(\alpha, x). \end{aligned}$$

If $\varphi_k^{-1}(f(\alpha)) = \{y\} \in \mathcal{A}$ and $\{y\} \oplus x = W \in \mathcal{V}$ then

$$\begin{aligned} h_k^1((h_k^1)^{-1}(f(\alpha)) \oplus x) &= h_k^1(\varphi_k^{-1}(f(\alpha)) \oplus x) = h_k^1(\{y\} \oplus x) = h_k^1(W) \\ &= h_k^0(W \oplus t) = F(\alpha_0, y + x + t) = F(\beta_0, y + x) \\ &= F(F(\beta_0, y), x) = F(f(\alpha), x) = F(\alpha, x). \end{aligned}$$

We have proved that every solution of the translation equation (0.1) satisfying the condition (2.10) can be obtained by Construction 2.2.28. □

Example 2.2.30. We will give an example of a solution using the Construction 2.2.28. Let $(G, +, 0, \leq) := (\mathbb{R}, +, 0, \leq)$. Let $X := [0, 2] \subset \mathbb{R}$ be the interval and, according to the point (1) of Construction 2.2.28, $f\colon X \to X$ be the identity function; so, $f(f(\alpha)) = f(\alpha) = \alpha$, for $\alpha \in X$. As in the point (2) of the considered construction, we define $K := \{1\}$, $f(X) = X = [0, 2] := X_1$. Let $\Delta_1^0 := \mathbb{R}^+$, $J_1 := [0, 1]$ and the invariant decomposition $\{E_{j1}\}_{j \in J_1}$ be desribed in the following way:

$$\forall j \in [0, 1[\, : E_{j1} := \{j\} \quad \text{and} \quad E_{11} := \mathbb{R}^+ \setminus [0, 1[.$$

The injection $h_1^0\colon \{E_{j1}\}_{j \in J_1} \to X_1$ is defined by $h_1^0(\{j\}) := j$, for $j \in [0, 1[$ and $h_1^0(\mathbb{R}^+ \setminus [0, 1[) := 1$. Therefore, $X_1^0 := h_1^0(\{E_{j1}\}_{j \in J_1}) = [0, 1]$. According to the point (2)(b), we define $A := [0, 1[$ and the bijection $\varphi_1\colon \mathcal{A} \to X_1^1$ where $\mathcal{A} = \{\{x\} : x \in A\}$ and $X_1^1 := X_1 \setminus X_1^0 =]1, 2]$ is defined by $\varphi_1(\{x\}) := 2 - x$. According to the point (3) of Construction 2.2.28, we have $\Delta_1^1 = \mathbb{R}^+$ and we can put $t := 0$, hence $\mathcal{V} = \mathbb{R}^+ \setminus [0, 1[$. Moreover, the function $h_1^1\colon \mathcal{A} \cup \mathcal{V} \to X_1$ is defined in the following way: $h_1^1(\{x\}) := 2 - x$, for $x \in [0, 1[$ and $h_1^1(\mathbb{R}^+ \setminus [0, 1[) := 1$. Using the point (4) of the Construction 2.2.28, we put

$$F(\alpha, x) := h_1^i((h_1^i)^{-1}(f(\alpha)) \oplus x), \quad \text{for } f(\alpha) \in X_1^i,\ i \in \{0, 1\}.$$

The obtained solution is the same as presented by B. Nowak in [43] and quoted in Example 2.2.25. This means that the solution F does not satisfy the condition (2.6).

Now, we will give an example of the solution not satisfying (2.10).

Example 2.2.31. Let $X := [0,1]$ and $(G,+,0,\leq) := (\mathbb{R},+,0,\leq)$. We define $F\colon [0,1]\times\mathbb{R}^+ \to [0,1]$ by the following way: $F(\alpha,0) := \alpha$, $F(\alpha,x) := 1$, for all $\alpha\in[0,1]$ and for all $x>0$. For this solution of the translation equation, we have $F_\alpha = \{\alpha,1\}$, for all $\alpha\in[0,1]$. It is evident that this solution does not satisfy the condition (2.10).

2.2.3. Semigroup of positive elements of linearly ordered and abelian group. In this subsection, the supposition, that the group $(G,+,\leq)$ is archimedean is replaced by the weaker supposition that G is abelian. For the solution $F\colon X\times G^+\to X$, the relation "$\sim$", defined by (2.14), is an equivalence relation. Moreover, let K denote a selection of the set $F(X\times G^+)/{\sim}$ and let X_k be the class for which $k\in K\cap X_k$. Therefore $F(X\times G^+)/{\sim} = \{X_k\}_{k\in K}$. Let us suppose that the solution $F\colon X\times G^+\to X$ satisfies two conditions, namely the condition (2.6) and the condition

$$\forall k\in K\ \exists\alpha_0\in X_k : [\forall\gamma\in C_{\alpha_0} : E_\gamma(\gamma) = E_{\alpha_0}(\alpha_0)] \tag{2.22}$$

where C_{α_0} is defined by (2.21). The paper [23] includes the theorem, which is a generalization of the Theorem 2.2.15. The Construction 2.2.33 presented below is a generalization of the Construction 2.2.16.

Theorem 2.2.32 ([23])**.** *The function $F\colon X\times G^+\to X$ is a solution of the translation equation* (0.1) *satisfying the conditions* (2.6) *and* (2.22) *if and only if F is obtained by the following construction.*

Construction 2.2.33.

(1) Let $f\colon X\to X$ be a function such that

$$\forall\alpha\in X : f(f(\alpha)) = f(\alpha).$$

(2) Let $f(X) = \bigcup_{k\in K} X_k$ be a union of nonempty disjoint sets X_k such that, for every $k\in K$, there exists an G^+-invariant and the *simple* decomposition $\{E_{jk}\}_{j\in J_k}$ of a final interval Δ_k of G, for which $G^+\subseteq\Delta_k$ and $\operatorname{card} J_k = \operatorname{card} X_k$.
(3) Let $h_k\colon \{E_{jk}\}_{j\in J_k}\to X_k$ be a bijection for $k\in K$.
(4) Let us define

$$F(\alpha,x) = h_k(h_k^{-1}(f(\alpha))\oplus x),\quad \text{for } f(\alpha)\in X_k,\ x\in G^+. \tag{2.23}$$

Remark 2.2.34. The word "simple" is the only difference between Constructions 2.2.16 and 2.2.33. It is natural by virtue of Remark 1.3.16 above and by Lemma 4 from the paper [23], in which it is proved that, for the archimedean group, every solution $F\colon X\times G^+\to X$ satisfies the condition (2.22). Assume that by Construction 2.2.33′, we understand the Construction 2.2.33, in which the word *simple* is omitted. In paper [23] it is also proved that every translation equation obtained by Construction 2.2.33′ satisfies (2.6).

We start with the proof of Theorem 2.2.32.

Proof. (⇒) Let $F\colon X\times G^+\to X$ be a solution of the translation equation (0.1) satisfying the conditions (2.6) and (2.22). Let us define $f\colon X\to X$ as follows:

$$f(\alpha)=F(\alpha,0),\quad \text{for } \alpha\in X.$$

For $\alpha\in X$, we have

$$f(f(\alpha))=f(F(\alpha,0))=F(F(\alpha,0),0)=F(\alpha,0)=f(\alpha).$$

Since

$$\forall(\alpha,x)\in X\times G^+ : F(\alpha,x)=F(F(\alpha,x),0)\in F(X,0)$$

then $f(X)=F(X,0)=F(X,G^+)$. We proved above that the relation "$\sim$" defined in $F(X\times G^+)$ by (2.14) is an equivalence relation and $G_{\alpha_0}=E_{\alpha_0}(\alpha_0)\cup E^{-1}_{\alpha_0}(\alpha_0)$ is a subgroup of G in the case of archimedean group. One can observe easily that the same proofs apply without this assumption. Let K denote a selection of the set $f(X)/\!\sim\,=F(X\times G^+)/\!\sim$ and let X_k be the class for which $k\in K\cap X_k$. Therefore, $F(X\times G^+)/\!\sim\,=\{X_k\}_{k\in K}$. Let $k\in K$. Let α_0 be the element occurring in the condition (2.22); so,

$$\forall\gamma\in C_{\alpha_0} : E_\gamma(\gamma)=E_{\alpha_0}(\alpha_0). \tag{2.24}$$

Let us consider two cases:

(A) the subgroup $G_{\alpha_0}=E_{\alpha_0}(\alpha_0)\cup E^{-1}_{\alpha_0}(\alpha_0)$ is unbounded,
(B) G_{α_0} is bounded.

Ad (A). By (2.24), we have $G_\gamma=G_{\alpha_0}$ for every $\gamma\in C_{\alpha_0}$. We will show that

$$\forall\gamma\in F_{\alpha_0} : G_\gamma=G_{\alpha_0}.$$

Let $\gamma\in F_{\alpha_0}{=}F(\{\alpha_0\}\times G^+)$. There exists $y\in G^+$ such that $F(\alpha_0,y)=\gamma$. Let $x\in E_{\alpha_0}(\alpha_0)$. We have

$$\gamma=F(\alpha_0,y)=F(F(\alpha_0,x),y)=F(\alpha_0,x+y)=F(F(\alpha_0,y),x)=F(\gamma,x),$$

hence, $E_{\alpha_0}(\alpha_0)\subseteq E_\gamma(\gamma)$, whence G_γ is unbounded and $G_{\alpha_0}\subseteq G_\gamma$. By the formula

$$\forall u\in G_\gamma\cap G^+ : F(\alpha_0,y+u)=F(F(\alpha_0,y),u)=F(\gamma,u)=\gamma=F(\alpha_0,y),$$

we get $G_\gamma\subseteq G_{\alpha_0}$ and $G_\gamma=G_{\alpha_0}$. From the above,

$$\forall\alpha,\gamma\in X_k : G_\alpha=G_\gamma=G_{\alpha_0}.$$

We will show that

$$\forall\alpha\in X_k : F_{\alpha_0} = F_\alpha = X_k.$$

Take $\alpha\in X_k$. There exists $\overline{x},\overline{y}\in G^+$ such that $F(\alpha_0,\overline{x}) = F(\alpha,\overline{y})$. Since $G_\alpha = G_{\alpha_0}$ is an unbounded subgroup of G, then there exist $u,v\in G^+$ such that $u+\overline{x}, v+\overline{y}\in G_{\alpha_0}\cap G^+ = G_\alpha\cap G^+$. Hence,

$$\begin{aligned} F(\alpha,\overline{y}+u) &= F(F(\alpha,\overline{y}),u) = F(F(\alpha_0,\overline{x}),u) = F(\alpha_0,\overline{x}+u) = \alpha_0,\\ F(\alpha_0,\overline{x}+v) &= F(F(\alpha_0,\overline{x}),v) = F(F(\alpha,\overline{y}),v) = F(\alpha,\overline{y}+v) = \alpha. \end{aligned}$$

Therefore, for every $x\in G^+$, we have

$$\begin{aligned} F(\alpha_0,x) &= F(F(\alpha,\overline{y}+u),x) = F(\alpha,\overline{y}+u+x),\\ F(\alpha,x) &= F(F(\alpha_0,\overline{x}+v),x) = F(\alpha_0,\overline{x}+v+x), \end{aligned}$$

whence, $F_{\alpha_0}\subseteq F_\alpha$ and $F_\alpha\subseteq F_{\alpha_0}$, so $F_\alpha = F_{\alpha_0} = X_k$.

Let the bijection h_k from the simple invariant decomposition $\{E_{\alpha_0}(\beta)\}_{\beta\in F_{\alpha_0}}$ of the final interval $\Delta_k := G^+$ of group G, to X_k, be defined as follows:

$$h_k(E_{\alpha_0}(\beta)) := \beta, \quad \text{for } \beta\in F_{\alpha_0}.$$

We will verify the statement that

$$\forall\alpha\in X\ \forall x\in G^+ : [f(\alpha)\in X_k \Rightarrow h_k(h_k^{-1}(f(\alpha))\oplus x) = F(\alpha,x)],$$

which means that, according the Construction 2.2.33, F has the form (2.23). So, let $f(\alpha)\in X_k$ and $x\in G^+$. Thus, we have

$$h_k(h_k^{-1}(f(\alpha))\oplus x) = h_k(E_{\alpha_0}(F(\alpha,0))\oplus x).$$

Since

$$F(\alpha_0,y+x) = F(F(\alpha_0,y),x) = F(F(\alpha,0),x) = F(\alpha,x),$$

for $y\in E_{\alpha_0}(F(\alpha,0))$, then $E_{\alpha_0}(F(\alpha,0))\oplus x = E_{\alpha_0}(F(\alpha,x))$. Therefore,

$$h_k(h_k^{-1}(f(\alpha))\oplus x) = h_k(E_{\alpha_0}(F(\alpha,0))\oplus x) = h_k(E_{\alpha_0}(F(\alpha,x))) = F(\alpha,x).$$

Ad (B). Let us introduce the following notation

$$F'_{\alpha_0} := F(\alpha_0, G^+\setminus C(G_{\alpha_0})),$$

where $C(G_{\alpha_0})$ is the smallest, convex subgroup of G including G_{α_0},

$$\begin{aligned} C'_{\alpha_0} &:= \{\alpha\in C_{\alpha_0} : F_{\alpha_0}\subsetneq F_\alpha\},\\ \mathcal{W}_0 :&= \{W\in G/G_{\alpha_0} : W\subseteq C(G_{\alpha_0})\} = C(G_{\alpha_0})/G_{\alpha_0} \end{aligned}$$

and define

$$\mathcal{W} := \mathcal{W}_0\cup\{E_{\alpha_0}(\beta)\}_{\beta\in F'_{\alpha_0}}\cup\{E_\beta^{-1}(\alpha_0)\}_{\beta\in C'_{\alpha_0}} \quad\text{and}\quad \Delta_k := \bigcup_{W\in\mathcal{W}} W.$$

We will show that the family $\{W : W \in \mathcal{W}\}$ is a simple invariant decomposition of the final interval Δ_k of G and $\Delta_k \supseteq G^+$. To prove that Δ_k is a final interval of G, it is sufficient to show the following implication: if for a certain $x \in G^+$ and $\beta \in C'_{\alpha_0}$ we have $-x \in E_\beta^{-1}(\alpha_0)$, then for $y \in G^+$ such that $x - y \in G^+$ the following alternative is true

$$y \in C(G_{\alpha_0}) \quad \text{or} \quad \exists \delta \in C'_{\alpha_0} : y \in E_\delta(\alpha_0). \tag{2.25}$$

Let so $x \in G^+$ and $-x \in E_\beta^{-1}(\alpha_0)$, for certain $\beta \in C'_{\alpha_0}$. Let $y \in G^+$ and $x - y \in G^+$. Put $\delta := F(\beta, x - y)$. We have

$$F(\delta, y) = F(F(\beta, x - y), y) = F(\beta, x) = \alpha_0.$$

If $\delta \in C'_{\alpha_0}$, then we have (2.25). Otherwise, there exists $\overline{x} \in G^+$ such that $F(\alpha_0, \overline{x}) = \delta$, hence $F(\alpha_0, \overline{x} + y) = F(F(\alpha_0, \overline{x}), y) = F(\delta, y) = \alpha_0$, whence $\overline{x} + y \in G_{\alpha_0} \subseteq C(G_{\alpha_0})$. Since $0 \le y \le \overline{x} + y$, then $y \in C(G_{\alpha_0})$ and (2.25). We have shown that Δ_k is a final interval of G. The inclusion $\Delta_k \supseteq G^+$ is obvious. We will prove now that the sets of family $\{W\}_{W\in\mathcal{W}}$ are disjoint, i.e. $W_1 \cap W_2 = \emptyset$, for $W_1, W_2 \in \mathcal{W}, W_1 \neq W_2$. The sets belonging to $\mathcal{W}_0 \cup \{E_{\alpha_0}(\beta)\}_{\beta \in F'_{\alpha_0}}$ and belonging to $\{E_{\alpha_0}(\beta)\}_{\beta\in F'_{\alpha_0}} \cup \{E_\beta^{-1}(\alpha_0)\}_{\beta \in C'_{\alpha_0}}$, are evidently disjoint. Therefore, we will consider only two cases:

(a) $\exists \beta_1, \beta_2 \in C'_{\alpha_0}, \beta_1 \neq \beta_2 : W_1 = E_{\beta_1}^{-1}(\alpha_0)$ and $W_2 = E_{\beta_2}^{-1}(\alpha_0)$,
(b) $\exists \beta \in C'_{\alpha_0} : W_1 = E_{\beta_1}^{-1}(\alpha_0)$ and $W_2 \in \mathcal{W}_0$.

Ad (a). Let us suppose $W_1 \cap W_2 \neq \emptyset$. Let $x \in G^+$ and $-x \in E_{\beta_1}^{-1}(\alpha_0) \cap E_{\beta_2}^{-1}(\alpha_0)$; so $F(\beta_1, x) = \alpha_0 = F(\beta_2, x)$. Since $\beta_1, \beta_2 \in X_k$, then we can assume $F_{\beta_2} \subseteq F_{\beta_1}$ (the same reasoning holds true for $F_{\beta_1} \subseteq F_{\beta_2}$). There exists $y \in G^+$ such that $F(\beta_1, y) = \beta_2$. Hence,

$$F(\alpha_0, y) = F(F(\beta_1, x), y) = F(\beta_1, x + y) = F(F(\beta_1, y), x) = F(\beta_2, x) = \alpha_0,$$

so $y \in E_{\alpha_0}(\alpha_0)$. The condition (2.22) implies $y \in E_{\beta_1}(\beta_1) = E_{\alpha_0}(\alpha_0)$, whence we get a contradiction $\beta_1 = F(\beta_1, y) = \beta_2$.

Ad (b). Let us suppose $W_1 \cap W_2 \neq \emptyset$, $x \in G^+$, $-x \in E_\beta^{-1}(\alpha_0)$ and $-x \in W_2 \in \mathcal{W}_0$. We have $F(\beta, x) = \alpha_0$ and

$$\exists y \in G^+ : y + x \in G_{\alpha_0} \cap G^+ = G_\beta \cap G^+.$$

Hence, $\beta = F(\beta, y + x) = F(F(\beta, x), y) = F(\alpha_0, y)$, so we get a contradiction with $\beta \in C'_{\alpha_0}$. One can observe easily that $\{W : W \in \mathcal{W}_0 \cup \{E_{\alpha_0}(\beta)\}_{\beta\in F'_{\alpha_0}}\}$ is an invariant decomposition of the final interval $\Delta := G^+ \cup C(G_{\alpha_0})$. We will verify now the statement that the family $\{W\}_{W\in\mathcal{W}}$ forms an invariant decomposition of Δ_k. It is sufficient to show that every set $E_\beta(\alpha_0)$, for $\beta \in C'_{\alpha_0}$, is a coset of G_{α_0} in G. Let so $\beta \in C'_{\alpha_0}$. By Theorem 1.3.3, the component $E_\beta(\alpha_0)$ of invariant decomposition $\{E_\beta(\gamma)\}_{\gamma\in F_\beta}$ of G^+ is a final interval of a coset of certain subgroup G_1 in G and $G_1 \supseteq G_\beta$. Since $G_\beta = G_{\alpha_0}$, then $G_1 \supseteq G_{\alpha_0}$. The formula

$$\alpha_0 = F(\beta, x) = F(\beta, x+u) = F(F(\beta, x), u) = F(\alpha_0, u),$$

for all $u \in G_1 \cap G^+$ and all $x \in E_\beta(\alpha_0)$, implies $G_1 \subseteq G_{\alpha_0}$, so $G_1 = G_{\alpha_0} = G_\beta$. By Theorem 1.3.3, the component $E_\beta(\alpha_0)$ is a coset G_{α_0} in G. Therefore, it is proved that the family $\{W\}_{W\in\mathcal{W}}$ forms an invariant decomposition of the final interval $\Delta_k \supseteq G^+$. Since the subgroup G_{α_0} is the smallest among bounded subgroup corresponding to components of decomposition, then invariant decomposition $\{W\}_{W\in\mathcal{W}}$ is simple.

Let the bijection h_k from the simple invariant decomposition $\{W\}_{W\in\mathcal{W}}$ to X_k be defined as follows:

$$\forall W \in \mathcal{W} : h_k(W) = \beta \text{ such that } \begin{cases} W \cap G^+ = E_{\alpha_0}(\beta) \\ \text{or} \\ W = E_\beta^{-1}(\alpha_0) \qquad\quad \text{for } \beta \in C'_{\alpha_0}. \end{cases}$$

We will verify the statement that

$$\forall \alpha \in X \ \forall x \in G^+ : [f(\alpha) \in X_k \Rightarrow h_k(h_k^{-1}(f(\alpha)) \oplus x) = F(\alpha, x)], \tag{2.26}$$

which means that, according the Construction 2.2.33, F has the form (2.23). So, let $f(\alpha) \in X_k$ and $x \in G^+$. We will consider two cases

(a) $f(\alpha) \in F_{\alpha_0}$,
(b) $f(\alpha) \in C'_{\alpha_0}$.

Ad (a). If $W = h_k^{-1}(f(\alpha))$, then $W \cap G^+ = E_{\alpha_0}(f(\alpha))$. If $W_1 = W \oplus x$, then $W_1 \cap G^+ = E_{\alpha_0}(F(f(\alpha), x))$. Hence,

$$\begin{aligned} h_k(h_k^{-1}(f(\alpha)) \oplus x) &= h_k(W \oplus x) = h_k(W_1) \\ &= F(f(\alpha), x) = F(F(\alpha, 0), x) = F(\alpha, x), \end{aligned}$$

so (2.26).

Ad (b). If $W = h_k^{-1}(f(\alpha))$, then $W = E_{f(\alpha)}^{-1}(\alpha_0)$. Let $W_1 = W \oplus x = E_{f(\alpha)}^{-1}(\alpha_0) \oplus x$. We will consider two subcases.

(i) If $W_1 = E_\beta^{-1}(\alpha_0)$, for a certain $\beta \in C'_{\alpha_0}$, then

$$h_k(h_k^{-1}(f(\alpha)) \oplus x) = h_k(W \oplus x) = h_k(W_1) = \beta$$

and $y - x \in E_\beta(\alpha_0)$ for all $y \in E_{f(\alpha)}(\alpha_0)$.

From the above

$$\forall y \in E_{f(\alpha)}(\alpha_0) : F(F(\alpha, x), y - x) = F(\alpha, y) = F(f(\alpha), y) = \alpha_0,$$

so $y - x \in E_{F(\alpha,x)}(\alpha_0)$ for all $y \in E_{f(\alpha)}(\alpha_0)$. Hence, $F(\alpha, x) = \beta$ and

$$h_k(h_k^{-1}(f(\alpha)) \oplus x) = \beta = F(\alpha, x).$$

(ii) If $W_1 = E\alpha_0(\beta)$ and $\beta \in F_{\alpha_0}$, then, similarly,

$$h_k(h_k^{-1}(f(\alpha)) \oplus x) = h_k(W \oplus x) = h_k(W_1) = \beta$$

and $x - y \in E_{\alpha_0}(\beta)$ for all $y \in E_{f(\alpha)}(\alpha_0)$.

From the above,

$$\beta = F(\alpha_0, x - y) = F(F(f(\alpha), y), x - y) = F(f(\alpha), x) = F(\alpha, x),$$

for a certain $y \in E_{f(\alpha)}(\alpha_0)$; so,

$$\beta = h_k(h_k^{-1}(f(\alpha)) \oplus x) = F(\alpha, x).$$

That was to have been proven for the "only if" part.

($\Leftarrow$) It is sufficient to prove that the solution of the translation equation $F\colon X \times G^+ \to X$ obtained by Construction 2.2.33 satisfies the condition (2.22). Fix an $k \in K$. Let Δ_k, $\{E_{jk}\}_{j\in J_k}$, h_k be the parameters occurring in Construction 2.2.33. Let us observe that if $E \in \{E_{jk}\}_{j\in J_k}$ is a final interval of a coset of G_1 of G and $h_k(E) = \alpha$, then $G_1 = G_\alpha$. Indeed, since $E \oplus x = E$ for every $x \in G_1 \cap G^+$, then

$$F(\alpha, x) = h_k(h_k^{-1}(f(\alpha)) \oplus x) = h_k(h_k^{-1}(\alpha) \oplus x) = h_k(E \oplus x) = h_k(E) = \alpha,$$

for all $x \in G_1 \cap G^+$. Whence $G_1 \cap G^+ \subseteq G_\alpha \cap G^+$. Moreover, since $F(\alpha, x) = \alpha$, for every $x \in G_\alpha \cap G^+$, then

$$\forall x \in G_\alpha \cap G^+ : [F(\alpha, x) = h_k(E \oplus x) = \alpha = h_k(E)],$$

whence $E \oplus x = E$ for every $x \in G_\alpha \cap G^+$; so, $G_\alpha \cap G^+ \subseteq G_1 \cap G^+$ and $G_1 = G_\alpha$. By the supposition that invariant decomposition $\{E_{jk}\}_{j\in J_k}$ is simple, we have two possibilities:

(a) all components of invariant decomposition $\{E_{jk}\}_{j\in J_k}$ are intersections with Δ_k of cosets of G^* in G where G^* is an unbounded subgroup of G,

(b) there exists a component of invariant decomposition $\{E_{jk}\}_{j\in J_k}$ which is a final interval of a coset of G_1 in G where G_1 is the smallest among bounded subgroup of G, corresponding to components of decomposition.

Ad (a). Let us take an arbitrary component $E \in \{E_{jk}\}_{j\in J_k}$. Let $\alpha_0 := h_k(E)$. From the above, we have $G_{\alpha_0} = G^*$ and $G_\gamma = G^*$ for every $\gamma \in X_k$. Therefore,

$$\exists \alpha_0 \in X_k \ \forall \gamma \in C_{\alpha_0} : E_\gamma(\gamma) = E_{\alpha_0}(\alpha_0).$$

Ad (b). Let $\alpha_0 := h_k(E)$. From the above, we have $G_{\alpha_0} = G_1$. We will show that

$$\forall \gamma \in C_{\alpha_0} : E_\gamma(\gamma) = E_{\alpha_0}(\alpha_0).$$

Let us take $\gamma \in C_{\alpha_0}$. There exists $y \in G^+$ such that $F(\gamma, y) = \alpha_0$. Since, for every $z \in E_\gamma(\gamma)$, we have

$$F(\alpha_0, z) = F(F(\gamma, y), z) = F(\gamma, y + z) = F(F(\gamma, z), y) = F(\gamma, y) = \alpha_0,$$

then $E_\gamma(\gamma) \subseteq E_{\alpha_0}(\alpha_0)$. The strong inclusion $E_\gamma(\gamma) \subsetneq E_{\alpha_0}(\alpha_0)$ is impossible. In the opposite case we have $G_\gamma \subsetneq G_{\alpha_0} = G_1$ and the component $h_k^{-1}(\gamma) \in \{E_{jk}\}_{j \in J_k}$ is a final interval of a coset of G_γ in G, so we get a contradiction with the supposition that G_1 is the smallest among bounded subgroup of G, corresponding to components of decomposition.

That was to have been proven for Theorem 2.2.32. □

Example 2.2.35 ([23]). Using Construction 2.2.33, we will give an example of solution satisfying the conditions (2.6) and (2.22). Let $\mathbb{Z}$ denote the set of integers and $G := \{ax + b : a, b \in \mathbb{Z}\}$ be the group of linear polynomials defined in Example 1.3.12. Let us recall, that the semigroup of positive elements is

$$G^+ := \{ax + b : a > 0,\ b \in \mathbb{Z}\} \cup \mathbb{Z}^+$$

where $\mathbb{Z}^+ := \{a \in \mathbb{Z} : a \geq 0\}$. Let $X := (\mathbb{Z} \setminus \mathbb{Z}^+) \cup \{0, 1\}$. According to Construction 2.2.33, we will define the parameters, determining the solution $F\colon X \times G^+ \to X$ of the translation equation. Let $f\colon X \to X$ be the identity function. Let $K := \{k_0\}$. Since the set K is a singleton and $X_{k_0} = f(X) = X$, then we omit the index k_0 in parameters determining the solution. For the final interval $\Delta := G$ of G; let us define the simple G^+-invariant decomposition $\{E_j\}_{j \in J}$ as follows: $J := X = (\mathbb{Z} \setminus \mathbb{Z}^+) \cup \{0, 1\}$ and

$$\forall j \in J : E_j := \begin{cases} j \cdot x + \mathbb{Z} = \{jx + a : a \ \in \mathbb{Z}\} & \text{for } j \neq 1, \\ G^+ \setminus \mathbb{Z} & \text{for } j = 1. \end{cases}$$

We define the bijection $h\colon \{E_j\}_{j \in J} \to X$ by $h(E_j) := j$ for all $j \in J$. According to the point (4) of Construction 2.2.33, we put

$$\forall \alpha \in X\ \forall w \in G^+ : F(\alpha, w) := h(h^{-1}(\alpha) \oplus w).$$

The obtained solution $F\colon X \times G^+ \to X$, can be expressed as follows

$$\forall (\alpha, w) \in X \times G^+ : F(\alpha, w) := \begin{cases} \alpha & \text{when } w \in \mathbb{Z}^+, \\ \alpha + a & \text{when } w = ax + b \text{ and } \alpha + a < 1, \\ 1 & \text{when } w = ax + b \text{ and } \alpha + a \geq 1. \end{cases}$$

Example 2.2.36 ([23])**.** Using Construction 2.2.33′, we will give an example of a solution satisfying the conditions (2.6) and not satisfying the condition (2.22). The group is the same as in the previous example. Let

$$X := \{a_{00}, a_{10}, a_{20}, a_{21}, a_{40}, a_{41}, a_{42}, a_{43}, \dots , a_{2^n 0}, a_{2^n 1}, \dots , a_{2^n 2^n - 1}, \dots \}.$$

Let us denote $\mathbb{Z}^+_{2^i} := \{2^i \cdot a : a \in \mathbb{Z}^+\}$ and $\mathbb{Z}_{2^i} := \{2^i \cdot a : a \in \mathbb{Z}\}$, for $i \in \{0, 1, \dots\}$. According to Construction 2.2.33′, we will define the parameters, determining the solution $F\colon X \times G^+ \to X$ of the translation equation. Similarly, as in the previous example, we assume that $f\colon X \to X$ is the identity function and the set K is a singleton. For the final interval $\Delta := G$ of G, let us define the (not simple) G^+-invariant decomposition $\{E_j\}_{j\in J}$ as follows: $J := X$ and

$$\forall j \in J : E_j := \begin{cases} G^+ \setminus (\mathbb{Z} \cup (\mathbb{Z} + x)) & \text{for } j = a_{00}, \\ \mathbb{Z}_{2^l} + m - (l-1)\cdot x & \text{for } j = a_{2^l m},\ l \in \{0,1,\dots\}, \\ & m \in \{0, 1, \dots , 2^l - 1\}. \end{cases}$$

The G^+-invariant decomposition $\{E_j\}_{j\in J}$ is the same, not the simple invariant decomposition of $\Delta = G$, as described in the Example 1.3.14.

We define the bijection $h\colon \{E_j\}_{j\in J} \to X$ by $h(E_j) := j$ for all $j \in J$. Define

$$\forall \alpha \in X\ \forall w \in G^+ : F(\alpha, w) := h(h^{-1}(\alpha) \oplus w).$$

Let us refer by $(i)_m$ to the remainder of division of i by m, for $i, m \in \mathbb{Z}^+$. The obtained solution $F\colon X \times G^+ \to X$, can be expressed as follows:

$$F(\alpha, w) := \begin{cases} a_{00} & \text{when } \alpha = a_{00}, \\ a_{2^{i-k}(m+l)_{2^{i-k}}} & \text{when } \alpha = a_{2^i m}, \\ & w \in (\mathbb{Z}_{2^i} - k + l + k\cdot x) \cap G^+, \\ & k \in \mathbb{Z}^+,\ i-k > 0,\ 0 \le l < 2^{i-k}, \\ a_{10} & \text{when } \alpha,\ w \text{ as above and } i-k = 0, \\ a_{00} & \text{when } \alpha,\ w \text{ as above and } i-k < 0. \end{cases}$$

for all $(\alpha, w) \in X \times G^+$.

The condition (2.22) is not satisfied for this solution. Indeed, one can observe that we have

$$E_{a_{2^i 0}}(a_{2^i 0}) = \mathbb{Z}^+_{2^i}, \quad \text{for } i = 1, 2, \dots$$

Problem 2.2.37 ([23])**.** The following implication is still not proved: if the solution of the translation equation $F\colon X \times G^+ \to X$ satisfies the condition (2.6), F can be obtained by Construction 2.2.33′?

2.3. Brandt's groupoid, Ehresmann's groupoid and category. Non-associative structures.

The definitions of the algebraic structures presented below can be found in the paper [55].

Definition 2.3.1. A set K with binary operation $K \times K \supseteq R \ni (x, y) \mapsto x \cdot y \in K$, is called *category* if the following axioms are fulfilled:

(a) $\forall x, y, z \in K : [((x, y) \in R \wedge (y, z) \in R) \Rightarrow ((x, y \cdot z) \in R)]$,
(b) $\forall x, y, z \in K : [((x, y) \in R \wedge (y, z) \in R) \Rightarrow ((x \cdot y, z) \in R)]$,
(c) $\forall x, y, z \in K : [((y, z) \in R \wedge (x, y \cdot z) \in R) \Rightarrow ((x, y) \in R)]$,
(d) $\forall x, y, z \in K : [((x, y) \in R \wedge (x \cdot y, z) \in R) \Rightarrow ((y, z) \in R)]$,
(e) $\forall x, y, z \in K : [((x, y) \in R \wedge (y, z) \in R) \Rightarrow ((x \cdot y) \cdot z) = (x \cdot (y \cdot z))]$,
(f) $\forall x \in K \ \forall e_1, e_2 \in K_0 : [((x, e_1) \in R \wedge (x, e_2) \in R) \Rightarrow (e_1 = e_2)]$,
(g) $\forall x \in K \ \exists e_1, e_2 \in K_0 : [((e_1, x) \in R \wedge (e_2, x) \in R) \Rightarrow (e_1 = e_2)]$,
(h) $\forall x \in K \ \exists e_1, e_2 \in K_0 : [(x, e_1) \in R \wedge (e_2, x) \in R]$,

where the set

$$K_0 := \{e \in K : \forall x \in K \ [((x, e) \in R \Rightarrow x \cdot e = x) \wedge ((e, x) \in R \Rightarrow e \cdot x = x)]\}$$

is called *the set of identities of the category* $(K, \cdot)$.

We admit then the notation $x \cdot y = z$ means the conjunction

$$(x, y) \in R \wedge x \cdot y = z.$$

Let us put $A := \{e \in K_0 : e \cdot e = e\}$. For $a \in A$, we define:

Definition 2.3.2 ([16]). The *left-sided stripe* ${}_aK$ (the *right-sided stripe* K_a, respectively) of the category $(K, \cdot)$, is the set:

$${}_aK := \{x \in K : a \cdot x = x\} \quad (K_a := \{x \in K : x \cdot a = x\}, \text{respectively}).$$

Definition 2.3.3. A category $(K, \cdot)$ is called *Ehresmann groupoid* if additionally (apart from axioms (a)–(h)) the following axiom is satisfied:

(i) $\forall x \in K \ \forall x \in {}_aK_b \ \exists y \in K : [(x, y) \in R \wedge (y, x) \in R \wedge x \cdot y = a \wedge y \cdot x = b]$ where ${}_aK_b := \{x \in K : a \cdot x = x \wedge x \cdot b = x\}$, for $a, b \in A$.

Definition 2.3.4. An Ehresmann groupoid is called *Brandt's groupoid* if additionally (apart from axioms (a)–(i)) the following axiom is satisfied:

(j) $\forall x, z \in K \ \exists y \in K : [(x, y) \in R \wedge (y, z) \in R]$.

The extremely complicated general construction of solutions of the translation equation on a category, is given in the paper [16]. The parameters of presented construction are involved in the so-called compatibility condition.

The particular form of this condition is presented above for a semigroup in (2.1). For the construction of solutions of the translation equation on a category, the quoted paper [16] contains description of invariant decompositions of left-sided stripe ${}_bK$ of category K. The paper [16] includes also the consideration of the possibilities of the elimination of the compatibility condition. The author showed that the elimination of the compatibility condition fails on the structures more general than Ehresmann's groupoid. The general solution of the translation equation with descriptions of suitable invariant decompositions on Brandt's and Ehresmann's groupoids are given in the papers [15], [49], [50]. We omit those anyway very complicated constructions. The general conclusion is: the problem of elimination of the compatibility condition is impossible to solve on the algebraic structures without inverses. In such cases, we can describe the classes of solutions through the constructions not including the compatibility condition as we can see above. For example: Constructions 2.2.16, 2.2.28, 2.2.33 as well as the classes of solutions described by Theorems 2.2.2, 2.2.10 and 2.2.11.

In the papers [11], [14] the authors investigate the translation equation on non-associative structures. This paper does not include such considerations. We have only the following remark.

Remark 2.3.5. The existence of the solution $F\colon X \times S \to X$ for which $F(\alpha_0, \cdot\,)$ is an injection, for a certain $\alpha_0 \in X$ implies the associativity of the operation in S. Indeed, we have

$$\begin{aligned} F(\alpha_0, (x \cdot y) \cdot z) &= F(F(\alpha_0, x \cdot y), z) = F(F(F(\alpha_0, x), y), z) \\ &= F(F(\alpha_0, x), y \cdot z) = F(\alpha_0, x \cdot (y \cdot z)), \end{aligned}$$

so $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. Hence, one can observe the following equivalence. The groupoid S is a semigroup if and only if there exist a nonempty set X and a solution $F\colon X \times S \to X$ of the translation equation (0.1) for which $F(\alpha_0, \cdot\,)$ is an injection for a certain $\alpha_0 \in X$. Indeed, if $(S, \cdot\,)$ is a semigroup with unity 1, we assume $X := S$, $F(\alpha, x) := \alpha \cdot x$ and $\alpha_0 := 1$. Otherwise, we can add the unity 1 to S and take X as S supplemented by 1 without changing the structure in S. The function F is defined on $X \times S$ by extended operation on X.

CHAPTER 3

CONSTRUCTIONS OF SOLUTIONS OF GENERALIZED TRANSLATION EQUATION

In this section we shall present constructions of solutions of the translation equation on the n-groups.

3.1. Hosszú's upshot and translation equation on n-group

Let $n \geq 2$ be a natural number and let X be an arbitrary set. By $(G, [\ldots])$ we refer to an arbitrary n-adic group (see [34]). The translation equation on the n-group $(G, [\ldots])$ is the following equation

$$\Phi(\Phi(\Phi(\ldots\Phi(\Phi(\alpha, x_1), x_2), \ldots), x_{n-1}), x_n) = \Phi(\alpha, [x_1 \ldots x_n]) \tag{3.1}$$

where $\Phi: X \times G \to X$. Due to Hosszú's conclusion [13], the n-group operation in G can be expressed by using a binary operation on G and some automorphism of it in the following way

$$[x_1 \ldots x_n] = x_1 \cdot \mu(x_2) \cdot u^2(x_3) \cdot \ldots \cdot \mu^{n-1}(x_n) \cdot a$$

where "$\cdot$" is a binary operation on G and μ is an automorphism of $(G, \cdot)$ and $a \in G$, $\mu(a) = a$ and $\mu^{n-1}(x) = a \cdot x \cdot a^{-1}$ (by μ^ν we refer to the ν-th iteration). Therefore, the equation (3.1) can be expressed in the form

$$\begin{aligned}\Phi(\Phi(\Phi(\ldots\Phi(\Phi(\alpha, x_1), x_2), \ldots), x_{n-1}), x_n)& \\ = \Phi(\alpha, x_1 \cdot \mu(x_2) \cdot u^2(x_3) \cdot \ldots \cdot \mu^{n-2}(x_{n-1}) \cdot a \cdot x_n).&\end{aligned} \tag{3.2}$$

If we set $a = e$ in (3.2) where e is the unit element of $(G, \cdot)$, then equation

$$\begin{aligned}\Phi(\Phi(\Phi(\ldots\Phi(\Phi(\alpha, x_1), x_2), \ldots), x_{n-1}), x_n)& \\ = \Phi(\alpha, x_1 \cdot \mu(x_2) \cdot u^2(x_3) \cdot \ldots \cdot \mu^{n-2}(x_{n-1}) \cdot x_n),&\end{aligned} \tag{3.3}$$

is the translation equation on these special n-groups. If additionally the automorphism μ is the identity on G, we obtain the equation

$$\Phi(\Phi(\Phi(\ldots\Phi(\Phi(\alpha,x_1),x_2),\ldots),x_{n-1}),x_n)=\Phi(\alpha,x_1\cdot x_2\cdot\ldots\cdot x_n), \tag{3.4}$$

which represents the translation equation on n-groups obtained from a binary group $(G,\,\cdot\,)$ by defining the n-group operation as follows:

$$[x_1\ldots x_n]=x_1\cdot\ldots\cdot x_n.$$

Evidently, the equation (3.1), as well as the equations (3.2)–(3.4), is a generalization of the translation equation (0.1). Let us observe that every solution $F\colon X\times G\to X$ of the translation equation (0.1) satisfies the equation (3.4) (for every $n\geq 2$). The converse is not true.

Example 3.1.1 ([27])**.** Let $n=3$, $X=\mathbb{R}$. Let $(G,\,\cdot\,,e):=(\mathbb{R}\setminus\{0\},\,\cdot\,,1)$ and $\Phi(\alpha,x):=-\alpha\cdot x$. It is easy to check that Φ satisfies equation (3.4) and Φ does not satisfy the translation equation (0.1).

Example 3.1.2 ([27])**.** Let $n=4$, $X=\{\gamma,\beta_0,\beta_1,\beta_2\}$. Let $(G,\,\cdot\,,e)$ be an arbitrary group. Define $\Phi\colon X\times G\to X$ by the formula

$$\Phi(\alpha,x):=\begin{cases}\beta_0 & \text{for } \alpha=\gamma,\\ \beta_{(i+1)_3} & \text{for } \alpha=\beta_i,\ i\in\{0,1,2\},\end{cases}$$

where $(i+1)_3$ denotes the remainder of division of $i+1$ by 3. Similarly, to the Example 3.1.1, the function Φ is a solution of equation (3.4) and Φ does not satisfy the translation equation (0.1).

Example 3.1.3 ([29])**.** Let $n=3$, $X=\mathbb{C}$ where $\mathbb{C}$ denotes the set of complex numbers. Let $(G,\,\cdot\,,e)=(\mathbb{C},+,0)$, $\mu(z)=\overline{z}$ and $\Phi(\alpha,z):=\overline{\alpha}+\overline{z}$. It is easy to check that Φ satisfies equation (3.3) and Φ does not satisfy the translation equation (0.1).

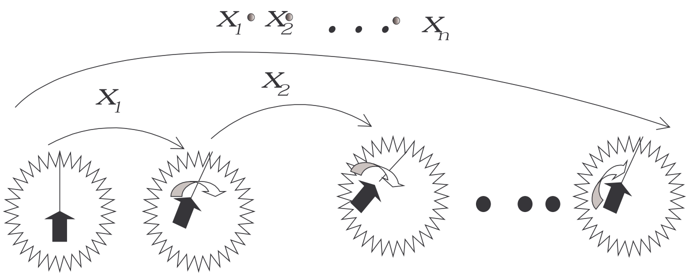

FIGURE 6

At the end of this subsection, we shall present a model of the equation (3.4). Let us consider a cogwheel with $n-1$ cogs. After the signal x_i the cogwheel moves on a distance x_i with one cog clockwise rotation. The Figure 6 illustrates this model.

Therefore, generalized translation equation describes the movement and rotation in contrast to the classical translation equation, which can have such kind of interpretation only by movement.

3.2. The equivalent problems of generalized translation equation

The problem of describing all solutions of the equation (3.2) is still open. The paper [28] contains the results in which a characterization of the solution of equation (3.2) by means of the following Problem 3.2.1 is given.

Problem 3.2.1 ([28])**.** Describe all solutions $F\colon X \times G \to X$ of the translation equation (0.1) satisfying the conditions

$$F(\alpha, e) = (h \circ f^{n-2}(\alpha)), \tag{3.6}$$

$$F(f(\alpha), x) = f(F(\alpha, \mu(x)) = h(F(\alpha, \mu(x) \cdot a) = F(h(\alpha), a \cdot x), \tag{3.7}$$

for $(\alpha, x) \in X \times G$, where $\mu \in \mathrm{Aut}(G, \cdot, e)$, $\mu(a) = a$, $\mu^{n-1}(a) = a \cdot x \cdot a^{-1}$, $x \in G$, and $f, h \in X^X$ are fixed functions such that

$$f \circ h = h \circ f, \tag{3.8}$$

$$f^{n-1} \circ h = f \quad \text{and} \quad f^{n-2} \circ h^2 = h. \tag{3.9}$$

The paper [28] includes the proofs of the following two theorems.

Theorem 3.2.2. *If $\Phi\colon X \times G \to X$ is a solution of the equation* (3.2) *and $f(\alpha) := \Phi(\alpha, e)$, $h(\alpha) := \Phi(\alpha, a^{-1})$, then the function $F\colon X \times G \to X$ given by the formula*

$$F(\alpha, x) := \Phi[f^{n-2}(\alpha), a^{-1} \cdot \mu(x)]$$

is the solution of the translation equation (0.1) *satisfying the conditions* (3.6)*,* (3.7)*, and*

$$f(F(\alpha, x)) = \Phi(\alpha, x). \tag{3.10}$$

Theorem 3.2.3. *Let f, h be the functions satisfying the conditions* (3.8)*,* (3.9)*. Let $F\colon X \times G \to X$ be a solution of the translation equation* (0.1) *satisfying the conditions* (3.6)*,* (3.7)*. The function $\Phi\colon X \times G \to X$ given by the formula* (3.10) *is the solution of the equation* (3.2) *fulfilling the conditions $\Phi(\alpha, e) = f(\alpha)$, $\Phi(\alpha, a^{-1}) = h(\alpha)$.*

In the papers [27], [29], it was shown that, in order to characterize all solutions of the equation (3.3), (3.4) respectively, it is necessary and sufficient to solve the particular cases of the Problem 3.2.1, namely the following Problem 3.2.1′ and 3.2.1″, respectively.

Problem 3.2.1′. Describe all solutions $F: X \times G \to X$ of the translation equation (0.1) satisfying the conditions:

$$F(\alpha, e), = f^{n-1}(\alpha), \tag{3.11}$$

$$F(f(\alpha), x) = f(F(\alpha, \mu(x)), \quad \text{for } (\alpha, x) \in X \times G, \tag{3.12}$$

where $\mu \in \operatorname{Aut}(G, \cdot, e)$, $\mu^{n-1}(x) = x$, $x \in G$, and $f \in X^X$ is a fixed function such that $f^n = f$.

Problem 3.2.1″. Describe all solutions $F: X \times G \to X$ of the translation equation (0.1) satisfying the conditions

$$F(\alpha, e) = f^{n-1}(\alpha), \tag{3.13}$$

$$F(f(\alpha), x) = f(F(\alpha, x), \quad \text{for } (\alpha, x) \in X \times G, \tag{3.14}$$

where $f \in X^X$ is a fixed function such that $f^n = f$.

The Problem 3.2.1″ is a particular case of the Problem 3.2.1′, which is a particular case of a Problem 3.2.1. The papers [27], [29] include the solutions of above mentioned problems 3.2.1″, 3.2.1′, i.e. the constructions of general solutions of equations (3.4) and (3.3), respectively.

Remark 3.2.4. If $F: X \times G \to G$ is a solution of the translation equation (0.1) and $F(\alpha, e) =: f(\alpha)$, then evidently $f(\alpha) = f^2(\alpha)$ and

$$F(f(\alpha), x) = F(F(\alpha, e), x) = F(\alpha, x) = F(F(\alpha, x), e) = f(F(\alpha, x)).$$

Therefore, if we put $n = 2$, then the Problem 3.2.1″ is the problem of solving the translation equation (0.1). This means, that every construction describing all solutions of the Problem 3.2.1″ or 3.2.1′ or 3.2.1 is a generalization of Construction 2.1.1.

3.3. Main constructions

Theorem 3.3.1 ([27, Theorem 3])**.** *Let $(G, \cdot, e)$ be a group and X be an arbitrary set. Let the function $f \in X^X$ satisfy the conditions: $f^n = f$, $n \geq 3$ and $f(\alpha) \neq f^l(\alpha)$, $1 < l < n$, $\alpha \in X$. All solutions $F: X \times G \to X$ of the translation equation* (0.1) *satisfying* (3.13), (3.14)*, and only those, are obtained by the following construction.*

Construction 3.3.2.

(1) Let us put $B := \{\{f(\alpha), f^2(\alpha), \ldots, f^{n-1}(\alpha)\} : \alpha \in X\}$ and decompose $B = \bigcup_{t \in T} X_t^0$ into a union of nonempty disjoint sets such that
(a) $\forall t \in T \ \exists G_t^0 \leq G : \operatorname{card} X_t^0 = \operatorname{card} G/G_t^0$,

(b) $\forall t \in T\ \exists G_t \lhd G_t^0 : (G_t^0 : G_t) = (n-1)/\nu(t)$,
where G_t^0/G_t is the cyclic group generated by $G_t \cdot x_t$ and $\nu\colon T \to \{1, \dots, n-1\}$ is a function such that $\nu(t)$ divides $n-1$ for every $t \in T$.

(2) For every t belonging to T let us refer by X_t^1 to a selection of the set X_t^0 and by S_t to a selection of the set

$$\{\{W, x_t W, x_t^2 W, \dots, x_t^{(n-1)/\nu(t)-1} W\} : W \in G/G_t\}.$$

(let us observe that by (1)(a), (1)(b) we have $\operatorname{card} X_t^1 = \operatorname{card} S_t$).

(3) For every $t \in T$, let

$$X_t := X_t^1 \cup f^{\nu(t)}(X_t^1) \cup f^{2\nu(t)}(X_t^1) \cup \ldots \cup f^{((n-1)/\nu(t)-1)\cdot\nu(t)}(X_t^1)$$

and let us define the bijection $g_t\colon G/G_t \to X_t$ in the following way:

$$g_t(W) := f^{n-1-j\nu(t)}(g_t^*(x_t^j \cdot W)),$$

if $x_t^j W \in S_t$ for $j \in \{0, \dots, (n-1)/\nu(t)-1\}$ where $g_t^*\colon S_t \to X_t^1$ is an arbitrary bijection.

(4) For $t \in T$ such that $\nu(t) > 1$ and for every $l \in \{1, \dots, \nu(t)-1\}$ we put $\overline{X}_t^l := f^l(X_t)$, $\overline{G}_t^l := G_t$, and define $\overline{g}_t^l\colon G/\overline{G}_t^l \to \overline{X}_t^l$ as

$$\overline{g}_t^l(W) := f^l(g_t(W)).$$

(5) We distinguish the family of sets $\{X_t : t \in T\} \cup \{\overline{X}_t^l : t \in T,\ l \in \{1, \dots, \nu(t)-1\}\}$ by $\{X_k\}_{k\in K}$ and the family of subgroups $\{G_t\colon t \in T\} \cup \{\overline{G}_t^l : t \in T,\ l \in \{1, \dots, \nu(t)-1\}\}$ by $\{G_k\}_{k\in K}$. We distinguish the family of bijections $\{g_t : t \in T\} \cup \{\overline{g}_t^l : t \in T,\ l \in \{1, \dots, \nu(t)-1\}\}$ by $\{g_k\}_{k\in K}$.

(6) We put $F(\alpha, x) := g_k(g_k^{-1}(f^{n-1}(\alpha)) \cdot x)$, when $f^{n-1}(\alpha) \in X_k$.

From now onwards, we will use the definitions (see [29]):

$$D_p := \{m \in \{1, \dots, p\} : m \text{ divides } p\}, \quad \text{for } p \in \mathbb{N},$$
$$A^m := \{\alpha \in X : f(\alpha) = f^{m+1}(\alpha) \text{ and } f(\alpha) \neq f^l(\alpha) \text{ for } 1 < l \leq m\}, \quad \text{for } m \in D_{n-1}.$$

Theorem 3.3.3 ([27, Theorem 4])**.** *Let $(G, \cdot, e)$ be a group and X an arbitrary nonempty set. Let the function $f \in X^X$ satisfy $f^n = f$ where $n \geq 2$ is a natural number. All solutions $F\colon X \times G \to X$ of the translation equation* (0.1) *satisfying conditions* (3.13), (3.14) (*i.e. all solutions of Problem 3.2.1″*), *and only those, can be obtained by the Construction* 3.3.4.

Construction 3.3.4.

(1) Let $\phi\colon A^1 \times G \to A^1$ be a solution of the translation equation (0.1), obtained by Construction 2.1.1, such that $\phi(\alpha, e) = f^{n-1}(\alpha)$.

(2) For every $m \in D_{n-1} \setminus \{1\}$, let $\psi_m\colon A^m \times G \to A^m$ be a solution of the translation equation (0.1) obtained by Construction 3.3.2.

(3) We put

$$F := \phi \cup \bigcup_{m \in D_{n-1} \setminus \{1\}} \psi_m. \tag{3.15}$$

Remark 3.3.5 ([27])**.** In Theorem 3.3.1 it was supposed that $n \geq 3$, but it is easy to see, that for $n = 2$ the Construction 3.3.2 is valid and is reducible to the Construction 2.1.1. Therefore, we can admit that all functions in (3.15), this means ϕ and ψ_m for $m \in D_{n-1} \setminus \{1\}$ are obtained by Construction 3.3.2. Therefore, the solution (3.15) can be presented in the form

$$F = \bigcup_{m \in D_{n-1}} \psi_m. \tag{3.16}$$

In the next theorem, including the Construction 3.3.8, which generalizes the constructions 2.1.1, 3.3.2, 3.3.4, the following definition is used.

Definition 3.3.6 ([29, Definition and Remark 12])**.** If G_k is a subgroup of the group G, $x_k \in G$, $m \in D_{n-1}$ and the conditions

$$\prod_{i=0}^{(n-1)/m-1} \mu^{n-1-im}(x_k) \in G_k \quad \text{and} \quad \mu^m(x_k^{-1} G_k x_k) = G_k$$

are satisfied, then we define

$$G/G_k^{\mu, x_k, d} := \{W \in G/G_k : \theta^*(W) = d\}, \quad \text{for } d \in D_{(n-1)/m}$$

where θ^* is defined by

$$\theta^*(W) := \min_{j \in \{1, \ldots, (n-1)/m\}} \theta(W, j) = W$$

and $\theta\colon G/G_k \times \{1, \ldots, (n-1)/m\} \to G/G_k$ is defined by

$$\theta(W, j) := \prod_{i=0}^{j-1} \mu^{n-1-im}(x_k) \cdot \mu^{n-1-jm}(W).$$

Theorem 3.3.7 ([29, Theorem 3])**.** *Let $(G, \cdot, e)$ be a group, μ an automorphism of G such that $\mu^{n-1} = \mathrm{id}_G$ (identity), X an arbitrary nonempty set and let the function $f \in X^X$ satisfy the equality $f^n = f$. All solutions $F\colon X \times G \to X$*

of the translation equation (0.1) *satisfying* (3.11), (3.12) (*i.e. all solutions of Problem* 3.2.1′)*, and only those, are obtained by the Construction* 3.3.8.

Construction 3.3.8.

(1) Let us put $B := \{\{f(\alpha), f^2(\alpha), \dots , f^{n-1}(\alpha)\} : \alpha \in X\}$. We decompose $B = \bigcup_{t\in T} X^0_t$ into a union of nonempty disjoint sets such that
(a) for all $t \in T$ there exist $G_t \le G$ and $x_t \in G$ such that

$$\prod_{i=0}^{(n-1)/\nu(t)-1} \mu^{n-1-i\nu(t)}(x_t) \in G_t \quad \text{and} \quad \mu^{\nu(t)}(x_t^{-1} G_t x_t) = G_t,$$

(b) $X^0_t \subset \bigcup_{d\in D_{(n-1)/\nu(t)}} \{\{f(\alpha), f^2(\alpha), \dots , f^{n-1}(\alpha)\} : \alpha \in A^{d\cdot\nu(t)}\}$,
(c) for $d \in D_{(n-1)/\nu(t)}$

$$d \cdot \operatorname{card}(X^0_t \cap \{\{f(\alpha), \dots , f^{n-1}(\alpha)\} : \alpha \in A^{d\cdot\nu(t)}\}) = \operatorname{card}(G/G_t^{\mu,x_t,d}),$$

where $\nu: T \to \{1, \dots , n-1\}$ is a function such that $\nu(t) \in D_{n-1}$ for every $t \in T$.

(2) For every t belonging to T let us refer by X^1_t to a selection of the set X^0_t and by S_t to a selection of the set

$$\Big\{\Big\{W, x_t\mu^{n-1-\nu(t)}(W), x_t\mu^{n-1-\nu(t)}(x_t)\mu^{n-1-2\nu(t)}(W), \dots ,\\ \prod_{i=0}^{(n-1)/\nu(t)-2} \mu^{n-1-i\nu(t)}(x_t)\mu^{\nu(t)}(W)\Big\} : W \in G/G_t\Big\}.$$

(let us observe that by (1)(b), (1)(c) we have $\operatorname{card} X^1_t = \operatorname{card} S_t$).

(3) For every $t \in T$, let

$$X_t := X^1_t \cup f^{\nu(t)}(X^1_t) \cup f^{2\nu(t)}(X^1_t) \cup \ldots \cup f^{((n-1)/\nu(t)-1)\cdot\nu(t)}(X^1_t)$$

and let us define the bijection $g_t: G/G_t \to X_t$ in the following way

$$g_t(W) := f^{n-1-j\nu(t)}\Big(g^*_t\Big(\prod_{i=0}^{j-1} \mu^{n-1-i\nu(t)}(x_t) \cdot \mu^{n-1-j\nu(t)}(W)\Big)\Big), \tag{3.17}$$

if

$$\prod_{i=0}^{j-1} \mu^{n-1-i\nu(t)}(x_t) \cdot \mu^{n-1-j\nu(t)}(W) \in S_t \quad \text{for } j \in \Big\{1, \dots , \frac{n-1}{\nu(t)}\Big\}$$

where $g^*_t: S_t \to X^1_t$ is an arbitrary bijection satisfying the condition

$$W \in G/G_t^{\mu,x_t,d} \Rightarrow g^*_t(W) \in X^1_t \cap A^{d\cdot\nu(t)} \text{ for } d \in D_{(n-1)/\nu(t)}. \tag{3.18}$$

(4) For $t \in T$ such that $\nu(t) > 1$ and for every $l \in \{1, \dots, \nu(t)-1\}$ we put

$$\overline{X}_t^l := f^l(X_t), \quad \overline{G}_t^l := \mu^{n-1-l}(G_t),$$

and define $\overline{g}_t^l\colon G/\overline{G}_t^l \to \overline{X}_t^l$ by $\overline{g}_t^l(W) := f^l(g_t(\mu^l(W)))$.

(5) Let us distinguish the family of sets $\{X_t : t \in T\} \cup \{\overline{X}_t^l : t \in T, l \in \{1, \dots, \nu(t)-1\}\}$ by $\{X_k\}_{k\in K}$ and the family of subgroups $\{G_t : t \in T\} \cup \{\overline{G}_t^l : t \in T, l \in \{1, \dots, \nu(t)-1\}\}$ by $\{G_k\}_{k\in K}$. Let us distinguish the family of bijections $\{g_t : t \in T\} \cup \{\overline{g}_t^l : t \in T, l \in \{1, \dots, \nu(t)-1\}\}$ by $\{g_k\}_{k\in K}$.

(6) Put $F(\alpha, x) := g_k(g_k^{-1}(f^{n-1}(\alpha)) \cdot x)$ when $f^{n-1}(\alpha) \in X_k$.

Example 3.3.9 (compare [27]). Let $(G, \cdot, e) := (\mathbb{Z}_{12}, +_{12}, 0)$ be the group of residues modulo 12 ($\mathbb{Z}_{12} = \{0, 1, \dots, 11\}$) and $n := 5$. Let us write $I_p := \{0, \dots, p\}$, for $p = 0, 1, \dots$. Let $X := \mathbb{Z}_{12} \cup \{\sigma_i^j : i \in I_3,\ j \in I_4\}$ and let $f \in X^X$ be defined by

$$f(\alpha) := \begin{cases} \alpha & \text{if } \alpha \in \mathbb{Z}_{12}, \\ \sigma^j_{(i+1)_4} & \text{if } \alpha = \sigma_i^j,\ i \in I_3,\ j \in I_4, \end{cases}$$

where $(i+1)_4$ denotes the remainder of division of $i+1$ by 4. Then $f^5 = f$.

According to the Construction 3.3.4 we have $D_4 = \{1, 2, 4\}$ and $A^1 = \mathbb{Z}_{12}$, A^2 is empty, $A^4 = X \setminus \mathbb{Z}_{12}$. Let us define (see Remark 3.3.5) $\psi_1\colon A^1 \times \mathbb{Z}_{12} \to A^1$ as follows

$$\psi_1(\alpha, x) := \alpha +_{12} x.$$

Evidently ψ_1 is a solution of the translation equation (0.1) and ψ_1 can be obtained by Construction 2.1.1 (or by Construction 3.3.2). Because ψ_2 is empty, we will define only $\psi_4\colon A^4 \times \mathbb{Z}_{12} \to A^4$ according to Construction 3.3.2.

We have here $B = \{\{\sigma_i^j : i \in I_3\} : j \in I_4\}$. Let $T := \{1, 2\}$. We decompose $B = X_1^0 \cup X_2^0$ into a union of disjoint sets: $X_1^0 := \{\{\sigma_i^j : i \in I_3\} : j \in I_1\}$ and $X_2^0 := \{\{\sigma_i^j : i \in I_3\} : j \in I_4 \setminus I_1\}$ and we take the subgroups $G_1^0 := \{2j : j \in I_5\}$, $G_2^0 := \{3j : j \in I_3\}$.

We define also $\nu\colon T \to \{1, 2, 3, 4\}$ as follows: $\nu(1) := 2$, $\nu(2) := 1$. In accordance with (1)(b) of Construction 3.3.2 we take the invariant subgroups $G_1 := \{4j : j \in I_2\}$, $G_2 := \{0\}$ of G_1^0, G_2^0 respectively, and we put $x_1 := 2$, $x_2 := 3$.

Using (2) of Construction 3.3.2 we set $X_1^1 := \{\sigma_0^0, \sigma_0^1\}$, $X_2^1 := \{\sigma_0^2, \sigma_0^3, \sigma_0^4\}$, $S_1 := \{\{4j : j \in I_2\}, \{4j+1 : j \in I_2\}\}$ and $S_2 := \{\{0\}, \{1\}, \{2\}\}$.

According to (3) of Construction 3.3.2, we define $X_1 = X_1^1 \cup f^2(X_1^1) = \{\sigma_0^0, \sigma_0^1\} \cup \{\sigma_2^0, \sigma_2^1\}$ and $X_2 = X_2^1 \cup f(X_2^1) \cup f^2(X_2^1) \cup f^3(X_2^1) = \{\sigma_i^j : i \in I_3,\ j \in I_4 \setminus I_1\}$.

We define the bijections $g_1^*\colon S_1 \to X_1^1$, $g_2^*\colon S_2 \to X_2^1$ by the following way: $g_1^*(\{4j+p : j \in I_2\}) := \sigma_0^p$, for $p \in I_1$, and $g_2^*(\{i\}) := \sigma_0^{2+i}$, for $i \in I_2$.

By (3.17), the bijections $g_1\colon G/G_1 \to X_1$, $g_2\colon G/G_2 \to X_2$ are defined as follows:

$$\begin{aligned} g_1(W) &:= f^{4-2j}(g_1^*(W+2j)), \quad \text{if } W+2j \in S_1, \text{ for } j \in I_1, \\ g_2(W) &= f^{4-j}(g_1^*(W+3j)), \quad \text{if } W+3j \in S_2, \text{ for } j \in I_3. \end{aligned}$$

Thus, we have

$$\begin{aligned} g_1(\{4j : j \in I_2\}) &= f^4(\sigma_0^0) = \sigma_0^0, & g_1(\{4j+1 : j \in I_2\}) &= f^4(\sigma_0^1) = \sigma_0^1, \\ g_1(\{4j+2 : j \in I_2\}) &= f^2(\sigma_0^0) = \sigma_2^0, & g_1(\{4j+3 : j \in I_2\}) &= f^2(\sigma_0^1) = \sigma_2^1 \end{aligned}$$

and

$$\begin{aligned} g_2(\{i\}) &= \sigma_0^{2+i}, \quad \text{for } i \in I_2, & g_2(\{i\}) &= \sigma_1^{i-1}, \quad \text{for } i \in I_5 \setminus I_2, \\ g_2(\{i\}) &= \sigma_2^{i-4}, \quad \text{for } i \in I_8 \setminus I_5, & g_2(\{i\}) &= \sigma_3^{i-7}, \quad \text{for } i \in I_{11} \setminus I_8. \end{aligned}$$

We use now (4) in Construction 3.3.2. We have $\nu(1) = 2 > 1$. Therefore, for $l \in \{1, \ldots, \nu(1)-1\} = \{1\}$ we put

$$\overline{X}_1^1 := f(X_1) = \{\sigma_1^0, \sigma_3^0, \sigma_1^1, \sigma_3^1\} \quad \text{and} \quad \overline{G}_1^1 := G_1.$$

The function $\overline{g}_1^1\colon G/G_1 \to \overline{X}_1^1$ is defined by $\overline{g}_1^1(W) = f(g_1(W))$.

According to (5) and (6) of the Construction 3.3.2, we put $K = \{1,2,3\}$ and distinguish $\{X_1, X_2, \overline{X}_1^1\}$ by $\{X_k : k \in K\}$ and $\{g_1, g_2, \overline{g}_1^1\}$ by $\{g_k : k \in K\}$.

We set $\psi_4(\alpha, x) := g_k(g_k^{-1}(\alpha) +_{12} x)$, for $\alpha \in A^4, x \in \mathbb{Z}_{12}$. As in (3.16) we define $F := \psi_1 \cup \psi_4$.

The next example is obtained by Construction 3.3.8.

Example 3.3.10 ([29])**.** Let $n := 4$, $X := \{\sigma, \beta_0, \beta_1, \beta_2\}$. Let $(G, \cdot, e) := (X, \circ, i)$ where $X := \{i = x_1, \ldots, x_{12}\}$, be the group of pair permutations of a set of 4 elements. The table of this group is the Table 1.

Let $f \in X^X$ be defined by $f(\sigma) = \sigma$, $f(\beta_i) = \beta_{(i+1)_3}$ where $(i+1)_3$ denotes the remainder of division of $i+1$ by 3. Then $f^4 = f$.

Let μ be the inner automorphism defined by the formula $\mu(x) := x_2 \circ x \circ x_3$. We have $\mu^2(x) := x_2 \circ x_2 \circ x \circ x_3 \circ x_3 = x_3 \circ x \circ x_2$ and $\mu^3(x) := x_2 \circ x_3 \circ x \circ x_2 \circ x_3 = x$.

According to the Construction 3.3.8 we put

$$B := \{\{f(\alpha), f^2(\alpha), f^3(\alpha)\} : \alpha \in X\} = \{\{\sigma\}, \{\beta_0, \beta_1, \beta_2\}\} = X_{t_0}^0,$$

$\circ$	i	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
i	i	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
x_2	x_2	x_3	i	x_{11}	x_9	x_4	x_{12}	x_7	x_{10}	x_5	x_6	x_8
x_3	x_3	i	x_2	x_6	x_{10}	x_{11}	x_8	x_{12}	x_5	x_9	x_4	x_7
x_4	x_4	x_{10}	x_8	x_5	i	x_{12}	x_2	x_{11}	x_6	x_7	x_3	x_9
x_5	x_5	x_7	x_{11}	i	x_4	x_9	x_{10}	x_3	x_{12}	x_2	x_8	x_6
x_6	x_6	x_9	x_{12}	x_{10}	x_3	x_7	i	x_4	x_{11}	x_8	x_2	x_5
x_7	x_7	x_{11}	x_5	x_8	x_{12}	i	x_6	x_{10}	x_2	x_4	x_9	x_3
x_8	x_8	x_4	x_{10}	x_{12}	x_7	x_3	x_{11}	x_9	i	x_6	x_5	x_2
x_9	x_9	x_{12}	x_6	x_2	x_{11}	x_{10}	x_5	i	x_8	x_3	x_7	x_4
x_{10}	x_{10}	x_8	x_4	x_3	x_6	x_5	x_9	x_2	x_7	i	x_{12}	x_{11}
x_{11}	x_{11}	x_5	x_7	x_9	x_2	x_8	x_3	x_6	x_4	x_{12}	i	x_{10}
x_{12}	x_{12}	x_6	x_9	x_7	x_8	x_2	x_4	x_5	x_3	x_{11}	x_{10}	i

TABLE 1

so $T = \{t_0\}$. Let $\nu(t_0) := 1$ and let us take the subgroup $G_{t_0} := \{i, x_2, x_3\}$, $x_{t_0} := x_2$. We have:

$$\begin{aligned}\prod_{i=0}^{(n-1)/\nu(t_0)-1} \mu^{n-1-i\nu(t_0)}(x_{t_0}) &= \prod_{i=0}^{2} \mu^{3-i}(x_2) = \mu^3(x_2)\mu^2(x_2)\mu(x_2)\\ &= x_2 \circ x_3 \circ x_2 \circ x_2 \circ x_2 \circ x_2 \circ x_3 = i \in G_{t_0},\\ \mu(G_{t_0}) &= G_{t_0}, \quad \mu(x_3 \circ G_{t_0} \circ x_2) = G_{t_0}\end{aligned}$$

and

$$\prod_{i=0}^{j-1} \mu^{3-i}(x_2) \circ \mu^{3-j}(G_{t_0}) = G_{t_0} \circ \prod_{i=1}^{3-j} \mu^{i}(x_3) \quad \text{for } j \in \{1,2\}.$$

Moreover, $D_{(n-1)/\nu(t_0)} = D_3 = \{1,3\}$, so

$$\begin{aligned}X_{t_0}^0 \subset \bigcup_{d\in\{1,3\}} \{\{f(\alpha), f^2(\alpha), f^3(\alpha)\} : \alpha \in A^d\} = \{\{\sigma\}, \{\beta_0, \beta_1, \beta_2\}\},\\ \operatorname{card}(X_{t_0}^0 \cap \{\{\sigma\}\}) = 1 = \operatorname{card}(G/G_{t_0}^{\mu,x_2,1}) = \operatorname{card}\{\{i, x_2, x_3\}\},\end{aligned}$$

and

$$\begin{aligned}3 \cdot \operatorname{card}(X_{t_0}^0 \cap \{\{\beta_0, \beta_1, \beta_2\}\}) = 3 = \operatorname{card}(G/G_{t_0}^{\mu,x_2,3})\\ = \operatorname{card}\{\{x_7, x_8, x_{12}\}, \{x_4, x_6, x_{11}\}, \{x_5, x_9, x_{10}\}\}.\end{aligned}$$

Using (2) of Construction 3.3.8 we take

$$X_{t_0}^1 := \{\sigma, \beta_0\} \quad \text{and} \quad S_{t_0} := \{\{i, x_2, x_3\}, \{x_7, x_8, x_{12}\}\}.$$

According to the point (3) of this construction we define

$$X_{t_0} := X_{t_0}^1 \cup f(X_{t_0}^1) \cup f^2(X_{t_0}^1) = \{\sigma, \beta_0\} \cup \{\sigma, \beta_1\} \cup \{\sigma, \beta_2\} = \{\sigma, \beta_0, \beta_1, \beta_2\}.$$

According to (3.18) we define the bijection $g_{t_0}^* : S_{t_0} \to X_{t_0}^1$ as follows

$$g_{t_0}^*(\{i, x_2, x_3\}) := \sigma, \; g_{t_0}^*(\{x_7, x_8, x_{12}\}) := \beta_0.$$

According to (3.17) we define the bijection $g_{t_0} \colon G/G_{t_0} \to X_{t_0}$ as follows

$$\begin{aligned} g_{t_0}(\{i, x_2, x_3\}) &= \sigma, \quad & g_{t_0}(\{x_7, x_8, x_{12}\}) &= \beta_0, \\ g_{t_0}(\{x_4, x_6, x_{11}\}) &= \beta_1, \quad & g_{t_0}(\{x_5, x_9, x_{10}\}) &= \beta_2. \end{aligned}$$

According to the point (5) of Construction 3.3.8, we put $K := \{1\}$ and we distinguish X_{t_0} by X_1, G_{t_0} by G_1 and g_{t_0} by g_1. As in the point (6) of Construction 3.3.8, we put

$$F(\alpha, x) := g_1(g_1^{-1}(f^3(\alpha)) \circ x) \quad \text{where } f^3(\alpha) \in X_1.$$

One can illustrate the solution with the Figure 7.

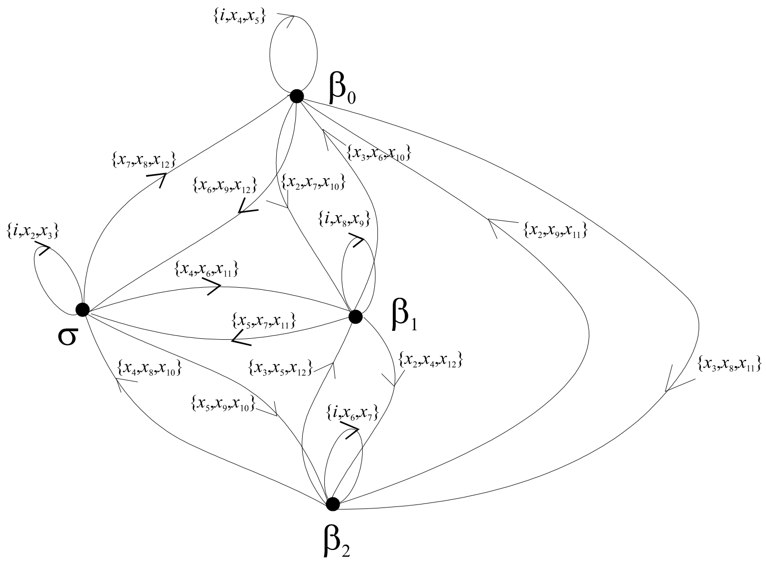

Figure 7

By $\alpha \xrightarrow{\{x,y,z\}} \beta$ we understand that $F(\alpha, x) = \beta,\ F(\alpha, z) = \beta,\ F(\alpha, y) = \beta$. One can easily verify that function $\Phi: X \times G \to X$ defined by the equality $\Phi(\alpha, x) = f(F(\alpha, x))$ is a solution of equation

$$\Phi(\Phi(\Phi(\Phi(\alpha, y_1), y_2), y_3), y_4) = \Phi(\alpha, y_1 \circ x_2 \circ y_2 \circ x_2 \circ y_3 \circ x_2 \circ y_4).$$

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