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Preface

The present volume contains papers selected from those submitted by mathematicians lecturing at the minisemester organized by the International Stefan Banach Mathematical Center in Warsaw with the cooperation of the Juliusz Schauder Center for Nonlinear Studies in Toruń. This minisemester was held during the period of September 22 – October 3, 1997 at Warsaw and it was devoted to topological methods in differential inclusions and optimal control problems.

The orginazers: Helena Frankowska, Lech Górniewicz, Marian Mrozek, Paolo Nistri, Sławomir Plaskacz invited for the plenary talks the internationally distinguished experts working actively in this field. Therefore, this meeting became very useful for a large group of young participants.

The intention of the editors is to provide a presentation of some of the most interesting results concerning areas discussed during the workshop. That is why the present publication consists mainly of research articles, but several survey or expository papers are included as well.

The contributions were received by the editors in Fall 1997 – Spring 1998 and refereed thereafter. They are grouped in three sections: differential inclusions (J. Andres), topological fixed-point theory (L. Górniewicz) and optimal control (P. Nistri) and authours are arranged in the alphabetical order of the names. The referee procesed in each group have been managed by the related editors.

The editors would like to express their gratitude to all the participants, the authors and other people who contributed to the program and activities of the minisemester. We are also indebted to the Banach Center, the Schauder Center for the organization and, in particular, to the Nicholas Copernicus University for the highly appriciated help to publish this volume.

J. Andres
L. Górniewicz
P. Nistri

Olomouc – Toruń – Florence
May 1998.

Differential Inclusions and Optimal Control
Lecture Notes in Nonlinear Analysis
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CONTROL AND OPTIMIZATION OF NONLOCAL STEADY-STATE PROBLEMS

Walter Allegretto and Paolo Nistri

1. Introduction

In a previous paper [1] the authors considered the problem of finding positive solutions to the nonlocal elliptic partial differential equation

$$-\nabla[a\nabla u + \vec{b}\,u] = [h - g\,\overline{u}]u \qquad \text{in } \Omega \tag{I}$$

with associated mixed boundary conditions

$$u = 0 \quad \text{on } \partial\Omega_D \qquad \text{and} \qquad a\frac{\partial u}{\partial n} + (\vec{b}\cdot\vec{n})u = 0 \quad \text{on } \partial\Omega_N \tag{II}$$

where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, $\partial\Omega_D$ closed and we require that regularity conditions hold at points of $N = \partial\Omega_D \cap \overline{\partial\Omega_N}$ (see [7], [8], [9], [11]). Ω is a smooth domain in $\mathbb{R}^n$. Here the nonlocal term $\overline{u}$ is given by

$$\overline{u}(x) = \int_\Omega B_\delta(x,y)u(y)\,dy$$

where $B_\delta(x,y) = B_\delta(|x-y|) \in C_0^\infty$ is a mollifier in $\mathbb{R}^n$, i.e., $\int_{\mathbb{R}^n} B_\delta(x,y)\,dy = 1$ for any x. $B_\delta(|x-y|) = 0$ if $|x-y| \geq \delta$, $B_\delta(|x-y|)$ bounded away from zero if $|x-y| < \mu < \delta$.

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The solutions of (I)-(II) represent the steady-states of an evolution equation similar to those previously introduced by [4], [5] and [2], (see also the extensive references therein).

The aim of this paper is to consider a related control problem to (I)-(II). Namely, motivated also by the results in [11], we consider here the problem

$$-\nabla[a\nabla u + \vec{b}\,u] = [h - g\,\overline{u}]u - f\sqrt{\varepsilon^2 + u^2}I(u > 0) \tag{1}$$

subject to mixed boundary conditions

$$u = u_d \geq 0 \quad \text{on } \partial\Omega_D \qquad \text{and} \qquad a\frac{\partial u}{\partial n} + (\vec{b} \cdot \vec{n})u = 0 \quad \text{on } \partial\Omega_N, \tag{2}$$

where $u \geq 0$ is the density of population of a species, $I(u > 0)$ is the characteristic function of the set $\{x \mid u(x) > 0\}$, $\varepsilon > 0$ and the term $f\sqrt{\varepsilon^2 + u^2}I(u > 0)$ is the so-called "harvesting" term. The function $f(x)$ represents the harvesting intensity, which is the control parameter, $a(x)$ the diffusion process, $\vec{b}(x) = (b_1(x)\dots b_n(x))$ a possible drift, $h(x)$ the intrinsic rate of growth of the species and $g(x)$ the crowding effect. The assumptions on all the above functions will be presented later in Section 1.

The parameter $\varepsilon > 0$ introduces a discontinuity in the harvesting term which makes it possible to have "dead zones": nonempty extinction regions $(u = 0) \subset \Omega$. This is fundamentally different from the problem considered in [1].

Following [10] we associate with (1)-(2) a cost functional to be maximized on the set of all the pairs (u, f) with $u \geq 0$ solution to (1)-(2) corresponding to f from a certain class of functions. For given $\alpha > 0$ the form of the cost functional is

$$J_\alpha(u, f) = \int_\Omega q_1 I(u \geq \alpha) + q_2 f u - q_3 f, \tag{3}$$

where $q_i(x)$, $i = 1, 2, 3$, are nonnegative weights to be chosen. The meaning of maximizing J_α is to maximize the region of Ω in which the density is bounded away from zero taking into account the economic benefit of the harvesting $q_2 f u$ and its cost $-q_3 f$.

We point out that (3) is slightly different from the functional considered in [10], where the cost functional contained the term $I(u > 0)$ instead of the term $I(u \geq \alpha)$, $\alpha > 0$. This modification is necessary since our approach does not allow us to prove the existence of the maximum even in very particular cases of the cost functional containing $I(u > 0)$.

The paper is organized as follows. In Section 1 we first prove in Theorem 1 the existence of nonnegative solutions of (1)-(2) with $u_d \equiv 0$ corresponding to a given control function $f \in L^\infty(\Omega)$, $0 \leq f < M$ a.e. in Ω, for some constant M. The positivity of such solutions in dependence of $\varepsilon > 0$ is also studied. Then in Theorem 2 the case of Dirichlet boundary conditions on both $\partial\Omega_D$ (with $u_d \geq 0$, $u_d \not\equiv 0$), and $\partial\Omega_N$ is considered. It is shown how the extinction region depends on the parameter $\varepsilon > 0$ for a given f.

In Section 2 we assume as the set of admissible controls the closure in the $L^2(\Omega)$-topology of the class of functions considered in Section 1. The existence of nonnegative solutions of (1)-(2) corresponding to all the admissible controls is proved by means of the results of Section 1.

Finally, the properties of the solution map S which associates to f the set of nonnegative solutions of (1)-(2) are investigated. The more relevant property of S, which allows us to solve the associated optimization problem is the closure of its graph in the $w\text{-}L^2(\Omega) \times L^\infty(\Omega)$ topology, where $w\text{-}L^2(\Omega)$ denotes the weak topology in $L^2(\Omega)$.

We would like to point out that the approach presented here in Section 1 for the existence results is quite different from that of [10]. In particular the presence in (1) of the nonlocal term $\overline{u}$ does not allow us to use any method based on order arguments as was shown earlier in [2]. Moreover, we consider here the case of mixed boundary conditions and the state equation is of more general form.

The optimization problem is also solved in a completely different way. In fact, in [10] the solution map is shown to be single-valued monotone and differentiable with respect to the control f. These properties permit the differentiation of the cost functional and the use of the steepest ascent algorithm in searching for a local maximizer of the cost functional. In our case the solution map is, in general, a multivalued one without any other relevant property than the closure of its graph in a suitable topology. In fact, the closure of the graph of S in the $w\text{-}L^2(\Omega) \times L^\infty(\Omega)$ topology will permit us to solve the proposed optimization problem.

2. Existence

We consider here the nonlocal problem

$$-\nabla[a\nabla u + \vec{b}\, u] = [h - g\,\overline{u}]u - f\sqrt{\varepsilon^2 + u^2}\, I(u > 0) \tag{1}$$

subject to mixed boundary conditions

$$u = u_d \geq 0 \quad \text{on } \partial\Omega_D \qquad \text{and} \qquad a\,\frac{\partial u}{\partial n} + (\vec{b}\cdot\vec{n})u = 0 \quad \text{on } \partial\Omega_N. \tag{2}$$

We search for nonnegative solutions of (1)-(2). As mentioned in the Introduction, (1)-(2) is a possible biological model for the steady-states of a species whose density of population is $u \geq 0$ in presence of a nonlocal term $\overline{u}(x) = \int_{\mathbb{R}^n} B_\delta(|x - y|)u(y)\, dy$ and of a discontinuous ($\varepsilon > 0$) harvesting term of which the function f is the control parameter.

We assume the following conditions:

(A) the function a is a piecewise smooth function, smooth near $\partial\Omega_N$, satisfying $0 < a_1 \leq a(x) \leq a_2$ for a.a. $x \in \Omega$; $f, g, h \in L^\infty(\Omega)$ with $g(x)$, $h(x) > c > 0$, $f \geq 0$ for a.a. $x \in \Omega$ and some constant c, u_d and $\vec{b} = (b_1, \ldots, b_n)$ are smooth functions in $\overline{\Omega}$.

(B) if $u_d \equiv 0$ the function f satisfies

$$\operatorname{ess}\inf_{x\in\Omega} [h(x) - f(x)] > \mu_1,$$

where μ_1 is the least eigenvalue of $-\nabla[a\nabla u + \vec{b}\,u]$ subject to (2).

To better understand the meaning of assumption (B) suppose $u > 0$, rewrite (1) in the form

$$-\nabla[a\nabla u + \vec{b}\,u] = [h - f - g\,\overline{u}]u - \frac{\varepsilon^2}{\sqrt{\varepsilon^2 + u^2} + u}\,f$$

and consider the second term on the right hand side as a perturbation for ε small. For $\varepsilon = 0$ and Dirichlet boundary conditions in [2] it has been shown that while (B) may be modified, it cannot be removed.

For convenience we introduce the linear operator $G : L^\infty(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ defined as follows

$$u = Gw, \qquad w \in L^\infty(\Omega)$$

if and only if u satisfies (1)-(2) with the right hand side replaced by w, i.e. $G = (-\nabla[a\nabla\,\cdot\; + \vec{b}\,\cdot\;])^{-1}$. By well known regularity results, see [11], it follows that G is a continuous and compact operator for α small.

We consider first the case when $u_d \equiv 0$, i.e., homogeneous boundary conditions. We have

Theorem 1. *Under assumptions* (A)-(B) *and* $u_d \equiv 0$ *there exist* $\varepsilon_0, \varepsilon_1 > 0$ *such that*

(a) *if* $0 \le \varepsilon < \varepsilon_0$ *then* (1)-(2) *has a nonnegative nontrivial solution;*
(b) *if* $\varepsilon > \varepsilon_1$ *then* (1)-(2) *has only the solution* $u \equiv 0$ *in* $L^\infty(\Omega)$.

Proof. (a) We draw on the results of [1] and [2], and thus only present a short proof for the reader's convenience emphasizing the few differences.

Without loss of generality, we assume that the left hand side of (1) is coercive, otherwise we add a linear term to both sides. Let $0 \le \lambda \le 1$ and consider first the equation

$$-\nabla[a\nabla u + \vec{b}\,u] = \lambda[h - f - g\,\overline{u}]u^+$$

subject to boundary conditions (2). If λ is small enough, the only solution is $u \equiv 0$. Indeed, solutions are nonnegative and satisfy

$$-\nabla[a\nabla u + \vec{b}\,u] - \lambda[h - f]u = -\lambda g\,\overline{u}\,u \le 0,$$

whence $u \equiv 0$ since the least eigenvalue of the operator on the left hand side is positive for λ small. Suppose thus $0 < \lambda_0 \le \lambda \le 1$ and that $0 \le u \in C^\alpha$, for some α, is a solution. Let a_n, h_n, f_n, g_n denote smooth approximations to a, h, f, g respectively. Without loss of generality, we assume that they satisfy the same L^∞ bounds as the original coefficients, and that they converge in L^p for any large p. We may also assume that $\vec{b}\cdot\vec{n} > 0$ on $\partial\Omega_N$ for if not, we replace u by

$v = e^{-w}u$, with $\nabla w \cdot \vec{n} \gg 0$ on $\partial\Omega_N$, chosen suitably in what follows. Consider now the linear eigenvalue problem

$$-\nabla[a_n \nabla u_n + \vec{b}\, u_n] - \lambda[h_n - f_n - g_n\, \overline{u}]u_n = k_n u_n$$

subject to conditions (2). Observe that k_n exists, and it is bounded above and below by constants independent of n. If we normalize the eigenvectors by $\|u_n\|_{L^\infty} = 1$, it follows that u_n is bounded in $H^{1,2} \cap C^\alpha$, see [11], for some $\alpha > 0$, and we may assume $0 \le u_n \to \omega$ weakly in $H^{1,2}$ and strongly in C^{α_0} for some $\alpha_0 < \alpha$. Since $\omega \ge 0$, and thus $\omega > 0$ by the maximum principle, we conclude that $k_n \to 0$ and $\omega \equiv u/\|u\|_{L^\infty}$. Finally, suppose u_n assumes its max in $\overline{\Omega}$ at P_n. In view of the boundary conditions, $P_n \notin \partial\Omega$ as there either $u_n = 0$ or $a(\partial u_n/\partial n) = -(\vec{b}\cdot\vec{n})u_n < 0$. It follows that $P_n \in \Omega$, and from the equation we get

$$\overline{u}\,(P_n) \le \frac{\lambda[h_n - f_n] + \operatorname{div}(\vec{b}) + k_n}{\lambda g_n} \le K$$

for a constant K independent of n, λ. Next suppose $P_n \to P$, then $\overline{u}\,(P) = \lim \overline{u}(P_n)$ and for any $Q \in \Omega$, $u(Q) \le u(P)$ by equicontinuity of the u_n. By a reflection process, see [11], and the generalized Harnack inequality [6], we conclude first that u is bounded in L^∞ and then in C^α for some α. Since $h(x) - f(x) > \mu_1$ for a.a. $x \in \Omega$, we also have that $\|u\|_{C^\alpha}$ cannot be too small and thus, as in [1] and [2], $\operatorname{Deg}(I-T, B_R - B_r, 0) = 1$ where B_ρ is the ball of radius ρ centered at zero in $C^\alpha(\overline{\Omega})$, and $T_\omega = G([h - f - g\overline{\omega}]\omega^+)$ subject to (2). Let $0 \le I_m(u) \le 1$ denote a smooth approximation to $I(u)$, $I_m(\xi) = 0$ if $\xi \le 0$, and consider the perturbation $Z(\omega) = G(\varepsilon^2 f I_m(\omega)/(\sqrt{\varepsilon^2 + \omega^2} + \omega))$ on $B_R - B_r$. If we choose $\varepsilon > 0$ small enough, independent of m, then $\operatorname{Deg}(I - T - Z, B_R - B_r, 0) = 1$ and thus there exists a solution $0 \le \omega_m$ of

$$-\nabla[a\nabla\omega_m + \vec{b}\,\omega_m] = [h - f + g\,\overline{\omega}_m]\omega_m - \frac{\varepsilon^2 f I_m(\omega_m)}{\sqrt{\varepsilon^2 + \omega_m^2} + \omega_m}.$$

We again have $\omega_m \to \omega$ weakly in $H^{1,2}$, strongly in C^α with $\omega \ge 0$, nontrivial. Since $\varepsilon^2 f I_m(\omega_m)/(\sqrt{\varepsilon^2 + \omega_m^2} + \omega_m)$ is bounded, we may take it to be weakly convergent in L^2 to a function z. If $\omega(x) > 0$, then $z = \varepsilon^2 f/(\sqrt{\varepsilon^2 + \omega^2} + \omega)$, while on the set $\{\omega = 0\}$ we have $\nabla\omega = 0$ almost everywhere. We assumed the coefficients are piecewise smooth and thus, on $\Omega - \Gamma$ we have $\omega \in H^{2,2}$, where Γ is a set of measure zero. We conclude that ω satisfies $-\nabla[a\nabla\omega + \vec{b}\omega] = [h - f + g\overline{\omega}]\omega + z$, a.e. on $\Omega - \Gamma$ whence $z = 0$ a.e. on the set $\{\omega = 0\}$.

(b) Again assume without loss of generality that $\vec{b}\cdot\vec{n} > 0$ on $\partial\Omega_N$. Suppose that $u \ge 0$, $u \not\equiv 0$, is a solution and select a constant $c_0 > 0$ such that the linear problem

$$-\nabla[a\nabla\omega + \vec{b}\,\omega] = [h - f - g\,\overline{u}]\omega + c_0\omega$$

subject to (2) does not have eigenvalue zero. Let once again a_n, h_n, f_n, g_n be smooth approximations and consider the problem

$$-\nabla[a_n\nabla\omega_n + \vec{b}\,\omega_n] - [h_n - f_n + c_0 - g_n\overline{u}]\omega_n = \left\{\frac{-\varepsilon^2 f}{\sqrt{\varepsilon^2 + u^2} + u} I(u > 0) - c_0 u\right\}_n,$$

where $\{\cdot\}_n$ denotes a mollifier. This problem has a solution $\omega_n \in C^\alpha$, and we may assume as before that $\omega_n \to u$ in C^{α_0} and weakly in $H^{1,2}$. Since $u \geq 0$, $u \not\equiv 0$, then $\max \omega_n > 0$ for n large, and $\overline{u}(P_n) < K$ where $\max \omega_n = \omega_n(P_n)$. We conclude as in (a), that $\overline{u}(P) \leq K_0$ if $u(P) = \max u$ and consequently that $u \leq K_1$ in Ω, by the generalized Harnack inequality, with K_1 independent of u, ε. We then have

$$-\nabla[a\nabla u + \vec{b}\,u] \leq \left(hK_1 - \frac{\varepsilon^2 f}{\sqrt{\varepsilon^2 + K_1^2} + K_1}\right) \operatorname{sign}(u).$$

For $\varepsilon > 0$ sufficiently large, the right hand side is nonpositive in Ω and thus $u \leq 0$ in Ω, i.e. $u \equiv 0$. □

We now pass to the case $u_d \geq 0$, $u_d \not\equiv 0$. In this case $u \equiv 0$ as a solution is impossible, we do not require condition (B), and we can show the existence of a nonnegative solution of (1)-(2), for any $\varepsilon > 0$, by using the earlier proof to show that $\operatorname{Deg}(I - T, B_R, 0) = 1$, where $T : C^\alpha(\Omega) \to C^\alpha(\Omega)$ is the compact operator composition of G with the Nemytskii operator generated by the right hand side of (1) regularized, and then passing to a limit. Briefly but specifically, suppose a, h, g, f are smooth. We consider

$$-\nabla[a\nabla u + \vec{b}\,u] = \lambda\left([h - f - g\,\overline{u}]u^+ - \frac{\varepsilon^2 f}{\sqrt{\varepsilon^2 + u^2} + u} I_m(u > 0)\right),$$

subject to $u = \lambda u_d$ on $\partial\Omega_D$ and the same natural boundary conditions on $\partial\Omega_N$, for $0 \leq \lambda \leq 1$. Again if λ is small, then $-\nabla[a\nabla u + \vec{b}\,u] - \lambda[h - f]u \leq 0$ and $u = \lambda u_d$ on $\partial\Omega_D$ shows that u is bounded in terms of u_d. Otherwise, either $\|u\|_{L^\infty} = \lambda\|u_d\|_{L^\infty}$ or we again have $\overline{u}(P) \leq K$ for a $P \in \overline{\Omega}$ at which u assumes its maximum. If P is away from $\partial\Omega_D \cap \overline{\partial\Omega_N}$, we proceed exactly as before to bound u in L^∞ and then in C^α for some α. Otherwise we map a neighborhood of P to a quarter sphere, and reflect the coefficients as before and u as an even function to the whole of the upper hemisphere. Since u_d extends as a C^α function, we use results in [6] to bound u in L^∞ and then in C^α. Observe that the L^∞ bound is independent of ε, m and the C^α bound is independent of m.

We now prove that for $\varepsilon > 0$ sufficiently large there exist extinction zones for the solutions of (1) found above subject to the Dirichlet boundary conditions $u = u_d \geq (\not\equiv)\ 0$ on $\partial\Omega_D$. We have the following result.

Theorem 2. *Assume* (A). *Let $\delta > 0$ be given. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon > \varepsilon_0$ then $\mu(u > 0) \leq \delta$.*

We need first the following.

Lemma 1. *Let $\{A_n\}$ be a sequence of measurable sets contained in Ω with $\mu(A_n) > \delta > 0$. Then there exists a subsequence and a function τ, $0 \leq \tau \leq 1$ such that $I(A_n) \to \tau$ in $L^1(\Omega)$ and $\int_\Omega \tau \geq \delta$.*

Proof. Let $f_n = I(A_n)$ for any $n \in \mathbb{N}$, $\{f_n\}$ is a sequence of bounded measurable functions and so, passing to a subsequence if necessary, $f_n \to \tau$ weakly in

$L^2(\Omega)$. Therefore by the Banach-Saks's Theorem $(1/n)\sum_{i=1}^n f_i \to \tau$ in $L^2(\Omega)$. But

$$\left\| \frac{1}{n}\sum_{i=1}^n f_i \right\|_{L^1(\Omega)} \geq \delta$$

and thus

$$\left\| \frac{1}{n}\sum_{i=1}^n f_i \right\|_{L^2(\Omega)} \geq \gamma > 0$$

for some $\gamma > 0$, which implies that $\tau \not\equiv 0$. Moreover, by passing again to a subsequence if necessary, $\{(1/n)\sum_{i=1}^n f_i\}$ converges a.e. in Ω to τ, thus τ takes only values in $[0,1]$ and $\|\tau\|_{L^1} \geq \delta$, whence $\mu(\tau \neq 0) \geq \delta$. □

We prove now Theorem 2.

Proof of Theorem 2. Suppose not, since $g\bar{\omega}\omega \geq 0$ we have

$$-\nabla[a\nabla\omega + \vec{b}\,\omega] - [h-f]\omega \leq -\frac{f\varepsilon^2}{\sqrt{\varepsilon^2+\omega^2}+\omega} I(\omega > 0)$$

and as mentioned above, $0 \leq \omega \leq k$ for some k independent of ε. I.e.,

$$-\nabla[a\nabla\omega + \vec{b}\,\omega] \leq [h-f]k - \frac{f\varepsilon^2}{\sqrt{\varepsilon^2+k^2}+k} I(\omega > 0).$$

Let $\varepsilon_n \to \infty$ and suppose $I(\omega_n > 0) > \delta$. Since $\omega_n \leq k$, we may assume $\omega_n \to \omega \geq 0$ weakly in $H^{1,2}$ and strongly in L^2. Let z_n solve

$$-\nabla[a\nabla z_n + \vec{b}\,z_n] = fI(\omega_n > 0)$$

with $z_n = 0$ on $\partial\Omega_D$, while r solves

$$-\nabla[a\nabla r + \vec{b}\,r] = [h-f]k$$

and $r = u_d$ on $\partial\Omega_D$. Since $I(\omega_n > 0) \to \tau$ in L^1 (and thus L^p for large p) then $z_n \to z$ in $C^\alpha(\overline{\Omega})$ for some $\alpha > 0$ with

$$-\nabla[a\nabla z + \vec{b}\,z] = f\tau.$$

Furthermore, since $f\tau \geq 0$ is nontrivial, $z > 0$ in Ω. Finally, for any given $M > 0$, $Mz_n + \omega_n \leq r$ if ε is large enough, i.e., $Mz + \omega \leq r$ whence $\omega < 0$ somewhere in Ω if M is large enough, contradicting $\omega \geq 0$. □

Remark 1. Observe that if f vanishes somewhere in Ω, then the previous arguments will show the result if $f\tau \not\equiv 0$, i.e., if $\mu\{f(x) = 0\} < \delta$.

3. Control and Optimization

As the set of admissible controls for problem (1)-(2) we consider

$$V = \{f \in L^\infty(\Omega) \mid 0 \le f(x) \le M \text{ for a.a. } x \in \Omega\},$$

where $M = \operatorname{ess\,inf}_{x\in\Omega}[h(x) - \mu_1]$ if $u_d \equiv 0$. Otherwise $M > 0$ is chosen for convenience. Obviously

$$V = \overline{\{f \in L^\infty(\Omega) \mid 0 \le f(x) < M \text{ for a.a. } x \in \Omega\}},$$

where the closure is in the $L^2(\Omega)$-topology. Since V is convex and closed it is weakly closed in the w-$L^2(\Omega)$ topology and vice versa. In Section 1 for $\varepsilon > 0$ as determined in Theorems 1 and 2, we proved that the set

$$S(f) = \{u \in L^\infty \mid u \ge 0 \text{ is a solution of (1)-(2)}\}$$

is a nonempty (compact) set for any $f \in L^\infty(\Omega)$ with $0 \le f(x) < M$ for a.a. $x \in \Omega$.

We can prove the following.

Theorem 3.

(a) *$S(f) \neq \emptyset$ for any $f \in V$;*
(b) *$S : V \to L^\infty(\Omega)$ has closed graph in the w-$L^2(\Omega) \times L^\infty(\Omega)$-topology;*
(c) *$S(V)$ is a compact set in $L^\infty(\Omega)$.*

Proof. (a) This is established in Section 1 for $0 \le f < M$. Otherwise, let $f_m \in V$ with $0 \le f_m < M$ such that $f_m \to f$ in $L^2(\Omega)$ weakly and suppose u_m solves (1), (2) with I replaced by I_m. We still have u_m uniformly bounded, and by a limit argument conclude the existence of a function $u \in S(f)$. Observe that $u \equiv 0$ is possible if $u_d \equiv 0$.

(b) Let $\{(u_n, f_n)\} \subset L^\infty(\Omega) \times V$, $u_n \in S(f_n)$ such that $u_n \to u$ in $L^\infty(\Omega)$, $u \ge 0$ and $f_n \to f \in V$ in $L^2(\Omega)$ weakly. The same arguments as in (a) show that $u \in S(f)$.

(c) Let $\{u_n\} \subset S(V)$ be any sequence and let $\{f_n\} \subset V$ be such that $u_n \in S(f_n)$. By passing to a subsequence if necessary we have $u_n \to u$ in $L^\infty(\Omega)$ and $f_n \to f$ in $L^2(\Omega)$ weakly and so $u \in S(f)$.

Remark 2. The previous results guarantee that the multivalued map S is upper-semicontinuous in the w-$L^2(\Omega) \times L^\infty(\Omega)$-topology (see [3], Corollary 1, p. 42).

Finally, we can solve our optimization problem. Consider the cost functional

$$J_\alpha(u, f) = \int_\Omega q_1 I(u \ge \alpha) + q_2 f u - q_3 f, \qquad \alpha > 0.$$

We assume that the nonnegative weights $q_i \in L^2(\Omega)$, $i = 1, 2, 3$. We can prove the following.

Theorem 4. *For any $\alpha > 0$, the cost functional $J_\alpha(u, f)$ attains a maximum on the set $\mathcal{S} = \{(u, f) \in L^\infty(\Omega) \times V \mid u \in S(f),\ f \in V\}$.*

Proof. Let $\{(u_n, f_n)\} \subseteq \mathcal{S}$ be a maximizing sequence. We have $u_n \to u_0$ in $L^\infty(\Omega)$ and $f_n \to f_0$ in $L^2(\Omega)$ weakly. Consider

$$\lim_{n\to\infty} \int_\Omega q_1 I(u_n \geq \alpha) + q_2 f_n u_n - q_3 f_n$$

$$\leq \limsup_{n\to\infty} \int_\Omega q_1 I(u_n \geq \alpha) + \limsup_{n\to\infty} \int_\Omega q_2 f_n u_n - q_3 f_n$$

$$\leq \int_\Omega \limsup_{n\to\infty} q_1 I(u_n \geq \alpha) + \limsup_{n\to\infty} \int_\Omega q_2 f_n u_n - q_3 f_n.$$

On the other hand,

$$\limsup_{n\to\infty} I(u_n \geq \alpha) \leq I(u_0 \geq \alpha)$$

a.e. in Ω. Rewriting the second term as

$$\int_\Omega [q_2 u_n - q_3] f_n = \int_\Omega [q_2 u_0 - q_3] f_n + \int_\Omega q_2 [u_n - u_0] f_n,$$

by our assumptions on q_2, q_3 we get

$$\lim_{n\to\infty} J_\alpha(u_n, f_n) \leq J_\alpha(u_0, f_0)$$

with $u_0 \in S(f_0)$ by Theorem 3.

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BOUNDED, ALMOST–PERIODIC AND PERIODIC SOLUTIONS OF QUASI–LINEAR DIFFERENTIAL INCLUSIONS

JAN ANDRES

1. Single–valued stimulation

Let us start by recalling two classical results which can be found e.g. in the book [D] (for the latter one cf. also [H1]).

Consider the system of ordinary differential equations

$$X' + AX = f(t, X), \tag{0}$$

where $X \in \mathbb{R}^n$, A is a constant hyperbolic $(n \times n)$-matrix (i.e. with nonzero real parts of the associated eigenvalues) and $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous.

Theorem 1 (P. Bohl). *System* (0) *admits* (*a unique*) *entirely bounded solution* $X(t)$, *namely*

$$\sup_{t\in(-\infty,\infty)} |X(t)| < \infty,$$

provided additionally

(i) $\sup_{t\in(-\infty,\infty)} |f(t, 0)| < \infty$,
(ii) $f(t, X)$ *is Lipschitzian in* X *with a sufficiently small Lipschitz constant.*

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Theorem 2 (G. I. Birjuk). *Let the assumptions of Theorem* 1 *be satisfied. If* $f(t, X)$ *is still (uniformly) almost–periodic (in the sense of H. Bohr) in a t-variable, uniformly w.r.t.* X *from any compact subset of* $\mathbb{R}^n$*, then system* (0) *possesses a (unique) almost–periodic solution.*

Remark 1. In particular, if $f(t+\omega, X) \equiv f(t, X)$, then (0) admits, under the same assumptions, an ω-periodic solution (see [D]). In fact, for the boundedness as well as for the periodicity result the Lipschitzianity can be replaced by a suitable growth restriction (see e.g. [AK]).

Hence, our main purpose is to give the multivalued analogies for the Carathéodory quasi–linear differential inclusions. Besides that, our attention will be paid to the appropriate methods, including the multiplicity criteria in the frame of the generalized Nielsen fixed–point theory. In the almost–periodic case, the new definitions of almost–periodic measurable multifunctions will be presented as correct.

2. Abstract existence and multiplicity results

Consider the system of differential inclusions

$$X' \in F(t, X), \tag{1}$$

where $F : I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a *set–valued Carathéodory mapping*, i.e.

(i) the set of values of F is nonempty, compact and convex for all $(t, X) \in I \times \mathbb{R}^n$;
(ii) the map $F(t, \cdot)$ is u.s.c. for a.a. $t \in I$;
(iii) the map $F(\cdot, X)$ is measurable for all $X \in \mathbb{R}^n$, i.e. for any open $U \in \mathbb{R}^n$ and every $X \in \mathbb{R}^n$ the set $\{t \in (-\infty, \infty) \mid F(\cdot, X) \cap U \neq \emptyset\}$ is measurable;

I is an arbitrary (possibly infinite) real interval.

By *a solution* $X(t)$ of (1), we always mean a locally absolutely continuous function $X(t)$ satisfying (1) for a.a. $t \in I$. The space of all locally absolutely continuous functions from I to $\mathbb{R}^n$ will be denoted by $AC_{loc}(I, \mathbb{R}^n)$.

Considering (1) with the constraint, namely

$$X \in S \subset C(I, \mathbb{R}^n),$$

where S is a nonempty subset, we start with the following existence result (see [A2], Theorem 3, [AGG], Corollary 2.34). Let us recall that the appropriate topology in $C(I, \mathbb{R}^n)$ is the one of the uniform convergence on compact subintervals of I.

Theorem 3 (Existence criterium I). *Consider the boundary value problem*

$$\begin{cases} X' \in F(t, X), \\ X \in S, \end{cases} \tag{2}$$

on a given interval $I \subset \mathbb{R}$, where $F : I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory map and S is a subset of $C(I, \mathbb{R}^n)$. Let $G : I \times \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a Carathéodory map such that $G(t, c, c) \subset F(t, c)$ for all $(t, c) \in I \times \mathbb{R}^n$. Assume that:

(i) *there exists a bounded retract $Q \subset C(I, \mathbb{R}^n)$ of $C(I, \mathbb{R}^n)$ such that the associated problem*

$$\begin{cases} X' \in G(t, X, q(t)), \\ X \in S \cap Q, \end{cases} \tag{3}$$

is solvable on I with an R_δ-set (i.e. an intersection of a decreasing sequence of compact contractible metric spaces) of solutions $T(q)$ for each $q \in Q$;

(ii) *there exists a locally (Lebesgue) integrable function $\alpha : I \to \mathbb{R}$ such that*

$$|G(t, X(t), q(t))| \le \alpha(t) \qquad a.e. \text{ in } I,$$

for any pair $(q, X) \in \Gamma_T$, where Γ_T denotes the graph of T;

(iii) $\overline{T(Q)} \subset S$.

Then problem (2) *has a solution.*

For the multiplicity results, it will be convenient to use the following definition (cf. [A3] and the references therein).

Definition 1. We say, that the mapping $T : Q \rightsquigarrow S$ is *retractible onto Q* if there is a retraction $r : P \to Q$, where P is an open subset of $C(I, \mathbb{R}^n)$ containing $Q \cup S$ and $p \in P \backslash Q$, $r(p) = q$ implies that $p \notin T(q)$.

Its advantage consists in the fact that, for a retractible mapping $T : Q \rightsquigarrow S$ onto Q with a retraction r in the sense of Definition 1, its composition with r, $r \circ T : Q \rightsquigarrow Q$, has a fixed point $\widehat{q} \in Q$ if and only if $\widehat{q} \in T(\widehat{q})$.

In [A3], the following statement has been proved.

Theorem 4 (Multiplicity criterium I). *Consider boundary value problem* (2) *on a given interval $I \subset \mathbb{R}$, where $F : I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory mapping and S is a subset of $C(I, \mathbb{R}^n)$. Let $G : I \times \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a Carathéodory mapping and assume that:*

(i) *there exists a (nonempty) compact, connected subset Q of $C(I, \mathbb{R}^n)$ such that, for any $q \in Q$, the set $T(q)$ of all solutions of the problem*

$$\begin{cases} X' \in G(t, X, q(t)), \\ X \in S \end{cases} \tag{4}$$

on $I \in \mathbb{R}$ is nonempty;

(ii) *$T(Q)$ is bounded in $C(I, \mathbb{R}^n)$, i.e. there exists a positive (single–valued) function $\phi : I \to \mathbb{R}^n$ such that $|\tau(q)| \le \phi(t)$ for all $t \in I$, $\tau \subset T(q)$ and $q \in Q$;*

(iii) *there exists a locally Lebesgue integrable function $\alpha : I \to \mathbb{R}$ such that*

$$|G(t, X(t), q(t))| \le \alpha(t) \qquad a.e. \text{ in } I,$$

for any pair $(q, X) \in \Gamma_T$, where Γ_T denotes the graph of T;

(iv) $\overline{T(Q)} \subset S$.

Assume, furthermore, that the (multivalued) operator $T : Q \rightsquigarrow S$, *related to problem* (4), *is retractible onto* Q *with a retraction* r *in the sense of Definition* 1, *having* R_δ*-values, for any* $q \in Q$ *and, if* T *is not single-valued, then assume particularly that* $\overline{T(Q)} \subset Q$. *At last, let*

$$G(t,c,c) \subset F(t,c)$$

take place a.e. in I, *for any* $c \in \mathbb{R}^n$. *Then the original problem* (2) *admits at least* $N(r|_{T(Q)} \circ T(\cdot))$ *solutions belonging to* Q, *where* $N(\cdot)$ *denotes the generalized Nielsen number (for the definition and more details see* [A3], [AGJ], [KM]).

Since the topological structure of the solution set to (3) or (4) plays an important role, we recall still some appropriate results in [AGG], Theorem 4.7, [A2], Lemma 7, [ADG], Theorem 4.

Proposition 1 ([AGG]). *Let* $F : I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ *be a Carathéodory mapping, where either* $I = [0,\infty)$ *or* $I = [0,\hat{t}]$, $\hat{t} \in (0,\infty)$, *and assume that*

$$|F(t,X)| \le \mu(t)(|X|+1) \tag{5}$$

for every $(t,X) \in I \times \mathbb{R}^n$, *where* $\mu : I \to [0,\infty)$ *is a suitable Lebesgue-integrable bounded function. Then the set* $T(q)$ *of solutions* $X(t)$ *of the global initial value problem for* (1), *i.e.* $X(t)$ *satisfying* (1) *a.e. in* I *and* $X(0) = X_0 \in \mathbb{R}^n$, *is an* R_δ*-set, for every* $X_0 \in \mathbb{R}^n$.

Summing up the conclusions of Proposition 1 (more precisely, of its parametrized modification — cf. also Remark 3 below) and Theorem 4, we can give immediately the following multiplicity criterium for the global initial value problems.

Theorem 5 (Multiplicity criterium II). *Let* $G : I \times \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ *be a Carathéodory product-measurable mapping, where either* $I = [0,\infty)$ *or* $I = [0,\hat{t}]$, $\hat{t} \in (0,\infty)$. *Assume, furthermore, that there exists a (nonempty) compact, connected subset* Q *of* $C(I,\mathbb{R}^n)$ *such that* $|G(t,X,q(t))| \le \mu(t)(|X|+1)$ *holds for every* $(t,X,q) \in I \times \mathbb{R}^n \times Q$, *and problem* (4) *has, for every* $q \in Q$, *a nonempty set of solutions* $T(q)$ *with the property* $\overline{T(Q)} \subset S$, *where* S *is a nonempty bounded subset of* $\{p(t) \in C(I,\mathbb{R}^n) \mid p(0) = X_0,\ X_0 \in \mathbb{R}^n\}$. *At last, let the mapping* $T : Q \rightsquigarrow S$ *be retractible onto* Q *with a retraction* r *in the sense of Definition* 1 *and, if* T *is not single-valued, then let* $\overline{T(Q)} \subset Q$ *be satisfied. Then the global initial value problem*

$$\begin{cases} X' \in F(t,X), \\ X \in Q \cap S \end{cases}$$

admits at least $N(r|_{T(Q)} \circ T(\cdot))$ *solutions, provided* $G(t,c,c) \subset F(t,c)$ *takes place a.e. in* I, *for any* $c \in \mathbb{R}^n$.

3. Results for quasi-linear differential inclusions

For boundary value problems, the situation becomes more delicate. The appropriate known results, concerning the topological structure of the solution set, are therefore related mostly to quasi – linear systems with the given boundary conditions, namely

$$\begin{cases} X' + A(t)X \in F(t,X), \\ X \in S, \end{cases} \tag{6}$$

where $F(t,X) : I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory function and $S \subset C(I,\mathbb{R}^n)$ is as above, but $A(t) : I \to \mathbb{R}^{n^2}$ is a single-valued bounded continuous $(n \times n)$-matrix. As we shall see, the study of the structure of the solution set related to the associated linearized problem, namely

$$\begin{cases} X' + A(t)X \in F(t,q(t)), \\ X \in S \cap Q, \end{cases} \tag{7}$$

for each $q(t) \in Q \subset C(I,\mathbb{R}^n)$, becomes significantly easier, especially when Q as well as S are nonempty, convex, and Q is bounded and closed in the given topology of the uniform convergence on compact subintervals of I.

Since, under the above assumptions, the solution set of problem (7) is convex (see [A2], Lemma 7) and, under the assumptions of Theorem 3 for $G(t,X,q) = -A(t)X + F(t,q)$, it is compact (see [A2], Theorem 2 and [AGG], Proposition 2.32), Theorem 3 simplifies as follows.

Theorem 6 (Existence criterium II). *Consider boundary value problem* (6) *on a given interval $I \subset \mathbb{R}$, where $F : I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory map, $A : I \to \mathbb{R}^{n^2}$ is a single-valued bounded continuous $(n \times n)$-matrix and $S \subset C(I,\mathbb{R}^n)$ is nonempty and convex. Let there exist a nonempty, convex, closed and bounded set $Q \subset C(I,\mathbb{R}^n)$ such that the associated linearized problem* (7) *admits for each $q(t) \in Q$ a solution and let $T(q)$ be the set of such solutions. Then the original problem* (6) *is solvable, provided only*

$$\sup_{|X| \le D} |F(t,X)| \le \alpha(t) \qquad \text{for a.a. } t \in I,$$

where $\alpha : I \to \mathbb{R}$ is a locally (Lebesgue) integrable function and D is a sufficiently big positive constant, and

$$\overline{T(Q)} \subset S. \tag{iii}$$

Remark 2. If S is closed in the given topology of the uniform convergence on compact subintervals of I, then (iii) is satisfied.

In the absence of convexity, the following slightly modified result can be very useful.

Proposition 2 ([ADG]). *Consider the problem*

$$\text{(8)}\qquad \begin{cases} X' + A(t)X \in F(t,X), \\ L(X) = \theta, \end{cases}$$

on a compact interval I, where A and F are as above, and $L : C(I,\mathbb{R}^n) \to \mathbb{R}^n$ is a linear operator such that the homogeneous problem

$$\text{(9)}\qquad \begin{cases} X' + A(t)X = 0, \\ L(X) = 0 \end{cases}$$

has only the trivial solution on I. If F is additionally Lipschitzian in X for a.a. $t \in I$ with a sufficiently small Lipschitz constant k (i.e. $h(F(t,X),F(t,Y)) \leq k|X-Y|$ for all X, $Y \in \mathbb{R}^n$ and a.a. $t \in I$, where $h(\,\cdot\,,\,\cdot\,)$ denotes the Hausdorff metric — see Chapter 5 below) and (5) is satisfied, then the solution set of (8) is a (nonempty) compact AR-space (i.e. more than R_δ-set). If the Lebesgue measure of the set $\{t \mid \dim F(t,X) < 1$ for some $X \in \mathbb{R}^n\}$ is still zero, where $\dim Y$ denotes the covering dimension of a space Y, then the solution set of (8) is infinite-dimensional.

Let us recall that a metric space M is an *AR-space* (absolute retract space) if, whenever it is a closed subset of another metric space N, then there exists a continuous retraction $r : N \to M$, $r(x) = x$ for $x \in M$. In particular, it is contractible and so connected.

Remark 3. Since the composition $F(t,q(t))$ of the above continuous in X multifunction F with any $q \in Q$, where Q is same as in Theorem 3 or Theorem 4, is measurable (see e.g. [ADTZ], p. 34]), the assertion of Proposition 2 is true for the linearized problem

$$\text{(10)}\qquad \begin{cases} X' + A(t)X \in F(t,q(t)), \\ L(X) = \theta, \end{cases}$$

even without the Lipschitzian restriction. On the other hand, if F is "only" u.s.c. in X then, in order to get the same for (10), we must assume (cf. [ADTZ], p. 34) measurability of the composition $F(t,q(t))$ either explicitly or (see [ADTZ], p. 36) the product — measurability for $F(t,X) : I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$.

Hence, applying Proposition 2 to Theorem 3, we arrive at

Theorem 7 (Existence criterium III). *Consider boundary value problem (8) on a compact interval I. Assume that $A : I \to \mathbb{R}^{n^2}$ is a single-valued continuous $(n \times n)$-matrix and $F : I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory product — measurable mapping satisfying (5). Furthermore, let $L : C(I,\mathbb{R}^n) \to \mathbb{R}^n$ be a linear operator such that problem (9) has only the trivial solution on I. At last, let there exist a bounded retract $Q \subset C(I,\mathbb{R}^n)$ of $C(I,\mathbb{R}^n)$ such that all (existing) solutions $T(q)$ of the associated linearized problem (10) belong to Q, for each $q \in Q$. Then the original problem (8) is solvable, provided $\overline{T(Q)} \subset \{X \in C(I,\mathbb{R}^n) \mid L(X) =$*

$\theta\}$. If F is additionally Lipschitzian in X for a.a. $t \in I$ with a sufficiently small Lipschitz constant and the Lebesgue measure of the set $\{t \mid \dim F(t,X) < 1$ for some $X \in \mathbb{R}^n\}$ is zero, then the solution set of (8) is an infinite dimensional AR-space.

Similarly, applying Proposition 2 to Theorem 4, we arrive at

Theorem 8 (Multiplicity criterium III). *Consider boundary value problem (8) on a compact interval I and for A, F, L assume the same as in Theorem 7. Then the original problem (8) has $N(r|_{T(Q)} \circ T(\cdot))$ solutions, provided there exists a (nonempty) compact, connected subset $Q \subset C(I,\mathbb{R}^n)$ of $C(I,\mathbb{R}^n)$ such that*

(i) *$T(Q)$ is bounded,*
(ii) *$T(q)$ is retractible onto Q with a retraction r in the sense of Definition 1 and, if $T(q)$ is not single-valued, then $\overline{T(Q)} \subset Q$,*
(iii) *$\overline{T(Q)} \subset \{X \in C(I,\mathbb{R}^n) \mid L(X) = \theta\}$,*

where $T(q)$ denotes the set of (existing) solutions of (10).

4. Bounded and periodic solutions

In order to prove the existence of a bounded solution of (1) by means of Theorem 6, it is sufficient to show the solvability of (7), where

$$Q = S = \{r(t) \in C(\mathbb{R},\mathbb{R}^n) \mid \sup_{t\in\mathbb{R}} |r(t)| \leq D\}$$

with a suitable sufficiently big constant D. This was done in [A2], where the following theorem has been obtained.

Theorem 9 (Boundedness result). *Let a continuous single-valued matrix function $A(t) : \mathbb{R} \to \mathbb{R}^{n^2}$ be bounded and a Carathéodory (multivalued) mapping $F(t,X) : \mathbb{R} \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be essentially bounded in t and can be for a.a. $t \in \mathbb{R}$ globally absolutely estimated by a single-valued continuous function $\Phi(X)$, i.e. $|F(t,X)| \leq \Phi(X)$ for all $X \in \mathbb{R}^n$ and a.a. $t \in \mathbb{R}$. Let, furthermore, system*

$$X' + A(t)X = 0$$

possesses on $I = (-\infty,\infty)$ an exponential dichotomy (for the definition and sufficient conditions see e.g. [C]). At last, let there exist a sufficiently small constant C such that

$$\limsup_{|X|\to\infty} \frac{\Phi(X)}{|X|} \leq C \tag{11}$$

is satisfied with Φ defined above. Then the inclusion

$$X' + A(t)X \in F(t,X) \tag{12}$$

admits an entirely bounded solution $X(t)$ *such that*

$$\sup_{t\in\mathbb{R}} |X(t)| \le D.$$

Remark 4 (Periodicity result). If still $A(t) \equiv A(t+\omega)$ and $F(t,X) \equiv F(t+\omega, X)$, then system (12) admits obviously an ω-periodic solution. This also follows from Theorem 7 jointly with additional information concerning the topological structure of the solution set, provided F is Lipschitzian in X with a sufficiently small Lipschitz constant.

As an example of application of Theorem 8, consider the Carathéodory system

$$\begin{aligned} x' + ax &\in e(t,x,y)y^{(1/m)} + g(t,x,y), \\ y' + by &\in f(t,x,y)x^{(1/n)} + h(t,x,y), \end{aligned} \tag{13}$$

where a, b are constants with $ab > 0$ and m, n are odd integers such that $\min(m,n) \ge 3$.

Let, furthermore, the multifunctions e, f, g, h be product-measurable and ω-periodic in a t-variable.

At last, let suitable positive constants $\delta_1, \delta_2, e_0, f_0, E_0, F_0, G, H$ exist such that

$$\left.\begin{aligned} |e(t,x,y)| \le E_0, \quad |f(t,x.y)| \le F_0 \\ |g(t,x,y)| \le G, \quad |h(t,x,y)| \le H \end{aligned}\right\} \quad \text{for a.a. } t \in [0,\omega] \text{ and all } (x,y) \in \mathbb{R}^2,$$

and

$$0 < e_0 \le e(t,x,y) \tag{14}$$

for $x \ge -\delta_1$, $y \ge \delta_2$ and a.a. t as well as for $x \le \delta_1$, $y \le -\delta_2$ and a.a. t,

$$0 < f_0 \le f(t,x,y) \tag{15}$$

for $x \ge \delta_1$, $y \le \delta_2$ and a.a. t as well as for $x \le -\delta_1$, $y \ge -\delta_2$ and a.a. t, or (14) for $x \le \delta_1$, $y \ge \delta_2$ and a.a. t as well as for $x \ge -\delta_1$, $y \le -\delta_2$ and a.a. t, (15) for $x \ge \delta_1$, $y \ge -\delta_2$ and a.a. t as well as for $x \le -\delta_1$, $y \le \delta_2$ and a.a. t.

Then we can proceed quite analogously to [A3], where the multifunctions have been however considered lower-semicontinuous in (x,y) for a.a. $t \in [0,\omega]$, to prove the following

Theorem 10 (Multiplicity result). *If the above constants* δ_1, δ_2, e_0, f_0, G, H *satisfy the inequalities*

$$\begin{cases} \dfrac{1}{|a|}|e_0\delta_2^{(1/m)} - G| > \delta_1 > \left(\dfrac{H}{f_0}\right)^n, \\[2ex] \dfrac{1}{|b|}|f_0\delta_1^{(1/n)} - H| > \delta_2 > \left(\dfrac{G}{e_0}\right)^m, \end{cases}$$

then system (13) *admits, under the above assumptions, at least three* ω*-periodic solutions.*

5. Almost-periodic multifunctions and selectors

Since we consider the Carathéodory differential inclusions, it seems natural to use one of the concepts of nonuniform almost-periodicity, when investigating the related almost-periodic problems. For our purposes, the generalized concepts of almost-periodicity in the sense of H. Weyl (cf. [A2]) and V. V. Stepanov (cf. [DS]) will be of a particular importance because of their effective applications.

At first, let us recall (see e.g. [L]) the classical definition of Weyl-like and Stepanov-like a.p. functions.

Definition 2. A locally Lebesgue integrable single-valued function $p(t)$ with nonempty values is called *a.p. in the sense of Weyl* (*or Stepanov*) if for every $\varepsilon > 0$ there exists a positive number $k = k(\varepsilon)$ such that in each interval of the length k there is at least one number τ satisfying

$$(16)\qquad \lim_{l\to\infty}\left[\sup_{a\in\mathbb{R}}\frac{1}{l}\left\{\int_a^{a+l}|p(t+\tau)-p(t)|\,dt\right\}\right]<\varepsilon$$
$$\left(\text{or }\ \sup_{a\in\mathbb{R}}\frac{1}{l}\left\{\int_a^{a+l}|p(t+\tau)-p(t)|dt\right\}<\varepsilon\qquad\text{for a fixed } l\right).$$

Remark 5. Since the Lebesgue integral is absolutely convergent, Definition 2 has a meaning, provided the (finite or infinite) limit, $\lim_{l\to\infty}[\,\cdot\,]$, exists. But this is always true (see [L], p. 221). On the other hand, we must understand that the space of all Weyl-like a.p. functions is not (in difference to Stepanov-like a.p. functions) complete with the above metric (see [L]).

If a (multivalued) essentially bounded vector map $P(t)\in\mathbb{R}^n$ with nonempty closed values is (*Lebesgue*) *measurable*, i.e. if for any open $U\subset\mathbb{R}^n$ the set

$$\{t\in(-\infty,\infty)\mid P(t)\cap U\neq\emptyset\}$$

is measurable, then we can replace (16) by

$$(17)\qquad \lim_{l\to\infty}\left[\sup_{a\in\mathbb{R}}\frac{1}{l}\left\{\int_a^{a+l}h(P(t+\tau),P(t))dt\right\}\right]<\varepsilon$$
$$\left(\text{or }\ \sup_{a\in\mathbb{R}}\frac{1}{l}\left\{\int_a^{a+l}h(P(t+\tau),P(t))dt\right\}<\varepsilon\qquad\text{for a fixed } l\right)$$

for the same goal, provided $h(P(\cdot+\tau),P(\cdot))$ is measurable, where $h(\cdot,\cdot)$ is the Hausdorff distance, namely

$$h(M,N):=\max\left\{h^+(M,N),h^-(M,N)\right\},$$

where

$$h^+:=\sup\{\rho(z,N)\mid z\in M\},\qquad h^-(M,N)=h^+(N,M),$$
$$\rho(z,N):=\inf\{|y-z|\mid y\in N\}.$$

Indeed. This is true, because $P(t)$ is known (see e.g. [ADTZ], p. 18 and the references therein) to be measurable if and only if there exists a sequence $\{p_n(t)\}$ of measurable selectors of $P(t)$ such that the following so called Castaing representation takes place, namely

$$P(t) = \overline{\bigcup_{n\in N} p_n(t)}.$$

Thus, the problem easily transforms to the single-valued one. The local convergence of the Lebesgue integral in (17) follows then by means of the well-known Lebesgue dominated theorem. The (finite or infinite) limit, $\lim_{l\to\infty}[\,\cdot\,]$, exists by the same reasons as in the single-valued case.

Hence we can give

Definition 3. An essentially bounded measurable (multivalued) function $P(t) \in \mathbb{R}^n$ with nonempty closed values is called *a.p. in the sense of Weyl* (*or Stepanov*) if for every $\varepsilon > 0$ there exists a positive number $k = k(\varepsilon)$ such that in each interval of the length k there is at least one number τ satisfying (17).

In the single-valued case, Definition 2 reduces to Definition 1. For the continuous (multivalued) functions, condition (17) can be simply rewritten into

$$h(P(t+\tau), P(t)) < \varepsilon \qquad \text{for all } t \in \mathbb{R}.$$

Proposition 3 ([DS]). *A Stepanov-like a.p. multifunction $P(t)$ possesses a family of Stepanov-like a.p. selectors $p_n(t)$ such that*

$$P(t) = \overline{\bigcup_{n\in N} p_n(t)}.$$

Although the stronger concept of Stepanov's almost-periodicity is certainly sufficient to guarantee a Weyl-like a.p. selector, it is not clear whether or not it is so for Weyl-like a.p. multifunctions. Moreover, since a periodic measurable multifunction possesses obviously a periodic measurable selector, we can introduce still another definition allowing us to deal with Weyl-like a.p. selectors.

Definition 4. A (multivalued) function $P(t) \in \mathbb{R}^n$ is called *selectionally a.p. in the sense of Weyl* if it can be written as the sum $P = P_1 + P_2$, where P_1 is a finite linear combination of essentially bounded measurable (multivalued) periodic functions with nonempty closed values and P_2 is a measurable essentially bounded function with nonempty closed values having the following property: for every $\varepsilon > 0$ there exists a positive number $k = k(\varepsilon)$ such that in each interval of the length k there is at least one number τ satisfying

$$\lim_{l\to\infty}\left[\sup_{a\in\mathbb{R}} \frac{1}{l}\left\{\int_a^{a+l} |P_2(t+\tau) - P_2(t)| dt\right\}\right] < \varepsilon, \tag{18}$$

where

$$\int_a^{a+l} |P_2(t+\tau) - P_2(t)|dt$$
$$= \left\{ \int_a^{a+l} |p_2(t+\tau) - p_2(t)|dt \mid p_2 \text{ is a mesurable selector of } P_2 \right\},$$

i.e. the integrals in (18) are considered in the sense of Aumann (see e.g. [ADTZ], p. 72 and the references therein).

Remark 6. Since a measurable multifunction with nonempty closed values is always measurably selectionable (see e.g. [ADTZ], p. 18), the integral set in (18) is nonempty (eventually a trivial singleton). Thus, every selectionally Weyl-like a.p. function $P(t)$ has (under the assumptions of Definition 4) a Weyl-like a.p. selector in the form of the sum of Weyl-like a.p. selectors of P_1 and P_2. Moreover, one can readily check that Definition 4 generalizes the one for (multivalued) periodic functions as well as Definition 2.

In the sequel, we need still to consider (because of applications) a.p. multifunctions containing a parameter, where the following kind of uniformity is necessary to take place.

Definition 5. An essentially bounded in t (multivalued) Carathéodory function $F(t, X) \in \mathbb{R}^n$ (i.e. measurable in t, upper semi-continuous for a.a. X and with nonempty, compact and convex values) is called *a.p. in the sense of Weyl (or Stepanov) uniformly w.r.t.* $X \in \mathbb{R}^n$, if for every $\varepsilon > 0$ and every $D > 0$ there exists a positive number $k = k(\varepsilon, D)$ such that in each interval of the length k there is at least one number τ satisfying

$$(19)\qquad \lim_{l\to\infty}\left[\sup_{a\in\mathbb{R}} \frac{1}{l}\left\{\int_a^{a+l} h(F(t+\tau, X), F(t, X))dt\right\}\right] < \varepsilon$$
$$\left(\text{or } \sup_{a\in\mathbb{R}} \frac{1}{l}\left\{\int_a^{a+l} h(F(t+\tau, X), F(t, X))dt\right\} < \varepsilon \text{ for a fixed } l\right),$$

for $|X| \le D$, where $h(\cdot\,,\cdot)$ in (19) denotes the Hausdorff distance.

Definition 6. An essentially bounded in t (multivalued) Carathéodory function $F(t, X) \in \mathbb{R}^n$ is called *selectionally a.p. in the sense of Weyl,* uniformly w.r.t. $X \in \mathbb{R}^n$, if it can be written as the sum $F = F_1 + F_2$, where F_1 is a finite linear combination of (multivalued) Carathéodory functions which are essentially bounded and periodic in t and (Lipschitz-) continuous in X for a.a. $t \in \mathbb{R}$, and F_2 is a Carathéodory multifunction which is essentially bounded in t and (Lipschitz-) continuous in X for a.a. $t \in \mathbb{R}$ (cf. Proposition 4 below) and satisfies (cf. (18))

$$(20)\qquad \lim_{l\to\infty}\left[\sup_{a\in\mathbb{R}} \frac{1}{l}\left\{\int_a^{a+l} |F_2(t+\tau, X) - F_2(t, X)|dt\right\}\right] < \varepsilon,$$

for $|X| \leq D$. The integrals are again understood in the sense of Aumann.

Remark 7. Under the assumptions of Definitions 5 and 6, a Carathéodory function $F(t, X)$ is known (see e.g. [ADTZ], p. 35) to be weakly superpositionally measurable. It means that the Nemytskii operator $F(t, X(t))$, where $X(t)$ is a continuous single-valued function, possesses a (Lebesgue) measurable selector. If $F(t, .)$ is additionally continuous (i.e. also lower-semi-continuous) for a.a. $t \in \mathbb{R}$, then $F(t, X)$ is even product-measurable, and consequently superpositionally measurable (see e.g. [ADTZ], p. 34) and having a measurable selector (see e.g. [ADTZ], p. 18). It means that the Nemytskii operator $F(t, X(t))$, where $X(t)$ is again a continuous single-valued function, becomes measurable.

We conclude this section by the following proposition which is crucial (jointly with Definitions 5 and 6) for the applications to differential inclusions. For more details see [ADTZ], pp. 24-25 and the references therein.

Proposition 4. *Let $F(t, X) : \mathbb{R}^{n+1} \rightsquigarrow \mathbb{R}^n$ be a multivalued function such that:*

(i) *$F(t, \cdot)$ with nonempty closed convex values is (Lipschitz-) continuous (with a sufficiently small Lipschitz constant L), namely*

$$h(F(t, X), F(t, Y)) \leq L|X - Y| \qquad \text{for a.a. } t \in \mathbb{R},$$

where $h(\cdot, \cdot)$ denotes the Hausdorff distance,

(ii) *$F(\cdot, X)$ is measurable with nonempty closed values.*

Let, furthermore, $X(t) : \mathbb{R} \to \mathbb{R}^n$ be a single-valued continuous map and $Y(t)$ be a measurable selector of $F(\cdot, X(\cdot))$ (which exists according to Remark 7), i.e. $Y(t) \subset F(t, X(t))$ for a.a. $t \in \mathbb{R}$. Then there exists a Carathéodory selector f of F such that $f(t, \cdot)$ is (Lipschitz-) continuous (with not necessarily the same, but sufficiently small Lipschitz constant (cf. [AC], *p. 77)) and satisfies $Y(t) = f(t, X(t))$ for a.a. $t \in \mathbb{R}$.*

Remark 8. In view of Proposition 3 or Remark 7, it is clear that if $F(t, X)$ is additionally Stepanov-like (see Definition 5) or selectionally Weyl-like (see Definition 6) a.p. in t-variable, uniformly w.r.t. $X \in \mathbb{R}^n$, then the Carathéodory selector $f \subset F$ in the conclusion of Proposition 4 can be also considered a.p. in the sense of Weyl, uniformly w.r.t. $X \in \mathbb{R}^n$.

6. Almost-periodic solutions

Finally, consider the following special form of inclusion (12), namely

$$X' + AX \in F(t, X), \tag{21}$$

where the matrix $A(t) \equiv A$ is constant and $F : \mathbb{R} \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory mapping which is Lipschitzian in X for a.a. t and essentially bounded in t.

We shall still assume that F is either Stepanov-like (see Definition 5) or selectionally Weyl-like (see Definition 6) a.p. in t, uniformly w.r.t. $X \in \mathbb{R}^n$.

Thus, according to Propositions 3 and 4 (see also Remark 8), there exists a Carathéodory selector $f \subset F$ which is Lipschitzian in X for a.a. t and Weyl-like a.p. in t, uniformly w.r.t. $X \in \mathbb{R}^n$. Since the required smallness of the Lipschitz constant for F implies also the one for $f \subset F$ (see again Proposition 4), the almost-periodicity problem for (21) can be reduced to the one for (0), i.e.

$$X' + AX = f(t, X),$$

where $f \subset F$ is such a selector.

Using the boundedness result in Theorem 9, we have proved in [A2] the following

Theorem 11 (A.-p. result). *Let the following assumptions be satisfied:*

(i) *a (single-valued) constant $(n \times n)$-matrix A is hyperbolic, i.e. all the associated eigenvalues have nonzero real parts;*

(ii) *a (multivalued) Carathéodory map $F(t,X) : \mathbb{R} \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is essentially bounded in t and Lipschitz-continuous in X for a.a. $t \in \mathbb{R}$ with a sufficiently small constant;*

(iii) *$F(t,X)$ is either Stepanov-like or selectionally Weyl-like a.p. in t uniformly w.r.t. $X \in \mathbb{R}^n$, (see Definitions 5 and 6).*

Then inclusion (21) admits an a.-p. solution in the sense of Weyl.

7. Concluding remarks

In the single-valued case, as we could see, Theorem 9 improves Theorem 1 of P. Bohl and so does Remark 4 w.r.t. Remark 1 (cf. also [A1], [AK]). Similarly, Theorem 11 generalizes Theorem 2.

In the case of inclusions, there are only few papers of the other authors devoted to the existence of bounded solutions (see e.g. [CMZ]), almost-periodic solutions (see e.g. [H2]) or to the multiplicity results (see e.g. [B]). On the other hand, those related to the existence of periodic solutions are rather frequent (see e.g. [BGP], [DGP] and the references therein).

It remains an open question, whether Weyl-like a.p. multifunctions possess Weyl-like a.p. selectors and if the interval I in Proposition 2 can be noncompact. If so, then the conclusions of Theorems 7, 8 and 11 could be significantly improved.

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NONCOMPACT VERSION OF THE MULTIVALUED NIELSEN THEORY AND ITS APPLICATION TO DIFFERENTIAL INCLUSIONS

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0. Introduction

The purpose of this paper is to proceed furthermore in our investigations initiated in [AGJ], where the multivalued Nielsen theory has been developed for admissible maps (in the sense of [G1]) on compact connected ANR-spaces. Since, for example, the Poincaré self-maps on tori generated by the Carathéodory vector fields belong to this class, one could obtain in such a way multiplicity results for differential inclusions.

In [A1] multiple bounded solutions have been proved again in the frame of Nielsen theory, but using another approach. More concretely, such problems were transformed to those for the lower estimate of fixed points of the related operators on bounded, compact, connected neighbourhood retracts of Fréchet spaces. Although such operators are composed by those with R_δ-values on compact connected ANRs, and subsequently the appropriate Nielsen number gives a lower estimate of fixed points (i.e. only a particular case of more general coincidence points as in [AGJ]), the compactness of ANRs is unpleasant for applications because of infinite dimensional function spaces.

Therefore, we decided to avoid this difficulty by considering a rather general class of admissible self-maps with only a certain amount of compactness, so called compact absorbing contractions (shortly, CAC), but on arbitrary connected ANR-spaces. Since the generalized Lefschetz number is well-defined for

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these mappings (see e.g. [FG], [GR]), the results in [AGJ] could be elaborated in a desired manner. The final theory seems to be the most general of all their analogies, including the single-valued case treated in [Sc].

On this basis, we have developed here the effective method for obtaining the multiplicity criteria to a large family of multivalued boundary value problems on possibly infinite intervals. In a certain sense, our method extends therefore also those in [AGG] for solving the sole existence problems. As an illustrating example, the existence of at least two entirely bounded solutions has been shown to a planar system of inclusions with a semicontinuous right hand-side.

1. Topological preliminaries

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ on the category of metric spaces.

We shall use also the notation $H(\cdot\,,\mathbb{Z})$ for the Čech homology functor with integer coefficients $\mathbb{Z}$. A nonempty space X is called *acyclic* if we have

$$H_i(X) = \begin{cases} \mathbb{Q} & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

We recall that a compact space X is said to be an *R_δ-set* if it is an intersection of a decreasing sequence of compact contractible spaces. It follows from the continuity property of the functor H that every R_δ-space is acyclic; in particular, every compact contractible space is acyclic.

A single-valued continuous map $p : X \to Y$ is called a *Vietoris map* if p is proper and, for every $x \in X$, the set $p^{-1}(x)$ is acyclic.

An u.s.c. map $\varphi : X \multimap Y$ is called *acyclic* if $\varphi(x)$ is acyclic for every $x \in X$.

For a multivalued map $\varphi : X \multimap Y$, we shall consider the graph $\Gamma_\varphi\{(x,y) \mid y \in \varphi(x)\}$ of φ and two natural projections:

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y,$$

where $p_\varphi(x,y) = x$, $q_\varphi(x,y) = y$.

Observe that if $\varphi : X \multimap X$ is acyclic, then $p_\varphi : \Gamma_\varphi \to X$ is a Vietoris map, and we have $\varphi(x) = q_\varphi(p_\varphi^{-1}(x))$ for every $x \in X$.

The above observation allows us to give the following (cf. [G1], [G2])

Definition 1.1. A multivalued map $\varphi : X \multimap Y$ is called *admissible* if there exist a compact metric space Γ and two (single-valued) continuous maps $p : \Gamma \to X$ and $q : \Gamma \to Y$ such that

(i) p is a Vietoris map,
(ii) $\varphi(x) = q(p^{-1}(x))$ for every $x \in X$;

then (p,q) is called a *selected pair* for φ and we write $(p,q) \subset \varphi$.

Recall (see e.g. [G1]) that any acyclic map $\varphi : X \multimap X$ is admissible, and so we have $(p_\varphi, q_\varphi) \subset \varphi$.

Lemma 1.2 ([G1]). *If $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$ are admissible maps, then the composition $\psi \circ \varphi : X \multimap Z$ of φ and ψ is also admissible.*

Remark 1.3. In what follows, by a multivalued map we shall mean a pair (p, q), $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$, of single-valued maps with p to be a Vietoris map.

Since any admissible map can be represented by an associated pair (p, q), for such a pair $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ we let:

$$C(p, q) = \{y \in \Gamma \mid p(y) = q(y)\}$$

and

$$\mathrm{Fix}(p, q) = \{x \in X \mid x \in q(p^{-1}(x))\}.$$

Of course, $C(p, q) \neq \emptyset$ if and only if $\mathrm{Fix}(p, q) \neq \emptyset$. Roughly speaking, the set $C(p, q)$ is bigger than $\mathrm{Fix}(p, q)$ in general.

We recommend [Gr] and [FG], [GR] for the notion of generalized Lefschetz number and its properties. Note that for a pair (p, q), $X \xleftarrow{p} \Gamma \xrightarrow{q} X$, its generalized Leschetz number is denoted by $\Lambda(p, q)$ and we let

$$\Lambda(p, q) = \Lambda(q_* \circ p_*^{-1}),$$

provided $\Lambda(q_* \circ p_*^{-1})$ is well-defined; in that case (p, q) is called a Lefschetz pair.

Now, we recall some elementary facts concerning absolute neighbourhood retracts.

Definition 1.4. A space Y is called *an absolute retract* (*an absolute neighbourhood retract*) whenever, for any metrizable X and closed $A \subset X$, each $f : A \to Y$ is extendable over X (over an open neighbourhood U of A in X).

We use then the notation: $Y \in \mathrm{AR}$ ($Y \in \mathrm{ANR}$).

The following theorem characterizes ARs (ANRs) in terms of retraction property (up to a homeomorphism):

Theorem 1.5. *A metrizable space is an* AR (*an* ANR) *if and only if it is a retract of* (*some open subset of*) *some normed space.*

Proposition 1.6. *If $X \in$ ANR and U is an open subset of X, then $U \in$* ANR.

Until the end of this section, a pair (p, q), $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$, representing a multivalued map will be denoted by φ, i.e., by a multivalued map $\varphi : X \multimap Y$ we always understand the pair (p, q) of the above type ($\varphi(x) = q(p^{-1}(x))$).

Since p is Vietoris and q is continuous, our multivalued map φ is always u.s.c. with compact values (see [G1] or [G2]).

A multivalued map $\varphi : X \multimap Y$ is called *compact* if

$$\overline{\varphi(X)} = \overline{\bigcup_{x \in X} \varphi(x)}$$

is a compact subset of Y. In what follows, $\varphi \in \mathbb{K}(X)$ denotes $\varphi : X \multimap X$ being compact.

Following [FG], we shall define several classes of maps of a great importance for our work.

Definition 1.7. A map $\varphi : X \rightsquigarrow X$ is called *locally compact* if for each $x \in X$ there exists an open subset V of X such that $x \in V$ and the restriction $\varphi|_V$ of f to V is compact.

All mappings considered in this section are assumed to be locally compact.

Definition 1.8. A map $\varphi : X \rightsquigarrow X$ is said to be a *compact absorbing contraction* if there exists an open subset U of X such that $\overline{\varphi(U)}$ is a compact subset of U and $X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(U)$.

We use the notation: $\varphi \in \mathrm{CAC}(X)$.

Definition 1.9. A map $\varphi : X \rightsquigarrow X$ is called *eventually compact* if there exists an iterate $\varphi^n : X \rightsquigarrow X$ of φ such that φ^n is compact.

We use the notation: $\varphi \in \mathrm{EC}(X)$.

Definition 1.10. A map $\varphi : X \rightsquigarrow X$ is called a *compact attraction* if there exists a compact subset K of X such that, for each open neighbourhood V of K, we have $X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(V)$. The compact set K is then called an *attractor* for φ.

We use the notation: $\varphi \in \mathrm{CA}(X)$.

Definition 1.11. A map $\varphi : X \rightsquigarrow X$ such that

$$\bigcup_{n=1}^{\infty} \{\varphi^n(x)\} \qquad \text{is relatively compact for every } x \in X$$

is called *asymptotically compact* if the set $C_\varphi = \bigcap_{i=0}^{\infty} \varphi^i(X)$ is a non-empty, relatively compact subset of X. The set C_φ is then called the *center* of φ .

We use the notation: $\varphi \in \mathrm{ASC}(X)$.

The following diagram shows the relations between the above classes of mappings:

$$\begin{array}{ccccc}
 & & & & \mathrm{CAC}(X) \\
 & & & & \cup \\
 & & \mathrm{EC}(X) & \subset & \mathrm{CA}(X) \\
 & & \cup & & \cup \\
\mathbb{K}(X) & \subset & EC(X) \cap \mathrm{ASC}(X) & \subset & ASC(X).
\end{array}$$

However, the following important question still remains open:

Open problem. *Is there any of the reverse inclusions to those in the above diagram to be true as well?*

We recall the main result proved in [FG].

Theorem 1.12. *Let $X \in \mathrm{ANR}$ and $\varphi \in \mathrm{CAC}(X)$. Then φ is a Lefschetz map and if $\Lambda(\varphi) \neq 0$, then $\mathrm{Fix}(\varphi) \neq \emptyset$.*

From Theorem 1.12 it follows immediately that

Corollary 1.13. *If $X \in \mathrm{AR}$ and $\varphi \in \mathrm{CAC}(X)$, then $\Lambda(\varphi) = 1$ and, in particular, $\mathrm{Fix}(\varphi) \neq \emptyset$.*

2. Nielsen number for CAC-maps

It has been shown in [FG] (cf. Theorem 1.12 above) that, for any multivalued CAC-map from an ANR-space, the Lefschetz number $\Lambda(p,q) \in \mathbb{Z}$ is defined and $\Lambda(p,q) \neq 0$ implies the existence of a coincidence point $z \in \Gamma$ $(p(z) = q(z))$ of the pair (p,q).

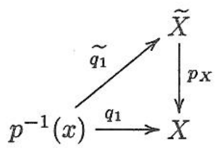
On the other hand, we have constructed in [AGJ] the Nielsen number $N(p,q)$ for a class of multivalued selfmaps on a compact ANR. $N(p,q)$ is a non-negative integer, a homotopy invariant and $C(p,q) \geq N(p,q)$.

In this section, we generalize this construction: we drop out the compactness assumption imposed on X by replacing (p,q) to be a CAC.

As in the single-valued case, the definition of a Nielsen number is done in two stages: At first, $C(p,q)$ is split into disjoint classes (Nielsen classes) and then we define essential classes.

Fix a universal covering $p_x : \widetilde{X} \to X$. We define $\widetilde{\Gamma} = \{(\widetilde{x}, z) \in \widetilde{X} \times \Gamma \mid p_x(\widetilde{x}) \supset p(z)\}$ (pullback) and the map $\widetilde{p} : \widetilde{\Gamma} \to \widetilde{X}$ by $\widetilde{p}(\widetilde{x}, z) = \widetilde{x}$.

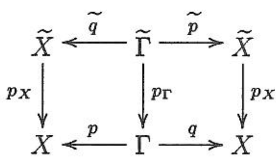
Property A. *For any $x \in X$, the restriction $q_1 : q_{1\,p^{-1}(x)} : p^{-1}(x) \to X$ admits a lift $\widetilde{q}$, making the diagram*



commutative.

Remark 2.1. Note that a sufficient condition of guaranteeing property A is, for example, that $p^{-1}(x)$ is an ∞-proximally connected set, for every $x \in X$ (see [KM], [G2]). It is well-known (see [G2]) that any ∞-proximally connected subset of an ANR-space is an R_δ-set.

Lemma 2.2. *If (p,q) satisfies $(CAC + A)$ then there is a lift $\widetilde{q} : \Gamma \to \widetilde{X}$ making the diagram*



commutative.

Proof. Notice that the assumptions (3.1), (3.2), (3.3) in [AGJ] are satisfied. Let

$$\theta_X = \{\alpha : \widetilde{X} \to \widetilde{X} \mid p_X \alpha = p_X\}$$

denote the group of covering transformations of the covering $\widetilde{X}$. Similarly we define θ_Γ.

The lifts $\widetilde{p},\widetilde{q}$ define homomorphisms:

$$\widetilde{p}^{\,!}:\theta_X\to\theta_\Gamma \quad \text{by the formula} \quad p^!(\alpha)(\widetilde{x},z)=(\alpha\widetilde{x},z)$$

and

$$\widetilde{q}_!:\theta_\Gamma\to\theta_X \quad \text{by the equality} \quad \widetilde{q}\cdot\alpha=\widetilde{q}_!(\alpha)\cdot\widetilde{q}.$$

Let us recall that θ_X is isomorphic with $\pi_1(X)$ and if (p,q) represents a single-valued map (i.e. $qp^{-1}(x)=\varrho(x)$ for a single-valued map $\varrho:X\to X$), then $\widetilde{q}_!\widetilde{p}^{\,!}:\theta_X\to\theta_X$ is equal to the homomorphism $\widetilde{\varrho}_!:\theta_X\to\theta_X$ given by $\widetilde{\varrho}\cdot\alpha=\widetilde{\varrho}_!(\alpha)\cdot\widetilde{\varrho}$, where $\widetilde{\varrho}$ is given by the formula $\widetilde{\varrho}(\widetilde{x})=\widetilde{q}\widetilde{p}^{-1}(\widetilde{x})$. However, the homomorphism $\widetilde{\varrho}_!$ corresponds to the induced map $\varrho_\#:\pi_1(X)\to\pi_1(X)$.

Thus, the composition $\widetilde{q}_!\widetilde{p}^{\,!}:\theta_X\to\theta_X$ can be considered as a generalization of the induced homotopy homomorphism. □

Property B. *There is a normal subgroup $H\subset\theta_X$ of finite index (θ_X/H-finite), invariant under the homomorphism $q_!p^!$ ($q_!p^!(H)\subset H$).*

Remark 2.3. In particular, if X is a connected space such that the fundamental group $\pi_1(X)$ of X is abelian and finitely generated, then X satisfies property B (see [Sp]). Note also that if (p,q) is admissibly homotopic to a single-valued map f, then property B holds true (see [AGJ]).

Let us notice that (CAC + A + B) makes the diagram

$$\begin{array}{ccccc}
\widetilde{X}_H & \xleftarrow{\widetilde{q}_H} & \widetilde{\Gamma}_H & \xrightarrow{\widetilde{p}_H} & \widetilde{X}_H \\
{\scriptstyle p_{XH}}\downarrow & & \downarrow{\scriptstyle p_{\Gamma H}} & & \downarrow{\scriptstyle p_{XH}} \\
X & \xleftarrow{p} & \Gamma & \xrightarrow{q} & X
\end{array}$$

commutative, where $p_{XH}:\widetilde{X}_H\to X$ is a covering corresponding to the normal subgroup $H\Delta\theta_X\approx\pi_1X$ and $\widetilde{\Gamma}_H$ is a pullback. As above, we can define homomorphisms $\widetilde{p}^!_H:\theta_{XH}\to\theta_{\Gamma H}$, $\widetilde{q}_{H!}:\theta_{\Gamma H}\to\theta_{XH}$, where $\theta_{XH}=\{\alpha:\widetilde{X}_H\to\widetilde{X}_H \mid p_{XH}\alpha=p_{XH}\}$.

Lemma 2.4 ([AGJ], Lemma 5.1). *We have:*

(i) $C(p,q)=\bigcup_{\alpha\in\theta_{XH}}p_{\Gamma H}C(\widetilde{p}_H,\alpha\widetilde{q}_H)$,

(ii) *if $p_{\Gamma H}C(\widetilde{p}_H,\alpha\widetilde{q}_H)\cap p_{\Gamma H}C(\widetilde{p}_H,\beta\widetilde{q}_H)$ is not empty, then there exists a $\gamma\in\theta_{XH}$ such that $\beta=\gamma\cdot\alpha\cdot(\widetilde{q}_{H!}\widetilde{p}^!_H\gamma)^{-1}$,*

(iii) *the sets $p_{\Gamma H}C(\widetilde{p}_H,\alpha\widetilde{q}_H)$ are either disjoint or equal.*

Thus, $C(p,q)$ splits into disjoint subsets $p_{\Gamma H}C(\widetilde{p}_H,\alpha\cdot\widetilde{q}_H)$ called Nielsen classes modulo a subgroup H.

Now, we shall define essential classes. We consider the diagram

$$\begin{array}{ccccc}
\widetilde{X}_H & \xleftarrow{\alpha\widetilde{q}_H} & \widetilde{\Gamma}_H & \xrightarrow{\widetilde{p}_H} & \widetilde{X}_H \\
{\scriptstyle p_{XH}}\downarrow & & \downarrow{\scriptstyle p_{\Gamma H}} & & \downarrow{\scriptstyle p_{XH}} \\
X & \xleftarrow{p} & \Gamma & \xrightarrow{q} & X
\end{array}$$

Lemma 2.5. *The multivalued map* $(\widetilde{p}_H, \widetilde{q}_H)$ *is a* CAC.

Proof. $\widetilde{X}$ is a metric ANR, because it is locally ANR (see [Hu]). Since $\widetilde{p}_{\Gamma H}$ is a homeomorphism between $\widetilde{p}_H^{-1}(x)$ and $\widetilde{p}^{-1}(px)$, $\widetilde{p}_H$ is Vietoris. If $U \subset X$ satisfies the definition of the CAC for (p, q), then $\widetilde{U} = p_{XH}^{-1}(U)$ satisfies the same for $(\widetilde{p}_H, \alpha\widetilde{q}_H)$. To see the last relation, we note that

$$\operatorname{cl} \widetilde{\varphi}(\widetilde{U}) \subset \operatorname{cl}(p_{XH}^{-1}(\varphi(U))) \subset \operatorname{cl}(p_{XH}^{-1}(\operatorname{cl}(\varphi(U)))).$$

Since $\operatorname{cl}(\varphi(U))$ is compact and covering p_{XH} is finite, $p_{XH}^{-1}(\operatorname{cl}(\varphi(U)))$ is also compact. Thus, so is $\operatorname{cl} \widetilde{\varphi}(\widetilde{U})$. □

Definition 2.6. A Nielsen class mod H of the form $p_{\Gamma H} C(\widetilde{p}_H, \alpha\widetilde{q}_H)$ is called *essential* if $\Lambda(\widetilde{p}_H, \alpha\widetilde{q}_H) \neq 0$.

By Lemma 6.5 in [AGJ], this definition is correct, i.e., if

$$p_{\Gamma H} C(\widetilde{p}_H, \alpha\widetilde{q}_H) = p_{\Gamma H} C(\widetilde{p}_H, \beta\widetilde{q}_H),$$

then

$$\Lambda(\widetilde{p}_H, \alpha\widetilde{q}_H) = \Lambda(\widetilde{p}_H, \beta\widetilde{q}_H).$$

Definition 2.7. The number of essential classes of (p, q) mod a subgroup H is called the *H-Nielsen number* and is denoted by $N_H(p, q)$.

Now, we can give two main theorems of this section.

Theorem 2.8. *A multivalued map* (p, q) *satisfing* (CAC $+A+B$) *has at least* $N_H(p, q)$ *coincidence points.*

Proof. We show that each essential H-Nielsen class is nonempty. Consider an essential class $p_{\Gamma H} C(\widetilde{p}_H, \alpha\widetilde{q}_H)$. Then $\Lambda(\widetilde{p}_H, \alpha\widetilde{q}_H) \neq 0$ implies a point $\widetilde{z} \in C(\widetilde{p}_H, \alpha\widetilde{q}_H)$, by which $p_{\Gamma H} C(\widetilde{p}_H, \alpha\widetilde{q}_H)$ is nonempty as required. □

Theorem 2.9. $N_H(p, q)$ *is a homotopy invariant* (*with respect to the homotopies satisfying* (CAC $+A+B$)).

Proof. Let the map (p_t, q_t) be such a homotopy. It is enough to see that the class $p_{\Gamma H} C(\widetilde{p}_{0H}, \alpha\widetilde{q}_{0H})$ is essential if and only if the same is true for the class $p_{\Gamma H} C(\widetilde{p}_{1H}, \alpha\widetilde{q}_{1H})$. However, this is implied by the eguality of Lefschetz numbers

$$\Lambda(\widetilde{p}_{0H}, \alpha\widetilde{q}_{0H}) = \Lambda(\widetilde{p}_{1H}, \alpha\widetilde{q}_{1H}).$$

3. Application to differential inclusions

Now, we will apply the Nielsen theory developed in the foregoing section for obtaining the multiplicity results to differential inclusions

$$X' \in F(t, X), \tag{1}$$

where $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a set-valued (upper) Carathéodory mapping (for the definition and more details see e.g. [G2], [AGG]) and J is an arbitrary (possibly infinite) real interval. By a solution $X(t)$ of (1), we always mean a locally absolutely continuous function $X(t)$ satisfying (1) for a.e. $t \in J$.

Considering (1) with the constraint, namely

$$X \in S \subset C(J, \mathbb{R}^n), \tag{2}$$

where S is a nonempty subset, we start with the following essential result (see [A1] Theorem 2, [AGG] Proposition 2.32). Let us recall that the appropriate topology in $C(J, \mathbb{R}^n)$ is the one of the uniform convergence on compact subintervals of J.

Lemma 3.1. *Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a Carathéodory mapping and assume that:*

(i) *there exists a subset Q of $C(J, \mathbb{R}^n)$ such that, for any $q \in Q$, the set $T(q)$ of all solutions of the problem*

$$\begin{cases} X' \in G(t, X, q(t)), \\ X \in S, \end{cases} \tag{3}$$

on $J \in \mathbb{R}$ is nonempty,

(ii) *$T(Q)$ is bounded in $C(J, \mathbb{R}^n)$, i.e. there exists a positive (single-valued) function $\phi : J \to \mathbb{R}^n$ such that $|\tau(t)| \leq \phi(t)$ for all $t \in J$, $\tau \in T(q)$ and $q \in Q$,*

(iii) *there exists a locally Lebesgue integrable function $\alpha : J \to \mathbb{R}^n$ such that*

$$|G(t, X(t), q(t))| \leq \alpha(t) \qquad \text{a.e. in } J,$$

for any pair $(q, X) \in \Gamma_T$, where Γ_T denotes the graph of T.

Then $T(Q)$ is a relatively compact subset of $C(J, R^n)$. Moreover, the multivalued operator $T : Q \rightsquigarrow S$ is u.s.c. with compact values if still

(iv) $\overline{T(Q)} \subset S$.

It will be also convenient to use the following definition.

Definition 3.2. We say that the mapping $T : Q \rightsquigarrow U$ is retractible onto Q, where U is an open subset of $C(J, \mathbb{R}^n)$ containing Q, if there is a retraction $r : U \to Q$ and $p \in U \setminus Q$, $r(p) = q$ implies that $p \notin T(q)$.

Its advantage consists in the fact that, for retractible mapping $T : Q \rightsquigarrow U$ onto Q with a retraction r in the sense of Definition 3.2, its composition with r, $r|_{T(Q)} \circ T : Q \rightsquigarrow Q$, has a coincidence point $\widehat{q} \in Q$ if and only if $\widehat{q}$ is a coincidence point of T.

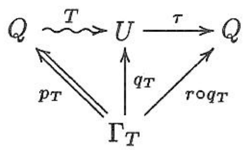
The following statement characterizes the matter.

Theorem 3.3. *Let the assumptions of Lemma* 3.1 *be satisfied, where Q is a closed connected subset of $C(J, \mathbb{R}^n)$ with a finitely generated abelian fundamental group. Assume, furthermore, that the operator $T : Q \rightsquigarrow U$, related to problem* (3), *is retractible onto Q with a retraction r in the sense of Definition* 3.2 *and with R_δ-values. At last, let*

$$G(t, c, c) \subset F(t, c) \tag{4}$$

take place a.e. in J, for any $c \in \mathbb{R}^n$. Then the original problem (1)-(2) *admits at least $N(\tau|_{T(Q)} \circ T(\cdot))$ solutions belonging to Q.*

Sketch of proof. By the hypothesis, Q is a connected (metric) ANR-space with a finitely generated abelian fundamental group and $T(q)$ is an R_δ-mapping. Since T is also, according to Lemma 3.1, u.s.c. and such that $\overline{T(Q)}$ is compact, $r \circ T$ is compact, admissible and consequently a CAC-mapping. This follows from the commutativity of the following diagram:



where (p_T, q_T) is a pair of natural projections of the graph Γ_T and p_T is Vietoris (for more details see [G1]). Therefore, according to Theorem 2.8 (see also Remarks 2.1 and 2.3), $(p_T, r|_{T(Q)} \circ T)$ admits at least $N(r|_{T(Q)} \circ T(\cdot))$ coincidence points. Because of Definition 3.2, they represent the solution of (3) and, in view of (4), they also satisfy the original problem (1)-(2). □

Since the topological structure of the solution set to (3) plays an important role, we recall still another slightly modified result in [ADG].

Lemma 3.4. *Consider the problem*

$$\begin{cases} X' + A(t)X \in F(t, X), \\ L(X) = \Theta, \end{cases} \tag{5}$$

on a compact interval J, where $A : J \to \mathbb{R}^{n^2}$ is a single-valued bounded continuous $(n \times n)$-matrix function, $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory function and $L : C(J, \mathbb{R}^n) \to \mathbb{R}^n$ is a linear operator such that the homogeneous problem

$$\begin{cases} X' + A(t)X = 0, \\ L(X) = 0 \end{cases} \tag{6}$$

has only the trivial solution on J. Assume, additionally, that F is Lipschitzian in X for a.a. $t \in J$ with a sufficiently small Lipschitz constant and

$$|F(t, X)| \leq \mu(t)(|X| + 1) \tag{7}$$

holds for every $(t, X) \in J \times \mathbb{R}^n$*, where* $\mu : J \to [0, \infty)$ *is a suitable Lebesgue-integrable bounded function. Then the solution set of* (5) *is a (nonempty) compact* AR*-space (i.e. more than* R_δ*-set).*

Since the composition $F(t, q(t))$ of the above continuous in X multifunction F with any $q \in Q$, where Q is the same as in Theorem 3.3, is measurable (see e.g. [ADTZ] p. 34), the assertion of Lemma 3.4 is true for the linearized problem

$$(8) \qquad \begin{cases} X' + A(t)X \in F(t, q(t)), \\ L(X) = \Theta, \end{cases}$$

even without the Lipschitzian restriction. On the other hand, if F is "only" u.s.c. in X then, in order to get the same for (8), we must assume (see [ADTZ] p. 36) the product-measurability for $F(t, X) : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$.

Hence, applying Lemma 3.4 to Theorem 3.3, we arrive at

Theorem 3.5. *Consider boundary value problem* (5) *on a compact interval* J*. Assume that* $A : J \to \mathbb{R}^{n^2}$ *is a single-valued continuous* $(n \times n)$*-matrix and* $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ *is a Carathéodory product-measurable mapping satisfying* (7)*. Furthermore, let* $L : C(J, \mathbb{R}^n) \to \mathbb{R}^n$ *be a linear operator such that problem* (6) *has only the trivial solution on* J*. Then the orginal problem* (5) *has* $N(r|_{T(Q)} \circ T(\cdot))$ *solutions, provided there exists a closed connected subset* Q *of* $C(J, \mathbb{R}^n)$ *with a finitely generated abelian fundamental group such that*

(i) $T(Q)$ *is bounded,*
(ii) $T(q)$ *is retractible onto* Q *with a retraction* r *in the sense of Definition* 3.2*,*
(iii) $\overline{T(Q)} \subset \{X \in AC(J, \mathbb{R}^n) : L(X) = \Theta\}$*,*

where $T(q)$ *denotes the set of (existing) solutions to* (8)*.*

Remark 3.6. In the single-valued case, we can assume the unique solvability of the associated linearized problem, and therefore we can consider problems on not nessesarily compact intervals, when applying Theorem 3.3 directly.

4. Nontrivial example

Consider the Carathéodory system

$$(9) \qquad \begin{cases} x' + ax \in e(t, x, y)y^{(1/m)} + g(t, x, y), \\ y' + by \in f(t, x, y)x^{(1/n)} + h(t, x, y), \end{cases}$$

where a, b are positive numbers and m, n are odd integers with $\min(m, n) \geq 3$. Let suitable positive constants E_0, F_0, G, H exist such that

$$|e(t, x, y)| \leq E_0, \qquad |f(t, x, y)| \leq F_0$$
$$|g(t, x, y)| \leq G, \qquad |h(t, x, y)| \leq H$$

hold for a.a. $t \in (-\infty, \infty)$ and all $(x, y) \in \mathbb{R}^2$.

Futhermore, assume the existence of positive constants $e_0, f_0, \delta_1, \delta_2$ such that

$$0 < e_0 \leq e(t, x, y) \tag{10}$$

for $x \geq -\delta_1$, $y \geq \delta_2$ and a.a. t as well as for $x \leq \delta_1$, $y \leq -\delta_2$ and a.a. t, jointly with

$$0 < f_0 \leq f(t, x, y) \tag{11}$$

for $x \geq \delta_1$, $y \leq \delta_2$ and a.a. t as well as for $x \leq -\delta_1$, $y \geq -\delta_2$ and a.a. t.

Another possibility is that (10) holds for $x \leq \delta_1$, $y \geq \delta_2$ and a.a. t as well as for $x \geq -\delta_1$, $y \leq -\delta_2$ and a.a. t and that (11) holds at the same time for $x \geq \delta_1$, $y \geq -\delta_2$ and a.a. t as well as for $x \leq -\delta_1$, $y \leq \delta_2$ and a.a. t.

As a constraint S, consider at first the periodic boundary condition

$$(x(0), y(0)) = (x(\omega), y(\omega)). \tag{12}$$

More precisely, we take $S = Q = Q_1 \cap Q_2 \cap Q_3$, where

$$Q_1 = \{q(t) \in C([0,\omega], \mathbb{R}^2) \mid \|q(t)\| := \max\{\max_{t\in[0,\omega]} |q_1(t)|, \max_{t\in[0,\omega]} |q_2(t)|\} \leq D\},$$
$$Q_2 = \{q(t) \in C([0,\omega], \mathbb{R}^2) \mid \min_{t\in[0,\omega]} |q_1(t)| \geq \delta_1 > 0 \text{ or } \min_{t\in[0,\omega]} |q_2(t)| \geq \delta_2 > 0\},$$
$$Q_3 = \{q(t) \in C([0,\omega], \mathbb{R}^2) \mid q(0) = q(\omega)\};$$

the constants δ_1, δ_2, D will be specified below.

Important properties of the set Q can be expressed as follows.

Lemma 4.1. *The set Q defined above satisfies:*

(i) *Q is a closed connected subset of $C([0,\omega], \mathbb{R}^2)$,*
(ii) *$Q \in$ ANR,*
(iii) *$\pi_1(Q) = \mathbb{Z}$.*

Proof. Since Q is an intersection of closed sets Q_1, Q_2, Q_3, we conclude that Q is a closed subset of $C([0,\omega], \mathbb{R}^2)$ as well. The connectedness follows from the proof of (iii) below.

For (ii), it is enough to show that Q is neighbourhood retract of Q_3. Hence, let $\varepsilon > 0$ be such that $\delta_1 - \varepsilon > 0$ and $\delta_2 - \varepsilon > 0$.

Defining

$$U = \{q \in Q_3 \mid \max\{\max_{t\in[0,\omega]} |q_1(t)|, \max_{t\in[0,\omega]} |q_2(t)|\} < D + \varepsilon \text{ and }$$
$$[\min_{t\in[0,\omega]} |q_1(t)| > \delta_1 - \varepsilon \text{ or } \min_{t\in[0,\omega]} |q_2(t)| > \delta_2 - \varepsilon]\},$$

U is obviously an open neighbourhood of Q in Q_3.

Now, we will define the retraction $r : U \to Q$. Let us take

$$A = \{(x, y) \in \mathbb{R}^2 \mid \max(|x|, |y|) \leq D \text{ and } [|x| \geq \delta_1 \text{ or } |y| \geq \delta_2]\}$$

and

$$V = \{(x,y) \in \mathbb{R}^2 \mid \max(|x|,|y|) < D + \varepsilon \text{ and } [|x| > \delta_1 - \varepsilon \text{ or } |y| > \delta_2 - \varepsilon]\}.$$

There exists a retraction $r_0 : V \to A$.

Now, notice that for every $q \in U$ and every $t \in [0,\omega]$ we have $q(t) \in V$. Define $r : U \to Q$,

$$r(q)(t) = r_0(q(t)).$$

It is easy to see that r is a desired retraction and the proof of (ii) is complete.

At last, we will show (iii). It is obvious that $\pi_1(A) = \mathbb{Z}$, where A is defined above. At the same time, $A = Q \cap \mathbb{R}^2$, when regarding $\mathbb{R}^2$ as a subspace of constant functions of Q_3. For (iii), it is sufficient to show that A is a deformation retract of Q.

We define

$$\rho : Q \times [0,1] \to A$$

by the formula

$$\rho(q,\lambda) = (\lambda q_1 + (1-\lambda)\overline{q_1}, \lambda q_2 + (1-\lambda)\overline{q_2}),$$

where $q = (q_1, q_2) \in Q$ and $\overline{q_1} = q_1(0)$, $\overline{q_2} = q_2(0)$. One can readily check that ρ is a deformation retraction, which completes the proof of our lemma. □

Besides (9) consider still its embedding into

$$\begin{cases} x' + ax \in [(1-\mu)e_0 + \mu e(t,x,y)]y^{1/m} + \mu g(t,x,y), \\ y' + by \in [(1-\mu)f_0 + \mu f(t,x,y)]x^{1/n} + \mu h(t,x,y), \end{cases} \tag{13}$$

where $\mu \in [0,1]$ and observe that (13) reduces to (9) for $\mu = 1$.

The associated linearized system to (13) takes for $\mu \in [0,1]$ the form

$$\begin{cases} x' + ax \in [(1-\mu)e_0 + \mu e(t,q_1(t),q_2(t))]q_2(t)^{1/m} + \mu g(t,q_1(t),q_2(t)), \\ y' + by \in [(1-\mu)f_0 + \mu f(t,q_1(t),q_2(t))]q_1(t)^{1/n} + \mu h(t,q_1(t),q_2(t)), \end{cases} \tag{14}$$

or equivalently

$$\begin{cases} x' + ax = [(1-\mu)e_0 + \mu e_t]q_2(t)^{1/m} + \mu g_t, \\ y' + by = [(1-\mu)f_0 + \mu f_t]q_1(t)^{1/n} + \mu h_t, \end{cases} \tag{15}$$

where $e_t \subset e(t,q_1(t),q_2(t))$, $f_t \subset f(t,q_1(t),q_2(t))$, $g_t \subset g(t,q_1(t),q_2(t))$, $h_t \subset h(t,q_1(t),q_2(t))$ are measurable selectors. These exist, because the Carathéodory functions e,f,g,h are weakly selectionally measurable (see e.g. [ADTZ]).

It is well-known that problem (12)-(15) has, for each $q(t) \in Q$ and every fixed quadruple of selectors e_t, f_t, g_t, h_t, a unique solution $X(t) = (x(t),y(t))$ namely

$$X(t) = \begin{cases} x(t) = \displaystyle\int_0^\omega G_1(t,s)[((1-\mu)e_0 + \mu e_s)q_2(s)^{1/m} + \mu g_s]ds, \\ y(t) = \displaystyle\int_0^\omega G_2(t,s)[((1-\mu)f_0 + \mu f_s)q_1(s)^{1/n} + \mu h_s]ds, \end{cases}$$

where

$$G_1(t,s) = \begin{cases} \dfrac{e^{-a(t-s+\omega)}}{1-e^{-a\omega}} & \text{for } 0 \le t \le s \le \omega, \\ \dfrac{e^{-a(t-s)}}{1-e^{-a\omega}} & \text{for } 0 \le s \le t \le \omega, \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{e^{-b(t-s+\omega)}}{1-e^{-b\omega}} & \text{for } 0 \le t \le s \le \omega, \\ \dfrac{e^{-b(t-s)}}{1-e^{-b\omega}} & \text{for } 0 \le s \le t \le \omega. \end{cases}$$

Since the solution set $T_\mu(q)$, $\mu \in [0,1]$, of (12)-(14) is for every fixed $q \in Q$, according to Lemma 3.4, a compact AR-set, provided Carathéodory multifunctions e, f, g, h are product-measurable, $T_\mu(q)$ has R_δ-values. Moreover, $T_\mu(q)$ is, in view of Lemma 3.1, u.s.c. with compact values and such that $\overline{T_\mu(Q)}$ is compact for every $\mu \in [0,1]$. Thus, $T_\mu(q)$ will be a particular case of a CAC-self-mapping if $T_\mu(Q) \subset Q$. In order to verify that $\overline{T_\mu(Q)} \subset S = Q$, it is sufficient to prove just that $T_\mu(Q) \subset Q$, $\mu \in [0,1]$, because $S = Q$ is closed. Hence, the Nielsen number $N(T_\mu)$ is well-defined for every $\mu \in [0,1]$, provided only product-measurability of e, f, g, h and $T_\mu(Q) \subset Q$.

Since $X(0) = X(\omega)$, i.e. $T_\mu(Q) \subset Q_3$, it remains to prove that $T_\mu(Q) \subset Q_1$ as well as $T_\mu(Q) \subset Q_2$. Let us consider the first inclusion. In view of

$$\min_{t,s\in[0,\omega]} G_1(t,s) \ge \frac{e^{-a\omega}}{1-e^{-a\omega}} > 0 \quad \text{and} \quad \min_{t,s\in[0,\omega]} G_2(t,s) \ge \frac{e^{-b\omega}}{1-e^{-b\omega}} > 0,$$

we obtain for the above solution $X(t)$ that

$$\begin{aligned} \max_{t\in[0,\omega]} |x(t)| &\le \max_{t\in[0,\omega]} \int_0^\omega |G_1(t,s)| \, |[(1-\mu)e_0 + \mu e_s]q_2(s)^{1/m} + \mu g_s| \, ds \\ &\le [(e_0+E_0)D^{1/m} + G] \int_0^\omega G_1(t,s)\, ds = \frac{1}{a}[(e_0+E_0)D^{1/m} + G] \end{aligned}$$

and

$$\begin{aligned} \max_{t\in[0,\omega]} |y(t)| &\le \max_{t\in[0,\omega]} \int_0^\omega |G_2(t,s)| \, |[(1-\mu)f_0 + \mu f_s]q_1(s)^{1/n} + \mu h_s| \, ds \\ &\le [(f_0+F_0)D^{1/n} + H] \int_0^\omega G_2(t,s)\, ds = \frac{1}{b}[(f_0+F_0)D^{1/n} + H]. \end{aligned}$$

Because of

$$\begin{aligned} \|X(t)\| &= \max\{\max_{t\in[0,\omega]} |x(t)|, \max_{t\in[0,\omega]} |y(t)|\} \\ &\le \max\left\{\frac{1}{a}[(e_0+E_0)D^{1/m} + G], \frac{1}{b}[(f_0+F_0)D^{1/n} + H]\right\}, \end{aligned}$$

a sufficiently large constant D certainly exists such that $\|X(t)\| \leq R$, i.e. $T_\mu(Q) \subset Q_1$, independently of $\mu \in [0,1]$ and e_t, f_t, g_t, h_t.

For the inclusion $T_\mu(Q) \subset Q_2$, we proceed quite analogously.

Assuming that $q(t) \in Q_2$, we have

$$\text{either } \min_{t\in[0,\omega]} |q_1(t)| \geq \delta_1 > 0 \qquad \text{or } \min_{t\in[0,\omega]} |q_2(t)| \geq \delta_2 > 0.$$

Therefore, we obtain for the above solution $X(t)$ that (see (10))

$$\begin{aligned}\min_{t\in[0,\omega]} |x(t)| &= \min_{t\in[0,\omega]} \int_0^\omega |G_1(t,s)|\,|[(1-\mu)e_0 + \mu e_s]q_2(s)^{1/m} + \mu g_s|\,ds \\ &\geq |e_0\delta_2^{1/m} - G| \int_0^\omega G_1(t,s)\,ds = \frac{1}{a}|e_0\delta_2^{1/m} - G| > 0,\end{aligned}$$

provided $G < e_0\delta_2^{1/m}$, for $q_1 \geq -\delta_1$, $q_2 \geq \delta_2$ as well as for $q_1 \leq \delta_1$, $q_2 \leq -\delta_2$ (or another alternative as above) or (see (11))

$$\begin{aligned}\min_{t\in[0,\omega]} |y(t)| &= \min_{t\in[0,\omega]} \int_0^\omega |G_2(t,s)|\,|[(1-\mu)f_0 + \mu f_s]q_1(s)^{1/n} + \mu h_s|ds \\ &\geq |f_0\delta_1^{1/n} - H| \int_0^\omega G_2(t,s)ds = \frac{1}{b}|f_0\delta_1^{1/n} - H| > 0,\end{aligned}$$

provided $H < f_0\delta_1^{1/n}$, for $q_1 \geq \delta_1$, $q_2 \leq \delta_2$ as well as for $g_1 \leq -\delta_1$, $q_2 \geq -\delta_2$ (or another alternative as above).

So, in order to prove that $X(t) \in Q_2$, we need to fulfil simultaneously the following inequalities

$$(16)\qquad \begin{cases} (1/a)|e_0\delta_2^{1/m} - G| \geq \delta_1 > (H/f_0)^n \\ (1/b)|f_0\delta_1^{1/n} - H| \geq \delta_2 > (G/e_0)^m. \end{cases}$$

Let us observe that the "amplitudes" of the multifunctions g, h must be sufficiently small. On the other hand, if e_0 and f_0 are sufficiently large (for fixed quantities a, b, G, H), then we can easily find δ_1, δ_2 satisfying (16).

After all, if there exist constants δ_1, δ_2 obeying (16), then we arrive at $X(t) \in Q_2$, i.e., $T_\mu(Q) \subset Q_2$, independently of $\mu \in [0,1]$ and e_t, f_t, g_t, h_t. This already means that $T_\mu(Q) \subset Q$, $\mu \in [0,1]$, as required.

Now, since all the assumptions of Theorem 3.5 are satisfied, problem (12)-(13) possesses at least $N(T_\mu(\cdot))$ solutions belonging to Q, for every $\mu \in [0,1]$. In particular, problem (9)-(12) has $N(T_1(\cdot))$ solutions, but according to the invariantness under homotopy, $N(T_1(\cdot)) = N(T_0(\cdot))$. So, it remains to compute the Nielsen number $N(T_0(\cdot))$ for the operator $T_0 : Q \to Q$, where

$$(17)\qquad T_0(q) = \left(e_0 \int_0^\omega G_1(t,s)q_2(s)^{1/m}ds, f_0 \int_0^\omega G_2(t,s)q_1(s)^{1/n}ds\right).$$

Hence, besides (17) consider still its embedding into the one-parameter family of operators

$$T^{\nu}(q) = \nu T_0(q) + (1-\nu) r \circ T_0(q), \qquad \nu \in [0,1],$$

where $r(q) := (r(q_1), r(q_2))$ and

$$r(q_i) = q_i(0) \qquad \text{for } i = 1,2.$$

One can readily check that $r : Q \to Q \cap \mathbb{R}^2$ is a retraction and $T_0(\overline{q}) : Q \cap \mathbb{R}^2 \to Q$ is retractible onto $Q \cap \mathbb{R}^2$ with the retraction r in the sense of Definition 3.2. Thus, $r \circ T_0(\overline{q}) : Q \cap \mathbb{R}^2 \to Q \cap \mathbb{R}^2$ has a fixed point $\widehat{q} \in Q \cap \mathbb{R}^2$ if and only if $\widehat{q} = T_0(\widehat{q})$. Moreover, $r \circ T_0(q) : Q \to Q \cap \mathbb{R}^2$ has evidently a fixed point $\widehat{q} = Q \cap \mathbb{R}^2$ if and only if $\widehat{q} = T_0(\widehat{q})$. So, the investigation of fixed points for $T^0(q) = r \circ T_0(q)$ turns out to be equivalent with the one for $T^0(\overline{q}) : Q \cap \mathbb{R}^2 \to Q \cap \mathbb{R}^2$.

Since, in view of invariantness under homotopy, we have

$$N(T_1(\cdot)) = N(T_0(\cdot)) = N(T^1(\cdot)) = N(T^0(\cdot)),$$

where

$$T^0(q) = \left(\frac{e_0 e^{-a\omega}}{1-e^{-a\omega}} \int_0^{\omega} e^{as} q_2(s)^{1/m} ds, \frac{f_0 e^{-b\omega}}{1-e^{-b\omega}} \int_0^{\omega} e^{bs} q_1(s)^{1/n} ds \right)$$

and

$$T^0(\overline{q}) = \left(\frac{e_0}{a} \overline{q_2}^{(1/m)}, \frac{f_0}{b} \overline{q_1}^{(1/n)} \right) \qquad \text{for } \overline{q} = (\overline{q_1}, \overline{q_2}) = (q_1(0), q_2(0)) \in Q \cap \mathbb{R}^2,$$

it remains to estimate $N(T^0(\cdot))$. It will be useful to do it by passing to a simpler finite-dimensional analogy, namely by the direct computation of fixed points of the operator

$$T^0(\overline{q}) : Q \cap \mathbb{R}^2 \to Q \cap \mathbb{R}^2,$$

belonging to different Nielsen classes.

There are two fixed points $\widehat{q}_+ = (\widehat{q}_1, \widehat{q}_2)$ and $\widehat{q}_- = (-\widehat{q}_1, -\widehat{q}_2)$ in $Q \cap \mathbb{R}^2$, where

$$\widehat{q}_1 = \left(\frac{e_0}{a}\right)^{(mn/mn-1)} \left(\frac{f_0}{b}\right)^{(1/mn-1)},$$

$$\widehat{q}_2 = \left(\frac{e_0}{a}\right)^{(m/mn-1)} \left(\frac{f_0}{b}\right)^{(mn/mn-1)}.$$

These fixed points belong to different Nielsen classes, because any path u connecting them in $Q \cap \mathbb{R}^2$ and its image $T^0(u)$ are not homotopic in the space $Q \cap \mathbb{R}^2$. Thus, according to the equivalent definition of the Nielsen number due to F. Wecken (see e.g. [Sc]), $N(T^0(\overline{q})) = 2$. By means of the reduction property which is true here (see [A2]), we have moreover

$$N(T_1(\,\cdot\,)) = N(T^0(\,\cdot\,)) = N(T^0(\overline{q})) = 2$$

and so, according to Theorem 3.5, system (9) admits at least two solutions belonging to Q, provided suitable positive constants δ_1, δ_2 exist satisfying (16) and e, f, g, h are product-measurable.

In fact, system (9) possesses at least three solutions satisfying (12), when the sharp inequalities appear in (16), by which the lower boundary of Q becomes fixed-point free. Indeed. Since

$$\Lambda(T_1(\,\cdot\,), Q) = \Lambda(T^0(\,\cdot\,), Q) = \lambda(T^0(\overline{q}), Q \cap \mathbb{R}^2)$$

holds for the generalized and ordinary Lefschetz numbers (see [G1]) and

$$|\lambda(T^0(\overline{q}), Q \cap \mathbb{R}^2)| = N(T^0(\overline{q})),$$

according to [BBPT], we obtain

$$|\Lambda(T_1(\,\cdot\,), Q)| = 2.$$

Futhermore, since for the self-map $T_1(\,\cdot\,)$ on the convex set $Q_1 \cap Q_3$ such that $\overline{T_1(Q_1 \cap Q_3)}$ is compact we have

$$\Lambda(T_1(\,\cdot\,), Q_1 \cap Q_3) = 1$$

(see [G1]), it follows from the additivity, excision and existence properties of the fixed-point index (see [BK]) that the mapping $T_1(\,\cdot\,)$ has the third coincidence point in $\overline{Q_1 \cap Q_3 \setminus Q}$ representing a solution of problem (9)-(12) and belonging to $Q_1 \setminus Q$.

As we could see, problem (9)-(12) admits at least two solutions in $Q_1 \cap Q_2$ for an arbitrary $\omega > 0$. Futhermore, because of rescaling (9), when replacing t by $t + (\omega/2)$, there are also two solutions of (9) satisfying $X(-\omega/2) = X(\omega/2)$ for an arbitrary $\omega > 0$ and belonging to $Q_1 \cap Q_2$.

Therefore, according to the intuitively clear Lemma 3.2 in [AGG] and by the obvious geometrical reasons, related to the appropriate subdomains of $Q_1 \cap Q_2$ system (9) possesses at least two entirely bounded solutions in $Q_1 \cap Q_2$.

Of cource, because of replacing t by $(-t)$ the same result holds for (9) with negative constants a, b as well.

Finally, let us consider again system (9), where a, b, m, n are the same but e, f, g, h are this time l.s.c. in (x, y) for a.a. $t \in (-\infty, \infty)$ multifunctions with the same estimates as above. Since each such mapping e, f, g, h has, under our regularity assumptions including the product-measurablity, a Carathéodory selector (see e.g. [R]), the same assertion must be also true in this new situation.

So, after summing up the above conclusions, we can give finally

Theorem 4.2. *Let suitable positive constants δ_1, δ_2 exist such that the inequalities*

$$\begin{cases} \dfrac{1}{|a|}|e_0\delta_2^{1/m} - G| \geq \delta_1 > \left(\dfrac{H}{f_0}\right)^n, \\ \dfrac{1}{|b|}|f_0\delta_2^{1/n} - H| \geq \delta_2 > \left(\dfrac{G}{e_0}\right)^m \end{cases} \tag{18}$$

are satisfied for constants e_0, f_0, G, H estimating the product-measurable semicontinuous multifunctions e, f, g, h as above, for constants a, b with $ab > 0$ and for odd integers m, n with $\min(m,n) \geq 3$. Then system (9) *admits at least two entirely bounded solutions. In particular, if the multifunctions e, f, g, h are still ω-periodic in t, then system* (9) *admits at least three ω-periodic solutions, provided the sharp inequalities appear in* (18).

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ON THE SEMILINEAR MULTI-VALUED FLOW UNDER CONSTRAINTS AND THE PERIODIC PROBLEM

RALF BADER

1. Semilinear differential inclusions under constraints

Let $A : D(A) \to E$ be a closed, linear, densely defined operator on a Banach space E (in general unbounded) being the infinitesimal generator of a C_0-semigroup $\{U(t)\}_{t\geq 0}$. Let D be a convex subset of E and let $F : [0,T] \times D \to 2^E \setminus \emptyset$ be a multi-valued mapping. Given $x_0 \in D$ we consider the initial value problem

$$(1) \qquad \begin{cases} x'(t) \in Ax(t) + F(t, x(t)), \\ x(0) = x_0. \end{cases}$$

A continuous mapping $x : [0,T] \to D$ is called a mild solution of (1) if x satisfies the integral equation

$$(2) \qquad x(t) = U(t)x_0 + \int_0^t U(t-s)f(s)ds \qquad \text{for every } t \in [0,T],$$

where

$$f \in N_F(x) := \{g \in L^1([0,T], E) \mid g(t) \in F(t, x(t)) \text{ a.e. on } [0,T]\}.$$

Let for $x \in D$ denote by

$$T_D(x) := \left\{ y \in E \mid \liminf_{h\to 0, h>0} \frac{d(x+hy, D)}{h} = 0 \right\}$$

the tangent cone to D at x and let χ be the Hausdorff measure of noncompactness (MNC). With these notations we can formulate:

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Proposition 1. *Let E be a Banach space, $D \subset E$ closed and convex and let A be the generator of a C_0-semigroup $\{U(t)\}_{t \geq 0}$ leaving the set D invariant. Assume that $F : [0,T] \times D \to 2^E \setminus \emptyset$ is upper semi-continuous, bounded with closed, convex values and that*

$$F(t,x) \cap T_D(x) \neq \emptyset \qquad \text{for each } t \in [0,T] \text{ and } x \in D. \tag{3}$$

Then for each $x_0 \in D$ there exists a mild solution of (1) *provided that for bounded $\Omega \subset D$*

$$\lim_{h \to 0,\ h>0} \chi(F((t-h,t+h) \cap [0,T] \times \Omega)) \leq k(t)\chi(\Omega) \tag{4}$$

for each $t \in [0,T]$ with $k \in L^1([0,T],\mathbb{R})$ or the semigroup $\{U(t)\}_{t \geq 0}$ is compact.

The set D is an invariant set of $\{U(t)\}_{t \geq 0}$, if and only if for each $t \geq 0$ we have $U(t)D \subset D$. Recall that this condition can be characterized solely in terms of the generator A (see [10]).

The existence of solutions of (1) was shown in [3] (see also [12]) under analogous assumptions as in Proposition 1, except that instead of our seperated boundary conditions, i.e. D is an invariant set of $\{U(t)\}_{t \geq 0}$ and (3), it was assumed that

$$F(t,x) \cap T_D^U(x) \neq \emptyset \qquad \text{for each } t \in [0,T] \text{ and } x \in D, \tag{5}$$

where

$$T_D^U(x) := \left\{ y \in E \mid \liminf_{h \to 0, h>0} \frac{d(U(h)x + hy, D)}{h} = 0 \right\}. \tag{6}$$

Now Proposition 1 follows since, under the given assumptions, $y \in T_D(x)$ implies $y \in T_D^U(x)$. Recall that from convexity of D we have that $T_D(\cdot)$ is lower semicontinuous with closed convex values and thus, by the Michael selection theorem, there is a continuous selection g of $T_D(\cdot)$ with $g(x) = y$. Then apply the known fact that for single-valued continuous maps the seperated conditions imply (5) (see [10]).

2. Topological characterization of the solution set

The purpose of this section is to show the following

Theorem 2. *Let the conditions of Theorem 1 be fulfilled and assume in addition that D is bounded and has nonempty interior. Then for each $x_0 \in D$ the set of mild solutions $S(x_0)$ of* (1) *is an R_δ-set, i.e. the intersection of a decreasing sequence of compact absolute retracts* (*see* [9]).

We construct maps $F_n : [0,T] \times D \to 2^E \setminus \emptyset$ such that

(i) $F(t,x) \subset F_{n+1}(t,x) \subset F_n(t,x)$ on $[0,T] \times D$ for each $n \geq 1$,

(ii) $d_H(F_n(t,x),F(t,x)) \to 0$ on $[0,T] \times D$ as $n \to \infty$, where d_H denotes the Hausdorff distance and

(iii) the maps $(t,x) \mapsto F_n(t,x) \cap T_D(x)$ have continuous selections f_n by the Michael selection theorem*.

These maps f_n now, in turn, can be approximated by locally lipschitz $\widetilde{f}_n$ being still selections of $T_D(\,\cdot\,)$ since D has nonempty interior. Using the maps $\widetilde{f}_n$ one shows that the set $S_n(x_0)$ of solutions to (1) with F_n instead of F is contractible. Moreover, using standard methods occuring in the existence theory of (1) we see that $\chi_0(S_n(x_0)) \to 0$, where χ_0 denotes the Hausdorff MNC in $C([0,T],E)$. Hence, as it was observed in [3], it follows that $S(x_0)$ is indeed an R_δ-set in view of Hyman's result (see [9]).

We would like to mention that the same conclusion for the semilinear system was obtained in [5] but without constraints.

3. Periodic solutions of the semilinear system

Let $F : [0,\infty) \times D \to 2^E \setminus \emptyset$ be T-periodic, i.e. $F(t,x) \subset F(t+T,x)$ for every $t \in [0,\infty)$ and every $x \in D$. We will be concerned with the existence of T-periodic, mild solutions to

$$x'(t) \in Ax(t) + F(t,x(t)). \tag{7}$$

In the sequel we give some existence principles under varying conditions on D, F and A similar to those given in [11] for single-valued perturbations F.

Theorem 3. *Let E be a Banach space and $D \subset E$ closed, convex and bounded with* $\operatorname{int} D \neq \emptyset$. *Let A be the generator of a C_0-semigroup $\{U(t)\}_{t\geq 0}$ of type $(1,\omega)$ leaving the set D invariant. Assume that $F : [0,\infty) \times D \to 2^E \setminus \emptyset$ is upper semi-continuous, bounded, T-periodic with closed, convex values such that* (3) *holds. Then the periodic problem* (7) *has a solution in each of the following cases:*

(i) *$t \mapsto U(t)$ is continuous with respect to the norm in $\mathcal{L}(E)$ for $t > 0$, $\omega T + 4\int_0^T k(s)\,ds < 0$ and* (4) *holds,*

(ii) *E is separable and $\omega T + \int_0^T k(s)\,ds < 0$ and* (4) *holds,*

(iii) *$U(t)$ is a compact semigroup.*

In view of our above obtained results the translation operator along trajectories is topologically admissible for the fixed point theory of multi-valued maps (see [8]). Each of the assumptions (i)-(iii) implies that this operator is a condensing map (see [2]). Hence the theorem follows by an application of an appropriate fixed point result (see e.g. [1]).

The consideration of semigroups of type $(1,\omega)$ does not seem to be a serious restriction since there is an equivalent norm on E such that this always can be achieved. Recall that our assumptions remain valid under equivalent renorming.

In the next result we consider also the case $\omega = 0$.

*here we have used lower semi-continuity of $T_D(\,\cdot\,)$ giving the reason why we have not worked with the cone $T_D^U(x)$ given in (6); $T_D^U(\,\cdot\,)$ is not lower semi-continuous.

Theorem 4. *Let the suppositions in front of Theorem 3 be fufilled. Then the periodic problem (7) has a solution provided*

$$0 \in D, \quad \omega \le 0, \quad F \text{ is compact and } \quad 1 \in \varrho(U(T)).$$

($\varrho(U(T))$ denotes the resolvent set of $U(T)$). For $\varepsilon > 0$ we consider $y' \in Ay - \varepsilon y + F(t, y)$ and obtain by Theorem 3 existence of a periodic solution x_ε to the perturbed equation. Assumption $1 \in \varrho(U(T))$ comes into play when proving convergence of x_ε as $\varepsilon \to 0+$ to a solution of (7).

In [6] there is an example showing that without the condition "$1 \in \varrho(U(T))$" the theorem may be false (in this example there is actually $A = 0$).

In the remaining results we would like to dispense with the assumption $\operatorname{int} D \ne \emptyset$.

Theorem 5. *Theorem 4 remais true if the condition "D has nonempty interior" is replaced by the condition that the metric retraction on D exists, i.e. there is a continuous map $r : E \to D$ such that $|r(x) - x| = d(x, D)$.*

Reduction to the above results is possible since the map $F(\cdot, r(\cdot))$ has (4) with $T_D(\cdot)$ replaced by $T_{D_\delta}(\cdot)$ where $D_\delta := \{x \in E \mid d(x, D) \le \delta\}$ is a set with nonempty interior.

Let us note that in case E is uniformly convex each closed, convex and bounded set posesses the metric retraction.

In the case when D is compact and convex we can use similar arguments and we have

Theorem 6. *Let E be a Banach space and $D \subset E$ compact, convex and let A be the generator of a C_0-semigroup leaving D invariant. Assume that $F : [0, \infty) \times D \to 2^E \setminus \emptyset$ is upper semi-continuous, T-periodic with closed, convex values such that (4) holds. Then the periodic problem (7) has a solution.*

4. Equilibria

Let $F : D \to 2^E \setminus \emptyset$ and consider the autonomous equation

$$x'(t) \in Ax(t) + F(x(t)). \tag{8}$$

A stationary solution to (8), i.e. a point $x_0 \in D(A) \cap D$ satisfying $0 \in Ax_0 + F(x_0)$ is called *an equilibrium of* (8).

From the results on periodic solutions we can derive sufficient conditions on the existence of equilibria. Here we would like to state the result which follows from Theorem 6 and generalizes the well-known result from Browder [4] on equilibria.

Theorem 7. *Let E be a Banach space and $D \subset E$ compact, convex and let A be the generator of a C_0-semigroup leaving D invariant. Assume that $F : D \to 2^E \setminus \emptyset$ is upper semi-continuous with closed, convex values and such that*

$$F(x) \cap T_D(x) \ne \emptyset \qquad \text{for each } x \in D.$$

Then (7) has an equilibrium.

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DIRICHLET PROBLEMS WITH VARIABLE BOUNDARY DATA

Dorota Bors and Stanisław Walczak

0. Introduction

In this paper we consider the Dirichlet problem described by systems of ordinary differential equations of the form

$$u''(t) = G_u(t, u(t)) \tag{0.1}$$

with variable boundary data

$$u(0) = a_s, \qquad u(\pi) = b_s, \tag{0.2}$$

where $u(\cdot) \in H^1(I, \mathbb{R}^n)$, $I = [0, \pi]$, $a_s, b_s \in \mathbb{R}^n$ and $G = G(t, u)$ is a scalar function defined on $I \times \mathbb{R}^n$.

We say that problem (0.1)-(0.2) is well-posed, in the sense of Hadamard, if for any boundary data (0.2), there exists a unique solution $u = u(t; a_s, b_s)$ and this solution continuously depends on a_s and b_s. The question of the existence and uniqueness of solutions for Dirichlet problem (0.1)-(0.2) was considered in many papers and monographs (cf. [6] and references therein). The problem of the continuous dependence on boundary data for scalar equations was investigated in papers [3], [4], [5], [8] (see also references therein). In these papers the authors, basing themselves on direct methods, prove some sufficient conditions under

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which a solution of the Dirichlet problem belongs to the C^2 class and continuously depends on boundary data (a_s, b_s). In our paper we investigate system (0.1)-(0.2) using variational methods. Without going into details, the main result of this paper may be formulated as follows: if the function G satisfies some growth conditions, then for any (a_s, b_s), there exists a solution $u_s(\cdot) \in H_0^1(I, \mathbb{R}^n)$ and $u_s(\cdot)$ tends to $u_0(\cdot)$ in the norm topology of H^1 provided that $\{(a_s, b_s)\}$ tends to (a_0, b_0) in $\mathbb{R}^n \times \mathbb{R}^n$. Thus we prove that problem (0.1)-(0.2) is well-posed in the sense of Hadamard.

Our paper is divided into two parts. In the first part we consider the coercive case, in the second one the case related to the Mountain Pass Theorem.

I. Coercive case

1. Formulation of the problem and the principal lemma

Let $I = [0, \pi]$. By H^1 we shall denote the space $H^1 = H^1(I, \mathbb{R}^n) = \{u(\cdot) \mid I \to \mathbb{R}^n \mid u(\cdot)$ is absolutely continuous on I and $u'(\cdot) \in L^2(I, \mathbb{R}^n)\}$. The norm in H^1 is given by the formula

$$\|u\| = |u(0)| + \left(\int_I |u'(t)|^2 dt\right)^{(1/2)}.$$

Let $H_0^1 = H_0^1(I, \mathbb{R}^n)$ denote the subspace of H^1 of all functions u such that $u(0) = u(\pi) = 0$ with the norm

$$\|u\| = \left(\int_I |u'(t)|^2 dt\right)^{(1/2)}.$$

Consider a system

$$\begin{aligned} &u''(t) = G_u(t, u(t)), \\ &u(0) = a_s, \ u(\pi) = b_s, \end{aligned} \tag{1.0}$$

where $G : I \times \mathbb{R}^n \to \mathbb{R}$, $u(\cdot) \in H^1$, $a_s, b_s \in \mathbb{R}^n$ and $|a_s| \le \overline{a}$, $|b_s| \le \overline{b}$ for some $\overline{a}, \overline{b} > 0$. If we put $u(t) = v(t) + \alpha_s(t)$ where $\alpha_s(t) = (1/\pi)(b_s - a_s)t + a_s$, then we obtain an equivalent system

$$\begin{aligned} &v''(t) = G_v(t, v(t) + \alpha_s(t)), \\ &v(0) = v(\pi) = 0. \end{aligned} \tag{1.1}$$

We shall impose the following conditions on the function G:

(1.2) the functions G, G_u are measurable with respect to t for any $u \in \mathbb{R}^n$ and continuous with respect to u for $t \in I$ a.e.;

(1.3) for any $r > 0$, there exists $h \in L^1(I, \mathbb{R}^+)$ such that

$$|G(t,u)| \leq h(t), \qquad |G_u(t,u)| \leq h(t)$$

for $t \in I$ a.e., $|u| \leqslant r$;

(1.4) there exist a constant $b < 1/2$ and some functions $\beta \in L^2(I, \mathbb{R}^n)$, $\gamma \in L^1(I, \mathbb{R})$, such that

$$G(t,u) \leq b|u|^2 - \langle \beta(t), u\rangle - \gamma(t)$$

for $t \in I$ a.e.

It is easy to see that system (1.1) is the Euler-Lagrange's equation for the functional of action

$$F_s(v) = F_{\alpha_s}(v) = \int_I \left(\frac{1}{2}|v'(t) + e_s|^2 - G(t, v(t) + \alpha_s(t))\right) dt, \tag{1.5}$$

where $e_s = (1/\pi)(b_s - a_s)$, $s = 0, 1, 2, \ldots$, and $v(\cdot) \in H_0^1(I, \mathbb{R}^n)$. Under assumptions (1.2), (1.3), functional (1.5) is well-posed and $F_s(\cdot)$ is of class C^1 (cf. [7]). Moreover, the Fréchet derivative of $F_s(\cdot)$ is given by the formula

$$\langle F_s'(v), h\rangle = \int_I (\langle v'(t) + e_s, h'(t)\rangle - \langle G_v(t, v(t) + \alpha_s(t)), h(t)\rangle)dt$$

for any $h \in H_0^1$.

Remark 1.1. Directly from assumptions (1.2), (1.3) and theorem 10.8i (cf. [2]) it follows that functional (1.5) is lower semicontinuous with respect to the weak topology of H_0^1.

Denote by V_s the set of all minimizers of the functional $F_s(\cdot)$, i.e.

$$V_s = \{v \in H_0^1 \mid F_s(v) = \min F_s(z),\ z \in H_0^1\}.$$

We say that a set $\widehat{V}_0 \subset H_0^1$ is the upper limit of the sequence of sets V_s, $s = 1, 2, \ldots$, if and only if any point $z \in \widehat{V}_0$ is a cluster point (with respect to the norm topology of H_0^1) of some sequence $\{v_s\}$ where $v_s \in V_s$ for $s = 1, 2, \ldots$. The upper limit of V_s will be denoted by $\limsup V_s = \widehat{V}_0$ (cf. [1]).

We shall prove the following

Lemma 1.1 (Principal lemma). *If*

1° *the function G satisfies assumptions* (1.2),(1.3) *and* (1.4),

2° *the sequences of functionals $F_s(\cdot)$ and $F_s'(\cdot)$ tend uniformly to $F_0(\cdot)$ and $F_0'(\cdot)$, respectively, on any ball $B_r = \{v(\cdot) \in H_0^1 \mid \|v\| \leq r\}$ with $r > 0$,* *then*

(a) *for any boundary data* a_s, b_s*, the set* V_s *of minimizers of the functional* $F_s(\cdot)$ *is nonempty and compact in the strong topology of* H_0^1*,* $s = 0, 1, 2, \ldots$*,*

(b) *there exists a ball* $B_\varrho \subset H_0^1$ *such that* $V_s \subset B_\varrho$ *for* $s = 0, 1, 2, \ldots$ *and any sequence* $\{v_s\}$*,* $v_s \in V_s$*, is compact in the strong topology of* H_0^1*,*

(c) $\limsup V_s$ *is a nonempty set and* $\limsup V_s \subset V_0$*, where* V_0 *is the set of minimizers of the functional* $F_0(\cdot)$ *(if* V_s *is a singleton, i.e.* $V_s = \{v_s\}$ *for* $s = 0, 1, 2, \ldots$*, then* $\lim v_s = v_0$*),*

(d) *for any* $\varepsilon > 0$*, there exists* $S > 0$ *such that* $V_s \subset V_0 + \varepsilon B_1$ *for all* $s > S$*.*

Proof. From (1.4) and the Poincaré inequality we get

$$\begin{aligned} F_s(v) &= \int_I \left(\frac{1}{2}|v'(t) + e_s|^2 - G(t, v(t) + \alpha_s(t))\right) dt \\ &\geq \left(\frac{1}{2} - b\right)\|v\|^2 - C_1\|v\| - C_2 = y(\|v\|), \end{aligned} \tag{1.6}$$

where b is described in (1.4) and C_1, C_2 are some constants. Since $(1/2) - b > 0$, the functional $F_s(\cdot)$ is coercive and weak lower semicontinuous (cf. Remark 1.1). This implies that there exists at least one minimizer v_s of $F_s(\cdot)$. Thus V_s is a nonempty set. The trinomial $y(\cdot)$ does not depend on s, so by (1.3), we have the following inclusions

$$V_s \subset \{v(\cdot) \in H_0^1 \mid y(\|v\|) \leq D\} \subset B_\varrho \tag{1.7}$$

for some $D > 0$, $\varrho > 0$ and $s = 0, 1, 2, \ldots$.

Let us fix, for a moment s ($s = 0, 1, 2, \ldots$), and let $\{v_l\} \subset V_s$ be a sequence. We have proved that V_s is bounded in H_0^1 (cf. (1.7)) and weakly closed. Passing, if necessary, to a subsequence, we may assume that v_l tends to some $v_0 \in V_s$ weakly in H_0^1. Since $F_s'(v_l) = 0$ for $l = 0, 1, 2, \ldots$, we get

$$\begin{aligned} 0 &= \langle F_s'(v_l), v_l - v_0\rangle - \langle F_s'(v_0), v_l - v_0\rangle = \|v_l - v_0\|_{H_0^1}^2 \\ &\quad + \int_I \langle G_v(t, v_0(t) + \alpha_s(t)) - G_v(t, v_l(t) + \alpha_s(t)), v_l(t) - v_0(t)\rangle dt. \end{aligned} \tag{1.8}$$

Since H_0^1 is compactly embedded into C^0, we get that the last component in (1.8) tends to zero if $l \to \infty$. Thus equality (1.8) implies that $v_l \to v_0$ strongly in H_0^1.

In this way we have proved that the set V_s is compact in the strong topology of H_0^1 for any $s = 0, 1, 2, \ldots$. Let $\{v_s\} \subset H_0^1$ be a sequence such that $v_s \in V_s$ for $s = 1, 2, \ldots$. Since $V_s \subset B_\varrho$, $s = 1, 2, \ldots$, for some $\varrho > 0$ (cf. (1.7)), we may assume without loss of generality that v_s tends to some $v \in B_\varrho$ weakly in H_0^1. Let us notice that $v_s \to v$ strongly in H_0^1. Indeed, by direct calculations we get

$$(1.9)\qquad \begin{aligned}\langle F_0'(v_s) - F_0'(v), v_s - v\rangle &= \|v_s - v\|^2_{H_0^1} \\ &+ \int_I \langle G_v(t, v(t)) - G_v(t, v_s(t)), v_s(t) - v(t)\rangle dt.\end{aligned}$$

It is easy to see that assumption (2°) of our lemma implies that $F_0'(v_s) \to 0$. Taking account (1.9) in a similar way as in equality (1.8) and by the embedding theorem we can demonstrate that $v_s \to v$ in the strong topology of H_0^1. We shall show that

$$v \in V_0 = \{\overline{v} \in H_0^1 \mid F_0(\overline{v}) = \min F_0(v),\ v \in H_0^1\}.$$

Denote by
$m_s = \min\{F_s(v) \mid v \in H_0^1\} = \min\{F_s(v) \mid v \in B_\varrho\}$ $s = 0, 1, 2, \ldots$. Since $F_s(\cdot)$ tends uniformly to $F_0(\cdot)$ on B_ϱ (by assumption (2°)), we obtain that $m_s \to m_0$. Suppose that v does not belong to V_0. For any $v_0 \in V_0$, we have

$$(1.10)\qquad m_s - m_0 = F_s(v_s) - F_0(v_0) = (F_s(v_s) - F_0(v_s)) + (F_0(v_s) - F_0(v_0)).$$

If $s \to \infty$, we get

$$\begin{gathered} m_s - m_0 \to 0, \qquad F_s(v_s) - F_0(v_s) \to 0, \\ F_0(v_s) - F_0(v_0) \to F_0(v) - F_0(v_0) > 0. \end{gathered}$$

Thus we have got a contradiction with (1.10). It means that $v \in V_0$ and $\limsup V_s \subset V_0$. We have shown that any sequence $\{v_s\}$, $v_s \in V_s$, for $s = 1, 2, \ldots$ is compact with respect to the norm topology of H_0^1. It is easy to notice that conditions (c) and (d) of lemma are equivalent. Thus the proof is completed. □

2. Continuous dependence on boundary data

Further, we shall consider a situation when the sequence of boundary data $\{(a_s, b_s)\}$ converges to some $(a_0, b_0) \in \mathbb{R}^n \times \mathbb{R}^n$. Without loss of generality we may assume that $(a_0, b_0) = (0, 0)$. Basing ourselves on the principal lemma, we shall prove some sufficient conditions for the continuous (or semicontinuous) dependence on boundary data of the solutions of variational and boundary value problems (1.1) and (1.5). We shall prove the following

Theorem 2.1. *If*

1° *the sequence* $\{(a_s, b_s)\}$ *tends to* $(0, 0)$ *in* $\mathbb{R}^n \times \mathbb{R}^n$,

2° *the function* G *satisfies basic assumptions* (1.2), (1.3) *and* (1.4), *then*

(a) *the set* V_s *of minimizers of the functional* $F_s(\cdot)$ *defined by* (1.5) *is nonempty and compact in the strong topology of* H_0^1 *for* $s = 0, 1, 2, \dots$,

(b) *there exists a ball* B_ϱ, $0 < \varrho < \infty$, *such that* $V_s \subset B_\varrho$ *for* $s = 0, 1, 2, \dots$ *and any sequence* $\{v_s\}$, $v_s \in V_s$, $s = 1, 2, \dots$, *is compact in the norm topology of* H_0^1,

(c) $\limsup V_s$ *is a nonempty set and* $\limsup V_s \subset V_0$ (*if* V_s *are singletons, i.e.* $V_s = \{v_s\}$, *then* $\lim v_s = v_0$),

(d) *for any* $\varepsilon > 0$, *there exists* $S > 0$ *such that* $V_s \subset V_0 + \varepsilon B_1$ *for all* $s > S$.

Proof. It is enough to show that the sequences $F_s(\cdot)$ and $F_s'(\cdot)$ tend uniformly to $F_0(\cdot)$ and $F_0'(\cdot)$, respectively, on any ball B_r. Suppose that $F_s(\cdot)$ does not tend uniformly to $F_0(\cdot)$ for some $r > 0$. Thus there exist an $a > 0$ and some sequence $\{v_s\} \subset B_r$, such that

$$|F_s(v_s) - F_0(v_s)| > a, \quad s = 1, 2, \dots . \tag{2.1}$$

Since the space H_0^1 is compactly embedded into C^0, we may assume that v_s tends to some v_0 uniformly on I. We have

$$\begin{aligned} |F_s(v_s) - F_0(v_s)| &= \bigg| \int_I \Big(\frac{1}{2}|v_s'(t) + e_s|^2 - G(t, v_s(t) + \alpha_s(t)) \Big) dt \\ &\quad - \int_I \Big(\frac{1}{2}|v_s'(t)|^2 - G(t, v_s(t)) \Big) dt \bigg| \le |e_s|\sqrt{\pi}\|v_s' - v_0'\|_{L^2} \\ &\quad + |e_s|\sqrt{\pi}\|v_0'\|_{L^2} + \frac{1}{2}|e_s|^2 + \int_I |G(t, v_s(t)) - G(t, v_0(t))| dt \\ &\quad + \int_I |G(t, v_0(t) + \alpha_0(t)) - G(t, v_s(t) + \alpha_s(t))| dt. \end{aligned}$$

Since $(a_s, b_s) \to (0, 0)$ and $v_s(t) \to v_0(t)$ uniformly for $t \in I$, the right-hand side of the inequality tends to null. So, we have got a contradiction with (2.1). This means that $F_s(\cdot)$ tends to $F_0(\cdot)$ uniformly on B_r. Now, let us suppose that $F_s'(\cdot)$ does not tend to $F_0'(\cdot)$ uniformly on B_r. This implies that there exist some sequences $\{v_s\} \subset B_r$ and $\{h_s\}$, $\|h_s\| \le 1$, such that

$$\langle F_s'(v_s) - F_0'(v_s), h_s \rangle \ge A \qquad \text{for some } A > 0. \tag{2.2}$$

It is easy to calculate that

$$
\begin{aligned}
|\langle F_s'(v_s) - F_0'(v_s), h_s\rangle| &= \Big| \int_I ((v_s'(t) + e_s, h_s'(t)) - (v_s'(t), h_s'(t)))dt \\
&\quad + \int_I ((G_v(t, v_s(t)), h_s(t)) - (G_v(t, v_s(t) + \alpha_s(t)), h_s(t)))dt \Big| \\
&\leq \int_I |e_s|\,|h_s'(t)|dt + \int_I |G_v(t, v_s(t)) - G_v(t, v_s(t) + \alpha_s(t))|\,|h_s(t)|dt \\
&\leq \sqrt{\pi}\Big(|e_s| + \int_I |G_v(t, v_s(t)) - G_v(t, v_0(t))|dt \\
&\quad + \int_I |G_v(t, v_0(t) + \alpha_0(t)) - G_v(t, v_s(t) + \alpha_s(t))|dt \Big).
\end{aligned}
$$

By a similar motivation as above, we get a contradition with inequality (2.2). Thus we have shown that $F_s(\cdot)$ and $F_s'(\cdot)$ tends uniformly on B_r to $F_0(\cdot)$ and $F_0'(\cdot)$, respectively. Applying Lemma 1.1, we complete the proof. □

Let us return to boundary value problem (1.1). Denote by $\overline{V_s}$ a set of solutions of (1.1). For the convex functional, the set $\overline{V_s}$ coincides with the set of minimizers of $F_s(\cdot)$. Thus Theorem 2.1 implies

Corollary 2.2. *If*

1° *the assumptions of Theorem* 2.1 *are satisfied,*

2° *the functional* $F_s(\cdot)$ (*cf.* 1.5) *is convex for any* a_s, b_s (*it enough to assume that* $G(t, \cdot)$ *is convex for* $t \in I$ *a.e.*), *then the conclusions* (a)-(d) *of Theorem* 2.1 *hold with* V_s *replaced by* $\overline{V_s}$.

Finally, let us return to original problem (1.0), i.e.

$$
\begin{aligned}
u''(t) &= G_u(t, u(t)), \\
u(0) &= a_s, \; u(\pi) = b_s
\end{aligned}
\tag{2.3}
$$

with the functional of action

$$
\varphi(u) = \int_I \left(\frac{1}{2}|u'(t)|^2 - G(t, u(t))\right)dt, \tag{2.4}
$$

where $u(\cdot) \in H^1(I, \mathbb{R}^n)$ and $u(0) = a_s$, $u(\pi) = b_s$. Denote by $\overline{U_s}$, $s = 0, 1, 2, \ldots$, a set of solutions of problem (2.3). It easy to see that $\overline{U}_s = \overline{V_s} + \alpha_s$. Thus, directly from Theorem 2.1 we get

Corollary 2.3. *If*

1^0 *the sequence* $\{(a_s, b_s)\}$ *tends to some* (a_0, b_0) *in* $\mathbb{R}^n \times \mathbb{R}^n$,

2^0 *the function* G *satisfies assumptions* (1.2)-(1.4) *and the functional* $\varphi(\cdot)$ (*cf.* 2.4) *is convex*,

then all conditions (a)-(d) *of Theorem* 2.1 *hold with* V_s *replaced by* $\overline{U_s}$, $s = 0, 1, 2, \ldots$.

II. Superlinear case

In the second part of our paper we shall consider boundary value and variational problems related to the Mountain Pass Theorem. To begin with let us recall some definitions.

3. Definitions and the principal lemma

Let $F : E \to \mathbb{R}$ be a functional of class C^1 defined on a real Banach space E. A point $v \in E$ is called a critical point of F if $F'(v) = 0$ and, moreover, the number $c = F(v)$ is referred to as a critical value. We say that the functional F satisfies the Palais-Smale conditions (P.S.) if any sequence $\{v_s\} \subset E$ such that $F'(v_s) \to 0$ and $|F(v_s)| < C$ for some $C > 0$ is compact in the strong topology of E. In this part we shall use the following version of the Mountain Pass Theorem (cf. [7], [9]).

Theorem 3.1. *If*

1° *there exist* $v_0 \in E$, $v_1 \in E$ *and a bounded neighbourhood* B *of* v_0, *such that* $v_1 \in E \setminus \overline{B}$,

2° $\inf_{\partial B} F > \max\{F(v_0), F(v_1)\}$,

3° $c = \inf_{g \in M} \max_{s \in [0,1]} F(g(s))$ *where* $M = \{g \in C([0,1], E) \mid g(0) = v_0,\ g(1) = v_1\}$,

4° F *satisfies the* (*P.S.*) *condition*,

then c *is a critical value and* $c > \max\{F(v_0), F(v_1)\}$.

By M_r we shall denote a family of continuous mappings $g : [0,1] \to B_r$, $g(0) = v_0$, $g(1) = v_1$, $B_r = \{v \in E \mid \|v\| < r,\ r > 0\}$, $v_1 \in B_r$. Let $F_s(\cdot) : E \to \mathbb{R}$, $s = 0, 1, 2, \ldots$, be a sequence of functionals of class C^1. Put

$$c_s(r) = \inf_{g \in M_r} \max_{\tau \in [0,1]} F(g(\tau)). \tag{3.1}$$

Denote by V_s a set of critical points which correspond to the critical value $c_s(r)$, i.e.

$$V_s(r) = \{v \in B_r \mid F_s(v) = c_s(r) \text{ and } F_s'(v) = 0\}, \qquad s = 0, 1, 2, \ldots \tag{3.2}$$

Now, we shall prove the following

Lemma 3.3. *If*

1° *the functional $F_0(\cdot)$ satisfies the (P.S.) condition,*

2° *the sequences $F_s(\cdot)$, $F_s'(\cdot)$, $s = 1, 2, \dots$, tend uniformly on the ball B_r to $F_0(\cdot)$, $F_0'(\cdot)$, respectively,*

3° *the sets $V_s(r)$ defined by (3.2) are nonempty for $s = 0, 1, 2, \dots$,*

then

(a) *any sequence $\{v_s\}$, $v_s \in V_s(r)$, $s = 1, 2, \dots$, is compact,*

(b) $\limsup V_s(r)$ *is a nonempty set and* $\limsup V_s(r) \subset V_0(r)$,

(c) *for any $\varepsilon > 0$, there exists $S > 0$ such that $V_s(r) \subset V_0(r) + \varepsilon B_1$ for $s > S$.*

Proof. Let us notice that $\lim c_s(r) = c_0(r)$, where $c_s(r)$ are described in (3.1). Indeed, by assumption (2°), we have

$$\begin{aligned} c_s(r) &= \inf_{g \in M_r} \max_{\tau \in [0,1]} [(F_s(g(\tau)) - F_0(g(\tau))) + F_0(g(\tau))] \\ &\le \inf_{g \in M_r} \max_{\tau \in [0,1]} (\varepsilon + F_0(g(\tau))) = \varepsilon + c_0(r) \end{aligned}$$

for an arbitrary $\varepsilon > 0$ and for k sufficiently large. Similarly, $c_0(r) \le \varepsilon + c_s(r)$. Thus

$$\lim c_s(r) = c_0(r). \tag{3.3}$$

Let $\{v_s\}$ be an arbitrary sequence such that $v_s \in V_s(r)$ for $s = 1, 2, \dots$. By assumption (2°) and (3.3), we have

$$A_s = F_0(v_s) - F_s(v_s) = F_0(v_s) - c_s(r) \to 0$$

and

$$\lim F_0(v_s) = \lim(A_s + c_s(r)) = \lim A_s + \lim c_s(r) = c_0(r).$$

Similarly, $0 = \lim(F_s'(v_s) - F_0'(v_s)) = -\lim F_0'(v_s)$ because $F_s'(v_s) = 0$. We have shown that the sequence $F_0(v_s)$ is bounded and $F_0'(v_s) \to 0$. Assumption (1°) implies that $\{v_s\}$ is compact. Let $\widetilde{v}$ be a cluster point of this sequence. Passing, if necessary, to a subsequence, we may assume that $v_s \to \widetilde{v}$. Suppose that $\widetilde{v} \notin V_0(r)$, i.e. $F_0(\widetilde{v}) \ne c_0$ or $F_0'(\widetilde{v}) \ne 0$. Let us notice that the inequality $F_0'(\widetilde{v}) \ne 0$, is false. Indeed, by assumption (2°), we have

$$F_0'(\widetilde{v}) = \lim F_0'(v_s) = \lim(F_0'(v_s) - F_s'(v_s)) = 0.$$

Let us put $\alpha = F_0(\widetilde{v}) - F_0(v_0)$, where $v_0 \in V_0(r)$. Of course, $\alpha \ne 0$. We have

$$\begin{aligned} c_s(r) - c_0(r) &= F_s(v_s) - F_0(v_0) \\ &= [F_s(v_s) - F_0(v_s)] + [F_0(v_s) - F_0(\widetilde{v})] + [F_0(\widetilde{v}) - F_0(v_0)] \\ &= [F_s(v_s) - F_0(v_s)] + [F_0(v_s) - F_0(\widetilde{v})] + \alpha. \end{aligned}$$

By (3.3) and assumption (2°), we have $c_s(r)-c_0(r) \to 0$, $F_s(v_s)-F_0(v_s) \to 0$ and $F_0(v_s) - F_0(\widetilde{v}) \to 0$. Since $\alpha \neq 0$, we have got a contradiction. Thus $\widetilde{v} \in V_0(r)$. We have proved that $\limsup V_s(r)$ is a nonempty set and $\limsup V_s(r) \subset V_0(r)$. It is easy to notice that (a) and (b) imply condition (c). □

4. Formulation of the problem

In this part we consider a system

$$u''(t) = G_u(t, u(t)), \qquad u(0) = a_s,\ u(\pi) = b_s, \tag{4.0}$$

where $G : I \times \mathbb{R}^n \to \mathbb{R}$, $u(\cdot) \in H^1$. If we put $u(t) = v(t) + \alpha_s(t)$, where $\alpha_s(t) = (1/\pi)(b_s - a_s)t + a_s$, then we obtain an equivalent system

$$v''(t) = G_v(t, v(t) + \alpha_s(t)), \qquad v(0) = v(\pi) = 0. \tag{4.1}$$

By $F_s(\cdot)$ we shall denote the functional of action of system (4.1)

$$F_s(v) = \int_I \left(\frac{1}{2}|v'(t) + e_s|^2 - \frac{1}{2}|e_s|^2 - G(t, v(t) + \alpha_s(t)) + G(t, \alpha_s(t))\right) dt, \tag{4.2}$$

where $e_s = (1/\pi)(b_s - a_s)$, $s = 0, 1, 2, \ldots$, and $v(\cdot) \in H_0^1(I, \mathbb{R}^n)$.

We shall assume that

(4.3) the functions $G(t, u)$, $G_u(t, u)$ are measurable with respect to $t \in I$ for any $u \in \mathbb{R}^n$, continuous with respect to $u \in \mathbb{R}^n$ for $t \in I$ a.e., $|a_s| \le \overline{a}$, $|b_s| \le \overline{b}$ for some $\overline{a}, \overline{b} > 0$ and for all s, and $G(t, 0) = 0$ for $t \in I$ a.e.,

(4.4) for any $r > 0$, there exists $h \in L^\infty(I, \mathbb{R}^+)$ such that

$$|G(t, u)| \le h(t), \qquad |G_u(t, u)| \le h(t)$$

for $t \in I$ a.e., $|u| \le r$,

(4.5) there exist $p > 2$, $a > 0$ and $R > 0$, such that

$$a < pG(t, u) \le \langle G_u(t, u), u\rangle$$

for $t \in I$ a.e., and $|u| \ge R$,

(4.6) there exist $\varrho > 0$ and $b < 1$, such that

$$G(t, u) \le \frac{b}{2}|u|^2$$

for $|u| \le \varrho$ and $t \in I$ a.e.

For given a_s and b_s, we define

$$c_s = \inf_{g \in M} \max_{\tau \in [0,1]} F_s(g(\tau)), \tag{4.7}$$

where the functional $F_s(\cdot)$ is defined by (4.2) and M is a family of continuous mappings $g : [0,1] \to H_0^1(I, \mathbb{R}^n)$ such that $g(0) = v_0$, $g(1) = v_1(\cdot)$, where $v_1(\cdot)$ is a given function. Let c_s be the given number defined by (4.7). For fixed (a_s, b_s), denote

$$V_s = \{v \in H_0^1 \mid F_s(v) = c_s \text{ and } F_s'(v) = 0\}. \tag{4.8}$$

Remark 4.1. The functional $F_s(\cdot)$ defined by (4.2) is continuously differentiable on H_0^1 (cf. [7], Theorem 1.4).

5. Continuous dependence on boundary data

Basing ourselves on Lemma 3.3 and the Mountain Pass Theorem (cf. Theorem 3.1), we shall prove some sufficient conditions for the continuous dependence on boundary data of the solution of a variational and boundary value problem of forms (4.0) and (4.1).

First, we prove

Lemma 5.1. *If the function G satisfies conditions* (4.3), (4.4) *and* (4.5), *then there exists a ball $B_r \subset H_0^1$ such that $V_s \subset B_r$ for any a_s, b_s, where V_s is the set of critical points, given by* (4.8).

Proof. Note that the set of critical values is bounded from above. Indeed,

$$\begin{aligned}
c_s &= \inf_{g \in M} \max_{\tau \in [0,1]} F_s(g(\tau)) \le \max_{\tau \in [0,1]} F_s(\tau v_1) \\
&= \max_{\tau \in [0,1]} \int_I \left(\frac{1}{2} |\tau v_1'(t) + e_s|^2 - \frac{1}{2} |e_s|^2 - G(t, \tau v_1(t) + \alpha_s(t)) + G(t, \alpha_s(t)) \right) dt \\
&\le \max_{\tau \in [0,1]} \int_I \left(|\tau v_1'(t)|^2 + \frac{1}{2} |e_s|^2 - \frac{a}{p} + G(t, \alpha_s(t)) \right) dt \\
&\le \int_I \left(|v_1'(t)|^2 + \frac{1}{2} |e_s|^2 - \frac{a}{p} + G(t, \alpha_s(t)) \right) dt \\
&\le \|v_1\|^2 + \left(\frac{1}{2} |e_s|^2 - \frac{a}{p} \right) \pi + c = \widetilde{c}.
\end{aligned}$$

For any a_s, b_s and $v \in V_s$, we have (by (4.4), (4.5))

$$\begin{aligned}
p\widetilde{c} \geq pc_s &= pF_s(v) - \langle F_s'(v), v + \alpha_s\rangle \\
&\geq \frac{p-2}{2}\|v\|^2 + \frac{p-2}{2}\int_I (2(v'(t), e_s) + |e_s|^2)dt + \int_{I+} a\, dt \\
&\quad + \int_{I^-} (-2p - R)h(t)dt \\
&= \frac{p-2}{2}\|v\|^2 + (p-2)\int_I (v'(t), e_s)dt + C_1 = \frac{p-2}{2}\|v\|^2 + C,
\end{aligned}$$

where

$$I^+ = \{t \mid |v + \alpha_s| \geq R\}, \qquad I^- = \{t \mid |v + \alpha_s| < R\}.$$

Since $p - 2 > 0$, for $r = \sqrt{2(p\widetilde{c} - C)/(p-2)}$, we have $\|v\| \leq r$, i.e. $V_s \subset B_r$. □

Basing ourselves on the above lemma, we prove

Theorem 5.2. *If*

1° *the function G satisfies assumptions* (4.3)-(4.6),

2° *the sequence $\{(a_s, b_s)\}$ tends to $(a_0, b_0) = (0,0)$ in $\mathbb{R}^n \times \mathbb{R}^n$,*

then

(a) *the set V_s of critical points of the functional $F_s(\cdot)$, defined by* (4.8), *is nonempty for $s = 0, 1, 2, \ldots$, $v = 0$ does not belong to V_s and any sequence $\{v_s\}$, $v_s \in V_s$, $s = 1, 2, \ldots$, is compact,*

(b) $\limsup V_s$ *is a nonempty set and* $\limsup V_s \subset V_0$ *(if V_s are singletons, i.e. $V_s = \{v_s\}$, then* $\lim v_s = v_0$*),*

(c) *for any $\varepsilon > 0$, there exists $S > 0$ such that $V_s \subset V_0 + \varepsilon B_1$ for all $s > S$.*

Proof. Similarly as in the proof of Theorem 2.1 from Part 1 one can show that $F_s(\cdot)$ and $F_s'(\cdot)$ tend uniformly to $F_0(\cdot)$ and $F_0'(\cdot)$, respectively, provided that $(a_s, b_s) \to (0,0)$. Basing oneself on assumptions (4.4) and (4.5), one can prove that, for any (a_s, b_s), the functional $F_s(\cdot)$, $s = 0, 1, 2, \ldots$, satisfies the (P.S.) condition. Indeed, let $\{v_k\}$ be a sequence such that $\{F_s(v_k)\}$ is bounded and $F_s'(v_k) \to 0$ as $k \to 0$. We have $|F_s(v_k)| \leq C_1$ and $\|F_s'(v_k)\| \leq C_2$ for s, $k \in \mathbb{N}$ and for some C_1, $C_2 > 0$. By assumption (4.4), (4.5) we get

$$\begin{aligned}
pC_1 + C_2\|v_k\| + C_2\|\alpha_s\| &\geq pC_1 + C_2\|v_k + \alpha_s\| \\
&\geq pF_s(v_k) - \langle F_s'(v_k), v_k + \alpha_s\rangle \geq \frac{p-2}{2}\|v_k\|^2 + C.
\end{aligned}$$

Thus we obtain

$$\|v_k\|^2 \leq \frac{2}{p-2}(pC_1 + C_2\|v_k\| + C_2\|\alpha_s\| + C) \qquad \text{for } k \in \mathbb{N}.$$

The above inequality implies that the sequence $\{v_k\}$ is bounded and we may assume without loss of generality that v_k tends to some v_0 weakly in H_0^1. Since the space H_0^1 is compactly embedded into C^0, then v_k tends to v_0 uniformly on I. Of course,

$$\langle F_s'(v_k) - F_s'(v_0), v_k - v_0\rangle \to 0 \qquad \text{as } k \to 0.$$

By direct calculations we get

$$\langle F_s'(v_k) - F_s'(v_0), v_k - v_0\rangle = \int_I |v_k'(t) - v_0'(t)|^2 dt$$
$$- \int_I \langle G_v(t, v_k(t) + \alpha_s(t)) - G_v(t, v_0(t) + \alpha_s(t)), v_k(t) - v_0(t)\rangle dt.$$

It is easy to show that the above integrals tend to null as v_k tend to v_0 in H_0^1. Thus $\|v_k - v_0\|^2 \to 0$ as $k \to 0$. This means that $v_k \to v_0$ strongly in H_0^1. We have shown that, for any $s = 0, 1, 2, \ldots$, $F_s(\cdot)$ satisfies the (P.S.) condition. Now, we shall show that there exists v_1 which satisfies the assumptions of the Mountain Pass Theorem (cf. Theorem 3.1) and the choice of v_1 does not depend on the boundary values a_s, b_s. Similarly as in Theorem 3.3 (cf. [6]) we can prove the inequality

$$G(t, \tau\widetilde{v}) \ge \tau^p G(t, \widetilde{v}) \ge \frac{a}{p}\tau^p$$

for $t \in [0, \pi]$ a.e., $\tau \ge 1$ and $|\widetilde{v}| = R$, where a and p are described in (4.5). For any $v \in \mathbb{R}^n$ such that $|v + \alpha_s| \ge R$, we put $\tau = (|v + \alpha_s|)/R$, $\widetilde{v} = R/(|v+\alpha_s|)(v + \alpha_s)$ and get, by direct calculations,

$$G(t, \tau\widetilde{v}) = G(t, v + \alpha_s) \ge \frac{a}{p}\frac{|v + \alpha_s|^p}{R^p} = \widehat{a}|v + \alpha_s|^p$$

for $t \in [0, \pi]$ a.e. and $\widehat{a} = a/(pR^p) > 0$ (because $a > 0$, $p > 2$, $R > 0$). Thus $G(t, v + \alpha_s) \ge \widehat{a}|v + \alpha_s|^p - h(t)$ for $t \in [0, \pi]$ a.e. and $v \in \mathbb{R}^n$, where the function $h(\cdot)$ satisfies assumption (4.4). Consequently, if $v \in H_0^1$, and $v \ne 0$ is fixed and $l > 0$, then

$$F_s(lv) = \int_I \left(\frac{1}{2}|lv'(t) + e_s|^2 - \frac{1}{2}|e_s|^2 - G(t, lv(t) + \alpha_s(t)) + G(t, \alpha_s(t))\right) dt$$
$$\le \frac{l^2}{2}\|v\|^2 + lC_1\|v\| - \widehat{a}l^p 2^p \int_I |v(t)|^p dt + C_2$$

for some C_1, $C_2 > 0$. Since $\widehat{a} > 0$, $p > 2$, we have

$$F_s(lv) \to -\infty \qquad \text{as } l \to \infty.$$

Therefore, there exist $l_0 > 0$, $\overline{R} > 0$ such that $\|l_0 v\| > \overline{R}$ and $F_s(l_0 v) < 0$. This means that $v_1 = l_0 v \notin \overline{B}_{\overline{R}}$. We have shown that there exists v_1 which does not depend on the boundary data a_s, b_s and a ball with centre at the origin such that v_1 does not belong to the closure of the ball. Similarly as in Theorem 3.3 (cf. [6]) one can prove that $\inf_{\partial B} F_s > \max\{F_s(0), F_s(v_1)\} = F_s(0) = 0$, where B is some ball with centre at the origin and with radius $\varrho_1 > 0$. The Mountain Pass Theorem with $v_0 = 0$, $E = H_0^1$ and $c = c_s$ (cf. (4.7)) implies that, for any a_s, b_s, the set of critical points corresponding to the critical value c_s is not empty, i.e.

$$V_s = \{v \in H_0^1 \mid F_s(v) = c_s \text{ and } F_s'(v) = 0\} \neq \emptyset.$$

Moreover,

$$c_s = \inf_{g \in M} \max_{\tau \in [0,1]} F_s(g(\tau)) > \max\{F_s(0), F_s(v_1)\} = 0.$$

Thus $v = 0$ does not belong to V_s for $s = 0, 1, 2, \ldots$. By Lemma 5.1, there exists a ball $B_r \subset H_0^1$ such that $V_s(r) = V_s$, where the sets $V_s(r)$ and V_s are defined by (3.2) and (4.8), respectively. Applying Lemma 3.3, we get assertions (b) and (c) of our theorem. □

Let us return to the superlinear boundary value problem for the system of O.D.E. which is described in (4.1). Denote by $\overline{V_s}$ a set of nontrivial solutions of problem (4.1) which correspond to the critical value c_s, i.e. $F_s(v) = c_s$. Of course, $\overline{V_s} = V_s$ and directly from Theorem 5.2 we get

Corollary 5.3. *If the assumptions of Theorem 5.2 are satisfied, then*

(a) *the set $\overline{V_s}$ is nonempty for $s = 0, 1, 2, \ldots$, $v = 0$ does not belong to $\overline{V_s}$ and any sequence $\{v_s\}$, $v_s \in \overline{V_s}$, $s = 1, 2, \ldots$, is compact,*

(b) $\limsup \overline{V_s}$ *is a nonempty set and* $\limsup \overline{V_s} \subset \overline{V_0}$ *(if $\overline{V_s}$ are singletons, i.e. $\overline{V_s} = \{v_s\}$, then $\lim v_s = v_0$),*

(c) *for any $\varepsilon > 0$, there exists $S > 0$ such that $\overline{V_s} \subset \overline{V_0} + \varepsilon B_1$ for all $s > S$.*

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ON DISCONTINUOUS DIFFERENTIAL EQUATIONS

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1. Introduction

Consider the Cauchy problem for an ordinary differential equation

$$\dot{x} = g(t,x), \qquad x(0) = \bar{x}, \qquad t \in [0,T]. \tag{1.1}$$

When g is continuous, the local existence of solutions is provided by Peano's theorem. Several existence and uniqueness results are known also in the case of a discontinuous right hand side [7]. We recall here the classical theorem of Carathéodory [8]:

Theorem A. *Let $g : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ be a bounded function.*

(i) *If the map $t \mapsto g(t,x)$ is measurable for each x and the map $x \mapsto g(t,x)$ is continuous for each t, then the Cauchy problem* (1.1) *has at least one solution.*

(ii) *If the map $t \mapsto g(t,x)$ is measurable for each x and the map $x \mapsto g(t,x)$ is Lipschitz continuous for each t, with a uniform Lipschitz constant, then the Cauchy problem* (1.1) *has a unique solution, depending Lipschitz continuously on the initial data $\bar{x}$.*

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By a solution of (1.1) we mean an absolutely continuous function $x : [0,T] \mapsto \mathbb{R}^n$ such that

$$x(t) = \bar{x} + \int_0^t g\big(t, x(t)\big)\, dt \qquad \text{for all } t \in [0,T]. \tag{1.2}$$

More recent results rely on the notions of directional continuity and of bounded directional variation of a vector field. More precisely, given a closed convex cone $\Gamma \subset \mathbb{R}^m$, we say that a (possibly discontinuous) map $\phi : \mathbb{R}^m \mapsto \mathbb{R}^n$ is *directionally continuous* if at each point $p \in \mathbb{R}^m$ one has

$$\lim_{p' \to p,\ p' - p \in \Gamma} \phi(p') \;=\; \phi(p). \tag{1.3}$$

We say that the map ϕ has *bounded directional variation* if

$$\sup\left\{ \sum_{i=1}^{N} |\phi(p_i) - \phi(p_{i-1})| \;\Big|\; N \geq 1,\ p_i - p_{i-1} \in \Gamma \text{ for every } i \right\} < \infty. \tag{1.4}$$

The following existence and uniqueness results were proved in [1], [9] and in [2], respectively.

Theorem B. *Let* $|g(t,x)| \leq L < M$ *for all* t, x.

(i) *Assume that* g *is directionally continuous in the direction of the cone*

$$\Gamma \doteq \{(t,x) \mid |x| \leq Mt\} \subset \mathbb{R}^{n+1}. \tag{1.5}$$

Then the Cauchy problem (1.1) *admits at least one solution.*

(ii) *Assume that* g *has bounded directional variation in the direction of the cone* Γ *in* (1.5). *Then the Cauchy problem* (1.1) *has a unique solution, depending Lipschitz continuously on the initial data* $\bar{x}$.

Further uniqueness results can be found in [3] and [4]. Concerning existence, see also the interesting work [10].

Aim of the present paper is to prove two theorems on the existence and the uniqueness of solutions to the autonomous Cauchy problem

$$\dot{x} = f(x), \qquad x(0) = \bar{x} \in \mathbb{R}^m, \qquad t \in [0,T], \tag{1.6}$$

extending the classical results of Carathéodory. As a preliminary, we observe that the equation (1.1) can be rewritten as an autonomous problem on $\mathbb{R}^{n+1}$, introducing the variable $x_0 = t$ and the vector field $f(x_0, x) = (1, g(x_0, x))$. Under the assumptions of Theorem A, the function f can jump across the hyperplanes of the form $x_0 =$ constant. These hyperplanes are certainly transversal to f. Namely, taking the inner product of f with their normal vector, one trivially has

$$f \cdot n = (1, g(x_0, x)) \cdot (1, 0) \equiv 1.$$

The next theorem shows that this transversality condition is indeed the key ingredient toward the existence of solutions.

Theorem 1. *Assume that f in* (1.6) *has the form*

$$f(x) \doteq F(g_1(\tau_1(x), x), \dots, g_N(\tau_N(x), x)), \tag{1.7}$$

where:

(i) *Each map $\tau_i : \mathbb{R}^m \mapsto \mathbb{R}$ is continuously differentiable. Each $g_i : \mathbb{R} \times \mathbb{R}^m \mapsto \mathbb{R}$ is a Carathéodory function, i.e. measurable in t and continuous in x. Moreover, $F : \mathbb{R}^N \mapsto \mathbb{R}^m$ is continuous.*

(ii) *For some compact set $K \subset \mathbb{R}^m$, at every point x one has*

$$f(x) \in K, \qquad \nabla\tau_i(x) \cdot z > 0 \quad \text{for every } z \in K. \tag{1.8}$$

Then the Cauchy problem (1.6) *has at least one solution.*

Remark 1. The assumption (ii) can be easily recognized as a transversality condition. Indeed, by $(1.8)_1$ every trajectory of (1.6) satisfies the differential inclusion $\dot{x} \in K$. Hence by $(1.8)_2$ this trajectory must cross transversally any hypersurface of the form $\tau_i(x) = \text{constant}$. According to the definition (1.7), these are the surfaces across which f can jump.

To guarantee the existence of solutions, some kind of transversality condition is necessary, as shown by obvious counterexample

$$\dot{x} = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x \geq 0, \end{cases} \qquad x(0) = 0.$$

Our second result is concerned with the uniqueness and continuous dependence of solutions.

Theorem 2. *Assume that f in* (1.6) *has the form*

$$f(x) \doteq g(\tau(x), x), \tag{1.9}$$

where:

(i) *The function $\tau : \mathbb{R}^m \mapsto \mathbb{R}$ is continuously differentiable. The map $g : \mathbb{R} \times \mathbb{R}^m \mapsto \mathbb{R}^m$ is measurable w.r.t. t and uniformly Lipschitz continuous w.r.t. x.*

(ii) *There exists a compact convex set K such that*

$$f(x) \in K, \quad \nabla\tau(x) \cdot z > 0 \qquad \text{for every } x \in \mathbb{R}^m, \ z \in K. \tag{1.10}$$

Moreover, the gradient $\nabla\tau$ has bounded directional variation w.r.t. the cone $\Gamma \doteq \{\lambda z \mid \lambda \geq 0, \ z \in K\}$.

Then the Cauchy problem (1.6) *has a unique solution, depending on the initial data* $\bar{x}$ *in a locally Lipschitz continuous way.*

Remark 2. In the case where $m = n + 1$, $x = (x_0, \dots, x_n)$ and $\tau(x) \equiv x_0$, the above theorem reduces to part (ii) of Theorem A. Roughly speaking, in Carathéodory's theorem one allows jumps across the hyperplanes $x_0 =$ constant. On the other hand, in Theorem 2 we allow jumps across the hypersurfaces $\tau(x) =$ constant, provided that these surfaces are transversal to the vector field f and the direction of their tangent planes does not wiggle too much.

The reader should also notice that in Theorem 2 the assumption of bounded directional variation is placed on the gradient $\nabla\tau$. This situation is quite different from part (ii) of Theorem B, where one assumes that the vector field f itself has bounded directional variation.

Remark 3. In Theorems 1 and 2, the scalar functions τ, τ_i were assumed to be $\mathcal{C}^1$. This assumption simplifies some technical aspects of the proofs, but may likely be relaxed. We conjecture that the same results hold if τ, τ_i are only assumed Lipschitz continuous, and the conditions (1.8), (1.10) are duly reformulated in terms of Clarke generalized gradients [5].

Remark 4. If in Theorem 2 we drop the key assumption that the directional variation of $\nabla\tau$ be bounded, then the uniqueness of solutions may fail. This will be illustrated by an example in the last section of this paper. On the other hand, the uniqueness result stated in [4] allows f to have discontinuities along a set of lines whose slopes have unbounded directional variation. However, the validity of this theorem relies on the very special structure of f, linked to the solution of a scalar conservation law.

2. Proof of Theorem 1

It is not restrictive to assume that $\bar{x} = 0$ and that K is convex: otherwise, one can simply shift the coordinates and replace K by its convex closure. Define the Picard operator $u \mapsto \mathcal{P}u$

$$(\mathcal{P}u)(t) \doteq \int_0^t F(g_1(\tau_1(u(s)), u(s)), \cdots, g_N(\tau_N(u(s)), u(s)))\, ds. \tag{2.1}$$

We will prove that this operator is continuous on the compact set

$$\mathcal{U} \doteq \left\{ u : [0,T] \mapsto \mathbb{R}^m \mid u(0) = 0,\ \frac{u(t) - u(s)}{t - s} \in K \text{ for all } t > s \right\}. \tag{2.2}$$

Let $\varepsilon > 0$ be given. Applying the theorem of Scorza-Dragoni [11] to each map g_i, $i = 1, \dots, N$, we obtain the existence of a closed set J_i with

$$\text{meas}(\mathbb{R} \setminus J_i) \le \varepsilon, \tag{2.3}$$

such that g_i is continuous restricted to the set $J_i \times \mathbb{R}^m$. Define the closed set

$$A \doteq \{x \in \mathbb{R}^m \mid \tau_i(x) \in J_i \text{ for every } i = 1, \dots, N\}.$$

By (1.7), the composed map f is continuous restricted to A. Without loss of generality, K is convex. Using the extension theorem of Dugundji [6] (see page 188) we now construct a continuous map $\widetilde{f} : \mathbb{R}^m \mapsto K$ such that $\widetilde{f} = f$ on A.

Call $|K| \doteq \max_{z \in K} |z|$. By (2.2), every function $u \in \mathcal{U}$ thus takes values inside the closed ball

$$X \doteq \{x \in \mathbb{R}^m \mid |x| \leq T|K|\}.$$

By $(1.8)_2$ and the continuity of the gradients $\nabla\tau_i$, there exists a strictly positive δ_0 such that

$$\nabla\tau_i(x) \cdot z \geq \delta_0 > 0 \qquad \text{for all } x \in X, \ z \in K, \tag{2.4}$$

because the sets X, K are compact. As a consequence, for each $u \in \mathcal{U}$ the maps $t \mapsto \tau_i(u(t))$ from $[0, T]$ into $\mathbb{R}$ are strictly increasing. Namely,

$$\frac{d\tau_i\big(u(t)\big)}{dt} = \nabla\tau_i \cdot \dot{u} \geq \delta_0 > 0. \tag{2.5}$$

For a fixed $u \in \mathcal{U}$, call $I_u \subset [0, T]$ the set of times t such that $u(t) \notin A$, i.e.

$$I_u \doteq \{t \in [0, T] \mid \tau_i(u(t)) \notin J_i \text{ for some } i = 1, \cdots, N\}.$$

Because of (2.3) and (2.5), the measure of I_u satisfies

$$\text{meas}(I_u) \leq \frac{N\varepsilon}{\delta_0}. \tag{2.6}$$

To prove the continuity of $\mathcal{P}$, call $\widetilde{\mathcal{P}}$ the Picard operator corresponding to the function $\widetilde{f}$, i.e.

$$(\widetilde{\mathcal{P}}u)(t) \doteq \int_0^t \widetilde{f}(u(s))ds.$$

Clearly, $\widetilde{\mathcal{P}}$ is continuous, hence for any fixed $u \in \mathcal{U}$ there exists a $\delta > 0$, such that

$$\|\widetilde{\mathcal{P}}v - \widetilde{\mathcal{P}}u\| \leq \varepsilon \qquad \text{whenever } v \in \mathcal{U}, \ \|v - u\| \leq \delta. \tag{2.7}$$

We now observe that the difference between the Picard operators $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ is small. Indeed, for every $v \in \mathcal{U}$, (2.6) implies

$$\|\widetilde{\mathcal{P}}v - \mathcal{P}v\| \leq \sup_{x \in X} |f(x) - \widetilde{f}(x)| \cdot \text{meas}(I_v) \leq 2|K| \cdot \frac{N\varepsilon}{\delta_0}. \tag{2.8}$$

Together, (2.7) and (2.8) yield

$$\|\mathcal{P}v - \mathcal{P}u\| \leq \|\mathcal{P}v - \widetilde{\mathcal{P}}v\| + \|\widetilde{\mathcal{P}}v - \widetilde{\mathcal{P}}u\| + \|\widetilde{\mathcal{P}}u - \mathcal{P}u\| \leq \varepsilon + 4|K|\frac{N\varepsilon}{\delta_0}, \tag{2.9}$$

for every $v \in \mathcal{U}$ with $\|v-u\| \leq \delta$. Since $\varepsilon > 0$ in (2.9) was arbitrary, this shows that the Picard operator $u \mapsto Pu$ is continuous, mapping the compact set $\mathcal{U}$ into itself. By applying the Schauder fixed point theorem we thus obtain the existence of a solution to the Cauchy problem (1.6).

3. Proof of Theorem 2

For any given $\bar{x} \in \mathbb{R}^m$, the existence of a solution follows from Theorem 1. The main part of the proof consists in showing that, given a radius $R > 0$ and any two solutions

$$\begin{aligned} \dot{x}(t) &= f(x(t)), \qquad x(0) = x_0, \\ \dot{y}(t) &= f(y(t)), \qquad y(0) = y_0, \end{aligned} \tag{3.1}$$

with $|x_0|, |y_0| \leq R$, one has the estimate

$$|x(t) - y(t)| \leq C_R \cdot |x_0 - y_0| \qquad t \in [0, T], \tag{3.2}$$

for some constant C_R depending only on f and R. The uniqueness of solutions is an obvious consequence of (3.2). The proof is given in four steps.

STEP 1. We first study the case where f, in addition to the assumptions (i) and (ii) in Theorem 2, is piecewise smooth. More precisely, we assume that f has the form

$$f(x) = g_k(x) \qquad \text{if } \tau_k \leq \tau(x) < \tau_{k+1}, \tag{3.3}$$

for some increasing sequence of times $\{\tau_k \mid k \in \mathbb{Z}\}$. Here the functions g_k have uniformly bounded $\mathcal{C}^1$ norm, say with

$$\sup_{x,k} |g_k(x)| \leq C_0, \qquad \sup_{x,k} |D_x g_k(x)| \leq C_1 \tag{3.4}$$

for some constants C_0, C_1. Under these additional regularity assumptions, the uniqueness of solutions of (1.6) is clear. Our aim is to derive the uniform estimate (3.2) by studying the evolution of infinitesimal tangent vectors.

Consider a one-parameter family of solutions

$$\dot{x}^\varepsilon(t) = f(x^\varepsilon(t)), \qquad x^\varepsilon(0) = x_0^\varepsilon, \tag{3.5}$$

regarded as small perturbations of a reference solution $x^0(\cdot) = x(\cdot)$. Define the first order tangent vector

$$\mathbf{v}(t) = \lim_{\varepsilon \to 0+} \frac{x^\varepsilon(t) - x(t)}{\varepsilon}. \tag{3.6}$$

Call t_k, $k \in \mathbb{Z}$, the times where the reference solution $x(\cdot)$ crosses the hypersurfaces $\tau(x) = \tau_k$. By (1.10), all these crossings are transversal. According to the standard theory of piecewise smooth differential equations [7, 8], if the limit (3.6) exists at time $t = 0$, then the tangent vector $\mathbf{v}$ is well defined for all $t \in [0, T]$, $t \neq t_k$, $k \in \mathbb{Z}$. The time evolution of $\mathbf{v}$ is governed by the linear equation

$$\dot{\mathbf{v}}(t) = D_x g_k(x(t)) \cdot \mathbf{v}(t) \qquad \text{for } t \in]t_k, t_{k+1}[\,, \tag{3.7}$$

together with impulses at the crossing times t_k. To describe the linear impulse at time t_k, call

$$\mathbf{n}_k \doteq \frac{\nabla\tau(x(t_k))}{|\nabla\tau(x(t_k))|}$$

the unit normal vector to the surface $\tau = \tau_k$ at the point $x(t_k)$. Moreover, define (fig. 1)

$$f_k \doteq \lim_{t \to t_k^+} f(x(t)) = g_k(x(t_k)), \qquad \widetilde{f}_k \doteq \lim_{t \to t_{k+1}^-} f(x(t)) = g_k(x(t_{k+1})), \tag{3.8}$$

$$\mathbf{v}_k \doteq \lim_{t \to t_k^+} \mathbf{v}(t), \qquad \widetilde{\mathbf{v}}_k \doteq \lim_{t \to t_{k+1}^-} \mathbf{v}(t). \tag{3.9}$$

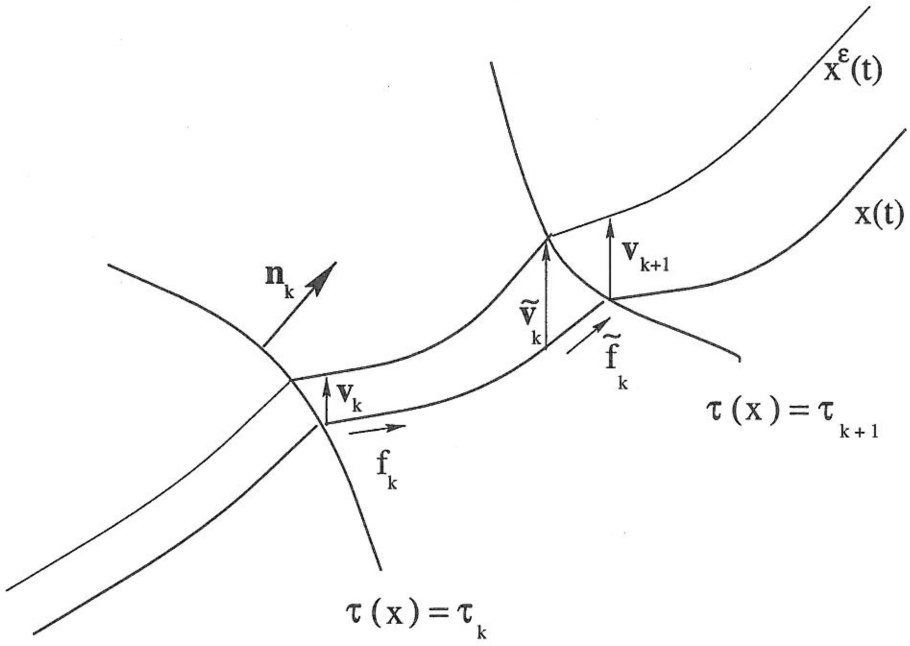

Figure 1

With the above notations, an elementary computation shows that, at the crossing time t_k, the values $\mathbf{v}(t_k+) = \mathbf{v}_k$ and $\mathbf{v}(t_k-) = \widetilde{\mathbf{v}}_{k-1}$ satisfy the linear relation

$$\mathbf{v}_k = \widetilde{\mathbf{v}}_{k-1} + (f_k - \widetilde{f}_{k-1})\frac{\widetilde{\mathbf{v}}_{k-1}\cdot \mathbf{n}_k}{\widetilde{f}_{k-1}\cdot \mathbf{n}_k}. \tag{3.10}$$

Our next goal is to derive a priori bounds on the size of $\mathbf{v}$. In the following, with the Landau symbol $\mathcal{O}(1)$ we denote a quantity whose norm is uniformly bounded. The bound may depend on T, on the constants C_0, C_1 in (3.4) and $\delta_0 > 0$ in (2.4), and on the total directional variation of $\nabla\tau$, but not on the particular solution $x(\cdot)$ in (3.1).

Recalling (3.8), from (3.4) we deduce

$$\widetilde{f}_k - f_k = g_k(x(t_{k+1})) - g_k(x(t_k)) = \mathcal{O}(1)(t_{k+1} - t_k). \tag{3.11}$$

Moreover, recalling (3.9), from (3.7) we deduce

$$\widetilde{\mathbf{v}}_k - \mathbf{v}_k = \mathbf{v}(t_{k+1}^-) - \mathbf{v}(t_k^+) = \mathcal{O}(1)(t_{k+1} - t_k)|\mathbf{v}_k|. \tag{3.12}$$

In the following, we use the superscripts ^{N_k} and ^{T_k} to denote the components of a vector which are normal and tangent to the surface $\tau(x) = \tau_k$, respectively. More precisely, we set

$$\mathbf{v}_k^{N_k} \doteq (\mathbf{v}_k\cdot\mathbf{n}_k)\mathbf{n}_k, \qquad \mathbf{v}_k^{T_k} = \mathbf{v}_k - \mathbf{v}_k^{N_k}. \tag{3.13}$$

The same notations are used for f_k. In addition, for every integer k, we define

$$w_k \doteq \frac{\mathbf{v}_k\cdot\mathbf{n}_k}{f_k\cdot\mathbf{n}_k} \qquad z_k \doteq \mathbf{v}_k^{T_k} - f_k^{T_k} w_k. \tag{3.14}$$

The quantities $\widetilde{w}_k$ and $\widetilde{z}_k$ are defined similarly. By (1.10), the quantities $|f_k^{N_k}| = f_k\cdot\mathbf{n}_k$ are uniformly positive. We thus have the estimates

$$|w_k| = \mathcal{O}(1)\cdot|\mathbf{v}_k|, \qquad |z_k| = \mathcal{O}(1)\cdot|\mathbf{v}_k|, \tag{3.15}$$

$$\begin{aligned} |\mathbf{v}_k| &\le |\mathbf{v}_k^{N_k}| + |\mathbf{v}_k^{T_k}| \le |f_k^{N_k}|\,|w_k| + |z_k| + |f_k^{T_k}|\,|w_k| \\ &= \mathcal{O}(1)\cdot(|w_k| + |z_k|). \end{aligned} \tag{3.16}$$

Bounds on the size of $\mathbf{v}$ can thus be obtained from estimates on w_k and z_k.

From (3.13) and (3.10) it follows

$$\begin{aligned} w_k &= \frac{\mathbf{v}_k\cdot\mathbf{n}_k}{f_k\cdot\mathbf{n}_k} \\ &= \frac{1}{f_k\cdot\mathbf{n}_k}\left[\widetilde{\mathbf{v}}_{k-1}\cdot\mathbf{n}_k + (f_k\cdot\mathbf{n}_k - \widetilde{f}_{k-1}\cdot\mathbf{n}_k)\frac{\widetilde{\mathbf{v}}_{k-1}\cdot\mathbf{n}_k}{\widetilde{f}_{k-1}\cdot\mathbf{n}_k}\right] \\ &= \frac{\widetilde{\mathbf{v}}_{k-1}\cdot\mathbf{n}_k}{\widetilde{f}_{k-1}\cdot\mathbf{n}_k} = \widetilde{w}_{k-1} + \mathcal{O}(1)\cdot|\widetilde{\mathbf{v}}_{k-1}|\,|\mathbf{n}_k - \mathbf{n}_{k-1}|. \end{aligned} \tag{3.17}$$

In addition, using (3.11) and (3.12) we deduce

$$\begin{aligned}\widetilde{w}_{k-1} - w_{k-1} &= \frac{\widetilde{\mathbf{v}}_{k-1} \cdot \mathbf{n}_{k-1} - \mathbf{v}_{k-1} \cdot \mathbf{n}_{k-1}}{\widetilde{f}_{k-1} \cdot \mathbf{n}_{k-1}} \\ &\quad + \left[\frac{\mathbf{v}_{k-1} \cdot \mathbf{n}_{k-1}}{\widetilde{f}_{k-1} \cdot \mathbf{n}_{k-1}} - \frac{\mathbf{v}_{k-1} \cdot \mathbf{n}_{k-1}}{f_{k-1} \cdot \mathbf{n}_{k-1}} \right] = \mathcal{O}(1)(t_k - t_{k-1})\, |\mathbf{v}_{k-1}|. \end{aligned} \tag{3.18}$$

Together, (3.12), (3.17) and (3.18) yield

$$w_k = w_{k-1} + \mathcal{O}(1) \cdot \{|\mathbf{n}_k - \mathbf{n}_{k-1}| + |t_k - t_{k-1}|\} |\mathbf{v}_{k-1}|. \tag{3.19}$$

Similar estimates can be obtained for the component z_k, namely

$$\begin{aligned} z_k - \widetilde{z}_{k-1} &= [\mathbf{v}_k^{T_k} - f_k^{T_k} w_k] - [\widetilde{\mathbf{v}}_{k-1}^{T_{k-1}} - \widetilde{f}_{k-1}^{T_{k-1}} \widetilde{w}_{k-1}] \\ &= [\widetilde{\mathbf{v}}_{k-1}^{T_k} + (f_k^{T_k} - \widetilde{f}_{k-1}^{T_k}) w_k - f_k^{T_k} w_k] - [\widetilde{\mathbf{v}}_{k-1}^{T_{k-1}} - \widetilde{f}_{k-1}^{T_{k-1}} \widetilde{w}_{k-1}] \\ &= (\widetilde{\mathbf{v}}_{k-1}^{T_k} - \widetilde{\mathbf{v}}_{k-1}^{T_{k-1}}) + f_k^{T_k} (\widetilde{w}_{k-1} - w_k) + (\widetilde{f}_{k-1}^{T_{k-1}} - \widetilde{f}_{k-1}^{T_k}) \widetilde{w}_{k-1} \\ &\quad + (f_k^{T_k} - \widetilde{f}_{k-1}^{T_k})(w_k - \widetilde{w}_{k-1}) = \mathcal{O}(1) |\mathbf{n}_k - \mathbf{n}_{k-1}|\, |\widetilde{\mathbf{v}}_{k-1}|, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \widetilde{z}_{k-1} - z_{k-1} &= (\widetilde{\mathbf{v}}_{k-1}^{T_{k-1}} - \mathbf{v}_{k-1}^{T_{k-1}}) + (f_{k-1}^{T_{k-1}} - \widetilde{f}_{k-1}^{T_{k-1}}) w_{k-1} \\ &\quad + \widetilde{f}_{k-1}^{T_{k-1}} (w_{k-1} - \widetilde{w}_{k-1}) = \mathcal{O}(1) |t_k - t_{k-1}|\, |\mathbf{v}_{k-1}|. \end{aligned} \tag{3.21}$$

Therefore,

$$z_k = z_{k-1} + \mathcal{O}(1) \cdot \{|\mathbf{n}_k - \mathbf{n}_{k-1}| + |t_k - t_{k-1}|\} |\mathbf{v}_{k-1}|. \tag{3.22}$$

Introducing the scalar quantity $y_k \doteq |w_k| + |z_k|$, from (3.19), (3.22) and (3.16) we deduce

$$y_k \le (1 + \mathcal{O}(1) \cdot \{|\mathbf{n}_k - \mathbf{n}_{k-1}| + |t_k - t_{k-1}|\}) y_{k-1}.$$

By induction on k, for any integers $p < q$ we obtain

$$y_q \le \exp\{C' \cdot \sum_{k=p+1}^{q} (|\mathbf{n}_k - \mathbf{n}_{k-1}| + |t_k - t_{k-1}|)\} y_p, \tag{3.23}$$

for some constant C'. Recalling the assumption on the directional variation of $\nabla\tau$, we now have

$$\sum_k |t_k - t_{k-1}| \le T, \qquad \sum_k |\mathbf{n}_k - \mathbf{n}_{k-1}| \le \text{const.} \tag{3.24}$$

From (3.23) and (3.24), since by (3.15)-(3.16) the quantities y_k are uniformly equivalent to the corresponding norms $|\mathbf{v}(t_k)|$, we finally obtain the estimate

$$|\mathbf{v}(t)| \le C_R \cdot |\mathbf{v}(s)| \qquad \text{for every } 0 \le s < t \le T, \tag{3.25}$$

valid for some constant C_R and any reference trajectory $x(\cdot)$ taking values inside the ball $B = \{x \in \mathbb{R}^m \mid |x| \leq R + T|K|\}$. Observe that C_R may depend on R through the quantity

$$\delta_0 \doteq \min\{\nabla\tau(x) \cdot z \mid z \in K, \ |x| \leq R + T|K|\}. \tag{3.26}$$

STEP 2. Relying on the uniform bounds (3.25) on tangent vectors, it is easy to derive the estimate (3.2) in the piecewise smooth case. Indeed, let the initial data x_0, y_0 be given. Choose R so that $|x_0|, |y_0| \leq R$. We then construct a one-parameter family of solutions $x^\theta : [0,T] \mapsto \mathbb{R}^m$, satisfying

$$\dot{x}^\theta(t) = f(x^\theta(t)), \quad x^\theta(0) = \theta y_0 + (1-\theta)x_0, \qquad \theta \in [0,1].$$

Defining the tangent vectors

$$\mathbf{v}^\theta(t) \doteq \lim_{\varepsilon \to 0} \frac{x^{\theta+\varepsilon}(t) - x^\theta(t)}{\varepsilon},$$

for all $t \in [0,T]$, from (3.25) it follows

$$\begin{aligned} |y(t) - x(t)| &\leq \int_0^1 \left| \frac{d}{d\theta} x^\theta(t) \right| d\theta = \int_0^1 |\mathbf{v}^\theta(t)|\, d\theta \leq C_R \int_0^1 |\mathbf{v}^\theta(0)|\, d\theta \\ &= C_R |y_0 - x_0|, \end{aligned} \tag{3.27}$$

proving (3.2). In this piecewise smooth case, the evolution equation in (1.6) thus generates a uniformly Lipschitz continuous flow. In the following, to denote the unique solution of the Cauchy problem (1.6), we shall use the semigroup notation

$$x(t) = S_t \bar{x}.$$

We recall that, if $w : [0,T] \mapsto \mathbb{R}^m$ is any Lipschitz function, one has the error estimate

$$|w(t) - S_t w(0)| \leq L \cdot \int_0^t \left(\liminf_{h \to 0+} \frac{|w(\tau+h) - S_h w(\tau)|}{h} \right) d\tau, \qquad t \in [0,T], \tag{3.28}$$

where L is the Lipschitz constant of the semigroup w. r. t. the initial data. In particular, if w solves the perturbed equation

$$\dot{w}(t) = f(w(t)) + e(t), \tag{3.29}$$

and satisfies the bounds $|w(t)| \leq R + t|K|$, then from (3.27) and (3.28) we deduce

$$|w(t) - S_t w(0)| \leq C_R \cdot \int_0^t |e(s)|\, ds \qquad t \in [0,T]. \tag{3.30}$$

STEP 3. The general case will be treated using an approximation procedure. Fix any radius R arbitrarily large, and define

$$\mathcal{U} \doteq \left\{ u : [0,T] \mapsto \mathbb{R}^m \mid |u(0)| \leq R, \ \frac{u(t)-u(s)}{t-s} \in K \text{ for all } t > s \right\}. \tag{3.31}$$

Lemma 1. *Let f, g, τ be as in Theorem 2, and let R and $\varepsilon > 0$ be given. Then there exists a piecewise smooth function $\widehat{f} : \mathbb{R}^m \mapsto K$ of the form*

$$\widehat{f}(x) = \widehat{g}_k(x) \quad \text{if} \quad \tau_k \leq \tau(x) < \tau_{k+1}, \tag{3.32}$$

with the following properties. Each $\hat{g}_k$ is smooth, and its Lipschitz constant satisfies

$$\mathrm{Lip}(\widehat{g}_k) \leq \sup_{t\in\mathbb{R}} \mathrm{Lip}(g(t, \cdot)). \tag{3.33}$$

Moreover, the Picard operators determined by f and $\hat{f}$ are close, namely

$$\sup_{u\in\mathcal{U}} \int_0^T |\widehat{f}(u(t)) - f(u(t))| \, dt \leq \varepsilon. \tag{3.34}$$

To construct $\widehat{f}$, we first apply the theorem of Scorza-Dragoni [11] to the Carathéodory function g and obtain a closed set J with $\mathrm{meas}(\mathbb{R} \setminus J) \leq \varepsilon$, such that the restriction of g to $J \times \mathbb{R}^m$ is continuous. The complement of J is an open set, which can be written as a disjoint union of countably many open intervals, say $]a_\nu, b_\nu[$, $\nu \geq 1$. We then define

$$g^*(t,x) = \begin{cases} \pi_K(g(t,x)) & \text{if } t \in J, \\ \theta \cdot \pi_K(g(b_\nu, x)) + & \text{if } t = \theta b_\nu + (1-\theta) a_\nu \text{ for} \\ \quad +(1-\theta) \cdot \pi_K(g(a_\nu, x)) & \text{some } \nu \geq 1, \ 0 < \theta < 1, \end{cases} \tag{3.35}$$

where π_K denotes the orthogonal projection on the compact convex set K. By (3.35), the function g^* is continuous in t and Lipschitz continuous in x. More precisely

$$\sup_{t\in\mathbb{R}} \mathrm{Lip}(g^*(t, \cdot)) = \sup_{t\in J} \mathrm{Lip}(g(t, \cdot)). \tag{3.36}$$

Moreover, recalling that $|K| = \max_{z\in K} |z|$, for any $u \in \mathcal{U}$ we have

(3.37)
$$\int_0^T |g^*(\tau(u(t)), u(t)) - g(\tau(u(t)), u(t))| dt \leq 2|K| \cdot \mathrm{meas}\{t \mid \tau(u(t)) \notin J\}$$
$$\leq 2|K|\varepsilon/\delta_0,$$

where $\delta_0 > 0$ is the constant in (3.26). We now choose a small $\delta^* > 0$ and, for each $k \in \mathbb{Z}$, we define $\tau_k \doteq k\delta^*$ and let $\widehat{g}_k$ be a mollification of the function $g^*(k\delta^*, \cdot)$. We then define

$$\widehat{g}(t, x) \doteq \widehat{g}_k(x) \qquad \text{if } \tau_k \le t < \tau_{k+1},$$

and let $\widehat{f}$ be as in (3.32). From (3.36) it thus follows (3.33). If δ^* is sufficiently small and the mollification kernel is sufficiently close to the identity, this construction yields

$$\int_0^T |g^*(\tau(u(t)), u(t)) - \widehat{g}(\tau(u(t)), u(t))|\, dt \le \varepsilon \qquad \text{for all } u \in \mathcal{U}. \tag{3.38}$$

Since $\varepsilon > 0$ was arbitrary, (3.37) and (3.38) together yield Lemma 1.

STEP 4. We can now conclude the proof of Theorem 2. Let x, y be any two solutions, as in (3.1). To prove the estimates (3.2), let $\varepsilon > 0$ be given, choose $R \doteq \max\{|x_0|, |y_0|\}$ and construct a function $\widehat{f}$ according to Lemma 1. According to STEP 2, the semigroup $\widehat{S}$ generated by the evolution equation $\dot{x} = \widehat{f}(x)$ is Lipschitz continuous, more precisely

$$|\widehat{S}_t x_* - \widehat{S}_t y_*| \le C_R \cdot |x_* - y_*| \qquad \text{whenever } |x_*|, |y_*| \le R,\ t \in [0, T], \tag{3.39}$$

for some constant C_R not depending on ε. Define the quantities e_x and e_y as

$$\begin{aligned} e_x(t) &\doteq f(x(t)) - \widehat{f}(x(t)), \\ e_y(t) &\doteq f(y(t)) - \widehat{f}(y(t)), \end{aligned} \qquad t \in [0, T]. \tag{3.40}$$

The functions x and y are thus solutions to

$$\begin{aligned} \dot{x}(t) &= \widehat{f}(x(t)) + e_x(t), \qquad & x(0) &= x_0, \\ \dot{y}(t) &= \widehat{f}(y(t)) + e_y(t), \qquad & y(0) &= y_0. \end{aligned}$$

By (3.39) we can now use (3.30) and deduce

$$\begin{aligned} |y(t) - x(t)| &\le |y(t) - \widehat{S}_t y_0| + |\widehat{S}_t y_0 - \widehat{S}_t x_0| + |\widehat{S}_t x_0 - x(t)| \\ &\le C_R \cdot \int_0^t |e_y(s)|\, ds + C_R |y_0 - x_0| + C_R \cdot \int_0^t |e_x(s)|\, ds \\ &\le 2C_R \varepsilon + C_R |y_0 - x_0| \end{aligned} \tag{3.41}$$

for every $t \in [0, T]$. Indeed, by (3.40) and (3.34) it follows

$$\int_0^T |e_y(s)|\, ds < \varepsilon, \qquad \int_0^T |e_x(s)|\, ds < \varepsilon.$$

Since ε was arbitrary, from (3.41) we deduce (3.2).

4. A counterexample

In Theorem 2, the assumption that $\nabla\tau$ has bounded directional variation is essential for the uniqueness of the solutions, as shown by the following counterexample.

Consider a function $g : \mathbb{R} \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that

$$(4.1) \quad g(t,x) = g(t) \doteq \begin{cases} (1,1), & \text{if } t \in \left]\frac{2k+1}{2k(k+1)}, \frac{1}{k}\right], \\ (1,-1), & \text{if } t \in \left]\frac{1}{k+1}, \frac{2k+1}{2k(k+1)}\right], \end{cases} \qquad \text{for any } k \geq 1.$$

In the plane with coordinates (x_1, x_2), define the sequences of points P_k, P'_k, Q_k and Q'_k by setting (fig. 2)

$$(4.2) \quad \begin{aligned} P_k &\doteq \left(\frac{1}{k}, 0\right), & Q_k &\doteq \left(\frac{2k+1}{2k(k+1)}, \frac{-1}{2k(k+1)}\right), \\ P'_k &\doteq \left(\frac{1}{k}, \frac{1}{2k}\right), & Q'_k &\doteq \left(\frac{4k+1}{4k(k+1)}, \frac{2k-1}{4k(k+1)}\right). \end{aligned}$$

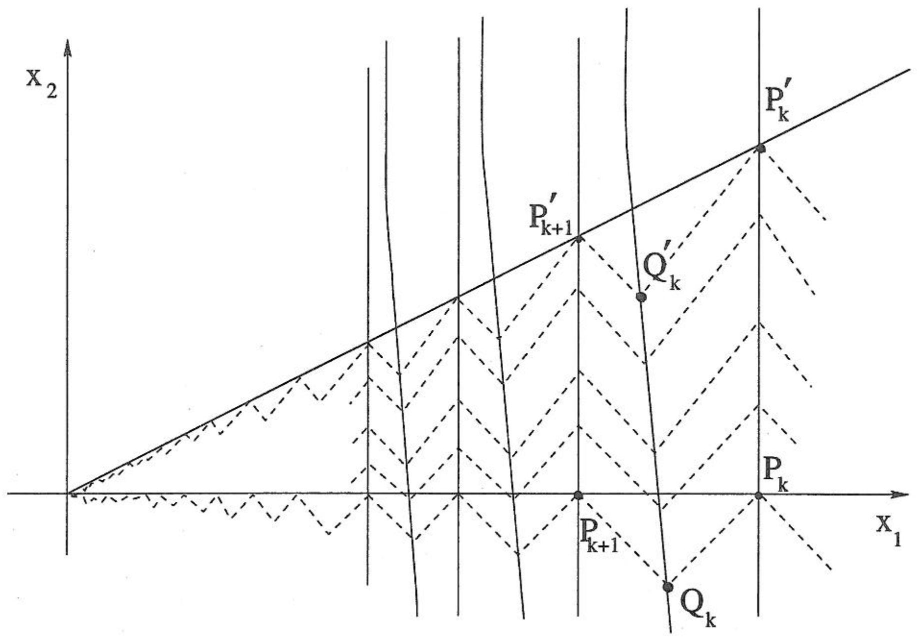

Figure 2

We can now construct a $\mathcal{C}^1$ function $\tau : \mathbb{R}^2 \mapsto \mathbb{R}$ with

$$(4.3) \qquad \nabla\tau(x) \cdot (1,1) > 0 \qquad \nabla\tau(x) \cdot (1,-1) > 0 \qquad \text{for all } x \in \mathbb{R}^2$$

and such that, for every integer $k \geq 1$,

$$(4.4) \qquad \tau(x) \doteq \begin{cases} \dfrac{1}{k}, & \text{along the segment joining } P_k, P'_k, \\ \dfrac{2k+1}{2k(k+1)}, & \text{along the segment joining } Q_k, Q'_k. \end{cases}$$

Letting $f(x) \doteq g(\tau(x))$, and defining

$$K \doteq \overline{co}\{(1,1);(1,-1)\} = \{(1,\lambda) \mid |\lambda| \leq 1\},$$

all of the assumptions in Theorem 2 are satisfied, except the one on the directional variation of $\nabla\tau$. Indeed, at all points P_k the gradient $\nabla\tau$ is parallel to the vector $(1,0)$. On the other hand, at each point Q_k this gradient is parallel to the vector $(1,\ 1/(2k+1))$. Since $\nabla\tau$ is continuous and never vanishes, its total variation in the direction of the cone $\Gamma \doteq \{(x_1,x_2) \mid |x_2| \leq x_1\}$ cannot be bounded.

From the definitions (4.1)-(4.4) it follows that, for each $k \geq 1$,

(4.5)

$$f(x) \doteq \begin{cases} (1,1) & \text{on the quadrilateral with vertices } P_k, P'_k, Q'_k, Q_k, \\ (1,-1) & \text{on the quadrilateral with vertices } Q_k, Q'_k, P'_{k+1}, P_{k+1}. \end{cases}$$

One can easily check that the Cauchy problem on $\mathbb{R}^2$

$$\dot{x} = f(x), \qquad x(0) = (0,0)$$

has two distinct solutions (Figure 2). Namely, one solution passing through all the points Q_k, P_k, $k \geq 1$, and a second solution passing through all the points Q'_k, P'_k.

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UNIQUENESS OF OPTIMAL TRAJECTORIES AND THE NONEXISTENCE OF SHOCKS FOR HAMILTON-JACOBI-BELLMAN AND RICCATI PARTIAL DIFFERENTIAL EQUATIONS

CHRISTOPHER I. BYRNES AND HÉLÈNE FRANKOWSKA

1. Introduction

In this paper, we study a Bolza problem arising in optimal control

$$\text{minimize } \int_{t_0}^{T} L(x(t),u(t))\,dt + \varphi(x(T)) \tag{1}$$

subject to

$$\begin{cases} x' = f(x) + g(x)u(t), & u(t) \in \mathbb{R}^m \\ x(t_0) = x_0, \end{cases} \tag{2}$$

where $t_0 \in [0,T]$, $g : \mathbb{R}^n \to \mathbb{M}_{n\times m}$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$.

Under appropriate smoothness hypothesis it can be shown that any optimal trajectory-control pair $(\overline{x},\overline{u})$ of the above problem verifies the maximum principle. There exists an absolutely continuous function $\overline{p} : [t_0,T] \to \mathbb{R}^n$ such that

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$(\overline{x}, \overline{p})$ solves the Hamiltonian system

$$(3)\qquad \begin{cases} x' = \dfrac{\partial H}{\partial p}(x,p), & x(t_0) = x_0 \\ -p' = \dfrac{\partial H}{\partial x}(x,p), & p(T) = -\nabla\varphi(\overline{x}(T)), \end{cases}$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(4)\qquad H(x,p) = \sup_u \{\langle p, f(x) + g(x)u\rangle - L(x,u)\}.$$

The above system, in general, does not have a unique solution because the initial condition for $p(\cdot)$ at t_0 is not known. For this very reason, the necessary condition for optimality given by the maximum principle is not sufficient.

Thanks to Proposition 3.2 below, the above system may be rewritten in a more familiar form of the Pontryagin principle involving an adjoint equation and a maximum condition. It may however happen that even for smooth f, g, L the Hamiltonian H is non differentiable.

As it will be shown below, $\overline{p}(\cdot)$ may be chosen in such way that $-\overline{p}(t_0)$ is equal to the gradient with respect to x of the value function $V : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ associated to the above problem provided $(\partial V/\partial x)(t_0, x_0)$ does exist. We may consider then the Cauchy problem

$$(5)\qquad \begin{cases} x' = \dfrac{\partial H}{\partial p}(x,p), & x(t_0) = x_0 \\ -p' = \dfrac{\partial H}{\partial x}(x,p), & p(t_0) = -\dfrac{\partial V}{\partial x}(t_0, x_0). \end{cases}$$

When ∇H is locally Lipschitz, then there exists at most one solution to the above problem and in this way the necessary condition becomes sufficient. Certainly, we do not have the above sufficient conditions, when $V(t_0, \cdot)$ is not differentiable at x_0.

In [7] it was proved that in the context of the Mayer problem (i.e. $L \equiv 0$) it is enough to pick any

$$p(t_0) \in -\Big(\operatorname*{Limsup}_{x \to x_0} \frac{\partial V}{\partial x}(t_0, x)\Big) \setminus \{0\}$$

to obtain the optimal design similar to (5), where Limsup denotes the upper limit of sets (see for instance [1]).

An alternative geometric approach for quantifying the nonuniqueness in the initial condition $p(t_0)$ was studied in [4] and begins with consideration of the canonical Hamiltonian system

$$(6)\qquad \begin{cases} x' = \dfrac{\partial H}{\partial p}(x,p), & x(t) = x_0 \\ p' = -\dfrac{\partial H}{\partial x}(x,p), & p(t) = p_0 \end{cases}$$

and the final transversality condition

$$p(T) = -\nabla\varphi(x(T)). \tag{7}$$

If one assumes that the Hamiltonian system (6) is complete, then the flow defined via

$$\Phi\left(t, \begin{pmatrix} x(0) \\ p(0) \end{pmatrix}\right) = \begin{pmatrix} x(t) \\ p(t) \end{pmatrix},$$

where $(x,p)(\cdot)$ solves (6) for initial conditions $(x(0), p(0))$, which is always defined for small t is, in fact, defined for all t. In particular, if the Hamiltonian H is C^k, $k \geq 1$, then

$$\Phi : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

is C^k and, for each t, the map

$$\Phi_t(\cdot) = \Phi(t, \cdot)$$

is a C^k diffeomorphism

$$\Phi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}.$$

If φ is at least C^2, then the subset M_T

$$M_T = \{(x,p) \mid p = -\nabla\varphi(x)\}$$

defined by the tranversality conditions (7) is a closed, connected, smooth submanifold of dimension n. Therefore, for $t \in [T - t_0, T]$, the subset

$$M_t = \Phi_{T-t}(M_T)$$

is a closed, connected smooth submanifold of $\mathbb{R}^{2n}$ consisting of those pairs $(x(t), p(t))$ which are initial conditions, for initial time t and for trajectories $(x(\cdot), p(\cdot))$ of the system (6), satisfying (7). In particular, to say there is a unique value of the costate variable for every x and every initial time t is to say

$$M_t = \operatorname{graph}(-\pi(x,t)), \qquad x \in \mathbb{R}^n,\ t \in [0,T], \tag{8}$$

where $\pi : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is a (single-valued) mapping. Thus, it is of particular interest to find conditions under which M_t will always be the graph of a function. The mapping π is, of course, intimately related to the construction of an optimal control (see Corollary 2.2). Since M_t is connected, such a π would be necessarily continuous.

Moreover, to say (8) holds for a C^1 function π defined on $\mathbb{R}^n \times [t_0, T]$ is to say (see e.g. [4]) that π satisfies

$$\frac{\partial \pi}{\partial t} = \frac{\partial H}{\partial x}(x, -\pi) - \frac{\partial \pi}{\partial x}\frac{\partial H}{\partial p}(x, -\pi) \tag{9}$$

together with the side constraint

$$\pi(x,T) = \nabla\varphi(x). \tag{10}$$

This "Riccati" Partial Differential Equation has, of course, a long history going back to [8], who introduced this system in the case $f(x) = 0$, $g(x) = \mathrm{Id}$ as the *fundamental equations* of the calculus of variations. It was extended to optimal control problems in the context of invariant imbedding (see e.g. [20] and the bibliography contained therein). In this context, it follows from the HJB equation (11) given below that, when the value function V is C^2, the mapping $\pi = \nabla_x V$ is a solution of (9). Conversely, when M_t is always the graph of a C^1 function, the value function is C^2.

Moreover, the smoothness properties of M_t can be used to improve upon known regularity for value functions. At one extreme, using Sard's Theorem, a generalization of the Poincaré Lemma and the fact that M_t is a Lagrangian submanifold, one can also show (see [4]), that (8) holds for a continuous function π if, and only if, $\pi = \nabla_x V$ for the value function V of (1)-(2), which is then necessarily C^1.

On the other hand, whenever H and φ are at least C^k and the submanifolds M_t are graphs, we can conclude from the variational equation that the value function is at least C^k. In particular, if all data is C^∞ or C^ω then, under these conditions, V is C^∞ or C^ω. Such higher order regularity results are particularly important in getting error estimates for the various schemes for obtaining approximating polynomials to the solutions of nonlinear Hamilton-Jacobi-Bellman equations (see e.g. [18]).

The second point of view on (8) follows from the Maximum Principle. Indeed it can be proved that for every $p_0 \in \mathrm{Limsup}_{x\to x_0}(\partial V/\partial x)(t_0,x)$ the solution $(x(\cdot),p(\cdot))$ of

$$\begin{cases} x' = \dfrac{\partial H}{\partial p}(x,p), & x(t_0) = x_0 \\ p' = -\dfrac{\partial H}{\partial x}(x,p), & p(t_0) = -p_0 \end{cases}$$

is such that $x(\cdot)$ is optimal and $p(\cdot)$ is the corresponding costate (see [11]). Thus (8) yields, $\mathrm{Limsup}_{x\to x_0}(\partial V/\partial x)(t_0,x)$ is a singleton and, by [13], $V(t_0,\cdot) \in C^1$. Consequently, also $\pi(\cdot,t) = \nabla_x V(t,\cdot)$.

The aims of our work are

(a) to show the equivalence between differentiability of value function, uniqueness of optimal solutions to Bolza problem and the absence of shocks in the method of characteristics for the Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V}{\partial t} + H\left(x, -\frac{\partial V}{\partial x}\right) = 0, \qquad V(T,\cdot) = \varphi(\cdot), \tag{11}$$

(b) to derive a general criterion which gives necessary and sufficient conditions for the absence of shocks for a general class of nonlinear systems in terms of the matrix Riccati differential equation

$$(12)\quad P' + \frac{\partial^2 H}{\partial p \partial x}(x(t),p(t))P + P\frac{\partial^2 H}{\partial x \partial p}(x(t),p(t)) + P\frac{\partial^2 H}{\partial p^2}(x(t),p(t))P + \frac{\partial^2 H}{\partial x^2}(x(t),p(t)) = 0, P(T) = -\varphi''(x(T)),$$

whose solution $P(\cdot)$ may escape to infinity in a finite time $t < T$. Namely, we prove that the optimal solution is unique (or, equivalently, $V \in C^1$) if and only if the *characteristic system*

$$(13)\quad \begin{cases} x' = \dfrac{\partial H}{\partial p}(x,p), & x(T) = x_T \\ -p' = \dfrac{\partial H}{\partial x}(x,p), & p(T) = -\nabla\varphi(x_T) \end{cases}$$

verifies the following property: if (x_1,p_1), (x_2,p_2) solve the above system for some final conditions x_T^1, x_T^2 and $x_1(t_0) = x_2(t_0)$, then also $x_1(T) = x_2(T)$, (and consequently, $x_1 = x_2$, $p_1 = p_2$) i.e. that M_{t_0} is the graph of a single-valued function.

We next show that if for every extremal pair (x,p) (i.e., a solution to (13)), the matrix Riccati equation (12) has no nonnegative escape time (i.e. its solution is well defined on $[0,T]$), then M_{t_0} is the graph of a single-valued function.

We also provide some sufficient conditions for the nonexistence of shocks. Examples of systems satisfying such conditions include many nonlinear mechanical systems and are given in Section 2.

Finally, we show that even when the value function V is merely Lipschitz, each optimal trajectory enters immediately into the domain of differentiability of V, provided the Hamiltonian is strictly convex in the last variable, which may be helpful to eliminate characteristics of HJB equation which are not related to the gradient of the value function.

2. Statements of results

Consider the minimization problem

$$(\mathrm{P})\qquad \text{minimize} \int_{t_0}^{T} L(x(t),u(t))dt\,\varphi(x(T))$$

over solution-control pairs (x,u) of control system

$$\begin{cases} x'(t) = f(x(t)) + g(x(t))u(t), & u \in L^1(t_0,T;\mathbb{R}^m) \\ x(t_0) = x_0, \end{cases}$$

where $t_0 \in [0,T]$, $x_0 \in \mathbb{R}^n$ and the mappings $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g : \mathbb{R}^n \mapsto \mathbb{M}_{n\times m}$, $L : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$, $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ are given.

We associate to these data the *Hamiltonian* H defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$H(x,p) = \sup_u (\langle p, f(x) + g(x)u\rangle - L(x,u))$$

and observe that if for some (x,p) the element $\overline{u} \in \mathbb{R}^m$ is so that

$$H(x,p) = \langle p, f(x)\rangle + \langle p, g(x)\overline{u}\rangle - L(x,\overline{u})$$

and if L is differentiable with respect to u, then

$$g(x)^\star p = \frac{\partial L}{\partial u}(x,\overline{u}). \tag{15}$$

If the Hamiltonian H is differentiable, then we say that the *Hamiltonian system*

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(x(t), p(t)) \\ -p'(t) = \dfrac{\partial H}{\partial x}(x(t), p(t)) \end{cases}$$

is *complete* if for all x_0, $p_0 \in \mathbb{R}^n$ it has the unique solution (x,p) defined on $\mathbb{R}$ and satisfying $x(0) = x_0$, $p(0) = p_0$.

We denote by $x(\,\cdot\,; t_0, x_0, u)$ the solution to (14) starting at time t_0 from the initial condition x_0 and corresponding to the control $u(\,\cdot\,)$.

The value function associated to this problem is defined by

$$V(t_0, x_0) = \inf_{u \in L^1(t_0,T)} \int_{t_0}^T L(x(t; t_0, x_0, u), u(t))dt + \varphi(x(T; t_0, x_0, u)),$$

when (t_0, x_0) range over $[0,T] \times \mathbb{R}^n$.

We now describe a class of nonlinear systems, including a variety of nonlinear mechanical systems, for which the value function of the Bolza problem is indeed smooth, and for which there is a unique optimal control expressed via state feedback.

Example 2.1. Consider a C^2 nonlinear system

$$\begin{cases} x' = f(x) + g(x)u, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^k \\ y = h(x), & y \in \mathbb{R}^m, \end{cases}$$

which is loseless, i.e., there exists a C^2 positive definite function W on $\mathbb{R}^n$ such that the dissipative equality holds for all measurable controls $u(\,\cdot\,)$:

$$W(x(t)) - W(x(0)) = \int_0^t \langle u(s), y(s)\rangle ds.$$

This equality can be interpreted as an equality between the energy spended in moving from the state $x(0)$ to the state $x(t)$, $W(x(t)) - W(x(0))$ and the energy supplied to the system, which is the integral of the product $\langle u, y\rangle$. Such systems, together with a wide range of physical examples, are discussed in detail in [23]. For example, a nonlinear mechanical system with potential energy $E(x)$ and kinetic energy $\|x'\|^2$ may be expressed as a loseless system

$$\begin{cases} x_1' = x_2, \\ Mx_2' = -E'(x) + u, \\ x_1 = x, \end{cases}$$

where $x_2 = x'$, $y = h(x_1, x_2) = x_2$, and

$$W(x_1, x_2) = E(x_1) + \|Mx_2\|^2.$$

In general, for a loseless system, an interest is in the class of optimal control problems defined by the explicit performance measure

$$J(x_0, u) = \int_0^T (\|u\|^2 + \|y\|^2)dt + W(x(T)),$$

a problem which has an appealing interpretation as a minimum energy problem for the case of nonlinear mechanial systems.

In general, one can derive an infinitesimal form of the dissipative equality,

$$L_f W \equiv 0, \qquad L_g W = h^T,$$

which is often referred to as the Kalman-Yakubivitch-Popov (KYP) Lemma. Moreover, from the KYP Lemma, one immediately checks that W is a solution to the Hamilton-Jacobi-Bellman equation and that, in fact, the following three assertions hold:

(i) The value function $V(t, x) = W(x)$ is continuously differentiable,
(ii) For every (t_0, x_0) there exists a unique optimal trajectory,
(iii) For all $t \in [0, T]$, the submanifold M_t is a graph; in fact $M_t = \mathrm{gr}(-\nabla W)$.

Finally, since the Lagrangian is strictly convex in u, one sees that the optimal synthesis is given by the smooth feedback law $u = -h(x)$ (we refer to [21] for a similar treatment of the infinite horizon problem for the rigid body model of a robot arm, as well as to [22] for a treatment of the H^∞-control problem).

In this paper we impose the following assumptions:

(H_1) f and g are locally Lipschitz,

(H_2) For every $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ an optimal solution of (P) does exist and $V : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}$ is locally Lipschitz,

(H_3) $L(x, \cdot)$ is continuous, convex and for some $c > 0$

$$\forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m, \ L(x, u) \geq c\|u\|^2.$$

Furthermore, for all $r > 0$, there exists $k_r \geq 0$ such that

$$\forall u \in \mathbb{R}^m, \ L(\cdot, u) \text{ is } k_r\text{-Lipschitz on } B_r(0),$$

(H_4) $f, g, L(\cdot, u)$ are differentiable and $\varphi \in C^1$,

(H_5) The Hamiltonian H is differentiable, $\nabla H(\cdot, \cdot)$ is locally Lipschitz and the Hamiltonian system (16) is complete.

Remarks. (a) Assumption (H_2) holds for instance under linear growth conditions on data (see [14], [12]). We provide a sufficient condition for it in Section 4.

(b) The completeness assumption in (H_5) is verified for instance if the mapping $\nabla H(\cdot,\cdot)$ is locally Lipschitz and has a linear growth:

$$\exists M_1 \geq 0,\ \forall\, x, p \in \mathbb{R}^n,\ \|\nabla H(x,p)\| \leq M_1(\|x\| + \|p\| + 1).$$

(c) (H_4) does not imply in general that the Hamiltonian H is differentiable.

Theorem 2.1. *Assume that* (H_1)-(H_5) *hold true. Then the following three statements are equivalent:*

(i) *The value function V is continuously differentiable.*
(ii) *For every $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$ the optimal trajectory of problem* (P) *is unique.*
(iii) *For the Hamiltonian system*

$$\text{(17)} \qquad \begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(x(t), p(t)), & x(T) = x_T, \\ -p'(t) = \dfrac{\partial H}{\partial x}(x(t), p(t)), & p(T) = -\nabla\varphi(x_T) \end{cases}$$

define the set

$$M_t := \{(x(t), p(t)) \mid (x,p) \text{ solves (17) on } [t,T] \text{ for some } x_T \in \mathbb{R}^n\}.$$

Then M_t is equal to the graph of a continuous function $-\pi_t : \mathbb{R}^n \mapsto \mathbb{R}^n$. Furthermore, if (iii)) *holds true, then $\pi_t(\cdot) = (\partial V/\partial x)(t,\cdot)$ and a solution (x,p) to* (17) *restricted to $[t_0, T]$ satisfies: x is optimal for problem* (P) *with $x_0 = x(t_0)$ and p is the corresponding co-state of the Pontryagin maximum principle. In particular, $p(t) = -(\partial V/\partial x)(t, x(t))$ for all $t \in [0,T]$.*

The proof of the above theorem is given in the next section.

Remark. Under all assumptions of Theorem 2.1 suppose that for every x, $L(x,\cdot)$ is differentiable and $(\partial L/\partial u)(x,\cdot)$ is injective. Then, by the maximum principle, the (equivalent) statements (i)-(iii) are equivalent to

(iv) For every $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$ there exists a unique optimal control $\overline{u}(\cdot)$ solving the problem (P). Furthermore, if z denotes the corresponding optimal trajectory, then

$$\text{(18)} \qquad \forall t \in [t_0, T],\ \overline{u}(t) = -\left(\frac{\partial L}{\partial u}(z(t),\cdot)\right)^{-1}\left(g(z(t))^\star \frac{\partial V}{\partial x}(t, z(t))\right).$$

Equation (18) provides an optimal synthesis in the following sense: First, (18) gives the following formula

$$\forall (t,x) \in [0,T] \times \mathbb{R}^n,\ u(t,x) = -\left(\frac{\partial L}{\partial u}(x,\cdot)\right)^{-1}\left(g(x)^\star \frac{\partial V}{\partial x}(t,x)\right) \tag{19}$$

for a unique optimal control law. Secondly, this law generates the unique optimal trajectory emanating from each fixed initial condition. We note that (19) expresses the optimal control law as a function, not necessarilly smooth or continuous, of the state. Nontheless, existence of an optimal synthesis $u(t,x)$ implies uniqueness of optimal trajectories and hence, according to Theorem 2.1, smootheness of the value function — regardless of the method actually used to construct the optimal synthesis.

Corollary 2.2 (Feedback Law). *Under all of the assumptions of Theorem 2.1, suppose that an optimal synthesis $u(t,x)$ exists. Then, the value function is C^1 and the optimal feedback law $u = u(t,x)$ satisfies the relation*

$$-g(x)^\star \frac{\partial V}{\partial x}(t,x) + \frac{\partial L}{\partial u}(x, u(t,x)) = 0. \tag{20}$$

Moreover, if L is C^2 in (x,u) and strictly convex in u, then the optimal synthesis is expressible as a feedback law (19).

Our next series of results give sufficient conditions for M_t to be the graph of a C^1 function. In particular, in the light of Theorem 2.1, it can also be viewed as either a uniqueness result or as giving sufficient conditions for the value function to be at least C^2.

In general, when the data are smooth, M_t is always a smooth manifold and one can compute the onset of shocks by computing the times for which tangent spaces of the submanifolds M_t acquire a vertical component. Since at the final time T, the tangent space is always a graph

$$T_{x(T)}(M_T) = \text{graph}(-\varphi''(x(T))$$

one can propose to propagate the tangent space backward in time, as before, by a matrix Riccati equation with final condition

$$P(T) = -\varphi''(x(T).$$

Theorem 2.3. *Assume that $\varphi, H \in C^2$ and that* (H_5) *holds true. Then the following two statements are equivalent:*

(i) *$\forall t \in [0,T]$, M_t is the graph of a C^1 function from an open set $\mathcal{D}(t)$ into $\mathbb{R}^n$,*

(ii) *$\forall (x,p)$ solving* (17) *on $[0,T]$, the matrix Riccati equation*

$$\begin{cases} P' + \dfrac{\partial^2 H}{\partial p \partial x}(x(t),p(t))P + P\dfrac{\partial^2 H}{\partial x \partial p}(x(t),p(t)) \\ \qquad +P\dfrac{\partial^2 H}{\partial p^2}(x(t),p(t))P + \dfrac{\partial^2 H}{\partial x^2}(x(t),p(t)) = 0, \\ P(T) = -\varphi''(x(T)) \end{cases} \tag{21}$$

has a solution on $[0,T]$. *Furthermore, if* (H_2) *holds true, then in the statement* (i), $\mathcal{D}(t) = \mathbb{R}^n$.

(See [11] for the proof of a more general statement).

Corollary 2.4. *Suppose* H, φ *are* C^2, *that* (H_5) *holds true and that*

(1) *H is concave in x;*

(2) *φ is convex.*

Then for all $t \in [0,T]$ *a* C^1 *function* $\pi(\cdot, t)$ *exists such that*

$$M_t = \mathrm{graph}(-\pi(\cdot, t)).$$

We note that conditions (1), (2) are satisfied by the standard LQ problems in classical optimal control. Moreover, the above provides a new proof for LC theory; i.e., linear problems with nonlinear but convex terminal cost.

We next deduce from the variational equation of ODE that for C^∞ (or C^ω) data, to say V is C^2 is to say V is C^∞ (or C^ω), giving a further amplification of Theorem 2.1.

Convention 3.1. For the Bolza problem (1)-(2), we assume from now on that the Hamiltonian (4) is C^r, with r at least 2 and with locally Lipschitz second derivatives. Our notation also includes the cases $r = \infty$ and $r = \omega$. We also assume that the terminal cost, φ, is C^l for $l \geq 2$. Setting $k = \min\{l-1, r-1\}$, we say that the data of the problem are C^k. Finally, we assume that the Hamiltonian system is complete, so that the flow Φ is C^k and defined for all t and x.

We are particularly interested in what degree of smoothness the value function enjoys; e.g. for computational reasons it is useful to know whether optimal controls will be smooth, or just continuous. One approach to this question lies in the geometry of the submanifolds M_t; e.g. can one represent M_t, for $t \in [0,T]$, via

$$M_t = \mathrm{graph}(-\pi(\cdot, t)) \tag{22}$$

for π a C^1 function defined on $\mathbb{R}^n \times [0,T]$. One consequence of this assumption follows immediately from the implicit function theorem as in [4]:

Lemma 2.5. *If* π *is a* C^1 *function satisfying* (22) *on* $\mathbb{R}^n \times [0,T]$, *then* π *is* C^k.

The proof reposes on showing that the submanifold

$$N = \{(x,p,t) \mid p = -\pi(x,t)\},$$

which is C^1 is, in fact, C^k since Φ and M_T are C^k. One then applies the implicit function theorem to the map

$$P : N \mapsto \mathbb{R}^{n+1}$$

defined via

$$P(x,p,t) = (x,t).$$

By Theorem 2.1, if π satisfies (22), then the value function V of (1)-(2) satisfies

$$\pi(x,t) = \frac{\partial V}{\partial x}(t,x).$$

In particular, as a corollary to Theorem 2.1 and Lemma 2.5, we deduce the following regularity result.

Theorem 2.6. *Assume $k \geq 2$. To say the value function V is C^2 on $\mathbb{R}^n \times [t_0, T]$ is to say that the value function is C^k on $R^n \times [t_0, T]$. In particular, for C^∞ (or for analytic) data, the value function is C^∞ (or analytic), whenever it is C^2.*

Example 2.2 (Vector Burger Equation). Consider again the control system

$$x' = u, \qquad x, u \in \mathbb{R}^n$$

and the cost functional

$$J(x_0, u) = \frac{1}{2}\int_0^T u(t)^2\,dt + \varphi(x(T))$$

for some arbitrary but fixed C^k function φ. An analysis of this problem boils down to the existence of solutions to the Riccati PDE; i.e., to the vector Burgers' equation

$$\frac{\partial \pi}{\partial t} = \frac{\partial \pi}{\partial x}\cdot \pi, \qquad \pi(\,\cdot\,, T) = \nabla\varphi(\,\cdot\,).$$

According to Theorem 2.4 it will have a global solution if φ is convex, in harmony with the classical analysis of Burgers' equation.

Furthermore, an analysis of the Riccati PDE in this example shows that global existence and uniqueness for optimal controls is equivalent to convexity of φ.

Remark. For the above example, convexity of the terminal cost can also be shown to be necessary by an analysis of the ordinary Riccati differential equation. However, even for linear problems with quadratic integrands, positive semidefiniteness of φ'' is not necessary for existence and uniqueness of an optimal control. In this sense, Theorem 2.4 is as sharp for nonlinear systems and the classical theory as it is for linear systems, as our next example shows.

Example. Consider the problem

$$\min \frac{1}{2}\int_0^T (x(t)^2 + u(t)^2)\,dt - \frac{1}{2}\,x^2(T)$$

subject to the constraint

$$x' = u, \qquad x(0) = x_0.$$

This problem has a concave terminal cost, but the Riccati ordinary differential equation

$$p' = 1 - p^2$$

always has a solution when integrated backward in time from the final condition $P(T) = 1$, which in turn is equivalent to the nonexistence of shocks for the corresponding state-costate system. According to Theorem 2.1, the optimal control, which is in fact given by

$$u_\star(t) = p(t) = -\frac{\partial V}{\partial x}(t, x(t))$$

is unique.

We end this section by a result stating that even when V is merely Lipschitz, the optimal trajectories avoid points of nondifferentiability of V.

Theorem 2.7. *Assume* (H_1)-(H_5) *and that* $H(x, \cdot)$ *is strictly convex. Let* $(\overline{x}, \overline{u})$ *be a trajectory-control pair of the system* (14). *If* $\overline{x}$ *is an optimal solution to the Bolza problem, then for all* $t \in]t_0, T]$, V *is differentiable at* $(t, \overline{x}(t))$.

3. Proofs of results from Section 2

Our arguments rely heavily on the following extension of the Pontryagin maximum principle.

Theorem 3.1. *Assume that* (H_1)-(H_5) *hold true and let* $(\overline{x}, \overline{u})$ *be an optimal solution-control pair of* (P). *Then* $\overline{x} \in C^1$ *and there exists a continuously differentiable* $p : [t_0, T] \mapsto \mathbb{R}^n$ *such that* $(\overline{x}, p)$ *solves the Hamiltonian system*

$$\text{(23)} \qquad \begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(x(t), p(t)), & x(t_0) = x_0, \\ -p'(t) = \dfrac{\partial H}{\partial x}(x(t), p(t)), & p(T) = -\nabla \varphi(\overline{x}(T)), \\ & p(t_0) \in -\partial_+ V_x(t_0, x_0), \end{cases}$$

where $\partial_+ V_x(t_0, x_0)$ *denotes the superdifferential of* $V(t_0, \cdot)$ *at* x_0 *(see* [1], [15]). *Furthermore,*

$$\forall t \in]t_0, T], \ (H(\overline{x}(t), p(t)), -p(t)) \in \partial_+ V(t, \overline{x}(t)).$$

To prove the above, we need the following result.

Proposition 3.2. *Assume that H is differentiable. Then*

$$\frac{\partial H}{\partial p}(x,p) = \{f(x)+g(x)u \mid \langle p, f(x)+g(x)u\rangle - L(x,u) = H(x,p)\}$$

and

$$\frac{\partial H}{\partial x}(x,p) = \Big\{ f'(x)^\star p + (g'(x)u)^\star p - \frac{\partial L}{\partial x}(x,u) \mid \langle p, f(x)+g(x)u\rangle - L(x,u) = H(x,p)\Big\}.$$

See for instance [16] for the proof.

Proof. Fix $v \in \mathbb{R}^n$ and let $h_k \to 0+$, $v_k \to v$ be such that

$$\begin{aligned}\mathcal{D}_\downarrow V_x(t_0,x_0)(v) &:= \limsup_{h\to 0+, v'\to v} \frac{V(t_0, x_0+hv') - V(t_0,x_0)}{h}\\ &= \lim_{k\to\infty} \frac{V(t_0,x_0+h_k v_k) - V(t_0,x_0)}{h_k}.\end{aligned}$$

Consider the solution $x_k(\cdot)$ to the system

$$\begin{cases} x'(t) = f(x(t)) + g(x(t))\overline{u}(t),\\ x(t_0) = x_0 + h_k v_k \end{cases}$$

and define $\psi : [0,T]\times\mathbb{R}^n \mapsto \mathbb{R}^n$ by $\psi(t,x) = f(x)+g(x)\overline{u}(t)$. Then the sequence $(x_k - \overline{x})/h_k$ converge to the solution $w(\cdot)$ of the linear system

$$w'(t) = \frac{\partial \psi}{\partial x}(t,\overline{x}(t))w, \qquad w(t_0) = v. \tag{24}$$

Let $X(\cdot)$ denote the fundamental solution of

$$X'(t) = \frac{\partial\psi}{\partial x}(t,\overline{x}(t))X(t), \qquad X(t_0) = \mathrm{Id}\,.$$

Then $w(t) = X(t)v$ for all $t \in [t_0,T]$. Thus

$$\begin{aligned}&\mathcal{D}_\downarrow V_x(t_0,x_0)(v)\\ &\le \limsup_{k\to\infty} \frac{\int_{t_0}^T (L(x_k(t),\overline{u}(t)) - L(\overline{x}(t),\overline{u}(t)))dt + \varphi(x_k(T)) - \varphi(\overline{x}(T))}{h_k}\\ &= \Big\langle \int_{t_0}^T X(t)^\star \frac{\partial L}{\partial x}(\overline{x}(t),\overline{u}(t))dt + X(T)^\star \nabla\varphi(\overline{x}(T)), v\Big\rangle.\end{aligned}$$

Let $p(\cdot)$ denote the solution of the adjoint system

$$\begin{cases} -p' = \dfrac{\partial \psi}{\partial x}(t, \overline{x}(t))^* p - \dfrac{\partial L}{\partial x}(\overline{x}(t), \overline{u}(t)) \\ p(T) = -\nabla \varphi(\overline{x}(T)). \end{cases}$$

Consequently, for all $v \in \mathbb{R}^n$

$$\mathcal{D}_{\downarrow} V(t_0, x_0)(v) \leq \langle -p(t_0), v \rangle$$

and so $p(t_0) \in -\partial_+ V_x(t_0, x_0)$. By the maximum principle for a.e. $t \in [t_0, T]$,

$$\langle p(t), f(\overline{x}(t)) + g(\overline{x}(t))\overline{u}(t) \rangle - L(\overline{x}(t), \overline{u}(t)) = H(\overline{x}(t), p(t)).$$

Since H is differentiable we deduce from Proposition 3.2 that $(\overline{x}, p)$ solves the Hamiltonian system (23).

To prove the last claim fix $\tau \in]t_0, T]$. By Proposition 3.2

$$\langle p(\tau), \overline{x}'(\tau) \rangle - L(\overline{x}(\tau), \overline{u}(\tau)) = H(\overline{x}(\tau), p(\tau)) \quad \text{and} \quad \overline{x}'(\tau) = \psi(\tau, \overline{x}(\tau)).$$

Fix $v \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ and denote by x_h the solution to

$$x' = \psi(t, x), \qquad x(\tau) = \overline{x}(\tau) + hv.$$

By the variational equation, $(x_h - \overline{x})/h$ converge uniformly when $h \to 0+$ to the solution w of the linear system (24), with t_0 replaced by τ and therefore

$$\begin{aligned}
&\limsup_{h \to 0+} \frac{V(\tau + h\alpha, \overline{x}(\tau) + h(\alpha \overline{x}'(\tau) + v))) - V(\tau, \overline{x}(\tau))}{h} \\
&\quad = \limsup_{h \to 0+} \frac{V(\tau + h\alpha, \overline{x}(\tau + h\alpha) + hw(\tau + h\alpha)) - V(\tau, \overline{x}(\tau))}{h} \\
&\quad \leq \limsup_{h \to 0+} \frac{1}{h} \biggl(\varphi(x_h(T)) + \int_{\tau + h\alpha}^T L(x_h(s), \overline{u}(s)) ds - \varphi(\overline{x}(T)) \\
&\quad\quad - \int_\tau^T L(\overline{x}(s), \overline{u}(s)) ds \biggr) \\
&\quad = \langle \nabla \varphi(\overline{x}(T)), w(T) \rangle + \int_\tau^T \left\langle \frac{\partial L}{\partial x}(\overline{x}(s), \overline{u}(s)), w(s) \right\rangle ds - \alpha L(\overline{x}(\tau), \overline{u}(\tau)) \\
&\quad = \langle -p(\tau), v \rangle - \alpha L(\overline{x}(\tau), \overline{u}(\tau)) = \alpha H(\overline{x}(\tau), p(\tau)) + \langle -p(\tau), \alpha \overline{x}'(\tau) + v \rangle .
\end{aligned}$$

Since v, α are arbitrary the last statement follows.

Lemma 3.3. *Assume* (H$_1$)-(H$_5$) *and that* (P) *has a unique optimal trajectory* $z(\cdot)$. *Then the set*

$$\partial_x^\star V(t_0, x_0) := \operatorname*{Limsup}_{x_i \to x_0, t_i \to t_0} \left\{ \frac{\partial V}{\partial x}(t_i, x_i) \right\}$$

is a singleton. Consequently, $V(t_0, \cdot)$ *is differentiable at* x_0.

Proof. Let $\overline{p}_1, \overline{p}_2 \in \partial_x^\star V(t_0, x_0)$ and $(t_k^i, x_k^i) \to (t_0, x_0)$, $i = 1, 2$ be such that

$$\lim_{k \to \infty} \frac{\partial V}{\partial x}(t_k^i, x_k^i) = \overline{p}_i.$$

Consider an optimal solution-control pairs (z_k^i, u_k^i) of (P) with (t_0, x_0) replaced by (t_k^i, x_k^i). From Theorem 3.1 it follows that there exist absolutely continuous functions p_k^i such that, for all k and $i = 1, 2$, (z_k^i, p_k^i) solves the following problem

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(x(t), p(t)), & x(t_k^i) = x_k^i \\ -p'(t) = \dfrac{\partial H}{\partial x}(x(t), p(t)), & p(T) = -\nabla \varphi(z_k^i(T)), \; p(t_k^i) = -\dfrac{\partial V}{\partial x}(t_k^i, x_k^i). \end{cases}$$

We extend (z_k^i, p_k^i) on the time interval $[0, t_k^i]$ as the solution to the Hamiltonian system

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(x(t), p(t)), & x(t_k^i) = x_k^i \\ -p'(t) = \dfrac{\partial H}{\partial x}(x(t), p(t)), & p(t_k^i) = p_k^i(t_k^i). \end{cases}$$

By (H$_5$), (z_k^i, p_k^i) converge uniformly to the unique solution (z^i, p^i) of the Hamiltonian system

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(x(t), p(t)), & x(t_0) = x_0 \\ -p'(t) = \dfrac{\partial H}{\partial x}(x(t), p(t)), & p(t_0) = \overline{p}_i \end{cases}$$

for $i = 1, 2$.

We claim that z^i is optimal. Indeed,

$$V(t_k^i, x_k^i) = \varphi(z_k^i(T)) + \int_{t_k^i}^T L(z_k^i(s), u_k^i(s)) ds.$$

Set $u_k^i(s) = 0$ for all $s \in [0, t_k^i[$. Using (H$_3$) we deduce that the sequence u_k^i is bounded in $L^2(0, T)$. Taking a subsequence and keeping the same notations, we may assume that for $i = 1, 2$, $\{u_k^i\}$ converge weakly in $L^2(0, T; \mathbb{R}^m)$ to some $u^i \in L^2(0, T; \mathbb{R}^m)$. On the other hand, by (H$_3$)

$$\liminf_{k \to \infty} \int_{t_k^i}^T L(z_k^i(t), u_k^i(t)) dt + \varphi(z_k^i(T)) \geq \int_{t_0}^T L(z^i(t), u^i(t)) dt + \varphi(z^i(T)).$$

Since V is locally Lipschitz, (z^i, u^i) is optimal for (P). Thus, from the uniqueness of optimal trajectory we deduce that $z^1 = z^2 = z$. Consequently, p^i solves the Cauchy problem

$$p'(t) = -\frac{\partial H}{\partial x}(z(t), p(t)), \qquad p(T) = -\nabla\varphi(z(T)).$$

So, by uniqueness, $p^1(t_0) = p^2(t_0)$. The last statement of our lemma follows from [13] p. 33.

Proof of Theorem 2.1. Assume first that (i) holds true. Fix $0 \le t_0 < T$, $x_0 \in \mathbb{R}^n$ and let $\overline{x}$ be an optimal solution to problem (P). Then, by Theorem 3.1 there exists $p(\cdot)$ such that $(\overline{x}, p)$ solves the system

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(x(t), p(t)), & x(t_0) = x_0 \\ -p'(t) = \dfrac{\partial H}{\partial x}(x(t), p(t)), & ip(t_0) = -\dfrac{\partial V}{\partial x}(t_0, x_0). \end{cases}$$

Since the solution to such system is unique, we deduce (ii).

Conversely, assume that (ii) holds true. Fix $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$. Then, by Lemma 3.3 $\partial_x^\star V(t_0, x_0)$ is a singleton. We claim that the set

$$\partial^\star V(t_0, x_0) := \operatorname*{Limsup}_{(t,x)\to(t_0,x_0)} \{\nabla V(t, x)\}$$

is a singleton. Indeed let $(p_t, p_x) \in \partial^\star V(t_0, x_0)$ and the sequence $(t_i, x_i) \to (t_0, x_0)$ be such that $\nabla V(t_i, x_i) \to (p_t, p_x)$. Then $\{p_x\} = \partial_x^\star V(t_0, x_0)$ and it is classical that V satisfies at (t_i, x_i) the Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V}{\partial t}(t_i, x_i) + H\left(x_i, -\frac{\partial V}{\partial x}(t_i, x_i)\right) = 0.$$

Taking the limit we get

$$p_t = H(x_0, -p_x).$$

So p_t is uniquely defined. From [13], p. 33, V is differentiable at (t_0, x_0). Since $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ is arbitrary and $\partial^\star V(t_0, x_0)$ is singleton, we deduce that V is continuously differentiable on $[0, T] \times \mathbb{R}^n$.

Assume next that (iii) holds true. Fix $t \in [0, T]$ and let $x \in \mathbb{R}^n$ be such that $V(t, \cdot)$ is differentiable at x. By Theorem 3.1,

$$\left(x, \frac{\partial V}{\partial x}(t, x)\right) \in \text{Graph}(\pi_t). \tag{25}$$

Fix $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$, $p_x \in \partial_x^\star V(t_0, x_0)$ and let $(t_i, x_i) \to (t_0, x_0)$ be such that

$$\frac{\partial V}{\partial x}(t_i, x_i) \to p_x.$$

Let (z^i, p^i) denote the solution to

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(x(t), p(t)), & x(t_i) = x_i \\ -p'(t) = \dfrac{\partial H}{\partial x}(x(t), p(t)), & p(t_i) = -\dfrac{\partial V}{\partial x}(t_i, x_i). \end{cases}$$

From Theorem 3.1 we know that $p_i(T) = -\nabla\varphi(z_i(T))$. By ($H_5$), (z^i, p^i) converge to the unique solution (z, p) of the Hamiltonian system

$$\begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(x(t), p(t)), & x(t_0) = x_0, \\ -p'(t) = \dfrac{\partial H}{\partial x}(x(t), p(t)), & p(t_0) = -p_x \end{cases}$$

and $p(T) = -\nabla\varphi(z(T))$. We proved that

$$(x_0, \partial_x^\star V(t_0, x_0)) \subset \text{Graph}(\pi_{t_0}).$$

Thus $\partial_x^\star V(t_0, x_0)$ is a singleton. In the same way as before we deduce that V is continuously differentiable.

It remains to show that (i) yields (iii). For this aim fix $t_0 \in [0, T]$ and define the continuous mapping $\Psi : \mathbb{R}^n \mapsto \mathbb{R}^n$ in the following way:

For all $x_0 \in \mathbb{R}^n$ consider the solution (x, p) to the system

$$\text{(26)} \qquad \begin{cases} x'(t) = \dfrac{\partial H}{\partial p}(x(t), p(t)), & x(t_0) = x_0, \\ -p'(t) = \dfrac{\partial H}{\partial x}(x(t), p(t)), & p(t_0) = -\dfrac{\partial V}{\partial x}(t_0, x_0) \end{cases}$$

and set $\Psi(x_0) = x(T)$. By Theorem 3.1 we know that $p(T) = -\nabla\varphi(x(T))$. Thus $(x(T), p(T)) \in \text{Graph}(-\nabla\varphi)$. In particular this yields that Ψ is injective. By the Invariance of Domain Theorem $\Psi(\mathbb{R}^n)$ is open. Thus also the set

$$\{(\Psi(x_0), -\nabla\varphi(\Psi(x_0))) \mid x_0 \in \mathbb{R}^n\} \text{ is open in } \text{Graph}(-\nabla\varphi).$$

It is easy to check that the above set is also closed in Graph$(-\nabla\varphi)$. So it coincides with Graph$(-\nabla\varphi)$. Hence, by uniqueness of solution to the Hamiltonian system (17), $\text{Graph}(\pi_{t_0}) = \text{Graph}((\partial V/\partial x)(t_0, \cdot))$. The proof is complete.

Proof of Theorem 2.7. By Theorem 3.1 for all $\tau \in]t_0, T]$, $\partial_+ V(\tau, \overline{x}(\tau)) \neq \emptyset$. We claim that

$$\forall (p_t, p_x) \in \partial_+ V(\tau, \overline{x}(\tau)), \qquad -p_t + H(\overline{x}(\tau), -p_x) = 0.$$

Indeed let $u \in U$, $(p_t, p_x) \in \partial_+ V(\tau, \overline{x}(\tau))$. Consider the solution $y(\cdot)$ to the system

$$y' = f(y) + g(y)u, \qquad y(\tau) = \overline{x}(\tau).$$

By the dynamic programming principle

$$0 \le \limsup_{h\to 0+} \left[\frac{V(\tau+h, y(\tau+h)) - V(\tau,\overline{x}(\tau))}{h} + \frac{1}{h}\int_\tau^{\tau+h} L(y(s),u)ds\right]$$
$$\le p_t + \langle p_x, f(\overline{x}(\tau)) + g(\overline{x}(\tau))u\rangle + L(\overline{x}(\tau),u).$$

Because $u \in U$ is arbitrary, $-p_t + H(\overline{x}(\tau), -p_x) \le 0$. On the other hand

$$\frac{1}{h}\left(\overline{x}(\tau-h) - \overline{x}(\tau), \int_{\tau-h}^{\tau} L(\overline{x}(s),\overline{u}(s))ds\right) \to (-\overline{x}'(\tau), L(\overline{x}(\tau),\overline{u}(\tau))),$$

when $h \to 0+$. Thus

$$0 = \limsup_{h\to 0+} \frac{V(\tau-h,\overline{x}(\tau-h)) - V(\tau,\overline{x}(\tau)) - \int_{\tau-h}^{\tau} L(\overline{x}(s),\overline{u}(s))ds}{h}$$
$$\le -p_t + \langle -p_x, \overline{x}'(\tau))\rangle - L(\overline{x}(\tau),\overline{u}(\tau)) \le -p_t + H(\overline{x}(\tau), -p_x),$$

which proves our claim.

Since $H(x,\cdot)$ is strictly convex, then it follows from the above that for all $t > t_0$, $\partial_+ V(t,\overline{x}(t))$ is a singleton. Theorem 3.1 implies that for all $\tau \in]t_0, T]$, the optimal trajectory to (P) with t_0, x_0 replaced by τ, $\overline{x}(\tau)$ is unique. Hence, by Lemma 3.3, for all $\tau \in]t_0,T]$, $\partial_x^\star V(\tau,\overline{x}(\tau))$ is a singleton. Exactly as in the proof of Theorem 2.1 we deduce that $V(\cdot,\cdot)$ is differentiable at $(\tau,\overline{x}(\tau))$.

4. Appendix: Lipschitz continuity of the value function

We assume:

($\mathcal{H}_1$) f and g are locally Lipschitz and either

$$\exists M \ge 0,\ \forall x \in \mathbb{R}^n,\ \|f(x)\| \le M(\|x\|+1),\ \|g(x)\| \le M(\|x\|+1).$$

or that for some $C > 0$

$$\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m,\ L(x,u) \ge c\|x\|^2$$

and the system

$$x' = f(x)$$

is complete,

($\mathcal{H}_2$) $\liminf_{\|x\|\to\infty} \varphi(x) = +\infty$,

($\mathcal{H}_3$) $L(x,\cdot)$ is continuous, convex and for some $c > 0$

$$\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m,\ L(x,u) \ge c\|u\|^2.$$

Furthermore, for all $r > 0$, there exists $k_r \ge 0$ such that

$$\forall u \in \mathbb{R}^m,\ L(\cdot,u) \text{ is } k_r\text{-Lipschitz on } B_r(0),$$

($\mathcal{H}_4$) $f, g, L(\cdot,u)$ are differentiable and $\varphi \in C^1$,

($\mathcal{H}_5$) The Hamiltonian H is differentiable, its gradient $\nabla H(\cdot,\cdot)$ is locally Lipschitz and the Hamiltonian system (16) is complete.

Proposition 4.1. *If $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ hold true and φ is lower semicontinuous, then for every $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$ the problem* (P) *has an optimal solution.*

Proof. By $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ for every $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$ we have $V(t_0, x_0) < \infty$. Consider solution-control pairs (x_k, u_k) to (14) where $u_k \in L^1(t_0, T; \mathbb{R}^m)$ such that

$$\int_{t_0}^{T} L(x_k(t), u_k(t))dt + \varphi(x_k(T))$$

converge to infimum in (P). Then, by $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ the sequences $\{x_k(T)\}_{k\geq 1}$ and, therefore, $\{\|u_k\|_{L^2}\}_{k\geq 1}$ are bounded. Thus also $\{\|u_k\|_{L^1}\}_{k\geq 1}$ is bounded.

Hence, by $(\mathcal{H}_1)$ and Gronwall's lemma, $\{\|x_k\|_\infty\}_{k\geq 1}$ is bounded. Since x_k is the solution to (14) corresponding to u_k, we deduce that the sequence $\{x_k'\}_{k\geq 1}$ is bounded in $L^2(t_0, T; \mathbb{R}^n)$. Taking a subsequence and keeping the same notations, we may assume that u_k converge weakly in $L^2(t_0, T; \mathbb{R}^m)$ to some $u \in L^2(t_0, T; \mathbb{R}^m)$ and that x_k' converge weakly in $L^2(t_0, T; \mathbb{R}^n)$ to some $y \in L^2(t_0, T; \mathbb{R}^n)$. Define the absolutely continuous function $x(\cdot)$ by

$$\forall t \in [t_0, T], \; x(t) = x_0 + \int_{t_0}^{t} y(s)ds.$$

Then $x'(t) = y(t)$ almost everywhere in $[t_0, T]$. Furthermore, using that for all $t \in [t_0, T]$,

$$x_k(t) = x_0 + \int_{t_0}^{t} x_k'(s)ds$$

and that x_k' converge to y weakly in $L^2(t_0, T; \mathbb{R}^n)$ we deduce that

$$\forall t \in [t_0, T], \; \lim_{k\to\infty} x_k(t) = x(t). \tag{27}$$

Fix any $t_0 \leq t_1 \leq t_2 \leq T$. From the Hölder inequality,

$$\|x_k(t_2) - x_k(t_1)\| \leq \int_{t_1}^{t_2} \|x_k'(t)\|dt \leq \sup_{t_0\leq t\leq T} \|f(x_k(t))\|(t_2 - t_1)$$
$$+ \sup_{t_0\leq t\leq T} \|g(x_k(t))\| \times \|u_k\|_{L^2}\sqrt{t_2 - t_1}.$$

This implies that $\{x_k\}_{k\geq 1}$ is a family of equicontinuous functions and therefore x_k converge uniformly to x. Using that for all $t \in [t_0, T]$,

$$x_k(t) \;=\; x_0 + \int_{t_0}^{t} [f(x_k(s)) + g(x_k(s))u_k(s)]ds$$

and taking the limit, we finally obtain that for all $t \in [t_0, T]$,

$$x(t) = x_0 + \int_{t_0}^{t} [f(x(s)) + g(x(s))u(s)]ds.$$

Thus $x(\cdot)$ is a solution to (14). On the other hand, by $(\mathcal{H}_3)$

$$\liminf_{k\to\infty} \int_{t_0}^{T} L(x_k(t), u_k(t))dt + \varphi(x_k(T)) \geq \int_{t_0}^{T} L(x(t), u(t))dt + \varphi(x(T)).$$

Consequently (x, u) is an optimal solution-control pair.

Proposition 4.2. *If* $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ *hold true, then* V *is locally Lipschitz on* $[0,T]\times\mathbb{R}^n$.

Proof. Fix $(t_0,x_0)\in[0,T]\times\mathbb{R}^n$ and let $(z,\overline{u})$ be an optimal solution to (P). By Theorem 3.1, z is Lipschitz. We claim that for every $\varepsilon>0$ there exists $\overline{M}>0$ such that for all $(t_1,x_1)\in[0,T]\times\mathbb{R}^n$ with $(t_1-t_0)+\|x_0-x_1\|\le\varepsilon$, every optimal solution-control pair (z_1,u_1) of problem (P) with (t_0,x_0) replaced by (t_1,x_1), satisfies $\|u_1\|_{L^2}\le\overline{M}$.

Indeed fix $\varepsilon>0$, $(t_1,x_1),(z_1,u_1)$ as above.

CASE 1. $t_1\ge t_0$. Denote by x_2 the solution to

$$\begin{cases} x'(t)=f(x(t))+g(x(t))\overline{u}(t)\\ x(t_1)=x_1.\end{cases}\tag{28}$$

Then for some $l>0$ depending only on ε

$$\sup_{t\in[t_1,T]}\|x_2(t)-z(t)\|\le l\|x_1-z(t_1)\|.$$

Hence $\|x_2\|_\infty$ is bounded by a constant depending only on ε and therefore

$$\varphi(z_1(T))+\int_{t_1}^T L(z_1(t),u_1(t))dt\le\varphi(x_2(T))+\int_{t_1}^T L(x_2(t),\overline{u}(t))dt\le M(\varepsilon)$$

for some $M(\varepsilon)\ge 0$ independent of (t_1,x_1). Hence, by assumptions $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ also $\|z_1(T)\|$ is bounded by a constant depending only on $M(\varepsilon)$ and so does $\|\varphi(z_1(T))\|$. Consequently, by $(\mathcal{H}_3)$, $\|u_1\|_{L^2}^2\le\overline{M}(\varepsilon)$ for some $\overline{M}(\varepsilon)$ independent of (t_1,x_1).

CASE 2. $t_1<t_0$. We extend $\overline{u}$ on $[t_1,t_0]$ by setting $\overline{u}(t)=0$ for all $t\in[t_1,t_0]$. Denote by x_2 the solution to (28). Then

$$\varphi(z_1(T))+\int_{t_1}^T L(z_1(t),u_1(t))dt\le\varphi(x_2(T))+\int_{t_1}^T L(x_2(t),\overline{u}(t))dt.$$

Reasoning exactly as in Step 1, we end the proof of our claim. The above claim, $(\mathcal{H}_1)$ and Gronwall's inequality imply that for some $M_1(\varepsilon)>0$ depending only on ε, $\sup_{t\in[t_1,T]}\|z_1(t)\|\le M_1(\varepsilon)$.

Fix next $(t_1,x_1),(t_2,x_2)$ in $[0,T]\times\mathbb{R}^n$ such that $\|(t_i,x_i)-(t_0,x_0)\|\le\varepsilon$. It is not restrictive to assume that $V(t_1,x_1)\le V(t_2,x_2)$. Let $(\overline{x}_i,\overline{u}_i)$ be an optimal solution-control pair of (P) with (t_0,x_0) replaced by (t_i,x_i).

By Theorem 3.1 there exists $p_1\in C^1(t_1,T;\mathbb{R}^n)$ such that $(\overline{x}_1,p_1)$ solves the system

$$\begin{cases} x'(t)=\dfrac{\partial H}{\partial p}(x(t),p(t)), & x(t_1)=x_1\\ -p'(t)=\dfrac{\partial H}{\partial x}(x(t),p(t)), & p(T)=-\nabla\varphi(\overline{x}_1(T)).\end{cases}$$

From $(\mathcal{H}_5)$ we deduce that p_1 is bounded in $C(t_1, T; \mathbb{R}^n)$ by a constant depending only on $\|\overline{x}_1(T)\|$. Thus $\|p_1\|_\infty$ is bounded by a constant depending only on ε. But this yields that $\overline{x}_1(\cdot)$ is Lipschitz with the constant l depending only on ε.

We first assume that $t_2 \geq t_1$. Denote by x_3 the solution to

$$\begin{cases} x'(t) = f(x(t)) + g(x(t))\overline{u}_1(t) \\ x(t_2) = x_2. \end{cases} \tag{29}$$

Then for some l_1 depending only $\int_{t_2}^T \|\overline{u}_1(t)\|dt$ we have

$$\sup_{t\in[t_2,T]} \|x_3(t) - \overline{x}_1(t)\| \leq l_1\|x_2 - \overline{x}_1(t_2)\|.$$

Consequently,

$$\sup_{t\in[t_2,T]} \|x_3(t) - \overline{x}_1(t)\| \leq l_1(\|x_2 - x_1\| + l|t_2 - t_1|).$$

Thus, for some $L(\varepsilon) > 0$, $l(\varepsilon) > 0$ and $k > 0$

$$\begin{aligned} V(t_2, x_2) \leq & \varphi(x_3(T)) + \int_{t_2}^T L(x_3(t), \overline{u}_1(t))dt \\ \leq & \varphi(\overline{x}_1(T)) + \int_{t_1}^T L(\overline{x}_1(t), \overline{u}_1(t))dt \\ & + L(\varepsilon)\|x_3(T) - \overline{x}_1(T)\| + k\int_{t_2}^T \|x_3(t) - \overline{x}_1(t)\|dt \\ = & V(t_1, x_1) + l(\varepsilon)(\|x_2 - x_1\| + t_2 - t_1) \end{aligned}$$

and therefore, $0 \leq V(t_2, x_2) - V(t_1, x_1) \leq l(\varepsilon)(\|x_2 - x_1\| + t_2 - t_1)$.

It remains to consider the case $t_2 < t_1$. We extend $\overline{u}_1$ on $[t_2, t_1]$ by setting $\overline{u}_1(t) = 0$ for all $t \in [t_2, t_1]$. Denote by x_3 the solution to (29). Then $\|x_3\|_\infty$ is bounded by a constant depending only on ε and

$$\begin{aligned} V(t_2, x_2) \leq & \varphi(x_3(T)) + \int_{t_2}^T L(x_3(t), \overline{u}_1(t))dt \\ \leq & V(t_1, x_1) + \varphi(x_3(T)) - \varphi(\overline{x}_1(T)) + \int_{t_2}^{t_1} L(x_3(t), 0)dt \\ & + \int_{t_1}^T (L(x_3(t), \overline{u}_1(t)) - L(\overline{x}_1(t), \overline{u}_1(t)))dt. \end{aligned}$$

We next observe that

$$\|x_3(t_1) - x_2\| \leq L_1(t_1 - t_2)$$

for some L_1 depending only on ε and by the Gronwall inequality for some $l_2 > 0$, depending only on $\|\overline{u}_1\|_{L_1}$,

$$\sup_{t\in[t_1,T]} \|x_3(t) - \overline{x}_1(t)\| \leq l_2\|x_3(t_1) - x_1\|.$$

Consequently for some $M_1 > 0$ depending only on ε

$$V(t_2, x_2) \leq V(t_1, x_1) + M_1(\|x_2 - x_1\| + t_1 - t_2).$$

So we finally conclude that also in this case

$$0 \leq V(t_2, x_2) - V(t_1, x_1) \leq M_1(\|x_2 - x_1\| + |t_1 - t_2|).$$

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ON SOME SECOND-ORDER NECESSARY CONDITIONS FOR DIFFERENTIAL INCLUSION PROBLEMS

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1. Introduction

In this paper we study the following problem

$$\text{minimize}\, g(x(T))$$

over the solutions of the differential inclusion

$$x' \in F(t,x) \qquad \text{a.e. } ([0,T]),\ x(0) \in X_0,\ x(T) \in X_1.$$

First-order necessary optimality conditions for this problem are well known ([2], [3], [4], [5], [7], [8], [9], [10], etc.). An approach concerning second-order necessary optimality conditions has been proposed by Zheng ([11]) by reducing the problem to a finite-dimensional minimization problem and applying known optimality conditions ([1]). An important tool here is a study of the Clarke normal cone to the reachable set using proximal analysis.

The aim of the present paper is to obtain the same result, but under another constraint qualification concerning the optimal solution. The alternative constraint qualification proposed allows to improve the hypothesis concerning the multifunction $F(\cdot,\cdot)$. In our approach no convexity of $F(\cdot,\cdot)$ is required, which is a basic hypothesis in [11]. On the other hand our proof is easier than the one in [11], which is done first in the case of bounded differential inclusions

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and then, using a reduction method ([9]), for bounded differential inclusions. We directly obtain the result for unbounded differential inclusions.

Our constraint qualification is in terms of the adjoint of the variational inclusion associated to our problem

$$w'(t) \in C_{z'(t)}F(t, \cdot)(z(t), w(t)),$$

where $C_{z'(t)}F(t, \cdot)(z(t), \cdot)$ is the Clarke set-valued derivative of $F(t, \cdot)$ at point $(z(t), z'(t))$, instead of the hamiltonian inclusion

$$(-p', x') \in \delta H(t, x, p), \qquad H(t, x, p) = \sup_{e \in F(t,x)} \langle p, e \rangle$$

used in [11]. The price we pay in our approach is the assumption that the Clarke set-valued derivative of $F(t, \cdot)$ at $(z(t), z'(t))$ is Lipschitz.

It is well known that no satisfactory relationships between the two inclusion are known, but in particular cases it is more convenient to use the variational inclusion than the hamiltonian one ([7]). In general, even for smooth control systems, H is merely Lipschitz. Hence we are led to differentiate H in one generalized way or another; it isn't clear how we can solve the nonsmooth hamiltonian inclusion. At the same time in examples ([7]), the hamiltonian necessary conditions are less powerfull than that concerning the adjoint variational inclusion.

On the other hand it is known the open question raised by Clarke ([4]) concerning the validity of the hamiltonian first-order necessary conditions for problems involving nonconvex differential inclusions.

Moreover, our constraint qualification it is closely related to the "surjectivity conditions" of Frankowska ([5]), under which are obtained first-order (normal) necessary conditions for optimality.

The paper is organized as follows: in Section 2 we recall some preliminary results to be used in the next section and in Section 3 we prove our main result.

2. Preliminary results

Let $X \subset R^n$ and $x \in cl(X)$ (the closure of X).

Definition 2.1.

(a) the quasitangent cone to X at x is defined by

$$Q_x X = \{v \in R^n \mid \forall s_m \to 0+, \exists v_m \to v : x + s_m x_m \in X\},$$

(b) the second-order quasitangent set to X at x relative to $v \in Q_x X$ is defined by

$$Q^2_{(x,v)} = \{w \in R^n \mid \forall s_m \to 0+, \exists w_m \to w : x + s_m v + s_m^2 w_m \in X\},$$

(c) Clarke's tangent cone to X at x is defined by

$$C_x X = \left\{ v \in R^n \mid \forall (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists y_m \in X : \frac{y_m - x_m}{s_m} \to v \right\}.$$

For the basic properties of these sets we refer to [1].

We recall that two cones $C_1, C_2 \subset R^n$ are said to be *separable* if there exists $q \in R^n \backslash \{0\}$ such that

$$\langle q, v\rangle \le 0 \le \langle q, w\rangle \qquad \forall v \in C_1,\ w \in C_2.$$

We denote by C^+ the positive dual cone of $C \subset R^n$

$$C^+ = \{q \in R^n \mid \langle q, v\rangle \ge 0,\ \forall\, v \in C\}.$$

The negative dual cone of $C \subset R^n$ is $C^- = -C^+$.

For a mapping $g(\cdot) : X \subset R^n \to R$ which is not differentiable, the classical (Fréchet) derivative is replaced by some generalized directional derivatives. We use the following notations

$$D_{\uparrow} g(x; y) = \liminf_{(y',\theta)\to(y,0+)} \frac{g(x+\theta y') - g(x)}{\theta},$$

$$D^2_{\uparrow} g(x, y, z) = \liminf_{(z',\theta)\to(z,0+)} \frac{g(x+\theta y + \theta^2 z') - g(x) - \theta D_{\uparrow} g(x, y)}{\theta^2}.$$

When $g(\cdot)$ is of class C^2 one has

$$D_{\uparrow} g(x, y) = g'(x)^T y, \qquad D^2_{\uparrow} g(x, y, z) = g'(x)^T z + \frac{1}{2} y^T g''(x) y.$$

Theorem 2.2 ([11]). *Let $g : R^n \to R$ be Lipschitzean in some open set containing z and let S_1, S_2 be nonempty sets of R^n containing z. If z solves the following minimization problem*

$$\text{minimize}\, g(x) \ \textit{over all}\ x \in S_1 \cap S_2$$

and also satisfies the constraint qualification

$$(C_z S_1)^- \cap (C_z S_2)^+ = \{0\},$$

then we have the first-order necessary condition

$$D_{\uparrow} g(z, v) \ge 0 \qquad \forall v \in Q_z S_1 \cap Q_z S_2.$$

Furthermore, if equality holds for some v_0, then we have the second-order necessary condition

$$D_{\uparrow} g(z, v_0, w) \ge 0 \qquad \forall w \in Q^2_{(z,v_0)} S_1 \cap Q^2_{(z,v_0)} S_2.$$

Corresponding to each type of tangent cone, say $\tau_x X$, one may introduce a *set-valued directional derivative* of a multifunction $G(\cdot) : X \subset R^n \to \mathcal{P}(R^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in \text{graph}(G)$ as follows

$$\tau_y G(x, v) = \{w \in R^n \mid (v, w) \in \tau_{(x,y)} \text{graph}(G)\}, \qquad v \in \tau_x X.$$

Let $A : R^n \to \mathcal{P}(R^n)$ be a set-valued map. A is called *closed (respectively, convex) process* if graph$(A(\cdot))$ is a closed (respectively, convex) cone.

The adjoint process $A^* : R^n \to \mathcal{P}(R^n)$ of the closed convex process A is defined by

$$A^*(p) = \{q \in R^n \mid \langle q, v\rangle \leq \langle p, v'\rangle \ \forall (v, v') \in \text{graph}\, A(\cdot)\}$$

If $G(\cdot) : R^n \to \mathcal{P}(R^n)$ is a set-valued map, $I = [0, T]$ and $z(\cdot) \in AC(I, R^n)$ is an absolutely continuous map that satisfies $z'(t) \in G(z(t))$ a.e. $x(I)$ then the directional derivatives $Q_{z'(t)}G(z(t); \cdot)$ is closed process and $C_{z'(t)}G(z(t); \cdot)$ is closed convex process.

The second-order quasitangent derivative of G at (x, u) relative to $(y, v) \in Q_{(x,u)}(\text{graph}(G(\cdot))$ is a set-valued map $Q^2_{(u,v)}G(x, y, \cdot)$ defined similary by ([1])

$$\text{graph}\, Q^2_{(u,v)}G(x, y; \cdot) = Q^2_{((x,u)(y,v))}(\text{graph}\, G(\cdot)).$$

In what follows, we consider the following problem

$$\text{(2.1)} \quad \text{minimize}\{g(x(T)) \mid x'(t) \in F(t, x(t)) \text{ a.e. } (I),\ x(0) \in X_0,\ x(T) \in X_1\},$$

where $X_0, X_1, \subset R^n$ are given closed subsets, $g(\cdot) : R^n \to R$ is a given locally-Lipschitz map and $F(\cdot, \cdot)$ is a set-valued map from $I \times R^n$ into R^n.

Let $z(\cdot) \in AC(I, R^n)$ be a solution of (2.1). We assume that $F(\cdot, \cdot)$ satisfies the following hypothesis

Hypothesis 2.3.

(i) $\forall (t, x) \in I \times R^n$ $F(t, x)$ *is closed and nonempty.*
(ii) $\forall x \in R^n$ $F(\cdot, x)$ *is a measurable set-valued map.*
(iii) $\exists \varepsilon_0 > 0$, $L(\cdot) \in L^1(I, R)$ *such that for almost* $t \in I$, $F(t, \cdot)$ *is* $L(t)$-*Lipschitz on* $z(t) + \varepsilon_0 B$.

We shall denote by $S_F(\tau, t_0, X_0)$ the set of all absolutely continuous solutions of the differential inclusion

$$x' \in F(t, x) \qquad \text{a.e. } ([t_0, \tau])\ x(t_0) \in X_0$$

and we shall use the following notation for its reachable set

$$R_F(\tau, t_0, X_0) = \{x(\tau) \mid x(\cdot) \in S_F(\tau, t_0, X_0)\}.$$

A first-order approximation of the reachable set $R_F(T,0,X_0)$ at $z(T)$ has been studied in [7].

Theorem 2.4 ([7]). *Assume Hypothesis* 2.3 *and let* $R_1^Q(T)$ *be the reachable set of the differential inclusion:*

$$v'(t) \in Q_{z'(t)}(F(t,\,\cdot\,))(z(t);v(t)) \qquad a.e.\ (I),\ v(0) \in Q_{z(0)}X_0. \tag{2.2}$$

Then we have $R_1^Q(T) \subset Q_{z(T)}R_F(T,0,X_0)$.

A second-order approximation of the reachable set $R_F(T,0,X_0)$ at $z(T)$ relative to $\overline{y}(T) \in R_1^Q(T)$ has been disscused in [11].

Hypothesis 2.5. *Hypothesis* 2.3 *is satisfied and the integrable function* $L(\,\cdot\,)$ *in Hypothesis* 2.3 *can be chosen such that: for every* $v(\,\cdot\,) \in AC(I,R^n)$ *satisfying*

$$(v(t),v'(t)) \in Q_{(z(t),z'(t))}(\operatorname{graph} F(t,\,\cdot\,))$$

there exists a constant $a_0 > 0$ *such that*

$$d(z'(t)+av'(t),F(t,z(t)+av(t))) \le a^2L(t) \qquad \forall\ 0 \le a \le a_0.$$

Theorem 2.6 ([11]). *Assume Hypothesis* 2.5, *let* $\overline{v}(\,\cdot\,)$ *satisfy* (2.2) *and let* $R_2^Q(T)$ *be the reachable set of the differential inclusion*

$$w'(t) \in Q^2_{(z'(t),\overline{y}'(t))}F(t,z(t),\overline{v}(t);w(t)) \qquad a.e.\ (I)\ w(0) \in Q^2_{(z(T),\overline{y}(T))}X_0.$$

Then we have $R_2^Q(T) \subset Q^2_{(z(T),\overline{v}(T))}R_F(T,0,X_0)$.

3. The main results

Let $Q(x) = \{p(T) \mid (-p',x') \in \delta H(t,x,p),\ p(0) \in LN_{x(0)}X_0\}$, where $H(t,x,p) = \max\{\langle p,e\rangle \mid v \in F(t,x)\}$ is the Hamiltonian associated with the multifunction F, $\delta H(t,x,p)$ is the Clarke subgradient set of $H(t,\,\cdot\,,\,\cdot\,)$ with respect to (x,p) and LN_xX is the limiting proximal normal cone to X at x (cf. [9]).

When Hypothesis 2.5 is satisfied, F has convex images and is integrably sub-Lipschitz in large at every point $(t,z(t))$ in graph$(z(\,\cdot\,))$ (see [9], [11]) in [11] the following result is proved.

Theorem 3.1 ([11]). *Let* $z(\,\cdot\,)$ *be a solution to* (2.1) *and satisfy the constraint qualification*

$$\overline{co}Q(z) \cap (C_{z(T)}X_1)^+ = \{0\}. \tag{3.1}$$

Then we have the first-order necessary condition

$$D_\uparrow g(z(T),y(T)) \ge 0 \qquad \forall y(T) \in R_1^Q(T) \cap Q_{z(T)}X_1.$$

Furthermore, if equality holds for some $\overline{y}(T)$ then we have the second-order necessary condition

$$D_{\uparrow}g(z(T),\overline{y}(T),w(T)) \geq 0, \qquad \forall w(T) \in R_2^Q(T) \cap Q^2_{(z(T),\overline{y}(T))}X_1.$$

In what follows we obtain the same result, but under another constraint qualification. We need first another first-order approximation of the reachable set $R_F(T,0,X_0)$, similar to the one in Theorem 2.4, but obtained in terms of the variational inclusion defined by the Clarke directional derivative of the set-valued map.

The next result is most probably known, but in the absence of references we give here the proof.

Theorem 3.2. *Assume Hypothesis 2.3, let $C_0 \subset Q_{z(0)}X_0$ be a closed convex cone and let $R_1^C(T)$ be the reachable set of the differential inclusion*

$$w'(t) \in C_{z'(t)}F(t,\cdot)(z(t),w(t)) \qquad a.e.\ (I)\ w(0) \in C_0. \tag{3.2}$$

Then we have $R_1^C(T) \subset C_{z(T)}R_F(T,0,X_0)$.

Proof. Let $w \in R_1^C(T)$, $s_k \to 0+$, $z_k \to z(T)$, $z_k \in R_F(T,0,X_0)$. It follows that $z_k = z_k(T)$ with $z_k(\cdot) \in S_F(T,0,X_0)$.

Using the Lipschitzianity of $F(t,\cdot)$ one has

$$|z_k(t)-z(t)| \leq |z_k(T)-z(T)| + \left|\int_T^t L(s)|z_k(s)-z(s)|ds\right|.$$

According to the Bellman-Gronwall inequality we obtain

$$|z_k(t)-z(t)| \leq |z_k(T)-z(T)| \exp\left(\left|\int_0^T L(s)ds\right|\right)$$

and hence $z_k(\cdot) \to z(\cdot)$ in $C(I,R^n)$.

Since $|z_k'(t)-z'(t)| \leq L(t)|z_k(t)-z(t)|$, a.e. (I) we deduce that $z_k'(t) \to z'(t)$ a.e. (I).

On the other hand $w \in R_1^C(T)$; so $w = w(T)$ with $w(\cdot)$ solution to (3.2). We apply Theorem 8.4.1 in [1], page 322 (see also Lemma 2.9 in [6]) and we find that there exists $v_k(\cdot) \in L^1(I,R^n)$ such that

$$v_k(\cdot) \to w'(\cdot) \qquad \text{in } L^1(I,R^n),$$
$$z_k'(t) + s_k v_k(t) \in F(t, z_k(t) + s_k w(t)).$$

We define $y_k(t) = z_k(t) + s_k \int_0^t v_k(s)ds$ and note that we have

$$\begin{aligned} d(y_k'(t), F(t, y_k(t))) &= d\Big(z_k'(t) + s_k v_k(t), F(t, z_k(t) + s_k \int_0^t v_k(s)ds\Big) \\ &\leq d\Big(F(t, z(t) + s_k w(t)), F(t, z_k(t) + s_k \int_0^t v_k(s)ds\Big) \\ &\leq L(t) s_k \Big| w(t) - \int_0^t v_k(s)ds \Big|. \end{aligned}$$

Therefore, by the Lebesque dominated convergence theorem, one has

$$\lim_{k\to\infty} \frac{1}{s_k} \int_0^T d(y_k'(t), F(t, y_k(t)))dt = 0.$$

By Filippov's theorem (e.g. [6]) there exists $x_k(\cdot) \in AC(I, R^n)$ solution to $x' \in F(t, x)$ with $x_k(0) = z_k(0)$ and

$$\frac{|x_k(T) - y_k(T)|}{s_k} \leq \left(\exp \int_0^T L(s)ds\right) \frac{1}{s_k} \int_0^T d(y'k(t), F(t, y_k(t))dt \to 0$$

as $k \to \infty$ and thus $((x_k(T) - z_k(T))/s_k) \to w(T)$ and the proof is complete.

Remark 3.3. A related result is stated without proof in [6], namely any solution of (3.2) with $z(t) = z_0$ $\forall t \in I$ is contained in $C_{z_0} S_F(T, 0, z_0)$.

We are now able to prove our main result.

Theorem 3.4. *Let $X_0, X_1 \subset R^n$ be given nonempty closed sets, let $g(\cdot) : R^n \to R$ be a locally-Lipschitz function. Let $C_0 \subset Q_{z(0)} X_0$ be a closed convex cone and let $z(\cdot)$ be an optimal solution for problem (2.1) such that: Hypothesis 2.5 is satisfied, there exists $k(\cdot) \in L^\infty(I, R)$ such that $C_{z'(t)} F(t, \cdot)(z(t), \cdot)$ is $k(t)$-Lipschitz $\forall t \in I$ and the following constraint qualification is satisfied*

$$\begin{aligned} (3.3) \quad \{-q(T) \mid q(\cdot) \in W^{1,\infty}(I, R^n),\ q'(t) \in -(C_{z'(t)} F(t, \cdot)(z(t), \cdot))^* q(t), \\ q(0) \in C_0^+\} \cap (C_{z(T)} X_1)^+ = \{0\}. \end{aligned}$$

Then we have the first-order necessary condition

$$D_\uparrow g(z(T), y(T)) \geq 0 \qquad \forall y(T) \in R_1^Q(T) \cap Q_{z(T)} X_1.$$

Furthermore, if equality holds for some $\overline{y}(T)$, then we have the second-order necessary condition

$$D_\uparrow g(z(T), \overline{y}(T), w(T)) \geq 0 \qquad \forall w(T) \in R_2^Q(T) \cap Q^2_{(z(T), \overline{y}(T))} X_1.$$

Proof. We apply Theorem 2.2 with $S_1 = R_F(T, 0, X_0)$ and $S_2 = X_1$. According with Theorem 3.2 $R_1^C(T) \subset C_{z(T)} R_F(T, 0, X_0)$, hence

$$(3.4) \qquad (C_{z(T)} R_F(T, 0, X_0))^+ \subset (R_1^C(T))^+.$$

Since $C_{z'(t)}F(t,\cdot)(z(t),\cdot)$ is $k(t)$-Lipschitz $\forall t \in I$ with $k(\cdot) \in L^\infty(I,R)$ we apply Lemma 3.5 in [7] to find that

$$(3.5)\quad (R_1^C(T))^+ = \{q(T) \mid q(\cdot) \in W^{1,\infty}(I,R^n) \\ q'(t) \in -(C_{z'(t)}F(t,\cdot)(z(t),\cdot))^* q(t), q(0) \in C_0^+\}.$$

From (3.3), (3.4) and (3.5) we infer that

$$(C_{z(T)}R_F(T,0,X_0))^- \cap (C_{z(T)}X_1)^+ = \{0\}.$$

Finally, Theorems 2.2, 2.4 and 2.6 yields the required result.

Remark 3.5. It is known (Remark 4.10 in [7]) that if $A(\cdot,\cdot) : I \times R^n \to \mathcal{P}(R^n)$ is a closed convex process, $k(t)$-Lipschitz, with $k(\cdot) \in L^\infty(I,R)$, $A(t,v) \subset Q_{z'(t)}F(t,\cdot)(z(t),v)$, $\forall v \in R^n$ such that $\mathrm{Dom}\, A^*(t,\cdot)$ is a subspace of R^n and $A^*(t,\cdot)$ is linear on its domain (in particular, this assumption is satisfied by smooth control systems) then, if $q(\cdot)$ is a solution of the adjoint differential inclusion

$$(3.6)\qquad q'(t) \in -A^*(t,q(t)) \qquad q(0) \in C_0^+$$

then the map $q_1(t) = -q(t)$ satisfies

$$(3.7)\qquad (-q_1(t), z'(t)) \in \delta H(t, z(t), q_1(t)).$$

So, in this particular case our constraint qualification (3.3) is stronger than the constraint qualification (3.1).

It is also known that in more general cases we do not know to compare solution of (3.6) and (3.7).

Remark 3.6. Obviously, if there exists $w(\cdot) \in AC(I,R^n)$ solution of (3.2) such that $w(t) \in \mathrm{Int}(C_{z(T)}X_1)$, then (3.3) is satisfied. This condition is similar to the "surjectivity hypothesis" in [5], which requires the existence of a solution of the variational inclusion with its value at the final point T to belongs to the interior of the cone of interior directions to $R_F(t,0,X_0)$ at $z(T)$; assumption under which are obtained necessary optimality conditions in a normal form for problem (2.1).

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A UNIFIED TOPOLOGICAL POINT OF VIEW FOR INTEGRO-DIFFERENTIAL INCLUSIONS

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1. Introduction

In this paper we develop a fixed point approach which can be used in the field of integro-differential multivalued problems. Our point of view is more synthetic than the previous works in this direction (see for instance [5], [8], [11], [14], or [13]) and the embedding of these differential problems into the Theory of Index applied to Multivalued Condensing Operators (see [4]) is complete and new. In particular we had to build a specific measure of noncompactness Ψ in the space of continuous functions from the interval $[0,d]$ to the Banach space E.

The fixed point problems $P_E(f)$ considered in this work depend upon a multivalued operator f from $[0,d] \times E$ to E.

Our Theorem 1 ensures that the superposition operator F defined from f in the next section is u.s.c. Ψ-condensing with compact acyclic values while Theorem 2 states a general nonlinear averaging principle when some periodicity condition on f in its first variable is required.

We do not detail the proofs of Theorem 2 and various applications mentioned here and we refer the reader to [6] for this task.

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This paper is organized as follows. In Section 2 we give the fundamental notations and assumptions. The measure of noncompactness Ψ and the superposition operator F are constructed and studied in Section 3. The averaging principle and general examples of fixed point problems $P_E(f)$ taken into account in this paper can be found in Section 4.

2. Assumptions and notations

Let E be an arbitrary Banach space, $d > 0$ and f be a multivalued operator from $[0,d] \times E$ to E.

In the following $C([0,d],E)$ is the space of continuous functions from $[0,d]$ to E endowed with the supremum norm (denoted by $\|\cdot\|_\infty$). For all $K \subset E$ we denote by $\mathbb{L}^1_w([0,d],K)$ the set

$$\{g \in \mathbb{L}^1([0,d],E) \mid g(t) \in K, \text{ a.e. } t \in [0,d]\}, \tag{1}$$

endowed with the relative weak topology of $\mathbb{L}^1([0,d],E)$. If J is a subset of $[0,d]$ the notation 1_J stands for the characteristic function of J.

In the sequel we suppose that there exists an operator S from $\mathbb{L}^1([0,d],E)$ to $C([0,d],E)$ such that:

(a1) There is a constant $M > 0$ such that for all $g,h \in \mathbb{L}^1([0,d],E)$, the following inequality holds,

$$\|S(g)(t) - S(h)(t)\| \leq M \int_0^t \|g(\tau) - h(\tau)\| d\tau \qquad \text{with } 0 \leq t \leq d.$$

(a2) For every compact convex subset K of E, the operator S is sequentially continuous from $\mathbb{L}^1_w([0,d],K)$ to $C([0,d],E)$.

(a3) If S is not linear for all $g_0 \in \mathbb{L}^1([0,d],E)$ there is a continuous map ψ_{g_0} from $[0,d] \times C([0,d],E)$ to $C([0,d],E)$ satisfying

$$S(1_{[0,\theta]}g + 1_{[0,d]}g_0) = \psi_{g_0}(\theta, Sg)$$

for every $g \in \mathbb{L}^1([0,d],E)$ and $\theta \in [0,d]$.

Our assumptions, denoted by $\mathcal{F}$, with regard to f are :

(f1) The multivalued operator f maps $[0,d] \times E$ into compact convex subsets of E.

(f2) For all $x \in E$, $f(\cdot,x)$ has a strongly measurable selection on $[0,d]$.

(f3) For almost all $t \in [0,d]$, the map $x \longmapsto f(t,x)$ is u.s.c.

(f4) For every bounded $\Omega \subset E$ there exists $\nu \in \mathbb{L}^1([0,d],\mathbb{R})$ such that $\|f(t,x)\| = \sup\{\|y\| \mid y \in f(t,x)\} \leq \nu(t)$ for almost all $t \in [0,d]$ and all $x \in \Omega$. Also there exists $k \in \mathbb{L}^1([0,d],\mathbb{R})$, such that for all bounded subset Λ of E, the following relation holds,

$$\chi(f(t,\Lambda)) \leq k(t)\chi(\Lambda) \qquad t \in [0,d]. \tag{2}$$

Recall that the Hausdorff measure of noncompactness χ_Z of the linear normed space Z is defined on each bounded subset Ω of Z by

$$\chi(\Omega) = \inf\{\varepsilon > 0 \mid \Omega \text{ has a finite } \varepsilon\text{-net in } Z\}. \tag{3}$$

In the sequel we will write χ instead of χ_E.

We denote by $P_E(f)$ the problem of finding fixed points in $C([0,d],E)$ of the superposition operator from $C([0,d],E)$ into itself

$$F = S \circ \mathrm{sel}_f \tag{4}$$

where $\mathrm{sel}_f(x)$ is for all $x \in C([0,d],E)$ the (nonvoid) set of strongly measurable selections of $f(\cdot, x(\cdot))$.

A fundamental example of such an abstract problem, is provided by the following Cauchy problem $CP_E^{A(t)}(x^0,f)$ denoted by $CP_E^A(x^0,f)$ when the family $A(t)$ does not depend upon t,

$$CP_E^{A(t)}(x^0,f) = \begin{cases} \dot{x}(t) \in A(t)x(t) + f(t,x(t)), & \\ x(0) = x^0, & x^0 \in E,\ t \in [0,d]. \end{cases} \tag{5}$$

In this case the operator S will be the mild solution operator. In other words for all $g \in \mathbb{L}^1([0,d],E)$, $S(g)$ stands for the solution (which must be unique) of the problem $CP_E^{A(t)}(x^0,g)$. We will see later in Section 4 that this mild solution operator S satisfies our assumptions (a1), (a2) and (a3) for a wide class of families $(A(t))$ and spaces E.

3. Measures of noncompactness and condensing operators

3.1. A specific measure of noncompactness. We start with the following definition adapted from [1] and [4] where $\overline{\mathrm{co}}(\Omega)$ stands for the closed convex hull of Ω.

Definition 1. A function Φ defined on the set of all bounded subsets of the Banach space X with values in some partially ordered set $(Y,\leq)$ is called a *measure of noncompactness* if $\Phi(\overline{\mathrm{co}}(\Omega)) = \Phi((\Omega))$ for all bounded subsets $\Omega \subset X$.

Now we introduce the announced measure of noncompactness Ψ in $C([0,d],E)$, which plays a crucial role in this work. We set, for all bounded subsets Ω of $C([0,d],E)$,

$$\Psi(\Omega) = \max_{D \in \mathcal{D}(\Omega)} \Big(\sup_{t \in [0,d]} e^{-Lt}\chi(D(t)),\ \lim_{\delta \to 0} \sup_{x \in D} \max_{|t-s| \leq \delta} \|x(t) - x(s)\| \Big). \tag{6}$$

The notation $\mathcal{D}(\Omega)$ stands for the set of countable subsets of Ω, $D(t)$ is the set of the cross sections $x(t)$ for $x \in D$ and $L \geq 0$ is a constant that we shall appropriatly choose in the theorems. We see that Ψ has its values in the ordered space $\mathbb{R}^2$ with respect to the usual partial order.

First we remark that Ψ is well defined, namely, there is a countable subset $D_0 \subseteq \Omega$ satisfying

$$\Psi(\Omega) = (\sup_{t\in[0,d]} e^{-Lt}\chi(D_0(t)), \lim_{\delta\to 0}\sup_{x\in D_0}\max_{|t-s|\leq\delta} \|x(t) - x(s)\|). \tag{7}$$

Indeed the sup in $\mathbb{R}$ are countably obtained and a countable union of countable subsets is still countable.

Second we have to check that Ψ is a measure of noncompactness. Let Ω be a bounded subset of $C([0,d],E)$, let $D_0 \in \mathcal{D}(\Omega)$ be such that (7) holds and $D_1 \in \mathcal{D}(\overline{\text{co}}(\Omega))$ such that (7) holds with $\overline{\text{co}}(\Omega)$ in place of Ω. Then, we have $D_1 \subseteq \overline{\text{co}}(\Delta_0)$ for some $\Delta_0 \in \mathcal{D}(\Omega)$. If we denote by $\text{pr}_2(\Psi)$ the second coordinate of Ψ we can check easily $\text{pr}_2(\Psi(\overline{\text{co}}(\Omega))) = \text{pr}_2(\Psi(\Omega))$. Moreover for all $t \in [0,d]$ we have

$$\chi(D_1(t)) \leq \chi((\overline{\text{co}}(\Delta_0))(t)) \leq \chi(\overline{\text{co}}(\Delta_0(t))) = \chi(\Delta_0(t)).$$

And therefore, it comes $\Psi(\overline{\text{co}}(\Omega)) \leq \Psi(\Omega)$. But clearly, we have $\Psi(\overline{\text{co}}(\Omega)) \geq \Psi(\Omega)$, since possibly we can replace D_1 by $D_1 \cup D_0$. Then the equality $\Psi(\overline{\text{co}}(\Omega)) = \Psi(\Omega)$ holds for all bounded subset Ω of $C([0,d],E)$.

The additional properties below will allow us to apply the usual Fixed Point Toplogical Degree Theory (see [1] and [4]).

Proposition 1. *For all bounded subsets Ω, $\Omega_1 \subset C([0,d],E)$ we have:*

(i) *If $\Omega \subset \Omega_1$ then $\Psi(\Omega) \leq \Psi(\Omega_1)$.*
(ii) *If K is compact in $C([0,d],E)$ then $\Psi(\Omega \cup K) = \Psi(\Omega)$.*
(iii) *If $\Psi(\Omega) = 0$ then Ω is relatively compact in $C([0,d],E)$.*

Proof. Obvious. In particular (iii) is the Ascoli-Arzelà theorem. □

3.2. The solution concept. Let us recall that the superposition operator F has been defined by (4) in Section 2.

Definition 2. The fixed points of F are what we mean by *solutions* of $P_E(x^0, f)$.

For $0 < b \leq d$, we can define the solution notion on $[0,b]$ in the following way. If $g \in \mathbb{L}^1([0,d],E)$, $\rho_b(g)$ will be the restriction of g to $[0,b]$, and if $h \in \mathbb{L}^1([0,b],E)$ we will extend h on $[0,d]$, by setting $e_b(h)(t) = h(t)$ pour $t \in [0,b]$, at $e_b(h)(t) = 0$ pour $t \in]b,d]$. Let f_b be the restriction of f to $[0,b] \times E$. Then the solutions of $P_E(f_b)$ are the the fixed points in $C([0,b],E)$ of the operator $\rho_b \circ S \circ e_b \circ \text{sel}_{f_b}$. The set of solutions of $P_E(f_b)$ is denoted in the sequel by Σ_b^f.

3.3. Properties of the superposition operator. In order to study the property of the superposition operator F we will need the following definition.

Definition 3. Let Φ be a measure of noncompactness on the Banach space X. A multivalued operator H from X to X is said to be Φ-*condensing* if, for all bounded subset $\Omega \subset X$, the relation $\Phi(H(\Omega)) \geq \Phi(\Omega)$ implies that Ω is relatively compact in X.

Now, under our assumptions (a1), (a2), (a3) and (f1)-(f4), we are in position to state the main results of this section about the superposition operator F and the solution set of the abstract problem $P_E(f)$.

Theorem 1. *The multivalued operator F is u.s.c. with compact acyclic values and Ψ-condensing.*

As a consequence of this theorem we give the following abstract result.

Proposition 2. *In addition to assumption $\mathcal{F}$, suppose that Σ_d^f is a nonvoid bounded subset of $C([0,d],E)$ and that for all $b \in [0,d]$, we have $\rho_b(\Sigma_d^f) = \Sigma_b^f$, then* $\operatorname{ind}(\Sigma_d^f, F) = 1$, *where* ind *denotes the topological index.*

Before ending this subsection let us point out an interesting result which is a fundamental tool in the proof of Theorem 1. This result is an extension of Lemma 4 in [5].

Proposition 3. *Let $(g_n)_n$ be a sequence of functions in $\mathbb{L}^1([0,d],E)$. Assume that there exists q, $\mu \in \mathbb{L}^1([0,d])$ satisfying*

$$\sup_n \|g_n(t)\| \leq \mu(t) \quad \text{and} \quad \chi(\{g_n(t)\}_n) \leq q(t) \tag{8}$$

a.e. $t \in [0,d]$. Then we have for all $t \in [0,d]$

$$\chi(\{S(g_n)(t)\}_n) \leq 2M \int_0^t q(\tau)d\tau. \tag{9}$$

Remark 1. The factor 2 in the relation (9) can be dropped if E is separable.

3.4. Proofs of the results of the subsection 3.3.

Proof of Proposition 3. Clearly, from the Bochner integrability of the g_n there exist a separable subspace $Y \subseteq E$ and a subset $\Theta_0 \subset [0,d]$ of Lebesgue measure zero satisfying $\{g_n(t)\}_n \subseteq Y$, for all $t \in [0,d] \setminus \Theta_0$. Without loss of generality we may assume that the inequalities (8) hold for $t \in [0,d] \setminus \Theta_0$. Let $\varepsilon > 0$ and choose $\delta > 0$ such that for every measurable subset Θ of $[0,d]$ we have

$$|\Theta| \leq 2\delta \Longrightarrow \int_\Theta \mu(\tau)d\tau < \varepsilon. \tag{10}$$

where $|\Theta|$ stands for the Lebesgue measure of Θ. Choose also a constant M_1 satisfying $|\Theta_1| < \delta$ with

$$\Theta_1 = \{t \in [0,d] \mid \mu(t) > M_1\}.$$

So we have $\|g_n(t)\| \leq M_1$ for $n \in \mathbb{N}$ and $t \in [0,d] \setminus \Theta_0 \cup \Theta_1$.

From the second inequality in (8) for $t \in [0,d] \setminus \Theta_0$. It comes

$$\chi_Y(\{g_n(t)\}_n) \leq 2\chi(\{g_n(t)\}_n) \leq 2q(t).$$

The formula of the noncompactness measures for separable spaces (see [1]) yields

$$\chi_Y(\{g_n(t)\}_n) = \lim_{k\to\infty} \sup_n \operatorname{dist}(g_n(t), Y_k), \tag{11}$$

where for each integer k the set Y_k is a k-dimensional subspace of Y satisfying $Y_k \subseteq Y_{k+1}$ and $\overline{\bigcup_{k\in\mathbb{N}} Y_k} = Y$. It is clear that for $t \in [0,d] \setminus \Theta_0 \cup \Theta_1$ it is possible to take in the formula (11) the closed balls of Y_k denoted by $\overline{B_k} = \overline{B_k}(0, 2M_1)$ instead of the whole space Y_k.

Let us introduce the measurable functions α_n^k from $[0,d]$ to $[0,+\infty[$ defined by $\alpha_n^k(t) = \operatorname{dist}(g_n(t), \overline{B_k})$. Then the functions β_k from $[0,d]$ to $[0,+\infty[$ such that $\beta_k(t) = \sup_n \alpha_n^k(t)$ are measurable too.

Therefore, by virtue of the Egorov's Theorem there is a set $\Theta_2 \subseteq [0,d] \setminus \Theta_0 \cup \Theta_1$ with $|\Theta_2| < \delta$ and an integer k_0 such that we have $\beta_k(t) \leq 2q(t) + \varepsilon$, for $t \in [0,d] \setminus \bigcup_{j=0}^{j=2} \Theta_j$ and $k \geq k_0$. Thus it comes

$$\operatorname{dist}(g_n(t), \overline{B_k}) < 2q(t) + \varepsilon \tag{12}$$

for $n \in \mathbb{N}$, $t \in [0,d] \setminus \bigcup_{j=0}^{j=2} \Theta_j$ and $k \geq k_0$. Since the functions g_n are measurable on $[0,d]$ each of them is a pointwise limit of step functions. Hence there exist a set $\Theta_3 \subseteq [0,d]$ with $|\Theta_3| = 0$ and step functions h_n from $[0,d]$ to E satisfying

$$\|g_n(t) - h_n(t)\| < \varepsilon \tag{13}$$

for $n \in \mathbb{N}$, $t \in [0,d] \setminus \Theta_3$. Then the inequalities (12) and (13) provide

$$\operatorname{dist}(h_n(t), \overline{B_k}) < 2q(t) + 2\varepsilon$$

for $n \in \mathbb{N}$, $k \geq k_0$ and $t \in [0,d] \setminus \Theta$ with $\Theta = \bigcup_{j=0}^{j=3} \Theta_j$. In other words there exist step functions g_n^k from $[0,d]$ to $\overline{B_k}$ satisfying for the same n, t, and k,

$$\|g_n(t) - g_n^k(t)\| < 2q(t) + 3\varepsilon. \tag{14}$$

Notice that we have $|\Theta| < 2\delta$ and that without loss of generality we may assume $g_n^k(t) = 0$, whenever $t \in \Theta$.

Let us fix $k > k_0$. Then (see [7]) the sequence $(g_n^k)_n$ is relatively compact in $\mathbb{L}_w^1([0,d], \overline{B_k})$. Therefore, from the hypothesis (a2) the sequence $(S(g_n^k))_n$

is relatively compact in $C([0,d],E)$. Hence for every $t \in [0,d]$ the sequence $(S(g_n^k)(t))_n$ is relatively compact in E. Let us estimate $\|S(g_n)(t) - S(g_n^k)(t)\|$.

Thanks to the assumption (a1) we have

$$\|S(g_n)(t) - S(g_n^k)(t)\| \leq M \int_0^t \|g_n(\tau) - g_n^k(\tau)\| d\tau.$$

In view of the relations (14) and (10) this inequality gives

$$\begin{aligned}\|S(g_n)(t) - S(g_n^k)(t)\| &\leq M \int_{[0,t]\setminus\Theta} \|g_n(\tau) - g_n^k(\tau)\| d\tau \\ &\quad + M \int_{[0,t]\cap\Theta} \|g_n(\tau)\| d\tau \\ &\leq M \int_{[0,t]\setminus\Theta} (2q(\tau) + 3\varepsilon) d\tau + M\varepsilon \\ &\leq 2M \int_0^t q(\tau) d\tau + M\varepsilon(3d+1).\end{aligned}$$

Then, for $\gamma = 2M\int_0^t q(\tau)d\tau + M\varepsilon(3d+1)$, the relatively compact set $\{S(g_n^k)(t)\}_n$ forms a γ-net of the set $\{S(g_n)(t)\}_n$ proving the lemma due to the arbitrary choice of ε. □

Proof of Theorem 1. Theorem 1 will be a synthesis of the next lemmas 2, 5 and 6.

Lemma 1. *The operator* sel_f *is strongly-weakly closed from* $C([0,d],E)$ *to* $\mathbb{L}^1([0,d],E)$.

Proof. This lemma comes from the Masur's theorem and the assumptions (f1) and (f3). A complete proof can be found for instance in [10]. □

Lemma 2. *The multivalued operator F is u.s.c. with compact values.*

Proof. It sufficies to show that F is closed with compact values.

(a) First, let us prove that the multivalued operator F is closed. Let $(x_n)_n$ be a sequence converging towards x_∞ in $C([0,d],E)$, and let $h_n \in F(x_n)$ such that $(h_n)_n$ converges towards h_∞ in $C([0,d],E)$. We have $h_n = S(g_n)$ with g_n measurable and $g_n(t) \in f(t, x_n(t))$ a.e. $t \in [0,d]$. From the assumption (f4) on f and since $x_n([0,d])$ is bounded it comes that $g_n \in \mathbb{L}^1([0,d],E)$. Moreover, using again (f4), we see that the sequence $(g_n)_n$ is uniformly integrable on $[0,d]$ and that the set $\{g_n(t) \mid n \in \mathbb{N}\}$ is relatively compact in E. By the Diestel's theorem on the weak convergence in $\mathbb{L}^1([0,d],E)$ (see [7]) $(g_n)_n$ is relatively weakly compact in $\mathbb{L}^1([0,d],E)$. Let g_∞ a weak cluster value of $(g_n)_n$. From Lemma 1 it results

$$g_\infty \in \mathrm{sel}_f(x_\infty). \tag{15}$$

Now the assumption (a2) gives

$$h_\infty = S(g_\infty). \tag{16}$$

The relations (15) and (16) yield $h_\infty \in F(x_\infty)$.

(b) Prove now that the multivalued operator F has compact values. Let $x \in C([0,d],E)$. We are going to prove that the set $F(x)$ is sequentially compact in $C([0,d],E)$. In this goal let $(x_n)_n$ be a sequence in $F(x)$. Then we have $x_n = S(g_n)$ with g_n strongly measurable and $g_n(t) \in f(t,x(t))$ a.e. $t \in [0,d]$. As in the part (a) of this proof it comes $g_n \in \mathbb{L}^1([0,d],E)$ and there is a subsequence $(g_{n_k})_k$ such that $(g_{n_k})_k$ converges weakly in $\mathbb{L}^1([0,d],E)$ towards some $g_\infty \in \mathrm{sel}_f(x)$. It follows from (a2) that the sequence $(x_{n_k})_k$ converges towards $S(g_\infty)$. Since $S(g_\infty)$ belongs to $F(x)$, we have shown that $(x_n)_n$ has a convergent subsequence in $F(x)$. The proof is now complete. □

Definition 4. We will say that the sequence $(g_n)_n$ in $\mathbb{L}^1([0,d],E)$ is *semicompact* if the sequence $(g_n(t))_n$ is compact in E for almost all $t \in [0,d]$ and if there is a function $\mu \in \mathbb{L}^1([0,d])$ satisfying $\sup_n \|g_n\| \le \mu$ in $\mathbb{L}^1([0,d])$.

Remark 2. From the Diestel's theorem given in [7] it results that a semicompact sequence $(g_n)_n$ is weakly precompact in $\mathbb{L}^1([0,d],E)$.

Lemma 3. *Let $(g_n)_n$ be a semicompact sequence in $\mathbb{L}^1([0,d],E)$. Then, for every $\varepsilon > 0$ there are a compact subset $K_\varepsilon \subset E$ and an ε-net in $\mathbb{L}^1([0,d],E)$ of $\{g_n\}_n$ formed by functions with values in K_ε.*

Proof. Exactly as in the proof of Proposition 3 we can build functions g_n^k with values in compact subsets $\overline{B_k}$ such that the relation (14) holds with this time $q(t) = 0$ and $\varepsilon/(3d+1)$ instead of ε. Then, we easily verify that we have $\int_0^d \|g_n(\tau) - g_n^k(\tau)\| d\tau \le \varepsilon$. □

Lemma 4. *Let $(g_n)_n$ be a semicompact sequence in $\mathbb{L}^1([0,d],E)$. Then, the sequence $(S(g_n))_n$ is precompact in $C([0,d],E)$.*

Proof. Let $\varepsilon > 0$. By virtue of Lemma 3 there is a compact subset $K_\varepsilon \subset E$ and a function g_n^ε such that we have $\int_0^d \|g_n(\tau) - g_n^\varepsilon(\tau)\| d\tau \le \varepsilon$ for each $n \in \mathbb{N}$. Thanks to the condition (a2) and the Remark 2 the sequence $(S(g_n^\varepsilon))_n$ is precompact in $C([0,d],E)$. So using (a1) we see that the precompact subset $\{S(g_n^\varepsilon)\}_n$ is a $M\varepsilon$-net in $C([0,d],E)$ of the set $\{S(g_n)\}_n$. That ends the proof. □

Lemma 5. *The multivalued operator F is Ψ-condensing.*

Proof. Let

$$C(L) = \sup_{t \in [0,d]} e^{-Lt} \int_0^t k(\tau) e^{L\tau} d\tau.$$

Since we have $\lim_{L\to+\infty} C(L) = 0$, it is possible to choose L such that

$$2MC(L) < 1. \tag{17}$$

Let $\Omega \subset C([0,d],E)$ be a bounded subset of $C([0,d],E)$ satisfying

$$\Psi(F(\Omega)) \geq \Psi(\Omega). \tag{18}$$

We have to prove that Ω is relatively compact in $C([0,d],E)$. Remark that $F(\Omega)$ is bounded in $C([0,d],E)$ and that for each countable subset $D' \subseteq F(\Omega)$ there is a countable subset $D'' \subseteq \Omega$ satisfying $D' = F(D'')$. Consequently there is a bounded countable subset $D_0'' \subseteq \Omega$ such that $F(D_0'')$ achieves the maximum in the definition of $\Psi(F(\Omega))$. For all countable subset Γ of $C([0,d],E)$ denote by $\operatorname{cont}(\Gamma)$ the following expression

$$\operatorname{cont}\Gamma = \lim_{\delta\to 0} \sup_{x\in\Gamma} \max_{|t-s|\leq\delta} \|x(t) - x(s)\|.$$

Then the relation (18) becomes

$$\begin{aligned} \sup_{t\in[0,d]} e^{-Lt}\chi(D(t)) &\leq \sup_{t\in[0,d]} e^{-Lt}\chi(F(D_0''(t))) \\ \operatorname{cont}(D) &\leq \operatorname{cont}(F(D_0''(t))) \end{aligned} \tag{19}$$

for all $D \in \mathcal{D}(\Omega)$. From assumption (f4) it follows

$$\begin{aligned} \chi(\mathrm{sel}_f(D_0'')(\tau)) &\leq k(\tau)\chi(D_0''(\tau)) \\ &\leq k(\tau)e^{L\tau} \sup_{\tau\in[0,d]} e^{-L\tau}\chi(D_0''(\tau)). \end{aligned} \tag{20}$$

Now according to (20) Proposition 3 applies and yields for $t \in [0,d]$,

$$\begin{aligned} e^{-Lt}\chi(F(D_0''(t))) &\leq 2Me^{-Lt}\int_0^t k(\tau)e^{L\tau}d\tau \sup_{\tau\in[0,d]} e^{-L\tau}\chi(D_0''(\tau)) \\ &\leq 2MC(L) \sup_{\tau\in[0,d]} e^{-L\tau}\chi(D_0''(\tau)). \end{aligned}$$

This last inequality combined with (17) and the first relation of (19) with D'' instead of D gives

$$\chi(D_0''(\tau)) = 0 \tag{21}$$

for all $\tau \in [0,d]$. From (f4) and the relation (21) it results $\chi(f(\tau, D_0''(\tau))) = 0$ a.e. $\tau \in [0,d]$. Hence $\mathrm{sel}_f\, D_0''$ is semicompact. Therefore, by Lemma 4, $F(D_0'')$ is precompact in $C([0,d],E)$. Then, we deduce $\Psi(F(\Omega)) = 0$ and in view of (18) $\Psi(\Omega) = 0$, that is Ω is relatively compact in $C([0,d],E)$. □

Lemma 6. *The values taken by the multivalued operator F are contractile and thus acyclic.*

Proof. Let $x \in C([0,d],E)$ and $y_0 \in F(x)$. Then we have $y_0 = S(g_0)$ for some $g_0 \in \mathrm{sel}_f(x)$. The assumption (a3) shows that the map H

$$(\lambda, y) \longmapsto \psi_{g_0}((1-\lambda)d, y) = H(\lambda, y)$$

from $[0,1] \times F(x)$ to $C([0,d],E)$ is continuous and satisfies

$$H(0,y) = y \quad \text{and} \quad H(1,y) = y_0.$$

But writing $y = S(g)$ for $g \in \mathrm{sel}_f(x)$, and using again (a3) we see that we have

$$H(\lambda, y) = S(1_{[0,(1-\lambda)d]}g + 1_{[(1-\lambda)d,d]}g_0)$$

and that $h = 1_{[0,(1-\lambda)d]}g + 1_{[(1-\lambda)d,d]}g_0$ belongs to $\mathrm{sel}_f(x)$. Therefore, $H(\lambda, y) \in F(x)$. In other words H is a retraction from $F(x)$ to $\{y_0\}$. Consequently, $F(x)$ is contractile and thus acyclic. □

Proof of Proposition 2. Let

$$W = \{(t,v) \in [0,d] \times E \mid v = x(t) \text{ with } x \in \Sigma_d^f\}.$$

For $\eta > 0$ let $V_\eta = [0,d] \times (\mathrm{pr}_2 W + \eta\mathcal{B})$, where $\mathcal{B} = \mathcal{B}(0,1)$ is the open ball of center 0 and radius 1 in E and pr_2 is the second projection. Let $\varepsilon > 0$ and

$$\mu(t,v) = \begin{cases} 1 & \text{if } (t,v) \in \overline{V_\varepsilon} \\ 0 & \text{if } (t,v) \notin \overline{V_{2\varepsilon}}. \end{cases}$$

With the help of the Dugundgi's Theorem extend μ to a continuous function from $[0,d] \times E$ to $[0,1]$. We can easily verify that the operator $\widetilde{f}$ from $[0,d] \times E$ to E defined by

$$\widetilde{f}(t,v) = \mu(t,v)f(t,v)$$

satisfies the assumptions (f1)-(f4). Then by setting

$$\widetilde{F} = S \circ \mathrm{sel}_{\widetilde{f}}$$

we get from Theorem 1 that $\widetilde{F}$ is u.s.c. Ψ-condensing with compact acyclic values in $C([0,d],E)$. Since $\overline{V_{2\varepsilon}}$ is bounded from (f4) and (a1) there is a constant $C > 0$ such that

$$||S(0)|| + \int_0^d ||\widetilde{g}(\tau)||d\tau \leq C \tag{22}$$

for all $\widetilde{g} \in \mathrm{sel}_{\widetilde{f}}(x)$ and all $x \in C([0,d],E)$. From (a1) it is clear that the relation (22) insures $||\widetilde{F}(x)||_\infty \leq C$ for all $x \in C([0,d],E)$. Let $C_1 > C$ such that the compact set Σ_d^f is included in the open ball $B_1 = B(0,C_1)$ of center 0 and radius

C_1 in $C([0,d],E)$. This choice insures that $\widetilde{F}$ maps B_1 into itself and has no fixed point on the boundary of B_1. So one has

$$\mathrm{Ind}_{B_1}\widetilde{F}=1. \tag{23}$$

This last equality comes from fundamental properties of the topological index. Thanks to (23) it remains to prove $\mathrm{Ind}_{B_1}\widetilde{F}=\mathrm{Ind}_{B_1}F$.

In this goal it suffices to prove $\mathrm{Fix}\,\widetilde{F}=\mathrm{Fix}\,F(=\Sigma_d^f)$, where the symbol Fix stands for the set of fixed points. Of course we have $\mathrm{Fix}\,F\subseteq\mathrm{Fix}\,\widetilde{F}$. We are going to prove the converse. Let $x\in\mathrm{Fix}\,\widetilde{F}$. Then applying (a1) and taking into account the relations (22), (23) and the definition of $\widetilde{f}$ we can find some $b\in\,]0,d]$ such that the restriction $\rho_b(x)$ belongs to Σ_b^f. So by hypothesis there is $x_b\in\Sigma_d^f$ satisfying $\rho_b(x)=\rho_b(x_b)$. Define

$$\Theta=\{b\in[0,d]\mid\rho_\beta(x)=\rho_\beta(x_\beta)\text{ for all }\beta\in[0,b]\}.$$

Clearly, Θ is a nonempty open set of $[0,d]$. Thanks to the compactness of Σ_d^f in $C([0,d],E)$ the set Θ is closed in $[0,d]$. Then we can conclude $\Theta=[0,d]$ and $x=x_d$ for some $x_d\in\Sigma_d^f$. That ends the proof. □

4. Examples

This paragraph does not contains any proof. In particular the proof of Theorem 2 is too much long to be given in this short paper. The reader is refered to [6] for the proofs.

4.1. The Cauchy problems $CP_E^A(x^0,f)$. Let A be a multivalued unbounded operator from E to E. We consider the mild solution operator S previously defined. In this subsection we give general examples of operators A and spaces E such that the mild solution operator S satisfies the assumptions (a1), (a2) and (a3). These examples include with some improvements the cases set out in [5], [8], or [14]. We notice that the assumption (a1) appears as a weak form of the classical Benilan's integral inequalities (see [3] or [2]) and it is always satisfied whenever $A-\omega I$ is dissipative. We emphasize that in our presentation, the assumption (a1) is invariant under a change of the norm into an equivalent norm. The assumption (a3) is also always satisfied when $A-\omega I$ is m-dissipative, since in this event we have $\psi_{g_0}(\theta,S(g))=S(g)$ on $[0,\theta]$ (really this is a general fact from (a1)) and since $\psi_{g_0}(\theta,S(g))$ restricted to $[\theta,d]$ is the mild solution of the following Cauchy problem

$$\begin{aligned}&\dot{x}(t)\in Ax(t)+g_0(t),\\&x(0)=S(g)(\theta),\qquad t\in[\theta,d].\end{aligned}$$

We can check easily that the continuity with respect to the initial data for such a problem insures the required continuity of ψ_{g_0}. Thus, we must concentrate our attention on the condition (a2).

The following proposition provides general classes of couples (A, E) such that the mild solution operator S satisfies (a1), (a2) and (a3).

Proposition 4. *Suppose that the unbounded operator $A-\omega I$ is m-dissipative in the Banach space E. Then the mild solution operator S satisfies* (a1), (a2) *and* (a3) *in each of the following cases.*

(i) *We have a decomposition $A = A_1 + B$, where A_1 generates a strongly continuous semigroup of bounded linear operators and B is a single valued locally Lipschitz operator.*
(ii) *The space E is smooth and A generates a compact semigroup.*
(iii) *The space E is uniformly smooth and A generates a strongly equicontinuous semigroup.*
(iv) *The space E is smooth reflexive and such that the (normalized) duality mapping is sequentially weak-weak* continuous. In addition, A generates a weakly equicontinuous semigroup of bounded (nonlinear) operators.*

Remark 3. The previous proposition contains some improvements of known results about the Cauchy problem. For instance in [5] the existence of solutions for the problem $CP_E^A(x^0, f)$ when A generates a compact semigroup was proved in the case where the space E is strictly convex.

4.2 Non autonomous semilinear problems. This subsection deals with the general semilinear problems $CP_E^{A(t)}(x^0, f)$ where the family of linear unbounded operators $(A(t))_{t\in[0,d]}$ generates an evolution operator U in the following sense (with $\Delta = \{(t,s) \in [0,d]^2 \mid s \leq t\}$):

(U1) For each $(t,s) \in \Delta$ the operator $U(t,s)$ is linear and bounded from E to E.
(U2) For all $x \in E$ the function $(t,s) \longmapsto U(t,s)x$ is continuous in Δ.
(U3) For all $(t,s),(s,r) \in \Delta$, the relations $U(t,s) \circ U(s,r) = U(t,r)$ and $U(t,t) = I$ hold.

The operator S defined for all $t \in [0,d]$ and $g \in \mathbb{L}^1([0,d], E)$ by the formula

$$S(g)(t) = U(t,0)x^0 + \int_0^t U(t,\tau)g(\tau)d\tau$$

enjoys the properties (a1), (a2) and (a3) of the Section 2.

Remark 4. By applying Theorem 1 we obtain a straightforward extension of Theorem 3 in [13] and of a lot of classical results on semilinear evolution problems (see [11])

4.3. Multivalued Volterra problems. The previous subsection is a particular case of the more general problems existence of solutions of the following inclusion

$$x(t) \in L(t) + \int_0^t k(t, \tau, f(\tau, x(\tau)))d\tau, \qquad t \in [0, d],$$

where L is a continuous function from $[0, d]$ to E and the kernel k is a single-valued function from $\Delta \times E$ to E, where $\Delta = \{(t, s) \in [0, d]^2 \mid s \leq t\}$.

For instance suppose that the following conditions hold.

(V1) The operator k is continuous in its first variable;
(V2) The function $\tau \longmapsto k(t, \tau, g(\tau))$ is integrable on $[0, t]$ for each $t \in [0, d]$ and each $g \in \mathbb{L}^1([0, d], E)$;
(V3) $\|k(t, \tau, y) - k(t, \tau, z)\| \leq M\|y - z\|$ for $(t, \tau) \in \Delta$, $y \in E$;
(V4) For every compact subset K of E there is a positive function $\mu \in \mathbb{L}^1([0, d])$ such that for all $t \in [0, d]$, and all $z \in K$, we have $\|k(t, \tau, z)\| \leq \mu(\tau)$ a.e. $\tau \in [0, d]$.

By taking

$$S(g)(t) = L(t) + \int_0^t k(t, \tau, g(\tau))d\tau$$

for $t \in [0, d]$ and $g \in \mathbb{L}^1([0, d], E)$ we see easily that the assumption (a1) holds. We can prove that the assumption (a2) is satisfied if and only if S is sequentially closed from $\mathbb{L}^1_w([0, d], K)$ to $C([0, d], E)$ for every compact subset $K \subset E$. Lastly, an abstract version of (a3) (see [6]) is suitable for particular classes of kernels.

4.4. Time dependent subdifferentials. In this subsection we deal with nonlinear evolution equations of the form $CP_E^{A(t)}(x^0, f)$ in a real Hilbert space E, where $A(t) = -\partial\phi(t, \cdot)$ is for almost all $t \in [0, d]$ and for $t = 0$, the subdifferential of a proper lower semicontinuous convex function from X into $]-\infty, +\infty]$. We assume that x^0 belongs to the closure in E of the domain D_0 of $\partial\phi(0, x^0)$. We suppose that the Yotsutani conditions hold (see [15]). Then it was shown in [15] that for all $g \in \mathbb{L}^1([0, d], E)$, the problem $CP_E^{A(t)}(x^0, f)$ has a unique (strong) solution $S^{x^0}(g)$. It is obvious that S^{x^0} satisfies (a1) and (a3). Since for all compact subset K of E, estimations in [15] or [12] show that the set $S^{x^0}(\mathbb{L}^1([0, d], K))$ is equicontinuous in $C([0, d], E)$ if $x^0 \in D_0$. It is now easy to see that (a2) holds.

Remark 5. Contrary to [12], we do not assume ϕ of compact type. However by using Theorem 1 we obtain immediately that the solution set of $CP_E^{A(t)}(x^0, f)$ is non empty and compact.

4.5. Statement of a general nonlinear averaging principle. In Theorem 2 we need stronger assumptions about f denoted by $\mathcal{F}'$, namely:

(f1') The multivalued operator f from $\mathbb{R}^+ \times E$ to E is T-periodic with respect to its first variable for some $T > 0$.

(f2') The restriction of f to $[0, T] \times E$ satisfies (f1) and (f2) with T in place of d.

(f3') For all $x_0 \in E$ and all $\varepsilon > 0$, there exists $\eta > 0$ such that $\|x - x_0\| \leq \eta$ implies, $f(t, x) \subset f(t, x_0) + \varepsilon B_E$ for almost all $t \geq 0$, where B_E is the unit ball of E.

(f4') There is $k \in \mathbb{R}$ such that for all bounded subsets Λ of E we have, $\chi(f([0, T] \times \Lambda)) \leq k\chi(\Lambda)$.

We define for $\varepsilon > 0$ and $x \in E$,

$$f_\varepsilon(t, x) = f\left(\frac{t}{\varepsilon}, x\right) \quad \text{and} \quad f_0(x) = \left\{\frac{1}{T}\int_0^T g(\tau)d\tau \mid g \in \mathrm{sel}_f\, x\right\}.$$

Then, we obtain the following principle.

Theorem 2. *Suppose that $\mathcal{F}'$ holds. Assume that $\Sigma_d^{f_0}$ is a nonvoid bounded subset of $C([0, d], E)$ and that for all $b \in [0, d]$, we have $\rho_b(\Sigma_d^{f_0}) = \Sigma_b^{f_0}$. Then, for all $\delta > 0$ there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$ the solution set $\Sigma_d^{f_\varepsilon}$ is non empty and included in $\Sigma_d^{f_0} + \delta B$, where B is the unit ball in $C([0, d], E)$.*

Remark 6. The conditions about $\Sigma_d^{f_0}$ in Theorem 2 as well as in Proposition 2 (with f in place of f_0) are fullfilled for instance if there exists some $\mu \in \mathbb{L}^1([0, d], \mathbb{R})$ satisfying: $\|f_0(t, x)\| \leq \mu(t)(1 + \|x\|)$ a.e. $t \in [0, d]$, $\forall x \in E$. We underline that in Theorem 2 these assumptions are only related to f_0 and not to f_ε for $\varepsilon > 0$.

This last statement is an abstract extension of the N. N. Bogulubov analogous result on classical ordinary differential equations. The classical methods do not apply in our framework since we do not assume the uniqueness of the solutions and the validity of the Duhamel's formula. Other applications of the Topological Degree Theory to the periodic solution problem were given in [9] and [10]. In those papers A were a linear operator.

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THE BORSUK — ULAM THEOREM FOR APPROXIMABLE MAPS

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1. Introduction

Let S^n denote the unit sphere in the Euclidean space R^{n+1}. The famous Borsuk-Ulam theorem states that for every continuous map $f : S^n \to R^n$ there exists a point $x \in S^n$ such that $f(x) = f(-x)$. It has been proved by Borsuk over sixty years ago. In 1985 H. Steinlein noted in his survey [S] 457 papers dealing with various generalizations and applications of that theorem. After that many new papers have appeared in the literature. In particular, there are generalizations of the Borsuk's result to various classes of multivalued maps and more general group actions than the antipodal one (see e.g. [GG], [I]). They applied mainly some algebraic topology tools like homology.

One of the most natural and easy techniques of extending theorems from single-valued maps to multivalued maps is the graph approximation approach. It has been initiated by J. von Neumann in the thirties and then studied by many authors (see [G] and the references therein). Let us say that at least convex-valued u.s.c. maps and their compositions admit arbitrary close graph approximations.

In this note we prove a modest version of Munkholm's result [M]. He gave a lower estimate for a covering dimension of the coincidence set for $f : S^{2n-1} \to R^m$ with respect to a free action of a cyclic group Z_{p^α} (see Theorem 3.1). We were

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able to prove that for an approximable multivalued map the coincidence set is at least nonempty, or infinite, if n is sufficiently large with respect to m (see Theorem 3.2 and Corollary 3.3).

Let us notice that using different methods and the result of [V] we have obtained another generalization of Borsuk-Ulam theorem, more in the spirit of [B].

2. Multivalued maps

Let X and Y be two topological spaces and assume that for every point $x \in X$ a non-empty closed subset $\varphi(x)$ of Y is given. In this case we say that $\varphi : X \to Y$ is a *multivalued map* from X to Y.

We associate with φ the *graph* Γ_φ by putting:

$$\Gamma_\varphi = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}.$$

If $A \subset X$ is a subset, then the *image* of A is the set $\varphi(A) = \bigcup_{a \in A} \varphi(a)$.

For a subset $B \subset Y$ we can define two types of a *counter-image*:

$$\varphi^{-1}(B) = \{x \in X \mid \varphi(x) \subset B\},$$
$$\varphi_+^{-1}(B) = \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}.$$

If $\varphi = f$ is single-valued, then $\varphi^{-1}(B)$ and $\varphi_+^{-1}(B)$ coincide.

A multivalued map $\varphi : X \to Y$ is *upper semicontinuous* (*u.s.c.*) provided for each open subset $B \subset Y$ the small counter-image $\varphi^{-1}(B)$ is an open subset of X; φ is *lower semicontinuous* (*l.s.c.*) provided for any open subset $B \subset Y$ the big counter-image $\varphi_+^{-1}(B)$ is open in X.

We shall concentrate here on u.s.c. maps. They have many properties analogous to continuous maps.

Proposition 2.1. *Let $\varphi : X \to Y$ be an u.s.c. map. Then the graph Γ_φ of φ is a closed subset of $X \times Y$. Moreover, if the values of φ are compact sets, then the image $\varphi(A)$ of a compact set A is compact.*

For other general properties of multivalued maps see e.g. in [AC].

One of the most useful techniques in the theory of multivalued maps is the idea of single-valued approximations.

Let A be a subset of a metric space (X, d_X) and let $\varepsilon > 0$. By *ε-neighbourhood* of A in X we denote the set

$$O_\varepsilon(A) = \{x \in X \mid \exists y \in X \; d_X(x, y) < \varepsilon\}.$$

If (X, d_X) and (Y, d_Y) are two metric spaces, then in the Cartesian product $X \times Y$ we consider the max-metric, i.e.

$$d_{X\times Y}((x,y),(u,v)) = \max\{d_X(x,u), d_Y(y,v)\}$$

for $x, u \in X$ and $y, v \in Y$.

Definition 3.1. Let $\varphi : X \to Y$ be a multivalued map and let $\varepsilon > 0$. A continuous map $f : X \to Y$ is called an *ε-approximation* (*on the graph*) of φ if

$$\Gamma_f \subset O_\varepsilon(\Gamma_\varphi).$$

The following is an easy excercise.

Proposition 2.2. *A continuous map $f : X \to Y$ is an ε-approximation of a multivalued map $\varphi : X \to Y$ if and only if $f(x) \in O_\varepsilon(\varphi(O_\varepsilon(x)))$ for each $x \in X$.*

Definition 3.2. An u.s.c. map $\varphi : X \to Y$ is *approximable* (*on the graph*) provided it has compact values and for every $\varepsilon > 0$ there exists a continuous map $f : X \to Y$ which is an ε-approximation of φ.

We denote the class of all approximable maps from x to Y by $A(X, Y)$.

3. Main result

Let G be a cyclic group of a prime power order $d = p^\alpha$, $p \geq 3$. Let C be the standard linear representation of G, i.e. C is a space of complex numbers and G acts on C by the formula

$$gz = \exp(2\pi\,\mathrm{id}^{-1}),$$

where $g \in G$ is a fixed generator and $z \in C$.

We denote by C^n the Cartesian product of n copies of the representation C and by $S(C^n)$ the unit sphere in C^n. Clearly, G acts freely on $S(C^n)$. For a continuous map $f : S(C^n) \to R^m$ into the m-dimensional Euclidean space we define *the coincidence set*

$$A_f = \{x \in S(C^n) \mid f(x) = f(g^i x),\ i = 1, \ldots, d-1\}.$$

By definition A_f is an invariant subset of $S(C^n)$. The following significant result has been proved by Munkholm in [M].

Theorem 3.1. *Let G be a cyclic group of an odd prime power order p^α. For every continuous map $f : S(C^n) \to R^m$ the covering dimension of A_f is greater than or equal to*

$$(n-1) - (p^\alpha - 1)m - [m(\alpha - 1)p^\alpha - (m\alpha + 2)p^{\alpha-1} + m + 3].$$

□In particular, the set A_f is nonempty if n is large comparatively to m.

Our aim is to prove a version of the above result for multivalued maps which are approximable on graphs. Our theorem states as folows.

Theorem 3.2. *Let G be a cyclic group of an odd prime power order $d = p^\alpha$. Let $\varphi : S(C^n) \to R^m$ be a multivalued map such that $\phi \in A(S(C^n), R^m)$. Then the set*

$$A_\varphi = \{x \in S(C^n) \mid \varphi(x) \cap \varphi(gx) \cap \dots \cap \varphi(g^{d-1}x) \neq \emptyset\}$$

is nonempty whenever

$$n - 1 \geq (p^\alpha - 1)m + [m(\alpha - 1)p^\alpha - (m\alpha + 2)p^{\alpha-1} + m + 3]. \tag{1}$$

Proof. For each $k \in N$ there exists a $(1/k)$-approximation of φ by a single-valued map $f_k : S(C^n) \to R^m$. It is a consequence of Theorem 3.1 that all sets A_{f_k} are nonempty provided condition (1) is satisfied. Therefore we can choose $x_k \in A_{f_k}$ for every natural number k. With no loss of generality we may suppose that the sequence of points (x_k) converges to a point $x_0 \in S(C^n)$. Since $\varphi(S(C^n)) \subset R^m$ is compact and the sequence of maps (f_k) consists of approximations of φ we claim that all sets $f_k(S(C^n))$, $k \in N$ are included in a bounded neighbourhood of $\varphi(S(C^n))$. Thus we can also suppose that the sequence of points $(c_k) = (f_k(x_k))$ converges to some point $c \in R^m$.

Now, for each $k \in N$ there are points $\overline{x}_k \in S(C^n)$ and $\overline{y}_k \in R^m$, $(\overline{x}_k, \overline{y}_k) \in \Gamma_\varphi$ such that

$$\operatorname{dist}(x_k, \overline{x}_k) < \frac{1}{k} \quad \text{and} \quad \operatorname{dist}(c_k, \overline{y}_k) < \frac{1}{k}.$$

Therefore $(x_0, c) \in \Gamma_\varphi$ since $\Gamma_\varphi \subset S(C^n) \times R^m$ is closed.

Let $\{x_k, x_k^1 = gx_k, \dots, x_k^{d-1} = g^{d-1}x_k\}$ be the whole orbit of x_k. By continuity of the action of G on $S(C^n)$ we obtain

$$x_0^i = \lim_{k\to\infty} x_k^i = \lim_{k\to\infty} g^i x_k = g^i \lim_{k\to\infty} x_k = g^i x_0$$

for $i = 1, \dots, d-1$. Using the same arguments as above one easily proves that $(x_0^i, c) \in \Gamma_\varphi$ which means that

$$c \in \varphi(x_0) \cap \varphi(gx_0) \cap \ldots \cap \varphi(g^{d-1}x_0)$$

and thus x_0 is an element of A_φ. □

As a consequence of Theorem 3.2 we obtain the following

Corollary 3.3. *Let G be a cyclic group of an odd prime power order p^α. Let $\varphi : S(C^n) \to R^m$ be a multivalued map approximable on the graph. If*

$$n - 3 \geq (p^\alpha - 1)m + [m(\alpha - 1)p^\alpha - (m\alpha + 2)p^{\alpha-1} + m + 3] \tag{2}$$

then the set A_φ consists of infinitely many points.

Proof. Suppose on a contrary that there is a map $\varphi : S(C^n) \to R^m$ such that A_φ is finite. Let $\psi : S(C^n) \setminus A_\varphi \to R^m$ be a restriction od φ. It follows from the definition of A_φ that $S(C^n) \setminus A_\varphi$ is an invariant subset of $S(C^n)$ and the set A_ψ is empty. On the other hand one can easily define a G-equivariant embedding (not necessarily an inclusion) $\iota : S(C^{n-1}) \to S(C^n) \setminus A_\varphi$. Consider a composition $\xi = \varphi \circ \iota : S(C^{n-1}) \to R^m$ which is obviously a multivalued mapping approximable on the graph. From inequality (2) it follows that $n-1$ satisfies (1). Thus by Theorem 3.2 $A_\xi \neq \emptyset$ which is a contradiction. □

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SEMICONTINUOUS SOLUTIONS OF HAMILTON-JACOBI-BELLMAN EQUATIONS WITH STATE CONSTRAINTS

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1. Introduction

This paper is devoted to Hamilton-Jacobi-Bellman equation under state constraints arising in optimal control of Bolza problem:

$$\text{minimize} \int_{t_0}^{T} L(s, x(s), u(s))ds + g(x(T))$$

over solutions of the control system

$$x' = f(s, x, u(s)), \qquad u(s) \in U, \ x(t_0) = x_0, \ x(t) \in \overline{\Omega}.$$

It is well known that the value function of the above problem may be discontinuous even if all data are smooth. Still it satisfies a Hamilton-Jacobi equation in a generalized sense. In Capuzzo-Dolcetta and Lions [6] it was shown that if a "controllability" assumption of Soner's type [15] holds true on the boundary of Ω, then the value function is continuous and is a unique viscosity solution to the Hamilton-Jacobi equation:

$$(1) \qquad -V_t + H(t, x, -V_x) = 0, \qquad V(T, \cdot) = g(\cdot)$$

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(see [7] for the corresponding definitions of viscosity solutions). The controllability condition in the case of bounded Ω with smooth boundary means that for some $\rho > 0$:

$$\forall\, x \in \partial\Omega,\ \inf_{u\in U}\langle n_x, f(t,x,u)\rangle \leq -\rho,$$

where n_x denotes the outer normal to $\overline{\Omega}$. This condition was introduced in Soner [15] to treat an infinite horizon problem. When the boundary of Ω is nonsmooth, Ishii and Koike [13], still in the contexte of the infinite horizon problem, proposed an extension of this condition in terms of properties of solutions to the control system. Rewritten in the language of tangents, their condition is

$$\forall\, x \in \partial\Omega,\ \exists\, u \in U,\ f(x,u) \in \mathrm{Int}(C_{\overline{\Omega}}(x)),$$

where $C_{\overline{\Omega}}(x)$ is the Clarke tangent cone to $\overline{\Omega}$ at x. This condition also yields continuity of the value function and uniqueness of solution to a corresponding Hamilton-Jacobi equation.

The natural question do arise, what happens if such condition is violated. When we wish the domain of the value function to be the whole $\overline{\Omega}$, then an easy argument implies that a necessary condition for it is

$$\forall\, x \in \partial\Omega,\ \inf_{u\in U}\langle n_x, f(t,x,u)\rangle \leq 0.$$

If we do not have the strict inequality, then the value may be discontinuous, as we show in Example 1. Still it is possible to get uniqueness of generalized solutions to Hamilton-Jacobi equation by adding the assumption

$$\forall\, x \in \partial\Omega,\ \sup_{u\in U}\langle n_x, f(t,x,u)\rangle > 0.$$

Note that it means that from every boundary point of Ω starts a trajectory leaving immediately $\overline{\Omega}$, while Soner's assumption means that a trajectory enters immediately into Ω. In this sense our condition in the smooth case looks like a complementary to Soner's: one can easily get examples where only Soner's or only our condition hold true. It would be interesting to study the case, where at each point of the boundary one or another conditions are satisfied. When Ω is nonsmooth, our assumption becomes

$$\forall\, x \in \partial\Omega,\ \exists\, u \in U, -f(x,u) \in \mathrm{Int}(C_{\overline{\Omega}}(x)),$$

which is again complementary to the one of Ishii-Koike.

Under the above assumption we show that the value function of the constrained Bolza problem is the unique solution to (1) in the following sense:

$$\begin{aligned}
&\forall\,(t,x)\in(0,T)\times\Omega,\ \forall\,(p_t,p_x)\in\partial_-W(t,x), -p_t+H(t,x,-p_x)=0,\\
&\forall\,(t,x)\in(0,T)\times\partial\Omega,\ \forall\,(p_t,p_x)\in\partial_-W(t,x), -p_t+H(t,x,-p_x)\geq 0,\\
&\forall\,(t,x)\in(0,T]\times\overline{\Omega},\quad \liminf_{(s,y)\to(t^-,x),\ y\in\Omega} W(s,y)=W(t,x),\\
&\forall\,x\in\overline{\Omega},\quad \liminf_{(s,y)\to(0+,x)} W(s,y)=W(t,x).
\end{aligned}$$

The proofs of the above result are based on viability theory. The dynamic programming principle implies in the usual way that the value function satisfies the above properties. To prove the opposite statement, using results of Rockafellar [14] and Guseinov, Subbotin, Ushakov [12] we rewrite the above Hamilton-Jacobi equation in terms of viability and invariance properties of the epigraph of its solution (restricted to a corresponding set) similarly to [9].

2. Viability and invariance

Let $K\subset R^n$ be a nonempty subset and $x\in K$. The contingent cone $T_K(x)$ to K at x is defined by

$$v\in T_K(x)\Longleftrightarrow \liminf_{h\to 0+}\frac{\mathrm{dist}(x+hv,K)}{h}=0.$$

The polar cone T^- to a subset $T\subset R^n$ is given by

$$T^-=\{v\in R^n\mid \forall w\in T\,,\ \langle v,w\rangle\leq 0\}.$$

A locally compact subset K of R^n is a viability domain of a set-valued map $G:R^n\mapsto R^n$ if for every $x\in K$

$$G(x)\cap T_K(x)\neq\emptyset.$$

The following formulation summarize several versions of the viability theorem (comp. [1]).

Theorem 1. *Suppose that $G:R^n\rightsquigarrow R^n$ is an upper semicontinuous map with compact convex values. Let $H(x,p)=\sup_{g\in G(x)}\langle g,p\rangle$. For a locally compact subset $K\subset R^n$ the following conditions are equivalent:*

(i) *K is a viability domain of G;*

(ii) *$H(x,-n)\geq 0$ for every $x\in K$ and every $n\in[T_K(x)]^-$;*

(iii) *for every $x_0\in K$ there is $T>0$ and a solution $x:[0,T)\to K$ to the Cauchy problem*

$$\begin{cases} x'(t) \in G(x(t)), \\ x(0) = x_0. \end{cases} \tag{2}$$

A locally compact subset $K \subset R^n$ is called a backward invariance domain of a set-valued map $G : R^n \mapsto R^n$ if for every $x \in K$

$$-G(x) \subset T_K(x).$$

Theorem 2. *Assume that $G : R^n \rightsquigarrow R^n$ is a locally lipschitz continuous map with nonempty compact values and K is a locally compact subset of R^n. Then the following conditions are equivalent:*

(i) *K is a backward invariance domain of G,*
(ii) *$H(x, -n) \leq 0$ for every $x \in K$, $n \in [T_K(x)]^-$,*
(iii) *for every $x_0 \in K$ there exists $T > 0$ such that for every solution $x(\cdot)$ to (2) we have $x(t) \in K$ for $t \in [-T, 0]$.*

3. Main result

We assume that $\Omega \subset R^n$ is open, U is a metric space and

$$\begin{cases} f : [0,T] \times R^n \times U \to R^n, \; L : [0,T] \times R^n \times U \to [0,\infty) \\ \text{are bounded continuous maps,} \end{cases} \tag{3}$$

$$\begin{cases} f(t,x,u), \; L(t,x,u) \text{ are } t,x\text{-locally lipschitz continuous} \\ \text{uniformly in } u, \end{cases} \tag{4}$$

$$\begin{cases} \text{for every } (t,x) \in [0,T] \times \overline{\Omega} \text{ there is } u \in U \text{ such that} \\ f(t,x,u) \in T_{\overline{\Omega}}(x), \end{cases} \tag{5}$$

$$\begin{cases} \{(f(t,x,u), L(t,x,u) + r) \in R^n \times R \mid u \in U, \; r \geq 0\} \\ \text{is closed and convex for every } t \in [0,T], \; x \in \overline{\Omega}, \end{cases} \tag{6}$$

$$g : \overline{\Omega} \to R \cup \{+\infty\} \qquad \text{is a lower semicontinuous function.} \tag{7}$$

Let $u : [a,b] \to U$ be a measurable control, $t_0 \in [a,b]$. We denote by $x(\cdot\,; t_0, x_0, u)$ the unique solution to the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t), u(t)) \\ x(t_0) = x_0, \end{cases}$$

defined on the interval $[a,b]$. Let us denote by $A(t_0,x_0) = \{u : [t_0,T] \to U \mid x(t; t_0, x_0, u) \in \overline{\Omega} \text{ for every } t \in [t_0,T]\}$ the set of admissible controls from the initial condition $(t_0, x_0) \in [0,T] \times \overline{\Omega}$. The value function $V : [0,T] \times \overline{\Omega} \to R \cup \{+\infty\}$ for the Bolza problem is given by

$$V(t_0,x_0) = \inf_{u \in A(t_0,x_0)} \int_{t_0}^{T} L(s, x(s; t_0, x_0, u), u(s))ds + g(x(T; t_0, x_0, u)).$$

Having the dynamics f and the integral cost function L of the control system we define set-valued maps $F^+,\ F^- : [0,T]\times R^n\times R \mapsto R\times R^n\times R$ by

$$\begin{aligned}F^+(t,x,v) &= \{(1, f(t,x,u), L(t,x,u)+r) \mid u\in U,\\ &\qquad\qquad r\in[0, M-L(t,x,u)]\},\\ F^-(t,x,v) &= \{(1, f(t,x,u), -L(t,x,u)-r) \mid u\in U,\\ &\qquad\qquad r\in[0, M-L(t,x,u)]\},\end{aligned} \tag{8}$$

where M is a bound on L.

If (3)-(6) hold true then F^+, F^- are lipschitz continuous bounded maps and they have convex compact values.

We denote by $S(t_0,x_0)$ the set of solutions to the Cauchy problem

$$\begin{cases} z'(t)\in F^+(z(t)),\\ z(t_0) = (t_0, x_0, 0)\end{cases}$$

defined on the interval $[t_0,T]$. Let $S_v(t_0,x_0) = \{z\in S(t_0,x_0) \mid z(t)\in[t_0,T]\times \overline{\Omega}\times R$ for every $t\in[t_0,T]\}$. It is easy to check that $z\in S_v(t_0,x_0)$ if and only if there exist $u\in A(t_0,x_0)$ and a measurable $\eta : [t_0,T]\to[0,+\infty)$ such that $L(s, x(s;t_0,x_0,u), u(s)) + \eta(s) \le M$ and

$$z(t) = \left(t, x(t;t_0,x_0,u), \int_t^T (L(s,x(s;t_0,x_0,u),u(s)) + \eta(s))ds\right).$$

Proposition 3. *Assume that* (3)-(7) *hold true. Then for every* $(t_0,x_0)\in[0,T]\times\overline{\Omega}$ *there is* $\overline{u}\in A(t_0,x_0)$ *such that*

$$V(t_0,x_0) = \int_{t_0}^T L(s,\overline{x}(s),\overline{u}(s))ds + g(\overline{x}(T)),$$

where $\overline{x}(s) = x(s;t_0,x_0,\overline{u})$.

Proof. Since $S_v(t_0,x_0)\subset C([t_0,T],R^n)$ is a compact nonempty set and the function $\varphi : S_v(t_0,x_0)\to R\cup\{+\infty\}$, $\varphi(z) = g(x(T)) + v(T)$, where $z(t) = (t,x(t),v(t))$, is lower semicontinuous, there is an optimal solution $\overline{z}\in S_v(t_0,x_0)$ such that $\varphi(\overline{z}) = \inf\{\varphi(z) \mid z\in S_v(t_0,x_0)\}$. If $\overline{u}\in A(t_0,x_0)$, $\eta : [t_0,T]\to[0,M]$ corespond to $\overline{z}$, then $\eta = 0$ and

$$V(t_0,x_0) = \int_{t_0}^T L(s,\overline{x}(s),\overline{u}(s))ds + g(\overline{x}(T)),$$

where $\overline{x}(s) = x(s;t_0,x_0,\overline{u})$ (i.e., $\overline{u}$ is an optimal control). □

Proposition 4. *Assume that* (3)-(7) *hold true. Then the value function* $V : [0,T]\times\overline{\Omega} \to R$ *is lower semicontinuous and* V *is left continuous along trajectories of control system, i.e. if* $u \in A(t_0, x_0)$ *then the function* $t \to V(t, x(t; t_0, x_0, u))$ *is left continuous. Moreover,*

$$\liminf_{t_n\to 0^+, y_n\to y, y_n\in\overline{\Omega}} V(t_n, y_n) = V(0, y)$$

for every $y \in \overline{\Omega}$.

Proof. Let $t_n \to t_0$ and $x_n \to x_0$. We choose $\overline{z}_n \in S_v(t_n, x_n)$ such that $\varphi(\overline{z}_n) = V(t_n, x_n)$. Passing to a subsequence we obtain $\overline{z}_{n_k}$ tending to a solution $\overline{z} \in S_v(t_0, x_0)$. Obviously $\overline{x}_{n_k}(T) \to \overline{x}(T)$ and $\overline{v}_{n_k}(T) \to \overline{v}(T)$. Thus $V(t_0, x_0) \leq \varphi(\overline{z}) \leq \liminf_{k\to\infty} \varphi(\overline{z}_{n_k}) = \liminf_{k\to\infty} V(t_{n_k}, x_{n_k})$, which implies that V is lower semicontinuous.

Fix $\tau \in (t_0, T]$ and a sequence $t_n \to \tau^-$. We denote by $x_u(\cdot) = x(\cdot\,; t_0, x_0, u)$ the solution corresponding to the control u. Let $\overline{u} \in A(\tau, x_u(\tau))$ be an optimal control. Setting

$$u_n(t) = \begin{cases} u(t) & \text{for } t \in [t_n, \tau), \\ \overline{u}(t) & \text{for } t \in [\tau, T] \end{cases}$$

we obtain $x(s; t_n, x_u(t_n), u_n) = x(s; t_m, x_u(t_m), u_m)$ for $s \in [t_n, T] \cap [t_m, T]$ and $x(s; t_n, x_u(t_n), u_n) = x(s; \tau, x_u(\tau), \overline{u})$ for $s \in [\tau, T]$. Hence

$$\begin{aligned} \limsup_{n\to\infty} V(t_n, x_u(t_n)) &\leq \lim_{n\to\infty} \int_{t_n}^{T} L(s, x(s; t_n, x_u(t_n), u_n), u_n(s))ds \\ &\quad + g(x(T; t_n, x_u(t_n), u_n)) \\ &= \int_{\tau}^{T} L(s, x(s; \tau, x_u(\tau), \overline{u}), \overline{u}(s))ds \\ &\quad + g(x(T; \tau, x_u(\tau), \overline{u})) = V(\tau, x_u(\tau)). \end{aligned}$$

Combining it with the lower semicontinuity we obtain that V is left continuous along trajectories.

Fix $y \in \overline{\Omega}$ and let $\overline{u} : [0,T] \to U$ be an optimal control for initial conditions $(0, y)$. Then

$$V(s, x(s; 0, y, \overline{u})) = V(0, y) - \int_0^s L(\tau, x(\tau; 0, y, \overline{u}), \overline{u})d\tau.$$

Clearly $\lim_{s\to 0^+} V(s, x(s; 0, y, \overline{u})) = V(0, y)$. This and lower semicontinuity of V end the proof. □

Proposition 5. *Assume that* (3)-(7)) *hold true. Then the epigraph of the value function over* $[0,T)\times\overline{\Omega}$, *i.e. the set* $K_0 = \{(t,x,v) \mid t\in[0,T),\ x\in\overline{\Omega},\ v\geq V(t,x)\}$ *is a viability domain for the set-valued map* F^-.

Proof. Fix $(t_0,x_0)\in[0,T)\times\overline{\Omega}$. By Proposition 3 there is an optimal control $\overline{u}\in A(t_0,x_0)$, i.e.

$$V(t_0,x_0)=\int_{t_0}^{T} L(s,\overline{x}(s),\overline{u}(s))ds+g(\overline{x}(T)),$$

where $\overline{x}(\cdot)=x(\cdot\,;t_0,x_0,\overline{u})$. If $v_0\geq V(t_0,x_0)$ then the function

$$z(t)=\left(t,\overline{x}(t),v_0+\int_{t_0}^{t}-L(s,\overline{x}(s),\overline{u}(s))ds\right)$$

is a solution to $z'\in F^-(z)$. Observe that

$$v_0+\int_{t_0}^{t}-L(s,\overline{x}(s),\overline{u}(s))ds\geq\int_{t}^{T} L(s,\overline{x}(s),\overline{u}(s))ds+g(\overline{x}(T))\geq V(t,\overline{x}(t)).$$

Thus, $z(t)\in K_0$ for $t\in[t_0,T)$. By Viability Theorem (Theorem 1), we obtain the desired conclusion. □

Proposition 6. *Assume that* (3)-(7) *hold true. Then the epigraph*

$$D_T=\{(t,x,v)\mid t\in(0,T],\ x\in\Omega \text{ and } v\geq V(t,x)\}$$

of the function V *restricted to* $(0,T]\times\Omega$ *is a backward invariance domain for* F^-.

Proof. Let $t_0\in(0,T]$, $x_0\in\Omega$ and $v_0\geq V(t_0,x_0)$. There is $\varepsilon>0$ such that for every measurable $u:[t_0-\varepsilon,t_0]\to U$ the solution $x_u:[t_0-\varepsilon,t_0]\to R^n$ of the Cauchy problem

$$\begin{cases} x'(t)=f(t,x(t),u(t)),\\ x(t_0)=x_0\end{cases}$$

satisfies $x_u(t)\in\Omega$ for $t\in[t_0-\varepsilon,t_0]$. Let

$$v_u(t)=v_0+\int_{t_0}^{t}-L(s,x_u(s),u(s))ds$$

and $\overline{u}\in A(t_0,x_0)$ be an optimal control. Fix $t_1\in[t_0-\varepsilon,t_0]$ and let $x_1=x_u(t_1)$. Define

$$u_1(s)=\begin{cases} u(t) & \text{for } t\in[t_1,t_0],\\ \overline{u}(t) & \text{for } t>t_0.\end{cases}$$

We have $u_1 \in A(t_1, x_1)$ and

$$\begin{aligned} V(t_1,x_1) &\le \int_{t_1}^{T} L(s,x(s;t_1,x_1,u_1),u_1(s))ds + g(x(T;t_1,x_1,u_1)) \\ &= \int_{t_1}^{t_0} L(s,x(s;t_1,x_1,u_1),u_1(s))ds + V(t_0,x_0) \le v_u(t_1). \end{aligned} \tag{9}$$

If $z(t)$ is a solution to the differential inclusion $z'(t) \in F^-(z(t))$ defined on the interval $[t_0 - \varepsilon, t_0]$ and $z(t_0) = (t_0, x_0, v_0)$, then there is a control $u : [t_0 - \varepsilon, t_0] \to U$ and a measurable function $\eta : [t_0 - \varepsilon, t_0] \to [0, M]$ such that

$$z(t) = \left(t, x_u(t), v_u(t) - \int_{t_0}^{t} \eta(s)ds\right).$$

By (9), we obtain $z(t) \in \mathrm{Epi}(V)$. From Theorem 2, we get the conclusion. □

Proposition 7. *Suppose that* (3)-(7) *hold true and* $W : [0,T] \times \overline{\Omega} \to R \cup \{+\infty\}$ *is a lower semicontinuous function. If* $W(T,x) \ge g(x)$ *for* $x \in \overline{\Omega}$ *and* $K = \{(t,x,v) \mid t \in [0,T), x \in \overline{\Omega}, v \ge W(t,x)\}$ *is a viability domain of* F^- *then*

$$W(t_0,x_0) \ge V(t_0,x_0)$$

for every $(t_0,x_0) \in [0,T] \times \overline{\Omega}$.

Proof. By Theorem 1, there is a K-viable solution $z : [t_0, t_1) \to R^{n+2}$ of the Cauchy problem $z' \in F^-(z)$, $z(t_0) = (t_0, x_0, W(t_0, x_0))$. There is a control $u \in A(t_0, x_0)$ such that $z(t) = (t, x_u(t), v_u(t))$, where $x_u(t) = x(t; t_0, x_0, u)$ and

$$v_u(t) = W(t_0,x_0) - \int_{t_0}^{t} (L(s,x_u(s),u(s)) + \eta(s))ds$$

for a nonnegative measurable function η. Setting $\eta = 0$ we get another K-viable solution denoted again by z. We have

$$v_1 = \lim_{t \to t_1^-} v_u(t) \ge \liminf_{t \to t_1^-} W(t, x_u(t)) \ge W(t_1, x_u(t_1)).$$

The solution $z(\cdot)$ can be extended onto the interval $[t_0, T]$ and

$$W(t_0,x_0) - \int_{t_0}^{T} L(s,x_u(s),u(s))ds \ge W(T,x_u(T)) \ge g(x_u(T)).$$

Hence

$$V(t_0,x_0) \le \int_{t_0}^{T} L(s,x_u(s),u(s))ds + g(x_u(T)) \le W(t_0,x_0),$$

which completes the proof. □

The following assumption plays the crucial role in the remainder of the paper

$$\forall t \in (0,T],\ \forall x \in \partial\Omega,\ \exists u \in U,\ -f(t,x,u) \in \mathrm{Int}(C_{\overline{\Omega}}(x)),$$

where $C_{\overline{\Omega}}(x)$ denotes the Clarke tangent cone to $\overline{\Omega}$ at x. Using Proposition 13 (page 425) in [3] we obtain

Proposition 8. *If $w \in \mathrm{Int}(C_{\overline{\Omega}}(x))$, then there is $R > 0$ such that for every $y \in \overline{\Omega} \cap B(x,R)$ we have*

$$y + (0,R)B(w,R) \subset \Omega.$$

Corollary 9. *Assume that* (3)-(7) *and* (10) *hold true. Then for every* $(t_0, y_0) \in (0,T] \times \overline{\Omega}$ *there is a sequence* (t_n) *tending to* t_0 *from the left and a sequence* $(y_n) \subset \Omega$ *convergent to* y_0 *such that* $\lim_{n\to\infty} V(t_n, y_n) = V(t_0, y_0)$.

Proof. By (10), there is $\overline{u} \in U$ such that $-f(t_0, y_0, \overline{u}) \in \mathrm{Int}(C_{\overline{\Omega}}(y))$. There is $\varepsilon > 0$ such that $\overline{x}(t) \in \Omega$ for $t \in (t_0 - \varepsilon, t_0)$, where $\overline{x}$ is a solution to the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t), \overline{u}), \\ x(t_0) = y_0. \end{cases}$$

Setting $y_n = \overline{x}(t_n)$, $t_n \to t_0^-$, by Proposition 4, we obtain $\lim_{n\to\infty} V(t_n, y_n) = V(t_0, y_0)$. □

Proposition 10. *Assume that* (3)-(7) *and* (10) *hold true. Let* $W : [0,T] \times \overline{\Omega} \to R \cup \{+\infty\}$ *be a lower semicontinuous function such that*

$$\liminf_{t_n \to t_0^-,\, y_n \to y_0,\, y_n \in \Omega} W(t_n, y_n) = W(t_0, y_0) \qquad \text{for } (t_0, y_0) \in (0,T] \times \overline{\Omega}.$$

If the set $D = \{(t,x,v) \mid t \in (0,T),\ x \in \Omega \text{ and } v \geq W(t,x)\}$ *is a backward invariance domain of* F^- *and* $W(T,x) \leq g(x)$ *for* $x \in \overline{\Omega}$, *then*

$$W(t_0, x_0) \leq V(t_0, x_0)$$

for every $(t_0, x_0) \in [0,T] \times \overline{\Omega}$.

Proof. Fix $\overline{t} \in (0,T)$, $\overline{x} \in \overline{\Omega}$, $u \in A(\overline{t}, \overline{x})$. Let $x_u(s) = x(s; \overline{t}, \overline{x}, u)$ and $v_u : [t_0, T] \to R$ solves the Cauchy problem

$$\begin{cases} v'(t) = -L(t, x_u(t), u(t)), \\ v(T) = g(x_u(T)). \end{cases}$$

We show that

$$W(\overline{t}, x_u(\overline{t})) \leq v_u(\overline{t}). \tag{11}$$

Let

$$t_0 = \inf\{t \in [\bar{t}, T] \mid \sup_{s\in[t,T]} (W(s, x_u(s)) - v_u(s)) \leq 0\}.$$

There is a sequence $(s_n) \subset [t_0, T]$, $s_n \to t_0$ such that $W(s_n, x_u(s_n)) \leq v_u(s_n)$. Since W is lower semicontinuous we have

$$W(t_0, x_u(t_0)) \leq \liminf_{n\to\infty} W(s_n, x_u(s_n)) \leq \lim_{n\to\infty} v_u(s_n) = v_u(t_0).$$

We claim that $t_0 = \bar{t}$. Indeed suppose for a moment that $t_0 > \bar{t}$.

We denote $y_0 = x_u(t_0)$. Let $\bar{u} \in U$ be such that (10) holds true and set $w = -f(t_0, y_0, \bar{u})$. By Proposition 8, there is $R \in (0,1)$ such that $y + (0,R)B(w,R) \subset \Omega$ for every $y \in \overline{\Omega} \cap B(y_0, R)$. It follows that if $y \in \overline{\Omega} \cap B(y_0, R)$, $r < R$ and

$$|x - (y + rw)| < rR \tag{12}$$

then $x \in \Omega$. There are sequences $(t_n) \subset (\bar{t}, t_0)$, $(y_n) \subset \Omega$ such that $t_n \to t_0$, $y_n \to y_0$ and $W(t_n, y_n) \to W(t_0, y_0)$. Let M be an upper bound of $|f(t,x,u)|$. We set $\varepsilon_n = (3/R)(|y_n - y_0| + (t_0 - t_n))$ Moreover, we choose $\varepsilon \in (0, t_0 - \bar{t}\,)$ such that

$$(M+2)(e^{l\varepsilon} - 1) < \frac{R}{3}, \tag{13}$$

where l is a lipschitz constant to $f(\cdot,\cdot,u)$ and $L(\cdot,\cdot,u)$. Define $u_n : [t_0 - \varepsilon, t_n] \to U$ by

$$u_n(s) = \begin{cases} \bar{u} & \text{for } s \in [t_n - \varepsilon_n, t_n], \\ u(s + \varepsilon_n + (t_0 - t_n)) & \text{for } s \in [t_0 - \varepsilon, t_n - \varepsilon_n]. \end{cases}$$

Define $x_n(s) = x(s; t_n, y_n, u_n)$ for $s \in [t_0 - \varepsilon, t_n]$. We show that for sufficiently large n, $x_n(s) \in \Omega$ for $s \in [t_0 - \varepsilon, t_n]$. For $s \in [t_n - \varepsilon_n, t_n]$ we have

$$\begin{aligned} |x_n(s) - (y_n + (t_n - s)w)| &= \left| \int_s^{t_n} f(\tau, x_n(\tau), \bar{u}) - f(t_0, y_0, \bar{u}) d\tau \right| \\ &\leq \int_s^{t_n} l(t_0 - \tau) + l(M(t_n - \tau) + |y_n - y_0|) d\tau \\ &\leq \left(\frac{1}{2} l(M+1)\varepsilon_n + l((t - t_n) + |y_n - y_0|) \right)(t_n - s). \end{aligned}$$

For n sufficiently large we have $(1/2)l(M+1)\varepsilon_n + l((t-t_n)+|y_n-y_0| < R$. By (12), we obtain $x_n(s) \in \Omega$. Now, we take $s \in [t_0-\varepsilon, t_n-\varepsilon_n]$. Then

$$\begin{aligned}
|x_n(s) - (x_u(s+\varepsilon_n+(t_0-t_n)) + \varepsilon_n w)| &= \left| y_n + \int_{t_n}^{t_n-\varepsilon_n} f(\tau, x_n(\tau), \overline{u}) d\tau \right. \\
&\quad + \int_{t_n-\varepsilon_n}^{s} f(\tau, x_n(\tau), u(\tau+\varepsilon_n+(t_0-t_n))) d\tau \\
&\quad \left. - \left(y_0 + \int_{t_0}^{s+\varepsilon_n+(t_0-t_n)} f(\tau, x_u(\tau), u(\tau)) d\tau + \varepsilon_n w \right) \right| \\
\leq |y_n - y_0| &+ \int_{t_n-\varepsilon_n}^{t_n} |f(\tau, x_n(\tau), \overline{u}) + w| d\tau \\
&\quad + \int_{s}^{t_n-\varepsilon_n} |f(\tau+(t_0-t_n)+\varepsilon_n, x_u(\tau+(t_0-t_n)+\varepsilon_n), \\
&\quad u(\tau+(t_0-t_n)+\varepsilon_n)) - f(\tau, x_n(\tau), u(\tau+(t_0-t_n)+\varepsilon_n))| d\tau \\
\leq |y_n - y_0| &+ \int_{t_n-\varepsilon_n}^{t_n} l((t_0-\tau) + |y_n-y_0| + M(t_n-\tau)) d\tau \\
&\quad + \int_{s}^{t_n-\varepsilon_n} l((t_0-t_n)+\varepsilon_n) + l(|x_u(\tau+(t_0-t_n)+\varepsilon_n) - x_n(\tau)| d\tau \\
\leq |y_n - y_0| &+ \left[l((t_0-t_n)+|y_n-y_0|) + \frac{\varepsilon_n}{2} l(M+1) \right] \varepsilon_n \\
&\quad + \int_{s}^{t_n-\varepsilon_n} l((t_0-t_n)+\varepsilon_n+|y_0-y_n|+\varepsilon_n M) e^{l(t_n-\varepsilon_n-\tau)} d\tau \\
\leq \Bigg\{ \frac{|y_0-y_n|}{\varepsilon_n} &+ l((t_0-t_n)+|y_0-y_n|) + \frac{\varepsilon_n}{2} l(M+1) \\
&\quad + \left[\frac{(t_0-t_n)+|y_0-y_n|}{\varepsilon_n} + (M+1) \right] (e^{l\varepsilon}-1) \Bigg\} \varepsilon_n.
\end{aligned}$$

Hence, for sufficently large n

$$|x_n(s) - (x_u(s+\varepsilon_n) + \varepsilon_n w)| \leq R\varepsilon_n.$$

By (12), we obtain $x_n(s) \in \Omega$ for $s \in [t_0-\varepsilon, t_n-\varepsilon_n]$. Since D is a backward invariance domain of F^- and $x_n(t) \in \Omega$ for $t \in [t_0-\varepsilon, t_n]$, then $z_n(t) \in D$ for $t \in [t_0-\varepsilon, t_n]$, where $z_n(t) = (t, x_n(t), W(t_n, y_n) + \int_{t_n}^{t} -L(s, x_n(s), u_n(s)) ds)$. Hence

$$W(t, x_n(t)) \leq W(t_n, y_n) + \int_{t}^{t_n} L(s, x_n(s), u_n(s)) ds.$$

We have

$$\begin{aligned}
&\left| \int_t^{t_n} L(s, x_n(s), u_n(s))ds - \int_t^{t_0} L(s, x_u(s), u(s))ds \right| \\
&\quad \le \left| \int_t^{t_n-\varepsilon_n} L(s, x_n(s), u_n(s))ds - \int_{t+\varepsilon_n+(t_0-t_n)}^{t_0} L(s, x_u(s), u(s))ds \right| \\
&\quad + \int_{t_n-\varepsilon_n}^{t_n} |L(s, x_n(s), u_n(s))|ds + \int_t^{t+\varepsilon_n+(t_0-t_n)} |L(s, x_u(s), u(s))|ds \\
&\quad \le \int_t^{t_n-\varepsilon_n} l(\varepsilon_n + (t_0 - t_n)) + l|x_n(s) - x_u(s + \varepsilon_n + (t_0 - t_n))|ds \\
&\quad + M(2\varepsilon_n + (t_0 - t_n)) \\
&\quad \le \varepsilon l(\varepsilon_n + (t_0 - t_n)) + \int_t^{t_n-\varepsilon_n} (\varepsilon_n|w| + R\varepsilon_n)ds + M(2\varepsilon_n + (t_0 - t_n)).
\end{aligned}$$

The expression in the last line tends to zero as $n \to \infty$. Thus

$$\begin{aligned}
W(t, x_u(t)) &\le \liminf_{n\to\infty} W(t, x_n(t)) \\
&\le \lim_{n\to\infty} W(t_n, y_n) + \int_t^{t_n} L(s, x_n(s), u_n(s))ds \\
&= W(t_0, y_0) + \int_t^{t_0} L(s, x_u(s), u(s))ds \le v_u(t)
\end{aligned}$$

which completes the proof of (11). Since u was arbitrary,

$$W(\bar{t}, \bar{x}) \le \inf_{u\in A(\bar{t},\bar{x})} \int_{\bar{t}}^T L(s, x_u(s), u(s))ds + g(x_u(T)).$$

It remains to consider the case $\bar{t} = 0$. Consider a sequence $t_n \to 0^+$, $y_n \to y$, $y_n \in \overline{\Omega}$ such that $V(0, y) = \lim_{n\to\infty} V(t_n, y_n)$. Then $W(t_n, y_n) \le V(t_n, y_n)$. Since W is lower semicontinuous, we get

$$W(0, y) \le V(0, y).$$

□

Let $W : [0, T] \times \overline{\Omega} \to R \cup \{+\infty\}$ be a lower semicontinuous function. We extend W to $\widetilde{W}$ onto $[0, T] \times R^n$ setting $\widetilde{W}(t, x) = +\infty$ for $x \notin \overline{\Omega}$. If $\widetilde{W}(t_0, x_0) < +\infty$ then the subdifferential $\partial_-\widetilde{W}(t_0, x_0)$ of $\widetilde{W}$ at (t_0, x_0) is defined by

$$\left\{ p \in R^{n+1} \mid \liminf_{(t,x)\to(t_0,x_0)} \frac{\widetilde{W}(t, x) - \widetilde{W}(t_0, x_0) - \langle p, (t - t_0, x - x_0)\rangle}{|(t - t_0, x - x_0)|} \ge 0 \right\}.$$

The subdifferential $\partial_- W(t_0, x_0)$ of W at $(t_0, x_0) \in (0,T) \times \overline{\Omega}$ relative to $\overline{\Omega}$ is given by

$$\left\{ p \in R^{n+1} \mid \liminf_{(t,x)\to(t_0,x_0), x\in\overline{\Omega}} \frac{W(t,x) - W(t_0,x_0) - \langle (p, (t,x) - (t_0,x_0)\rangle}{|(t-t_0, x-x_0)|} \geq 0 \right\}.$$

Obviousily $\partial\widetilde{W}(t_0,x_0) = \partial W(t_0,x_0)$. It is well known that $p \in \partial\widetilde{W}(t_0,x_0)$ if and only if there exists a function $\varphi \in C^1(R^{n+1})$ such that $p = \nabla\varphi(t_0,x_0)$ and $W - \varphi$ has a local minimum at (t_0,x_0) relative to $(0,T)\times\overline{\Omega}$. Moreover, $p \in \partial_-\widetilde{W}(t_0,x_0)$ if and only if $(p,-1) \in [T_{\mathrm{Epi}(W)}(t_0,x_0,W(t_0,x_0)]^-$. To characterize the value function as a generalized solution to the Hamilton-Jacobi equation we shall use the following version of Rockafellar's result (see Lemma 4.2 in [9]).

Lemma 11. *Let* $(p,0) \in \left[T_{\mathrm{Epi}(W)}(t_0,x_0,W(t_0,x_0))\right]^-$ *be such that* $p \neq 0$. *Then for every* $\varepsilon > 0$, *there exist* $t_\varepsilon \in [0,T]$, $x_\varepsilon \in \overline{\Omega}$, $p_\varepsilon \in R^{n+1}$ *and* $q_\varepsilon < 0$ *such that* $q_\varepsilon \to 0$ *as* $\varepsilon \to 0^+$ *and*

$$|t_\varepsilon - t_0| \leq \varepsilon,\ |x_\varepsilon - x_0| \leq \varepsilon,\ |p_\varepsilon - p| \leq \varepsilon, \quad (p_\varepsilon, q_\varepsilon) \in [T_{\mathrm{Epi}(W)}(t_\varepsilon, x_\varepsilon, W(t_\varepsilon,x_\varepsilon))]^-.$$

We define $H : (0,T) \times \overline{\Omega} \times R^n \to R$ by

$$H(t,x,p) = \sup_{u\in U} \langle f(t,x,u), p\rangle - L(t,x,u).$$

We summarise obtained results in

Theorem 12. *Assume that* (3)-(7) *and* (10) *hold true. Then for a function* $W : [0,T] \times \overline{\Omega} \to R \cup \{+\infty\}$ *the following conditions are equivalent:*

(a) $W = V$;

(b) *W is a lower semicontinuous function such that* $W(T,\cdot) = g(\cdot)$,

$$\liminf_{t_n\to t^-, y_n\to y, y_n\in\Omega} W(t_n,y_n) = W(t,y)$$

for every $(t,y) \in (0,T] \times \overline{\Omega}$ *and:*

$$\text{(14)} \qquad \begin{cases} \{(t,x,v) \mid t \in [0,T),\ x \in \overline{\Omega},\ v \geq W(t,x)\} \\ \textit{is a viability domain of } F^- , \end{cases}$$

$$\text{(15)} \qquad \begin{cases} \{(t,x,v) \mid t \in (0,T],\ x \in \Omega,\ v \geq W(t,x)\} \\ \textit{is a backward invariance domain of } F^-; \end{cases}$$

(c) *W is a lower semicontinuous function such that* $W(T,\cdot) = g(\cdot)$,

$$\liminf_{t_n\to t^-,\ y_n\to y,\ y_n\in\Omega} W(t_n,y_n) = W(t,y)$$

for every $(t,y) \in (0,T] \times \overline{\Omega}$,

$$\liminf_{t_n\to 0^+,\ y_n\to y,\ y_n\in\overline{\Omega}} W(t_n,y_n) = W(0,y)$$

for $y \in \overline{\Omega}$ and:

$$\left\{ \begin{array}{l} \forall (t,x) \in (0,T) \times \partial\Omega,\ \forall (n_t, n_x) \in \partial_- W(t,x) \\ -n_t + H(t,x,-n_x) \geq 0; \end{array} \right. \tag{16}$$

$$\left\{ \begin{array}{l} \forall (t,x) \in (0,T) \times \Omega,\ \forall (n_t, n_x) \in \partial_- W(t,x) \\ -n_t + H(t,x,-n_x) = 0. \end{array} \right. \tag{17}$$

Proof. By Propositions 4, 5, 6, we obtain the implication (a) $\Longrightarrow$ (b). Assume (b). By Propositions 7 and 10, we get

$$W(t,x) = V(t,x) \quad \text{for} \quad (t,x) \in [0,T] \times \overline{\Omega}.$$

We prove next that (a) $\Longrightarrow$ (c).

From Proposition 4 we obtain the desired regularity of W. The remaining properties of W follow from (b) (which holds true by the previous part of the proof). Fix $t \in (0,T)$ and $x \in \overline{\Omega}$. If $(n_t, n_x) \in \partial_- W(t,x)$ then $(n_t, n_x, -1) \in [T_{\mathrm{Epi}(W)}(t,x,W(t,x))]^-$. Observe that

$$\sup_{u \in U} \langle (1, f(t,x,u), -L(t,x,u)), (-n_t, -n_x, 1) \rangle = -n_t + H(t,x,-n_x). \tag{18}$$

Viability Theorem 1 ((i)$\Longrightarrow$(ii)) and (14) yield

$$\sup_{u \in U} \langle (1, f(t,x,u), -L(t,x,u)), (-n_t, -n_x, 1) \rangle \geq 0.$$

According to (18)

$$\forall (t,x) \in (0,T) \times \overline{\Omega},\ \forall (n_t, n_x) \in \partial_- W(t,x) \mid -n_t + H(t,x,-n_x) \geq 0. \tag{19}$$

From Invariance Theorem 2 and (15) we obtain

$$\forall (t,x) \in (0,T) \times \Omega,\ \forall (n_t, n_x) \in \partial_- W(t,x) \mid -n_t + H(t,x,-n_x) \leq 0. \tag{20}$$

Combining it with (19) we get (16), (17).

Now, we prove that (c) $\Longrightarrow$ (a).

STEP 1. We show that (c) implies

$$\left\{ \begin{array}{l} K = \{(t,x,v) \mid t \in (0,T),\ x \in \overline{\Omega},\ v \geq W(t,x)\} \\ \text{is a viability domain of } F^-, \end{array} \right. \tag{21}$$

$$\left\{ \begin{array}{l} D = \{(t,x,v) \mid t \in (0,T),\ x \in \Omega,\ v \geq W(t,x)\} \\ \text{is a backward invariance domain of } F^-; \end{array} \right. \tag{22}$$

First observe that (19), (20) hold true. To prove (21) it sufficies to verify the condition (ii) in Viability Theorem 1 for K and F^-, i.e.

$$\sup_{u \in U} \langle (1, f(t,x,u), -L(t,x,u)), (-p_t, -p_x, -p_v) \rangle \geq 0 \tag{23}$$

for $(p_t, p_x, p_v) \in [T_{\mathrm{Epi}(W)}(t, x, W(t,x))]^-$, $(t,x) \in (0,T) \times \overline{\Omega}$. If $p_v < 0$ then $((p_t/-p_v), (p_x/-p_v)) \in \partial_- W(t,x)$. By (19), we obtain

$$\frac{-p_t}{-p_v} + H\left(t, x, \frac{-p_x}{-p_v}\right) \geq 0,$$

which implies (23). If $p_v = 0$ then by Lemma 11, there are $t_n \to t$, $x_n \to x$, $v_n \to v$, $p_{t,n} \to p_t$, $p_{x,n} \to p_x$, $p_{v,n} \to 0$, $p_{v,n} < 0$ such that

$$(p_{t,n}, p_{x,n}, p_{v,n}) \in [T_{\mathrm{Epi}(W)}(t_n, x_n, v_n)]^-.$$

By (19), we have

$$\sup_{u \in U} \langle (1, f(t_n, x_n, u), -L(t_n, x_n, u)), (-p_{t,n}, -p_{x,n}, -p_{v,n}) \rangle \geq 0.$$

This and assumptions (3), (4), (6) imply (23).

To obtain (22) we have to verify the statment (ii) in Invariance Theorem 2 for the domain D and the right hand side F^-, i.e.

$$\sup_{u \in U} \langle (1, f(t,x,u), -L(t,x,u)), (-p_t, -p_x, -p_v) \rangle \leq 0 \tag{24}$$

for $(p_t, p_x, p_v) \in \left[T_{\mathrm{Epi}(W)}(t, x, W(t,x))\right]^-$, $(t,x) \in (0,T) \times \Omega$. If $p_v < 0$ then $((p_t/-p_v), (p_x/-p_v)) \in \partial_- W(t,x)$. By (20), we obtain

$$\frac{-p_t}{-p_v} + H\left(t, x, \frac{-p_x}{-p_v}\right) \leq 0,$$

which yields (24). If $p_v = 0$, then by Lemma 11, there are $t_n \to t$, $x_n \to x$, $v_n \to v$, $p_{t,n} \to p_t$, $p_{x,n} \to p_x$, $p_{v,n} \to 0$, $p_{v,n} < 0$ such that

$$(p_{t,n}, p_{x,n}, p_{v,n}) \in [T_{\mathrm{Epi}(W)}(t_n, x_n, v_n)]^-.$$

By (20), we have

$$\sup_{u \in U} \langle (1, f(t_n, x_n, u), -L(t_n, x_n, u)), (-p_{t,n}, -p_{x,n}, -p_{v,n}) \rangle \leq 0.$$

We fix u and pass to the limit with n. Using (3), (4), (6) we obtain (24).

STEP 2. Applying Propositions 7, 10 with the time interval $[0,T]$ replaced by $[t,T]$ with $t > 0$ we get

$$W(t,x) = V(t,x) \qquad \text{for } (t,x) \in (0,T] \times \overline{\Omega}.$$

For $t = 0$ and $y \in \overline{\Omega}$ we have

$$W(0,y) = \liminf_{t \to 0^+, x \to y, x \in \overline{\Omega}} W(t,x) = \liminf_{t \to 0^+, x \to y, x \in \overline{\Omega}} V(t,x) = V(0,y),$$

which completes the proof. □

Example 1. We set $T = 1$, $\Omega = \{(x,y) \mid x < 0 \text{ or } y < 0\}$, $U = [0,1]$, $f(t,x,y,u) = (u, 1-u)$, $L(t,x,y,u) = u$ and $g = 0$. We can easily see that Ω, $f(t,x,y,u)$ satisfy (10). The value function $V : \overline{\Omega} \to R$ is given by

$$V(t,x,y) = \begin{cases} 1-t+y & \text{if } x > 0 \text{ and } t-1 < y \leq 0, \\ 0 & \text{elsewhere in } [0,1] \times \overline{\Omega}. \end{cases}$$

The function V is a unique discontinuous solution of the Hamilton-Jacobi equation

$$-V_t + H(t,(x,y), -(V_x, V_y)) = 0,$$

where

$$H(t,(x,y),(p_1,p_2)) = \begin{cases} p_2 & \text{if } p_2 - p_1 + 1 \geq 0, \\ p_1 - 1 & \text{if } p_2 - p_1 + 1 < 0 \end{cases}$$

satisfying the terminal condition $V(1,x) = 0$.

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ON TOPOLOGICAL DIMENSION OF A SET OF SOLUTIONS OF FUNCTIONAL INCLUSIONS

Boris D. Gel'man

The first work devoted to evaluation of topological dimension of a set of fixed points of multi-valued maps was a paper [13]. Some other results in this direction were proved in [5], [10], [14], [15].

An evaluation on a global dimension of a set of solutions of the Cauchy problem for differential inclusions was obtained for the first time in the paper [10]. However, in this work restrictions on the differential inclusion were imposed too strong.

In the present paper some common theorems of topological dimension of a set of fixed points of multivalued maps are proved and we give applications of these theorems to an evaluation of local and global dimension of a set of solutions of the Cauchy problem for the Carateódory differential inclusions and for some functional equation. The obtained theorems are new and specify results of the paper [10].

1. Main facts of the theory of multivalued maps

Let Y be a subset of a Banach space E, we shall denote
by $K(Y)$ the set of all nonempty compact subsets in Y;
by $Kv(Y)$ the set of all nonempty compact convex subsets in Y.

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The multivalued map (m-map) metric space X in metric space Z is a correspondence associating to each point $x \in X$ a nonempty subset $F(x) \subset Z$ which is called an image of the point x.

Hereinafter, if images of m-map F are compact sets, we shall denote it as follows, $F : X \to K(Z)$. Similarly, the denotation $F : X \to Kv(Z)$ means that images $F(x)$ are convex compact sets.

Everywhere hereinafter the m-maps are denoted by capital letters, and single-valued by lower case ones.

Definition 1.1. M-map $F : X \to K(Z)$ is called *upper semicontinuous in a point* $x_0 \in X$, if for any open set $V \subset Z$, $V \supset F(x_0)$, there exists an open neighbourhood U of a point x_0 such that $F(U) \subset V$. If F is semicontinuous above in each point $x \in X$ then it is called *upper semicontinuous.*

Definition 1.2. M-map $F : X \to K(Z)$ is called *lower semicontinuous in a point* $x_0 \in X$ if for any open set $V \subset Z$ such that $F(x_0) \cap V \neq \phi$ there exists an open neighbourhood U of the point $x_\circ$ such that $F(x) \cap V \neq \phi$ for any $x \in U$. If F is semicontinuous below in each point $x \in X$ then it is called *lower semicontinuous.*

Definition 1.3. M-map $F : X \to K(Z)$ is called *continuous* if it is simultaneously semicontinuous above and below.

Definition 1.4. The continuous map $f : X \to Z$ is called a *continuous selection of m-map* F if for any point $x \in X$ the inclusion $f(x) \in F(x)$ is fulfilled.

Definition 1.5. M-map $F : X \to K(Z)$ is called *completely continuous*, if:

(1) F is semicontinuous above;
(2) the set $\overline{F(B)}$ is compact in Z for any bounded set $B \subset X$.

Definition 1.6. M-map $F : X \to K(Z)$ is called *strong completely continuous*, if:

(1) F is continuous m-map;
(2) the set $\overline{F(B)}$ is compact in Z for any bounded set $B \subset X$.

Let E be a Banach space, $X \subset E$, $F : X \to K(E)$ is a some m-map. A point x_0 will be called a *fixed point of the m-map* F if $x_0 \in F(x_0)$.

We shall explain some results about rotation of completely continuous multivalued vector fields with convex compact images in a Banach space.

Let U be a bounded open set in E and $F : \overline{U} \to Kv(E)$ be a completely continuous m-map. By a completely continuous multivalued vector field (as a mv-field) generated F, we call m-map $\Phi = i - F : \overline{U} \to Kv(E)$ defined by a condition: $\Phi(x) = x - F(x)$.

A mv-field Φ will be called non-singular on ∂U if $0 \notin \Phi(x)$ for any $x \in \partial U$.

The set of all completely continuous nondegenerate on ∂U of mv-fields will be denoted by $\Re(\overline{U}, \partial U)$.

Definition 1.7. The mv-fields $\Phi_0, \Phi_1 \in \Re(\overline{U}, \partial U)$ are called homotopic ($\Phi_0 \sim \Phi_1$) if there exists a completely continuous map $\Psi : \overline{U} \times [0,1] \to Kv(E)$ such that:

(a) $\Phi_0(x) = x - \Psi(x, 0)$, $\Phi_1(x) = x - \Psi(x, 1)$ for any $x \in \overline{U}$;
(b) $x \notin \Psi(x, \lambda)$ for any $x \in \partial U$, $\lambda \in [0,1]$.

Let Z be a group of integers. There exists the map $\gamma : \Re(\overline{U}, \partial U) \to Z$, satisfying the following conditions:

(1) if $\gamma(\Phi, \overline{U}) \neq 0$ then there exists a point $x_0 \in U$ such that $\Phi(x_0) \ni 0$;
(2) if $\Phi(x) \ni x - x_0$ for any $x \in \partial U$ then

$$\gamma(\Phi, \overline{U}) = \begin{cases} 1 & \text{if } x_0 \in U, \\ 0 & \text{if } x_0 \notin \overline{U}; \end{cases}$$

(3) if $\Phi_0 \sim \Phi_1$ then $\gamma(\Phi_0, \overline{U}) = \gamma(\Phi_1, \overline{U})$;
(4) if $\Phi_0(x) \subset \Phi_1(x)$ for any $x \in \overline{U}$ then $\gamma(\Phi_0, \overline{U}) = \gamma(\Phi_1, \overline{U})$;
(5) let $\{U_j\}_{j \in J}$ be a set of open nonintersected subsets of U such that $\overline{U} = \bigcup_{j \in J} \overline{U_j}$; if $\Phi \in \Re(\overline{U_j}, \partial U_j)$ for any $j \in J$ then $\gamma(\Phi, \overline{U_j}) \neq 0$ only for a finite number $j \in J$ and $\gamma(\Phi, \overline{U}) = \sum_{j \in J} \gamma(\Phi, \overline{U_j})$.

The number $\gamma(\Phi, \overline{U})$ is called rotation of a completely continuous mv-field Φ on the boundary of the domain U.

Proof of the existence of such a map γ and the study of its properties is contained, for example, in papers [3], [4].

The proof of these and other properties of multivalued maps can be found in the bibliographies [4], [8], [9].

2. Topological dimension of a set of fixed points of multivalued map

Let E be a Banach space, U be a bounded open set in E and $F : \overline{U} \to Kv(E)$ be a completely continuous m-map, $\Phi = i - F \in \Re(\overline{U}, \partial U)$. We shall denote by $N(\Phi, \overline{U})$ the set of fixed points F, i.e.

$$N(\Phi, \overline{U}) = \{x \in \overline{U} \mid x \in F(x)\} = \{x \in \overline{U} \mid 0 \in \Phi(x)\}.$$

It is obvious that $N(\Phi, \overline{U})$ is a compact set in E. We shall study dimension dim of this set. The basic properties of dimension dim are explained, for example, in [1], [6].

The following statement playing important role in further constructions can be put forward.

Lemma 2.1. *Let X be a metric compact space, $\dim(X) \leq n-1$. If E is a Banach space and $T : X \to Kv(E)$ is a lower semicontinuous m-map, satisfying conditions:*

(1) $T(x) \ni 0$ *for any* $x \in X$;
(2) $\dim(T(x)) \geq n$ *for any* $x \in X$,

then there exists a single-valued continuous map $f : X \to E$ such that $f(x) \neq 0$, $f(x) \in F(x)$ for any $x \in X$.

The proof of this lemma is contained in [13].

Basing on this lemma we shall prove the following statement.

Theorem 2.2. *Let $F : \overline{U} \to Kv(E)$ be a completely continuous m-map, $\Phi = i - F \in \Re(\overline{U}, \partial U)$. Let the following conditions be fulfilled:*

(a) $\gamma(\Phi, \overline{U}) \neq 0$;
(b) *there exists an open neighbourhood V, $N(\Phi, \overline{U}) \subset V \subset U$, and lower semicontinuous m-map $G : V \to Kv(E)$, $\dim(G(x)) \geq n$, $G(x) \subset F(x)$, for any $x \in V$.*

If $x \in G(x)$ for any $x \in N(\Phi, \overline{U})$ then $\dim(N(\Phi, \overline{U}) \geq n$.

Proof. We assume then to the contrary that $\dim(N(\Phi, \overline{U})) \leq n-1$. Let us consider the m-map $G_1 = i - G : V \to Kv(E)$. Contraction of this map $\widehat{G}_1 = G_1|_{N(\Phi,\overline{U})}$ satisfies the conditions of Lemma 2.1, therefore, there exists the selection $\widehat{\varphi} = i - \widehat{f} : N(\Phi, \overline{U}) \to E$ that $0 \neq \widehat{\varphi}(x) \in \widehat{G}_1(x)$ for any $x \in N(\Phi, \overline{U})$. By the Michal theorem about a continuous selection (see, for example, [11]), there exists a continuous selection $\varphi : V \to E$, $\varphi(x) \in G_1(x)$, for any $x \in V$ that$\varphi|_{N(\Phi,\overline{U})} = \widehat{\varphi}$. It is obvious, that $\varphi(x) \neq 0$, for any $x \in V$, since $0 \notin G_1(x)$, if $x \notin N(\Phi, \overline{U})$.

For any $x \in V$ we have the following inclusions: $\varphi(x) \in G_1(x) \subset \Phi(x)$.

We shall consider a new mv-field $\Phi_1 : \overline{U} \to Kv(E)$, defined by a condition:

$$\Phi_1(x) = \begin{cases} \varphi(x) & \text{if } x \in V, \\ \Phi(x) & \text{if } x \notin V. \end{cases}$$

It is obvious, that $0 \notin \Phi_1(x) \subset \Phi(x)$ for any $x \in \overline{U}$ and Φ_1 is a completely continuous mv-field. Then, $\gamma(\Phi_1, \overline{U}) = \gamma(\Phi(x), \overline{U}) \neq 0$, therefore, there exists a point x_0 such, that $\Phi_1(x_0) \ni 0$. However, it contradicts the construction of the m-map Φ_1. The obtained contradiction proves the theorem. □

Corollary 2.3. *Let $F : \overline{U} \to Kv(E)$ be a strongly, completely continuous m-map. If $\Phi = i - F \in \Re(\overline{U}, \partial U)$, $\gamma(\Phi, \overline{U}) \neq 0$ and $dim(F(x)) \geq n$ for any $x \in U$ then* $\dim(N(\Phi, \overline{U})) \geq n$.

The proof implies from Theorem 2.2.

3. Topological dimension of a set of solutions of the Cauchy problem

Let G be an open area in $R^1 \times R^n$ such that $[t_0, t_0 + h] \times B[x_0, r] \subset G$. Let $F : G \to Kv(R^n)$ be a m-map satisfying the following conditions:

(1) $F(t, \cdot) : B[x_0, r] \to Kv(R^n)$ is continuous m-map for almost all $t \in [t_0, t_0 + h]$;
(2) $F(\cdot, x) : [t_0, t_0 + h] \to Kv(R^n)$ is measurable for all $x \in B[x_0, r]$;
(3) there exist integrable in the Lebesgue sense non-negative functions

$$a, b : [t_0, t_0 + h] \to R^1$$

such that for any $x \in B[x_0, r]$ the following inequality is fulfilled:

$$\max_{y \in F(t,x)} ||y|| \leq a(t) + b(t)||x||$$

for almost all $T \in [t_0, t_0 + h]$.

We shall consider the following problem:

$$\dot{x} \in F(t, x),$$
$$x(t_0) = x_0.$$

By solution of this problem on the interval $[t_0, t_0 + d]$, $0 < d \leq h$, we shall call an absolutely continuous function $x : [t_0, t_0 + d] \to R^n$ such that $\dot{x}(t) \in F(t, x(t))$ almost everywhere and $x(t_0) = x_0$.

We denote by $\Sigma([t_0, t_0 + d], x_0)$ the set of solutions of this problem on the interval $[t_0, t_0 + d]$. It is known (see, for example, [4]) that $\Sigma([t_0, t_0 + d], x_0)$ is a nonempty subset for rather small d.

Let $U \subset C_{[t_0, t_0 + d]}$ be an open ball, defined by the condition:

$$U = \{x = x(\cdot) \in C_{[t_0, t_0+d]} \mid ||x_0 - x|| < r\}.$$

We consider the following m-maps (integral operators)

$$\Im_F(x) = \{y = y(\cdot) \in L^1_{[t_0, t_0+d]} \mid y(t) \in F(t, x(t)) \text{ for a.a. } t \in [t_0, t_0 + d]\},$$

$$\Phi(x)(t) = \left\{x_0 + \int_{t_0}^{t} y(\tau) d\tau \mid y(\cdot) \in \Im_F(x)\right\}.$$

Proposition 3.1. *M-map Φ on $\overline{U}$ satisfies the following conditions:*

(a) $\Phi(x) \neq \phi$ *for any* x;
(b) $\Phi(x) \subset Kv(C_{[t_0,t_0+d]})$;
(c) *the function* $x = x(\cdot) \in \overline{U}$ *is a fixed point of m-map* Φ *if and only if it is a solution of the Cauchy problem and* $\|x_0 - x(\cdot)\| \leq r$.

Proof of this statement is well known and is explained, for example, in [4].

We shall study a continuity of m-map Φ, and for this purpose we need the following lemma.

Lemma 3.2. *Let m-map* $F : G \to Kv(R^n)$ *satisfy the mentioned suppositions, then for any* $\delta > 0$, *there exists such compact set* $\Delta_\delta \subset [a,b]$, *that* $\mu([a,b] \setminus \Delta_\delta) \leq \delta$ *and the contraction* F *on* $\Delta_\delta \times B[x_0, r]$ *is a continuous m-map.*

The proof of this lemma will be carried out similarly to the proof of the appropriate theorem in [12] (see also [4]).

Theorem 3.3. *The operator Φ is strongly completely continuous m-map.*

Proof. As F is a continuous m-map on second variable, in particular, it is upper semicontinuous on this variable. It is well known that in this case the integral operator Φ is upper semicontinuous. We need to prove only lower semicontinuity of this operator.

Let V be an arbitrary open set in space $C_{[a,b]}$, $\widehat{x} = \widehat{x}(t) \in \overline{U}$ and $\widehat{y} \in V \cap \Phi(\widehat{x})$. Then there exists $\varepsilon_0 > 0$ such that the open neighbourhood $U_{\varepsilon_0}(\widehat{y}) \subset V$. By the definition of map Φ, $\widehat{y}(t) = x_0 + \int_a^t u(s)ds$, where $u(s) \in F(s, \widehat{x}(s))$, for almost all $s \in [a,b]$.

We shall prove that there exists positive number η such that as soon as $\|x - \widehat{x}\| < \eta$ then $V \cap \Phi(x) \neq \phi$.

We shall denote: $a_0 = \int_a^b a(s)ds$, $b_0 = \int_a^b b(s)ds$, $N = a_0 + b_0(r + \|x_0\|)$. We select now compact set Δ, satisfying Lemma 2.5 so that $\mu([a,b] \setminus \Delta) < \varepsilon_0/6N$.

Let us denote by $\widehat{F}$ a contraction of m-map F on set $\Delta \times B[x_0, r]$. If the map $\widehat{F}$ has compact images then from a continuity of this m-map follows its continuity as a map in the metric space $(K(R^n), h)$, where h is the Hausdorff metric (see, for example, [4]). If any continuous map of a compact set in metric space is uniformly continuous map then there exists such a number $\eta > 0$ that as soon as

$$|t_1 - t_2| < \eta, \; \|x_1 - x_2\| < \eta, \qquad (t_i, x_i) \in \Delta \times B[x_0, r], \; i = 1, 2,$$

then

$$h(F(t_1, x_1), F(t_2, x_2)) < \frac{\varepsilon_0}{3(b-a)}.$$

We consider the map $x = x(t) \in \overline{U}$ such that $\|x - \widehat{x}\| < \eta$. Then the set

$$M(t) = \widehat{F}(t, x(t)) \cap U_{(\varepsilon_0/(3(b-a)))}(u(t)) \neq \phi$$

for almost all $t \in \Delta$. It is easy to see that the m-map $M : \Delta \to R^n$ is a measurable one. Then M has the measurable selector $v_0 = v_0(t)$, i.e. v_0 is a measurable map and $v_0(t) \in M(t)$ for almost all $t \in \Delta$.

Let $v_1 = v_1(t)$ be an arbitrary measurable selection of m-map

$$P : ([a, b] \setminus \Delta) \to R^n, \qquad P(t) = F(t, x(t)).$$

We consider measurable map

$$v(t) = \begin{cases} v_0(t) & \text{when } t \in \Delta, \\ v_1(t) & \text{when } t \in [a, b] \setminus \Delta. \end{cases}$$

It is obvious, that $v(t) \in F(t, x(t))$ at almost all $t \in [a, b]$. Let $y(t) = x_0 + \int_a^t v(s)ds$. It is obvious, that $y \in \Phi(x)$. We will show now, that $y \in U_{\varepsilon_0}(\widehat{y})$. Really,

$$\begin{aligned} \|y - \widehat{y}\| &\leq \int_a^b \|u(s) - v(s)\|ds \\ &\leq \int_\Delta \|u(s) - v(s)\|ds + \int_{[a,b]\setminus\Delta} \|u(s) - v(s)\|ds \\ &\leq \mu(\Delta)\frac{\varepsilon_0}{3(b-a)} + \mu([a, b] \setminus \Delta)2N < \varepsilon_0. \end{aligned}$$

Therefore, $y \in (\Phi(x) \cap V)$ and that proves the theorem. □

Lemma 3.4. *Let $F : [a, b] \to Kv(R^n)$ by a measurable m-map, there exists a measurable set $A \subset [a, b]$ such that for any $t \in A$, $\dim(F(t)) \geq 1$. If $\mu(A) > 0$, for any whole positive number m, there exist the measurable selections $\{x_i(\cdot)\}_{i=1}^m$ m-maps F linearly independent on $[a, b]$.*

Proof of this lemma is contained in [10].

Theorem 3.5. *Let F satisfy the conditions* (1), (2), (3). *Let the set*

$$A = \{t \in [t_0, t_0 + h] \mid \dim((F(t, x)) \geq 1, \textit{ for any } x \in B[x_0, r]\}$$

be measurable and

$$\lim_{h\to 0} \frac{\mu(A \cap [t_0, t_0 + h])}{h} > 0.$$

Then there exists such a number β_0 that for any β, $0 < \beta \leq \beta_0$, the set

$$\Sigma = \Sigma([t_0, t_0 + \beta], x_0) \neq \phi,$$

and the dimension $\dim(\Sigma) = \infty$.

Proof. Let us consider $\widehat{a}(t) = \int_{t_0}^{t} a(s)ds$ and $\widehat{b}(t) = \int_{t_0}^{t} b(s)ds$. They are continuous, non-negative, monotonically growing functions, and $\widehat{a}(t_0) = \widehat{b}(t_0) = 0$. We choose $\beta_1 > 0$ so small, that $\widehat{a}(\beta_1)/(1 - \widehat{b}(\beta_1)) < r$. Let $U \subset C_{[t_0, t_0+\beta]}$, $0 < \beta \leq \beta_1$, be a set defined by a condition:

$$U = \{x = x(\cdot) \mid \|x_0 - x(\cdot)\| < r\}.$$

We consider an integral operator $\Phi : \overline{U} \to Kv(C_{[t_0, t_0+\beta]})$. If $z(\cdot) \in \Phi(x)$ and $z(t) = x_0 + \int_{t_0}^{t} y(s)ds$, where $y(s) \in F(s, x(s))$ at almost all $s \in [t_0, t_0 + \beta]$. Then

$$\|z - x_0\| = \left\| \int_{t_0}^{t} y(s)ds \right\| \leq \int_{t_0}^{\beta_1} a(s)ds + r \int_{t_0}^{\beta_1} b(s)ds < r.$$

Therefore, $\Phi : \overline{U} \to Kv(U)$ and it has not fixed points on ∂U.

Since

$$\lim_{h \to 0} \frac{\mu(A \cap [t_0, t_0 + h])}{h} > 0,$$

then there exists a number $\beta_2 > 0$ such that for any $\beta > 0$ and $\beta \leq \beta_2$, $\mu(A \cap [t_0, t_0 + \beta]) > 0$.

Let $\beta_0 = \min\{\beta_1, \beta_2\}$, $0 < \beta \leq \beta_0$. Then $\Phi : \overline{U} \subset C_{[t_0, t_0+\beta]} \to Kv(U)$ is strongly, completely continuous m-map.

We show that for any natural number m and for any $x \in \overline{U}$ the topological dimension $\dim(\Phi(x)) \geq m$. For this purpose, due to convexity of the set $\Phi(x)$, it is enough to prove existence $m + 1$ of the linearly independent point $z_0(\cdot), \cdots, z_m(\cdot) \in \Phi(x)$.

As due to Lemma 3.4 m-maps $F_x(\cdot) = F(\cdot, x(\cdot)) : [t_0, t_0 + \beta] \to Kv(R^n)$ has $m+1$ linearly independent selection $\{y_i(\cdot)\}_{i=0}^{m}$, then $z_i(t) = x_0 + \int_{t_0}^{t} y_i(s)ds$, $i = 0, 1, \cdots, m$, lay in an image $\Phi(x)$ and are linearly independent.

Then, due to Corollary 2.3, we have:

$$\dim(N(i - \Phi, \overline{U})) = \dim(\Sigma) \geq m$$

for any m. Therefore, $\dim(\Sigma) = \infty$, as was to be proved. □

4. Local properties of dimension of the set of fixed points of multivalued maps

We shall consider local properties of dimension of the set $N(\Phi, \overline{U})$. Let U be a bounded open area in E and $\Phi = i - F \in \Re(\overline{U}, \partial U)$.

Lemma 4.1. *Let X be a metric space and $F : X \to Kv(E)$ be a lower semicontinuous m-map. Then for any point $x_0 \in X$ and any integer $n \leq \dim(F(x_0))$ there exists $\varepsilon > 0$ such that for any point $x \in U_\varepsilon(x_0)$ an inequality is carried out: $\dim(F(x)) \geq n$.*

The proof follows from the Michael Theorem.

Theorem 4.2. *Let $F : \overline{U} \to Kv(E)$ be a strongly, completely continuous m-map and $x_0 \in U$ be a fixed point of m-map F. If there exists an open neighbourhood $W \subset U$ of the point x_0 and completely, continuous selection $f : W \to E$ m-maps F such, that:*

(1) *the point x_0 is a unique singular point of a completely, continuous field $\varphi = i - f$;*
(2) $\mathrm{ind}(\varphi, x_0) \neq 0$*;*

then for any $\varepsilon > 0$, the dimension

$$\dim(N_{x_0,\varepsilon}(i - F, \overline{U})) \geq \dim(F(x_0)),$$

where $N_{x_0,\varepsilon}(i - F, \overline{U}) = \{x \in N(i - F, \overline{U}) \mid \|x - x_0\| \leq \varepsilon\}$.

Proof. Since the dimension dim is monotone (see, for example, [1]), it is enough to prove this inequality only for rather small $\varepsilon > 0$.

Note, that due to lower semicontinuity of m-map F, for any number $n \leq \dim(F(x_0))$, there exists $\varepsilon_0 > 0$ and neighbourhood $U_{\varepsilon_0}(x_0) \subset \overline{U_{\varepsilon_0}(x_0)} \subset W$ such that for any $x \in U_{\varepsilon_0}(x)$ the dimension $\dim(F(x)) \geq n$.

We shall consider ε, $0 < \varepsilon < \varepsilon_0$. Let $V = U_{\varepsilon_0}(x_0)$, $\widehat{F} : \overline{V} \to Kv(E)$ be defined by the condition:

$$\widehat{F}(x) = \begin{cases} F(x) & \text{if } \|x - x_0\| < \varepsilon, \\ \dfrac{\|x - x_0\| - \varepsilon}{\varepsilon_0 - \varepsilon} f(x) + \dfrac{\varepsilon_0 - \|x - x_0\|}{\varepsilon_0 - \varepsilon} F(x) & \text{if } \varepsilon \leq \|x - x_0\| \leq \varepsilon_0. \end{cases}$$

It is obvious, that $\widehat{F}(x) \subset F(x)$ for any $x \in \overline{V}$, $\widehat{F}$ is completely continuous m-map, and the following inclusion is fulfilled:

$$N(i - \widehat{F}, \overline{V}) \subset N(i - F, \overline{V}) \subset N_{x_0,\varepsilon_0}(i - F, \overline{U}).$$

Note also, that $\dim(\widehat{F}(x)) = \dim(F(x))$ for any $x \in V$. As $\widehat{F}(x) = f(x) \neq x$ for any $x \in \partial V$ then all conditions of Corollary 2.3 are carried out, hence,

$$\dim(N_{x_0,\varepsilon_0}(i - F, \overline{U})) \geq \dim(N(i - \widehat{F}, \overline{V})) \geq n,$$

it proves the theorem since number n was taken arbitrary. □

Corollary 4.3. *Let F be a strongly completely continuous m-map, let there exist such an open neighbourhood W of the point $x_0 \in N(i-F,\overline{U})$, $W \subset U$, that for any $x \in W$ the point $x_0 \in F(x)$. Then*

$$\dim(N_{x_0,\varepsilon}(i-F,\overline{U})) \geq \dim(F(x_0))$$

for any $\varepsilon > 0$.

Proof. As a selection f we take the map $f : W \to E$ such that $f(x) = x_0$. It is obvious, that x_0 is a unique fixed point f and $\operatorname{ind}(i - f, x_0) = 1$. Now the validity of this statement follows from Theorem 4.2

Let $\Phi = i - F : \overline{U} \to Kv(E)$ be a strongly completely, continuous mv-field, $x_0 \in N(\Phi,\overline{U}) \subset U$. We shall consider function $\beta(x) = \rho(x_0, F(x))$.

Theorem 4.4. *If there exists $\varepsilon_0 > 0$ such that for any $x \in \overline{U}$, $\|x - x_0\| < \varepsilon_0$, the following inequality is fulfilled*

$$\beta(x) \leq k\|x - x_0\|, \qquad k \in [0,1),$$

then $\dim(N_{x_0,\varepsilon}(\Phi, B) \geq \dim(F(x_0))$ for any $\varepsilon > 0$.

For the proof of this theorem we need the following lemma.

Lemma 4.5. *Let conditions of the Theorem 4.4 be fulfilled, then for any number k_1, $k < k_1 < 1$, there exists a continuous map $f : U_{\varepsilon_0}(x_0) \to E$ satisfying conditions:*

(a) *f is a continuous selection F,*
(b) *$\|x_0 - f(x)\| \leq k_1\|x - x_0\|$ for any $x \in U_{\varepsilon_0}(x_0)$.*

Proof. Let the number k_1 satisfy an inequality $k < k_1 < 1$. We consider the number $\lambda \in (0, (k_1 - k)/k]$ and the function $\alpha(x) = (1 + \lambda)\beta(x)$.

Then the m-map $F_1 : U_{\varepsilon_0}(x_0) \to Kv(E)$ is defined as follows

$$F_1(x) = \{y \in F(x) \mid \|y - x_0\| \leq \alpha(x)\}.$$

It is obvious, that $F_1(x) \neq \phi$ for any x and is a convex closed set. It is easy to prove also that m-map F_1 is lower semicontinuous. Therefore, it has a continuous selection f.

Then for any $x \in U_{\varepsilon_0}(x_0)$ an inequality is fair:

$$\|x_0 - f(x)\| \leq \alpha(x) = (1 + \lambda)\beta(x) \leq k_1\|x - x_0\|.$$

The lemma is proved.

Proof of Theorem 4.4. It is enough to check fulfillness of conditions of Theorem 4.2, where f satisfies two conditions of Lemma 4.5.

It is obvious, that $\varphi(x) = x - f(x) \neq 0$ for any $x \neq x_0$, and $\varphi(x_0) = 0$. And, $||\varphi(x)|| \geq (1 - k_1)||x - x_0||$ for any x. Note, that $\text{ind}(\varphi, x_0) = 1$, i.e. f maps a ball $B[x_0, \varepsilon]$ into itself. Therefore, all conditions of Theorem 4.2 are fulfilled, so it proves the theorem. □

Corollary 4.6. *Let the completely continuous mv-field $\Phi = i - F : \overline{U} \to Kv(E)$ be nonsingular on ∂U and satisfy conditions:*

(a) $N(\Phi, \overline{U}) \neq \phi$,

(b) *F is contracting, i.e. for any $x, y \in \overline{U}$ an inequality is carried out:*

$$h(F(x), F(y)) \leq k||x - y||, \qquad 0 \leq k < 1.$$

Then for any point $x_0 \in N(\Phi, \overline{U})$ and any number $\varepsilon > 0$ we have:

$$\dim(N_{x_0,\varepsilon}(\Phi, \overline{U})) \geq \dim(F(x_0)).$$

Proof. Let $x_0 \in N(\Phi, \overline{U})$, i.e. x_0 is a fixed point of m-map F. The following inequality takes place:

$$\rho(x_0, F(x)) \leq h(F(x_0), F(x)) \leq k||x - x_0||$$

for any $x \in \overline{U}$. If the field Φ is continuous and completely continuous, then validity of this statement follows from Theorem 4.4. □

5. Local dimension of the set of solutions of the Cauchy problem for differential inclusions

We apply the theorems, proved in the previous paragraph, to study local dimension of a solutions set of the Cauchy problem for one class of differential inclusions.

Let G be an open area in $R^1 \times R^n$ such that $[t_0, t_0 + h] \times B[x_0, r] \subset G$. Let $F : G \to Kv(R^n)$ be a m-map satisfying the following conditions:

(1) $F(t, \cdot) : B[x_0, r] \to Kv(R^n)$ is Lipschitzian with a constant α for almost all $t \in [t_0, t_0 + h]$,

(2) $F(\cdot, x) : [t_0, t_0 + h] \to Kv(R^n)$ is measurable for all $x \in B[x_0, r]$,

(3) there exist integrable functions $a, b : [t_0, t_0 + h] \to R^1$ in the Lebesgue sense such that for any $x \in B[x_0, r]$ the following inequality is carried out:

$$\max_{y \in F(t,x)} ||y|| \leq a(t) + b(t)||x||$$

for almost all $T \in [t_0, t_0 + h]$.

We consider the following problem:

$$\dot{x} \in F(t,x),$$
$$x(t_0) = x_0.$$

Let us denote by $\Sigma([t_0, t_0+d], x_0)$ a set of solutions of this problem on the interval $[t_0, t_0 + d]$.

Let $U \subset C_{[t_0,t_0+d]}$ be an open ball defined by the condition

$$U = \{x = x(\cdot) \in C_{[t_0,t_0+d]} \mid \|x_0 - x\| < r\}.$$

As well as in Section 3, we consider the following integral operators: $\Im_F$ and Φ.

Proposition 5.1. *The m-map Φ is Lipschitzian on $\overline{U}$.*

The proof of this statement is explained in [10].

Theorem 5.2. *Let F satisfy the conditions* (1), (2), (3). *Let the set*

$$A = \{t \in [t_0, t_0 + h] \mid \dim(F(t,x) \geq 1, \text{ for any } x \in B[x_0, r]\}$$

is measurable and

$$\lim_{h \to 0} \frac{\mu(A \cap [t_0, t_0 + h])}{h} > 0.$$

Then, there exists such a number β_0 that for any β, $0 < \beta \leq \beta_0$, the set

$$\Sigma = \Sigma([t_0, t_0 + \beta], x_0) \neq \phi$$

and for any $\varepsilon > 0$ and any solution $y \in \Sigma$ dimensionality $\dim(\Sigma_{y,\varepsilon}) = \infty$, *where*

$$\Sigma_{y,\varepsilon} = \{z \in \Sigma \mid \|z - y\| \leq \varepsilon\}.$$

Proof. We consider $\widehat{a}(t) = \int_{t_0}^{t} a(s)ds$ and $\widehat{b}(t) = \int_{t_0}^{t} b(s)ds$. They are continuous, non-negative, monotonically increasing function and $\widehat{a}(t_0) = \widehat{b}(t_0) = 0$. We choose $\beta_1 > 0$ so small, that

$$\frac{\widehat{a}(\beta_1)}{1 - \widehat{b}(\beta_1)} < r.$$

Let $U \subset C_{[t_0,t_0+\beta]}$, $0 < \beta \leq \beta_1$ be a set defined by the condition:

$$U = \{x = x(\cdot) \mid \|x_0 - x(\cdot)\| < r\}.$$

We consider an integral operator $\Phi : \overline{U} \to Kv(C_{[t_0,t_0+\beta]})$. If $z(\cdot) \in \Phi(x)$ and

$$z(t) = x_0 + \int_{t_0}^{t} y(s)ds,$$

where $y(s) \in F(s, x(s))$ at almost all $s \in [t_0, t_0 + \beta]$ then

$$\|z - x_0\| = \left\| \int_{t_0}^{t} y(s)ds \right\| \leq \int_{t_0}^{\beta_1} a(s)ds + r \int_{t_0}^{\beta_1} b(s)ds < r.$$

Therefore, $\Phi : \overline{U} \to Kv(U)$ and has not fixed points on ∂U.

We select β_2 so small that $\alpha\beta_2 < 1$, where α by a constant of the Lipschitz of m-map F. Since

$$\lim_{h \to 0} \frac{\mu(A \cap [t_0, t_0 + h]}{h} > 0,$$

then there exists such a number $\beta_3 > 0$, that for any β, $0 < \beta \leq \beta_3$ we get $\mu(A \cap [t_0, t_0 + \beta]) > 0$. Let $\beta_0 = \min\{\beta_1, \beta_2, \beta_3\}$, $0 < \beta \leq \beta_0$. Then $\Phi : \overline{U} \subset C_{[t_0, t_0+\beta]} \to Kv(U)$ is completely continuous contracting m-map.

We show now that for any natural number m and for any $x \in \overline{U}$ topological dimension $\dim(\Phi(x) \geq m$. For this purpose, due to convexity of a set $\Phi(x)$, it is enough to prove existence $m + 1$ of a linearly independent point $z_0(\cdot), \cdots, z_m(\cdot) \in \Phi(x)$.

As due to Lemma 3.4 the m-maps $F_x(\cdot) = F(\cdot, x(\cdot)) : [t_0, t_0+\beta] \to Kv(R^n)$ has linearly independent selections $\{y_i(\cdot)\}_{i=0}^{m}$, then

$$z_i(t) = x_0 + \int_{t_0}^{t} y_i(s)ds, \qquad i = 0, 1, \cdots, m,$$

lay in an image $\Phi(x)$ and they are linearly independent.

Then, due to Corollary 4.6 we have:

$$\dim(N_{y,\varepsilon}(i - \Phi, \overline{U})) = \dim(\Sigma_{y,\varepsilon}) \geq \dim(\Phi(y)) \geq m$$

for any m. Therefore, $\dim(\Sigma_{y,\varepsilon}) = \infty$, as was to be proved. □

6. Of dimension of a set of solutions of some operator equation

Let E_1, E_2 be two Banach spaces $a : E_1 \to E_2$ be a continuous linear surjective operator and $f : E_1 \to E_2$ be a completely continuous operator. We consider the following equation:

$$a(x) = f(x).$$

We denote by $N(a, f)$ the set of solutions of this equation i.e.

$$N(a, f) = \{x \in E_1 \mid a(x) = f(x)\}.$$

In the [15] the following theorem is proved.

Theorem 6.1. *If the set* $f(E_1)$ *is rather compact, then the set* $N(a, f)$ *is not empty and* $\dim(N(a, f)) \geq \dim(a^{-1}(0))$.

We will be interested in local dimension of the set $N(a, f)$ in the neighbourhood of some fixed solution of the given equation.

Let $x_0 \in N(a, f)$ be some fixed solution of our equation and $y_0 = f(x_0) = a(x_0) \in E_2$. We consider multivalued map $a^{-1} : E_2 \to E_1$. It is known that this map is lower semicontinuous therefore it has the continuous selector $g : E_2 \to E_1$ such that $g(y_0) = x_0$. The following theorem takes place.

Theorem 6.2. *Let the multivalued map* a^{-1} *have a continuous selection* g *satisfying conditions:*

(1) $g(y_0) = x_0$,
(2) *the point* y_0 *is an isolated fixed point of the completely continuous map* $p = f \circ g : E_2 \to E_2$,
(3) *index* $\operatorname{ind}(i - p; y_0) \neq 0$.

Then

$$\dim(N_{x_0,\varepsilon}(a, f)) \geq \dim(a^{-1}(0)),$$

where $N_{x_0,\varepsilon}(a, f) = \{x \in N(a, f) \mid \|x - x_0\| \leq \varepsilon\}$.

Proof. Let number n be such that $0 \leq n \leq \dim(a^{-1}(0))$. Then in the subspace $\operatorname{Ker}(a) = \{x \in E_1 \mid a(x) = 0\}$ there exist n linearly independent, unit vectors $e_1, e_2, \dots, e_n$. We shall consider the space $E_3 = E_2 \times R^n$ and multivalued map $F : E_3 \to Kv(E_3)$ defined by a condition:

$$F(y, u) = \left\{(z, v) \mid z = f\left(g(y) + \sum_{i=1}^{n} \xi_i e_i\right),\ v \in R^n,\ \|v\| \leq \eta\right\},$$

where $u = (\xi_1, \xi_2, \dots, \xi_n)$. It is easy to note that this multivalued map is strongly, completely continuous.

Easily to check also that the point (y, u) is a fixed point of map F, in the only case, when the following two conditions are carried out:

(1) $u \in R^n$, $\|u\| \leq \eta$,
(2) $a(g(y) + \sum_{i=1}^n \xi_i e_i) = f(g(y) + \sum_{i=1}^n \xi_i e_i) = y$.

Remark that map $\widehat{g} : E_3 \to E_1$, $\widehat{g}(y, u) = g(y) + \sum_{i=1}^n \xi_i e_i$, is a bijection. Really, if $(y_1, u_1) \neq (y_2, u_2)$ then $\widehat{g}(y_1, u_1) \neq \widehat{g}(y_2, u_2)$. Thus, different fixed points of the map F due to the map $\widehat{g}$ different solutions of our initial equation correspond. It allows, for studying the dimension of a set of solutions of our equation, to apply the theorems of dimension of a set of fixed points of multivalued maps. We shall consider the selection of the multivalued map F defined by a condition:

$$b(y,u) = \left(f\left(g(y) + \sum_{i=1}^{n} \xi_i e_i\right), 0\right).$$

It is obvious that $b : E_3 \to E_2 \times 0 \subset E_3$. Fixed points of this map are pairs $(y, 0)$, where the point y is a fixed point of the map $p = f \circ g : E_2 \to E_2$. Due to conditions of the theorem, the point $(y_0, 0)$ is an isolated singular point of a completely continuous field of vectors $i - b$, and, due to the theorem of a contraction, $\operatorname{ind}(i - b, (y_0, 0)) = \operatorname{ind}(i - p, y_0) \neq 0$.

Let U be an arbitrary limited open neighbourhood in space E_2, $y_0 \in U$. We shall consider an open set $W = U \times U_\eta(0) \subset E_3$, where $U_\eta(0)$ by an open ball in space R^n. It is obvious that the map F is defined on $\overline{W}$ and satisfies the conditions of Theorem 4.2 Therefore, $\dim(N_{(y_0,0),\delta}(i - F, \overline{W})) \geq n$ for any $\delta > 0$.

We shall fix arbitrary positive number ε and we shall show that the set $N_{x_0,\varepsilon}(a, f)$ contains a compact set of dimension greater or equal to n. As the map g is continuous in a point y_0, then there exists such $\delta > 0$ that $\|g(y_0) - g(y)\| < \varepsilon/2$ as $\|y - y_0\| \leq \delta$. Without loss of generality it is possible to consider that $\delta \leq \varepsilon/2$.

It is obvious that the set $N_{(y_0,0),\delta}(i - F, \overline{W})$ is compact, and as map $\widehat{g}$: $N_{(y_0,0),\delta}(i - F, \overline{W}) \to E_1$ is a bijection, then $\widehat{g}$ is a homeomorphism on a range of values. Due to mentioned constructions, $\widehat{g}(N_{(y_0,0),\delta}(i - F, \overline{W})) \subset N(a, f)$, and if $(y, u) \in N_{(y_0,0),\delta}(i - F, \overline{W})$,

$$\|x_0 - \widehat{g}(y,u)\| = \left\|x_0 - g(y) - \sum_{i=1}^{n} \xi_i e_i\right\| \leq \|g(y_0) - g(y)\| + \sum_{i=1}^{n} \xi_i \leq \varepsilon,$$

i.e.

$$\widehat{g}(N_{(y_0,0),\delta}(i - F, \overline{W})) \subset N_{x_0,\varepsilon}(a, f).$$

This inclusion proves the theorem. □

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ACCESSIBLE POINTS AND SUBMANIFOLDS OF MECHANICAL SYSTEMS WITH SET-VALUED FORCES ON RIEMANNIAN MANIFOLDS

YURI GLIKLIKH

In this paper we put together the results, published in [5]-[13], on qualitative behaviour of solutions of the second order differential inclusions on Riemannian manifolds given in terms of covariant derivatives. We study the question of whether or not two points m_0 and m_1 in the configuration space can be connected by a solution. Those inclusions appear naturally, e.g., in non-linear mechanics through the approach to investigation of differential equations with discontinuous right-hand sides, suggested by A. F. Filippov (here we modify the approach in such a way that we cover the case of mechanical systems with discontinuous forces on Riemannian manifolds). Note that the equation of motion of a mechanical system with control may also be presented in terms of differential inclusions of the same type, and for such systems the problem under consideration means whether they have property of global controllability.

For mechanical systems with non-holonomic constraints the natural question is to achieve a certain submanifold in the configuration space. We consider this problem at the end of the paper.

It is a well-known fact that for a second order differential equation on the Euclidean space there exists a trajectory joining two given points provided that the right-hand side of the differential equation is bounded and continuous. More precisely, for any two points m_0 and m_1 and any interval $[a, b]$, there exists a solution $m(t)$ such that $m(a) = m_0$ and $m(b) = m_1$. When the right-hand side is linearly bounded with respect to velocities, some similar results are known for

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small intervals of time. The situation becomes much more complex for a non-linear configuration space. In Section 2 below we illustrate this by three examples of mechanical systems on two-dimensional sphere. In the first example, the force field is smooth and independent of time and velocity (and so it is bounded). However, none of the trajectories beginning at the North Pole reaches the South Pole. In the second example, the force field is still bounded, autonomous, and smooth but now depends on the velocity. In this case there is no trajectory connecting any two antipodal points on the sphere. In the third example, we consider a gyroscopic force on S^2. (Hence, the force field is linear in velocity.) The behavior of trajectories in this system turns out to be quite similar to the second example.

The difference in the behavior of trajectories on flat and "curved" configuration spaces has a deep geometric reason. Notice that antipodal points on two-dimensional sphere are conjugate along all geodesic curves joining them. Below we show that if the (set-valued upper semicontinuous and having convex closed images) force field is bounded or has linear growth in velocity, then for any two points m_0 and m_1, there exists a trajectory joining m_0 and m_1 provided that the points are not conjugate along some geodesic. The fact that conjugate points are absent on flat manifolds (e.g. in Euclidean spaces) leads to the classical result mentioned above (obviously continuous single-valued force satisfies the above conditions). Analogues of these statements for systems with constraints are also considered.

For studying the qualitative behaviour of solutions we use some analogue of integral operators on manifolds, based on Riemannian parallel translation (see [11]-[13] for details). Alternative idea, to use the theory of set-valued dynamical systems, leads to various set-valued maps with complicated structure of images[1] and cannot be applied here.

1. Mechanical systems with discontinuous forces and systems with control. Differential inclusions on manifolds

Consider a mechanical system with a discontinuous force field. Such fields appear, for example, in systems with dry friction, switching, or with motion in several media having different resistance forces, etc. When the configuration space is linear, the following method (suggested by A. F. Filippov) is often used to study systems with a discontinuous force: first, one extends the discontinuous force field to a set-valued vector field with convex images, and then passes from a differential equation of motion to a corresponding differential inclusion. In this section, we generalise this method to non-linear configuration spaces.

Note that the equation of motion of a mechanical system with control may also be reduced to a differential inclusion. In this case, the set-valued force, a

[1] Note the class of aspherical set-valued maps appeared in dynamical systems. This class was discovered in [15], in [2] the topological degree and Lefschetz number were constructed for it. Recently this class together with the degree was rediscovered and named uv-maps (see e.g. [14]).

subset in every tangent space, is formed by all values of the force for all possible values of the controlling parameter.

Consider a locally bounded vector field f on a finite-dimensional manifold M. The vector field f is not assumed continuous, nor even measurable. For any point m_0, let us define a subset $R(m_0) \subset T_{m_0}M$ as follows. The set $R(m_0)$ is formed by the limits of all sequences $f(m_k)$ as $m_k \to m_0$ with $m_k \neq m_0$. It is easy to see that

$$R(m_0) = \bigcap_{\varepsilon>0} \left\{ \mathrm{cl} \left[\left(\bigcup_{m \in U_\varepsilon} f(m) \right) \setminus f(m_0) \right] \right\},$$

where U_ε is the ε-neighborhood of the point m_0 and cl means the closure.

Definition 1.1. The set $F(m_0) = \overline{\mathrm{co}} R(m_0) \subset T_{m_0}M$, where $\overline{\mathrm{co}}$ denotes the convex hull, is called *the essential extension of the field f at m_0*.

The essential extension F is a set-valued mapping which assigns a subset in $T_{m_0}M$ to $m_0 \in M$. It is natural to call this mapping a set-valued vector field. Note that $F = f$ if f is continuous.

Theorem 1.2. *The set-valued vector field F is upper semicontinuous.*

Proof. Let $\delta > 0$ be a real number. Fix a metric ρ on TM which gives rise to a topology equivalent to that on the tangent bundle. Denote the δ-neighborhoods of $R(m_0)$ and $F(m_0)$ by $R^\delta(m_0)$ and $F^\delta(m_0)$, respectively. We prove that for any δ and any $m \in M$, there exists a neighborhood $U(m) \subset M$ of m such that $R(m') \subset R^\delta(m)$ for every $m' \in U(m)$ and, therefore, $F(m') \subset F^\delta(m)$.

By the definition of set $R(m)$, there exists a neighborhood $U(m)$ of m such that for all $m' \in U(m)$ we have $\rho(f(m'), R(m)) < \delta$. Then there exists an open neighborhood $V(m') \subset U(m)$ of the point m' such that the inequality $\rho(f(m''), R(m')) < \delta$ is satisfied for every $m'' \in V(m')$. Pick a sequence $m''_k \to m'$ in $V(m')$. We have

$$\lim \rho(f(m''_k), R(m)) = \rho \lim(f(m''_k), R(m)) < \delta.$$

Hence, $R(m') \subset R^\delta(m)$ and $F(m') \subset F^\delta(m)$. □

Now consider a mechanical system with the configuration space M and the kinetic energy $\mathrm{K}(X) = \langle X, X \rangle/2$, where $\langle\, , \rangle$ is the Riemannian metric on M. Let $\alpha(t, m, X)$ be a force field that we require to only be locally bounded in all variables. (Note that, as above, we do not assume that α is continuous or even measurable.) Consider the vector field $Z(m, X) + \alpha(t, m, X)^l$ (i.e., a second order differential equation on M; see [13]), where Z is the geodesic spray of the Levi-Civitá connection of $\langle\, , \rangle$ and $\alpha(t, m, X)^l$ is the vertical lift of $\alpha(t, m, X)$ to the point $(m, X) \in TM$. It is easy to see that the essential extension (with respect to all variables) of $Z(m, X) + \alpha(t, m, X)$ may be written in the form

$$Z(m, X) + \mathbb{A}(t, m, X)^l, \tag{1.1}$$

where $\mathbb{A}(t, m, X)^l$ is the vertical lift of the essential extension $A(t, m, X)$ of $\alpha(t, m, X)$ to the point (m, X). Note that $A(t, m, X) = \overline{\mathrm{co}} Q(t, m, X)$, where

$Q(t,m,X)$ is the set of limit points of all sequences $\alpha(t_k,m_k,X_k)$ such that $(t_k,m_k,X_k) \to (t,m,X)$, $X_k \in T_{m_k}M$, and $(t_k,m_k,X_k) \neq (t,m,X)$.

From now on, we focus on the differential inclusion in TM given by the formula

$$\dot\gamma(t) \in Z(\gamma(t)) + \mathbb{A}(t,\gamma(t))^l. \tag{1.2}$$

Definition 1.3. A solution of (1.2) is an *absolutely continuous curve* $\gamma(t)$ in TM which almost everywhere satisfies (1.2).

Alternatively, making use of covariant derivatives, we consider the following differential inclusion on M:

$$\frac{D}{dt}\dot m(t) \in \mathbb{A}(t,m(t),\dot m(t)). \tag{1.3}$$

Definition 1.4. A solution of (1.3) is a C^1-curve $m(t)$ in M such that $\dot m(t)$ is absolutely continuous and (1.3) is almost everywhere satisfied.

Taking into account (1.1) and the definition of covariant derivative D/dt, it is easy to check that (1.2) and (1.3) are equivalent. More precisely, this means that $m(t)$ is a solution of (1.3) if and only if $\dot m(t)$, regarded as a curve in TM, is a solution of (1.2).

Definition 1.5. A solution of (1.3) is called a trajectory of the mechanical system with a discontinuous force field $\mathbb{A}$.

It is easy to see that Definition 1.5 is justified from the physical point of view. As we have mentioned above, for a flat configuration space the reasons supporting the definition are discussed, for example, in [4].

The right hand side of (1.2) is an upper semicontinuous set-valued vector field with convex images. This implies that locally there exists a solution of the Cauchy problem for (1.2) (e.g., [1], [4]). Thus, for any initial conditions $m \in M$ and $X \in T_mM$, inclusion (1.3) has a solution on a sufficiently small interval.

Note that an interesting question for applications in physics is whether or not the local solution of (1.3) is unique. Certain uniqueness conditions are found in [4].

Another class of mechanical systems involving inclusions like (1.3) are mechanical systems with control. Let the force field $\alpha(t,m,X,u)$ depend on the parameter $u \in U$. We define the set-valued vector field $\mathbb{A}(t,m,X)$ on TM as

$$\mathbb{A}(t,m,X) = \bigcup_{u\in U} \alpha(t,m,X,u).$$

Now we have to assume that this field is upper semicontinuous and has closed convex images. The solution of (1.3) is a trajectory of the control system for a time-dependent control $u(t)$.

2. Examples

Example 1. Consider the mechanical system on the unit sphere S^2 in R^3 with the force field $\alpha(\mathbf{r}) = (-y, x, 0)$, where $\mathbf{r} = (x, y, z) \in S^2$. The motion of the system is given by the following equations in R^3:

$$\ddot{\mathbf{r}} = \alpha(\mathbf{r}) - 2K \cdot \mathbf{r}$$

or, equivalently,

$$\ddot{x} = -y - 2K \cdot x, \quad \ddot{y} = x - 2K \cdot y, \quad \ddot{z} = -2K \cdot z, \tag{2.1}$$

where

$$K = \frac{||\dot{\mathbf{r}}||}{2} = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2}$$

is the kinetic energy. To obtain these equations, one applies the d'Alembert principle (see, e.g., [16]) to the holonomic constraint $F(\mathbf{r}) = x^2 + y^2 + z^2$.

Denote the North and South Pole of the sphere by $N = (0, 0, 1)$ and $S = (0, 0, -1)$, respectively. Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be the trajectory of the system such that $\mathbf{r}(t_0) = S$ for some t_0 and $\dot{\mathbf{r}}(t_0) = V \neq 0$. Note that if $V = 0$, then $\mathbf{r}(t) \equiv S$. It is clear that $V \in T_S S^2$ must have the form $(X, Y, 0)$. We claim that the kinetic energy increases along $\mathbf{r}(t)$ until $\mathbf{r}$ hits the North or South Pole. By (2.1) we have

$$\dot{K}(\mathbf{r}(t)) = \dot{x}y + \dot{y}x \quad \text{and} \quad \ddot{K}(\mathbf{r}(t)) = x^2 + y^2.$$

Note also that $\dot{K}(\mathbf{r}(t_0)) = 0$. This means that $\dot{K}(\mathbf{r}(t)) > 0$, unless $\mathbf{r}(t) = S$ or $\mathbf{r}(t) = N$. In fact, the derivative $\dot{K}(\mathbf{r}(t))$ is also increasing. Since $\dot{K}(N) = \dot{K}(S) = 0$, we have $\dot{K}(\mathbf{r}(t)) \neq N$ for any $t > t_0$.

To clarify the geometric picture, consider the function $z(t) = z(\mathbf{r}(t))$. Let $t_1 > t_0$ be such that $\dot{z}(t_1) = 0$ and $z(t)$ is increasing on $[t_0, t_1]$. The last equation in (2.1) implies that $z(t_1) > 0$ and, as a consequence, $\ddot{z}(t_1) < 0$, i.e., $z(t_1)$ is a local maximum of $z(t)$. Since K is increasing along $\mathbf{r}(t)$, we see that $z(t_1) < 1$. In the same way, one may show that

$$\operatorname{sign} z(t_i) = (-1)^{i+1} \quad \text{and} \quad |z(t_i)| > |z(t_{i+1})|$$

for all points $t_1 < t_2 < \ldots$ such that $\dot{z}(t_i) = 0$.

Therefore, the trajectory $r(t)$ goes to the equator of S^2 and oscillates near it. In particular, the trajectory never reaches the point $N = (0, 0, 1)$.

Example 2. Let us replace the field α in the system of Example 1 by the force field

$$\Omega(\mathbf{r}, \dot{\mathbf{r}}) = \frac{[\dot{\mathbf{r}}, \mathbf{r}]}{1 + ||\dot{\mathbf{r}}||},$$

where $[\,,\,]$ is the vector product in R^3. The equation of motion of the mechanical system is as follows:

$$\ddot{\mathbf{r}} = \Omega(\mathbf{r}, \dot{\mathbf{r}}) - 2K \cdot \mathbf{r}. \tag{2.2}$$

A straightforward calculation shows that

$$\dot{K} = (\Omega(\mathbf{r}, \dot{\mathbf{r}}), \dot{\mathbf{r}}) - 2K \cdot (\mathbf{r}, \dot{\mathbf{r}}) = 0$$

along the solution of (2.2) (i.e., the force is always orthogonal to the velocity) and $\dot{\bar{b}} = 0$, where

$$\bar{b} = [\dot{\mathbf{r}}, \ddot{\mathbf{r}}] = -\frac{\|\dot{\mathbf{r}}\|^2 \cdot \mathbf{r}}{1 + \|\dot{\mathbf{r}}\|} - \|\dot{\mathbf{r}}\|^2 \cdot [\dot{\mathbf{r}}, \mathbf{r}].$$

Therefore, the kinetic energy $K = \|\dot{\mathbf{r}}\|^2/2$ is constant along the trajectory $r(t)$ and $r(t)$ lies in the plane orthogonal to the constant vector $\bar{b}$. In other words, the trajectory is the circle $(\mathbf{r}(t), \bar{b}) = \text{const}$ on S^2. Let us assume that there is a trajectory passing through two antipodal points. Then it must be a great circle on S^2 and, therefore, $(\mathbf{r}(t), \bar{b}) = 0$.

Let ϕ be the angle between $\mathbf{r}(t)$ and $\bar{b}$. A straightforward calculation (based on the explicit formulas for $\|\bar{b}\|$ and $(\mathbf{r}(t), \bar{b})$ and on the equality $\|\mathbf{r}(t)\| \equiv 1$) shows that

$$\cos\phi = \frac{\psi(\|\dot{\mathbf{r}}\|)}{\|\dot{\mathbf{r}}\|^2},$$

where $\psi : [0, \infty) \to R$ is a bounded function. Hence, $(\mathbf{r}(t), \bar{b})$ goes to zero as $K \to \infty$ assuming non-zero values only. This means that there is no trajectory in the system passing through two antipodal points. Note also that any two points which are not antipodal can be connected by a trajectory with sufficiently high kinetic energy.

Example 3. Replace the force $\Omega(\mathbf{r}, \dot{\mathbf{r}})$ of the preceding example by the gyroscopic force $A(\mathbf{r}, \dot{\mathbf{r}}) = [\dot{\mathbf{r}}, \mathbf{r}]$. The equation of motion of the new system is

$$\ddot{\mathbf{r}} = [\dot{\mathbf{r}}, \mathbf{r}] - 2K \cdot \mathbf{r}.$$

The analysis of this example is quite similar to that of Example 2. First, we prove that $\dot{K} = 0$ and $\dot{\bar{b}} = 0$, where $\bar{b} = [\dot{\mathbf{r}}, \ddot{\mathbf{r}}]$. This implies that the trajectory lies in the plane orthogonal to $\bar{b}$. If the trajectory were a great circle, so that $(\mathbf{r}, \bar{b}) = 0$, then this would give us the equality $[\mathbf{r}, \dot{\mathbf{r}}] = 0$, which is impossible.

3. The main result on accessible points

In the examples of Section 2, the points which could not be connected by a trajectory were conjugate along all geodesics. In this section, we prove that if two points are not conjugate along a geodesic and the force field is bounded (or has linear growth in velocities), then there exists a trajectory joining the points. We consider the general case with the set-valued force field $\mathbb{A}(t, m, X)$ which is upper semicontinuous and has convex images. Thus the trajectories are solutions of the differential inclusion (1.3). This general result yields, as a simple corollary, the existence of such a trajectory for a mechanical system with continuous α (see [6]).

Let M be a manifold, let $\langle\, ,\rangle$ be a complete Riemannian metric on M, and let m_0 and m_1 be points that are not conjugate along a geodesic $\mathbf{a}(t)$.

Theorem 3.1. *Assume that $\mathbb{A}(t,m,X)$ is upper semicontinuous, uniformly (in t, m and X) bounded by a constant $C > 0$, and $\mathbb{A}$ has convex images. Then there exists a constant $L = L(m_0, m_1, C, \mathbf{a}) > 0$ such that for any t_0, $0 < t_0 < L$ the points m_0 and m_1 can be connected by a solution of (1.3) with $m(0) = m_0$ and $m(t_0) = m_1$.*

Proof. In order to study the global behavior of solutions of (1.3), first we construct a certain analog of integral operator, acting on curves in the manifold M.

Let $m_0 \in M$, $I = [0,l]$ and let $v : I \to T_{m_0}M$ be a continuous curve.

Lemma 3.2. *There exists a unique C^1-curve $\gamma : I \to M$ such that $\gamma(0) = m_0$ and the tangent vector $(d/dt)\gamma(t) = \dot{\gamma}(t)$ is parallel to the vector $v(t) \in T_{m_0}M$ for every $t \in I$.*

The above curve in M, constructed from $v(t)$, will be denoted by $\mathcal{S}v(t)$. So, a certain operator $\mathcal{S}$, sending continuous curves $v(t)$ in $T_{m_0}M$ to C^1-curves in M, taking the value m_0 at $t = 0$, is well-posed. Obviously $\mathcal{S}$ is continuous as a map from the Banach space $C^0(I, T_{m_0}M)$ of continuous curves in $T_{m_0}M$ with C^0 topology into the Banach manifold $C^1(I, M)$ of C^1-curves in M with C^1 topology.

Lemma 3.3. *Assume that the point $m_1 \in M$ is not conjugate to m_0 along some geodesic of the Levi-Civitá connection on M. Then for any geodesic $\mathbf{a}(\cdot)$, $\mathbf{a}(0) = m_0$, $\mathbf{a}(1) = m_1$ along which m_0 and m_1 are not conjugate and for any number $K > 0$, there exists a constant $\bar{L}(m_0, m_1, K, \mathbf{a}) > 0$ with the following property: for any t_1, $0 < t_1 < \bar{L}(m_0, m_1, K, \mathbf{a})$, and for any curve $u(\cdot) \in \mathcal{U}_K \subset C^0([0,t_1], T_{m_0}M)$, there exists a unique vector $C_u \in T_{m_0}M$, such that $\mathcal{S}(u + C_u)(t_1) = m_1$, and C_u belongs to a bounded neighborhood of $t_1^{-1} \cdot \dot{\mathbf{a}}(0) \in T_{m_0}M$ and depends continuously on u.*

Proofs of Lemmas 3.2 and 3.3 are based on constructions of Riemannian geometry which are far from the topic of present paper. That is why we refer the reader to [11]-[13] where the proofs are given in details.

For any curve $v(\cdot) \in C^0(I, T_{m_0}M)$, consider $\mathbb{A}(t, \mathcal{S}v(t), (d/dt)\mathcal{S}v(t))$, the restriction of the set-valued vector field $\mathbb{A}$ to the curve $\mathcal{S}v(t)$. Fixing v, let us introduce the set-valued map $\Gamma_A \circ \mathcal{S}v : I \to T_{m_0}M$ such that the set $\Gamma_A \circ \mathcal{S}v(t)$ is obtained by parallel translation of $\mathbb{A}(t, \mathcal{S}v(t), (d/dt)\mathcal{S}v(t))$ to the point m_0 along $\mathcal{S}v(\cdot)$. Using the properties of parallel translation and the fact that $\mathbb{A}$ is upper semicontinuous, one can show that the map $\Gamma_A \circ \mathcal{S} : C^0(I, T_{m_0}M) \times I \to T_{m_0}M$ is upper semicontinuous too. Consider the set $\mathcal{P}\Gamma_A \circ \mathcal{S}$ formed by all measurable selections of $\Gamma_A \circ \mathcal{S}v$. Note that such selections do exist (see [1]). Since the field $\mathbb{A}$ is bounded, all elements of $\mathcal{P}\Gamma_A \circ \mathcal{S}$ are integrable. Let us define the set-valued map $\int \mathcal{P}\Gamma_A \circ \mathcal{S}$ with convex images in $C^0(I, T_{m_0}M)$ by the formula

$$\int \mathcal{P}\Gamma_A \circ \mathcal{S}(v) = \left\{ \int_0^t u(\tau)\, d\tau \mid u \in \mathcal{P}\Gamma_A \circ \mathcal{S} \right\}.$$

Lemma 3.4. *The set-valued map $\int \mathcal{P}\Gamma_A \circ \mathcal{S}$ sends bounded subsets of the space $C^0(I, T_{m_0}M)$ to compact ones.*

Proof. Since the metric $\langle\,,\rangle$ is complete, for any ball U_K in $C^0(I, T_{m_0}M)$ with radius K, the union of curves $\{(\mathcal{S}v, (d/dt)\mathcal{S}v) \mid v \in U_K\}$ lies in a compact subset of TM. Indeed, since $v \in U_K$ and the parallel translation preserves the norms, the curves form $\{(\mathcal{S}v \mid v \in U_K)\}$ have derivatives, bounded by K, and so the curves are equicontinuous and lie in a compact set of M. Then all sets $\mathbb{A}(t, \gamma, \dot{\gamma})$, where $\gamma \in \mathcal{S}U_K$, are uniformly bounded because the field $\mathbb{A}$ is bounded. As a consequence, since parallel translations preserve the norm, the sets $(\Gamma_A \circ \mathcal{S}v)(t)$ for $v \in U_K$ are also uniformly bounded, and so are all their measurable selections $\mathcal{P}\Gamma_A \circ \mathcal{S}v$.

Thus, continuous curves

$$u \in \bigcup_{v \in U_K} \left(\int \mathcal{P}\Gamma_A \circ \mathcal{S} \right) v$$

are uniformly bounded and equicontinuous. □

Lemma 3.5. *The map $\int \mathcal{P}\Gamma_A \circ \mathcal{S}$ is upper semicontinuous.*

Proof. It suffices to prove that the set-valued map $\int \mathcal{P}\Gamma_A \circ \mathcal{S}$ has a closed graph. In other words, that $v_k \to v_0$ and $u_k \to u_0$, where $u_k \in (\int \mathcal{P}\Gamma_A \circ \mathcal{S})v_k$, implies that $u_0 \in (\int \mathcal{P}\Gamma_A \circ \mathcal{S})v_0$, i.e., $\dot{u}_0 \in (\Gamma_A \circ \mathcal{S}v_0)(t)$ for almost all t. Since the map $\int \mathcal{P}\Gamma_A \circ \mathcal{S}$ sends bounded sets to compact ones, the map is upper semicontinuous, provided that it has a closed graph (see [1]).

Recall that the sets $(\Gamma_A \circ \mathcal{S}v_0)(t)$ are convex and the map $(\Gamma_A \circ \mathcal{S}v)(t)$ is upper semicontinuous in v and t. As a result, we have $\dot{u}_0 \in (\Gamma_A \circ \mathcal{S}v_0)(t)$. A detailed proof of a similar (and simpler) inclusion may be found in [1]. □

Let the constant $\bar{L}(m_0, m_1, Ct_1, a)$ be defined as in Lemma 3.3. If $\mathbb{A}(t, m, X)$ is bounded by a constant C, the inequality $t_1 < \bar{L}(m_0, m_1, Ct_1, \mathbf{a})$ holds for a sufficiently small t_1. Let us denote the supremum of all such t_1 by $L(m_0, m_1, Ct_1, a)$ and pick $t_0 < L(m_0, m_1, Ct_1, \mathbf{a})$. Without loss of generality, we may assume that the operator $\int \mathcal{P}\Gamma_A \circ \mathcal{S}$ acts on the space $C^0([0, t_0], T_{m_0}M)$. Consider the upper semicontinuous set-valued compact map

$$Bu = \left(\int \mathcal{P}\Gamma_A \circ \mathcal{S} \right) (u + C_u)$$

on the ball $U_{Ct_0} \subset C^0([0, t_0], T_{m_0}M)$, where C_u is defined in Lemma 3.3.

Since parallel translation preserves the norm, $B(U_{Ct_0}) \subset U_{Ct_0}$. Thus, B has a fixed point $u_0 \in Bu_0$ (see [1]). Let us show that

$$m(t) = \mathcal{S}(u_0(t) + C_{u_0})$$

is the desired solution. By definition, we have

$m(0) = m_0$ and $m(t_0) = m_1$;
$m(t)$ is a C^1-curve;
$\dot{m}(t)$ is absolutely continuous.

Note that $\dot{u}_0$ is a selection of $\Gamma_A \circ \mathcal{S}(u_0 + C_{u_0})$ because u_0 is a fixed point of B. In other words, the inclusion $\dot{u}_0(t) \in \Gamma_A \circ \mathcal{S}(u_0 + C_{u_0})(t)$ holds for all points t at which the derivative exists. Using the properties of the covariant derivative and the definition of u_0, one can show that $\dot{u}_0(t)$ is parallel to $D\dot{m}(t)/dt$ along $m(\cdot)$ and $\Gamma_A \circ \mathcal{S}(u_0 + C_{u_0})(t)$ is parallel to $\mathbb{A}(t, m(t), \dot{m}(t))$. Therefore,

$$\frac{D}{dt}\dot{m}(t) \in \mathbb{A}(t, m(t), \dot{m}(t)).$$

This completes the proof of Theorem 3.1. □

It is worth noticing that if m_0 and m_1 are not conjugate along several geodesics, then any of them can be used in the proof. Naturally, different geodesics can give rise to different solutions and constants L.

Assume that the configuration space M is compact and the metric $\langle\,,\rangle$ has a non-negative sectional curvature. Then there are no conjugate points on M. As follows from Theorem 3.1, there exists a constant $L > 0$ such that any two points can be connected by a trajectory $m(t)$ with $t \in [0, t_0]$ for any $t_0 > L$.

In particular, one evidently may take $L = \infty$ when M is flat. This means that the corresponding two-point boundary-value problem has a solution on any time interval.

Remark 3.6. By definition, $(D/dt)\dot{m}(t)$ is a measurable vector field along the solution. Thus, in the case where $\mathbb{A}(t, m, X)$ is the set of possible values of the control force, Theorem 3.1 gives a condition which guarantees the existence of a control sending m_0 to m_1. For example, Theorem 3.1 can be applied to systems with a delayed control force studied in [7]. Consider a mechanical system with a bounded continuous force $\alpha(t, m, X)$ and with a delayed control force. Assume that the possible values of the control force form the set $F \in T_{m_0}M$, where m_0 is the beginning of the trajectory, and the control starts in time $h > 0$ (to take into account the delay present in many realistic models). We also assume that $B_\varepsilon \subset F$, where B_ε is the ball of a small enough radius $\varepsilon > 0$. Taking into account the mechanical meaning of parallel translation discovered by J. Radon and described by W. Blaschke in [13], the motion of the system can be described by the differential inclusion

$$\frac{D}{dt}\dot{m}(t) \in \alpha(t, m(t), \dot{m}(t)) + \|\Xi(t),$$

where $\Xi(t) = 0$ for $t \in [0, h)$, $\Xi(t) = B_\varepsilon$ for $t > h$, and $\|$ means the parallel translation along the trajectory. It is shown that there exists a measurable control sending m_0 to m_1, provided that m_0 and m_1 are subject to the hypothesis of Theorem 3.1. The problem of existence of the optimal control satisfying this condition is also studied in [7]. □

Theorem 3.7. *Assume that the points m_0 and m_1 are not conjugate along a geodesic* $\mathbf{a}$ *such that* $\mathbf{a}(0) = m_0$ *and* $\mathbf{a}(1) = m_1$. *Let the upper semicontinuous set-valued vector field* $\mathbb{A}(t, m, X)$ *have convex images for all* t, m *and* X *and satisfy the inequalities*

$$||\mathbb{A}(t, m, X)|| < C + k||X||,$$

where C *and* k *are positive constants, and*

$$||\mathbb{A}(t, m, X)|| = \sup_{y \in \mathbb{A}(t,m,X)} ||y||.$$

Then there exists a constant $L(m_0, m_1, C, k, \mathbf{a}) > 0$ *such that for any* $0 < t_0 < L(m_0, m_1, C, k, \mathbf{a})$, *there exists a solution* $m(t)$ *of* (1.3) *with* $m(0) = m_0$ *and* $m(t_0) = m_1$.

We omit the proof of this result since it is quite similar to that of Theorem 3.1. (See [10].)

Remark 3.8. It should be noticed that, in contrast to Theorem 3.1, the assertion of Theorem 3.7 is local in time even on a flat configuration space. □

4. The case of systems with constraints

In this section we show how to generalize Theorems 3.1 and 3.7 to systems with constraints. In the framework of mechanics with constraints it is more natural to consider the question of whether or not a submanifold transverse to the union of least constrained geodesics, leaving a specified point, is accessible from the point. The author is grateful to Boris D. Gel'man for pointing out this problem.

Following L. Faddeev and Vershik (see e.g. [3] and [17]) we consider a constraint as a distribution (subbundle of tangent bundle) β on the configuration space (manifold) M. The constraint is called non-holonomic if β is non-integrable (non-involutive). Let M be a Riemannian manifold. Denote by $P : TM \to \beta$ the fiber-wise orthogonal projection. $(\bar{D}/dt) = P(D/dt)$ is called the covariant derivative of reduced connection on M. A curve $m(t)$ in M is called admissible if $\dot{m}(t) \in \beta$ for all t. An admissible curve $m(t)$ is called least constraint geodesic if $(\bar{D}/dt)\dot{m}(t) = 0$ (see details in [11] and [13]).

Let M be a complete Riemannian manifold equipped with a constraint β. Fix a point $m_0 \in M$. The non-holonomic exponential map $\exp^{\beta}_{m_0} : \beta_{m_0} \to M$ can be defined in the same manner as for a manifold without constraint. Namely, for $X \in \beta_{m_0}$, we set $\exp^{\beta}_{m_0}(X) = \gamma_X(1)$, where $\gamma_X(t)$ is the least constrained geodesic with $\gamma_X(0) = m_0$ and $\dot{\gamma}_X(0) = X$. It is clear that $\exp^{\beta}_{m_0}$ is a C^{∞}-smooth map.

Definition 4.1. A point $m_1 \in \exp^{\beta}_{m_0}(\beta_{m_0})$ is not conjugate to m_0 along the least constraint geodesic γ_X (where $\gamma_X(1) = m_1$) if the differential $d\exp^{\beta}_{m_0}$ has the maximum rank at $X \in \beta_{m_0}$.

In particular, this means that the image of $\exp^{\beta}_{m_0}$ is a smooth submanifold in a neighborhood of m_1 that is not conjugate to m_0. Moreover, $\exp^{\beta}_{m_0}$ is a diffeomorphism of a neighborhood of $X \in \beta_{m_0}$ onto the neighborhood of m_1 in the submanifold.

Assume that m_0 is not conjugate to m_1 along a least constrained geodesic γ_X. Let us specify a submanifold $N \subset M$, $m_1 \in N$, which is transversal to the image of $\exp^{\beta}_{m_0}$. (In other words, the sum of spaces $T_{m_0}N$ and $T_{m_0}\exp^{\beta}_{m_0}(\beta_{m_0})$ coincides with $T_{m_0}M$.)

Lemma 4.2. *Under the above hypothesis, for any $K > 0$, there exists a constant $\bar{L}(m_0, N, K, \gamma_X) > 0$ such that for $0 < t_1 < \bar{L}(m_0, N, K, \gamma_X)$ and for any continuous curve $u(t) \in U_K \subset C^0([0, t_1], \beta_{m_0})$, there exists a vector $C_u \in \beta_{m_0}$ satisfying the condition $\mathcal{S}^{\beta}(u + C_u)(t_1) \in N$. Furthermore, C_u is unique in a neighborhood of $t_1^{-1}X \in \beta_{m_0}$ and continuous in u.*

The Lemma is a natural generalization of Lemma 3.3. The only extra argument needed in proof is that the manifold N stays transversal to a C^1-small perturbation of the image of $\exp^{\beta}_{m_0} = \mathcal{S}^{\beta}(\cdot)(1)$ (see [9], [11] and [13] for details).

Let $\mathbb{A}$ be a set-valued vector field on M. As in Theorem 3.1, we assume that $\mathbb{A}$ is upper semicontinuous, bounded, and has convex images. The constraint motion with the force field $\mathbb{A}$ is described by the differential inclusion

$$\frac{\bar{D}}{dt}\dot{m}(t) \in P\mathbb{A}(t, m(t), \dot{m}(t)). \tag{4.1}$$

It is easy to see that the sets $P\mathbb{A}(t, m, X)$ are convex and the set-valued vector field $P\mathbb{A}$ is upper semicontinuous and bounded. Such a field can arise, for example, as a discontinuous force acting on the system, or the image of $P\mathbb{A}$ can be formed by all possible values of the control force.

Theorem 4.3. *Let $P\mathbb{A}$ be upper semicontinuous, bounded and have convex images. Then there exists a constant $L(m_0, N, C, \gamma_X) > 0$ such that for any t_0, $0 < t_0 < L(m_0, N, C, \gamma_X)$, there exists an admissible solution $m(t)$ of (4.1) which connects m_0 and N, i.e., $m(0) = m_0$ and $m(t_0) \in N$.*

The Theorem can be proved in the same way as Theorem 3.1. One only has to replace $\mathcal{S}$ and Γ by non-holonomic integral operators $\mathcal{S}^{\beta}$ and Γ^{β} (based on non-holonomic parallel translation, cf. construction of $\mathcal{S}$ and Γ above) and apply Lemma 4.2 instead of Lemma 3.3 (see [9], [11] and [13] for details).

Theorem 4.4. *Let the field $\mathbb{A}(t, m, X)$ be as in Theorem 3.7. There exists a constant $L(m_0, N, C, k, \gamma_X) > 0$ such that for any t_0, $0 < t_0 < L(m_0, N, C, k, \gamma_X)$, equation (4.1) has an admissible solution $m(t)$ satisfying the conditions $m(0) = m_0$ and $m(t_0) \in N$.*

The proof of the theorem is analogous to that of Theorem 3.7.

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LERAY-SCHAUDER TYPE THEOREMS AND EQUILIBRIUM EXISTENCE THEOREMS

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1. Introduction and preliminaries

In this paper, we study two main applications of a fixed point theorem due to the first author [I] related to convexly totally bounded sets. We first extend the Leray-Schauder theorem to topological vector spaces which are not necessarily locally convex. This new result can be used to derive some new or well-known fixed point theorems. Secondly, we deduce a variation of social equilibrium existence theorem of Debreu [D]. This is applied to results on saddle points, minimax theorems, and the Nash equilibria.

All topological vector spaces in this paper are assumed to be real Hausdorff spaces. Given a set X, $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X. In what follows, X and Y are two subsets of two topological vector spaces E and F, respectively. The boundary, the closure, the interior, and the convex hull of a subset X of E are denoted by $\partial X, \overline{X}$, $\operatorname{Int} X$, and $\operatorname{co} X$, respectively. For brevity, locally convex topological vector spaces are called locally convex spaces.

Definition 1.1. Let $T : X \to \mathcal{P}(Y)$ be a map.

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(1.1.1) T is said to be *upper semicontinuous* (u.s.c.) on X if the set $\{x \in X \mid T(x) \subset V\}$ is open in X whenever V is an open subset of Y.
(1.1.2) T is said to be *compact* if $T(X)$ is relatively compact in Y.
(1.1.3) A set $K \subset E$ is *convexly totally bounded* (c.t.b. for short), if for every neighborhood V of $0 \in E$ there exist a finite subset $\{x_i \mid i \in I\} \subset K$ ($|I| < +\infty$) and a finite family of convex sets $\{C_i \mid i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $K \subset \bigcup_{i \in I}(x_i + C_i)$.

Weber [W] gave another definition of the c.t.b. set:
(1.1.4) A set $K \subset E$ is *convexly totally bounded* (c.t.b. for short), if for every neighborhood V of $0 \in E$ there exist a finite subset $\{x_i \mid i \in I\} \subset E$ ($|I| < +\infty$) and a finite family of convex sets $\{C_i \mid i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $K \subset \bigcup_{i \in I}(x_i + C_i)$.

Observe that

Proposition 1.2. *The definitions* (1.1.3) *and* (1.1.4) *are equivalent.*

Proof. Let U, V be symmetric neighborhoods of 0 in E such that $U + U \subset V$. We can find $\{x_i\}_{i \in I}$ and $\{C_i\}_{i \in I}$ ($|I| < +\infty$) such that $K \subset \bigcup_{i \in I}(x_i + C_i)$, $x_i \in E$, and $C_i \subset U$ is convex for $i \in I$. Let $y_i \in K \cap (x_i + C_i)$. Then $K \subset \bigcup_{i \in I}(y_i + (x_i - y_i + C_i))$ and the convex set $x_i - y_i + C_i \subset -C_i + C_i \subset U + U \subset V$ for all $i \in I$. □

Proposition 1.3. *If a compact set K is c.t.b., then the compact set* $[0,1]K$ *is also c.t.b.*

Proof. Let V be a closed and circled neighborhood of $0 \in E$. By definition $K \subset \bigcup_{i \in I}(x_i + C_i)$ for some $x_i \in E$ and $C_i \subset V$, $i \in I$ ($|I| < +\infty$). Without loss of generality we may assume that C is closed. Observe that for each $r \in [0,1]$ and $i \in I$, $r(x_i + C_i) \cap K \subset r(x_i + C_i) \subset r(x_i + V) \subset rx_i + V$. Furthermore, we have $[0,1]K = \bigcup_{i \in I} \widetilde{C}_i$, where $\widetilde{C}_i = \bigcup_{r \in [0,1]} r(x_i + C_i) \cap K$ is a compact set. Thus $\widetilde{C}_i$ is covered by a finite number of sets of the form $rx_i + V$, and hence $\widetilde{C}_i \subset \bigcup_{j \in J}(r_j x_i + V)$ where $r_j \in [0,1]$, $|J| < +\infty$, $i \in I$. □

Proposition 1.4. *Every compact subset of a c.t.b. set is c.t.b.*

The following theorem is a special case of Theorem 4.3 in [I]:

Theorem 1.5. *Let X be a nonempty convex subset of a t.v.s. E and $T: X \to \mathcal{P}(X)$ be an u.s.c. map with closed convex values. If $\overline{T(X)}$ is a compact c.t.b. subset of X, then T has a fixed point $x_0 \in X$; that is $x_0 \in T(x_0)$.*

2. The Leray-Schauder type theorems

From Theorem 1.5 we deduce the following Leray-Schauder type theorem:

Theorem 2.1. *Let X be a closed subset of a t.v.s. E such that $0 \in \operatorname{Int} X$ and $T : X \to \mathcal{P}(E)$ be a compact u.s.c. map with closed convex values. If $\overline{T(X)}$ is a c.t.b. subset of E, then either*

(1) *T has a fixed point; or*

(2) *$\lambda x \in T(x)$ for some $\lambda > 1$ and $x \in \partial X$.*

Proof. Let $Y \subset X$ be defined by $Y = \{x \in X \mid x \in tT(x) \text{ for some } t \in [0,1]\}$. Y is nonempty since $0 \in Y$. Moreover, it is closed since T is u.s.c. and has closed values. Therefore, Y is compact since T is compact. Suppose that the Leray-Schauder condition is satisfied:

$$\text{(LS)} \qquad T(y) \cap \{\lambda y : \lambda > 1\} = \emptyset \qquad \text{for all } y \in \partial X.$$

Then $Y \cap \partial X = \emptyset$. Since X is completely regular, there exists a continuous function $r : X \to [0,1]$ such that $r(x) = 1$ for $x \in Y$ and $r(x) = 0$ for $x \in \partial X$.

Let $S : E \to \mathcal{P}(E)$ be defined by

$$S(x) = \begin{cases} r(x)T(x) & \text{if } x \in X, \\ \{0\} & \text{if } x \notin X. \end{cases}$$

Then S is convex-valued. Since T is compact and closed, so is S. Moreover, $\overline{S(E)}$ is a c.t.b. subset of E as a subset of $[0,1]\overline{T(X)}$. Therefore, by Theorem 1.5, S has a fixed point. Now $x \in S(x)$ implies $x \in Y$ and $r(x) = 1$. Therefore, $x \in X$ and $x \in T(x)$. This completes our proof. □

We recall a few definitions:

Definitions 2.2. Let $\mathcal{N}$ be the fundamental system of neighborhoods of 0 in E:

(2.2.1) a set $K \subset E$ is said to be *locally convex* if for every $x \in K$ and every $V \in \mathcal{N}$, there exists a $U \in \mathcal{N}$ such that $\operatorname{co}((x+U) \cap K) \subset x + V$, and

(2.2.2) a set $K \subset E$ is said to be *of Z type* (see [H]) if for every $V \in \mathcal{N}$ there exists a $U \in \mathcal{N}$ such that $\operatorname{co}(U \cap (K - K)) \subset V$.

The following are well-known:

Proposition 2.3. *In a locally convex space E every subset $K \subset E$ is of Z type and is a locally convex set.*

Proposition 2.4. *If $K \subset E$ is a compact set which is locally convex or of Z type, then it is c.t.b.*

Corollary 2.5. *Let X be a closed subset of a t.v.s. E such that $0 \in \operatorname{Int} X$ and $T : X \to \mathcal{P}(E)$ be a compact u.s.c. map with closed convex values. Suppose that one of the following holds:*

(i) *E is locally convex,*
(ii) *the set $\overline{T(X)}$ is locally convex,*
(iii) *the set $\overline{T(X)}$ is of Z type,*
(iv) *the set $\overline{T(X)}$ is c.t.b.*

Then the conclusion of Theorem 2 holds.

The case (i) includes many well-known results. Related theorems have been considered recently by Ben-El-Mechaiekh and Idzik [BI], S. Park [P1], [P2], and S. Park and J. A. Park [PP].

From Theorem 2.1 we can obtain a Schaefer type theorem, Birkhoff-Kellog type theorems and a fixed point theorem for non-selfmaps, as in our previous works (see [P1], [P2]).

We give only two results as follows:

Corollary 2.6. *Let E be a t.v.s. and $T : E \to \mathcal{P}(E)$ be a compact u.s.c. map with closed convex values. If $\overline{T(E)}$ is a c.t.b. subset of E, then T has a fixed point.*

Proof. Observe that $\partial E = \emptyset$. This also follows immediately from Theorem 1.5. □

Theorem 2.7. *Let X be a closed convex subset of a t.v.s. E and $T : X \to \mathcal{P}(E)$ is an u.s.c. map with closed convex values such that $T(\partial X) \subset X$. If $\overline{T(X)}$ is a compact c.t.b. subset of E, then T has a fixed point.*

Proof. If $\partial X = X$ the theorem follows from Theorem 1.5. If $\operatorname{Int} X \neq \emptyset$, then without loss of generality we may assume that $0 \in \operatorname{Int} X$. The condition $T(\partial X) \subset X$ implies that the Leray-Schauder condition (LS) is satisfied for the convex set X. Thus by Theorem 2.1 T has a fixed point. □

3. Equilibrium existence theorems

Let $\{X_i\}_{i \in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j \in I} X_j \quad \text{and} \quad X^i = \prod_{j \in I \setminus \{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the j-th coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: Its i-th coordinate is x_i and, for $j \neq i$, its j-th coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x onto X^i.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) = \{y_i \in X_i \mid [x^i, y_i] \in A\} \quad \text{and} \quad A(x_i) = \{y^i \in X^i \mid [y^i, x_i] \in A\}.$$

The following is a collectively fixed point theorem equivalent to Theorem 1.5:

Theorem 3.1. *Let $\{X_i\}_{i \in I}$ be a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and $T_i : X = \Pi_{i \in I} X_i \to \mathcal{P}(K_i)$ an u.s.c. map with closed convex values. If $K = \Pi_{i \in I} K_i$ is a c.t.b. subset of X, then there exists an $\widehat{x} \in K$ such that $\widehat{x}_i \in T_i(\widehat{x})$ for each $i \in I$.*

Proof. Define $T : X \to \mathcal{P}(K)$ by $T(x) = \Pi_{i \in I} T_i(x)$ for each $x \in X$. Then T is a compact u.s.c. map with closed convex values. Since $\overline{T(X)} \subset K$ is a compact c.t.b. subset of X, by Theorem 1.5, T has a fixed point $\widehat{x} \in K$; that is, $\widehat{x} \in T(\widehat{x})$ and $\widehat{x}_i \in T_i(\widehat{x})$. □

From Theorem 3.1, we have the following variation of the social equilibrium existence theorem of Debreu [D]:

Theorem 3.2. *Let $\{X_i\}_{i \in I}$, E_i, K_i be the same as in Theorem 3.1. Let $A_i : X^i \to \mathcal{P}(K_i)$ be u.s.c. maps with closed values, and $f_i, g_i : \mathrm{Gr}(A_i) \to \overline{\mathbf{R}}$ u.s.c. extended real-valued functions for each $i \in I$, where $\mathrm{Gr}(A_i)$ denotes the graph of A_i. Suppose that*

(1) *$g_i(x) \leq f_i(x)$ for all $x \in \mathrm{Gr}(A_i)$;*
(2) *$\varphi_i(x^i) = \max_{y \in A_i(x^i)} g_i(x^i, y)$ is a l.s.c. function of $x^i \in X^i$; and*
(3) *for each $i \in I$ and $x^i \in X^i$, the set*

$$M(x^i) = \{x_i \in A_i(x^i) \mid f_i(x^i, x_i) \geq \varphi_i(x^i)\} \qquad \textit{is convex.}$$

If $K = \Pi_{i \in I} K_i$ is a c.t.b. subset of X, then there exists an equilibrium point $\widehat{a} \in \mathrm{Gr}(A_i)$ for all $i \in I$; that is,

$$\widehat{a}_i \in A_i(\widehat{a}^i) \quad \textit{and} \quad f_i(\widehat{a}) = \max_{a_i \in A(\widehat{a}^i)} g_i(\widehat{a}^i, a_i) \qquad \textit{for all } i \in I.$$

Proof. For each $i \in I$, define a map $T_i : X \to \mathcal{P}(X_i)$ by

$$T_i(x) = \{y \in A_i(x^i) \mid f_i(x^i, y) \geq \varphi_i(x^i)\}$$

for $x \in X$. Then $T_i(x) \neq \emptyset$ by (1) since $A_i(x^i)$ is compact and $g_i(x^i, \cdot)$ is u.s.c. on $A_i(x^i)$. We show that $\mathrm{Gr}(T_i)$ is closed in $X \times X_i$. In fact, let $(x_\alpha, y_\alpha) \in \mathrm{Gr}(T_i)$ and $(x_\alpha, y_\alpha) \to (x, y)$. Then

$$f_i(x^i, y) \geq \overline{\lim_\alpha} f_i(x^i_\alpha, y_\alpha) \geq \overline{\lim_\alpha} \varphi_i(x^i_\alpha) \geq \underline{\lim_\alpha} \varphi_i(x^i_\alpha) \geq \varphi_i(x^i)$$

and, since $\mathrm{Gr}(A_i)$ is closed in $X^i \times X_i$, $y_\alpha \in A_i(x^i_\alpha)$ implies $y \in A_i(x^i)$. Hence $(x, y) \in \mathrm{Gr}(T_i)$. Therefore, T_i is u.s.c. with convex values $T_i(x) = M(x^i)$ by (3).

Now we apply Theorem 3.1. Then there exists an $\widehat{x} \in X$ such that $\widehat{x}_i \in T_i(\widehat{x})$ for all $i \in I$; that is, $\widehat{x}_i \in A_i(\widehat{x}^i)$ and $f_i(\widehat{x}_i, \widehat{x}_i) \geq \varphi_i(\widehat{x}^i)$. This completes our proof. □

From Theorem 3.2, we obtain a saddle point theorem and a minimax theorem.

Theorem 3.3. *Let X, Y be two compact convex c.t.b. subsets, each in a t.v.s., and $f : X \times Y \to \overline{\mathbf{R}}$ a continuous function. Suppose that for each $x_0 \in X$ and $y_0 \in Y$, the sets*

$$\{x \in X \mid f(x, y_0) = \max_{\zeta \in X} f(\zeta, y_0)\} \quad \textit{and} \quad \{y \in Y \mid f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta)\}$$

are convex. Then f has a saddle point $(x_0, y_0) \in X \times Y$; that is

$$\min_{\eta \in Y} f(x_0, \eta) = f(x_0, y_0) = \max_{\zeta \in X} f(\zeta, y_0).$$

Proof. Note that a saddle point is a particular case of an equilibrium point for two agents ($n = 2$) in Theorem 3.2 for $a = (a_1, a_2)$, $X_1 = X$, $X_2 = Y$, $A_1(a^1) = X$, $A_2(a^2) = Y$, $f_1(a) = g_1(a) = f(x, y)$, $f_2(a) = g_2(a) = -f(x, y)$. Note that condition (2) holds by Berge's theorem (see: [B], Theorem VI.3.2). □

Theorem 3.4. *Under the hypothesis of Theorem 3.3, we have the minimax inequality*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Proof. By Theorem 3.3, we have a saddle point $(x_0, y_0) \in X \times Y$ such that

$$\max_{x \in X} f(x, y_0) = f(x_0, y_0) = \min_{y \in Y} f(x_0, y).$$

Therefore,

$$\min_{y \in Y} \max_{x \in X} f(x, y) \leq \max_{x \in X} f(x, y_0) = f(x_0, y_0) = \min_{y \in Y} f(x_0, y) \leq \max_{x \in X} \min_{y \in Y} f(x, y).$$

On the other hand, we clearly have

$$\min_{y \in Y} \max_{x \in X} f(x, y) \geq \max_{x \in X} \min_{y \in Y} f(x, y).$$

Therefore, we have the conclusion. □

From Theorem 3.2, we have the following generalization of the Nash equilibrium theorem:

Theorem 3.5. *Let $\{X_i\}_{i \in I}$ be a family of compact convex sets, each in a t.v.s. E_i and for each i, $f_i : X \to \overline{\mathbf{R}}$ a continuous function such that*

(0) *for each $x^i \in X^i$ and each $\alpha \in \overline{\mathbf{R}}$, the set $\{x_i \in X_i \mid f_i(x^i, x_i) \geq \alpha\}$ is empty or convex.*

If $X = \Pi_{i\in I} X_i$ *is a c.t.b. subset of* $E = \Pi_{i\in I} E_i$, *then there exists a point* $\widehat{a} \in X$ *such that*

$$f_i(\widehat{a}) = \max_{y_i \in X_i} f_i(\widehat{a}^i, y_i) \qquad \textit{for all } i \in I.$$

Proof. We apply Theorem 3.2 with $f_i = g_i$ and $A_i : X^i \multimap X_i$ defined by $A_i(x^i) = X_i$ for $x^i \in X^i$. Then condition (2) of Theorem 3.2 follows from Berge's theorem, and the set in condition (3) is nonempty and convex by (0). Therefore, we have the conclusion. □

Finally, note that Theorems 3.2-3.4 generalize corresponding results of von Neumann, Kakutani, Nash, and von Neumann and Morgenstern; for the literature, see Debreu [D].

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STABILITY AND INSTABILITY OF PERIODIC SOLUTIONS OF A DAMPED WAVE EQUATION IN A THIN DOMAIN

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1. Introduction

In the previous papers [5] and [6] the authors showed the existence of periodic solutions

$$w^\varepsilon = \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix}$$

with respect to the time t of a damped wave system in the non-autonomous and autonomous cases respectively.

The considered system in the non-autonomous case is of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = v \\ \dfrac{\partial v}{\partial t} = \Delta_X u + \dfrac{\partial^2 u}{\partial Y^2} - \beta v - \alpha u + g(t, X, Y, u) \end{cases} \tag{1}$$

with Neumann boundary condition:

$$\frac{\partial u}{\partial \nu_\varepsilon} = 0 \qquad \text{on } \partial Q_\varepsilon, \tag{2}$$

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where α and β are positive constants, g is an appropriate smooth function T-periodic with respect to time t, and (X, Y) is a generic point of the "thin domain" $Q_\varepsilon = \Omega \times (0, \varepsilon) \subset \mathbb{R}^{N+1}$.

The method employed in [5] and [6] consists in assuming that the "reduced" problem at $\varepsilon = 0$ in the domain Ω admits an isolated T_0-periodic solution

$$w^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix},$$

$T_0 = T$ in the non-autonomous case, and then in searching for conditions under which this solution extends to one for the problem (1)-(2) in Q_ε. The main tool is the topological degree for nonlinear compact operators.

It must be observed that in the autonomous case, i.e. when g is independent of t in (1), the assumed T_0-periodic solution of the reduced problem is not isolated. To overcome this difficulty we normalize the unknown period $T > 0$, in general different from T_0, of the sought-after periodic solution of (1)-(2) by introducing T as a parameter in system (1) by means of the substitution $t \to (T_0/T)t$.

Furthermore, in this case additional assumptions are required on the linearized reduced system. Under these assumptions it is possible to prove the existence of a continuous functional $T = T(w)$, $w = \begin{pmatrix} u \\ v \end{pmatrix}$, such that $T(w^0) = T_0$ and such that w^0 is an isolated fixed point with topological index different from zero of the $T(\,\cdot\,)$-parametrized reduced problem.

The aim of this paper is to derive the stability properties of the periodic solution w^ε defined in Q_ε, for small $\varepsilon > 0$, from those of the T_0-periodic solution w^0 in Ω.

Specifically in the non-autonomous case, we will prove that if the latter is stable or unstable then for $\varepsilon > 0$ sufficiently small the former is also stable or unstable. This result will be obtained by considering the first order approximation L^ε of the Poincaré map V^ε associated to (1)-(2) at w^ε. It is well known (see for instance [2] and [3]) that if all the $\lambda \in \sigma(L^\varepsilon)$, the spectrum of L^ε, satisfy the inequality $|\lambda| \le q < 1$ then w^ε is stable. On the contrary, if there exists $\lambda \in \sigma(L^\varepsilon)$ such that $|\lambda| > 1$ then w^ε is unstable. In our case these situations can be treated by using the fact that L^ε is a condensing operator with constant $k < 1$ (see [7] and [8]) and the properties of its spectrum. In fact, it turns out (see [1]) that if $\lambda \in \sigma(L^\varepsilon)$ satisfies $|\lambda| > k + d$, whenever $d > 0$, then it is an eigenvalue of finite multiplicity.

The same will be done for the autonomous case to obtain orbital (in)stability of the T-periodic solution w^ε defined in Q_ε from that of the T_0-periodic solution w^0 defined in Ω. The additional problem in this case is that we must prove the simplicity of the eigenvalue 1 of the linearization of (1) around w^ε for $\varepsilon > 0$

sufficiently small, from that at $\varepsilon = 0$. Obviously, the assumptions for orbital (in)stability will concern all the other eigenvalues of the spectrum of the linearization.

The paper is organized as follows. In Section 2 we provide assumptions, definitions and preliminary results to be used in the sequel. In Section 3 we treat the non-autonomous case and in Section 4 the autonomous one.

2. Assumptions, definitions and preliminary results

We first consider the case when g depends on t and we assume the following conditions on $g : [0, T] \times \Omega \times [0, \varepsilon_0) \times \mathbb{R} \to \mathbb{R}$:

— g is of class C^1 jointly in the variables t, X, Y and u and it is T-periodic in $t : g(t + T, X, Y, u) \equiv g(t, X, Y, u)$. Moreover, g satisfies the following estimates:

$$|g_X(t, X, Y, u)| \leq a(1 + |u|^{\theta+1}),$$
$$|g_Y(t, X, Y, u)| \leq a(1 + |u|^{\theta+1}),$$
$$|g_u(t, X, Y, u)| \leq a(1 + |u|^{\theta}),$$

for all values of its arguments t, X, Y, u. Here $a > 0$ is a suitable constant and $\theta \in [0, \infty)$ if $N = 1$, $\theta \in [0, 2/(N - 1))$ if $N > 1$.

Observe that the growth rate θ is strictly less than the critical value $2/(N - 1)$. This is because the validity of the Sobolev compact embedding result is crucial in our approach.

Following [4], for fixed $\varepsilon > 0$ we introduce new variables $X = x$, $Y = \varepsilon y$. System (1) becomes

$$(3) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u + \dfrac{1}{\varepsilon^2} \dfrac{\partial^2 u}{\partial y^2} - \beta v - \alpha u + g(t, x, \varepsilon y, u) \end{cases}$$

and boundary condition (2) takes the form

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial Q,$$

where $Q = \Omega \times (0, 1)$ and ν denotes the outward unit normal vector to Q. We suppose that Ω is a C^2-smooth domain.

For the reader convenience, we now give the most relevant definitions which permit to rewrite (1)-(2) as a fixed point problem in a suitable space. More details can be found in [5].

Following [5] (which in turn follows Hale-Raugel [4]), we introduce the following Banach spaces when $\varepsilon > 0$. Let X^1_ε be the space $H^1(Q)$ with the norm

$$\left(\|u\|^2_{1Q} + \frac{1}{\varepsilon^2} \left\| \frac{\partial u}{\partial y} \right\|^2_{0Q} \right)^{1/2}.$$

Here and below, $\|\cdot\|_{0Q}$ denotes the norm in $L^2(Q)$ and $\|\cdot\|_{1Q}$ that in $H^1(Q)$. Let $U_\varepsilon(t)$ be the semigroup generated by the system of linear equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u + \dfrac{1}{\varepsilon^2} \dfrac{\partial^2 u}{\partial y^2} - \beta v - \alpha u, \end{cases}$$

with boundary condition (2). It is known (see [5]) that $U_\varepsilon(t)$ is a C_0-semigroup in the space

$$Y^1_\varepsilon = X^1_\varepsilon \times L^2(Q) \ni (u, v) = w.$$

One has the exponential estimate

$$\|U_\varepsilon(t)\|_{Y^1_\varepsilon \to Y^1_\varepsilon} \le ce^{-\gamma t}, \qquad (t \ge 0),$$

where $c, \gamma > 0$. By introducing for $\varepsilon > 0$ the linear operator

$$A_\varepsilon = \begin{pmatrix} 0 & I \\ \Delta_x + \dfrac{1}{\varepsilon^2} \dfrac{\partial^2}{\partial y^2} - \alpha & -\beta \end{pmatrix}$$

with Neumann boundary condition, we can write

$$U_\varepsilon(t) = e^{A_\varepsilon t}, \qquad t \ge 0.$$

In the sequel by a solution of any differential equation we mean a solution of the corresponding integral equation obtained by the variation-of-constants formula.

Now let $C_T(Y^1_\varepsilon)$ be the space of all continuous, T-periodic functions $w = \begin{pmatrix} u \\ v \end{pmatrix}$ from $\mathbb{R}$ into Y^1_ε with the usual norm

$$\|w\| = \sup_{t \in [0,T]} \|w(t)\|_{Y^1_\varepsilon}.$$

Define the following maps on $C_T(Y^1_\varepsilon)$:

$$f_\varepsilon(w)(t)(x, y) = \begin{pmatrix} 0 \\ g(t, x, \varepsilon y, u(t, x, y)) \end{pmatrix}$$

and

$$J_\varepsilon w(t) = U_\varepsilon(t)(I - U_\varepsilon(T))^{-1} \int_0^T U_\varepsilon(T - s) w(s)\, ds + \int_0^t U_\varepsilon(t - s) w(s)\, ds.$$

Then define

$$F_\varepsilon(w) = J_\varepsilon f_\varepsilon(w).$$

Using the Sobolev embedding theorem together with the theory of nonlinear Nemytskii operators, it is easy to show that F_ε maps $C_T(Y_\varepsilon^1)$ into itself and is completely continuous, i.e. it is continuous and it maps bounded sets into relatively compact sets. We give now the following

Definition 1. A fixed point of the completely continuous operator F_ε : $C_T(Y_\varepsilon^1) \to C_T(Y_\varepsilon^1)$ is a T-periodic solution of (1)-(2).

It is known that a fixed point of F_ε is always a T-periodic distributional solution of (1)-(2).

Next we pose the limit problem at $\varepsilon = 0$. Let $U_0(t)$ $(t \geq 0)$ be the semigroup generated by the linear system

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u - \beta v - \alpha u, \end{cases}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega.$$

Observe that $U_0(t) = e^{A_0 t}$, $t \geq 0$, where

$$A_0 = \begin{pmatrix} 0 & I \\ \Delta_x - \alpha & -\beta \end{pmatrix}$$

with Neumann boundary condition.

Let $w_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ be an element of $H^1(\Omega) \times L^2(\Omega)$. Then $U_0(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ is in $H^1(\Omega) \times L^2(\Omega)$ and one has the estimate

$$||U_0(t)||_{H^1(\Omega)\times L^2(\Omega)\to H^1(\Omega)\times L^2(\Omega)} \leq ce^{-\gamma t},$$

where $c, \gamma > 0$. Defining $i : \Omega \to Q$ by $i(x) = (x, 0)$, we obtain an inclusion $\mathcal{J} : H^1(\Omega) \times L^2(\Omega) \to Y_\varepsilon^1$ with $\mathcal{J}(u,v)(x,y) = (u(x), v(x))$. The map $\mathcal{J}$ is an isometry for all $0 < \varepsilon < \varepsilon_0$, and we identify $U_0(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ with the element $\mathcal{J}U_0(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ of Y_ε^1.

Define an operator F_0 on $C_T(H^1(\Omega) \times L^2(\Omega))$ as follows:

$$F_0(w) = J_0 f_0(w),$$

where J_0 has the same form as J_ε with $U_\varepsilon(t)$ replaced by $U_0(t)$ and

$$f_0(w)(t)(x) = \begin{pmatrix} 0 \\ g(t,x,0,u(t,x)) \end{pmatrix}.$$

Then $F_0 : C_T(H^1(\Omega) \times L^2(\Omega)) \to C_T(H^1(\Omega) \times L^2(\Omega))$ and it is completely continuous.

We identify the T-periodic solutions of the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u - \beta v - \alpha u + g(t,x,0,u) \end{cases} \tag{4}$$

together with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \tag{5}$$

with the fixed points of the operator F_0. The main result proved in [5] is the following existence result, here $\operatorname{ind}(\cdot\,,\,\cdot)$ indicates the topological index.

Theorem A. *If the problem* (4)-(5) *admits an isolated T-periodic solution $w^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in C_T(H^1(\Omega) \times L^2(\Omega))$ with $\operatorname{ind}(F_0, w^0) \neq 0$, then for sufficiently small $\varepsilon > 0$ the problem* (4)-(5) *admits a T-periodic solution $w^\varepsilon = \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix} \in C_T(Y_\varepsilon^1)$ and*

$$\|w^\varepsilon - \mathcal{J} w^0\|_{C_T(Y_\varepsilon^1)} \to 0 \qquad \text{as } \varepsilon \to 0.$$

The proof of Theorem A is mainly based on the following result, which we repeat here for the reader's convenience since it will be used in the next sections.

Lemma A. *Suppose that there exist $r > 0$, $\varepsilon_n \to 0$ and*

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} \in C_T(H^1(\Omega) \times L^2(\Omega))$$

such that the problem (1), (2) *admits T-periodic solutions*

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} \in C_T(Y_{\varepsilon_n}^1)$$

with $\left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} - \mathcal{J} \begin{pmatrix} u^* \\ v^* \end{pmatrix} \right\|_{C_T(Y_{\varepsilon_n}^1)} = r$. Then there exist a T-periodic solution $\begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}$ of (4)-(5) *and a subsequence $\left\{ \begin{pmatrix} u_{k_n} \\ v_{k_n} \end{pmatrix} \right\}$ of $\left\{ \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}$ such that*

$$\left\| \begin{pmatrix} u_{k_n} \\ v_{k_n} \end{pmatrix} - \mathcal{J} \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} \right\|_{C_T(Y_{\varepsilon_n}^1)} \to 0 \quad \text{as } n \to \infty$$

with

$$\left\| \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} - \begin{pmatrix} u^* \\ v^* \end{pmatrix} \right\|_{C_T(H^1(\Omega)\times L^2(\Omega))} = r.$$

Finally, for the autonomous case

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u + \dfrac{1}{\varepsilon^2}\dfrac{\partial^2 u}{\partial y^2} - \beta v - \alpha u + g(x, \varepsilon y, u) \end{cases} \tag{6}$$

together with Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial Q, \tag{7}$$

the substitution $t \to (T_0/T)t$ produces the dependence on T of all the operators introduced before. Specifically, as it can be easily verified we have

$$F_\varepsilon(T, w) = J_\varepsilon(T) f_\varepsilon(T, w),$$

where

$$f_\varepsilon(T, w)(t)(x, y) = \begin{pmatrix} 0 \\ (T/T_0) g(x, \varepsilon y, u(t, x, y)) \end{pmatrix},$$

$$J_\varepsilon(T) w(t) = U_\varepsilon(T, t)[I - U_\varepsilon(T, T_0)]^{-1} \int_0^T U_\varepsilon(T, T_0 - s) w(s)\, ds + \int_0^t U_\varepsilon(T, t - s) w(s)\, ds,$$

and $U_\varepsilon(T, t)$ is the semigroup genarated by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{T}{T_0} v, \\ \dfrac{\partial v}{\partial t} = \dfrac{T}{T_0}\left[\Delta_x u + \dfrac{1}{\varepsilon^2}\dfrac{\partial^2 u}{\partial y^2} - \beta v - \alpha u\right]. \end{cases}$$

Similar formulas hold for $\varepsilon = 0$. Obviously, a fixed point of $F_\varepsilon(T, \cdot) : C_{T_0}(Y_\varepsilon^1) \to C_{T_0}(Y_\varepsilon^1)$ for some $T > 0$ is a T-periodic solution of (6)-(7).

The following existence result was proved in [6].

Theorem B. *Suppose that the system*

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \Delta_x u - \beta v - \alpha u - g(x, 0, u) \end{cases}$$

together with

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega$$

has a T_0-periodic solution $w^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$ in the classical sense such that the linearized system

$$\frac{\partial \varphi}{\partial t} = \psi,$$
$$\frac{\partial \psi}{\partial t} = \Delta_x \varphi - \beta\psi - \alpha\varphi + g_u(x, 0, u^0(t, x))\varphi$$

has no T_0-periodic solutions which are linearly independent of $(\partial w^0/\partial t)$. Furthermore, we suppose that it does not possess any solution of the form $\widetilde{w}(t, x) + (t/T_0)w^0(t, x)$, where $\widetilde{w} = \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix}$ is T_0-periodic with respect to t. Then for sufficiently small $\varepsilon > 0$ problem (6)-(7) admits a T_ε-periodic solution $w^\varepsilon = \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix}$ with $T_\varepsilon \to T_0$ and

$$\|\widehat{w}^\varepsilon - \mathcal{J} w^0\|_{C_{T_0}(Y_\varepsilon^1)} \to 0 \qquad \text{as } \varepsilon \to 0,$$

where $\widehat{w}^\varepsilon(t) = w^\varepsilon((T_\varepsilon/T_0)t)$.

In the next two sections, for $\varepsilon > 0$ sufficiently small, we investigate the stability of the periodic solution w^ε of (1)-(2) (resp. (6)-(7)).

3. The non-autonomous case

For $\varepsilon > 0$ define the Poincaré map $V^\varepsilon : Y_\varepsilon^1 \to Y_\varepsilon^1$ associated to (1)-(2) as follows:

$$V^\varepsilon(p) = U_\varepsilon(T)p + \int_0^T U_\varepsilon(T - s) f_\varepsilon(w)(s)\, ds,$$

where $w(t) \in Y_\varepsilon^1$, $t \in [0, T]$, is a solution of (1)-(2) with $w(0) = p$ and T is the fixed period of the nonlinearity g.

As a direct consequence of Theorem A we have that for $\varepsilon > 0$ sufficiently small the Poincaré map has a fixed point $w^\varepsilon(0) \in Y_\varepsilon^1$ which represents the initial conditions of the T-periodic solution w^ε of (1)-(2).

Consider the linearization $L^\varepsilon : Y_\varepsilon^1 \to Y_\varepsilon^1$ of V^ε around w^ε:

$$L^\varepsilon q = U_\varepsilon(T)q + \int_0^T U_\varepsilon(T - s) f'_\varepsilon(w^\varepsilon(s))\psi(s)\, ds,$$

where $\psi(t) \in Y_\varepsilon^1$, $t \in [0, T]$, is the solution of the linearization of (1) around w^ε such that $\psi(0) = q$.

For $\varepsilon = 0$ we also define $V^0 : H^1(\Omega) \times L^2(\Omega) \to H^1(\Omega) \times L^2(\Omega)$ and the corresponding linearization L^0 around w^0. The linear map $L^\varepsilon : Y^1_\varepsilon \to Y^1_\varepsilon$ is k-condensing with respect to the measure of noncompactness of Kuratovskii generated by a suitable equivalent norm in the space Y^1_ε (see [8]). Moreover, in [7] this condensivity property was proved for some special measures of noncompactness defined by means of the Hausdorff measure of noncompactness.

It follows that (see [1]) for any $d > 0$ the points $\lambda \in \sigma(L^\varepsilon)$ for which $|\lambda| > k+d$ are eigenvalues of finite multiplicity.

In the sequel we will study the stability of w^ε by means of the properties of the spectrum of $L^0 : H^1(\Omega) \times L^1(\Omega) \to H^1(\Omega) \times L^2(\Omega)$ and the corresponding stability results [1]. In this section we assume the conditions of Theorem A.

We will first prove the following result.

Theorem 1. *Assume that for any $\lambda \in \sigma(L^0)$ we have that $|\lambda| < 1$. Then for sufficiently small $\varepsilon > 0$ the T-periodic solution w^ε of (1)-(2) is stable.*

Proof. Assume the contrary, then by [1] there exist two sequences $\varepsilon_n \to 0$ and $\lambda_n \in \sigma(L^{\varepsilon_n})$ such that $|\lambda_n| \geq 1$. For any $n \in \mathbb{N}$ let $q_n \in Y^1_{\varepsilon_n}$, $\|q_n\|_{Y^1_{\varepsilon_n}} = 1$ and $L^{\varepsilon_n} q_n = \lambda_n q_n$.

Put $e^{\mu_n T} = \lambda_n$, $n \in \mathbb{N}$, and let $\varphi_n(t) = e^{-\mu_n t}\psi_n(t)$, where $\psi_n(t)$ is the solution of the linearization of (1) around w^ε which we denote by

$$\dot{\psi} = A_{\varepsilon_n}\psi + B_{\varepsilon_n}(t)\psi$$

with $\psi_n(0) = q_n$. It turns out that φ_n is a T-periodic solution of the linear equation

$$\dot{\varphi} = (A_{\varepsilon_n} - \mu_n)\varphi + B_{\varepsilon_n}(t)\varphi. \tag{8}$$

In fact, we have $\varphi_n(T) = e^{-\mu_n T}\psi_n(T) = e^{-\mu_n T}\lambda_n q_n = q_n$. Thus $\varphi_n(T) = \varphi_n(0) = q_n$, moreover

$$\varphi_n(t) = e^{(A_{\varepsilon_n} - \mu_n)t} q_n + \int_0^t e^{(A_{\varepsilon_n} - \mu_n)(t-s)} B_{\varepsilon_n}(s)\varphi_n(s)\, ds,$$

and

$$\begin{aligned}\varphi_n(t+T) &= e^{(A_{\varepsilon_n}-\mu_n)(t+T)}q_n + \int_0^T e^{(A_{\varepsilon_n}-\mu_n)(t+T-s)}B_{\varepsilon_n}(s)\varphi_n(s)\,ds \\ &\quad + \int_T^{t+T} e^{(A_{\varepsilon_n}-\mu_n)(t+T-s)}B_{\varepsilon_n}(s)\varphi_n(s)\,ds \\ &= e^{(A_{\varepsilon_n}-\mu_n)t}\left[e^{(A_{\varepsilon_n}-\mu_n)T}q_n + \int_0^T e^{(A_{\varepsilon_n}-\mu_n)(T-s)}B_{\varepsilon_n}(s)\varphi_n(s)\,ds\right] \\ &\quad + \int_0^t e^{(A_{\varepsilon_n}-\mu_n)(t-\xi)}B_{\varepsilon_n}(\xi)\varphi_n(\xi+T)\,d\xi \\ &= e^{(A_{\varepsilon_n}-\mu_n)t}q_n + \int_0^t e^{(A_{\varepsilon_n}-\mu_n)(t-\xi)}B_{\varepsilon_n}(\xi)\varphi_n(\xi+T)\,d\xi.\end{aligned}$$

By the uniqueness of solutions of the Cauchy problem for the linearized equation we obtain that φ_n is a T-periodic solution of (8).

By Lemma A and Theorem A we get that $\varphi_n(t) \to \varphi_0(t)$ as $n \to \infty$, where $\varphi_0(t) \in H^1(\Omega) \times L^2(\Omega)$, $t \in [0,T]$, is a continuous T-periodic solution of

$$\dot{\varphi} = (A_0 - \mu_0)\varphi + B_0(t)\varphi,$$

the linearized system of (1) around w^0, with $\lambda_0 = \lim_{n\to\infty} \lambda_n$, $\|\varphi_0(0)\| = 1$ and $e^{-\mu_0 T} = \lambda_0$. This contradicts the fact that $|\lambda| < 1$ for any $\lambda \in \sigma(L^0)$.

We state now the instability result. We first give the following lemma.

Lemma 1. *Let* $q_0 \in H^1(\Omega) \times L^2(\Omega)$. *Then*

$$(\lambda I - L^\varepsilon)^{-1}\mathcal{J}q_0 \to \mathcal{J}(\lambda I - L^0)^{-1}q_0 \qquad as\ \varepsilon \to 0$$

uniformly with respect to $\lambda \in C$, *where* C *is a circle such that* $C \cap \sigma(L^0) = \emptyset$.

Proof. Note that as it is proved in Theorem 2 below, for sufficiently small $\varepsilon > 0$, $C \cap \sigma(L^\varepsilon) = \emptyset$ and so $(\lambda I - L^\varepsilon)^{-1}$ is well defined.

We argue by contradiction, then there exist $\delta_0 > 0$, $\varepsilon_n \to 0$ and $\lambda_n \to \lambda_0 \in C$, $\lambda_n \in C$, such that

$$\|(\lambda_n I - L^{\varepsilon_n})^{-1}\mathcal{J}q_0 - \mathcal{J}(\lambda_n I - L^0)^{-1}q_0\|_{Y^1_{\varepsilon_n}} \geq \delta_0. \tag{9}$$

Let $p_n = (\lambda_n I - L^{\varepsilon_n})^{-1}\mathcal{J}q_0$. We claim that the sequence $\{p_n\}$ is bounded, i.e. there exists $M > 0$ such that $\|p_n\|_{Y^1_{\varepsilon_n}} \leq M$ for any $n \in \mathbb{N}$.

In fact, assume $\|p_n\|_{Y^1_{\varepsilon_n}} \to \infty$ as $n \to \infty$ and recall that

$$\lambda_n p_n = L^{\varepsilon_n} p_n + \mathcal{J} q_0.$$

Let

$$\xi_n(t) = e^{A_{\varepsilon_n} t} p_n + \int_0^t e^{A_{\varepsilon_n}(t-s)} B_{\varepsilon_n}(s) \xi_n(s)\, ds,$$

hence $L^{\varepsilon_n} p_n = \xi_n(T)$. Put $z_n(t) = (\xi_n(t)/\|p_n\|_{Y^1_{\varepsilon_n}})$, thus

$$z_n(t) = e^{A_{\varepsilon_n} t} z_n(0) + \int_0^t e^{A_{\varepsilon_n}(t-s)} B_{\varepsilon_n}(s) z_n(s)\, ds$$

and

$$\lambda_n z_n(0) = z_n(T) + \frac{\mathcal{J} q_0}{\|p_n\|_{Y^1_{\varepsilon_n}}}.$$

Let $\lambda_n = e^{\mu_n T}$ and define

$$\eta_n(t) = e^{-\mu_n t} z_n(t).$$

Then

$$\eta_n(t) = e^{(A_{\varepsilon_n} - \mu_n)t} \eta_n(0) + \int_0^t e^{(A_{\varepsilon_n} - \mu_n)(t-s)} B_{\varepsilon_n}(s) \eta_n(s)\, ds \tag{10}$$

where

$$\eta_n(0) = \eta_n(T) + e^{-\mu_n T} \frac{\mathcal{J} q_0}{\|p_n\|_{Y^1_{\varepsilon_n}}}, \qquad \|\eta_n(0)\|_{Y^1_{\varepsilon_n}} = 1.$$

Therefore, calculating $\eta_n(0)$ from the previous two relations and substituting in (10) we obtain

$$\begin{aligned}\eta_n(t) = {} & e^{(A_{\varepsilon_n} - \mu_n)t} (I - e^{(A_{\varepsilon_n} - \mu_n)T})^{-1} \int_0^T e^{(A_{\varepsilon_n} - \mu_n)(T-s)} B_{\varepsilon_n}(s) \eta_n(s)\, ds \\ & + e^{(A_{\varepsilon_n} - \mu_n)t} (I - e^{(A_{\varepsilon_n} - \mu_n)t})^{-1} e^{-\mu_n T} \frac{\mathcal{J} q_0}{\|p_n\|_{Y^1_{\varepsilon_n}}} \\ & + \int_0^t e^{(A_{\varepsilon_n} - \mu_n)(t-s)} B_{\varepsilon_n}(s) \eta_n(s)\, ds.\end{aligned}$$

Letting $n \to \infty$, since $\|p_n\|_{Y^1_{\varepsilon_n}} \to \infty$, Lemma A and Theorem A yield $\eta_n(t) \to \eta(t)$ where η is a T-periodic solution of

$$\dot{\varphi} = (A_0 - \mu_0)\varphi + B_0(t)\varphi$$

with $\mu_0 \in C$, which contradicts the fact that $C \cap \sigma(L^0) = \emptyset$.

Finally, by using the boundedness of $\{p_n\}$ and the same arguments as before we can prove that

$$\lambda_n p_n = L^{\varepsilon_n} p_n + \mathcal{J} q_0 \to \lambda_0 p_0 = L^0 p_0 + q_0, \qquad \lambda_0 \in C,$$

which is a contradiction with (9).

We are now in the position to proving the following result.

Theorem 2. *Assume that there exists $\lambda_0 \in \sigma(L^0)$ such that $|\lambda_0| > 1$, then for sufficiently small $\varepsilon > 0$ the T-periodic solution w^ε of (1)-(2) is unstable.*

Proof. Since λ_0 is an eigenvalue of finite multiplicity (see [1]) there is a closed disc D centered at λ_0 which does not contain points of $\sigma(L^0)$ different from λ_0 and $\operatorname{dist}(\partial D, 0) > 1$.

Furthermore, for $\varepsilon > 0$ sufficiently small there are no points of $\sigma(L^\varepsilon)$ lying on $C = \partial D$. In fact, assume there exist sequences $\varepsilon_n \to 0$ and $\lambda_n \in \sigma(L^{\varepsilon_n})$ with $\lambda_n \in C$. Then

$$\lambda_n q_n = L^{\varepsilon_n} q_n \qquad \text{for some } q_n \in Y^1_{\varepsilon_n}$$

and passing to the limit as $n \to \infty$ by Theorem A we obtain

$$\widehat{\lambda}_0 q_0 = L^0 q_0$$

with $\widehat{\lambda}_0 \in C$, $q_0 \in H^1(\Omega) \times L^2(\Omega)$ which is a contradiction.

Therefore for $\varepsilon > 0$ sufficiently small the Riesz's projector

$$P_\varepsilon q = -\frac{1}{2\pi i} \int\limits_C (\lambda I - L^\varepsilon)^{-1} q \, d\lambda$$

is well defined.

Let $q_0 \in H^1(\Omega) \times L^2(\Omega)$ such that $\lambda_0 q_0 = L^0 q_0$, $\|q_0\| = 1$. By Lemma A we get

$$P_\varepsilon q_0 \ \to \ P_0 q_0 \neq 0 \quad \text{as } \varepsilon \to 0,$$

and from this a contradiction if we assume the existence of a sequence $\varepsilon_n \to 0$ with the property that from $\lambda_n \in \sigma(L^{\varepsilon_n})$ it follows $\lambda_n \notin D$. Indeed in this case $P_{\varepsilon_n} \equiv 0$ for any $n \in \mathbb{N}$.

4. The autonomous case

Following the lines of the previous section we first define for $\varepsilon > 0$ and $T > 0$ the Poincaré map $V^\varepsilon_T : Y^1_\varepsilon \to Y^1_\varepsilon$ associated to (6)-(7) as follows

$$V^\varepsilon_T(p) = U_\varepsilon(T, T_0)p + \int\limits_0^{T_0} U_\varepsilon(T, T_0 - s) f_\varepsilon(T, w)(s) \, ds,$$

where $w(t) \in Y^1_\varepsilon$, $t \in [0, T_0]$, is a solution of (6)-(7) with $w(0) = p$ and the operators are those defined in Section 2.

We then consider the linearization $L^\varepsilon_T : Y^1_\varepsilon \to Y^1_\varepsilon$ of V^ε_T around $\widehat{w}^\varepsilon(t) = w^\varepsilon((T_\varepsilon/T_0)t)$, where w^ε is a T_ε-periodic solution of (6)-(7), whose existence for sufficiently small $\varepsilon > 0$ is guaranteed by Theorem B, namely

$$L^\varepsilon_T q = U_\varepsilon(T, T_0)q + \int_0^{T_0} U_\varepsilon(T, T_0 - s) f'_\varepsilon(T, \widehat{w}^\varepsilon(s))\psi(s)\, ds,$$

where $\psi(t) \in Y^1_\varepsilon$, $t \in [0, T_0]$, is the solution of the linearization of (6) around $\widehat{w}^\varepsilon$, such that $\psi(0) = q$.

For $\varepsilon = 0$ we also define $V^0_T : H^1(\Omega) \times L^2(\Omega) \to H^1(\Omega) \times L^2(\Omega)$ and the corresponding linearization L^0_T around w^0. Consider the linear operator L^ε_T for T close to T_0. One can show that it has the condensivity properties of the operator L^ε of Section 3. Also its spectrum has the properties indicated for the spectrum of L^ε.

In the sequel we assume the conditions of Theorem B. Therefore, in particular, we assume the eigenvalue $1 \in \sigma(L^0_{T_0})$ is simple. In order to investigate orbital (in)stability of w^ε we need the following

Lemma 2. *If $\varepsilon > 0$ is sufficiently small the eigenvalue $1 \in \sigma(L^\varepsilon_{T_\varepsilon})$ is simple.*

Proof. We argue by contradiction, thus we assume that there exist sequences $\varepsilon_n \to 0$ and $q_n, q'_n \in Y^1_{\varepsilon_n}$ such that q_n, q'_n are linearly independent eigenvectors of $L^{\varepsilon_n}_{T_{\varepsilon_n}}$ with $\|q_n\|_{Y^1_{\varepsilon_n}} = \|q'_n\|_{Y^1_{\varepsilon_n}} = 1$ corresponding to $1 \in \sigma(L^{\varepsilon_n}_{T_{\varepsilon_n}})$.

For any $n \in \mathbb{N}$ we define a projection in $Y^1_{\varepsilon_n}$ as

$$P_n q = q - \langle q, q_n \rangle_n q_n,$$

where $\langle \cdot\,, \cdot \rangle_n$ denotes the scalar product in $Y^1_{\varepsilon_n} = X^1_{\varepsilon_n} \times L^2(Q)$ which generates the norm as it is defined in Section 2. Consider now

$$P_n q'_n = q'_n - \alpha_n q_n,$$

where $\alpha_n = \langle q'_n, q_n \rangle$. That is

$$q'_n = \alpha_n q_n + P_n q'_n.$$

From this we obtain

$$L^{\varepsilon_n}_{T_{\varepsilon_n}} q'_n = \alpha_n L^{\varepsilon_n}_{T_{\varepsilon_n}} q_n + P_n q'_n$$

or equivalently

$$P_n q'_n = L^{\varepsilon_n}_{T_{\varepsilon_n}} P_n q'_n.$$

Clearly $P_n q_n' \neq 0$, otherwise q_n, q_n' are linearly dependent. Therefore the previous equation can be rewritten as

$$\frac{P_n q_n'}{\|P_n q_n'\|_{Y^1_{\varepsilon_n}}} = L^{\varepsilon_n}_{T_{\varepsilon_n}} \frac{P_n q_n'}{\|P_n q_n'\|_{Y^1_{\varepsilon_n}}}.$$

Observe that $P_n q_n'$ is orthogonal to q_n. Now, passing to the limit as $n \to \infty$, by Lemma A (which still holds in the autonomous case) and Theorem B we obtain

$$\widehat{q}_0 = L^0_{T_0} \widehat{q}_0, \qquad \|\widehat{q}_0\|_{H^1(\Omega)\times L^2(\Omega)} = 1.$$

Furthermore,

$$\widehat{q}_0 = \frac{P_0 q_0}{\|P_0 q_0\|_{H^1(\Omega)\times L^2(\Omega)}},$$

where $q_0 \in H^1(\Omega) \times L^2(\Omega)$ is the normalized eigenvector of $L^0_{T_0}$ corresponding to the eigenvalue 1 and

$$P_0 q = q - \langle q, q_0 \rangle_0 \, q_0$$

is the projection in $H^1(\Omega) \times L^2(\Omega)$ defined by the usual norm $(\langle \cdot\,, \cdot \rangle_0)^{1/2}$ in this space which is the limit of the considered norm in $Y^1_{\varepsilon_n}$ as $n \to \infty$.

Since $\widehat{q}_0$ is orthogonal to q_0 we have a contradiction. Moreover by a similar procedure we can show that there is no adjoint vectors to q_n and so $1 \in \sigma(L^{\varepsilon_n}_{T_{\varepsilon_n}})$ is simple. This completes the proof.

To conclude it is sufficient to observe that we can repeat the same arguments employed in the non-autonomous case to establish the analogous of Theorems 1 and 2 for the orbital stability and instability respectively.

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POTENTIAL TYPE INCLUSIONS

Zoltán Kánnai and Peter Tallos

1. Introduction

In [3], Bressan, Cellina and Colombo (see also Ancona and Colombo [1]) proved the existence of solutions to upper semicontinuous differential inclusions

$$x'(t) \in F(x(t)), \qquad x(0) = x_0 \tag{1}$$

without convexity assumptions on the right-hand side. They replaced convexity with cyclical monotonocity, i.e. they assumed the existence of a proper convex potential function V with $F(x) \subset \partial V(x)$ at every point. This condition assures the L^2-norm convergence of the derivatives of approximate solutions thus, no convexity is needed to guarantee that the limit is in fact a solution.

Rossi [5] extended this result to problems with phase constraints (viable solutions), and Staicu [6] considered added perturbations on the right-hand side. Ultimately, both papers followed the method of [3].

In the present paper we relax the convexity assumption on the potential function V, namely we suppose that V is lower regular. That means a locally Lipschitz continuous function whose upper Dini directional derivatives coincide with the Clarke directional derivatives. Convex analysis subdifferentials are replaced by Clarke subdifferentials. We prove the existence of viable solutions with

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the presence of phase constraint. The proof retains the basic idea of the original paper [3]. A counterexample shows that lower regularity cannot be omitted.

2. Lower regular functions

Let X be a real Hilbert space and consider a locally Lipschitz continuous real valued function V defined on X. For every direction $v \in X$ the upper Dini derivative of V at $x \in X$ in the direction v is given by

$$D^+V(x;v) = \limsup_{t \to 0+} \frac{V(x+tv) - V(x)}{t},$$

and its generalized (Clarke) directional derivative at x in the direction v is defined by

$$V^\circ(x;v) = \limsup_{y \to x,\, t \to 0+} \frac{V(y+tv) - V(y)}{t}.$$

The directional derivative of V at x in the direction v (if it exists) will be denoted by $DV(x;v)$.

Definition 1. The locally Lipschitz continuous function V is said to be *lower regular* at x if for every direction v in X we have $D^+V(x;v) = V^\circ(x;v)$. We say that V is lower regular if it is lower regular at every point.

Let us note here that lower regular functions are not necessarily regular in the sense of Clarke [4]. Take for instance the function $f(x) = \log(1+x)$ on the real positive half line. Now think of a piecewise linear function V with alternating slopes $+1$ and -1, whose graph lies between f and $-f$. Whenever V reaches the graph of f or $-f$, it bounces back. Since for every $x > 0$, $|f'(x)| < 1$, it is obvious that V zigzags infinitely many times in every neighborhood of the origin. Finally, eliminate all corners of V lying on the graph of f by making the derivative turn from 1 into -1 smoothly. Keep the corners on the graph of $-f$. Clearly, such a V is Lipschitz continuous and it can easily be seen that $D^+V(0,1) = V^\circ(0,1) = 1$ and hence, V is lower regular at the origin. However, $DV(0,1)$ does not exist and therefore, V cannot be regular.

The intermediate (or adjacent) cone to the subset K at $x \in K$ is

$$I_K(x) = \left\{ v \in X \mid D^+d_K(x;v) = 0 \right\},$$

where d_K denotes the distance function, moreover

$$C_K(x) = \{ v \in X \mid d_K^\circ(x;v) = 0 \}$$

is the Clarke tangent cone to K at x. The following characterization of lower regular functions can be verified by a staightforward adaptation of the proof of Theorem 2.4.9 in [4].

Theorem 2. *The following two statements are valid for every x in X.*

(a) $I_{\text{epi}\,V}(x, f(x)) = \text{epi}\, D^+V(x;\,\cdot\,)$,

(b) *V is lower regular at x if and only if $I_{\text{epi}\,V}(x, f(x)) = C_{\text{epi}\,V}(x, f(x))$.*

For further characterizations we refer to Aubin and Frankowska [2], pp. 239.

Consider a lower regular function V and let x be a point in X. Suppose $\lambda > 0$ is a Lipschitz constant for V in a neighborhood of x. Let B stand for the closed unit ball in X. By $\partial V(x)$ we denote the Clarke subdifferential of V at x.

Lemma 1. *For every $0 \leq \varepsilon \leq \lambda$ and $v \in \partial V(x) + \varepsilon B$ the inequality*

$$\|v\|^2 \leq D^+V(x; v) + 2\varepsilon\lambda$$

holds true.

Proof. Take $u \in \partial V(x)$ with $\|u - v\| \leq \varepsilon$. Since for each $w \in X$ we have $\langle u, w\rangle \leq D^+V(x; w)$, by setting $w = v$ it follows

$$\begin{aligned} D^+V(x; v) &\geq \langle u, v\rangle \geq \|v\|^2 + \langle u - v, v\rangle \\ &\geq \|v\|^2 - \varepsilon\|v\| \geq \|v\|^2 - \varepsilon(\varepsilon + \lambda) \geq \|v\|^2 - 2\varepsilon\lambda \end{aligned}$$

that is the desired inequality. □

Lemma 2. *Suppose the function $f(t) = V(x + tv)$ is differentiable at $t = 0$ for some $x \in X$ and $v \in \partial V(x)$. Then $f'(0) = \|v\|^2$.*

Proof. Lower regularity of V at x implies that

$$\langle v, u\rangle \leq D^+V(x; u)$$

for each u in X. Applying this inequality with $u = v$ and $u = -v$ the lemma ensues. □

Lemma 3. *If $x : [0, T] \to X$ is absolutely continuous on the interval $[0, T]$ with $x'(t) \in \partial V(x(t))$ a.e., then*

$$(V \circ x)'(t) = \|x'(t)\|^2$$

for a.e. $t \in [0, T]$.

Proof. Let S be a set of measure zero such that both x and $V \circ x$ are differentiable on $[0, T] \setminus S$ moreover $x'(t) \in \partial V(x(t))$ at every $t \in [0, T] \setminus S$. Thus, if $t \in [0, T] \setminus S$ is given, there is a $\delta > 0$ such that $x(t + h) - x(t) - hx'(t) = r(h)$ for every $|h| < \delta$, where $\lim_{h \to 0} \|r(h)\|/h = 0$. Since a locally Lipschitz function on a compact set is globally Lipschitz continuous, we can assume that

$$|V(x(t + h)) - V(x(t) + hx'(t))| \leq \lambda\|r(h)\|,$$

whenever $|h| < \delta$. Consequently, the function $h \to V(x(t) + hx'(t))$ is differentiable at $h = 0$, and its derivative is the same as the derivative of $h \to V(x(t+h))$ at $h = 0$. Making use of Lemma 2, we obtain

$$(V \circ x)'(t) = \lim_{h \to 0} \frac{V(x(t) + hx'(t)) - V(x(t))}{h} = \|x'(t)\|^2$$

at each point $t \in [0, T] \setminus S$. □

3. The main result

Let K be a convex and locally compact subset of X and consider an upper semicontinuous set valued map F defined on K with nonempty closed images in X. Let us suppose that there exists a lower regular potential function V on X such that the tangential condition

$$T_K(x) \cap F(x) \cap \partial V(x) \neq \emptyset \tag{2}$$

holds true for every $x \in K$, where $T_K(x)$ denotes the tangent cone to K at x.

Let the point x_0 be given in K and consider the Cauchy problem

$$\begin{aligned} &x'(t) \in F(x(t)) \quad \text{a.e.} \\ &x(0) = x_0 \end{aligned} \tag{3}$$

with the phase constraint

$$x(t) \in K, \qquad t \geq 0. \tag{4}$$

Theorem 2. *Assume that the tangential condition* (2) *is valid. Then under the above conditions there exists a* $T > 0$ *such that the problem* (3), (4) *admits a solution on* $[0, T]$.

Choose $\varrho > 0$ such that $K_0 = K \cap (x_0 + 2\varrho B)$ is compact and V is Lipschitz continuous on $x_0 + 2\varrho B$ with Lipschitz constant $\lambda > 0$. Then $\partial V(x) \subset \lambda B$ for every $x \in K_0$. Set $T = \varrho/\lambda$ and $K_1 = K \cap (x_0 + \varrho B)$. Then no solution x starting from x_0 with

$$x'(t) \in F(x(t)) \cap \partial V(x(t)) \qquad \text{a.e.} \tag{5}$$

can leave the compact set K_1 on the interval $[0, T]$. Therefore, without loss of generality, we may assume that K is compact. Below we construct a solution to the problem (3), (4) that also solves (5).

We denote by S_T the solution set to the problem (3), (4) on the interval $[0, T]$. S_T will be regarded as a subset of the Banach space $W^{1,2}(0, T, X)$ of absolutely continuous functions equipped with the norm

$$\|x\| = \max_{t \in [0,T]} \|x(t)\| + \left(\int_0^T \|x'(t)\|^2 \, dt \right)^{1/2}.$$

Theorem 3. *Under the additional assumption*

$$F(x) \subset \partial V(x) \qquad \text{for each } x \in K$$

there exists a $T > 0$ *such that* S_T *is a nonempty compact subset in* $W^{1,2}(0, T, X)$.

4. Approximate solutions

Let $0 < \varepsilon < \varrho/\lambda$ be given. Select a vector

$$v \in T_K(x_0) \cap F(x_0) \cap \partial V(x_0).$$

Then we can find an $0 < h < \varepsilon$ such that $d_K(x_0 + hv) < \varepsilon h$. This implies the existence of a point y in K with $\|x_0 + hv - y\| < \varepsilon h$. Set

$$w = \frac{y - x_0}{h} \in \partial V(x_0) + \varepsilon B.$$

Then $\|v - w\| < \varepsilon$ and by Lemma 1,

$$\|w\|^2 \le D^+V(x_0; w) + 2\varepsilon\lambda .$$

Consequently, we can pick a $0 < \delta < h$ such that

$$\|w\|^2 \le \frac{V(x_0 + \delta w) - V(x_0)}{\delta} + 3\varepsilon\lambda .$$

In view of the convexity of K we immediately get $x_0 + tw \in K$ whenever $0 \le t \le \delta$. Now set $\tau = \delta$ and define x_ε on $[0, \tau]$ by

$$x_\varepsilon(t) = x_0 + tw, \qquad t \in [0, \tau].$$

Our construction proceeds as follows. Let us suppose that for some $\tau > 0$ we have defined x_ε with the following three properties:

$$\int_0^\tau \|x'_\varepsilon(t)\|^2 \, dt \le V(x_\varepsilon(\tau)) - V(x_0) + 3\varepsilon\lambda\tau, \tag{6}$$

and for a.e. $t \in [0, \tau]$

$$x'_\varepsilon(t) \in F(x_\varepsilon(t) + 2\lambda\varepsilon B) + \varepsilon B, \tag{7}$$

and finally, for each $t \in [0, \tau]$

$$x_\varepsilon(t) \in K. \tag{8}$$

Our construction starting from x_0 obviously fulfills these criteria. Keeping the previous notations take a vector

$$v \in T_K(x_\varepsilon(\tau)) \cap F(x_\varepsilon(\tau)) \cap \partial V(x_\varepsilon(\tau)),$$

then there exists an $0 < h < \varepsilon$ with $d_K(x_\varepsilon(\tau) + hv) < \varepsilon h$. Hence, we can find a point $y \in K$ such that $\|x_\varepsilon(\tau) + hv - y\| < \varepsilon h$. Put

$$w = \frac{y - x_\varepsilon(\tau)}{h} \in \partial V(x_\varepsilon(\tau)) + \varepsilon B.$$

Then clearly $\|v - w\| < \varepsilon$ and by exploiting Lemma 1 we have

$$\|w\|^2 \le D^+ V(x_\varepsilon(\tau); w) + 2\varepsilon\lambda.$$

This implies that a $0 < \delta < h$ can be chosen with

$$\|w\|^2 \le \frac{V(x_\varepsilon(\tau) + \delta w) - V(x_\varepsilon(\tau))}{\delta} + 3\varepsilon\lambda.$$

The convexity of K assures that $x_\varepsilon(\tau) + tw \in K$ whenever $\tau \le t \le \tau + \delta$. Define x_ε on $[\tau, \tau + \delta]$ by

$$x_\varepsilon(t) = x_\varepsilon(\tau) + tw, \qquad t \in [\tau, \tau + \delta].$$

It is easy to verify that x_ε satisfies conditions (6), (7) and (8) with τ replaced by $\tau + \delta$.

A straightforward application of Zorn's lemma shows that x_ε can be extended to $[0, T]$ with retaining the properties (6), (7) and (8) on the interval $[0, T]$.

5. Proof of the theorems

Proof of Theorem 2. Let $\varepsilon = 1/n$ and consider the sequence of approximate solutions x_n given in the preceding section. Assume that $\lambda > 0$ is a Lipschitz constant for V on the compact set K. Then by the construction

$$\|x_n'(t)\| \le \lambda + 1$$

for a.e. $t \in [0, T]$ and $\operatorname{graph} x_n$ is contained in K for each n. Therefore, we can select a subsequence, again denoted by x_n which uniformly converges to an absolutely continuous function x on $[0, T]$, moreover $x_n' \to x'$ weakly in $L^2(0, T, X)$.

By passing to the limit, standard arguments show that $x'(t) \in \partial V(x(t))$ a.e. Thus, in view of Lemma 3, we obtain

$$\int_0^T \|x'(t)\|^2 \, dt = \int_0^T (V \circ x)'(t) \, dt = V(x(T)) - V(x_0).$$

Hence, taking the limit in (6), we get

$$\limsup_{n\to\infty} \int_0^T \|x_n'(t)\|^2 \, dt \le \int_0^T \|x'(t)\|^2 \, dt$$

or in other words

$$\limsup_{n\to\infty} \|x_n'\|_{L^2} \le \|x'\|_{L^2}.$$

This latter relation combined with the weak convergence implies the L^2-norm convergence of the derivative sequence. Consequently, we may assume that x_n' converges to x' almost everywhere. On the other hand (7) can be rewritten as

$$(x_n(t), x_n'(t)) \in \operatorname{graph} F + \alpha_n(B \times B),$$

where $\alpha_n \to 0$ if $n \to \infty$. This relation together with (8) tell us that x is a solution to the problem (3), (4) on the interval $[0, T]$. □

Proof of Theorem 3. Consider a sequence x_n in S_T. Since the derivatives are uniformly bounded, without loss of generality we may assume that $x_n' \to x'$ weakly in $L^2(0, T, X)$ and $x_n \to x$ uniformly on $[0, T]$. By Lemma 3 we have

$$\int_0^T \|x(t)\|^2\, dt = V(x_n(T)) - V(x_0).$$

Since the right hand side of the above equality converges to $V(x(T)) - V(x_0)$, and by standard arguments $x'(t) \in \partial V(x(t))$, a repeated application of Lemma 3 gives us

$$\lim_{n\to\infty} \int_0^T \|x_n'(t)\|^2\, dt = \int_0^T \|x'(t)\|^2\, dt$$

and hence, $x_n' \to x'$ with respect to the $L^2(0, T, X)$-norm. From this point we can follow the patterns of the proof to Theorem 2 to get that x lies in S_T. This proves that S_T is a compact subset of $W^{1,2}(0, T, X)$. □

Examples. It is worth mentioning here that our Theorem 2 generalizes the result of [3]. Indeed, take the lower regular function V on the real line described in the example next to Definition 1. Consider the differential inclusion problem

$$x'(t) \in F(x(t)), \qquad x(0) = 0, \tag{9}$$

where the set valued map F is given by

$$F(x) = \begin{cases} \{V'(x)\} & \text{if the derivative exists,} \\ [-1, 1] & \text{if } x = 0, \\ \{-1, 1\} & \text{otherwise.} \end{cases}$$

It is easy to verify that F is upper semicontinuous, admits nonconvex values in every neighborhood of the origin and $F(x) \subset \partial V(x)$ at every point. However, it is obvious that there is no proper convex continuous function W with $F(x) \subset \partial W(x)$.

Finally, let us note that the lower regularity of the potential function V cannot be omitted. Consider for instance the Cauchy problem (9) with

$$F(x) = \begin{cases} \{1\} & \text{if } x < 0, \\ \{-1, 1\} & \text{if } x = 0, \\ \{-1\} & \text{if } x > 0, \end{cases}$$

that is the common example of an uppersemicontinuous map with no solutions. Although we have $F(x) \subset \partial V(x)$ at every point for $V(x) = -|x|$, the potential function V is apparently not lower regular at the origin.

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GRAPH-APPROXIMATION OF SET-VALUED MAPS. A SURVEY

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Since the early paper of von Neumann [34], approximation methods proved to be useful in many applications involving set-valued maps: mathematical economics, game theory [6], differential equations and inclusions [5], control theory and others. These methods, together with homological ones (see [22], [26]), play also an important role in the fixed point theory of set-valued maps. The so-called *graph-approximations* of upper semicontinuous multivalued mappings with nonconvex values are the main object of our interest. The existence of selections or uniform approximations for such maps is a rare phenonemenon.

In what follows a *space* is a *(Hausdorff) topological* space; a *map* is a *continuous* transformation of spaces. By a *set-valued map* φ of a space X into a space Y (denoted $\varphi : X \multimap Y$) we understand an *upper semicontinuous multivalued transformation* with *compact nonempty* values (see [14] or [22] for more details on set-valued maps) ([1]). If $f : X \to Y$ (resp. $\varphi : X \multimap Y$) is a map (resp. set-valued map), then $\mathrm{Gr}(f)$ (resp. $\mathrm{Gr}(\varphi)$) stands for the *graph* of f (resp. of φ), i.e. $\mathrm{Gr}(\varphi) = \{(x,y) \in X \times Y \mid y \in \varphi(x)\}$.

Let $\varphi : X \multimap Y$ be a set-valued map from a space X into a space Y and let $A \subset X$.

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[1] Many results of this paper are valid for maps with *closed* values; we assume compactness of values in order to avoid tedious distinctions between results valid for compact-valued maps but not true for maps with merely closed ones. It seems that the reader will be able to see instances for which compactness of values is superfluous.

Definition 1. Given a neighborhood $\mathcal{U}$ of the graph $\mathrm{Gr}(\varphi)$ in $X \times Y$ ([2]), we say that a map $f : A \to Y$ is a *$\mathcal{U}$-approximation of* φ provided

$$\mathrm{Gr}(f) \subset \mathcal{U}.$$

In case of metrizable spaces X and Y one may use epsilons instead of neigborhoods in Definition 1. Given a continuous function $\varepsilon : X \to (0, +\infty)$ we say that a map $f : A \to Y$ is an *ε-approximation* of φ if ([3])

$$\forall\, x \in A \;\; f(x) \in B(\varphi(B(x, \varepsilon(x))), \varepsilon(x)). \tag{1}$$

If A is compact, then one may replace ε-functions by positive constants and thus arrive the traditional notion of an ε-graph-approximation.

Proposition 1. *Let X, Y be metric spaces.*

(i) *For each neighborhood $\mathcal{U}$ of* $\mathrm{Gr}(\varphi)$*, there is a continuous function $\varepsilon : X \to (0, +\infty)$ such that any ε-approximation $f : A \to Y$ of φ is a $\mathcal{U}$-approximation of φ.*

(ii) *Conversely, given a function $\varepsilon : X \to (0, +\infty)$, there is a neighborhood $\mathcal{U}$ of* $\mathrm{Gr}(\varphi)$ *such that any $\mathcal{U}$-approximation $f : A \to Y$ of φ is an ε-approximation of φ.*

Let α be an open covering of $\mathrm{Gr}(\varphi)$. We say that $f : A \to Y$ is an *α-approximation of* φ if for all $p \in \mathrm{Gr}(f)$, there is $q \in \mathrm{Gr}(\varphi)$ such that both p and q lie in the same member of α (see [35]). It is clear that, for a covering α of $\mathrm{Gr}(\varphi)$, any $\mathcal{U}$-approximation $f : A \to Y$ of φ, where $\mathcal{U} = \bigcup_{V \in \alpha} V$, is an α-approximation; conversely, for a neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there exists its covering α such that any α-approximation $f : A \to Y$ of φ is a $\mathcal{U}$-approximation.

In case X, Y are subsets of topological vector spaces E and F, respectively, and given neighborhoods U and V of the origins in E and F, we say that $f : A \to Y$ is a *(U, V)-approximation* of φ if, for any $x \in A$, $f(x) \in [\varphi((x+U) \cap X) + V] \cap Y$ (see [11]). This is a direct generalization of the concept of an ε-approximation to the non-metrizable context (moreover, it may be easily generalized for uniform spaces X, Y). As in Proposition 1, we easily see that, for any U, V (as above), any $\mathcal{U}$-approximation $f : A \to Y$ of φ, where $\mathcal{U} = U \times V + \mathrm{Gr}(\varphi)$, is a (U, V) approximation of φ. Conversely, given a neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there are neighborhoods U and V of the origins such that a (U, V)-approximation $f : A \to Y$ of φ is a $\mathcal{U}$-approximation provided A is compact.

[2] In the sequel, we always speak of *open neighborhoods of* $\mathrm{Gr}(\varphi)$ in $X \times Y$.

[3] If (Z, d) is a metric space, $\delta > 0$ and $C \subset Z$, then $B(C, \delta) := \{z \in Z \mid \inf_{c \in C} d(z, c) < \delta\}$.

It seems, therefore, that the notion of $\mathcal{U}$-approximation is the simplest, the most general and perhaps the best.

Let again $\varphi : X \multimap Y$ be a set-valued map between spaces and let A, B be closed subsets of X such that $A \subset \operatorname{int} B$.

Definition 2. We say that φ is:

- *approximable* if, for each neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there exists a $\mathcal{U}$-approximation $f : X \to Y$ of φ;
- *ε-approximable* if X, Y are metrizable and, for each $\varepsilon > 0$, there is an ε-approximation $f : X \to Y$ of φ;
- *(U, V)-approximable* if X, Y are subsets of topological vector spaces and, for all neighborhoods U and V of the origins, there is a (U, V)-approximation $f : X \to Y$ of φ;
- *weakly relatively approximable over* A, i.e. for every neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ with the following property: if a $\mathcal{V}$-approximation $f : A \to Y$ of φ extends to a map $f' : N \to Y$, where N is a neighborhood of A in X, then there is a $\mathcal{U}$-approximation $F : X \to Y$ of φ such that $F|_A = f$;
- *relatively approximable over* A, i.e. for any neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that any $\mathcal{V}$-approximation $f : A \to Y$ of φ extends to a $\mathcal{U}$-approximation $F : X \to Y$ of φ ([4]).
- *relatively approximable over* (A, B) if, for every neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ with the following property: if $f : B \to Y$ is a $\mathcal{V}$-approximation of φ, then there is a $\mathcal{U}$-approximation $F : X \to Y$ of φ such that $F|_A = f|_A$.

Clearly if X, Y are metrizable and φ is approximable, then it is ε-approximable; the converse holds provided X is compact. Similar statement relates approximability to (U, V)-approximability.

In general, if φ is relatively approximable over A, then it is weakly relatively approximable over A. If Y is a neighborhood extensor with respect to the pair (X, A), then clearly the weak relative approximability of φ over A implies relative approximability. This holds, for instance, when A is a neighborhood retract in X.

Obviously if φ is weakly relatively approximable over A, then it is relatively approximable over (A, B) for any closed neighborhood B of A.

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[4]This notion has its counterpart in the metrizable case and the language of ε-approximability: one says that φ is *relatively ε-approximable over* A provided, for each $\varepsilon > 0$, there is $\delta > 0$ such that any δ-approximation $f : A \to Y$ extends to an ε-approximation $F : X \to Y$ of φ. Similarly one may define the notion of the relative (U, V)-approximability over A.

Sometimes approximability implies the existence of uniform approximations. For example we have the following result (comp. [25]).

Proposition 2. *Assume that $\varphi : X \multimap Y$, where X is paracompact and Y is a metric space, is approximable. If φ is additionally lower semicontinuous (i.e. the set $\{x \in X \mid \varphi(x) \subset F\}$ is closed for a closed $F \subset Y$) then for any $\varepsilon > 0$, there is a map $f : X \to Y$ such that $d(f(x), \varphi(x)) = \inf_{y\in\varphi(x)} d(f(x), y) < \varepsilon$ for all $x \in X$.*

It is clear that if no conditions are imposed on the values of a map φ, then the problem of the existence of sufficiently close approximations has hardly an answer. Here we shall deal with set-valued maps whose values satisfy some of the so-called UV-properties. Sets satisfying these properties have been intensively studied for many years by geometric topologists (see e.g. [7], [30]).

Definition 3. Let A be a closed subset of Y. We say that the inclusion $A \hookrightarrow Y$ has:

- UV^n-property ($n \geq 0$ is an integer) if each open neighborhood U of A in Y contains a neighborhood V of A such that any singular k-sphere, $0 \leq k \leq n$, in V is null-homotopic in U;
- UV^ω-property if it has UV^n-property for each $n \geq 0$;
- UV^∞-property if each neighborhood U of A (in Y), contains a neighborhood V of A such that V is contractible in U ([5]).

It is clear that properties defined above are rather properties of the embedding of a given compactum in the ambient space. For instance, a point has properties UV^n, $0 \leq n < \infty$, (resp. UV^ω, UV^∞) if and only if the ambient space Y is locally n-connected (resp. locally ∞-connected, locally contractible).

Clearly if $A \hookrightarrow Y$ has UV^n-property, $0 \leq n < \infty$, then it has UV^m-property for $0 \leq m \leq n$; if $A \hookrightarrow Y$ has UV^∞-property, then it has UV^ω-property.

The following proposition collects important examples of sets satisfying some UV-properties

Proposition 3. *For a compact subset A of an ANR set Y the following conditions are equivalent:*

- *A is an R_δ-set (i.e. it can be represented as the intersection of a decreasing sequence of compact AR-spaces)* ([6]);
- *$A \hookrightarrow Y$ has UV^∞-property (is a cell-like set)*;

[5] In a less popular terminology, due to Dugundji, a set $A \subset Y$ such that the inclusion $A \hookrightarrow Y$ satisfies UV^n- (resp. UV^ω-) property is called n-proximally conected (resp. ∞-proximally connected).

[6] In particular, any contractible subset of Y has UV^∞-property.

- *A may be represented as the intersection of a decreasing sequence of compact contractible sets;*
- *A is contractible in each of its neighborhoods (i.e. A is approximatively contractible);*
- *A has the shape of a point.*

Let $\varphi : X \to Y$ be a set-valued map.

Definition 4. Let $0 \leq n \leq \infty$ or $n = \omega$. We say that φ is a *UV^n-valued* map if for each $x \in X$, the inclusion $\varphi(x) \hookrightarrow Y$ has UV^n-property.

Suppose now that X, Y are metrizable. After Myshkis – [33] (see also [15] and [21]) we shall say that a map $\varphi : X \to Y$ is *aspheric in dimension* $k \geq 0$ if for each $x \in X$ and *all* $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that any singular k-sphere in $B(\varphi(x), \delta)$ is null-homotopic in $B(\varphi(x), \varepsilon)$ and *weakly aspheric in dimension* k if this holds for each $x \in X$ and *some* $\varepsilon > 0$. Finally, a map $\varphi : X \to Y$, where X is a finite polyhedron, $\dim X = n$ is *aspheric* if it is apsheric in dimension $0 \leq k \leq n-1$ and weakly aspheric in dimesnion n. It is clear that any aspheric map (defined on an n-dimesional finite polyhedron) is a UV^{n-1}-valued map.

Theorem 1. *Let $\varphi : X \multimap Y$ be a set-valued map of spaces.*

1. (Cellina 1969 – [18]) *If X is metrizable, Y is a metrizable locally convex space and the values of φ are convex, then φ is ε-approximable* ([7]) ([8]).
2. (Myshkis 1954 – [33]) *If X is an n-dimensional polyhedron, Y is metrizable and φ is an aspheric map, then φ is ε-approximable.*
3. (Mas-Collel 1974 – [32]) *If X is a finite polyhedron, φ compact contractible-valued and $Y = \mathbb{R}^n$, then φ is ε-approximable.*
4. (Anichini, Conti, Zecca 1985 – [3]) *The same result holds for any normed space Y.*
5. (Cannon 1975 – [17]) *If X is metrizable locally compact separable, Y is an ENR (Euclidean Neighborhood Retract) and φ is a UV^∞-valued map, then it is relatively approximable over any closed subset $A \subset X$.*
6. (Ancel 1985 – [1]) *If X is metrizable is countable dimensional or* $\dim(X \setminus A) < \infty$ *and φ is a UV^∞-valued map then φ is weakly relatively approximable over A.*

[7]The paper [19] provides an extensive discussion of various approximation results for convex valued maps satisfying less restrictive continuity assumptions.

[8]Beer 1988 – [13] shows that a similar results holds for a starshaped-valued φ.

7. (Górniewicz, Granas, Kryszewski 1989 – [23]) *If (X, A) is a finite polyhedral pair, Y is metrizable and φ is a UV^ω-valued map then φ is relatively ε-approximable over A* ([9]).
8. (1989 – [23]) *If X is a compact ANR, Y metrizable and φ is a UV^ω-valued map then φ is ε-approximable* ([10]).
9. (Bader, Gabor, Kryszewski 1993 – [9]) *If X and A are compact ANRs, Y is metrizable and φ is a UV^ω-valued map then φ is relatively ε-approximable over A.*

A result due to Ancel – [1] is by all means the most general one concerning UV^∞-valued maps. It appeared while the author investigated general properties of the so-called cell-like maps from the strictly topological viewpoint (extending earlier ideas of Lacher, Haver and others). Some later, mentioned above, results for UV^∞-valued maps are implied by Ancel's. However they were obtained independently, the starting point to these investigations was different and they were addressed to analysts rather. In [23], [27], [11] and [24] the authors were interested in the fixed-point theory implications of the existence of graph-approximations rather, assumptions concerning maps in these papers are, generally speaking, weaker then those of Ancel (note that UV^∞-property is stronger than UV^ω); however it seems that the authors have not been fully aware of Ancel's (or Cannon's) results. It should also be remarked that older approximation results obtained, for instance, by Myshkis or Mas-Collel has not been noticed for a long time and, as it seems, Ancel did not know about them.

Remark 1.

(i) It is no wonder that *all* above results concern maps with values satisfying some UV-properties: the approximability of a set-valued map is in a sense a sufficient condition for the map to have UV-values. Namely, one shows easily that if $\varphi : X \multimap Y$ is a set-valued map of metric spaces having the following property: for every compact ANR T with $\dim T \leq n+1$, for every sub-ANR A of T, every map $j : T \to X$ and every $\varepsilon > 0$, there is $\delta > 0$ such that any δ-approximation $f : A \to Y$ of $\varphi \circ j$ extends to an ε-approximation of $\varphi \circ j$, then φ is a UV^n-valued map.

[9] This result has been repeated by Ben-El-Mechaiekh and Deguire – [11] in the context of (UV)-approximability

[10] McLennan 1989 – [31] shows this result for contractible valued maps; Anichini, Conti, Zecca – [4] have shown it in case X is compact convex in a normed space, Y is a normed space and φ is UV^∞-valued while in [11] this result has been established in the context of (U, V)-approximability; Górniewicz, Lassonde [24] extended this result for a compact AANR.

(ii) The most important among stated above are results concerning approximability of maps defined on *finite* polyhedra. The finitness assumption was crucial in their proofs: it made a sort of finite constructions on skeleta possible. These constructions can not be done on infinite polyhedra.

(iii) It is important to note that Górniewicz, Lassonde [24] and Bader [8] (and also Ben-El-Mechaiekh, Deguire [11]), independently, observed that the ε-approximability result of Granas, Górniewicz and Kryszewski [23] (stated in Theorem 1 part (8) may be generalized to a metric space X (or a subset X of a topological vector space in [11]) being α-dominated by a finite (i.e. *compact*) polyhedron for any covering α of X ([11]). This observation led also to a remarkable simplification of the original proof from [23].

(iv) Let us note the following fact due to Górniewicz, Lassonde [24] (see also [11]):
if $\varphi_i : X \multimap Y$, $i = 1, 2, \ldots$, *is a decreasing sequence of set-valued approximable maps,* X *is compact, then* $\varphi = \bigcup_{i=1}^{\infty} \varphi_i$ *is approximable.*

It is clear that this observation enables to get further generalizations of results stated in Theorem 1.

Some of the above mentioned results concerning ε-approximability were generalized for maps being finite compositions of maps from above classes (Cannon 1975 – [17]; Górniewicz, Lassonde 1994 - [24]; Miklaszewski, Kryszewski 1989 – [27]) defined on *compact* domains. However, many facts concerning approximability extend to compositions (without assumptions involving compactness) – see [28]. Here is a sample result:

Proposition 4. *Let* $\varphi : X \multimap Y$ *be approximable and let* $\psi : Y \multimap Z$*. If*

(i) ψ *is single-valued; or*

(ii) Y *is paracompact,* ψ *is approximable and* φ *is proper* ([12]),

then $\psi \circ \varphi$ *is approximable. In both cases approximations of* $\psi \circ \varphi$ *are of the form* $g \circ f$*, where* $f : X \to Y$ *and* $g : Y \to Z$ *are sufficiently close approximations of* φ *and* ψ *(or* $g = \psi$ *in case* (i)*), respectively.*

These facts hold for ε-approximability only if X is compact (then φ is automatically proper).

[11] Recall that, given a covering α of X, we say that X is *α-dominated by a space P* if there are maps $s : X \to P$, $r : P \to X$ such that $r \circ s$ and the identity id_X are α-close (that is, for each $x \in X$, both x and $r(s(x))$ belong to the same member of α). Any compact ANR satisfies this property

[12] I.e. $\varphi^{-1}(K) := \{x \in X \mid \varphi(x) \cap K \neq \emptyset\}$ is compact for any comapct $K \subset Y$; any φ is proper if X is compact.

Approximation results for set-valued maps have been mainly applied in the fixed-point theory. Results of Myshkis [33] were used by Borisovich, Gliklikh [15] and Gliklikh [21] to define the Lefschetz number and topological degree of aspheric maps on subsets of $\mathbb{R}^n$. The fixed point index theory on compact ANRs was constructed in [23], [27] and [9] and on arbitrary ones in [10] (see also a survey [25]).

Finally observe that, save results of Cellina, Cannon and Ancel, the compactness of a domain was an essential and unavoidable component of assumptions. In what follows we shall deal with maps defined on *noncompact* domains and we shall establish far going generalizations of results collected in Theorem 1. All below results are taken from Kryszewski 1994 – [28].

The first one shows that in a finite-dimensional setting approximations always exist (no structural assumptions on the domain are necessary).

Theorem 2. *Let $0 \leq n < \infty$ and let $\varphi : X \multimap Y$ be a UV^n-valued map between spaces.*

1. *If X is paracompact, $\dim X \leq n+1$, then φ is approximable.*
2. *If X, Y are metrizable, Y is locally n-connected, A is closed in X and $\dim(X \setminus A) \leq n+1$ then φ is is relatively approximable over A.*

In 1994 Repovs, Semenov and Ščepin [35] proved a result similar to Theorem 2.1. They do not assume that Y is paracompact; instead the map φ has to be $*$-paracompact (this means hat it behaves well with respect to open coverings of X and Y). Unfortunately, not every set-valued map is such. $*$-paracompact are open set-valued maps (i.e. transforming open sets onto open ones) or set-valued maps between compact metric spaces). Under these assumptions, they prove that, for any coverings α of X and β of Y, there is an $\alpha \times \beta$-approximation of φ.

Moreover, there are some results (due to Şçepin, Repovs and Brodski) similar to Theorem 1, where Y is the Banach space but the assumption concerning φ is relaxed. Namely, they suppose that φ has the so-called UV^n-filtration (comp. Remark 1 (iv)).

Admitting infinite-dimensional domains we have to pose some structural assumptions. Recall that a simplicial complex K is *locally finite dimensional* if, for each vertex $v \in K$, there is an upper bound for dimensions of simplices having v as a vertex, i.e. $\sup\{\dim \sigma \mid v \text{ is a vertex of a simplex } \sigma \text{ in } K\} < \infty$.

Theorem 3. *Let $\varphi : X \multimap Y$ be a UV^ω-valued map from a locally finite dimensional polyhedron X (with the Whitehead topology) to a space Y. Then φ is approximable. If A is a closed subpolyhedron in X then φ is relatively* *approximable over A. Moreover, if A is an arbitrary closed subset of X then φ is weakly relatively approximable over A.*

In the situation of Theorems 2 and 3 the relative approximability over A is equivalent to weak relative approximability of φ over A. This is because the hypotheses of these theorems guarantee that every map $f : A \to Y$ has a neighborhood extension $f' : N \to Y$. In particular, the assumption of Theorem 2 (part 2) that Y is locally n-connected combined with the results of Eilenberg, Wilder [20] and Kuratowski [29] provides such an extension. In Theorem 3 a closed subpolyhedron A is neighborhood retract of X and such an extension exists for trivial reasons.

Using different methods of proof and applying one of the main results from [36] we have the following result concerning weak approximability.

Theorem 4. *Let $\varphi : X \multimap Y$ be a UV^ω-valued map from an ANR X to a metric space Y. If A is a closed subset of X and N is a neighborhood of A, then for every neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ and any selection $f : N \to Y$ of φ ([13]) there is a $\mathcal{U}$-approximation $F : X \to Y$ of φ such $F|_A = f|_A$.*

Setting $A = N = \emptyset$ in Theorem 4 the next approximation result follows.

Theorem 5. *A UV^ω-valued map $\varphi : X \multimap Y$ from an ANR X to a metric space Y is approximable.*

Unfortunately the author was unable to get a general result concerning relative approximability over the closed subset $A \subset X$ of φ in the setting of Theorem 5. However it is true for a separable or a locally compact ANR X and its sub-ANR A. To explain it we need the following notion.

Definition 5. Let α be an open cover of a space X and let $A \subset X$ be closed. We say that the pair (X, A) is *α-dominated by a pair (Z, C)* if there are maps

$$(X, A) \xrightarrow{p} (Z, C) \xrightarrow{r} (X, A)$$

such that $r \circ p$ and the identity id_X are α-close (i.e. for all $x \in X$, $r(p(x))$ and x belong to the same member of α).

We say that (X, A) is *homotopy α-dominated by* (Z, C) if it is α-dominated and there is a α-homotopy $h : X \times [0, 1] \to X$ (i.e. $\{h(\{x\} \times [0, 1])\}_{x \in X}$ refines α) such that $h_0 = 1_X$ and $h_1 = r \circ p$.

We say that (X, A) is *properly* (resp. *homotopy*) *α-dominated* if the map r is proper.

Finally, we say that (X, A) is (resp. *properly*) (resp. *homotopy*) *dominated*

[13] i.e., $f(x) \in \varphi(x)$ for all $x \in N$

if, for each covering α of X, (X, A) is (resp. properly) (resp. homotopy) α-dominated by a locally finite dimensional polyhedral pair.

In case $A = \emptyset$ (and $C = \emptyset$) we have the respective notions of X being (resp. properly) (resp. homotopy) dominated.

If (X, A) is an ANR-pair, then the (proper) α-domination, for each covering α of X implies the the (proper) homotopy α-domination for all α. In the early 1980's, Ancel [2] and Toruńczyk independently asked which ANRs are properly homotopy dominated. For instance, the standard construction of dominating polyhedra of ANR's yields that every separable or locally compact ANR, being strongly paracompact, is properly homotopy α-dominated by a locally finite, hence locally finite dimensional polyhedron for any open cover α; therefore separable or locally compact ANR are properly homotopy dominated. The same holds for an arbitary Hilbert manifold. The question whether every ANR satisfies this property remains open.

Theorem 6. *Suppose that X is a paracompact space, $A \subset X$ is closed and let $\varphi : X \multimap Y$ be UV^ω-valued map into a space Y.*

1. *If X is dominated then φ is approximable.*
2. *If X is properly homotopy dominated, $B \subset X$ is closed and $A \subset \operatorname{int} B$ then φ is relatively approximable over (A, B).*
3. *If (X, A) is a properly homotopy dominated ANR-pair, then φ is relatively approximable over A.*

Unfortunately the author is not sure whether the version of Theorem 5 part 3 holds for *any* (paracompact) pair (X, A).

Regarding graph-approximations as tools for studying properties of set-valued maps (in particular, the existence of their fixed points via homotopy invariants such as e.g. the fixed point index) the following concept and problem seem to be of importance.

Definition 6. Let $\varphi : X \to Y$ be a set-valued map between spaces We say that φ is *homotopy approximable* if for each neighborhood $\mathcal{U}$ of $\operatorname{Gr}(\varphi)$, there exists a neighbourhood $\mathcal{V}$ of $\operatorname{Gr}(\varphi)$ such that any $\mathcal{V}$-approximations $f, g : X \to Y$ of φ are joined by a homotopy $h : X \times [0,1] \to Y$ such that $h_t = h(\,\cdot\,, t)$ is a $\mathcal{U}$-approximation of φ for every $t \in [0,1]$.

Clearly, the notion of homotopy approximability may be generalized in way approximability was generalized.

It is natural to ask what are the sufficient conditions implying the homotopy approximability of a map φ. Such a homotopy approximable map φ may be adequately studied by means of (single-valued) approximations and their homotopy invariants like topological degree, fixed-point index and/or the Lefschetz number and, moreover, these invariants can be easily defined for φ. Without homotopy approximability of φ, for instance, the fixed-point indices of sufficiently close approximations of φ would not stabilize.

Theorem 1 part 9 implies that if X is a compact ANR, then a UV^ω-valued map $\varphi : X \to Y$ is homotopy approximable (this was actually proved in [23], but [9] 1996 provides a simpler direct proof). Also Theorems 2, 3 and 6 are designed in the way allowing to obtain homotopy approximability.

Theorem 7. *Let $\varphi : X \multimap Y$ be a UV^n-valued map between spaces, $0 \le n \le \omega$ and let $A \subset X$ be closed. If*

- *$n < \infty$, X is metrizable, $\dim(X \setminus A) \le n$, Y is metrizable locally n-connected; or*
- *X is locally finite dimensional polyhedron with the Whitehead topology, A is a closed subpolyhedron of X,*

then, given a neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that, for any $\mathcal{V}$-approximations $f, g : A \to Y$ and a (partial) homotopy $h : A \times [0,1] \to Y$ joining $f|_A$ to $g|_A$ and such that h_t is a $\mathcal{V}$-approximation of φ for all $t \in [0,1]$, there is a homotopy $H : X \times [0,1] \to Y$ joining f to g such that $H|_{A\times[0,1]} = h$ and H_t is a $\mathcal{U}$-approximation of φ for all $t \in [0,1]$. In particular if

- *$n < \infty$, X is metrizable and $\dim X \le n$, Y is metrizable locally n-connected, or*
- *If X is a locally finite dimensional polyhedron with the Whitehead toipology, $n = \omega$,*

then φ is homotopy approximable.

This result has a counterpart for spaces being properly dominated.

Theorem 8. *Suppose that a paracomapct space X is properly homotopy dominated and let $\varphi : X \multimap Y$ be a UV^ω-valued map into a space Y. Let $A, B \subset X$ be closed and $A \subset \operatorname{int} B$. For any neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that, given $\mathcal{V}$-approximations $f, g : X \to Y$ of φ and a homotopy $h : B \times [0,1] \to Y$ joining $f|_B$ to $g|_B$ such that h_t is a $\mathcal{V}$-approximation of φ for all $t \in [0,1]$, there is a homotopy $H : X \times [0,1] \to Y$ joining f to g such that $H|_{A\times[0,1]} = h|_{A\times[0,1]}$ and H_t is a $\mathcal{U}$-approximation of φ for all $t \in [0,1]$. In particular, φ is homotopy approximable.*

The same type of a reslt may be derived from Theorem 6 part 3.

Finally, let us recall a result which shows perhaps a future possible development of the theory.

Theorem 9 (Ben-El-Mechaiekh, Kryszewski 1997 – [12]). *Let X be a metric space and Y a normed one. Let $\varphi : X \multimap Y$ be a set-valued map with convex closed values and $\Phi : X \multimap Y$ a lower semicontinuous map also with closed and convex values. For any $\varepsilon > 0$, there exists an ε-approximation $f : X \to Y$ of φ such that f is a selection of Φ (i.e. $f(x) \in \Phi(x)$ for all $x \in X$).*

It is, perhaps, easy to generalize this result for the case X is paracomapct and Y is a metrizable locally convex space (or merely locally convex space and φ has compact values); then, for any neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there should exist a selection of Φ being a $\mathcal{U}$-approximation of φ. This type of a "controlled" approximation of set-valued maps is of importance in applications and could be probably carried over to larger classes of set-valued maps with nonconvex values.

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APPLICATIONS OF MARCZEWSKI FUNCTION TO MULTIFUNCTIONS

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1. Preliminaries

The Marczewski function (i.e. the characteristic function of a sequence of sets) allows us to derive some results on measurable and Carathéodory type functions from corresponding theorems for continuous functions. In this paper we survey our results on selections and approximations of multifunctions obtained by this method. By the use of the Marczewski function we get new proofs of well known theorems as well as some new results.

Throughout this paper $(T, \mathcal{T})$ is a measurable space, and X, Y metric spaces. By $\mathcal{B}(X)$ we denote the Borel σ-field on X, and by $\mathcal{T} \otimes \mathcal{B}(X)$ the product σ-field on $T \times X$. $\mathcal{P}(Y)$ stands for the family of all nonempty subsets of Y.

We say that a multifunction $\varphi : X \to \mathcal{P}(Y)$ is *lower* (*upper*) *semicontinuous* if for each open $V \subset Y$ the set $\varphi^-(V) = \{x \mid \varphi(x) \cap V \neq \emptyset\}$ (respectively, $\varphi^+(V) = \{x \mid \varphi(x) \subset V\}$) is open in X. A multifunction is *continuous* if it is lower and upper semicontinuous. A multifunction $\psi : T \to \mathcal{P}(Y)$ is *measurable*, if $\psi^-(V) \in \mathcal{T}$ for each open $V \subset Y$. Note that such a multifunction is called weakly measurable by Himmelberg [4]. We say that a function or a multifunction defined on $T \times X$ is *Carathéodory* if it is measurable in t and continuous in x.

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Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of T. By the *Marczewski function* of $\mathcal{A}$ we mean the mapping $\chi : T \to \{0,1\}^{\mathbb{N}}$, where $\{0,1\}^{\mathbb{N}}$ is the space of zero-one sequences, defined by $\chi(t) = (\chi_n(t))_{n\in\mathbb{N}}$; χ_n is the characteristic function of A_n. This function was introduced by Marczewski [11] in 1938, under the name of the characteristic function of a sequence of sets.

The space $\{0,1\}^{\mathbb{N}}$ is considered with the product topology, and the image $\chi(T) \subset \{0,1\}^{\mathbb{N}}$ with the induced topology. It is not difficult to see that for each $n \in \mathbb{N}$ the set $\chi(A_n)$ is closed-open in $\chi(T)$, and χ is measurable with respect to the σ-field $\sigma(\mathcal{A})$ generated by the family $\mathcal{A}$.

2. Measurable selections

The most known result on measurable selections is the following theorem of Kuratowski and Ryll-Nardzewski:

Theorem 2.1 ([10]; see also [4], Theorem 5.1). *Let $(T,\mathcal{T})$ be a measurable space, Y a separable metric space, and $\psi : T \to \mathcal{P}(Y)$ a measurable multifunction with complete values. Then ψ has a measurable selection.*

The first author [5] showed, that this result is a consequence of the zero-dimensional theorem of Michael:

Theorem 2.2 ([12], Theorem 2). *Let X be a zero-dimensional (in the sense of the covering dimension dim) paracompact space, Y a metric space, and $\varphi : X \to \mathcal{P}(Y)$ a multifunction. If φ is lower semicontinuous and complete-valued, then it has a continuous selection.*

Note that Michael assumed that Y is complete and φ has closed values. By the completion of Y we obtain such a version of his result.

Now we give the proof of Theorem 2.1 based on the use of the Marczewski function (cf. [5]).

Proof of Theorem 2.1. Let $(V_n)_{n\in\mathbb{N}}$ be a base of Y, and define $A_n = \psi^-(V_n)$. Since ψ is measurable, $A_n \in \mathcal{T}$. Let χ be the Marczewski function of $(A_n)_{n\in\mathbb{N}}$. Now we define the new multifunction $\Psi : \chi(T) \to \mathcal{P}(Y)$ by the formula $\psi(t) = \Psi(\chi(t))$, $t \in T$. We have to show that Ψ is well defined. If $\psi(t) \neq \psi(t')$ for some $t, t' \in T$, then there is $x \in \psi(t) \setminus \psi(t')$ (or $x \in \psi(t') \setminus \psi(t)$). It suffices to consider the first case. Since $\psi(t')$ is closed, there is $n \in \mathbb{N}$ such that $x \in V_n$ and $\psi(t') \cap V_n = \emptyset$. It means that $t \in \psi^-(V_n)$ and $t' \notin \psi^-(V_n)$. Consequently, $\chi(t) \neq \chi(t')$, which shows that Ψ is well defined. Moreover, Ψ is lower semicontinuous. Indeed, for each $n \in \mathbb{N}$

$$\Psi^-(V_n) = \chi(\psi^-(V_n)) = \chi(A_n)$$

is an open subset of $\chi(T)$. Being a separable and metrizable space with a closed-open base, $\chi(T)$ is zero-dimensional and paracompact. Hence, by Theorem 2.2, Ψ has a continuous selection $g : \chi(T) \to Y$. The function $f(t) = g(\chi(t))$ is a required measurable selection of ψ. It completes the proof.

In some applications we need measurable selections satisfying additional conditions. The following result of Schäl ([14], Theorem 2; [15], Proposition 9.4 and Theorem 12.1) is very useful in the stochastic optimization:

Theorem 2.3. *Let X be a separable metric space, $\varphi : T \to \mathcal{P}(X)$ a measurable multifunction with compact values and u a real-valued function on the graph of φ, $\mathrm{Gr}\varphi = \{(t,x) \mid x \in \varphi(t)\}$. Suppose u is the pointwise limit of a decreasing sequence (u_n) of real-valued functions on $\mathrm{Gr}\,\varphi$ such that u_n is $(\mathcal{T} \otimes \mathcal{B}(X)|_{\mathrm{Gr}\varphi})$-measurable, and $u_n(t,\cdot)$ is continuous on $\varphi(t)$ for every $t \in T$. Then there is a measurable selection $f : T \to X$ of φ such that*

$$u(t, f(t)) = \sup\{u(t,x) \mid x \in \varphi(t)\}$$

for every $t \in T$.

In [9] we give a new proof of this result, based on the use of the Marczewski function.

3. Carathéodory selections

In this section we give a general result on Carathéodory selections obtained by the application of the Marczewski function.

We say that a multifunction $\varphi : T \times X \to \mathcal{P}(Y)$ is *lower R-Carathéodory* if for each open $V \subset Y$

$$\varphi^-(V) = \bigcup\{A_n(V) \times U_n(V) \mid n \in \mathbb{N}\},$$

where $A_n(V) \in \mathcal{T}$ and $U_n(V) \subset X$ are open. It means that preimages of open sets can be represented as countable unions of measurable rectangles with open vertical sides. The notion of R-Carathéodory maps was introduced by the first author [5]. The letter "R" in this definition comes from these rectangles.

If a multifunction φ is lower R-Carathéodory, then it is $\mathcal{T} \otimes \mathcal{B}(X)$-measurable and lower semicontinuous in the second variable. In some cases we can inverse this implication.

Theorem 3.1 ([5], Main Lemma; [8], Theorem 3(i)). *Let X be a Polish space, and $\varphi : T \times X \to \mathcal{P}(Y)$ a multifunction. Suppose either $\mathcal{T}$ is complete* *with respect to a σ-finite measure or T is Polish and $\mathcal{T} = \mathcal{B}(T)$. If φ is lower semicontinuous in x and product-measurable, then it is lower R-Carathéodory.*

In order to state the result on Carathéodory selections we need another notion. Let $\mathcal{F}$ be a family of nonempty subsets of Y. We say that Y is *$\mathcal{F}$-selective* if for each metrizable X, each lower semicontinuous multifunction $\varphi : X \to \mathcal{F}$ has a continuous selection (cf. [5]). Different continuous selection theorems provide examples of pairs $(Y, \mathcal{F})$ such that Y is $\mathcal{F}$-selective.

The following theorem holds:

Theorem 3.2 ([5], Theorem 1). *Suppose X and Y are metric spaces, Y is separable, and $\mathcal{F}$ is a family of closed subsets of Y such that Y is $\mathcal{F}$-selective. Then each lower R-Carathéodory multifunction $\varphi : T \times X \to \mathcal{F}$ has a Carathéodory selection.*

Sketch of the proof. Let $(V_k)_{k\in\mathbb{N}}$ be a base of Y. Since φ is lower R-Carathéodory,

$$\varphi^-(V_k) = \bigcup\{A_n(V_k) \times U_n(V_k) \mid n \in \mathbb{N}\},$$

where $A_n(V_k) \in \mathcal{T}$ and $U_n(V_k) \subset X$ are open. Let $\mathcal{A} = \{A_n(V_k) \mid k, n \in \mathbb{N}\}$, and let χ be the Marczewski function of $\mathcal{A}$ (an enumeration of $\mathcal{A}$ is inessential). We define the new multifunction $\Phi : \chi(T) \times X \to \mathcal{P}(Y)$ by $\Phi(\chi(t), x) = \varphi(t, x)$. It can be shown that Φ is well defined and lower semicontinuous. Since Y is $\mathcal{F}$-selective, Φ has a continuous selection $F : \chi(T) \times X \to Y$. The function $f : T \times X \to Y$ given by $f(t, x) = F(\chi(t), x)$ is a Carathéodory selection of φ. It completes the proof.

Most of known results on Carathéodory selections are consequences of Theorem 3.2.

Corollary 3.3. *Let X be a Polish space, Y a separable Banach space, and $\varphi : T \times X \to \mathcal{P}(Y)$ a multifunction. Suppose either $\mathcal{T}$ is complete with respect to a σ-finite measure or T is Polish and $\mathcal{T} = \mathcal{B}(T)$. If φ is product-measurable, lower semicontinuous in x and closed convex-valued, then it has a Carathéodory selection.*

Proof. By the well known result of Michael ([12], Theorem 1) a Banach space is selective with respect to the family of all nonempty, closed and convex subsets. It follows from Theorem 3.1 that φ is lower R-Carathéodory. Now an application of Theorem 3.2 completes the proof.

Remark 1. For $\mathcal{T}$ complete this corollary is the result of Rybiński ([13], Theorem 2). For the Borel case cf. our Theorem 3 in [7], obtained by other methods.

Let $(S, \mathcal{S}, \mu)$ be a probability space, E a Banach space, and $L^1(S; E)$ the Banach space of all μ-integrable functions $f : S \to E$. A subset $Z \subset L^1(S; E)$ is called *decomposable* if for each f, $g \in Z$ and each $A \in \mathcal{S}$ the function $h : S \to E$ given by $h(s) = f(s)$ for $s \in A$ and $h(s) = g(s)$ for $s \in S \setminus A$, also belongs to Z.

The next corollary is a generalization of an unpublished result of A. Fryszkowski:

Corollary 3.4. *Let X be a separable metric space and $\varphi : T \times X \to \mathcal{P}(L^1(S; E))$ a multifunction. Suppose $(S, \mathcal{S}, \mu)$ is nonatomic and $L^1(S; E)$ is separable. If φ is lower R-Carathéodory and has closed and decomposable values, then it has a Carathéodory selection.*

Proof. By the result of Bressan and Colombo ([1], Theorem 3), for each separable metric space Y, each lower semicontinuous multifunction $\psi : Y \to \mathcal{P}(L^1(S; E))$ with closed and decomposable values has a continuous selection. Now it suffices to apply Theorem 3.2.

Remark 2. In general, the product measurability together with the lower semicontinuity with respect to the second variable do not suffice for the existence of Carathéodory selections. The first author [6] gave an example of a Borel multifunction $\varphi : T \times [0, 1] \to \mathcal{P}([0, 1])$ with compact convex values, which is lower semicontinuous in $x \in [0, 1]$ and has no Carathéodory selection. The space T in this example is a coanalytic subset of the plane.

4. Approximation of Carathéodory type multifunctions

In this section, following [8], we give two results on approximation of multifunctions measurable in one and semicontinuous in the second variable. We start with the following theorem:

Theorem 4.1 ([8], Theorem 5). *Let X be a separable metric space, Y a separable Banach space, and $\varphi : T \times X \to \mathcal{P}(Y)$ a closed convex-valued multifunction. Then the following conditions are equivalent:*

(i) *φ is lower R-Carathéodory,*

(ii) *there exists an increasing sequence (φ_n) of Carathéodory multifunctions $\varphi_n : T \times X \to \mathcal{P}(Y)$ with convex and compact values, such that*

$$\varphi(t, x) = \operatorname{cl} \bigcup \{\varphi_n(t, x) \mid n \in \mathbb{N}\}, \qquad t \in T, \ x \in X.$$

The proof of the implication (i)$\Rightarrow$(ii) is based on the use of the Marczewski function, and on the corresponding result for lower semicontinuous multifunctions.

We say that a multifunction $\varphi : T \times X \to \mathcal{P}(Y)$ is *upper R-Carathéodory* if for each open $V \subset Y$ there exist $A_n(V) \in \mathcal{T}$ and open $U_n(V) \subset X$ such that

$$\varphi^+(V) = \bigcup\{A_n(V) \times U_n(V) \mid n \in \mathbb{N}\}.$$

Note that an upper R-Carathéodory multifunction is product measurable and upper semicontinuous in x. We have the following analogue of Theorem 3.1:

Theorem 4.2 ([8], Theorem 3 (ii)). *Suppose X is a Polish space and either $\mathcal{T}$ is complete with respect to a σ-finite measure, or T is Polish and $\mathcal{T} = \mathcal{B}(T)$. If $\varphi : T \times X \to \mathcal{P}(Y)$ is upper semicontinuous in x, and for each open $V \subset Y$, $\varphi^+(V) \in \mathcal{T} \otimes \mathcal{B}(X)$, then φ is upper R-Carathéodory.*

The next theorem deals with the approximation by a decreasing sequence of multifunctions, which are measurable in the first variable and Lipschitzian in the second one.

Theorem 4.3 ([8], Theorem 6). *Let (X, ρ) be a separable metric space, Y a separable normed space, and $\varphi : T \times X \to \mathcal{P}(Y)$ a multifunction. Suppose φ is compact convex-valued and bounded, i.e. there is $M > 0$ such that $\sup\{\|y\| \mid y \in \varphi(t,x)\} \le M$ for all $t \in T$, $x \in X$. If φ is upper R-Carathéodory then there exist constants $L_n \ge 0$ and multifunctions $\varphi_n : T \times X \to \mathcal{P}(Y)$ with closed, convex and bounded values, which are measurable in t and for each $t \in T$, $x, y \in X$ satisfy the following conditions:*

(i) $\varphi(t,x) \subset \varphi_{n+1}(t,x) \subset \varphi_n(t,x)$,
(ii) $H(\varphi_n(t,x), \varphi(t,x)) \to 0$ *as* $n \to \infty$,
(iii) $H(\varphi_n(t,x), \varphi_n(t,y)) \le L_n \rho(x,y)$,

where H is the Hausdorff metric defined by the norm of Y. If Y is finite-dimensional, then the existence of such approximations (φ_n) implies, that φ is upper R-Carathéodory.

The proof of the existence of such (φ_n) is based on the use of the Marczewski function, and on the Gavioli result ([3], Theorem 2.1) on Lipschitzian approximations of upper semicontinuous multifunctions.

Remark 3. In the upper semicontinuous case the Marczewski function method works for multifunctions with compact values. Therefore in Theorem 4.3 we have to assume that φ is compact-valued. It would be interesting to obtain such parametrized Lipschitzian approximations for multifunctions with closed, convex and bounded values, as it was in the result of Gavioli.

Remark 4. Related results to our Theorem 4.3 were obtained by Moussaoui and El Arni (see [2], Theorems 4.1 and 4.2). Approximation of Carathéodory type multifunctions was also studied by Zygmunt [17] and Srivastava [16].

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CHARACTERIZATION OF APPROXIMATE SOLUTIONS TO CONVEX PROBLEMS OF CALCULUS OF VARIATIONS

Mohammed Moussaoui and Alberto Seeger

1. Introduction

The generalized Bolza problem of calculus of variations consists in minimizing the cost

$$F(x) := H(x(0), x(1)) + \int_0^1 L(t, x(t), \dot{x}(t))\, dt \tag{1.1}$$

over the space

$$X = A_p^n := A_p([0,1]; \mathbb{R}^n) \qquad 1 \leq p < +\infty$$

of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$ whose derivatives $\dot{x}$ belong to $L_p^n := L_p([0,1]; \mathbb{R}^n)$. The aim of this short note is to derive a necessary and sufficient condition for a trajectory $x \in X$ to be an approximate minimum of F.

To proceed further with our presentation, we need to fix some mathematical technicalities. It will be helpful if the reader is already familiar with the analysis of convex integral functionals, and with the duality theory for convex programs in abstract spaces (cf. Rockafellar [3]). In what follows, the space $X = A_p^n$ is paired with $Y = A_q^n$ by means of the bilinear form

$$\langle y, x \rangle := x(0) \cdot y(0) + \int_0^1 \dot{x}(t) \cdot \dot{y}(t)\, dt \qquad \forall (x, y) \in X \times Y,$$

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where q is the conjugate number of p (i.e., $p^{-1}+q^{-1}=1$), and the dot "$\cdot$" stands for the usual Euclidean product. L_p^n and L_q^n are paired in the usual way. The data functions H and L are assumed to satisfy the following measurability, convexity, and constraint qualification hypotheses :

(1.2) $H:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is proper convex lower-semicontinuous,

(1.3) the Lagrangian $L:[0,1]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is measurable, and $L(t,\cdot,\cdot)$ is proper convex lower-semicontinuousi for almost every a.e. $t\in[0,1]$,

(1.4) for all $(w,u)\in L_q^n\times L_p^n$, the negative part of the integrand $t\longmapsto L(t,w(t),u(t))$ is summable over $[0,1]$,

(1.5) here is some $x\in X$ such that H is finite at $(x(0),x(1))$ and the integral functional $(w,u)\in L_q^n\times L_p^n\longmapsto I_L(w,u):=\int_0^1 L(t,w(t),u(t))\,dt$ is continuous at $(x,\dot{x})$.

Under these assumptions, the cost function $F:X\longmapsto\mathbb{R}\cup\{+\infty\}$ is proper and convex. According to Rockafellar's work [2], a trajectory $x\in X$ is a minimum of F if and only if, there is an "adjoint" trajectory $\varphi\in X$ satisfying the *transversality condition*

$$(\varphi(0),-\varphi(1))\in\partial H(x(0),x(1)), \tag{1.6}$$

and the *Euler-Lagrange inclusion*

$$(\dot{\varphi}(t),\varphi(t))\in\partial L(t,x(t),\dot{x}(t)) \qquad \text{for a.e. } t\in[0,1]. \tag{1.7}$$

Here the symbol "∂" stands for the subdifferential mapping in the sense of convex analysis. A point that we would like to stress is that the cost function F may fail to have a minimum. This occurs when the system (1.6)-(1.7) is not solvable with respect to the pair $(x,\varphi)\in X\times X$. In this unfavourable context, it is natural to look for trajectories that minimize F only in an approximate sense. If the infimal-value

$$\operatorname*{Inf}_X F:=\operatorname{Inf}\{F(x)\mid x\in X\}$$

is finite, then it is possible to find a trajectory $x\in X$ that minimizes F within a tolerance level $\varepsilon>0$, i.e.

$$F(x)\le\varepsilon+\operatorname*{Inf}_X F.$$

Without computing explicitly the infimal-value $\operatorname{Inf}_X F$, how can we recognize such a trajectory? This question will be answered in a clear-cut manner.

2. Characterization of approximate solutions to the Bolza problem

The minimization of $F:X\to\mathbb{R}\cup\{+\infty\}$ requires to examine only the trajectories in $\operatorname{dom}F:=\{x\in X\mid F(x)<+\infty\}$. The first-order behavior of F

at a given $x \in \operatorname{dom} F$ is reflected by the set

$$\partial F(x) := \{y \in Y \mid F^*(y) + F(x) - \langle y, x\rangle = 0\},$$

known as the *subdifferential* of F at x. Here $F^* : Y \to \mathbb{R} \cup \{+\infty\}$ is the Legendre-Fenchel conjugate of F ([3]). The next lemma shows how to compute the conjugate of the cost function (1.1).

Lemma 1. *Under the assumptions* (1.2)-(1.5), *the conjugate of the cost function* (1.1) *is given by*

$$\text{(2.1)}\quad F^*(y) = \operatorname*{Min}_{\varphi \in X} \left\{ H^*(y(0) + \varphi(0), -\varphi(1)) + \int_0^1 L^*(t, \dot{\varphi}(t), \dot{y}(t) + \varphi(t))\, dt \right\} \quad \forall y \in \operatorname{dom} F^*.$$

Proof. Observe that $F = H \circ M + I_L \circ K$, where $M : A_p^n \to \mathbb{R}^n \times \mathbb{R}^n$ and $K : A_p^n \to L_q^n \times L_p^n$ are the continuous linear operator defined by

$$Mx = (x(0), x(1)) \quad \text{and} \quad Kx = (x, \dot{x}).$$

The constraint qualification condition (1.5) implies that $I_L \circ K$ is continuous at some point in $\operatorname{dom}(H \circ M)$. This fact allows us to write (cf. [1], p. 178)

$$F^*(y) = \operatorname*{Min}_{y_1 + y_2 = y} \{(H \circ M)^*(y_1) + (I_L \circ K)^*(y_2)\} \qquad \forall y \in \operatorname{dom} F^*.$$

By applying general calculus rules for computing conjugates (cf. [3], Theorem 19, [1], p. 179], and [3], Theorem 21), one obtains

$$(H \circ M)^*(y_1) = \operatorname{Min}\{H^*(c, d) \mid M^*(c, d) = y_1\},$$

and

$$(I_L \circ K)^*(y_2) = \operatorname{Min}\{I_{L^*}(s, r) \mid K^*(s, r) = y_2\}.$$

Here $M^* : \mathbb{R}^n \times \mathbb{R}^n \to A_q^n$ and $K^* : L_p^n \times L_q^n \to A_q^n$ are the adjoint operators of M and K, respectively. In other words, the equality $M^*(c, d) = y_1$ says that

$$y_1(0) = c + d, \ \dot{y}_1(t) = d \qquad \text{for a.e. } t \in [0, 1],$$

while the equality $K^*(s, r) = y_2$ takes the form

$$y_2(0) = \int_0^1 s(\tau)\, d\tau, \quad \dot{y}_2(t) = r(t) + \int_t^1 s(\tau)\, d\tau \qquad \text{for a.e. } t \in [0, 1].$$

By putting all these pieces together, one gets the expression

$$F^*(y) = \operatorname{Min}\{H^*(c, d) + I_{L^*}(s, r)\},$$

where the minimum is taken with respect to all $(c,d,s,r) \in \mathbb{R}^n \times \mathbb{R}^n \times L_p^n \times L_q^n$ such that

$$y(0) = c + d + \int_0^1 s(\tau)\,d\tau,$$
$$\dot{y}(t) = d + r(t) + \int_t^1 s(\tau)\,d\tau \qquad \text{for a.e. } t \in [0,1].$$

By introducing the "adjoint" trajectory

$$t \in [0,1] \longmapsto \varphi(t) = -\left[d + \int_t^1 s(\tau)\,d\tau\right],$$

one transforms the above minimization problem into the one appearing in (2.1). This completes the proof of the lemma.

With the help of Lemma 1 it is fairly simple to obtain a characterization of the set $\partial F(x)$. In fact, it is possible to characterize the more general set

$$\partial_\varepsilon F(x) = \{y \in Y : F^*(y) + F(x) - \langle y, x\rangle \le \varepsilon\},$$

known as the *ε-subdifferential* of F at x. The (exact) subdifferential $\partial F(x)$ corresponds to the limiting case $\varepsilon = 0$. In the next proposition one uses the notation

$$\Sigma(\alpha) := \left\{\sigma \in L_1[0,1] \mid \int_0^1 \sigma(t)\,dt = \alpha,\ \sigma(t) \ge 0 \quad \text{for a.e. } t \in [0,1]\right\}.$$

Proposition 2. *Let $x \in \operatorname{dom} F$ and $\varepsilon \ge 0$. Under the assumptions* (1.2)-(1.5), *the element $y \in Y$ belongs to $\partial_\varepsilon F(x)$ if and only if, there exist $\varphi \in X$, $\alpha \in [0,\varepsilon]$, and $\sigma \in \Sigma(\alpha)$, such that*

$$\begin{aligned} &(y(0) + \varphi(0), -\varphi(1)) \in \partial_{\varepsilon-\alpha} H(x(0), x(1)), \\ &(\dot{\varphi}(t), \dot{y}(t) + \varphi(t)) \in \partial_{\sigma(t)} L(t, x(t), \dot{x}(t)) \qquad \textit{for a.e. } t \in [0,1]. \end{aligned} \tag{2.2}$$

Proof. According to Lemma 1, the condition $F^*(y) + F(x) - \langle y, x\rangle \le \varepsilon$ is equivalent to the existence of some $\varphi \in X$ such that

$$\begin{aligned} H^*(y(0) + \varphi(0), -\varphi(1)) + \int_0^1 L^*(t, \dot{\varphi}(t), \dot{y}(t) + \varphi(t))\,dt + H(x(0), x(1)) \\ + \int_0^1 L(t, x(t), \dot{x}(t))\,dt - x(0)\cdot y(0) - \int_0^1 \dot{x}(t)\cdot\dot{y}(t)\,dt \le \varepsilon. \end{aligned}$$

But, the last inequality can be written in the form

$$[H^*(y(0)+\varphi(0),-\varphi(1))+H(x(0),x(1))-x(0)\cdot(y(0)+\varphi(0))$$
$$+x(1)\cdot\varphi(1)]+\int_0^1 [L^*(t,\dot\varphi(t),\dot y(t)+\varphi(t))+L(t,x(t),\dot x(t))$$
$$-x(t)\cdot\dot\varphi(t)-\dot x(t)\cdot(\dot y(t)+\varphi(t))]\,dt\le\varepsilon.$$

Due to the Young-Fenchel inequality [1], p. 172, the expressions between square brackets are nonnegative. This means that the above equality holds if and only if

$$H^*(y(0)+\varphi(0),-\varphi(1))+H(x(0),x(1))-x(0)\cdot(y(0)+\varphi(0))$$
$$-x(1)\cdot\varphi(1)\le\varepsilon-\alpha,$$

and

$$L^*(t,\dot\varphi(t),\dot y(t)+\varphi(t))+L(t,x(t),\dot x(t))-x(t)\cdot\dot\varphi(t)$$
$$-\dot x(t)\cdot(\dot y(t)+\varphi(t))\le\sigma(t)\qquad\text{for a.e. } t\in[0,1],$$

for some $\alpha\in[0,1]$ and $\sigma\in\Sigma(\alpha)$. This completes the proof.

By setting $\varepsilon=0$ in the statement of Proposition 2 one obtains straightforwardly a characterization of the subdifferential $\partial F(x)$.

Corollary 3. *Let $x\in\operatorname{dom}F$. Suppose the hypotheses (1.2)-(1.5) hold. Then, $y\in\partial F(x)$ if and only if, there is an adjoint trajectory $\varphi\in X$ such that*

$$(y(0)+\varphi(0),-\varphi(1))\in\partial H(x(0),x(1)),$$
$$(\dot\varphi(t),\dot y(t)+\varphi(t))\in\partial L(t,x(t),\dot x(t))\qquad\textit{for a.e. } t\in[0,1].$$

In the context of the present note, the main merit of Proposition 2 is yielding a characterization of the set

$$\varepsilon\text{-argmin}\,F:=\{x\in X: F(x)\le\varepsilon+\operatorname*{Inf}_X F\}$$

of ε-minima of F.

Theorem 4. *Let $x\in\operatorname{dom}F$ and $\varepsilon\ge0$. Under the assumptions (1.2)-(1.5), the trajectory $x\in X$ is an ε-minimum of the cost function F if and only if, there exist $\varphi\in X$, $\alpha\in[0,\varepsilon]$, and $\sigma\in\Sigma(\alpha)$, such that*

(i) (*approximate transversality condition*)

$$(\varphi(0),-\varphi(1))\in\partial_{\varepsilon-\alpha}H(x(0),x(1))$$

(ii) (*approximate Euler-Lagrange inclusion*)

$$(\dot\varphi(t),\varphi(t))\in\partial_{\sigma(t)}L(t,x(t),\dot x(t))\qquad\textit{for a.e. } t\in[0,1].$$

Proof. Since $\operatorname{Inf}_X F = -F^*(0)$, the ε-minima of F are obtained by writing the condition $0 \in \partial_\varepsilon F(x)$.

As way of illustration of Theorem 4, consider the problem

$$(\mathrm{P}) \qquad \begin{cases} \text{Minimize } \int_0^1 t^2[\sin^2(2\pi t) + (\dot{x}(t))^2]^{1/2}\,dt \\ \text{subject to } x \in A_2^1,\ x(0) = 0,\ x(1) = 1. \end{cases}$$

In this case

$$H(a,b) = \begin{cases} 0 & \text{if } (a,b) = (0,1) \\ +\infty & \text{if } (a,b) \neq (0,1), \end{cases}$$
$$L(t,\alpha,\beta) = t^2[\sin^2(2\pi t) + \beta^2]^{1/2}.$$

As a matter of calculus, one gets :

$$\forall \eta \geq 0,\ \partial_\eta H(a,b) = \begin{cases} \mathbb{R}^2 & \text{if } (a,b) = (0,1) \\ \emptyset & \text{if } (a,b) \neq (0,1), \end{cases}$$

$$(\gamma,\delta) \in \partial_{\sigma(t)} L(t,\alpha,\beta) \iff \begin{cases} \gamma = 0, \text{ and there is some } \theta_t \in \mathbb{R} \text{ such that} \\ [\theta_t^2 + \delta^2]^{1/2} \leq t^2, \\ t^2[\sin^2(2\pi t) + \beta^2]^{1/2} \leq \theta_t \sin(2\pi t) + \beta\delta + \sigma(t). \end{cases}$$

By applying Theorem 4, one arrives (after a short simplification) at the following conclusion: $x \in A_2^1$ solves (P) within a tolerance level $\varepsilon > 0$ if and only if, $x(0) = 0$, $x(1) = 1$, and there exist $\sigma \in \Sigma(\varepsilon)$, and a function $\theta : [0,1] \to \mathbb{R}$ such that

$$|\theta(t)| \leq t^2,\ t^2[\sin^2(2\pi t) + (\dot{x}(t))^2]^{1/2} \leq \theta(t)\sin(2\pi t) + \sigma(t) \qquad \text{for a.e. } t \in [0,1].$$

The last condition, yields the inequality

$$t^2[\sin^2(2\pi t) + (\dot{x}(t))^2]^{1/2} \leq t^2|\sin(2\pi t)| + \sigma(t) \qquad \text{for a.e. } t \in [0,1],$$

from where one deduces that the infimal-value of the problem (P) must be

$$\int_0^1 t^2|\sin(2\pi t)|\,dt = \frac{3}{4\pi} - \frac{1}{\pi^3} = 0,20648\ldots.$$

This infimal-value is not attained.

3. Conclusions.

The underlying ideas behind the proofs of Lemma 1 and Proposition 2 are somehow hidden in a previous work by Seeger [4]. The paper [4] is concerned with the sensitivity analysis of the infimal-value function associated to a perturbed convex problem of calculus of variations. Modified forms of the Euler-Lagrange inclusion and the transversality condition are instrumental in the writing of the sensitivity results established in [4]. The issue discussed in this note is, however, different. We provide here a simple and independent discussion on the characterization of approximate solutions.

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CONTINUOUS SELECTIONS OF NON-LOWER SEMICONTINUOUS NONCONVEX-VALUED MAPPINGS

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1. Introduction

While lower semicontinuity of mappings with closed convex values is *sufficient* for the existence of continuous singlevalued selections, it is of course, not *necessary*. For example, one can start by an arbitrary continuous singlevalued mapping $f : X \to Y$ and then define $F(x)$ to be a subset of Y such that $f(x) \in F(x)$. Then F admits the selection f, but there are no continuity type restrictions for F. A very natural problem immediately arises. Namely, to find a weaker version of lower semicontinuity which preserves the existence of singlevalued selections. If we can find a lower semicontinuous selection G of a given convex-valued mapping F, then Michael's techniques can be used to find a continuous selection f of a lower semicontinuous mapping $Cl(\mathrm{conv}(G))$ (see [7] or [14]). Moreover, any selection of $Cl(\mathrm{conv}(G))$ will automatically be a selection of F. The situation is more complicated for the case of nonconvex-valued mappings F.

The notion of the function of nonconvexity of a closed subset of a Banach space was first introduced in [11]. In this paper we consider mappings F whose

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values $F(x)$ have some common non-decreasing majorant $\alpha : (0, \infty) \to [0,1)$ for their functions of nonconvexity. In this situation, we have in general, no information about "nonconvexity" of the values $G(x)$ for a lower semicontinuous selection G of F. So we replace the property "F admits a lower semicontinuous selection" by the property "F admits a sufficiently large family of lower semicontinuous selections". The formalisation of the last property leads us to introduce some new classes of non-lower semicontinuous mappings.

We denote by $D(y, r)$ the open ball of radius r, centered at an arbitrary point y of a metric space Y. For any subset $A \subset Y$, we put $D(A, r) = \bigcup\{D(y, r) \mid y \in A\}$ and $D(A, \infty) = Y$. For two multivalued mappings F_1 and F_2 from X into Y we denote by $F_1 \cap F_2$ the mapping $x \mapsto F_1(x) \cap F_2(x)$. For a multivalued mapping $F : X \to Y$ into a metric space Y and for a real-valued mapping $d : X \to (0, \infty)$ we denote by $D(F, d)$ the multivalued mapping $x \mapsto D(F(x), d(x))$. For a closed-valued mapping $F : X \to Y$ into a metric space Y and for a real-valued mapping $\varepsilon : X \to (0, \infty)$ we say that a continuous singlevalued mapping $f : X \to Y$ is an *ε-selection of F*, whenever $\varepsilon(\cdot)$ is a strong majorant of $\text{dist}(f(\cdot), F(\cdot))$, i.e. $\varepsilon(x) > \text{dist}(f(x), F(x))$, for every $x \in X$. We use the term *function* for singlevalued mappings with values from $\mathbb{R}$.

Definition 1.1. Let $\lambda : (0, \infty) \to [1, \infty)$ be any function. Then a closed-valued mapping $F : X \to Y$ to a metric space Y is said to be an *LS_λ-mapping* if for every continuous function $\varepsilon : X \to (0, \infty)$ and every continuous ε-selection f of F, the multivalued mapping $\text{clos}(F \cap D(f, \lambda(\varepsilon) \cdot \varepsilon))$ admits a lower semicontinuous selection.

Each lower semicontinuous mapping F is an LS_1-mapping because the intersection $F \cap D(f, \varepsilon)$ is lower semicontinuous. Clearly, each LS_λ-mapping is an LS_μ-mapping, whenever $\lambda \le \mu$. Next, LS_∞-mappings are exactly those which admit lower semicontinuous selections. We chose the notation LS_λ-mapping as an abbreviation for "mappings, having lower semicontinuous selections with respect to the λ-enlargement of open balls".

Theorem 1.2. *Let $\alpha : (0, \infty) \to [0, 1)$ and $\lambda : (0, \infty) \to [1, \infty)$ be any functions such that λ is locally bounded at the origin and $t \mapsto \alpha(\lambda(t) \cdot t) \cdot \lambda(t)$ has a nondecreasing strong majorant $M : (0, \infty) \to [0, 1)$. Then every LS_λ-mapping from a paracompact space X into a Banach space Y admits a singlevalued continuous selection, whenever $\alpha(\cdot)$ is a majorant of the set of functions of nonconvexity of values $F(x)$, for every $x \in X$.*

For *constants* α and λ, the hypotheses of Theorem 1.2 are guaranteed by the inequality $\alpha \cdot \lambda < 1$. For the constant λ it suffices to assume that the set of all functions of nonconvexity of values $F(x)$, $x \in X$, has a nondecreasing majorant $\alpha : (0, \infty) \to [0, 1)$. Various weakenings of lower semicontinuity of convex-valued mappings for which a continuous singlevalued selections exist (as in the classical situation) have been intensively studied in the series of papers [1], [2], [5], [6], [9] (see also [10] and [14], § II.3). Most of them are related to the behaviour of a different kind of derived mappings (F', F_0, F_ε) of a given mapping. For the class of so-called *quasi-lower semicontinuous* mappings (see Definition 2.2 below), the derived mapping F' in the sense of Brown [3] is the largest possible lower semicontionuous selection of F. For *convex-valued* quasi l.s.c. mappings and for a *constant* λ, a property somewhat similar to our Definition 1.1 was obtained in [9]. We state the following fact related to Definition 1.1:

Theorem 1.3. *Every quasi-lower semicontinuous mapping of a paracompact space into a complete metric space is an LS_λ-mapping, for each continuous real-valued function $\lambda : (0, \infty) \to (1, \infty)$.*

We derive the following theorem from Theorems 1.2 and 1.3.

Theorem 1.4. *Let $\alpha : (0, \infty) \to [0, 1)$ be a nondecrasing function. Then every quasi-lower semicontinuous mapping F from a paracompact space X into a Banach space admits a singlevalued continuous selection, whenever $\alpha(\cdot)$ is a majorant of the set of functions of nonconvexity of values $F(x)$, for every $x \in X$.*

We list some special cases of Theorem 1.4. For $\alpha(\cdot) \equiv 0$ and any l.s.c. mapping F it yields the Michael convex-valued selection theorem [7]. For $\alpha(\cdot) \equiv q < 1$ and any l.s.c. mapping F we get the Michael paraconvex-valued selection theorem [8]. For $\alpha(\cdot) \equiv 0$ and weakly Hausdorff l.s.c. F (respectively, weakly l.s.c. or quasi l.s.c. F) it gives the DeBlasi-Myjak's (respectively, Przeslawski-Rybinski's or Gutev's) selection theorem [2], [5], [6], [9]. For any nondecreasing function α and for any l.s.c. F it yields a theorem proved earlier by these authors [11], [17]. As an application to the theory of fixed-points of multivalued contractions we can also obtain the following generalization of Ricceri's result [15], in the spirit of the Rybinski paper [16].

Theorem 1.5. *Let X be a paracompact space, Y a Banach space and $X \times Y$ a paracompact space. Suppose that for a multivalued mapping $F : X \times Y \to Y$ and some constants α and γ from $[0, 1)$ the following properties hold:*

(a) *Functions of nonconvexity of all values $F(x, y)$ are less than or equal to α,*
(b) *Each mapping $F(x, \cdot)$ is a γ-contraction,*
(c) *Each mapping $F(\cdot, y)$ is quasi-lower semicontinuous, and*
(d) $\alpha + \gamma < 1$.

Then there exists a singlevalued continuous mapping $f : X \times Y \to Y$ such that for every $x \in X$, the restriction $f(x, \cdot)$ is a retraction onto the fixed-point set of the contraction $F(x, \cdot)$.

2. Preliminaries

We begin by a construction of a function of nonconvexity. For any nonempty closed subset $P \subset Y$ of a Banach space Y and for any open r-ball $D_r \subset Y$, we define the *relative precision* of an approximation of P by elements of D_r as follows:

$$\delta(P, D_r) = \sup \left\{ \frac{\operatorname{dist}(q, P)}{r} \mid q \in \operatorname{conv}(P \cap D_r) \right\}.$$

Clearly, for a convex set P with nonempty intersection $P \bigcap D_r$, the equality $\delta(P, D_r) = 0$ means that this intersection is a convex subset of P.

Definition 2.1. For a nonempty closed subset $P \subset Y$ of a Banach space Y, *the function $\alpha_P(\cdot)$ of nonconvexity of P* associates to each positive number r the following nonnegative number:

$$\alpha_P(r) = \sup\{\delta(P, D_r) \mid D_r \text{ runs over all open } r\text{-balls}\}.$$

Clearly, the identical equality $\alpha_P(\cdot) \equiv 0$ is equivalent to *convexity* of the closed set P. Following Michael [8], the set P is said to be *q-paraconvex*, whenever the number q is a majorant of the function $\alpha_P(\cdot)$. A selection theorem for q-paraconvex valued l.s.c. maps, $q < 1$, was proved in [8]. For a possible substitute of a suitable *function* $q(\cdot)$ instead of the *constant* see [11]. For examples of classes of closed sets with nice functions of nonconvexity see [12], [13], [19].

The notion of quasi lower semicontinuity (respectively, weak lower semicontinuity) of a multivalued mapping was introduced in [5], [6] (respectively, in [9]). Recall, that for a multivalued mapping $F : X \to Y$, the preimage $F^{-1}(A)$, $A \subset Y$, is defined as $\{x \in X \mid F(x) \cap A \neq \emptyset\}$ and for topological spaces X and Y, a mapping F is said to be *lower semicontinuous* if preimages of open sets are open sets.

Definition 2.2. A multivalued mapping $F : X \to Y$ of a topological space X into a metric space (Y, ρ) is said to be *quasi lower semicontinuous* if for every triple $(x, U(x), \varepsilon)$, where $x \in X$, $U(x)$ is a neighborhood of x and $\varepsilon > 0$, there exists a point $q(x) \in U(x)$ such that for every $y \in F(q(x))$, the point x belongs to the interior of the set $F^{-1}(D(y, \varepsilon))$.

Clearly, each l.s.c. map is quasi l.s.c.: it suffices to put $q(x) = x$. For examples of quasi l.s.c., non l.s.c. mappings see [6], [10]. Possibly, one of the simplest examples is given by the mapping $F : X \to [0, \infty)$, $F(x) = [0, l(x)]$, where

$l : X \to [0, \infty)$ is an arbitrary singlevalued locally positive function. We need two Gutev's theorems [6]. Recall that for a multivalued mapping $F : X \to Y$ between topological spaces its *derived* mapping $F' : X \to Y$ is defined by setting $F'(x)$ to be equal to the set of all $y \in F(x)$, for which x belongs to the interior of the preimage (with respect to F) of every neighborhood of y (see [3]).

Theorem 2.3. *Let $F : X \to Y$ be a closed valued quasi lower semicontinuous mapping of a topological space X into a complete metric space (Y, ρ). Then the derived mapping $F' : X \to Y$ is a lower semicontinuous selection of F with nonempty closed values. Moreover, if $G : X \to Y$ is a lower semicontinuous selection of F, then G is also a selection of F'.*

Theorem 2.4. *A mapping $F : X \to Y$ of a topological space X into a complete metric space (Y, ρ) is quasi lower semicontinuous if and only if for every triple $(x, U(x), \varepsilon)$, where $x \in X$, $U(x)$ is a neighborhood of x and $\varepsilon > 0$, there exists a point $q(x) \in U(x)$ such that $F(q(x)) \subset D(F'(x), \varepsilon)$.*

Finally, for each function $M : (0, \infty) \to [0, 1)$ we define the following sequence of functions:

$$M_0(t) \equiv 1,\ M_1(t) = M(t),\ \ldots,\ M_{n+1}(t) = M(M_n(t) \cdot t) \cdot M_n(t), \ldots$$

Lemma 2.5. *Let $M : (0, \infty) \to [0, 1)$ be a nondecreasing function. Then for every positive τ, the series $\sum_{n=0}^{\infty} M_n(t)$ uniformly converges on the interval $(0, \tau)$.*

3. Proof of Theorem 1.2

Under assumptions of the theorem, let $F : X \to Y$ be a given LS_λ-mapping. Then F is an LS_∞-mapping and, hence, has a lower semicontinuous closed-valued selection, say G. Let $f_0 : X \to Y$ be an arbitrary singlevalued continuous mapping. Then the distance $d(x) = \operatorname{dist}(f_0(x), G(x))$ is an upper semicontinuous real-valued function on the paracompact space X. By the Dowker theorem, the function $d(\cdot)$ has a strong continuous singlevalued majorant, say $\varepsilon : X \to (0, \infty)$. Clearly, f_0 is an ε-selection of F. Now, for every natural number n we put:

$$R_n(x) = M_n(\varepsilon(x)) \cdot \varepsilon(x), \qquad r_n(x) = \lambda(R_n(x)) \cdot R_n(x),$$

where $M : (0, \infty) \to [0, 1)$ is a fixed nondecreasing majorant of the function

$$t \mapsto \alpha(\lambda(t) \cdot t) \cdot \lambda(t)$$

and functions $M_n(\cdot)$ are defined above, before Lemma 2.5. Due to the continuity of the mapping $\varepsilon : X \to (0, \infty)$ and due to Lemma 2.5, for every $x \in X$, there exists its neighborhood $U(x)$ such that the series $\sum_{n=0}^{\infty} R_n(\cdot)$ uniformly converges on $U(x)$. Similarly, the series $\sum_{n=0}^{\infty} r_n(\cdot)$ uniformly converges on $U(x)$, because of local boundedness of the function $\lambda(\cdot)$.

Let us construct a sequence of singlevalued continuous mappings $f_n : X \to Y$ with the properties that for each natural n and for each $x \in X$:

(a_n) $\quad d_n(x) = \operatorname{dist}(f_n(x), F(x)) < R_n(x)$; and

(b_n) $\quad \operatorname{dist}(f_{n+1}(x), f_n(x)) \leq r_n(x)$.

We then see from (b_n) that there exists a pointwise limit $f = \lim_{n\to\infty} f_n$ and that f is a locally (and, hence globally) continuous mapping, due to the local uniform convergence of the series $\sum_{n=0}^{\infty} R_n(\cdot)$ and $\sum_{n=0}^{\infty} r_n(\cdot)$. The closedness of $F(x)$ and inequalities (a_n) imply that f is a selection of F.

So, the mapping f_0 was constructed so that the inequality (a_0) holds. Suppose that for some $n > 0$, we have mappings $f_0, f_1, \ldots, f_n$ for which the inequalities $(a_0), (a_1), \ldots, (a_n)$ and $(b_0), (b_1), \ldots, (b_{n-1})$ hold. By (a_n), the mapping f_n is an R_n-selection of F. Moreover, each nondecreasing mapping $M : (0,\infty) \to [0,1)$ has a continuous majorant $M_1 : (0,\infty) \to [0,1)$, i.e. without loss of generality one can assume that $M(\cdot)$ from the hypotheses of the theorem is a continuous function. Hence, the functions $R_n(x) = M_n(\varepsilon(x)) \cdot \varepsilon(x)$ are also continuous and it is possible to use Definition 1.1 of LS_λ-mappings, which directly shows that the mapping

$$x \to Cl(F(x) \cap D(f_n(x), \lambda(R_n(x)) \cdot R_n(x))) = Cl(F(x) \cap D(f_n(x), r_n(x)))$$

admits a lower semicontinuous selection, say G_n. By the classical Michael selection theorem [7], the mapping $Cl(\operatorname{conv}(G_n))$ admits a singlevalued continuous selection, say f_{n+1}. Then

$$f_{n+1}(x) \in Cl(\operatorname{conv}(G_n(x))) \subset Cl(D(f_n(x), r_n(x))),$$

i.e. the inequality (b_n) holds. Now, using Definition 2.1 of the function of nonconvexity for open balls $D(f_n(x), r_n(x))$ and remembering that $\alpha(\lambda(t)\cdot t)\cdot\lambda(t) < M(t)$ for all positive t, we see that:

$$\begin{aligned}
\operatorname{dist}(f_{n+1}(x), F(x)) &\leq \alpha_{F(x)}(r_n(x)) \cdot r_n(x) \\
&\leq \alpha(\lambda(R_n(x)) \cdot R_n(x)) \cdot \lambda(R_n(x)) \cdot R_n(x) \\
&< M(R_n(x)) \cdot R_n(x) \\
&= M(M_n(\varepsilon(x)) \cdot \varepsilon(x)) \cdot M_n(\varepsilon(x)) \cdot \varepsilon(x)) \\
&= M_{n+1}(\varepsilon(x)) \cdot \varepsilon(x) = R_{n+1}(x),
\end{aligned}$$

i.e. the inequality (a_{n+1}) holds. Theorem 1.2 is thus proved.

Remark. Clearly,

$$\operatorname{dist}(f_0(x), f(x)) \le \sum_{n=0}^{\infty} r_n(x).$$

4. Proofs of Theorems 1.3-1.5

The initial step of the proofs represents the following lemma, which resulted from our discussions with Gutev.

Lemma 4.1. *Let $F : X \to Y$ be a quasi lower semicontinuous mapping of a topological space X into a complete metric space (Y, ρ), $f : X \to Y$ a singlevalued continuous mapping and $c(\cdot)$ a strong majorant for the distance function $d = \operatorname{dist}(f, F)$. Suppose that the interval-valued mapping $x \mapsto (d(x), c(x))$, $x \in X$, admits a singlevalued continuous selection. Then for every $x \in X$, the intersection $F'(x) \cap D(f(x), c(x))$ is nonempty.*

Proof. Let $s : X \to (0, \infty)$ be a continuous mapping such that $d(x) < s(x) < c(x)$, for every $x \in X$. Pick a point $x \in X$ and put $\varepsilon = (c(x) - s(x))/2$. Let $V = V(x)$ be a neighborhood of x such that the restriction of $s(\cdot)$ onto V is less than $(c(x) + s(x))/2$. Due to the continuity of f, find a neighborhood $U = U(x)$ such that $f(x) \in D(f(z), \varepsilon)$, for every $z \in U$. We can apply Theorem 2.4 to the triple $(x, V \cap U, \varepsilon)$, i.e. we can find a point $q(x) \in V \cap U$ such that

$$F(q(x)) \subset D(F'(x), \varepsilon).$$

By invoking the inequality $d < s$, we see that

$$f(q(x)) \in D(F(q(x)), s(q(x))) \subset D(F'(x), s(q(x)) + \varepsilon).$$

Hence the inequality $s(q(x)) < (c(x) + s(x))/2$, implies that

$$f(x) \in D(f(q(x)), \varepsilon) \subset D(F'(x), s(q(x)) + 2\varepsilon) \subset D(F'(x), c(x)),$$

i.e. the distance between $f(x)$ and $F'(x)$ is less than $c(x)$.

Proof of Theorem 1.3. Let $F : X \to Y$ be a quasi lower semicontinuous mapping of a topological space X into a complete metric space (Y, ρ), $f : X \to Y$ a singlevalued continuous ε-selection of F, for some continuous function $\varepsilon : X \to (0, \infty)$, and $\lambda : (0, \infty) \to (1, \infty)$ a singlevalued continuous function. Then for the (continuous!) strong majorant $c(x) = \lambda(\varepsilon(x)) \cdot \varepsilon(x)$ of the distance function $d(x) = \operatorname{dist}(f(x), F(x))$, there exists an obvious continuous function $s(\cdot)$ which separates $d(\cdot)$ and $c(\cdot)$. Namely, $s(x) = \varepsilon(x)$. Lemma 4.1 shows that the mapping $G = F' \cap D(f, c)$ has nonempty values. But the derived mapping F' is a selection of F. Hence, G is a selection of the mapping $F \cap$

$D(f,c)$. Lower semicontinuity of G follows from the lower semicontinuity of F' (see Theorem 2.3), from continuity of f, and from continuity of $c(\cdot)$. Thus we conclude that the mapping $x \mapsto F(x) \cap D(f(x), \lambda(\varepsilon(x)) \cdot \varepsilon(x))$ admits a lower semicontinuous selection. Theorem 1.3 is thus proved.

Proof of Theorem 1.4. Because of Theorems 1.2 and 1.3 it suffices to check the following simple fact:

Lemma 4.2. *For every nondecreasing function* $\alpha : (0, \infty) \to [0,1)$, *there exists a continuous function* $\lambda : (0,\infty) \to (1,\infty)$ *such that the function* $\alpha(\lambda(t) \cdot t) \cdot \lambda(t)$ *has a nondecreasing strong majorant* $M : (0,\infty) \to [0,1)$.

Proof. It is easy to find a *continuous* nondecreasing majorant $\beta : (0,\infty) \to [0,1)$ of the function $\alpha(\cdot)$ such that $\lim_{t\to\infty} \beta(t) = 1$. Let $\beta(\cdot) < \gamma(\cdot) < M(\cdot) < 1$ and the functions $\gamma(\cdot)$ and $M(\cdot)$ be both continuous and nondecreasing. We claim that $\lambda(\cdot)$ can then be defined as follows:

$$\lambda(t) = \frac{1}{2} \cdot \left(1 + \min\left\{\frac{M(t)}{\gamma(t)}, \frac{\beta^{-1}(\gamma(t))}{t}\right\}\right).$$

Clearly, $\lambda(\cdot)$ is continuous and greater than 1. Moreover,

$$\lambda(t) \cdot t < \beta^{-1}(\gamma(t)),$$
$$\alpha(\lambda(t) \cdot t) \leq \beta(\lambda(t) \cdot t) < \gamma(t)$$

and

$$\alpha(\lambda(t) \cdot t) \cdot \lambda(t) < \gamma(t) \cdot \lambda(t) < M(t)$$

due to the choice of $\lambda(t)$. Lemma 4.2 (and hence Theorem 1.4) are thus proved.

Sketch of the proof of Theorem 1.5. First, we refer to [16] for the proof that the hypotheses (b) and (c) together imply the quasi lower semicontinuity of the mapping F in *two* variables and, moreover, of the composition $F(x, h(x,y))$, for each continuous $h : X \times Y \to Y$. Second, (d) implies that $\gamma/(1-\alpha) < 1$ and hence for some numbers $M \in (\alpha, 1)$ and $\lambda > 1$, we have that $\gamma/(1-M) < 1$ and $\gamma \cdot \lambda/(1-M) < 1$.

Now the special case of the selection Theorem 1.2, when α, λ and M are constants, works for the α-paraconvex valued mapping $F_0 = F$ and we can find a selection of F_0, say f_1. Moreover, starting by $f_0(x,y) = y$, we have (see Remark after proof of Theorem 1.2),

$$\operatorname{dist}(f_0(x,y), f_1(x,y)) \leq \sum_{n=0}^{\infty} r_n(x,y) = \lambda \cdot \sum_{n=0}^{\infty} M^n \cdot \varepsilon(x,y) = \frac{\lambda}{1-M} \cdot \varepsilon(x,y),$$

for some continuous singlevalued $\varepsilon : X \times Y \to (0,\infty)$.

Put $F_1(x,y) = F_0(x, f_1(x,y))$ and let us estimate the distance between f_1 and F_1:

$$\begin{aligned}\operatorname{dist}(f_1(x,y), F_1(x,y)) &\leq H_{\text{dist}}(F_0(x,y), F_0(x, f_1(x,y))) \\ &\leq \gamma \cdot \operatorname{dist}(f_0(x,y), f_1(x,y)) \\ &< \gamma \cdot \frac{\mu}{1-M} \cdot \varepsilon(x,y); \qquad \lambda < \mu.\end{aligned}$$

Hence f_1 is an ε_1-selection of F_1 with

$$\varepsilon_1(x,y) = \gamma \cdot \frac{\mu}{1-M} \cdot \varepsilon(x,y).$$

Reapplying Theorem 1.2, we find a selection of F_1, say f_2, with

$$\operatorname{dist}(f_1(x,y), f_2(x,y)) \leq \gamma \cdot \frac{\lambda}{1-M} \cdot \varepsilon_1(x,y) < \gamma^2 \cdot \frac{\mu^2}{(1-M)^2} \cdot \varepsilon(x,y).$$

Continuation of this procedure yields the estimate

$$\operatorname{dist}(f_n(x,y), f_{n+1}(x,y)) < q^{n+1} \cdot \varepsilon(x,y), \qquad q = \frac{\gamma \cdot \mu}{1-M}.$$

Having $\gamma \cdot \lambda/(1-M) < 1$, it is clear that we can assume that $\mu > \lambda$ and $q < 1$.

Remark. For the *functions* α and γ of nonconvexity and contractivity one can replace the hypotheses (d), i.e. the numerical inequality $\alpha + \gamma < 1$ by some *functional* expression.

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ON MAXIMALITY PROPERTY

STEFAN ROLEWICZ

In the classical convex analysis an essential role is played by the following characterization of a subdifferential as a multifunction.

Theorem A (Minty [1]). *A subdifferential of a convex function $f(x)$ is a maximal cyclic monotone multifunction.*

Theorem B (Rockafellar [4]). *Every maximal cyclic monotone multifunction is a subdifferential of a certain convex function. Moreover this function is uniquely determined up to the constant.*

Since applications in optimization in the last years the theory of so called Φ-convexity was strongly developed (see for example [2], [5]).

It is not difficult to show repeating the classical considerations that

Theorem A-Φ ([2], Prop. 1.1.9). *A Φ-subdifferential of a Φ-convex function $f(x)$ is a cyclic monotone multifunction.*

Theorem B-Φ ([2], Prop. 1.1.11). *Every maximal cyclic monotone multifunction is a Φ-subdifferential of a certain Φ-convex function.*

Unfortunately Theorem A-Φ is weaker than Theorem A since in general Φ-subdifferential of a Φ-convex function $f(x)$ is a cyclic monotone multifunction but not necessary maximal ([2], Example 1.1.10). Similarly Theorem B-Φ is weaker than Theorem B since we are not able to prove the second part.

The aim of this note is to give sufficient conditions on the class Φ such that every Φ-subdifferential of a Φ-convex function $f(x)$ is a maximal cyclic monotone multifunction. If it holds we say that the class Φ has *maximality property*. The

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classical result of Minty [1] says that the class $\Phi = X^*$ of all continuous linear functionals over a Banach space X has maximality property.

Now we shall recall fundamental notions of Φ-convex analysis.

Let an arbitrary set X, called later the space, be given. Let Φ ba class of real valued functions. Let $f(x)$ be a function defined on X with values in the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

A function $\phi \in \Phi$ will be called a *Φ-subgradient* of the function $f(x)$ at a point x_0 if

$$f(x) - f(x_0) \geq \phi(x) - \phi(x_0) \tag{1}$$

for all $x \in X$. The set of all Φ-subgradients of the function f at a point x_0 we shall call *Φ-subdifferential* of the function f at a point x_0 and we shall denote it by $\partial_\Phi f|_{x_0}$.

Observe that the order of real numbers induces the order on real-valued functions. Namely, we shall write $g \leq f$ without the argument if $g(x) \leq f(x)$ for all $x \in X$. For a given function $f(x)$ we shall denote by

$$f^\Phi(x) = \sup\{\phi(x) + c \mid \phi \in \Phi,\ c \in \mathbb{R},\ \phi + c \leq f\}. \tag{2}$$

The function $f^\Phi(x)$ is called *Φ-convexification* of the function f. If $f^\Phi(x) = f(x)$, i.e.

$$f(x) = \sup\{\phi(x) + c \mid \phi \in \Phi,\ c \in \mathbb{R},\ \phi + c \leq f\}, \tag{3}$$

we say that the function f is *Φ-convex.*

Let X, Y be two spaces. Let $\Gamma : X \to 2^Y$ be a multifunction, i.e. the mapping of the set X into subsets of Y. By the *domain* of Γ, $\mathrm{dom}(\Gamma)$, we shall call the set of those x, that $\Gamma(x) \neq \emptyset$,

$$\mathrm{dom}(\Gamma) = \{x \in X \mid \Gamma(x) \neq \emptyset\}.$$

By the *graph* of Γ, $G(\Gamma)$, we shall call the set of those $(x, y) \in X \times Y$, that $y \in \Gamma(x)$.

We say that a multifunction Γ mapping X into Φ is *monotone* if for $\phi_x \in \Gamma(x)$, $\phi_y \in \Gamma(y)$ we have

$$\phi_x(x) + \phi_y(y) - \phi_x(y) - \phi_y(x) \geq 0. \tag{4}$$

In particular, when X is a linear space, and Φ is a linear space consisting of linear functionals $\phi(x) = \langle \phi, x \rangle$, then we can rewrite (4) in the classical form

$$\langle \phi_x - \phi_y, x - y \rangle \geq 0. \tag{5}$$

A multifunction Γ mapping X into Φ is called *n-cyclic monotone*, if for arbitrary $x_0, x_1, \ldots, x_n = x_0 \in X$ and $\phi_{x_i} \in \Gamma(x_i)$, $i = 0, 1, 2, \ldots, n$, we have

$$\sum_{i=1}^{n} [\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_{i-1}}(x_i)] \geq 0. \tag{6}$$

A multifunction Γ mapping X into Φ is called *cyclic monotone* if it is n-cyclic monotone for $n = 2, 3, \dots$. Of course, just from the definition a multifunction Γ is monotone if and only if it is 2-cyclic monotone.

It is easy to give example of monotone multifunction which is not cyclic monotone. Such example a reader can find in the book of Phelps [3].

A monotone (cyclic monotone) multifunction $\Gamma(x)$ is called *maximal monotone* (resp. *maximal cyclic monotone*) if for each monotone (resp. cyclic monotone) multifunction $\Gamma_1(x)$ such that $\Gamma(x) \subset \Gamma_1(x)$ for all x (in other words such that the graph of Γ, $G(\Gamma)$, is contained in the graph of Γ_1, $G(\Gamma_1)$), we have $\Gamma(x) = \Gamma_1(x)$ for all $x \in X$. It is easy to see that a monotone multifunction $\Gamma(x)$ is maximal monotone if and only if for arbitrary $x, y \in X$ and for arbitrary $\phi_x \in \Gamma(x)$ if

$$\phi_x(x) + \psi(y) - \phi_x(y) - \psi(x) \geq 0 \tag{7}$$

implies that $\psi \in \Gamma(y)$. Observe that a maximal monotone multifunction which is simultaneously cyclic monotone is a maximal cyclic monotone.

Ordering multifunctions by inclusions of graphs and using Kuratowski-Zorn Lemma we obtain that there are always maximal monotone (resp. maximal cyclic monotone) multifunctions and, even more, each monotone (resp. cyclic monotone) multifunction Γ can be embedded in a maximal monotone (resp. maximal cyclic monotone) multifunction $\Gamma_{\max}$. Of course in general, $\Gamma_{\max}$ is not uniquely determined.

Different as in the classical case there are Φ-convex functions such that their Φ-subdifferential $\partial^{\Phi} f|_x$ is not a *maximal* cyclic monotone multifunction.

It is a natural question to find a condition on the class Φ, which warrants that a Φ-subdifferential of a Φ-convex function as a multifunction of x is a *maximal* cyclic monotone multifunction. If a class Φ has this property we say that the class Φ has *maximality property.*

Proposition 1. *Suppose that for each Φ-convex function $f(x)$ and for arbitrary $x, y \in X$ there is $\phi(\cdot) \in \Phi$, which is simultaneously a Φ-subgradient of the function $f(\cdot)$ at the point x and a Φ-subgradient of the function $f(\cdot)$ at the point y. Then the class Φ has maximality property.*

Proof. By Theorem A-Φ the Φ-subdifferential of the function $f(\cdot)$, $\partial^{\Phi} f|_x$ is cyclic monotone multifunction as a multifunction of x. We need to show maximality. Let $\Gamma(x)$ be a monotone multifunction such that $\partial^{\Phi} f|_x \subset \Gamma(x)$ for all $x \in X$. Fix $x \in X$ and suppose that there is $\psi \in \Gamma(x)$ such that $\psi \notin \partial^{\Phi} f|_x$. It means that there is $y \in X$ such that

$$\psi(y) - \psi(x) > f(y) - f(x). \tag{8}$$

By our assumption there is $\phi(\cdot) \in \Phi$, which is simultaneously a Φ-subgradient of the function $f(\cdot)$ at the point x and a Φ-subgradient of the function $f(\cdot)$ at the point y. Since monotonicity of Γ we obtain that

$$\psi(x) - \psi(y) + \phi(y) - \phi(x) \geq 0. \tag{9}$$

By (9) and (8) it follows that

$$\phi(y) - \phi(x) \geq \psi(y) - \psi(x) > f(y) - f(x), \tag{10}$$

i.e. ϕ is not a Φ-subgradient of the function $f(\cdot)$ at the point x, which contradicts our assumption.

As a consequence we obtain

Corollary 2. *Let X be an arbitrary set. Let Φ denote the set of all real-valued functions defined on X. Then the class Φ has maximality property.*

Corollary 3. *Let X be an arbitrary set. Let Φ be a class of real-valued functions defined on X such that for arbitrary $\phi, \psi \in \Phi$, $c \in R$, $\max[\phi, \psi] \in \Phi$ and $\phi + c \in \Phi$. Suppose that for all $x \in X$ the Φ-subdifferential of the function $f(\cdot)$, $\partial^\Phi f|_x$ is not empty. Then the the class Φ has maximality property.*

Proof. Take arbitrary $x, y \in X$. Let $\phi_x(\cdot) \in \Phi$ be a Φ-subgradient of the function $f(\cdot)$ at the point x and a let $\phi_y(\cdot) \in \Phi$ be a Φ-subgradient of the function $f(\cdot)$ at the point y. It is easy to see that $\phi(\cdot) = \max[\phi_x(\cdot) - \phi_x(x) + f(x), \phi_y(\cdot) - \phi_y(y) + f(y)]$ is simultaneously a Φ-subgradient of the function $f(\cdot)$ at the point x and a Φ-subgradient of the function $f(\cdot)$ at the point y.

Corollary 4. *Let X be a metric space. Let Φ denote the set of all continuous real-valued functions defined on X. Then the class Φ has maximality property.*

Corollary 5. *Let X be a metric space. Let Φ denote the set of all Lipschitz real-valued functions defined on X. Then the class Φ has maximality property.*

Corollary 6. *Let X be a metric space. Let Φ denote the set of Lipschitz functions defined on X, such that the Lipschitz constant is non-greater (or smaller) than given $c > 0$. Then the class Φ has maximality property.*

Proposition 1 have also some approximative version

Proposition 7. *Suppose that for each Φ convex function $f(x)$ and for arbitrary $x, y \in X$, $\varepsilon > 0$, there is $\phi(\cdot) \in \Phi$, which is simultaneously an ε-Φ-subgradient of the function $f(\cdot)$ at the point x, i.e.*

$$f(y) - f(x) \geq \phi(y) - \phi(x) - \varepsilon \tag{11}$$

for all $x \in X$ and a Φ-subgradient of the function $f(\cdot)$ at the point y. Then the class Φ has maximality property.

Proof. By Theorem A-Φ the Φ-subdifferential of the function $f(\cdot)$, $\partial^\Phi f|_x$ is cyclic monotone multifunction as a multifunction of x. We need to show maximality. Let $\Gamma(x)$ be a monotone multifunction such that $\partial^\Phi f|_x \subset \Gamma(x)$ for all $x \in X$. Fix $x \in X$ and suppose that there is $\psi \in \Gamma(x)$ such that $\psi \notin \partial^\Phi f|_x$. It means that there is $y \in X$ such that

$$\psi(y) - \psi(x) > f(y) - f(x). \tag{8}$$

Let ε be chosen in such way that

$$0 < \varepsilon < [\psi(y) - \psi(x)] - [f(y) - f(x)]. \tag{12}$$

By our assumption there is $\phi(\cdot) \in \Phi$, which is simultaneously a ε-Φ-subgradient of the function $f(\cdot)$ at the point x and a Φ-subgradient of the function $f(\cdot)$ at the point y. Since monotonicity of Γ we obtain that

$$\psi(x) - \psi(y) + \phi(y) - \phi(x) \geq 0. \tag{9}$$

By (12) and (14) it follows that

$$\phi(y) - \phi(x) \geq \psi(y) - \psi(x) > f(y) - f(x) + \varepsilon, \tag{13}$$

i.e. ϕ is not an ε-Φ-subgradient of the function $f(\cdot)$ at the point x, which contradicts our assumption.

As a consequence we get

Corollary 8. *Let X be a compact (locally compact) space. Let Φ_0 denote the space of all continuous real-valued functions defined on X. Let Φ denote the subset of Φ_0 dense in uniform (resp. uniform on compact set) topology. Then the class Φ has maximality property.*

Corollary 9. *Let X be a closed set in $\mathbb{R}^n$. Let Φ denote the set of all polynomials restricted to X. Then the class Φ has maximality property.*

Problem 10. *Does every finite class Φ has maximality property?*

Problem 11. *Does every linear class Φ has maximality property?*

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CONSTRUCTION OF SOLUTIONS IN DIFFERENTIAL GAMES OF PURSUIT-EVASION

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The paper concerns differential games of pursuit-evasion with geometrical constraints on players' controls. The problem of constructing solutions in these games is investigated. The theory of differential games was formed and advanced, to a considerable extent, by works of R. Isaacs, W. Fleming, L. S. Pontryagin, N. N. Krasovskii, A. I. Subbotin. The monograph of R. Isaacs "Differential Games" published in 60's played important role in the development of the theory of differential games. In the series of works by W. Fleming, the existence of equilibrium situation in differential games was proved. For the linear differential games of pursuit, L. S. Pontryagin developed special construction of the alternated integral. In the well-known monograph of N. N. Krasovskii and A. I. Subbotin "Positional Differential Games", differential games are considered in general setting. The feature of this approach is that solving strategies for players can be found as positional ones.

The paper is devoted to the solutions of differential games in the framework of the approach presented in monograph N. N. Krasovskii and A. I. Subbotin.

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1. Setting pursuit-evasion problem

We consider a conflict controlled system whose dynamics over a time segment $[t_0, \vartheta]$ is described by

$$\frac{dx}{dt} = f(t, x, u, v), \qquad x(t_0) = x_0, \tag{1.1}$$

where $x \in R^m$ is the phase vector of the system; u, v are control vectors of the first and second players restricted by constraints

$$u \in P, \quad v \in Q, \tag{1.2}$$

where P, Q are compacts in Euclidean spaces R^p and R^q.

It is supposed that the following conditions are satisfied:

I. The game takes place in a bounded closed region D of variables $(t, x) \in [t_0, \vartheta] \times R^m$. (Fig. 1)

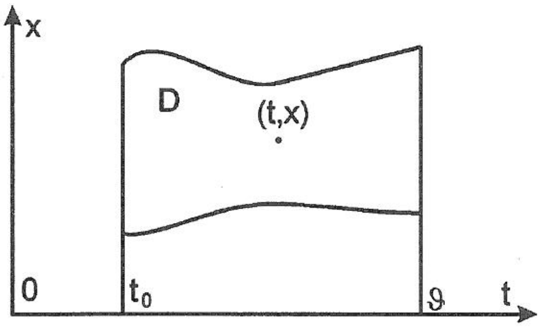

Figure 1

II. Vector-function $f(t, x, u, v)$ is continuous at $D \times P \times Q$ and satisfies the Lipschitz condition on variable x:

$$\|f(t, x^{(2)}, u, v) - f(t, x^{(1)}, u, v)\| \le L\|x^{(2)} - x^{(1)}\| \tag{1.3}$$

for all $(t, x^{(i)}, u, v) \in D \times P \times Q$, $i = 1, 2$.

III. For all $(t_0, x_0) \in D$ and Lebesque measurable functions $u(\cdot) : [t_0, \vartheta] \to P$, $v(\cdot) : [t_0, \vartheta] \to Q$, solution $x(t)$ of the equation

$$\dot{x} = f(t, x(t), u(t), v(t)), \qquad x(t_0) = x_0$$

satisfies the inclusion $(t, x(t)) \in D$ for $t \in [t_0, \vartheta]$.

IV. For any $(t, x) \in [t_0, \vartheta] \times R^m$ and $s \in R^m$ the equality is valid

$$\min_{u \in P} \max_{v \in Q} \langle s, f(t, x, u, v) \rangle = \max_{v \in Q} \min_{u \in P} \langle s, f(t, x, u, v) \rangle$$

Let's introduce positional strategies of the first and second players. Players assume to know the position $(t, x) = (t, x(t))$ of the game at each moment $t \in [t_0, \vartheta]$. A function $U = U(t, x)$, $(t, x) \in D$, $(U(t, x) \subset P, \forall(t, x) \in D)$ is called the positional strategy of the first player. A function $V = V(t, x)$, $(t, x) \in D$, $(V(t, x) \subset Q, \forall(t, x) \in D)$ is called the positional strategy of the second player. The concept of motion $x(t) = x(t, t_0, x_0, U, V)$, $t \in [t_0, \vartheta]$ generated by pair (U, V) of positional strategies of the first and second player can be accurately defined (see [7]).

Let's formulate the problem of pursuit-evasion. Let a closed target set M be selected in the phase space R^m.

The aim of the first player is to find such positional strategy $U = U(t, x)$ that ensures, for every strategy $V = V(t, x)$ of the second player, the attainment of the target set M at the time ϑ, namely, $x(\vartheta) \in M$. (Fig. 2)

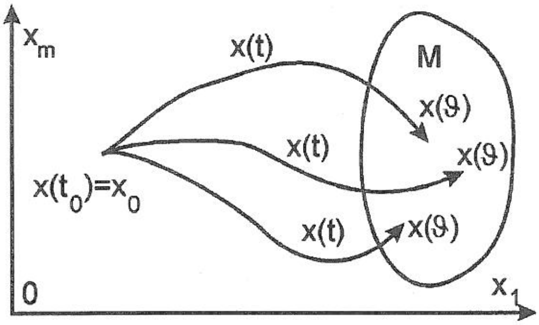

Figure 2

The setting of the pursuit problem is illustrated in figure 3.

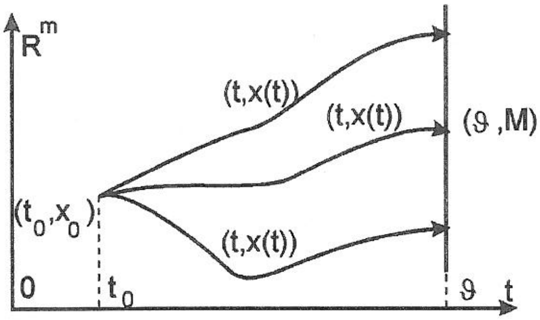

Figure 3

The aim of the second player is to find such positional strategy $V = V(t, x)$, that ensures, for every strategy $U = U(t, x)$ of first player, the evasion from all ε-neighborhood of the target set M $(\varepsilon > 0)$ at the time ϑ, namely, $x(\vartheta) \notin M_\varepsilon$. (Fig. 4)

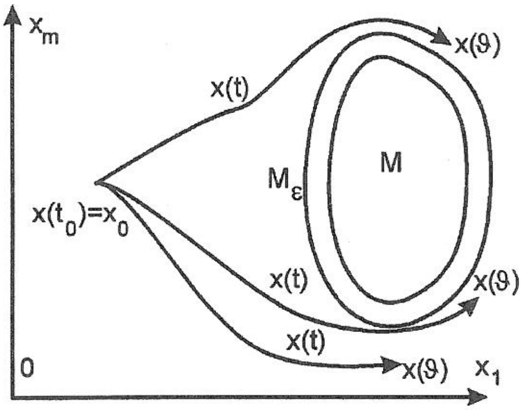

Figure 4

We shall mainly consider the pursuit problem. For solving problems of this kind, the principle of extremal aiming onto stable bridges was developed by N. N. Krasovskii and A. I. Subbotin in 70's. The heart of the principle can be described in the following way.

A set W in the space of positions (t, x) of the game is given such that it is embedded into the target set M at the time ϑ and possesses such property that for all positions (t_*, x_*) belonging to W and for the whole sufficiently small segment $[t_*, t^*]$, the first player can parry any control $v_* \in Q$ of second player which is constant on $[t_*, t^*]$. "Parry" means the choice by the first player of such programmed control $u(t)$ on the segment $[t_*, t^*]$ that, together with v_*, generates the motion $x(t)$ of the system

$$\frac{dx}{dt} = f(t, x, u(t), v_*), \qquad x(t_*) = x_* \tag{1.4}$$

getting to W at the moment t^*. (Fig. 5)

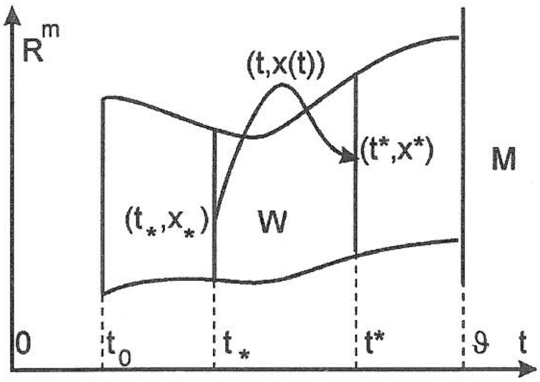

Figure 5

Such set W is called the *u-stable bridge.*

The principle of extremal aiming states that if the initial position (t_0, x_0) of the game belongs to u-stable bridge W, then there exists the positional strategy of the first player, solving the pursuit problem. The positional strategy extremal to the u-stable bridge W can be taken as the solving strategy.

Note that u-stable bridge W, as it is determined above, can be *not unique.*

According to the definition of stability, the set $W^0 = \bigcup W$ (where W are all-possible u-stable bridges) is also u-stable bridge and the maximal one. (Fig. 6)

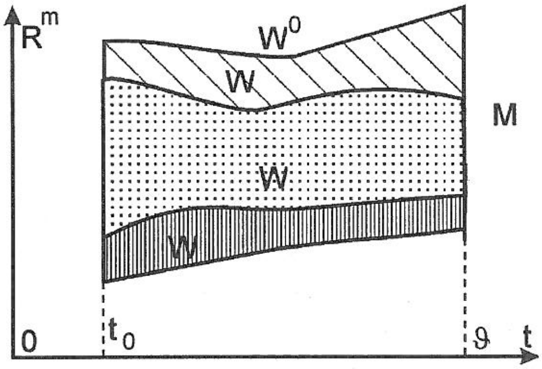

Figure 6

The well-known theorem on an alternative states: if initial position $(t_0, x_0) \in W^0$, then the pursuit problem is solvable for it, and if $(t_0, x_0) \notin W^0$, then the evasion problem is solvable for it. Thus, if, by some way, we succeed to allocate the maximal u-stable bridge W^0 in the space of positions (t, x) of the game, then we can answer the question whether the pursuit or evasion problem is solvable. Therefore, constructing maximal u-stable bridge W^0 is a very important problem. This is the main difficulty for solving the pursuit problem. Let us look how to overcome the difficulty, namely, how to construct u-stable bridge W^0.

Note, that it is possible to allocate exactly the bridge W^0 only for simple pursuit-evasion problems. For example, for controlled system with dynamics and constraints

$$\begin{aligned}
\frac{dx}{dt} &= A(t)x + B(t)u - B(t)v, \\
x[t_0] &= x_0, \qquad\qquad t \in [t_0, \vartheta], \\
P &= \{u \in R^p \mid \|u\| \le \mu\}, \\
Q &= \{v \in R^q \mid \|v\| \le \nu\} \qquad (\nu < \mu < \infty)
\end{aligned}$$

and the target set M be a circle with radius $r \in R$, the set W is determined by analytical formulae

$$W^0 = \{(t,x) \mid t \in [t_0, \vartheta],\ x \in W(t)\},$$
$$W(t) = \{x \in R^m \mid \max_{l:\ \|l\|=1} \varepsilon(t,x,l) \le 0\},$$
$$\varepsilon(t,x,l) = l'X(\vartheta,\tau)x + (\mu - \nu)\int_t^{\vartheta} \|l'BX(\vartheta,\tau)d\tau\| - r,$$

where the symbol $\|h\|$ means Euclidean norm of the vector h in the corresponding Euclidean space; $X(\vartheta,\tau)$ is a fundamental matrix of solutions of the system $dx/dt = A(t)x$.

In more complicated pursuit-evasion problems we don't know similar simple description. In this connection, the problem of approximate construction of maximal u-stable bridge W^0 arises.

Let us consider one method of approximate constructing of the bridge W^0. This method has its origin in the works of R. Bellman, W. Fleming, L. S. Pontryagin, N. N. Krasovskii. The idea is in substitution of segment $[t_0, \vartheta]$ with continuum power by finite partition $\Gamma = \{t_0, t_1, \dots, t_i, t_{i+1}, \dots, t_N = \vartheta\}$ and recurrent backward construction

$$\widetilde{W}(t_i) \leftarrow \widetilde{W}(t_{i+1}) \qquad (i = N-1, N-2, \dots, 0)$$

of the system $\{\widetilde{W}(t_i) \mid t_i \in \Gamma\}$ of sets $\widetilde{W}(t_i) \subset R^m$ which approximate the bridge W^0. (Fig. 7)

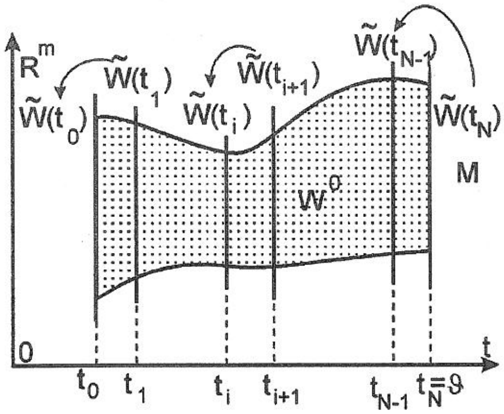

Figure 7

Let us regard in more details constructing approximating system of sets $\{\widetilde{W}(t_i) \mid t_i \in \Gamma\}$. In this connection, it would be noted that there exist several definitions of u-stable bridge. Even if all of them differ by the form, they are equivalent because of allocating the same stable bridges.

Let us formulate one definition, namely, unificational definition of u-stable bridge W ([8], [17], [18]). First, we introduce some auxiliary notions and designations:

$$H(t,x,l) = \max_{u\in P}\min_{v\in Q}\langle l, f(t,x,u,v)\rangle, \qquad (l \in R^m),$$
$$F(t,x) = co\{f \mid f = f(t,x,u,v),\ u \in P,\ v \in Q\},$$
$$S = \{l \mid l \in R^m,\ \|l\| = 1\}$$

where $co\{f\}$ is a convex hull of the set f, $\langle l, f\rangle$ is a scalar product of vectors l and f, $\|l\|$ is a Euclidean norm of the vector l. (Fig. 8)

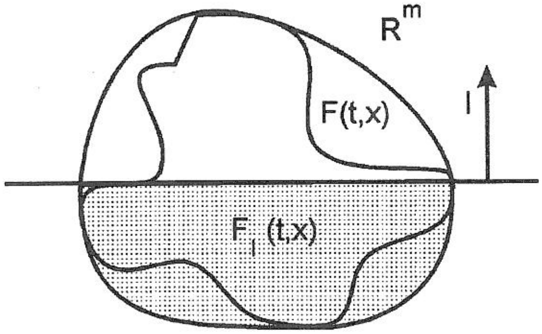

Figure 8

Now let us consider differential inclusions

$$\dot{x} \in F_l(t,x), \qquad x(t_*) = x_* \tag{1.5}$$

on the segment $[t_*, t^*]$, where l are vectors of unit sphere S.

The attainability set of differential inclusion (1.5) calculated at time t^* is denoted by symbol $X_l(t^*, t_*, x_*)$.

Definition 1. Let a set W in the space of positions (t,x) be given. We shall name the set W as u-stable if

$$W(\vartheta) = \{x \in R^m \mid (\vartheta, x) \in W\} \subset M$$

and for any pair t_*, t^*, $(t_0 \le t_* < t^* \le \vartheta)$, any point $x_* \in W(t)$ and any vector $l \in S$ the following expression takes place

$$W(t^*) \bigcap X_l(t^*, t_*, x_*) \neq \emptyset. \tag{1.6}$$

Here and further, $W(t) = \{x \in R^m \mid (t,x) \in W\}$. It is the cutset of the u-stable bridge.

Note, that in presented definition of u-stable bridge W, vectors l of unit sphere play a role of controls $v_* \in Q$, which must be parried by the first player.

Vectors l of S are interpreted as controlling vectors of the second player in the small game, which takes place on the segment $[t_*, t^*]$ with initial position (t_*, x_*). In the small game the second player assumes to select its control $l \in S$ the first, but the first player selects the point x^* from the attainability set $X_l(t^*, t_*, x_*)$.

The cutset $W(t^*)$ of the stable bridge W, corresponding to the last time t^* of the segment $[t_*, t^*]$ is a target set in the small game.

The position (t_*, x_*) is regarded as "good" for the first player in the small game, if for any choice $l \in S$ by the second player, the first player is able to select the point $x^* \in X_l(t^*, t_*, x_*)$ belonging to the target set $W(t^*)$.

From Definition 1 we can see, that any position $(t_0, x_0) \in W(t_*)$, $(t_* < \vartheta)$ is "good" for the first player in the sense of handling of the small game. Therefore, the bridge W consists of positions which are "good" for the first player. (Fig. 9)

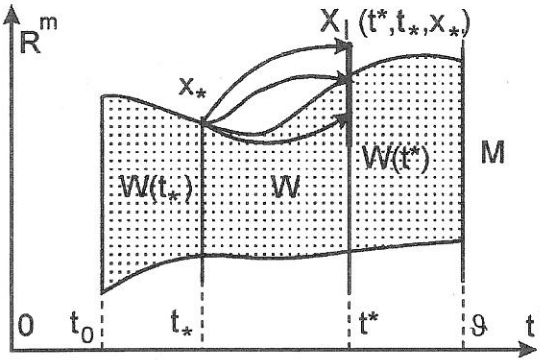

Figure 9

Now, we shall construct the approximating system of sets for maximal u-stable bridge.

2. Approximating system of sets

Let the segment $[t_0, \vartheta]$ be substituted by its discreet approximation, namely, by partition $\Gamma = \{t_0, t_1, \dots, t_i, t_{i+1}, \dots, t_N = \vartheta\}$, and the set M is substituted by its neighborhood M_ε $(\varepsilon > 0)$, and the sets from the Definition 1 are substituted by their linear approximations, which are sets $\widetilde{X}_l(t^*, t_*, x_*) = x_* + (t^* - t_*)\Gamma_l(t_*, x_*)$, $(t_0 \leq t_* < t^* \leq \vartheta)$.

Definition 2. The system of sets $\{\widetilde{W}(t_i) \mid t_i \in \Gamma\}$ is called the approximating system of sets, corresponding to the pair (Γ, ε), if it satisfies relations:

$$\widetilde{W}(t_N) = M_\varepsilon,$$
$$\widetilde{W}(t_i) = \{x[t_i] \in R^m \mid \widetilde{W}(t_{i+1}) \bigcap \widetilde{X}_l(t_{i+1}, t_i, x[t_i]) \neq \emptyset, \ \forall l \in S\}$$

According to the Definition 2, the approximating system of sets represents a collection of sets which are contained in R^m and correspond to times t_i of partition Γ. The collection results from the substitution of $[t_0, \vartheta]$ by partition Γ, the extension of M and replacement in formula (1.6) of t_*, t^* by t_i, t_{i+1}, the set $W(t^*)$ by $\widetilde{W}(t_{i+1})$, sets $X_l(t^*, t_*, x_*)$ by $\widetilde{X}_l(t_{i+1}, t_i, x[t_i])$, $l \in S$.

It is known ([17]) that $\lim_{\Delta\Gamma\downarrow 0,\ \varepsilon\downarrow 0}\{\widetilde{W}(t_i) \mid t_i \in \Gamma\} = W^0$. In other words, approximating sequence $\{\widetilde{W}(t_i) \mid t_i \in \Gamma\}$ converges to the maximal u-stable bridge W^0 (at that, it converges by above) at $\Delta(\Gamma) \downarrow 0$ and $\varepsilon \downarrow 0$. Here $\Delta(\Gamma)$ is the diameter of the partition Γ.

It would be noted that for convergence of the system $\{\widetilde{W}(t_i) \mid t_i \in \Gamma\}$ to W^0x , it is necessary that values $\Delta(\Gamma)$ and ε be connected in some way.

The question is, how to construct the system $\{\widetilde{W}(t_i) \mid t_i \in \Gamma\}$, corresponding to the pair (Γ, ε)?

The constructing must be realized through the backward procedure which can be illustrated by the scheme:

$$M_\varepsilon = \widetilde{W}(t_N) \longrightarrow \widetilde{W}(t_{N-1}) \longrightarrow \ldots \longrightarrow \widetilde{W}(t_{i+1}) \longrightarrow \widetilde{W}(t_i) \longrightarrow \ldots \longrightarrow \widetilde{W}(t_0).$$

At this stage, corresponding to segment $[t_i, t_{i+1}]$, the set $\widetilde{W}(t_i)$ is constructed as issue from the set $\widetilde{W}(t_{i+1})$ which was obtained at the previous stage:

$$\widetilde{W}(t_{i+1}) \longrightarrow \widetilde{W}(t_i).$$

Note, that there is a difficulty to construct the set $\widetilde{W}(t_i)$ using directly the definition.

The difficulty is that for verification whether the point $x[t_i]$ belongs to the set $\widetilde{W}(t_i)$ or not, it is necessary to test the realizability of relation

$$\widetilde{W}(t_{i+1}) \bigcap \widetilde{X}_l(t_{i+1}, t_i, x[t_i]) \neq \emptyset$$

for all $l \in S$.

For constructing of the set $\widetilde{W}(t_i)$ one can make localization procedure by putting, for every point $x[t_i] \in R^m$, a correspondence with a set (Fig. 10)

$$M_{t_{i+1}}([x_i]) = \widetilde{W}(t_{i+1}) \bigcap \widetilde{X}(t_{i+1}, t_i, x[t_i]),$$
$$\widetilde{X}(t_{i+1}, t_i, x[t_i]) = x[t_i] + \Delta_i F(t_i, x[t_i]), \qquad \Delta_i = t_{i+1} - t_i$$

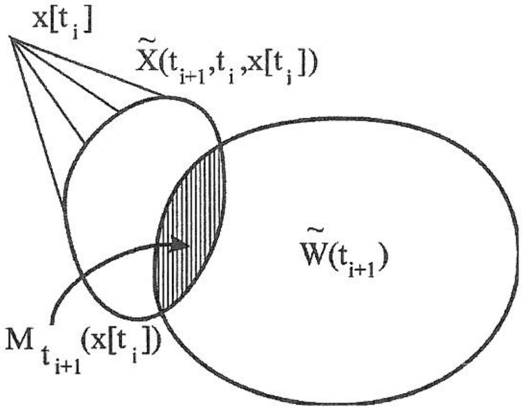

Figure 10

The set $M_{t_{i+1}}(x[t_i])$ can be treated as local target set in the local pursuit game on the segment $[t_i, t_{i+1}]$ with a target set $\widetilde{W}(t_{i+1})$ and the origin at the point $x[t_i]$.

Let's introduce the function

$$\varepsilon(t_i, x[t_i], l) = -\langle l, x[t_i]\rangle - \Delta_i H(t_i, x[t_i], l) + \rho_{M_{t_{i+1}}(x[t_i])}(l), \tag{2.1}$$

where

$$\rho_{M_{t_{i+1}}(x[t_i])}(l) = \begin{cases} \min\limits_{w \in M_{t_{i+1}}(x[t_i])} \langle l, w\rangle, & \text{if } M_{t_{i+1}}(x[t_i]) \neq \emptyset \\ +\infty & \text{if } M_{t_{i+1}}(x[t_i]) \neq \emptyset \end{cases}$$

Theorem 1. $\widetilde{W}(t_i) = \{x[t_i] \in R^m \mid \max_{l \in S} \varepsilon_{\Delta_i}(t_i, x[t_i], l) = 0\}$.

Theorem 1 gives us a functional description of the set $\widetilde{W}(t_i)$. It states that the point $x[t_i]$ belongs to the set W if and only if (Fig. 11)

$$\omega(t_i, x[t_i]) = \max_{l \in S} \varepsilon_{\Delta_i}(t_i, x[t_i], l) \leq 0.$$

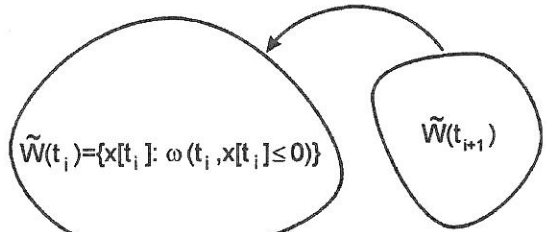

Figure 11

Note, that in a number of cases, the set $\widetilde{W}(t_i)$ can be easily built if its bound $\partial\widetilde{W}(t_i)$ is known. For example, in the case, when the set $\widetilde{W}(t_i)$ is convex. That is why the problem of allocation of the bound $\partial\widetilde{W}(t_i)$ in the space R^m is very interesting.

The question arises: what can we say about the functional description of the bound $\partial\widetilde{W}(t_i)$ of the set $\widetilde{W}(t_i)$?

Let us consider the case, when $\widetilde{W}(t_i)$ is a convex set and $int\widetilde{W}(t_i) \neq \emptyset$, and the following condition (E) is also satisfied:

(E) For all t, x, l $((t, x, l) \in D \times S)$ the inequality is valid:

$$\min_{(u,v)\in P\times Q} \langle l, f(t, x, u, v)\rangle < H(t, x, l) < \max_{(u,v)\in P\times Q} \langle l, f(t, x, u, v)\rangle.$$

The set $\Lambda_0(x[t_i]) = \partial M_{t_{i+1}}(x[t_i]) \bigcap \mathrm{int}\widetilde{X}(t_{i+1}, t_i, x[t_i])$ is put into correspondence with the point $x[t_i] \in R^m$. (Fig. 12)

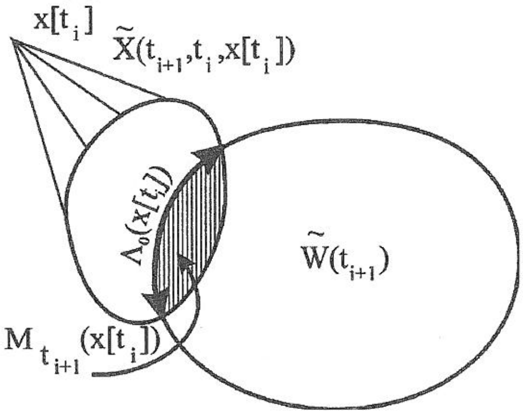

Figure 12

Also denote $L^0(x[t_i]) = \mathrm{cl}\{l \in S \mid \exists w \in \Lambda_0(x[t_i]),\ \langle l, w\rangle = \rho_{M_{t_{i+1}}(x[t_i])}(l)\}$ (see Fig. 13)

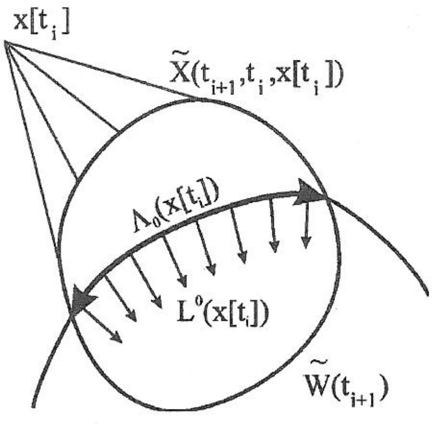

Figure 13

The statement on reduction holds:

Theorem 2. *Let* $\widetilde{W}(t_{i+1})$ *be a convex set,* $\operatorname{int}\widetilde{W}(t_{i+1}) \neq \emptyset$ *and the condition* (E) *is satisfied. Then* $x[t_i] \in \partial\widetilde{W}(t_{i+1})$ *if and only if*

$$\max_{l\in S} \varepsilon_{\Delta_i}(t_i, x[t_i], l) = \max_{l\in L^0(x[t_i])} \varepsilon_{\Delta_i}(t_i, x[t_i], l) = 0.$$

Accordingly the Theorem 2, while verification whether the point $x[t_i]$ belongs to the bound $\partial\widetilde{W}(t_i)$ or not, one would consider only vectors $l \in L^0(x[t_i])$ and calculate $\max_{l\in L^0(x[t_i])} \varepsilon_{\Gamma_i}(t_i, x[t_i], l)$.

Note that the Theorem 2 was efficiently used while computation of approximating system $\{\widetilde{W}(t_i) \mid t_i \in \Gamma\}$ in various examples on the plain and in 3-dimensional space was implemented.

Moreover, the ideology of the Theorem 2 was transferred to cases, when the set $\widetilde{W}(t_i)$ is not convex, but possesses the structure of supergraph of some function.

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Differential Inclusions and Optimal Control

September 22 - October 3, 1997

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LIST OF PARTICIPANTS

First week: September 22-26

1. Sergei Ageev (Brest University, Brest, Belarus)
2. Jan Andres (Palacký University, Olomuoc)
3. Diego Averna (University of Palermo, Palermo)
4. Grzegorz Bartuzel (Warsaw Technical University, Warszawa)
5. Dorota Bors (University of Łódź, Łódź)
6. Monika Bartkiewicz (University of Łódź, Łódź)
7. Krzysztof Bielawski (University of Gdańsk, Gdańsk)
8. Alberto Bressan (SISSA Institut, Trieste)
9. Daria Bugajewska (Adam Mickiewicz University, Poznań)
10. Dariusz Bugajewski (Adam Mickiewicz University, Poznań)
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13. Mieczysław Cichoń (Uniwersytet Adama Mickiewicza, Poznań)
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15. Aleksander Ćwiszewski Nicholas Copernicus University, Toruń)
16. Stanisław Domachowski (University of Gdańsk, Gdańsk)
17. Zdzisław Dzedzej (University of Gdańsk, Gdańsk)
18. Vladimir V. Filippov (Moscow State University, Moscow)
19. Jiri Fiser (Palacký University, Olomouc)
20. Helene Frankowska (University Paris-Dauphine, Paris)
21. Andrzej Fryszkowski (Warsaw Technical University, Warszawa)
22. Grzegorz Gabor (Nicholas Copernicus University, Toruń)
23. Marek Galewski (University of Łódź, Łódź)
24. Boris Gelman (Voronezh State University, Voronezh)

25. Yuri Gliklikh (Voronezh State University, Voronezh)
26. Lech Gorniewicz (Nicholas Copernicus University, Toruń)
27. Joachim Gwinner (Universitaet der Bundeswehr Muenchen, Neubiberg)
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29. Adam Idzik (Polish Academy of Science, Warszawa)
30. Jerzy Jezierski (Warsaw University of Agricalture, Warszawa)
31. Russel Johnson (University of Florence, Florence)
32. Mihaił Kamenski (Voronezh State University, Voronezh)
33. Wioletta Karpińska (University of Łódź, Łódź)
34. Zbigniew Karno (University of Bialystok, Bialystok)
35. Wojciech Kryszewski (Nicholas Copernicus University, Toruń)
36. Zbynek Kubacek (University of Bratislava, Bratislava)
37. Zygfryt Kucharski (University of Gdańsk, Gdańsk)
38. Anna Kucia (University of Silesia, Katowice)
39. Anna Kurdubska (Nicholas Copernicus University, Toruń)
40. Krzysztof Leśniak (Nicolas Copernicus University, Toruń)
41. Salvatore Marano (University of Catania, Catania)
42. Marek Majewski (University of Łódź, Łódź)
43. Dariusz Miklaszewski (Nicholas Copernicus University, Toruń)
44. Józef Myjak (University of L'Aquila, Coppito)
45. Paolo Nistri (University of Florenca, Florence)
46. Andrzej Nowak (University of Silesia, Katowice)
47. Valerii Obukhovskii (Voronezh State University, Voronezh)
48. Anna Ochal (Jagiellonian University, Kraków)
49. Czesław Olech (Polish Academy of Science, Warszawa)
50. Jolanta Olko (University of Silesia, Katowice)
51. Jerzy Ombach (Jagiellonian University, Kraków)
52. Aleksandra Orpel (University of Łódź, Łódź)
53. Radosław Pietkun (Nicholas Copernicus University, Toruń)
54. Slawomir Plaskacz (Nicholas Copernicus University, Toruń)
55. Beata Potaczek (University of Silesia, Katowice)
56. Bogdan Przeradzki (University of Łódź, Łódź)
57. Dusan Repovs (University of Ljubljana, Ljubljana)
58. Biagio Ricceri (University of Catania, Catania)
59. Aandrzej Rogowski (University of Łódź, Łódź)
60. Stefan Rolewicz (Polish Academy of Sciences, Warszawa)
61. Tadeusz Rzeżuchowski (Warsaw Technical University, Warszawa)
62. Pavel Semenov (Moscow State Pedagogical University, Moscow)
63. Andrzej Smajdor (University of Silesia, Katowice)

64. Wilhelmina Smajdor (University of Silesia, Katowice)
65. Robert Stańczy (University of Łódź, Łódź)
66. Joanna Szczawińska (University of Silesia, Katowice)
67. Aldona Szukała (Adam Mickiewicz University, Poznań)
68. Peter Tallos (Budapest University of Economics, Budapest)
69. Gabriele Villari (University of Florenca, Florence)
70. Stanisław Walczak (University of Łódź, Łódź)
71. Andrzej Wieczorek (Polish Academy of Sciences, Warszawa)

Second week: September 29 - October 3

1. Ralf Bader (Universität München, München)
2. Monika Bartkiewicz (Jagiellonian University, Kraków)
3. Grzegorz Bartuzel (Warsaw Technical University, Warszawa)
4. Semeon Bogatyi (Moscow State University, Moscow)
5. Dorota Bors (Jagiellonian University, Kraków)
6. Daria Bugajewska (Adam Mickiewicz University, Poznań)
7. Dariusz Bugajewski (Adam Mickiewicz University, Poznań)
8. Aurelian Cernea (Romanian Academy, Bucharest)
9. Waldemar Choduń (Technical University of Rzeszów, Rzeszów)
10. Miczysław Cichoń (Adam Mickiewicz University, Poznań)
11. Aleksander Ćwiszewski (Nicholas Copernicus University, Toruń)
12. Zdzisław Dzedzej (University of Gdańsk, Gdańsk)
13. Andrzej Fryszkowski (Warsaw Technical University, Warszawa)
14. Grzegorz Gabor (Nicholas Copernicus University, Toruń)
15. Marek Galewski (University of Łódź, Łódź)
16. Lech Górniewicz (Nicholas Copernicus University, Toruń)
17. V. V. Gorokhovik (Byelorussian Academy of Sciences, Minsk)
18. Dariusz Idczak (University of Łódź, Łódź)
19. Adam Idzik (Polish Academy of Sciences, Warszawa)
20. Zoltan Kanai (Budapest University of Economics, Budapest)
21. Zbigniew Karno (University of Białystok, Białystok)
22. Wioletta Karpińska (University of Łódź, Łódź)
23. Anna Kucia (Silesian University, Katowice)
24. Marek Majewski (University of Łódź, Łódź)
25. Mariusz Michta (Technical University of Zielona Góra, Zielona Góra)
26. Jerzy Motyl (Technical University of Zielona Góra, Zielona Góra)
27. Marian Mureasan (Babes-Bolyai University, Cluj-Napoca)

28. Anatoly D. Myshkis (Moscow State University, Moscow)
29. Andrzej Nowak (Silesian University, Katowice)
30. Andrzej Nowakowski (University of Łódź, Łódź)
31. Anna Ochal (Jagiellonian University, Kraków)
32. Czesław Olech (Polish Academy of Science, Warszawa)
33. Joanna Olko (Silesian University, Katowice)
34. Sławomir Plaskacz (Nicholas Copernicus University, Toruń)
35. Victor Plotnikov (Odessa State University, Odessa)
36. Beata Potaczek (Silesian University, Katowice)
37. Bogdan Przeradzki (University of Łódź, Łódź)
38. Marc Quincampoix (University of Bretagne Occidentale, Brest)
39. Andrzej Rogowski (University of Łódź, Łódź)
40. Stefan Rolewicz (Polish Academy of Sciences, Warszawa)
41. Longin Rybiński (Technical University of Zielona Góra, Zielona Góra)
42. Tadeusz Rzeżuchowski (Warsaw Technical University, Warszawa)
43. Patric Saint-Pierre (University Paris-Dauphine, Paris)
44. Alberto Seeger (University of Avignon, Avignon)
45. Robert Stańczy (University of Łódź, Łódź)
46. Joanna Szczawińska (Silesian University, Katowice)
47. Aldona Szuka (Adam Mickiewicz University, Poznań)
48. Peter Tallos (Budapest University of Economics, Budapest)
49. Victor Ushakov (Ural Section of Russian Academy of Science, Ekaterinburg)
50. Vladimir Veliov (Bulgarian Academy of Science, Sofia)
51. Andrzej Wieczorek (Polish Academy of Sciences, Warszawa)
52. Piotr Zabreiko (Byelorussian Academy of Sciences, Minsk)

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SCHEDULE

Monday, September 22

P. Nistri: *Control and optimization of nonlocal problems via topological methods,*
J. Andres: *Almost periodic and bounded solutions of differential inclusions,*
A. Marano: *Fixed points of multivalued contractions, some recent results,*
W. Kryszewski: *Some results on the graph approximation of set-valued maps,*
Z. Dzedzej: *On the solution set for some multivalued boundary value problems,*
G. Gabor: *Topological degree theory in locally convex spaces,*
D. Averna: *Existence of solutions for operator inclusions: a unified approach,*
S. Ageev: *Lyapunov's convexity theorem is not necessary in the selection theorems for decomposable valued mappings,*

Tuesday, September 23

A. Bressan: *Well-posedness of the Cauchy problem for discontinuous O.D.E's,*
J. Jezierski: *A Nielsen theory for a class of multivalued mappings,*
A. Fryszkowski: *Differential inclusion with the mixed right hand side,*
B. Ricceri: *On the topological dimension of the solution set of a class of nonlinear equations,*
D. Idczak: *Optimality conditions for some systems of the Dirichlet type,*
D. Bugajewska and D. Bugajewski: *Nonlinear equations, approximative derivative and Denjoy integral,*
M. Cichoń: *On perturbations of m-accretive differential inclusions,*

Wednesday, September 24

V. Filippov: *Equations and inclusions with discontinuities in spaces variables. Properties of the synthesis of optimal control,*

G. Villari: *A continuation lemma with applications to periodicaly forced Lienard equations in presence of a separatrix*,
D. Repovs: *Milyutin mappings and their applications*,
S. Walczak: *Variational and boundary value problems with controls*,
B. Gelman: *Surjectivity of multivalued maping and differential inclusions*,
A. Kucia and A. Nowak: *Applications of Marczewski function to multifunction*,
Y. Glicklich: *Achievement points and submanifolds of mechanical systems with multivalued forces on Riemannian manifolds*,
Z. Kubacek: *Aronszajn-type results for some differential equations on unbounded intervals*,

Thursday, September 25

H. Frankowska: *Relaxation of control systems under state constraints*,
R. Johnson: *The topological degree applied to thin domain problems*,
M. Kamenski: *Averaging principle for semilinear inclusions with noncompact semigroups*,
P. Semenov: *Controlled relaxation of convexity. Fixed-points theorems*,
J. Ombach: *Continuous and inverse shadowing for discrete dynamical systems*,
A. Rogowski: *On some generalization of the Krasnosielskiĭ theorem and its applications*,
A. Prykarpatsky: *Adiabatic chaos: problems and paradoxes*,

Friday, September 26

V. Obukhowskii: *On some problems of theory of functional cdifferential inclusions in Banach spaces*,
J. Gwinner: *A class of nonlinear evolution differential inclusions and their discretization*,
J. Myjak: *Iterated function systems; fractals and semifractals*,
Z. Kucharski: *On the Nielsen number*,

Monday, September 29

S. Rolewicz: *Φ-subdifferentials as set-valued mappings*,
M. Quincampoix: *Viability under uncertain initial state*,
A. D. Myshkis: *Differential equations with multi-dimensional time without condition of total integrability*,
S. Bogatyi: *Ljusternik-Schnirelmann theorem, Hausdorff-Banach-Tarski paradox, invariant measure*,
A. Cernea: *Optimal control of differential inclusions using derived cones*,

Tuesday, September 30

P. Zabreiko: *On differentiability of some classes of multifunctions,*

R. Bader: *Periodic solutions of semilinear differential inclusions in Banach spaces,*

A. Idzik: *Leray-Schauder type theorems,*

A. Wieczorek: *Reducing the search for a fixed point to nonlinear optimization problems: an economic case study,*

M. Muresan: *Several properties of the solutions to quasi-linear set-valued functions,*

Wednesday, October 1

T. Rzeżuchowski: *Boundary solutions of differential inclusions,*

V. N. Ushakov: *Construction of solutions in differential games of pursuit-evasion,*

P. Talos and Z. Kannai: *Potential-type inclusions,*

V. Gorokhovik: *Frechet differentiability of multifunctions,*

S. Seeger: *Sensitivity analysis in convex dynamical optimization,*

Thursday, October 2

P. Saint-Pierre: *Estimation of convergence and approximation of optimal synthesis for dynamical systems,*

V. Veliov: *Discretization methods for differential inclusions,*

A. Nowakowski: *Generalized field theory, synthesis and dynamical programming approach to optimality conditions,*

V. Plotnikov: *Differential inclusions and numerical asymptotic methods in the control problems,*

J. Motyl: *Stochastic inclusions — some existence problems,*

Friday, October 3

B. Przeradzki: *Travelling waves solutions to reaction-diffusion equations,*

J. Olko: *Selections of an iteration semigroup of linear set-valued functions,*

J. Szczawińska: *On Nemytski operator,*

D. Bugajewski: *Nonlinear equations in abstract spaces and axiomatic measures of noncompactness,*

M. Mrozek: *Multivalued mappings in time series analysis,*