Juliusz Schauder University Centre for Nonlinear Studies
Nicolaus Copernicus University

Existence and Structure of Solution Sets for Impulsive Differential Inclusions: a Survey

Smaïl Djebali
École Normale Suppérieure, Kouba, Algeria

Lech Górniewicz
Juliusz Schauder University Centre for Nonlinear Studies
Nicolaus Copernicus University of Toruń, Poland
Kazimierz Wielki University, Bydgoszcz, Poland

Abdelghani Ouahab
Sidi-Bel-Abbès University, Algeria

Toruń, 2012

ISSN 2082–4335

Recenzenci:
prof. dr hab. Andrzej Fryszkowski
prof. dr hab. Jerzy Motyl

Centrum Badań Nieliniowych im. Juliusza Schaudera
Uniwersytet Mikołaja Kopernika
ul. Chopina 12/18, 87-100 Toruń

Redakcja: tel. +48 (56) 611 34 28, faks: +48 (56) 622 89 79

e-mail: tmna@ncu.pl
http://www.cbn.ncu.pl

Wydanie pierwsze. Nakład 200 egz.

Dedicated to Professor Francesco S. De Blasi
on the occasion of his 70th birthday

ABSTRACT

In this survey paper, we present some existence results of mild solutions and study the topological structure of solution sets for first-order impulsive semilinear differential inclusions with initial value and periodic boundary conditions. Under various assumptions on the nonlinear term, we present several existence results for the Cauchy problem. We appeal to the topological fixed point theory as well as to some results and properties from multi-valued analysis, functional analysis, and measure of noncompactness. Further to the compactness of the solution sets, we prove some geometric properties about the structure of the solution sets such as AR, R_δ, contractibility, and acyclicity, corresponding to Aronszajn–Browder–Gupta type results. This is achieved by using some elements from algebraic topology and homology. Regarding the periodic boundary conditions, the problem is formulated as a fixed point problem either for an integral operator or for a Poincaré translation operator. In particular, one existence result relies on a new nonlinear alternative for compact *u.s.c.* maps defined in infinite-dimensional Banach spaces. Then, we investigate the topological structure of the solution sets. A continuous version of Filippov's theorem is provided and the continuous dependence of solutions on parameters in the both convex and the nonconvex cases are proved. More generally, a class of impulsive functional differential inclusions is considered and the same theory is developed. Finally, we consider the question of existence and the structure of solution sets for first-order impulsive differential inclusions in Fréchet space settings and the initial and terminal problems are considered. The results on the geometric structure of the solution sets are obtained via the method of the inverse system limit of some Banach spaces.

2010 *Mathematics Subject Classifications.* 34A37, 34A60, 34K30, 34K45, 47D60, 47H09, 47H10.

Key words and phrases. Impulsive functional differential inclusions, mild solution, periodic problem, infinity interval, Filippov's theorem, fixed point, solution set, compactness, AR, R_δ, contractibility, acyclicity.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Motivation

Differential equations with impulses were considered for the first time by Milman and Myshkis [110] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [77]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations (see for instance [1], [101], [102] and the references therein). These perturbations may be treated as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine, and biology. A comprehensive introduction to the basic theory is well developed in the monographs by Bainov and Simeonov [17], Laskshmikantham *et al.* [103], Samoilenko and Perestyuk [127] or the survey paper by Rogovchenko [126] or the more recent books [123], [131]. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions.

1.1.1. Ecological model with impulsive control strategy. For the impulsive model with distributed time delay, the authors of papers [92], [109], [129] have investigated some ecological models with distributed time delay and impulsive control strategy. The model can be described by the following differential equations:

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{k}\right) - a_2 x(t)y(t) - \frac{\beta a_1 x(t)z(t)}{b_1 + x(t) + c_1 z(t)},$$
$$t \neq nT,\ t \neq (n+l-1)T,$$

$$
\begin{aligned}
y'(t) &= dy(t)\int_{-\infty}^{t} F(t-s)x(s)\,ds - m_1 y(t), && t \neq nT,\ t \neq (n+l-1)T,\\
z'(t) &= \frac{e_1\beta a_1 x(t)z(t)}{b_1 + x(t) + c_1 z(t)} - m_2 z(t), && t \neq nT,\ t \neq (n+l-1)T,\\
\Delta x(t) &= -\delta_1 x(t), && t = (n+l-1)T,\\
\Delta y(t) &= -\delta_2 y(t), && t = (n+l-1)T,\\
\Delta z(t) &= -\delta_3 z(t), && t = (n+l-1)T,\\
\Delta x(t) &= 0,\\
\Delta y(t) &= 0,\\
\Delta z(t) &= p,
\end{aligned}
$$

where

- $x(t)$, $y(t)$, $z(t)$ are the densities of one prey and two predators at time t, respectively,
- $\Delta x(t) = x(t+) - x(t)$, $\Delta y(t) = y(t+) - y(t)$, $\Delta z(t) = z(t+) - z(t)$,
- $0 < l < 1$ is used to describe the intervals of time between the pulsed use of controls,
- r is the intrinsic growth rate, a_i $(i = 1, 2)$ are the cropping rate,
- e_1 denotes the efficiency with which resources are converted to new consumers,
- k is the carrying capacity of the prey,
- b_1 is a saturation constant, c_1 scales the impact of predator interference,
- m_i $(i = 1, 2)$ are the mortality rates for each predator,
- d denotes the product of the per-capita rate of predation and the rate of conversing prey into predator,
- β is the relative superiority of predator (z).

1.1.2. Impulsive Leslie predator-prey system. Leslie [106] introduced the famous Leslie predator-prey system

$$
\begin{cases}
x'(t) = x(t)[a - bx(t)] - p(x)y(t),\\
y'(t) = y(t)\left[e - f\dfrac{y(t)}{x(t)}\right],
\end{cases}
$$

where $x(t)$ and $y(t)$ stand for the population (the density) of the prey and the predator at time t, respectively, and $p(x)$ is the so-called predator functional response to prey. In biomathematics, when $p(x) = cx$, the functional response $p(x)$ is called type 1; when $p(x) = \frac{cx}{d+x}$, the functional response $p(x)$ is called type 2; when $p(x) = \frac{cx^2}{d+x^2}$, the functional response $p(x)$ is called type 3. In [137],

the authors consider a ratio-dependent Leslie predator-prey model with impulses

$$\begin{cases} x_1'(t) = x_1(t)\left[b(t) - a(t)x1(t) - \dfrac{c(t)x_1(t)x_2(t)}{h_2x_2^2(t) + x_1^2(t)}\right], \\ x_2'(t) = x_2(t)\left[e(t) - f(t)\dfrac{x_2(t)}{x_1(t)}\right], & t \neq t_k, \\ x_i(t_k^+) = (1 + h_k^i)x_i(t_k),\ x_i(0) > 0, & i = 1, 2, \end{cases}$$

where $x_i(t)$, $i = 1, 2$ denote the density of prey and predator at time t, respectively. $b, a, c, d, e, f, p \in C(\mathbb{R}, \mathbb{R}_+)$, $i = 1, 2$ are all ω-periodic functions of t; h^2 is a positive constant, denoting the constant of capturing half-saturation.

1.1.3. Pulse vaccination model. The pulse vaccination proposes to vaccinate a fraction p of the entire susceptible population in a single pulse, applied every τ years; the following standard SIR model and constant vaccination

$$\begin{cases} S'(t) = -\beta IS^2 + \mu - \mu S, \\ I'(t) = \beta IS^2 - (\gamma + \mu)S, \\ R'(t) = \gamma I - \mu R, \end{cases}$$

evolves from its initial state without being further affected by the vaccination schemes until the next pulse is applied, when the pulse vaccination is incorporated, the system may be rewritten as

$$\begin{cases} S'(t) &= -\beta IS^2 + \mu - \mu S, & t \neq n\tau, \\ I'(t) &= \beta IS^2 - (\gamma + \mu)S, & t \neq n\tau, \\ R'(t) &= \gamma I - \mu R, & t \neq n\tau, \\ S(n\tau^+) &= (1 - p)S(n\tau^-), & t = n\tau, \\ I(n\tau^+) &= I(n\tau^-), & t = n\tau, \\ R(n\tau^+) &= R(n\tau^-) + pS(n\tau^-), & t = n\tau. \end{cases}$$

where the fractions of the population that are susceptible, infectious, and recovered with immunity, are denoted by S, I and R, respectively. For more information about this model, see [89].

1.2. General presentation

Recently, the question of the existence of solutions and other mathematical aspects related to differential equations and inclusions have been extensively studied and have attracted much attention; in this direction, important contributions have been obtained so far (see the monographs and recent papers [11], [17], [20], [41]–[43], [66], [68], [70], [103], [117] among others). In 1890, Peano [122] proved that the Cauchy problem for ordinary differential equations

has local solutions although the uniqueness property does not hold in general. In case where the uniqueness does not hold, Kneser [97] proved in 1923 that the solution set is a continuum, i.e. closed and connected. In 1928, Hukuhara [90] used modern functional analytic approach. In 1942, Aronszajn [9] improved this result for differential inclusions in the sense that he showed that the solution set is in fact compact and acyclic, and he even specified this continuum to be an R_δ-set. An analogous result was obtained for differential inclusions with *u.s.c.* convex valued nonlinearities by several authors; we quote [22], [39], [52], [57], [63]–[65], [67].

Very recently, the topological and geometric structure of solution sets for impulsive differential inclusions on compact intervals were investigated in [41], [68], [69], [83] and contractibility, AR, acyclicity, and R_δ-sets properties are obtained. Also, the topological structure of solution sets for some Cauchy problems without impulses and posed on non-compact intervals were studied by various techniques in referencess [6], [16], [38], [39], [55], [95].

This paper surveys recent results on the existence, the geometric and topological structure of solution sets of a class of semi-linear differential inclusions in the setting of Banach and Fréchet spaces. It is organized as follows. In Chapter 2, we introduce all the background material needed in this survey paper such as multi-valued analysis, elements from functional analysis, some properties of C_0-semigroup, and also some results from homology and algebraic topology. Chapter 3 is devoted to establishing some existence results for impulsive differential inclusions; we use the nonlinear alternative of Leray–Schauder type (in the convex case), the Bressan–Colombo–Fryszkowski selection theorem, and the Covitz–Nadler fixed point theorem for contraction multi-maps in a generalized metric space (in the nonconvex case). The second part of Chapter 3 is concerned with the topological properties of the solution set. In Chapter 4, we discuss impulsive differential inclusions with periodic conditions and we examine the cases when the right hand side is either convex valued or nonconvex valued. The parameter-dependance of solutions is also investigated and a Filippov's theorem is proved. In Chapter 5, we extend some results of existence and compactness of solution sets for semi-linear functional impulsive Cauchy problem and Aronszajn-type results are proved. We present in Chapter 6 three existence theorems for differential inclusions posed on the positive half-line. Different kinds of growth of the nonlinearity F are considered in case F is *u.s.c.*, *l.s.c.*, Lipschitz or when it satisfies Nagumo-type condition. Also, we consider questions of existence and topological structure of solutions for the corresponding terminal problem and we investigate the geometric structure of solution sets (AR, R_δ, acyclicity, contractibility) in both the convex and the nonconvex cases. We employ the projective limit approach which consists in considering a Fréchet

space as a limit of inverse system of some Banach spaces. Finally, an application of topological and geometric structure to solve some impulsive boundary value problems is presented. We end this survey paper with some concluding remarks and a rich bibliography.

CHAPTER 2

BACKGROUND MATERIAL

2.1. Multivalued analysis

In this section, we recall some notations, definitions, and auxiliary results which will be used throughout this paper.

2.1.1. Generalities. Let $(E, |\cdot|)$ be a separable Banach space, $J = [0, b]$ a compact interval in $\mathbb{R}$ and $C(J, E)$ the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le b\}.$$

A function $y\colon J \to E$ is called measurable provided for every open subset $U \subset E$, the set $y^{-1}(U) = \{t \in J : y(t) \in U\}$ is Lebesgue measurable. A measurable function $y\colon J \to E$ is Bochner integrable if $|y|$ is Lebesgue integrable. For properties of the Bochner integral, we refer to Yosida [142]. In what follows, $L^1(J, E)$ denotes the Banach space of functions $y\colon J \to E$ which are Bochner integrable with norm

$$\|y\|_1 = \int_0^b |y(t)|\, dt.$$

$L^1_{\mathrm{loc}}([0, \infty), E)$ will refer to the space of locally Bochner-integrable functions $y\colon [0, \infty) \to E$, i.e. integrable on every compact subinterval of $[0, \infty)$.

Denote by $\mathcal{P}(E) = \{Y \subset E : Y \neq \emptyset\}$, $\mathcal{P}_{\mathrm{cl}}(E) = \{Y \in \mathcal{P}(E) : Y \text{ closed}\}$, $\mathcal{P}_{\mathrm{b}}(E) = \{Y \in \mathcal{P}(E) : Y \text{ bounded}\}$, $\mathcal{P}_{\mathrm{cv}}(E) = \{Y \in \mathcal{P}(E) : Y \text{ convex}\}$, and $\mathcal{P}_{\mathrm{cp}}(E) = \{Y \in \mathcal{P}(E) : Y \text{ compact}\}$.

Let X, Y two topological spaces, $F\colon X \to \mathcal{P}(Y)$ be a multi-map, and $D \subseteq Y$ be a set. The small preimage $F_+^{-1}(D)$ and the complete preimage $F_-^{-1}(D)$ of a set D are, respectively, the sets

$$F_+^{-1}(D) = \{x \in X : F(x) \subset D\} \quad \text{and} \quad F_-^{-1}(D) = \{x \in X : F(x) \cap V \neq \emptyset\}.$$

Let (X,d) and (Y,ρ) be two metric spaces and $F\colon X \to \mathcal{P}_{\mathrm{cl}}(Y)$ be a multi-map. A single-valued map $f\colon X \to Y$ is said to be a selection of G and we write $f \subset F$ whenever $f(x) \in F(x)$ for every $x \in X$.

Definition 2.1. A multi-map $F\colon X \to \mathcal{P}(Y)$ is said to be upper semi-continuous at the point $x_0 \in X$, if, for every open $W \subseteq Y$ such that $F(x_0) \subset W$, there exists a neighborhood $V(x_0)$ of x_0 such that $F(V(x_0)) \subset W$.

A multi-map is called *upper semi-continuous* (*u.s.c.* for short) on X if for each $x \in X$ it is *u.s.c.* at x.

Theorem 2.2 ([86])**.** *The following conditions are equivalent:*

(a) *the multi-map F is u.s.c.*
(b) *the set $F_+^{-1}(W)$ is open for every open set $W \subseteq Y$;*
(c) *the set $F_-^{-1}(Q)$ is closed for every closed $Q \subseteq Y$.*

Definition 2.3. A multi-map $F\colon X \to \mathcal{P}(Y)$ is said to be lower-continuous at the point $x_0 \in X$, if, for every open $W \subseteq Y$ such that $F(x_0) \cap W \neq \emptyset$, there exists a neighborhood $V(x_0)$ of x_0 with property that $F(x) \cap W \neq \emptyset$ for all $x \in V(x_0)$.

A multi-map is called *lower semi-continuous* (*l.s.c.* for short) provided that is lower semi-continuous at every point $x \in X$.

Theorem 2.4 ([86])**.** *The following conditions are equivalent:*

(a) *the multi-map F is l.s.c.*
(b) *the set $F_-^{-1}(W)$ is open for every open subset $W \subseteq Y$;*
(c) *the set $F_+^{-1}(Q)$ is closed for every closed subset $Q \subseteq Y$.*

The following two results are easily deduced from the limit set properties.

Lemma 2.5 (See, e.g. [13, Theorem 1.4.13])**.** *Let X, Y be two metric spaces. If $F\colon X \to \mathcal{P}_{\mathrm{cp}}(Y)$ is u.s.c., then for any $x_0 \in X$,*

$$\limsup_{x\to x_0} F(x) = F(x_0),$$

where

$$\limsup_{x\to x_0} F(x) = \Big\{ y \in Y : \liminf_{x\to x_0} d(y, F(x)) = 0 \Big\}.$$

Lemma 2.6 (See, e.g. [13, Lemma 1.1.9])**.** *Let $(K_n)_{n\in\mathbb{N}} \subset K \subset X$ be a sequence of subsets where K is compact in a separable Banach space X. Then*

$$\overline{\mathrm{co}}\Big(\limsup_{n\to\infty} K_n \Big) = \bigcap_{N>0} \overline{\mathrm{co}}\Big(\bigcup_{n\geq N} K_n \Big),$$

where $\overline{\mathrm{co}}\, A$ *refers to the closure of the convex hull of* A.

F is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $F(A)$ is relatively compact, i.e. there exists a relatively compact set $K \subset X$ such that $F(A) = \bigcup\{G(x),\ x \in A\} \subset K$. F is compact if $F(X)$ is relatively compact. It is called locally compact if, for each $x \in X$, there exists $U \in \mathcal{V}(x)$ such that $F(U)$ is relatively compact, where $\mathcal{V}(x)$ is a neighbourhood of x. F is quasicompact if, for every compact $A \subset X$, $F(A)$ is relatively compact.

Finally, for a multi-map $F\colon J \times E \to \mathcal{P}(E)$, denote

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,x)\}.$$

Consider the Hausdorff pseudo-metric distance

$$H_d\colon \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}^+ \cup \{\infty\}$$

defined by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(E), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space (see [96]).

Definition 2.7. A multivalued operator $N\colon E \to \mathcal{P}_{\mathrm{cl}}(E)$ is called

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad \text{for each } x, y \in E,$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

2.1.2. Closed graphs. Let X, Y be two Banach spaces and denote the graph of F by $\mathcal{G}r(F) = \{(x,y) \in X \times Y,\ y \in F(x)\}$.

Definition 2.8. G is closed if $\mathcal{G}r(F)$ is a closed subset of $X \times Y$, i.e. for every sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$, if $x_n \to x_*$, $y_n \to y_*$ as $n \to \infty$ with $y_n \in F(x_n)$, then $y_* \in F(x_*)$.

We recall some important results in connection with closed graphs.

Lemma 2.9 (See [40, Proposition 1.2])**.** *If* $F\colon X \to \mathcal{P}_{\mathrm{cl}}(Y)$ *is u.s.c., then* $\mathcal{G}r(F)$ *is a closed subset of* $X \times Y$. *Conversely, if* F *is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Lemma 2.10 ([104]). *Given a Banach space X, let $F : [0,b] \times X \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ be an L^1-Carathéodory multi-map such that for each $y \in C([0,b],X)$, $S_{F,y} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^1([0,b],X)$ into $C([0,b],X)$. Then the operator*

$$\begin{aligned}\Gamma \circ S_F \colon C([0,b],X) &\to \mathcal{P}_{\mathrm{cp,cv}}(C([0,b],X)),\\ y &\mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})\end{aligned}$$

has a closed graph in $C([0,b],X) \times C([0,b],X)$.

2.1.3. Measurable selections.

Definition 2.11. $F\colon J \times X \to \mathcal{P}(Y)$ is said:

(a) integrable if it has a summable selection $f \in L^1(J,X)$,
(b) integrably bounded, if there exists $q \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,z)\|_{\mathcal{P}} \leq q(t), \quad \text{for a.e. } t \in J \text{ and every } z \in X.$$

Definition 2.12. A multi-map F is called a Carathéodory function if

(a) the multi-map $t \mapsto F(t,x)$ is measurable for each $x \in X$;
(b) for a.e. $t \in J$, the map $x \mapsto F(t,x)$ is upper semi-continuous.

Furthermore, F is L^1-Carathéodory if it is further locally integrably bounded, i.e. for each positive r, there exists $h_r \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} \leq h_r(t), \quad \text{for a.e. } t \in J \text{ and all } |x| \leq r.$$

Definition 2.13. A multi-map $F\colon J \to \mathcal{P}(Y)$ is said measurable provided for every open $U \subset Y$, the set $F_+^{-1}(U)$ is Lebesgue measurable.

Lemma 2.14 ([33], [61]). *The mapping $F\colon J \to \mathcal{P}_{\mathrm{cl}}(Y)$ is measurable if and only if for each $x \in Y$, the function $\zeta\colon J \to [0,+\infty)$ defined by*

$$\zeta(t) = \mathrm{dist}(x, F(t)) = \inf\{\|x-y\| : y \in F(t)\}, \quad t \in J,$$

is Lebesgue measurable.

The following two lemmas are needed in this paper. The first one is the celebrated Kuratowski–Ryll–Nardzewski selection theorem.

Lemma 2.15 (See [61, Theorem 19.7]). *Let Y be a separable metric space and $F\colon J \to \mathcal{P}(Y)$ a measurable multi-map with nonempty closed values. Then F has a measurable selection.*

Lemma 2.16 (See [148, Lemma 3.2]). *Let $F\colon J \to \mathcal{P}(Y)$ be a measurable multi-map and $u\colon J \to Y$ a measurable function. Then for any measurable function $v\colon J \to (0, +\infty)$, there exists a measurable selection f_v of F such that, for almost every $t \in J$,*

$$|u(t) - f_v(t)| \leq d(u(t), F(t)) + v(t).$$

As a simple consequence, we have:

Corollary 2.17. *Let $F\colon J \to \mathcal{P}_{\mathrm{cp}}(Y)$ be a measurable multi-map and $u\colon J \to Y$ a measurable function. Then there exists a measurable selection f of F such that for almost every $t \in J$,*

$$|u(t) - f(t)| \leq d(u(t), F(t)).$$

For each $x \in C(J, E)$, the set

$$S_{F,x} = \{f \in L^1(J, E) : f(t) \in F(t, x(t)), \text{ for a.e. } t \in J\}$$

is known as the set of selection functions of the composition $F \circ x$.

Remark 2.18. (a) For each $x \in C(J, E)$, the set $S_{F,x}$ is closed whenever F has closed values. It is convex if and only if $F(t, x(t))$ is convex for almost eevery $t \in J$.

(b) From Theorem 5.10 in [147] (see also [104], [124] when E is finite-dimensional), we know that $S_{F,x}$ is nonempty if and only if the mapping $t \mapsto \inf\{|v| : v \in F(t, x(t))\}$ belongs to $L^1(J)$. It is bounded if and only if the mapping $t \mapsto \|F(t, x(t))\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t, x(t))\}$ belongs to $L^1(J)$; this particularly holds true when F is L^1-Carathéodory.

(c) For the sake of completeness, we refer also to Theorem 1.3.5 in [93] which states that $S_{F,x}$ contains a measurable selection whenever x is measurable and F is a Carathéodory function.

2.1.4. Decomposability and continuous selections. First, recall the well-known Michael's selection theorem.

Theorem 2.19. *Let X be a metric space, Y a Banach space and $F\colon X \to \mathcal{P}(Y)$ a l.s.c. multi-map with closed convex values. Then there exists $f\colon X \to Y$, a continuous selection of F.*

Now, we present a corresponding selection theorem for a class of multi-maps with decomposable values. Let X, Y be Banach spaces and $\mathcal{A}$ a family of subsets of X.

Definition 2.20. $\mathcal{A}$ is called $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$ where I is Lebesgue measurable in J and D is Borel measurable in X.

Definition 2.21. A subset $A \subset L^p(J,Y)$ $(p \geq 1)$ is decomposable if for all $u, v \in A$ and for every Lebesgue measurable set $I \subset J$, we have $u.\chi_I + v.\chi_{J\setminus I} \in A$, where χ_I stands for the characteristic function of the set I.

Let $F\colon J \times X \to \mathcal{P}_{\mathrm{cl}}(Y)$ be a multi-map. Assign to F the multivalued operator $\mathcal{F}\colon C(J,Y) \to \mathcal{P}(L^1(J,Y))$ defined by $\mathcal{F}(x) = S_{F,x}$. The operator $\mathcal{F}$ is called the Nemyts'kiĭ operator associated to F.

Definition 2.22. Let $F\colon J \times X \to \mathcal{P}_{\mathrm{cp}}(Y)$ be a multi-map. We say that F is of lower semi-continuous type (*l.s.c.* type) if its associated Nemyts'kiĭ operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Along with this definition, the following lemma is very useful.

Lemma 2.23 (See, e.g. [51])**.** *Let $F\colon J \times X \to \mathcal{P}_{\mathrm{cp}}(Y)$ be an integrably bounded multi-map satisfying*

($\mathcal{H}_{\mathrm{lsc}}$) *$F\colon J \times X \to \mathcal{P}(Y)$ is a nonempty compact valued multi-map such that*
(a) *the mapping $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;*
(b) *the mapping $x \mapsto F(t,x)$ is l.s.c. for almost every $t \in J$.*

Then F is of lower semi-continuous type.

Finally, we state a selection theorem due to Bressan, Colombo and Fryszkowski.

Lemma 2.24 (See [24], [53], [54])**.** *Let X be a separable metric space and let Y be a Banach space. Then every l.s.c. multivalued operator $N\colon X \to \mathcal{P}_{\mathrm{cl}}(L^1(J,Y))$ with closed decomposable values has a continuous selection.*

2.1.5. σ-selectionability. The following definitions and the result can be found in [61], [76] (see also [12, p. 86]). Let (X,d) and (Y,d') be two metric spaces.

Definition 2.25. We say that a map $F\colon X \to \mathcal{P}(Y)$ is σ-Ca-selectionable if there exists a decreasing sequence of compact valued *u.s.c.* maps $F_n\colon X \to \mathcal{P}(Y)$ satisfying:

(a) F_n has a Carathéodory selection, for all $n \geq 0$ (F_n are called *Ca*-selectionable),
(b) $F(x) = \bigcap_{n\geq 0} F_n(x)$, for all $x \in X$.

Definition 2.26. A single-valued map $f\colon J \times X \to Y$ is said to be measurable-locally-Lipschitz (mLL) if $f(\,\cdot\,,x)$ is measurable for every $x \in X$ and for every $x \in X$, there exists a neighborhood V_x of $x \in X$ and an integrable function $L_x\colon J \to [0,\infty)$ such that

$$d'(f(t,x_1), f(t,x_2)) \leq L_x(t) d(x_1,x_2), \quad \text{for every } t \in J \text{ and } x_1, x_2 \in V_x.$$

Definition 2.27. A multi-map $F\colon J \times X \to \mathcal{P}(Y)$ is mLL-selectionable if it has an mLL-selection.

Definition 2.28. We say that a multi-map $\phi\colon J \times E \to \mathcal{P}(E)$ with closed values is upper–Scorza–Dragoni if, given $\delta > 0$, there exists a closed subset $A_\delta \subset J$ such that the measure $\mu(J \setminus A_\delta) \leq \delta$ and the restriction ϕ_δ of ϕ to $A_\delta \times E$ is *u.s.c.*

Theorem 2.29 (See [61, Theorem 19.19])**.** *Let E, E_1 be two separable Banach spaces and let $F\colon J \times E \to \mathcal{P}_{\mathrm{cp,cv}}(E_1)$ be an upper-Scorza-Dragoni map. Then F is σ-*Ca*-selectionable, the maps $F_n\colon J \times E \to \mathcal{P}(E_1)$ $(n \in \mathbb{N})$ are almost upper semicontinuous and we have*

$$F_n(t,x) \subset \overline{\mathrm{co}}\Big(\bigcup_{x\in E} F_n(t,x)\Big),$$

where $\mathrm{co}\, A$ *stands for the convex hull of the set A. Moreover, if F is integrably bounded, then F is σ-*mLL*-selectionable.*

For further readings and details on multivalued analysis, we refer the reader to the books by Andres and Górniewicz [7], Aubin and Celina [12], Aubin and Frankowska [13], Deimling [40], Górniewicz [61], Hu and Papageorgiou [86], [87], Kamenskiĭ *et al.* [93], Kisielewicz [96], Smirnov [128], and Tolstonogov [134].

2.2. Background in algebraic topology

2.2.1. Basic notions. We recall some fundamental notions from algebraic topology. For details, we recommend [22], [71], [63], [62], [64], [60], [105]. In what follows (X,d) and (Y,d') stand for two metric spaces.

Definition 2.30. A set $A \subset X$ is called a contractible space provided there exists a continuous homotopy $h\colon A \times [0,1] \to A$ and $x_0 \in A$ such that:

(a) $h(x,0) = x$, for every $x \in A$,
(b) $h(x,1) = x_0$, for every $x \in A$,

i.e. if the identity map $A \to A$ is homotopic to a constant map (A is homotopically equivalent to a one-point space).

Note that any closed convex subset of X is contractible.

Definition 2.31. A compact nonempty metric space X is called an R_δ-set provided there exists a decreasing sequence of compact nonempty contractible metric spaces $(X_n)_{n\in\mathbb{N}^*}$ such that $X = \bigcap_{n=1}^{\infty} X_n$.

In particular, every compact contractible space is R_δ.

Definition 2.32. A subset $A \subset X$ is called a retract of X if there exists a continuous mapping $r\colon X \to A$ such that $r(x) = x$, for all $x \in A$.

Every retract is closed. The unit ball $A = \overline{B}(0,1)$ is a retract of the whole Banach space X through the radial retraction:

$$r(x) = \begin{cases} x & \text{if } x \in \overline{B}(0,1), \\ x/\|x\| & \text{otherwise.} \end{cases}$$

More generally, every closed, convex subset of a normed vector space is a retract. This follows from Dugundgi's extension theorem [71]:

Theorem 2.33. *Let X be a metric space, $A \subset X$ a closed subset, Y a normed vector space, and $f\colon A \to Y$ a continuous function. Then f has a continuous extension $\widetilde{f}$ over X such that $\widetilde{f}(X) \subset \operatorname{co}(f(A))$.*

The following notion, strictly connected with extendability, was first introduced by K. Borsuk [23].

Definition 2.34. A space X is called an absolute retract (in short $X \in \mathrm{AR}$) provided that for every space Y, every closed subset $B \subseteq Y$ and any continuous map $f\colon B \to X$, there exists a continuous extension $\widetilde{f}\colon Y \to X$ of f over Y, i.e. $\widetilde{f}(x) = f(x)$ for every $x \in B$. In other words, for every space Y and for any embedding $f\colon X \to Y$, the set $f(X)$ is a retract of Y. If the set $f(X)$ is a retract of U for every open neighborhood U of B in Y we say that $X \in \mathrm{ANR}$ and call X to be an absolute neighborhood retract.

By embedding, we mean any homeomorphism $h\colon X \to Y$ such that $h(X)$ is a closed subset of Y. Another characterization of AR spaces is given by:

Theorem 2.35. $X \in \mathrm{AR}$ *if and only if X is homeomorphic to a retract of a convex subset of a normed vector space.*

This follows from Arens–Eells embedding theorem which states that every metric space can be embedded isometrically in a closed subset of a normed vector space.

Since any two continuous function over an AR space are homotopic, we deduce that if $X \in \mathrm{AR}$, then X is contractible hence arcwise connected. Also, we have:

Lemma 2.36. *Let $X \in \mathrm{AR}$ and $\widetilde{X}$ be such that $X \simeq \widetilde{X}$ (i.e. X homeomorphic to $\widetilde{X}$). Then $\widetilde{X} \in \mathrm{AR}$.*

Proof. Let Y be a space and $B \subset Y$ a closed subset. Let $f\colon B \to \widetilde{X}$ be a continuous map and $\varphi\colon \widetilde{X} \to X$ be a homeomorphism. Since $X \in \mathrm{AR}$ there exists $\widetilde{g}\colon Y \to X$ which is a continuous extension of $\varphi \circ f$. Then $\widetilde{f} = \varphi^{-1} \circ \widetilde{g}\colon Y \to \widetilde{X}$ is a continuous extension of f. $\square$

Remark 2.37. Owing to Hyman [91], a compact, metric space is R_δ if it can be expressed as the intersection of compact, absolute retracts.

Example 2.38. Decomposable subsets of $L^1([0,b],E)$ are contractible when E is a Banach space. Moreover, any closed decomposable subset of $L^1([0,b],E)$ is AR (see [61, Remark (21.6)]).

Definition 3.39. A space A is closed acyclic if

(a) $H_0(A) = \mathbb{Q}$,
(b) $H_n(A) = 0$, for every $n > 0$,

where $H_* = \{H_n\}_{n\geq 0}$ is the Čech-homology functor with compact carriers and coefficients in the field of rationals $\mathbb{Q}$. In other words, a space A is acyclic if the map $j\colon \{p\} \to X$, $j(p) = x_0 \in A$, induces an isomorphism $j_*\colon H_*(\{p\}) \to H_*(A)$.

Remark 2.40. (a) From the continuity of Čech-homology functors, if X is a contractible compact space, then it is acyclic.

(b) An $R_\delta-$space is a compact, connected space acyclic with respect to the Čech-homology functor; it has the same homology as a one-point space.

Definition 2.41. A *u.s.c.* map $F\colon X \to \mathcal{P}(Y)$ is called acyclic if for each $x \in X$, $F(x)$ is compact, acyclic.

The next definitions were introduced in [64].

Definition 2.42. A metric space X is called acyclically contractible if there exists an acyclic homotopy $\Pi\colon X \times [0,1] \to \mathcal{P}(X)$ such that:

(a) $x \in \Pi(x,0)$, for every $x \in X$.
(b) $x_0 \in \Pi(x,1)$, for every $x \in X$ and for some $x_0 \in X$.

Remark 2.43. Clearly, any contractible space is acyclically contractible. In addition, any acyclic, compact metric space is acyclically contractible. Indeed, it is enough to put $\Pi(x,\alpha) = X$, for each $x \in X$ and $\alpha \in [0,1]$.

Definition 2.44. A metric space X is called R_δ-contractible if there exists a multivalued homotopy $\Pi\colon X \times [0,1] \to \mathcal{P}(X)$ which is *u.s.c.* and satisfies:

(a) $x \in \Pi(x,0)$, for every $x \in X$,
(b) $\Pi(x,1) = B$ for every $x \in X$ and for some R_δ subset $B \subset X$,
(c) $\Pi(x,\alpha)$ is an R_δ-set, for every $\alpha \in [0,1]$ and $x \in X$.

Remark 2.45. An R_δ-contractible space has the same homology as one-point space $\{p\}$; so it is acyclic, hence connected.

Collecting the above results, we deduce that, for any *compact* space, we have the inclusions:

$$\text{convex} \subset \text{AR} \subset \text{contractible} \subset R_\delta$$
$$\subset \text{acyclic} \subset \text{acyclically contractible connected.}$$

and

$$\text{convex} \subset \text{AR} \subset \text{contractible}$$
$$\subset R_\delta - \text{contractible} \subset \text{acyclic} \subset \text{acyclically contractible}.$$

2.2.2. Aronszajn–Browder–Gupta result. The following result is a fundamental result proved by Aronszajn in 1942 [9] and later improved by Browder and Gupta in 1969 [27].

Theorem 2.46. *Let X be a metric space, $(E, \|\cdot\|)$ be a Banach space and $f\colon X \to E$ be a proper map i.e. f is continuous and for every compact $K \subset E$, the set $f^{-1}(K)$ is compact. Assume further that, for each $\varepsilon > 0$, a continuous map $f_\varepsilon\colon X \to E$ is given and the following two conditions are satisfied:*

(a) *$\|f_\varepsilon(x) - f(x)\| < \varepsilon$, for every $x \in X$,*
(b) *for every $\varepsilon > 0$ and $u \in E$ such that $\|u\| \le \varepsilon$, the equation $f_\varepsilon(x) = u$ has exactly one solution x.*

Then the set $S = f^{-1}(\{0\})$ is an R_δ-set.

A useful tool to get approximate functions which satisfy condition (b) is given by the following result (see [60, Theorem 3.1], [132], or [146]) that turns out to be more suitable for Volterra integral equations.

Theorem 2.47. *Let $E = C([0,a], \mathbb{R}^m)$ be the Banach space of continuous maps with the usual max-norm and let $X = \overline{B(0,r)}$ be the closed ball in E. Let $F\colon X \to E$ be a compact map and $f\colon X \to E$ the associated compact vector field (i.e. $f(u) = u - F(u)$) such that*

(a) *there exists $x_0 \in \mathbb{R}^m$ such that for all $u \in B(0,r)$, $F(u)(0) = x_0$.*
(b) *For every $0 < \varepsilon \le a$ and every $u, v \in X$, if $u(t) = v(t)$ for each $t \in [0,\varepsilon]$, then $F(u)(t) = F(v)(t)$ for each $t \in [0,\varepsilon]$.*

Then there exists a sequence $f_n\colon X \to E$ of continuous proper mappings satisfying conditions (a)–(b) *in Theorem* 2.46.

2.2.3. Limits of inverse systems. Let us recall that an inverse system of Hausdorff topological spaces is a family $S = (X_\alpha, \pi_\alpha^\beta, J)$, where J is a poset directed by the relation $\le$, X_α is a Hausdorff topological space, for every $\alpha \in J$, and $\pi_\alpha^\beta\colon X_\beta \to X_\alpha$ is a continuous mapping, for each two elements $\alpha, \beta \in J$, such that $\alpha \le \beta$. Moreover, for each $\alpha \le \beta \le \gamma$, π_α^β satisfies $\pi_\alpha^\alpha = \mathrm{id}_{X_\alpha}$ and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$. By $\pi_\alpha^\beta \pi_\beta^\gamma$, it is meant the composite $\pi_\alpha^\beta \circ \pi_\beta^\gamma$. The following subset of the product $\prod_{\alpha \in J} X_\alpha$

$$\varprojlim S = \left\{ (x_\alpha) \in \prod_{\alpha \in J} X_\alpha : \pi_\alpha^\beta(x_\beta) = x_\alpha, \text{ for all } \alpha \le \beta \right\}$$

is called a limit (or projective limit) of the inverse system S. The inverse limit of the corresponding inverse system is just the product. The limit projective $\varprojlim S$ is also called the generalized intersection $\bigcap_{\alpha\in J} X_\alpha$ (see e.g. [48] or [94, p. 439]). An element of $\varprojlim S$ is called thread or fibre of the system S. One can see that if we denote by $\pi_\alpha\colon \varprojlim S \to X_\alpha$ a restriction of the projection $p_\alpha\colon \prod_{\alpha\in J} X_\alpha \to X_\alpha$ onto the αth axis, then we obtain that $\pi_\alpha^\beta \pi_\beta = \pi_\alpha$, for each $\alpha \le \beta$. Let us give an important example of inverse systems.

Example 2.48. For every $m \in \mathbb{N}$, let $C_m = C([0,m],\mathbb{R}^n)$ be the Banach space of all continuous functions on the closed interval $[0,m]$ into $\mathbb{R}^n$ and $C = C([0,\infty),\mathbb{R}^n)$ the Fréchet space of continuous functions. For $p \ge m$, consider the restriction maps $\pi_m^p\colon C_p \to C_m$ defined by $\pi_m^p(x) = x|_{[0,m]}$. It is easy to see that C is isometrically homeomorphic to the limit of the inverse system $\{C_m, \pi_m^p, \mathbb{N}\}$. The maps defined by $\pi_m\colon C \to C_m$, $\phi_m(x) = x|_{[0,m]}$ correspond to suitable projections.

Some useful properties of limits of inverse systems are summarized in the following

Proposition 2.49. *Let $S = \{X_\alpha, \pi_\alpha^\beta, J\}$ be an inverse system.*

(1) *The limit $\varprojlim S$ is a closed subset of $\Pi_\alpha X_\alpha$.*
(2) *If, for every $\alpha \in J$, X_α is*
   (a) *compact, then $\varprojlim S$ is compact,*
   (b) *compact and nonempty, then $\varprojlim S$ is compact and nonempty,*
   (c) *compact and acyclic and $\varprojlim S$ is nonempty, then $\varprojlim S$ is compact and acyclic,*
   (d) *metrizable and J is countable, then $\varprojlim S$ is metrizable.*

Part (c) is due to Gabor [55]. Proofs of Proposition 2.49 can be found in [4], [5], [7], [55]. In case J is countable, we have (see [55, Proposition 3.2]):

Proposition 2.50. *Let $S = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system such that X_n is an R_δ-set. Then $\varprojlim S$ is an R_δ set.*

Next, we introduce the notation of multi-maps of inverse systems. Suppose that two systems $S = \{X_\alpha, \pi_\alpha^\beta, J\}$ and $S' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, J'\}$ are given.

Definition 2.51. By a multi-map from the system S into the system S', we mean a family $\{\sigma, \varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma\colon J' \to J$, that is $\sigma(\alpha') \le \sigma(\beta')$, for $\alpha' \le \beta'$, and of multi-maps $\varphi_{\sigma(\alpha')}\colon X_{\sigma(\alpha')} \to \mathcal{P}(Y_{\alpha'})$ with nonempty values, defined for every $\alpha' \in J'$ and such that for each $\alpha' \le \beta'$

$$\pi_{\alpha'}^{\beta'} \varphi_{\sigma(\beta')} = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')}^{\sigma(\beta')}.$$

A map of systems $\{\sigma, \varphi_{\sigma(\alpha')}\}$ induces a limit map $\varphi\colon \varprojlim S \to \mathcal{P}(\varprojlim S')$ defined by

$$\varphi(x) = \prod_{\alpha' \in J'} \varphi_{\sigma(\alpha')}(x_{\sigma(\alpha')}) \cap \varprojlim S'.$$

In other words, a limit map is a map such that $\pi_{\alpha'}\varphi = \varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')}$, for every $\alpha' \in J$. In terms of countable inverse systems, $\varphi((x_n)) = \prod_{n=1}^{\infty} \varphi_n(x_n) \cap \varprojlim S'$.

Since a topology of the limit of an inverse system is the one generated by the base consisting of all sets of the form $\pi_\alpha(U_\alpha)$, where α runs over an arbitrary set cofinal in J ($K \subseteq J$ is called cofinal if for every $\alpha \in J$ there exists $\beta \in K$ such that $\beta \preceq \alpha$) and U_α are open subsets of the space X_α, it is easy to prove the following continuity property for limit maps.

Proposition 2.52 (See [5, Proposition 2.7])**.** *Let $S = \{X_\alpha, \pi_\alpha^\beta, J\}$ and $S' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, J'\}$ be two inverse systems and let $\varphi\colon \varprojlim S \to \mathcal{P}(\varprojlim S')$ be a limit map induced by the map $\{\sigma, \varphi_{\sigma(\alpha')}\}$. If, for every $\alpha' \in J'$, $\varphi_{\sigma(\alpha')}$ is*

(a) *u.s.c. with compact values, then φ is u.s.c.*
(b) *l.s.c., then φ is l.s.c.*
(c) *continuous, then φ is continuous.*

Regarding the structure of the fixed point sets of limit maps, we have (see [5, Theorem 2.8]):

Proposition 2.53. *Let $S = \{X_\alpha, \pi_\alpha^\beta, J\}$ be an inverse system and $\varphi\colon \varprojlim S \to \mathcal{P}\big(\varprojlim S\big)$ be a limit map induced by the map $\{\mathrm{id}, \varphi_\alpha\}$ where $\varphi_\alpha\colon X_\alpha \to \mathcal{P}(X_\alpha)$. Then the fixed point set of φ is a limit of the inverse system generated by the set* $\mathrm{Fix}(\varphi_\alpha)$. *In particular if the fixed point set of φ_α is compact acyclic (resp. R_δ); then φ is compact acyclic (resp. R_δ) as well.*

2.3. Measure of noncompactness

2.3.1. Definitions. First, we give definition and main properties on a measure of noncompactness. For more details, we refer to [2], [133], [93] and some references therein.

Definition 2.54. Let E be a Banach space and $(\mathcal{A}, \leq)$ a partially ordered set. A map $\beta\colon \mathcal{P}(E) \to \mathcal{A}$ is called a measure of noncompactness on E, MNC for short, if $\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega)$ for every bounded $\Omega \in \mathcal{P}(E)$.

Notice that if D is dense in Ω, then $\overline{\mathrm{co}}\,\Omega = \overline{\mathrm{co}}D$ and hence $\beta(\Omega) = \beta(D)$.

Definition 2.55. A measure of noncompactness β is called:

(a) monotone if $\Omega_0, \Omega_1 \in \mathcal{P}(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$,
(b) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E$, $\Omega \in \mathcal{P}(E)$,

(c) invariant with respect to the union with compact sets if $\beta(K\cup\Omega) = \beta(\Omega)$ for every relatively compact set $K \subset E$, and $\Omega \in \mathcal{P}(E)$,
(d) real if $\mathcal{A} = \overline{\mathbb{R}}_+ = [0, \infty]$ and $\beta(\Omega) < \infty$ for every bounded Ω,
(e) semi-additive if $\beta(\Omega_0 \cup \Omega_1) = \max(\beta(\Omega_0), \beta(\Omega_1))$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(E)$,
(f) lower-additive if β is real and $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(E)$,
(g) regular if the condition $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As example of MNC, consider the Kuratowski measure defined by $\alpha(A) = \inf D$ where

$$D = \left\{\varepsilon > 0 : A = \bigcup_{i=1}^{n} \Omega_i,\ \operatorname{diam}(\Omega_i) \leq \varepsilon,\ \text{for all } i = 1, \dots, n\right\}.$$

The Hausdorff MNC is defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

Recall that a bounded set $A \subset E$ has a finite ε-net if there exit ε and a finite subset $S \subset E$ such that $A \subset S + \varepsilon\overline{B}$, where $\overline{B}$ is a closed unit ball in E.

Other examples of MNCs are given by the following measures of noncompactness defined on the space of continuous functions $C([0, b], E)$ with values in a Banach space E:

(i) the modulus of fiber noncompactness

$$\varphi(\Omega) = \sup_{t\in[0,b]} \chi_E(\Omega(t)),$$

where χ_E is the Hausdorff MNC in E and $\Omega(t) = \{y(t) : y \in \Omega\}$;
(ii) the modulus of equicontinuity

$$\operatorname{mod}_C(\Omega) = \lim_{\delta\to 0} \sup_{y\in\Omega} \max_{|\tau_1-\tau_2|\leq\delta} |y(\tau_1) - y(\tau_2)|.$$

It should be mentioned that these MNCs satisfy all above-mentioned properties except regularity.

2.3.2. Compactness criteria.

Definition 2.56. Let $\mathcal{M}$ be a closed subset of a Banach space E and $\beta\colon \mathcal{P}(E) \to (\mathcal{A}, \leq)$ an MNC on E. A multivalued map $\mathcal{F}\colon \mathcal{M} \to \mathcal{P}_{\mathrm{cp}}(E)$ is said to be β-condensing if for every bounded $\Omega \subset \mathcal{M}$, the relation

$$\beta(\mathcal{F}(\Omega)) \leq \beta(\Omega),$$

implies the relative compactness of Ω.

Some important results on fixed point theory with MNCs are recalled hereafter (see e.g. [93] for the proofs and further details). The first one is a compactness criterion.

Lemma 2.57 (See [93, Theoremn 5.1.1]). *Let $N\colon L^1([a,b],E) \to C([a,b],E)$ be an operator satisfying the following conditions:*

($\mathcal{S}_1$) *N is ξ-Lipschitz: there exists $\xi > 0$ such that for every $f, g \in L^1([a,b],E)$*

$$|Nf(t) - Ng(t)| \le \xi \int_a^b |f(s) - g(s)|\, ds, \quad \textit{for all } t \in [a,b].$$

($\mathcal{S}_2$) *N is weakly-strongly sequentially continuous on compact subsets: for any compact $K \subset E$ and any sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b],E)$ such that $\{f_n(t)\}_{n=1}^{\infty} \subset K$ for almost every $t \in [a,b]$, the weak convergence $f_n \rightharpoonup f_0$ implies the strong convergence $N(f_n) \to N(f_0)$ as $n \to +\infty$.*

Then for every semi-compact sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([0,b],E)$, the image sequence $N(\{f_n\}_{n=1}^{\infty})$ is relatively compact in $C([a,b],E)$.

Lemma 2.58 (See [93, Theorem 5.2.2]). *Let an operator $N\colon L^1([a,b],E) \to C([a,b],E)$ satisfy conditions ($\mathcal{S}_1$) and ($\mathcal{S}_2$) together with*

($\mathcal{S}_3$) *There exits $\eta \in L^1([a,b])$ such that for every integrably bounded sequence $\{f_n\}_{n=1}^{\infty}$, we have*

$$\chi(\{f_n(t)\}_{n=1}^{\infty}) \le \eta(t), \quad \textit{for a.e. } t \in [a,b],$$

where χ is the Hausdorff MNC.

Then

$$\chi(\{N(f_n)(t)\}_{n=1}^{\infty}) \le 2\xi \int_a^b \eta(s)\, ds, \quad \textit{for all } t \in [a,b],$$

where ξ is the constant in ($\mathcal{S}_1$).

2.3.3. Weak-compactness in L^1.

Definition 2.59. A sequence $\{v_n\}_{n\in\mathbb{N}} \subset L^1(J,E)$ is said to be semi-compact if

(a) it is integrably bounded, i.e. there exists $q \in L^1(J,\mathbb{R}^+)$ such that

$$|v_n(t)|_E \le q(t), \quad \text{for a.e. } t \in J \text{ and every } n \in \mathbb{N},$$

(b) the image sequence $\{v_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in E for almost every $t \in J$.

We recall two weak compactness criteria that follow from the Dunford–Pettis theorem (see [142]).

Lemma 2.60 (See [93, Proposition 4.2.1]). *Every semi-compact sequence in $L^1(J,E)$ is weakly compact in $L^1(J,E)$.*

Lemma 2.61 (See [119, Corollary 6.4.11]). *Let $A \subset L^1(\Omega, E)$ be a bounded decomposable set with Ω finite-measurable and E reflexive. Then A is weakly relatively compact in $L^1(\Omega, E)$*

The following Mazur's lemma is well known in functional analysis:

Lemma 2.62 (See [111, Theorem 21.4]). *Let E be a normed space and $\{x_n\}_{n\in\mathbb{N}^*} \subset E$ be a sequence weakly converging to some limit $x \in E$. Then there exists a sequence of convex combinations $y_n = \sum_{k=n}^{\infty} \alpha_{n,k} x_k$ which converges strongly to x. Here $(\alpha_{n,k})_{k,n\in\mathbb{N}^*}$ is a double sequence satisfying: for all $n \in \mathbb{N}^*$ there exists $k_0(n) \in \mathbb{N}^*$ such that $\alpha_{n,k} = 0$ for all $k \geq k_0(n)$ and $\sum_{k=n}^{\infty} \alpha_{n,k} = 1$, for all $n \in \mathbb{N}^*$.*

2.4. C_0-semigroups

In all this subsection, $B(E)$ refers to the Banach space of linear bounded operators from E into E with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

Definition 2.63. A semigroup is a one parameter family $\{T(t) : t \geq 0\} \subset B(E)$ satisfying the conditions:

(a) $T(t) \circ T(s) = T(t+s), \quad \text{for } t, s \geq 0,$
(b) $T(0) = I,$

where I denotes the identity operator in E.

Definition 2.64. A linear semigroup $\{T(t)\}_{t\geq 0}$ is uniformly continuous if

$$\lim_{t\to 0^+} \|T(t) - I\|_{B(E)} = 0,$$

that is if

$$\lim_{|t-s|\to 0} \|T(t) - T(s)\|_{B(E)} = 0.$$

Definition 2.65. We say that the semigroup $\{T(t)_{t\geq 0}\}$ is strongly continuous (or a C_0-semigroup) if the map $t \to T(t)(x)$ is strongly continuous, for each $x \in E$, i.e.

$$\lim_{t\to 0^+} T(t)x = T(0)x = x, \quad \text{for all } x \in E.$$

Definition 2.66. Let $T(t)$ be a C_0-semigroup defined on E. The infinitesimal generator $A \in B(E)$ of $T(t)$ is the linear operator defined by

$$A(x) = \lim_{t \to 0^+} \frac{T(t)(x) - T(0)x}{t}, \quad \text{for } x \in D(A),$$

where $D(A) = \{x \in E \colon \lim_{t \to 0^+} (T(t)(x) - x)/t \text{ exists in } E\}$.

Let $A \colon E \to E$ be a linear operator not necessarily bounded. The following properties are classical (see Engel and Nagel [47], Pazy [121], Hill and Philips [84]).

Proposition 2.67. *If A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t \geq 0}$, then $D(A)$, the domain of A, is dense in X and A is a closed linear operator.*

Proposition 2.68. *A linear operator $A \colon D(A) \subset E \to E$ is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator. In this case, the semi-group can be defined by $T(t) = e^{At}$, $t \geq 0$.*

Proposition 2.69.

(a) *If $\{T(t)\}_{t \geq 0}$ is a C_0-semigroup of bounded linear operators, then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\|_{B(E)} \leq M e^{\omega t}, \quad \textit{for } t \geq 0.$$

(b) *Let $\{T(t)\}_{t \geq 0}$ be a uniformly continuous semigroup of bounded linear operators. Then there exists a constant $\omega \geq 0$ such that*

$$\|T(t)\|_{B(E)} \leq e^{\omega t}, \quad \textit{for } t \geq 0.$$

Definition 2.70. $\{T(t)\}_{t > 0}$ is said to be a compact if the $T(t)$ is a compact operator for every $t > 0$.

Proposition 2.71. *If $\{T(t)\}_{t > 0}$ is a compact C_0-semigroup then it is uniformly continuous, for $t > 0$.*

Notice however that if $\{T(t)\}_{t \geq 0}$ is a compact C_0-semigroup, then E is a finite-dimensional space.

2.5. Fixed point theorems

We recall some classical fixed point theorems we need to prove our existence results. The first one is the so-called nonlinear alternative of Leray and Schauder (see [71], [61]).

Lemma 2.72. *Let $(X, |\cdot|)$ be a normed space and $F\colon X \to \mathcal{P}_{\mathrm{cl,cv}}(X)$ a compact, u.s.c. multi-map. Then either one of the following conditions holds:*

(a) *F has at least one fixed point,*
(b) *the set $\mathcal{M} := \{x \in X,\ x \in \lambda F(x),\ \lambda \in (0,1)\}$ is unbounded.*

The single-valued version may be stated as follows:

Lemma 2.73. *Let X be a Banach space and $C \subset X$ a nonempty bounded, closed, convex subset. Assume U is an open subset of C with $0 \in U$ and let $G\colon \overline{U} \to C$ be a a continuous compact map. Then*

(a) *either there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda G(u)$,*
(b) *or G has a fixed point in $\overline{U}$.*

For contraction multi-maps, Covitz and Nadler proved the following

Lemma 2.74 (see [37], [33])**.** *Let (X, d) be a complete metric space. If $N\colon X \to \mathcal{P}_{\mathrm{cl}}(X)$ is a contraction, then* $\mathrm{Fix}\,(N) \neq \emptyset$.

The next result is concerned with β-condensing *u.s.c.* multi-maps.

Lemma 2.75 (See [93])**.** *Let $V \subset E$ be a bounded open neighborhood of zero and $N\colon \overline{V} \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ a β-condensing u.s.c. multi-map, where β is a nonsingular measure of noncompactness defined on subsets of E. If N satisfies the boundary condition*

$$x \notin \lambda N(x) \quad \textit{for all } x \in \partial V \textit{ and } 0 < \lambda < 1,$$

then the set $\mathrm{Fix}\,(N) = \{x \in V : x \in N(x)\}$ *is nonempty.*

Lemma 2.76. *Let W be a closed bounded convex subset of a Banach space E and $\mathcal{F}\colon W \to \mathcal{P}_{\mathrm{cp}}(W)$ be a closed β-condensing multi-map where β is a monotone* MNC *on E. Then* $\mathrm{Fix}\,(\mathcal{F})$ *is nonempty and compact.*

Finally, let E be a Fréchet space with the topology generated by a family of semi-norms $\|\cdot\|_n$ and corresponding distances $d_n(x,y) = \|x-y\|_n$ $(n \in \mathbb{N})$. Recall

Definition 2.78. A multi-map $F\colon E \to \mathcal{P}(E)$ is called an admissible contraction with constant $\{k_n\}_{n\in\mathbb{N}}$ if for each $n \in \mathbb{N}$, there exists $k_n \in (0,1)$ such that

(a) $H_{d_n}(F(x), F(y)) \leq k_n \|x-y\|_n$ for all $x, y \in E$, where H_d is the Hausdorff distance,
(b) for every $x \in E$ and every $\varepsilon > 0$, there exists $y \in F(x)$ such that

$$\|x-y\|_n \leq d_n(x, F(x)) + \varepsilon, \quad \text{for every } n \in \mathbb{N}.$$

A subset $A \subset E$ is bounded if for every $n \in \mathbb{N}$, there exists $M_n > 0$ such that $|x|_n \leq M_n$, for every $x \in A$. The following nonlinear alternative for multivalued contractions is brought from [50]:

Lemma 2.79. *Let E be a Fréchet space, $U \subset E$ an open neighborhood of the origin, and let $N\colon \overline{U} \to \mathcal{P}(E)$ be a bounded admissible multivalued contraction. Then either one of the following statements holds:*

(a) *N has a fixed point.*
(b) *There exists $\lambda \in [0,1)$ and $x \in \partial U$ such that $x \in \lambda N(x)$.*

Finally recall the Schauder–Tikhonov fixed point theorem:

Lemma 2.80 ([7])**.** *Let E be a locally convex space, C a convex closed subset of E and $N\colon C \to C$ is a continuous, compact map. Then N has at least one fixed point.*

CHAPTER 3

IMPULSIVE DIFFERENTIAL INCLUSIONS

Let $(E, |\cdot|)$ be a Banach space. Throughout this section, we consider the following impulsive problem for first-order semi-linear differential inclusions:

$$\begin{cases} (y' - Ay)(t) \in F(t, y(t)), & \text{a.e. } t \in J', \\ \Delta y_{t=t_k} = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = a, \end{cases} \tag{3.1}$$

where $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $J = [0, b]$, and $J' = J \setminus \{t_1, \dots, t_m\}$, $F\colon J \times E \to \mathcal{P}(E)$ is a multifunction, and $a \in E$. The operator A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t \geq 0}$ on a separable Banach space $(E, |\cdot|)$ (see Chapter 2), $I_k \in C(E, E)$ $(k = 1, \dots, m)$, and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively. We shall be mainly concerned with some existence results and structure of solution sets for problem (3.1). This is presented and developed in two subsections. In Section 3.1, we discuss some results of the existence of solutions for problem (3.1) and some properties of operators solutions. Section 3.2 is devoted to proving some geometric properties of solution sets such that acyclicity, AR, R_δ, and contractibility. Let $J_k = (t_k, t_{k+1})$, $k = 0, \dots, m$, and let y_k be the restriction of a function y to J_k. In order to define mild solutions for problem (3.1), consider the space

$$\begin{aligned} \mathrm{PC} = \{y\colon [0, b] \to E,\ y_k \in C(J_k, E),\ k = 0, \dots, m, \text{ such that} \\ y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k^-) = y(t_k) \text{ for } k = 1, \dots, m\}. \end{aligned}$$

Endowed with the norm

$$\|y\|_{\mathrm{PC}} = \max\{\|y_k\|_\infty,\ k = 0, \dots, m\},$$

PC is a Banach space where $y_k = y_{|J_k}$. Throughout this paper, the constants $M > 0$ and ω are as introduced in Proposition 2.67. A fundamental notation of solutions of problem (3.1) is given by

Definition 3.1. A function $y \in \mathrm{PC}$ is said to be a mild solution of problem (3.1) if there exists $v \in L^1(J, E)$ such that $v(t) \in F(t, y(t))$ almost eeverywhere on J, and

$$y(t) = T(t)a + \int_0^t T(t-s)v(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)).$$

3.1. Existence results

3.1.1. The convex case. Let $F\colon J \times E \to \mathcal{P}_{\mathrm{cp},cv}(E)$ be a Carathéodory multimap which satisfies some of the following assumptions:

($\mathcal{A}_1$) There exist a function $p \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\rho\colon [0,\infty) \to [0,\infty)$ such that

$$\|F(t,z)\|_{\mathcal{P}} \le p(t)\psi(|z|), \quad \text{for a.e. } t \in J \text{ and each } z \in E,$$

with

$$\int_0^b p(s)\,ds < \int_1^\infty \frac{du}{\psi(u)}.$$

($\mathcal{A}_2$) There exist constants $\overline{c}_k > 0$ and continuous functions $\phi_k\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|I_k(x)| \le c_k\phi_k(|x|) \quad \text{for each } x \in E,\ k = 1, \dots, m.$$

($\mathcal{A}_3$) E is a reflexive Banach space, the semigroup $\{T(t)\}_{t\ge 0}$ is uniformly continuous, and the semigroup $\{T(t)\}_{t>0}$ is compact in E.

($\mathcal{A}_4$) There exists $\overline{p} \in L^1([0,b], \mathbb{R}^+)$ such that for every bounded subset D in E

$$\chi(F(t,D)) \le \overline{p}(t)\chi(D),$$

and there exist $L_k > 0$, $k = 0, \dots, m$ such that

$$q_k := 2Me^{\omega t_{k+1}} \sup_{t\in[t_k,t_{k+1}]} \int_{t_k}^{t_{k+1}} e^{-L_k(t-s)}\overline{p}(s)\,ds < 1, \quad k = 0, \dots, m.$$

Here χ is the Hausdorff MNC and M, ω are as defined in Proposition 2.69.

Remark 3.2. Note that if $\{T(t)\}$ is compact for $t > 0$, then so it is for any $t' > t$ since $T(t') = T(t'-t) \circ T(t)$ and $T(t)$ is bounded.

Theorem 3.3. *Assume that F satisfies either $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ or $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_4)$. Then the set of solutions for problem* (3.1) *is nonempty and compact. Moreover, the operator solution $S\colon a \to \mathcal{P}(S(a))$ is u.s.c., where*

$$S(a) = \{y \in \mathrm{PC} \mid y \text{ is a mild solution of } (3.1)\}.$$

Proof. According to the hypotheses considered, the proof is split in two parts.

Part 1. Under assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$, the solution set is nonempty and compact.

Step 1. $S(a) \neq \emptyset$. Consider the operator $N\colon \mathrm{PC} \to \mathcal{P}(\mathrm{PC})$ defined for $y \in \mathrm{PC}$ by

$$N(y) = \Bigg\{ h \in \mathrm{PC} : \\ h(t) = T(t)a + \int_0^t T(t-s)v(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)),\ t \in J \Bigg\}$$

where $v \in S_{F,y} = \{u \in L^1(J,E)\colon u \in F(t,y(t))$, for almost every $t \in J\}$. Clearly, fixed points of the operator N are mild solutions of problem (3.1). Since, for each $y \in \mathrm{PC}$, the nonlinearity F takes convex values, the selection set $S_{F,y}$ is convex and then N has convex values.

CLAIM 1. N sends bounded sets into bounded sets.

Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that, for each $y \in B_q := \{y \in \mathrm{PC} : \|y\|_{\mathrm{PC}} \leq q\}$, one has $\|N(y)\|_{\mathrm{PC}} \leq l$. Let $y \in B_q$. Then for each $t \in [0,b]$,

$$N(y)(t) = T(t)a + \int_0^t T(t-s)f(y(s))\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-), \quad t \in [0,b].$$

We have, for each $t \in [0,b]$,

$$\begin{aligned} |N(y)(t)| &\leq e^{\omega b}|a| + e^{\omega b}\int_0^t |f(y(s))|\,ds + e^{\omega b}\sum_{k=1}^m c_k\phi_k(|y(t_k)|) \\ &\leq e^{\omega b}|a| + e^{\omega b}\|\varphi_q\|_{L^1} + e^{\omega b}\sum_{k=1}^m c_k \sup_{0\leq x\leq q} \phi_k(x) := l. \end{aligned}$$

CLAIM 2. To prove that $N(\mathcal{B}_q)$ is an equicontinuous set of PC, let $0 < \tau_1 < \tau_2 \leq b$, $y \in \mathcal{B}_q$, and $h \in N(y)$. Then there exists $v \in S_{F,y}$ such that

$$h(t) = T(t)a + \int_0^t T(t-s)v(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)), \quad t \in J.$$

Letting $d_k = \sup_{0\le r\le q} \phi_k(r)$, we obtain the estimates

$$\begin{aligned}|h(\tau_2) - h(\tau_1)| \le{}& |T(\tau_2)a - T(\tau_1)a| \\ &+ \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} p(s)\psi(q)\,ds \\ &+ \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} p(s)\psi(q)\,ds + \sum_{k=1}^{m} T(\tau_2 - \tau_1) I_k(y_k) \\ &+ \sum_{0<t_k<\tau_1} c_k d_k \|T(\tau_1 - t_k) - T(\tau_2 - t_k)\|_{B(E)}.\end{aligned}$$

Hence

$$\begin{aligned}|h(\tau_2) - h(\tau_1)| \le{}& \|T(\tau_2 - \tau_1) - \mathrm{id}\,\|_{B(E)} |a| \\ &+ e^{\omega b}\psi(q)\|T(\tau_2 - \tau_1) - \mathrm{id}\,\|_{B(E)} \int_0^{\tau_1} p(s)\,ds \\ &+ e^{\omega b}\psi(q) \int_{\tau_1}^{\tau_2} p(s)\,ds + e^{\omega b} \sum_{\tau_1<t_k<\tau_2} c_k d_k \\ &+ \|T(\tau_2 - \tau_1) - \mathrm{id}\|_{B(E)} \sum_{0<t_k<\tau_1} c_k d_k.\end{aligned}$$

Since $\{T(t)\}_{t\ge 0}$ is uniformly continuous, then $\|T(h) - \mathrm{id}\| \to 0$, as $h \to 0^+$. Thus the right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$. This proves the equicontinuity for the case where $t \ne t_i$, $i = 1, \dots, m$. So, it remains to examine the equicontinuity at $t = t_i$. Let

$$h_1(t) = T(t)a + \sum_{0<t_k<t} T(t - t_k) I_k(y(t_k)) \quad \text{and} \quad h_2(t) = \int_0^t T(t - s) v(s)\,ds.$$

To prove equicontinuity at $t = t_i^-$, fix $\delta_1 > 0$ such that $\{t_k : k \ne i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$. Then

$$h_1(t_i) = T(t_i)a + \sum_{0<t_k<t_i} T(t_i - t_k) I_k(y(t_k)) = T(t_i)a + \sum_{k=1}^{i-1} T(t_i - t_k) I_k(y(t_k)).$$

For $0 < \varepsilon < \delta_1$, we have the estimates

$$\begin{aligned}&|h_1(t_i - \varepsilon) - h_1(t_i)| \\ &\quad \le |(T(t_i - \varepsilon) - T(t_i))a| + \sum_{k=1}^{i-1} |[T(t_i - \varepsilon - t_k) - T(t_i - t_k)] I(y(t_k^-))| \\ &\quad \le e^{\omega(t_i - \varepsilon)} \|T(\varepsilon) - \mathrm{id}\|_{B(E)} |a| + \|T(\varepsilon) - \mathrm{id}\,\|_{B(E)} \sum_{k=1}^{i-1} e^{\omega(t_i - \varepsilon - t_k)} c_k d_k,\end{aligned}$$

where $d_k = \sup_{0\le r\le q} \phi_k(r)$. Again, the terms in the right-hand side tend to zero as $\varepsilon \to 0$. Moreover,

$$\begin{aligned}
&|h_2(t_i-\varepsilon) - h_2(t_i)| \\
&\le \psi(q)\|T(\varepsilon) - \mathrm{id}\,\|_{B(E)} \int_0^{t_i-\varepsilon} e^{\omega(t_i-\varepsilon-s)} p(s)|\,ds + \psi(q)\int_{t_i-\varepsilon}^{t_i} e^{\omega(t_i-s)} p(s)\,ds,
\end{aligned}$$

which tends to zero as $\varepsilon \to 0$. Now, define

$$\widehat{h}_0(t) = h(t), \quad t\in[0,t_1] \quad \text{and} \quad \widehat{h}_i(t) = \begin{cases} h(t), & t\in(t_i,t_{i+1}], \\ h(t_i^+), & t = t_i. \end{cases}$$

To prove equicontinuity at points $t = t_i^+$, let $\delta_2 > 0$ be such that

$$\{t_k : k \ne i\} \cap [t_i-\delta_2, t_i+\delta_2] = \emptyset.$$

Then

$$\widehat{h}(t_i) = T(t_i)a + \int_0^{t_i} T(t_i-s)v(s)\,ds + \sum_{k=1}^{i} T(t_i-t_k)I_k(y(t_k)).$$

For $0<\varepsilon<\delta_2$, we have the estimates

$$\begin{aligned}
&|\widehat{h}(t_i+\varepsilon) - \widehat{h}(t_i)| \\
&\quad\le |(T(t_i+\varepsilon) - T(t_i))a| + \psi(q)\int_0^{t_i} |[T(t_i+\varepsilon-s) - T(t_i-s)|p(s)\,ds \\
&\qquad + \sum_{k=1}^{i} |[T(t_i+\varepsilon-t_k) - T(t_i-t_k)]I(y(t_k^-))| \\
&\quad\le e^{\omega t_i}\|T(\varepsilon)-\mathrm{id}\|_{B(E)}|a| + \psi(q)\|T(\varepsilon)-\mathrm{id}\|_{B(E)}\int_0^{t_i} e^{\omega(t_i-s)}p(s)\,ds \\
&\qquad + \|T(\varepsilon)-\mathrm{id}\|_{B(E)}\sum_{k=1}^{i} e^{\omega(t_i-t_k)}c_k d_k.
\end{aligned}$$

The terms in the right-hand side tend to zero as $\varepsilon \to 0$.

Claim 3. The set $N(\mathcal{B}_q)(t)$, $t\ge 0$ is relatively compact in PC.

For $\varepsilon > 0$ and $t > 0$, let

$$\begin{aligned}
h_\varepsilon(t) &= T(t)a + \int_0^{t-\varepsilon} T(t-s)v(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)) \\
&= T(t)a + T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon)v(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)).
\end{aligned}$$

The compactness of the semigroup $\{T(t)\}_{t>0}$ (assumption $(\mathcal{A}_3)$) implies that the set $N_\varepsilon(\mathcal{B}_q)(t)$ is precompact, where N_ε is the fixed point operator obtained by replacing the function h by h_ε. Now,

$$|h(t) - h_\varepsilon(t)| \le \psi(q) \int_{t-\varepsilon}^{t} e^{(t-s)} p(s)\, ds \le \psi(q) e^{\omega\varepsilon} \int_{t-\varepsilon}^{t} p(s)\, ds$$

and the right-hand term tends to 0 uniformly in t as $\varepsilon \to 0^+$ since $p \in L^1(J, \mathbb{R}^+$. This proves the relative compactness of $N(\mathcal{B}_q)(t)$ for $t \ge 0$.

By the Arzelá-Ascoli theorem, we conclude that $N\colon \mathrm{PC} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathrm{PC})$ is a completely continuous operator.

Claim 4. A priori bounds on solutions. Let $y \in \mathrm{PC}$ be a solution of the nonlinear equation $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in [0, b]$,

$$y(t) = \lambda\biggl(T(t)a + \int_0^t T(t-s)v(s)\, ds + \sum_{0<t_k<t} T(t-t_k) I_k(y(t_k^-))\biggr),$$

where $v(s) \in F(s, y(s))$. This implies that, for each $t \in [0, t_1]$

$$|y(t)| \le e^{\omega t_1}|a| + e^{\omega t_1} \int_0^t p(s)\psi(|y(s)|)\, ds. \tag{3.2}$$

Let us take the right-hand side of the inequality (3.2) as $Z(t)$. Then we have

$$c = Z(0) = e^{\omega t_1}|a|, \quad Z'(t) = e^{\omega t_1} p(t)\psi(|y(t)|), \quad t \in [0, t_1].$$

Using the nondecreasing character of ψ we get

$$Z'(t) \le e^{\omega t_1} p(t)\psi(Z(t)), \quad t \in [0, t_1].$$

This implies for each $t \in [0, t_1]$ that

$$\int_{Z(0)}^{Z(t)} \frac{d\tau}{\psi(\tau)} \le e^{\omega t_1} \int_0^{t_1} p(s)\, ds < \int_{Z(0)}^{\infty} \frac{d\tau}{\psi(\tau)}.$$

$(\mathcal{A}_1)$ implies that there exists a constant K_0 such that $Z(t) \le K_0$, $t \in [0, t_1]$, and hence $|y(t)| \le K_0$, $t \in [0, t_1]$. Then, we have

$$\|y_1\|_\infty = \sup\{|y(t)| : t \in [0, t_1]\} \le K_0,$$

where K_0 depends only on t_1 and on the functions p and ψ. For $t \in (t_1, t_2]$, we have

$$y(t) = T(t-t_1)[y(t_1) + I_1(y(t_1))] + \int_{t_1}^{t} T(t-s)v(s)\, ds.$$

Then

$$\begin{aligned}|y(t)| &\le e^{\omega(t_2-t_1)}[|y(t_1)| + |I_1(y(t_1))|] + e^{\omega t_1}\int_{t_1}^{t} p(s)\psi(|y(s)|)\,ds\\ &\le e^{\omega(t_2-t_1)}[K_0 + c_1 d_0] + e^{\omega t_2}\int_{t_1}^{t} p(s)\psi(|y(s)|)\,ds,\end{aligned}$$

where $d_0 = \sup\{\phi_1(x) : 0 \le x \le K_0\}$. Then

$$|y(t)| \le e^{\omega(t_2-t_1)}[K_0 + c_1 d_0] + e^{\omega t_2}\int_{t_1}^{t} p(s)\psi(|y(s)|)\,ds. \tag{3.3}$$

Let us denote the right-hand side of the inequality (3.3) as $W(t)$. Then we have

$$c_1 = W_1(t_1) = e^{\omega t_1}[K_0 + d_0], \quad W_1'(t) = e^{\omega t_1} p(t)\psi(|y(t)|), \quad t \in (t_1, t_2].$$

Using the nondecreasing character of ψ, we find that

$$W_1'(t) \le e^{\omega t_1} p(t)\psi(W_1(t)), \quad t \in (t_1, t_2].$$

This implies for each $t \in (t_1, t_2]$ that

$$\int_{W_1(t_1)}^{W_1(t)} \frac{d\tau}{\psi(\tau)} \le e^{\omega t_1}\int_{t_1}^{t_2} p(s)\,ds < \int_{W_1(t_1)}^{\infty} \frac{d\tau}{\psi(\tau)}.$$

$(\mathcal{A}_1)$ implies that there exists a constant K_1 such that $W(t) \le K_1$, $t \in (t_1, t_2]$, and hence $|y(t)| \le K_1$, $t \in (t_1, t_2]$. Then, we have

$$\|y_1\|_\infty = \sup\{|y(t)| : t \in [t_1, t_2]\} \le K_1,$$

where K_1 depends only on t_1, t_2, M and on the functions p and ψ. We repeat this process in each interval $(t_i, t_{i+1}]$ to arrive at the estimates:

$$\|y_i\|_\infty = \sup\{|y(t)| : t \in [t_i, t_{i+1}]\} \le K_i, \quad i = 1, \dots, m.$$

Let $K_* = \sup\{K_i : i = 1, \dots, m\}$ and

$$U = \{y \in \mathrm{PC} : \|y\|_{\mathrm{PC}} < K_* + 1\}.$$

Then U is a open subset of PC and there is no $y \in \partial U$ such that $y = \lambda N(y)$ and $\lambda \in (0, 1)$. By Lemma 2.73, N has at least one fixed point y and then $S(a) \ne \emptyset$.

Step 2. $S(a)$ is a compact set in PC. Since from Step 1, $S(a)$ is bounded and $N\colon \mathrm{PC} \to \mathcal{P}(\mathrm{PC})$ is completely c ontinuous, the set $N(S(a))$ is relatively compact. Moreover, N being a fixed point operator, we have $S(a) \subset \overline{N(S(a))}$. Then it is sufficient to show that $S(a)$ is closed. For this, let $\{y_n\}_{n\in\mathbb{N}} \subset S(a)$ be a sequence converging to some limit y; then there exists $v_n \in S_{F,y_n}$ such that

$$y_n(t) = T(t)a + \int_0^t T(t-s)v_n(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k)), \quad t \in J.$$

Since F is L^1-Carathéodory multi-map with closed values, the selection set $S_{F,y}$ is closed and nonempty; by [119, Theorem 6.4.6] or [54, Theorem 40], this set is decomposable. Since E is reflexive, then Corollary 2.61 implies that $S_{F,y}$ is weakly relatively compact, hence sequentially weakly relatively compact by Eberlein's Theorem [142]. As a consequence, there exists a subsequence, still denoted $(v_n)_{n\in\mathbb{N}}$, which converges weakly to some limit $v(\,\cdot\,) \in L^1$. Mazur's Lemma implies the existence of a double sequence $(\alpha_{n,k})_{k,n\in\mathbb{N}^*}$ such that for all $n \in \mathbb{N}^*$, tehre exists $k_0(n) \in \mathbb{N}^*$ such that $\alpha_{n,k} = 0$ for all $k \geq k_0(n)$, $\sum_{k=n}^{\infty} \alpha_{n,k} = 1$, for all $n \in \mathbb{N}^*$, and the sequence of convex combinations $g_n(\,\cdot\,) = \sum_{k=n}^{\infty} \alpha_{n,k} v_k(\,\cdot\,)$ converges strongly to v in L^1. Since F takes convex values, using Lemma 2.6, we obtain that

$$\begin{aligned} v(t) &\in \bigcap_{n\geq 1} \overline{\{g_k(t),\ k \geq n\}}, \quad \text{a.e. } t \in J \\ &\subset \bigcap_{n\geq 1} \overline{\mathrm{co}}\{v_k(t),\ k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{\mathrm{co}}\{\bigcup_{k\geq n} F(t, y_k(t))\} = \overline{\mathrm{co}}(\limsup_{k\to\infty} F(t, y_k(t))). \end{aligned} \tag{3.4}$$

Moreover F is *u.s.c.* with compact values; then by Lemma 2.5, we have

$$\limsup_{n\to\infty} F(t, y_n(t)) = F(t, y(t)), \quad \text{for a.e. } t \in J.$$

This with (3.4) imply that $v(t) \in \overline{\mathrm{co}}\, F(t, y(t))$. Let

$$z(t) = T(t)a + \int_0^t T(t-s)v(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k), \quad t \in J.$$

The functions I_k, $k = 1, \dots, m$ being continuous, the Lebesgue dominated convergence theorem implies that

$$\|y_n - z\|_{\mathrm{PC}} \leq e^{\omega b} \int_0^b |g_n(s) - v(s)|\,ds + e^{\omega b} \sum_{k=1}^{m} |I_k(y_n(t_k) - I_k(y(t_k))| \to 0,$$

as $n \to \infty$. Hence $y(t) = z(t)$, $t \in J$, proving that $S(\,\cdot\,) \in \mathcal{P}_{\mathrm{cp}}(E)$.

Part 2. Under assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_4)$, the set $S(a)$ is nonempty and compact.

Step 1. Let $N_0\colon C([0,t_1],E) \to \mathcal{P}(C([0,t_1],E))$ be defined by

$$N_0(y) = \left\{ h \in C([0,t_1],E) : h(t) = T(t)a + \int_0^t T(t-s)v(s),\ t \in [0,t_1] \right\},$$

where

$$v \in S_{F,y} = \{v \in L^1([0,t_1],E) : v(t) \in F(t,y(t)) \text{ a.e. } t \in [0,t_1]\}.$$

Let

$$\mathcal{K}_0 := \{y \in C([0,t_1],E) : |y(t)| \le a_0(t),\ t \in [0,t_1]\},$$

where

$$a_0(t) = \Gamma^{-1}\left(\int_0^t \widehat{M}(s)\,ds\right), \qquad \Gamma(z) = \int_c^z \frac{du}{\psi(u)}$$

and

$$c = Me^{\omega t_1}|a| \quad \text{and} \quad \widehat{M}(t) = Me^{\omega t_1}p(t), \quad t \in [0,t_1].$$

It is clear that $\mathcal{K}_0$ is a closed bounded convex set in $C([0,t_1],E)$. If $h \in N_0(y)$, then there exists $v \in S_{F,y}$ such that

$$h(t) = T(t)a + \int_0^t T(t-s)v(s)\,ds, \quad t \in [0,t_1].$$

Claim 1. $N_0(\mathcal{K}_0) \subset \mathcal{K}_0$. We have

$$\begin{aligned} |h(t)| &\le Me^{\omega t_1}|a| + \int_0^t |T(t-s)|v(s)|\,ds \\ &\le Me^{\omega t_1}|a| + Me^{\omega t_1}\int_0^t p(s)\psi(|y(s)|)\,ds \\ &\le Me^{\omega t_1}|a| + \int_0^t \widehat{M}(s)\psi(a_0(s))\,ds. \end{aligned}$$

It follows that, for each $t \in [0,t_1]$,

$$|h(t)| \le c + \int_0^t a_0'(s)\,ds = a_0(t),$$

proving our claim.

Claim 2. The multi-map $N_0 : \mathcal{K}_0 \to \mathcal{P}(\mathcal{K}_0)$ has at least one fixed point. Since F is a multifunction with convex values, N_0 has convex values. Moreover N_0 has a closed graph. Indeed, let $\{y_n : n \in \mathbb{N}\} \subseteq \mathcal{K}_0$ be such that $y_n \to y_*$, $h_n \in N_0(y_n)$, and $h_n \to h_*$, as $n \to +\infty$. We shall prove that $h_* \in N_0(y_*)$. $h_n \in N_0(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that for almost every $t \in [0,t_1]$,

$$h_n(t) = T(t)a + \int_0^t T(t-s)v_n(s)\,ds.$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that for almost every $t \in [0,t_1]$, we have

$$h_*(t) = T(t)a + \int_0^t T(t-s)v_*(s)\,ds. \tag{3.5}$$

Since $\{y_n : n \in \mathbb{N}\} \subseteq \mathcal{K}_0$ and $\{v_n : n \in \mathbb{N}\} \subseteq F(t,y_n(t))$, assumption $(\mathcal{A}_1)$ implies that

$$|v_n(t)| \le p(t)\psi(a_0(t_1)), \quad t \in [0,t_1].$$

In addition, the set $\{v_n(t) : n \in \mathbb{N}\}$ is relatively compact for almost every $t \in J$ because assumption $(\mathcal{A}_4)$ both with the convergence of $\{y_n\}_{n\in\mathbb{N}}$ imply that

$$\chi(\{v_n(t) : n \in \mathbb{N}\}) \leq \chi(F(t, y_n(t)) \leq \overline{p}(t)\chi(y_n(t)) = 0.$$

Then the sequence $\{v_n : n \in \mathbb{N}\}$ is semi-compact, hence weakly compact in $L^1([0, t_1]; E)$ by Lemma 2.60, i.e. there exists $v_* \in L^1$ such that $\{v_n\}$ converges weakly to v_*. Finally, (3.5) follows from the Lebesgue dominated convergence theorem.

Claim 3. N_0 is a $\beta-$condensing operator for a suitable MNC β. For a bounded subset $D \subset \mathcal{K}_0$, let $\mathrm{mod}_C(D)$ be the modulus of quasi-equicontinuity of the set of functions D given by

$$\mathrm{mod}_C(D) = \lim_{\delta\to 0} \sup_{x\in D} \max_{|\tau_2-\tau_1|\leq\delta} |x(\tau_1) - x(\tau_2)|.$$

It is well known (see, e.g. Example 2.1.2 in [KaObZe]) that $\mathrm{mod}_C(D)$ defines an MNC in PC which satisfies all of the properties in the Definition 2.55 except regularity. Given the Hausdorff MNC χ, let γ_0 be the real MNC defined on bounded subsets on PC by

$$\gamma_0(D) = \sup_{t\in[0,t_1]} e^{-L_0 t}\chi(D(t)).$$

Finally, define the following MNC on bounded subsets of PC by

$$\beta_0(D) = \max_{D\in\Delta(\mathrm{PC})} (\gamma_0(D), \mathrm{mod}_C(D)),$$

where $\Delta(\mathrm{PC})$ is the collection of all denumerable subsets of B. Then the MNC β_0 is monotone, regular and nonsingular (see Example 2.1.4 in [93]). This measure is also used in [41], [30], [31], [59] in the discussion of semi-linear evolution differential inclusions. To show that N_0 is β_0-condensing, let $B \subset \mathcal{K}_0$ be a bounded set in $\mathcal{K}_0$ such that

$$\beta_0(B) \leq \beta_0(N_0(B)). \tag{3.6}$$

We will show that B is relatively compact. Let $\{y_n : n \in \mathbb{N}\} \subset B$ and let $N_0 = \widetilde{L_0} \circ S_F$, where $S_F\colon C([0, t_1], E) \to L^1([0, t_1], E)$ is defined by

$$S_F(y) = S_{F,y} = \{v \in L^1([0, t_1], E) : v(t) \in F(t, y(t)),\ \text{for a.e. } t \in [0, t_1]\}$$

and $\widetilde{L_0}\colon L^1([0, t_1], E) \to C([0, t_1], E)$ is defined by

$$\widetilde{L_0}(v)(t) = \int_0^t T(t-s)v(s)\,ds, \quad t \in [0, t_1].$$

Then

$$\begin{aligned}|\widetilde{L_0}v_1(t) - \widetilde{L_0}v_2(t)| &\leq \int_0^t \|T(t-s)\| \cdot |v_1(s) - v_2(s)|\, ds \\ &\leq Me^{wt_1} \int_0^t |v_1(s) - v_2(s)|\, ds.\end{aligned}$$

Moreover, each element h_n in $N_0(y_n)$ can be represented as

$$h_n = T(\,\cdot\,)a + \widetilde{L_0}(v_n), \quad v_n \in S_F(y_n).$$

First, using (3.6), we obtain that

$$\beta_0(\{h_n : n \in \mathbb{N}\}) \geq \beta_0(\{y_n : n \in \mathbb{N}\}).$$

Now from assumption $(\mathcal{A}_4)$, it holds that for almost every $t \in [0, t_1]$,

$$\begin{aligned}\chi(\{v_n(t) : n \in \mathbb{N}\}) &\leq \chi(F(t, \{y_n(t)\}_{n=1}^{\infty}) \\ &\leq \overline{p}(t) \sup_{-r \leq \theta \leq 0} \chi(\{y_n(t)\}_{n=1}^{\infty}) \leq \overline{p}(t) \sup_{0 \leq s \leq t} \chi(\{y_n(s)\}_{n=1}^{\infty}) \\ &\leq e^{L_0 t}\overline{p}(t) \sup_{0 \leq s \leq t} e^{-L_0 s}\chi(\{y_n(s)\}_{n=1}^{\infty}) \leq e^{L_0 t}\overline{p}(t)\gamma_0(\{y_n\}_{n=1}^{\infty}).\end{aligned}$$

From Lemmas 2.57 and 2.58, we deduce that

$$e^{-L_0 t}\chi(\{\widetilde{L_0}(v_n)(t)\}_{n=1}^{\infty}) \leq \gamma_0(\{y_n\}_{n=1}^{\infty}) \sup_{t \in [0,t_1]} 2Me^{wt_1} \int_0^t e^{-L_0(t-s)}\overline{p}(s)\, ds.$$

Therefore

$$\begin{aligned}\gamma_0(\{y_n\}_{n=1}^{\infty}) &\leq \gamma_0(\{h_n\}_{n=1}^{\infty}) \\ &= \sup_{t \in [0,t_1]} e^{-L_0 t}\chi(\{h_n(t)\}_{n=1}^{\infty} \leq q_0\gamma_0(\{y_n\}_{n=1}^{\infty}).\end{aligned} \tag{3.7}$$

Since $0 < q_0 < 1$, we infer that

$$\gamma_0(\{y_n\}_{n=1}^{\infty}) = 0. \tag{3.8}$$

Next, we show that $\mathrm{mod}_C(B) = 0$ i.e. the set B is equicontinuous. This is equivalent to showing that every $\{h_n\} \subset N_0(B)$ satisfies this property. Given a sequence $\{h_n\}$, there exist sequences $\{y_n\} \subset B$ and $\{v_n\} \subset S_{F,y_n}$ such that

$$h_n = T(\,\cdot\,)a + \widetilde{L_0}(v_n).$$

Back to (3.8), we infer that $\{y_n\}$ satisfies

$$\chi(\{y_n(t)\}) = 0, \quad \text{for a.e. } t \in [0, t_1].$$

Assumption $(\mathcal{A}_4)$ in turn implies that

$$\chi(\{v_n(t)\}) = 0, \quad \text{for a.e. } t \in [0, t_1].$$

From $(\mathcal{A}_1)$, the sequence $\{v_n\}$ is integrably bounded, hence semi-compact. Arguing as in Part 1, Step 2, we deduce that, up to a subsequence, $\{h_n\}$ is relatively compact. Therefore $\beta_0(\{h_n\}_{n=1}^{\infty}) = 0$ which implies that $\beta_0(\{y_n\}_{n=1}^{\infty}) = 0$. We have proved that B is relatively compact and so the map N_0 is β-condensing. From Lemma 2.75, we deduce that N_0 has at least point fixe denoted y_0. Moreover since $\mathrm{Fix}\,(N_0)$ is bounded, by Lemma 2.77, $\mathrm{Fix}\,(N_0)$ is compact.

Step 2. Let

$$\begin{aligned} C_1 &= \{y \in C((t_1, t_2], E) : y(t_1^+) \text{ exists}\}, \\ \mathcal{K}_1 &:= \{z \in C_1 : \|y_t\|_\infty \le a_1(t),\ t \in [t_1, t_2]\}, \end{aligned}$$

where

$$a_1(t) = \Gamma^{-1}\bigg(\int_{t_1}^{t} \widehat{M}(s)\,ds\bigg) \quad \text{and} \quad \Gamma(z) = \int_c^z \frac{du}{\psi(u)}.$$

Define the operator $N_1 \colon C_1 \to \mathcal{P}(C_1)$ by $Ny = \{h\}$ where

$$h(t) = \int_{t_1}^{t} T(t-s)v(s)\,ds + T(t-t_1)[y_0(t_1) + I_1(y_0(t_1^-))], \quad t \in [t_1, t_2],$$

and

$$v \in S_{F,y} = \{v \in L^1([t_1, t_2], E) : v(t) \in F(t, y(t)),\ \text{a.e. } t \in [t_1, t_2]\}.$$

We can easily check that $N_1(\mathcal{K}_1) \subset (\mathcal{K}_1)$. Thus we only prove that N_1 is a β-condensing operator. For a bounded subset $B \subset \mathcal{K}_1$, let $\mathrm{mod}_C(B)$ be the modulus of quasi-equicontinuous of the set of functions B, γ_1 be the real MNC defined on bounded subset on $\overline{U}$ by

$$\gamma_1(B) = \sup_{t \in [t_1, t_2]} e^{-L_1 t} \chi(B(t)),$$

and β_1 the MNC defined on $\mathcal{K}_1$ by

$$\beta_1(B) = \max_{\Delta(\mathcal{K}_1)} (\gamma_1(B), \mathrm{mod}_C(B)),$$

where $\Delta(\mathcal{K}_1)$ is the collection of all denumerable subsets of B. Let $B \subset \mathcal{K}_1$ be a bounded set in $\mathcal{K}_1$ such that

$$\beta_1(B) \le \beta_1(N_1(B)). \tag{3.9}$$

We will show that B is relatively compact. It is clear that h_n has the representation:

$$h_n(t) = \widetilde{L_1}(v_n(t)) + T(t-t_1)[y_0(t_1) + I_1(y_0(t_1))],$$

where $\widetilde{L_1}$ is as defined in Step 1, Claim 3. Then, we have the estimates

$$\begin{aligned}
&\chi(\{v_n(t) : n \in \mathbb{N}\}) \\
&\qquad \le \chi(F(t, \{(y_n)_n\}_{n=1}^{\infty} + T(t-t_1)[y_0(t_1) + I_1(y_0(t_1))] \\
&\qquad \le \chi(F(t, \{(y_n)_s\}_{n=1}^{\infty})) + Me^{\omega(t_2-t_1)}\chi(I_1(y_0(t_1))) \\
&\qquad \le \overline{p}(t) \sup_{-r\le\theta\le 0} \chi(\{y_n(t)\}_{n=1}^{\infty})) \le \overline{p}(t) \sup_{t_1\le s\le t} \chi(\{y_n(s)\}_{n=1}^{\infty})) \\
&\qquad \le e^{L_1 t}\overline{p}(t) \sup_{t_1\le s\le t} e^{-L_1 s}\chi(\{y_n(s)\}_{n=1}^{\infty})) \le e^{L_1 t}\overline{p}(t)\gamma_1(\{y_n\}_{n=1}^{\infty}).
\end{aligned}$$

From Lemmas 2.57 and 2.58, we deduce that

$$e^{-L_1 t}\chi(\{\widetilde{L_1}(v_n)(t)\}_{n=1}^{\infty}) \le \gamma_1(\{y_n\}_{n=1}^{\infty}) \sup_{t\in[t_1,t_2]} 2Me^{\omega t_2} \int_0^t e^{-L_1(t-s)}\overline{p}(s)\, ds.$$

Therefore

$$\gamma_1(\{y_n\}_{n=1}^{\infty}) \le \gamma_1(\{h_n\}_{n=1}^{\infty}) = \sup_{t\in[t_1,t_2]} e^{-L_1 t}\chi(\{h_n(t)\}_{n=1}^{\infty} \le q_1\gamma_1(\{y_n\}_{n=1}^{\infty}).$$

Since $0 < q_1 < 1$, it follows that $\gamma_1(\{y_n\}_{n=1}^{\infty}) = 0$. Arguing as in Step 1, we can show that $\mathrm{mod}_C(\{y_n\}_{n=1}^{\infty}) = 0$ and then $\beta_1(\{y_n\}_{n=1}^{\infty}) = 0$. Finally N_1 is β-condensing and from Lemma 2.75, we deduce that N_1 has a fixed point y_1 in $\mathcal{K}_1$ denoted by y_1. As in Step 1, we can prove that $\mathrm{Fix}\,(N_1)$ is a compact set.

Step 3. We continue this process taking into account that $y_m := y|_{[t_m,b]}$ is a solution of the problem

$$\begin{cases} y'(t) - Ay(t) \in F(t, y(t)), & \text{a.e. } t \in (t_m, b], \\ y(t_m^+) = y_{m-1}(t_{m-1}) + I_m(y_{m-1}(t_m^-)), \\ y(t_m) = y_{m-1}(t_m). \end{cases}$$

A solution y of problem (3.1) is ultimately defined by

$$y(t) = \begin{cases} y_0(t) & \text{if } t \in [0, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots \\ y_m(t) & \text{if } t \in (t_m, t_{m+1}]. \end{cases}$$

To sum up, we obtain that $S(a) = \bigcap_{k=0}^{k=m} \mathrm{Fix}\,(N_k)$, hence $\emptyset \ne S(a) \subset \mathcal{P}_{\mathrm{cp}}(\mathrm{PC})$. This completes the proof of Theorem 3.3. □

3.1.2. The nonconvex case. In this subsection, a selection theorem due to Bressan, Colombo, and Fryszkowski for lower semicontinuous multi-valued operators with nonempty closed decomposable values combined with the nonlinear alternative of Leray and Schauder are used to investigate the existence

of mild solutions for first order impulsive semi-linear differential inclusions with nonconvex valued Carathéodory right-hand side.

Theorem 3.4. *Suppose that $F\colon J\times E \to \mathcal{P}_{\mathrm{cp}}(E)$ satisfies assumptions $(\mathcal{H}_{\mathrm{lsc}})$ (see Lemma 2.23), $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ together with*

$(\mathcal{A}_3)'$ *the semigroup $\{T(t)\}_{t\geq 0}$ is uniformly continuous, and the semigroup $\{T(t)\}_{t>0}$ is compact.*

Then the semi-linear impulsive initial value problem (3.1) *has at least one mild solution.*

Proof. First, $(\mathcal{H}_{\mathrm{lsc}})$ and $(\mathcal{A}_1)$ imply by Lemma 2.23 that F is of lower semi-continuous type. Then from Lemma 24, there exists a continuous function $f colon \mathrm{PC} \to L^1([0,b],E)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \mathrm{PC}$. Consider the following problem:

$$\begin{cases} y'(t) - Ay(t) = f(y)(t), & t \in [0,b],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1,\dots,m, \\ y(0) = a. \end{cases} \tag{3.10}$$

Clearly, if $y \in \mathrm{PC}$ is a solution of problem (3.10), then y is a solution of problem (3.1). Thus, in order to transform problem (3.10) into a fixed point problem, we consider the operator $N\colon \mathrm{PC} \to \mathrm{PC}$ defined by:

$$N(y)(t) = T(t)a + \int_0^t T(t-s)f(y(s))\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \quad t \in [0,b].$$

We can show that N is completely continuous. Indeed, N sends bounded sets in PC into relatively compact sets. This is performed using $(\mathcal{A}_3)'$ (see the proof of Theorem 3.3, Part 1, Step 1). To prove that N is continuous, let $\{y_n\}$ be a sequence such that $y_n \to y$ in PC. Then

$$\begin{aligned} &|N(y_n(t)) - N(y(t))| \\ &\quad \leq Me^{\omega b}\int_0^t |f(y_n(s)) - f(y(s))|\,ds + Me^{\omega b}\sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k^-))| \\ &\quad \leq Me^{\omega b}\int_0^b |f(y_{n,s}) - f(y_s)|\,ds + Me^{\omega b}\sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k^-))|. \end{aligned}$$

Since the functions f and I_k, $k = 1,\dots,m$ are continuous, then

$$\begin{aligned} \|N(y_n) - N(y)\|_{\mathrm{PC}} \leq{}& Me^{\omega b}\|f(y_n(\,\cdot\,)) - f(y(\,\cdot\,))\|_{L^1} \\ &+ Me^{\omega b}\sum_{k=1}^m |I_k(y_n(t_k^-)) - I_k(y(t_k^-))| \to 0 \end{aligned}$$

as $n \to \infty$. With the Ascoli–Arzelá theorem, we conclude that $N\colon \mathrm{PC} \to \mathrm{PC}$ is completely continuous.

Arguing as in the proof of Theorem 3.3, Part I, Step 1, Claim 4, we prove the existence of U an open subset of PC such that there is no $y \in \partial U$ satisfying $y = \lambda N(y)$ and $\lambda \in (0,1)$. By Lemma 2.73, N has at least one fixed point y solution of problem (3.10) which is solution of (3.1). □

Next, we present a second result for problem (3.1) with a non-convex valued Carathéodory right-hand side. Our considerations are based on the fixed point theorem for contractive multi-valued operators (Lemma 2.74).

Theorem 3.5. *Assume that $F\colon J \times E \to \mathcal{P}_{\mathrm{cp}}(E)$ is a Carathéodory multimap such that*

($\mathcal{B}_1$) *there exists a function $l \in L^1(J, \mathbb{R}^+)$ such that, for almost every $t \in J$ and all $u, \overline{u} \in E$,*

$$H_d(F(t,u), F(t,\overline{u})) \le l(t)\|u - \overline{u}\|,$$

and

$$H_d(0, F(t,0)) \le l(t), \quad \textit{for a.e. } t \in J,$$

are satisfied. Then problem (3.1) *has at least one solution.*

Proof. The proof will be given in several steps.

Step 1. Consider the initial value problem

$$\begin{cases} y'(t) - Ay(t) \in F(t, y(t)), & \text{for a.e. } t \in [0, t_1], \\ y(0) = a. \end{cases} \tag{3.11}$$

To transform problem (3.11) into a fixed point problem, we introduce the operator $N_0\colon C([0,t_1],E) \to \mathcal{P}(C([0,t_1],E))$ defined by:

$$N_0(y) = \left\{ h \in C([0,t_1],E) : h(t) = T(t)a + \int_0^t T(t-s)v(s)\,ds, \text{ if } t \in [0,b] \right\},$$

where $v \in S_{F,y} = \{v \in L^1([0,t_1],E) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,t_1]\}$.

Next we will prove that N_0 has a fixed point.

Claim 1. $N_0(y) \in \mathcal{P}_{\mathrm{cl}}(C([0,t_1],E))$ for each $y \in C([0,t_1],E)$. Indeed, let $(y_n)_{n\ge 0} \in N_0(y)$ be such that $y_n \to \widetilde{y}$ in $C([0,t_1],E)$. Then there exists $v_n \in S_{F,y}$ such that for each $t \in [0,t_1]$

$$y_n(t) = T(t)a + \int_0^t T(t-s)v_n(s)\,ds.$$

From ($\mathcal{B}_1$), there exists a constant $k > 0$ such that $v_n(t) \le kl(t)$, for almost every $t \in [0,t_1]$, that is $(v_n(\,\cdot\,))$ is integrably bounded. Since F is *u.s.c.* with

respect to the second argument, then for each $\varepsilon > 0$, there exists some $n_0 > 0$ such that

$$v_n(t) \subset F(t, y_n(t)) \subset F(t, y(t)) + \varepsilon l(t) B(0,1), \quad \text{for a.e. } t \in [0, t_1].$$

Now, F has further compact values. Then, by Lemma 2.60, there exists a subsequence $v_{n_m}(\,\cdot\,)$ which converges weakly to $v(\,\cdot\,)$ as $m \to \infty$ and $v(t) \in F(t, y(t))$, for almost every $t \in [0, t_1]$. As a consequence, we obtain at the limit as $m \to +\infty$:

$$\widetilde{y}(t) = T(t)a + \int_0^t T(t-s)v(s))\, ds \quad \text{for a.e. } t \in [0, t_1].$$

So $\widetilde{y} \in N_0(y)$.

Claim 2. We claim that there exists $\gamma < 1$, such that

$$H_d(N_0(y), N_0(y_*)) \le \gamma \|y - \overline{y}\|_0, \quad \text{for each } y, \overline{y} \in C([0, t_1], E).$$

Let $y, \overline{y} \in C([0, t_1], E)$ and $h \in N_0(y)$. Then there exists $v(t) \in F(t, y(t))$ such that for each $t \in [0, t_1]$

$$h(t) = T(t)a + \int_0^t T(t-s)v(s)\, ds.$$

From $(\mathcal{B}_1)$, it follows that

$$H_d(F(t, y(t)), F(t, \overline{y}(t))) \le l(t)|y(t) - \overline{y}(t)|.$$

Hence there is $u \in F(t, y(t))$ such that

$$|v(t) - u| \le l(t)|y(t) - \overline{y}(t)|, \quad t \in [0, t_1].$$

Define the multi-map $U\colon [0, t_1] \to \mathcal{P}(E)$ by

$$U(t) = \{u \in C([0, t_1], E) : |v(t) - u| \le l(t)|y(t) - \overline{y}(t)|\}.$$

Since the multi-valued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see Proposition III.4 in [33]), there exists a function $\overline{v}(t)$ which is a measurable selection for V. So, $\overline{v}(t) \in F(t, \overline{y}(t))$ and

$$|v(t) - \overline{v}(t)| \le l(t)|y(t) - \overline{y}(t)|, \quad \text{for a.e. } t \in [0, t_1].$$

Let us define for each $t \in [0, t_1]$

$$\overline{h}(t) = T(t)a + \int_0^t T(t-s)\overline{v}(s)\, ds.$$

Let $\widehat{l}(t) = \int_0^t \widetilde{l}(s)\,ds$, $\widetilde{l}(t) = Me^{\omega b} l(t)$ and $\tau > 1$ be a real parameter. We have that $\overline{h} \in N_0(\overline{y})$ and

$$\begin{aligned}
|h(t) - \overline{h}(t)| &\le Me^{\omega b} \int_0^t l(s)|y(s) - \overline{y}(s)|\,ds \\
&\le \int_0^t \frac{1}{\tau}\widetilde{l}(s)e^{\tau\widehat{l}(s)}e^{-\tau\widehat{l}(s)} \sup_{s\in[0,t]} |y(s) - \overline{y}(s)|\,ds \\
&\le \int_0^t \frac{1}{\tau}\widetilde{l}(t)e^{\tau\widehat{l}(t)}\,ds\|y - \overline{y}\|_0 \\
&\le \frac{1}{\tau}\int_0^t (e^{\tau\widehat{l}(s)})'\,ds\|y - \overline{y}\|_0 \le \frac{1}{\tau}e^{\tau\widehat{l}(t)}\,ds\|y - \overline{y}\|_0.
\end{aligned}$$

Thus

$$e^{-\tau\widehat{l}(t)}|\overline{h}(t)(t) - \overline{h}(t)| \le \frac{1}{\tau}\|y - \overline{y}\|_0.$$

Therefore,

$$\|h - \overline{h}\|_0 \le \frac{1}{\tau}\|y - \overline{y}\|_0,$$

where $\|\cdot\|_0$ is the Bielecki-type norm on C_0 defined by

$$\|y\|_0 = \sup\{e^{-\tau\widehat{l}(t)}|y(t)| : t \in [0, t_1]\}.$$

By an analogous relation, obtained by interchanging the roles of h and $\overline{h}$, we obtain that

$$H_d(N_0(y), N_0(\overline{y})) \le \frac{1}{\tau}\|y - \overline{y}\|_0.$$

So, N_0 is a contraction and thus, by Lemma 2.74, N_0 has a fixed point. Then problem (3.11) has at least one solution. Denote this solution by y_0.

Step 2. Consider now the problem

$$\begin{cases} y'(t) - Ay(t) \in F(t, y(t)), & \text{for a.e. } t \in (t_1, t_2], \\ y(t_1^+) = y_0(t_1^-) + I_1(y_0(t_1)), \end{cases} \tag{3.12}$$

and let $C_1 = \{y \in C((t_1, t_2], E) : y(t_1^+) \text{ exists}\}$.

Consider the operator $N_1 \colon C_1 \to \mathcal{P}(C_1)$ defined by:

$$N_1(y) = \Big\{h \in C_1 : h(t) = T(t - t_1)[y_0(t_1^-) + I_1(y_0(t_1^-))] + \int_{t_1}^t T(t - s)v(s)\,ds, \\ t \in (t_1, t_2]\Big\},$$

where $v \in S_{F,y} = \{v \in L^1([0, t_1], E) : v \in F(t, y(t)),\ \text{a.e. } t \in [t_1, t_2]\}$.

By analogy with Step 1, we can prove that problem (3.12) has at least one solution. We denote this solution by y_1.

Step 3. We continue this process taking into account that $y_m := y|_{[t_m,b]}$ is a solution of the problem

$$\begin{cases} y'(t) - Ay(t) \in F(t, y(t)), & \text{for a.e. } t \in (t_m, b], \\ y(t_m^+) = y_{m-1}(t_{m-1}^-) + I_m(y_{m-1}(t_m^-)). \end{cases}$$

The solution y of problem (3.1) is ultimately defined by

$$y(t) = \begin{cases} y_0(t), & \text{if } t \in [0, t_1], \\ y_1(t), & \text{if } t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots \\ y_m(t), & \text{if } t \in (t_m, b]. \end{cases}$$

□

3.2. Structure of solution sets

We start with the first-order impulsive single-valued problem:

$$\begin{cases} y'(t) - Ay(t) = f(t, y(t)), & \text{a.e. } t \in J = [0, b] \setminus \{t_1, \dots, t_m\}, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = a, \end{cases} \tag{3.13}$$

where $f\colon J \times E \to E$ is a given function. Denote by $S(f, a)$ the set of all solutions of problem (3.13). We are in a position to state and prove an Aronszajn-type result for this problem. First, we list two assumptions:

($\mathcal{C}_1$) $f\colon J \times E \to E$ is a Carathéodory function.

($\mathcal{C}_2$) There exist a function $p \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\rho\colon [0, \infty) \to [0, \infty)$ such that

$$|f(t, x)| \le p(t)\rho(|x|), \quad \text{for a.e. } t \in J \text{ and each } x \in E$$

with

$$\int_0^b p(s)\, ds < \int_{M|a|e^{\omega b}}^{\infty} \frac{du}{\rho(u)}.$$

Then, we have

Theorem 3.6. *Assume that assumptions* ($\mathcal{C}_1$)–($\mathcal{C}_2$) *hold together with either* ($\mathcal{A}_2$), ($\mathcal{A}_3$)′ *or* ($\mathcal{A}_2$), ($\mathcal{A}_4$). *Then the set* $S(f, a)$ *is* R_δ.

Proof. Let $N\colon \mathrm{PC} \to \mathrm{PC}$ be defined by:

$$N(y)(t) = T(t - a) + \int_0^t T(t - s) f(s, y(s))\, ds + \sum_{0 < t_k < t} T(t - t_k) I_k(y(t_k)),$$

for $t \in (0, b]$. Thus $\mathrm{Fix}\,(N) = S(f, a)$. From Theorem 3.3, we know that $S(f, a) \neq \emptyset$ and there exists $\overline{M} > 0$ such that

$$\|y\|_{\mathrm{PC}} < \overline{M}, \quad \text{for every } y \in S(f, a).$$

Define

$$\widetilde{f}(t,x) = \begin{cases} f(t,x), & \text{if } |x| \le \overline{M}, \\ f\left(t, \dfrac{\overline{M}x}{|x|}\right), & \text{if } |x| \ge \overline{M}. \end{cases}$$

Since f is L^1-Carathéodory, the function $\widetilde{f}$ is Carathéodory and it is integrably bounded by $(\mathcal{C}_2)$. So there exists $h \in L^1(J, \mathbb{R}^+)$ such that

$$|\widetilde{f}(t,x)| \le h(t), \quad \text{for a.e. } t \text{ and all } x \in E. \tag{3.14}$$

Now, consider the modified problem

$$\begin{cases} y'(t) - Ay(t) = \widetilde{f}(t, y(t)), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = a. \end{cases}$$

We can easily prove that $S(f,a) = S(\widetilde{f}, a) = \operatorname{Fix}(\widetilde{N})$, where $\widetilde{N}\colon \mathrm{PC} \to \mathrm{PC}$ is defined by

$$\widetilde{N}(y)(t) = T(t)a + \int_0^t T(t-s)\widetilde{f}(s, y(s))\, ds + \sum_{0<t_k<t} T(t-t_k) I_k(y(t_k^-)), \quad t \in [t_0, b].$$

By the inequality (3.14) and the continuity of I_k, we deduce that

$$\|\widetilde{N}(y)\|_{\mathrm{PC}} \le M e^{\omega b}\left(|a| + \|h\|_{L^1} + \sum_{k=1}^m c_k \phi_k(\overline{M})\right) := R.$$

Then $\widetilde{N}$ is uniformly bounded. As in Theorem 3.3, we can prove that $\widetilde{N}\colon \mathrm{PC} \to \mathrm{PC}$ is compact which allows us to define the compact perturbation of the identity $\widetilde{G}(y) = y - \widetilde{N}(y)$ which is a proper map. From the compactness of $\widetilde{N}$, we can easily prove that all conditions of Theorems 2.47 and 2.46 are met. Therefore the solution set $S(\widetilde{f}, a) = \widetilde{G}^{-1}(0)$ is an R_δ set, hence acyclic space. □

Next, we and prove a more precise characterization of the geometric structure of $S(a)$.

Theorem 3.7. *Let $F\colon J \times E \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ be a Carathéodory and an* mLL-*selectionable multi-map which satisfies conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$. Then, for every $a \in E$, the set $S(a)$ is compact and contractible.*

Proof. Let $f \subset F$ be a measurable, locally Lipschitz selection and consider the single-valued problem (3.13). Then problem (3.13) has exactly one local solution $\overline{x}$ for every $a \in E$ (see e.g. [7, Theorem 2.4], [21] or [114, Theorems 3.3 and 3.5], [115]). Because of assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, this solution is wholly

defined over $[0,b]$. Furthermore, Theorem 3.3 implies that $S(a)$ is nonempty and compact. Define the homotopy $h\colon S(a)\times[0,1]\to S(a)$ by

$$h(y,\alpha)(t)=\begin{cases} y(t), & \text{for } 0\le t\le \alpha b,\\ \overline{x}(t), & \text{for } \alpha b<t\le b.\end{cases}$$

In particular,

$$h(y,\alpha)=\begin{cases} y, & \text{for } \alpha=1,\\ \overline{x}, & \text{for } \alpha=0.\end{cases}$$

To show that h is a continuous homotopy, let $(y_n,\alpha_n)\in S(a)\times[0,1]$ be such that $(y_n,\alpha_n)\to(y,\alpha)$, as $n\to\infty$. We shall prove that $h(y_n,\alpha_n)\to h(y,\alpha)$ as $n\to+\infty$. We have

$$h(y_n,\alpha_n)(t)=\begin{cases} y_n(t), & \text{for } t\in[0,\alpha_n b],\\ \overline{x}(t), & \text{for } t\in(\alpha_n b,b].\end{cases}$$

Three cases may occur.

(a) If $\lim_{n\to\infty}\alpha_n=0,]$ then by definition of h we have

$$h(y,0)(t)=\overline{x}(t),\quad \text{for } t\in[0,b].$$

Hence

$$\|h(y_n,\alpha_n)-h(y,\alpha)\|_{\mathrm{PC}}\le\|y_n-\overline{x}\|=\sup\{|y_n(t)-\overline{x}(t)|,\quad t\in[0,b]\},$$

which tends to 0 as $n\to+\infty$. The case when $\lim_{n\to\infty}\alpha_n=1$ is treated similarly.

(b) If $\alpha_n\ne 0$ and $0<\lim_{n\to\infty}\alpha_n=\alpha<1,]$ then we may distinguish between two sub-cases:

(i) Let $t\in[0,\alpha_n b]$. Since $y_n\in S(a)$, there exists $v_n\in S_{F,y_n}$ such that for $t\in[0,\alpha_n b]$

$$y_n(t)=T(t)a+\int_0^t T(t-s)v_n(s)\,ds+\sum_{0<t_k<t}T(t-t_k)I_k(y_n(t_k)).$$

Arguing as in the proof of Theorem 3.3, Part 1, Step 2, we can prove that there exists a subsequence $v_{n_m}(\,\cdot\,)$ such that $v_{n_m}(\,\cdot\,)$ converges weakly to a limit $v(\,\cdot\,)$ satisfying

$$v(t)\in F(t,y(t)),\quad \text{for a.e. } t\in[0,\alpha b].$$

Now $\{y_n\}$ converges to y; then there exists $R>0$ such that

$$\|y_n\|_{\mathrm{PC}}\le R.$$

Then assumption $(\mathcal{A}_1)$ implies that

$$|v_{n_m}(t)|\le p(t)\psi(R),\quad \text{for a.e. } t\in[0,\alpha b].$$

Hence $v \in L^1([0, \alpha b], E)$ and by using the continuity of I_k, we deduce that for $t \in [0, \alpha b]$

$$y(t) = T(t)a + \int_0^t T(t-s)v(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)).$$

(ii) If $t \in (\alpha_n b, b]$, then

$$h(y_n, \alpha_n)(t) = h(y, \alpha)(t) = \overline{x}(t).$$

Thus

$$\|h(y_n, \alpha_n) - h(y, \alpha)\|_{\mathrm{PC}} \to 0, \quad \text{as } n \to \infty.$$

Therefore h is a continuous function, proving that $S(a)$ is contractible to the point $\overline{x}$. □

Also, we have

Theorem 3.8. *Let $F\colon J \times E \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ be a Carathéodory and a* Ca*-selectionable multi-map which satisfies conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$. Then, for every $a \in E$, the set $S(a)$ is compact and R_δ-contractible.*

Proof. From Theorem 3.7, we know that $S(a)$ is compact. Now replace the single-valued homotopy $h\colon S(a) \times [0,1] \to S(a)$ in Theorem 3.7 by the multi-valued homotopy $\Pi\colon S(a) \times [0,1] \to \mathcal{P}(S_{[-r,b]}(a))$ defined by

$$\Pi(x, \alpha) = \{y \in S(f, \alpha b, x)\},$$

where $f \subset F$ and $S(f, \alpha b, x)$ is the solution set of the following problem

$$\begin{cases} (y' - Ay)(t) = f(t, y(t)), & \text{a.e. } t \in [\alpha b, b] \setminus \{t_1, t_2, \dots t_m\}, \\ y(t_k^+) - y(t_k) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(\alpha b) = x(\alpha b). \end{cases}$$

From the definition of Π, we have that $\Pi(x, 0) = S(f, 0, x)$ and $x \in \Pi(x, 1)$ for every $x \in S(a)$. From Theorem 3.6, we know that $\Pi(x, \alpha)$ is an R_δ set for each x, α. So, it remains to prove that $\Pi(\,\cdot\,,\,\cdot\,)$ is *u.s.c.* Since $\Pi(\,\cdot\,,\,\cdot\,)$ has nonempty compact values, we only check (see Lemma 2.9) that Π is locally compact and has a closed graph. This will be performed in two steps.

Step 1. Π is locally compact. This is divided in two sub-steps.

(a) The multi-map $\widetilde{S}\colon [0, b] \times E \to \mathcal{P}(\mathrm{PC})$ defined by

$$\widetilde{S}(u, a) = S(f, u, a)$$

is *u.s.c.* Here $S(f,u,a)$ refers to the solution set of the problem

$$\begin{cases} (y'-Ay)(t) &= f(t,y(t)), \quad \text{a.e. } t\in[u,b]\setminus\{t_1,\dots t_m\},\\ y(t_k^+)-y(t_k) &= I_k(y(t_k^-)), \quad k=1,\dots,m,\\ y(u) &= a. \end{cases}$$

On the contrary, assume that $\widetilde{S}$ is not *u.s.c.* at some point (t_0,a_0). Then there exists an open neighborhood U of $\widetilde{S}(t_0,a_0)$ in PC such that for every open neighborhood V at (t_0,a_0) in the metric space $[0,b]\times E$, there exists $(t_1,a_1)\in V$ such that $\widetilde{S}(t_1,a_1)\not\subset U$. Let $V_n=\{(t,\phi)\in[0,b]\times E\colon d((t,a),(t_0,a_0))<1/n\}$, for each $n=1,2,\dots$, where d denotes the product metric in $[0,b]\times E$. Then for each $n=1,2,\dots$, we get some $(t_n,a_n)\in V_n$ and $y_n\in\widetilde{S}(t_n,a_n)$ such that $y_n\notin U$. Define the map $F_{t_0,a_0}\colon \mathrm{PC}\to\mathrm{PC}$ by

$$F_{t_0,a_0}(x)(t)=T(t-t_0)a_0+\int_{t_0}^{t}T(t-s)f(s,x(s))\,ds+\sum_{t_0<t_k<t}T(t-t_k)I_k(x(t_k)),$$

for $t\in[t_0,b]$, and the compact perturbation of the identity

$$G_{t_0,a_0}(x)=x-F_{t_0,a_0}(x),\quad \text{for } x\in\mathrm{PC}.$$

By a simple calculation, for $x\in\Omega$, $t,t_0\in[0,b]$ and $a_0\in E$, we have

$$F_{t_0,a_0}(x)(t)=T(t-t_0)a_0(t_0)-F_{t_0,0}(x)(t_0)+F_{t_0,0}(x)(t).$$

By definition of G, we have, for $t\in[0,b]$

$$\begin{aligned} G_{t_0,a_0}(x)(t) &= x(t)-F_{t_0,a_0}(x)(t)\\ &= x(t)-[T(t-t_0)a_0(t_0)-F_{t_0,0}(x)(t_0)+F_{t_0,0}(x)(t)]\\ &= -T(t-t_0)a_0+x(t)-F_{t_0,0}(x)(t)+F_{t_0,0}(x)(t_0)\\ &= -T(t-t_0)a_0(t_0)+F_{t_0,0}(x)(t_0)+G_{t_0,0}(x)(t). \end{aligned}$$

Thus

$$G_{t_0,a_0}(x)(t)=-T(t-t_0)a_0(t)+F_{t_0,0}(x)(t_0)+G_{t_0,0}(x)(t).$$

Since F_{t_0,a_0} is a compact map (see Theorem 3.3 in [115]), the compact perturbation of the identity G_{t_0,a_0} is proper. Moreover $y_n\in\widetilde{S}(t_n,a_n)$. Then

$$y_n(t)=T(t-t_n)a_n+\int_{t_n}^{t}T(t-s)f(s,y_n(s))\,ds+\sum_{t_n<t_k<t}T(t-t_k)I_k(y_n(t_k)),$$

for $t\in[t_n,b]$. It follows that

$$0=G_{t_n,a_n}(y_n)(t)=-T(t-t_n)a_n+F_{t_n,0}(y_n)(t_n)+G_{t_n,0}(y_n)(t)$$

and

$$G_{t_0,a_0}(y_n)(t)=-T(t-t_0)a_0(t_0)+F_{t_0,0}(y_n)(t_0)+G_{t_0,0}(y_n)(t).$$

Then, we obtain by substraction the successive estimates

$$\begin{aligned}
&\|G_{t_0,a_0}(y_n)(t)\| \\
&\quad \le \|T(t-t_n)a_n - T(t-t_0)a_0(t_0)\| + \|G_{t_n,0}(y_n)(t_n) - G_{t_0,0}(y_n)(t_0)\| \\
&\quad = \|T(t-t_n)a_n - T(t-t_0)a_0\| + \|F_{t_n,0}(y_n)(t) - F_{t_0,0}y_n(t)\| \\
&\quad \le \|T(t-t_n)a_n - T(t-t_n)a_0\| + \|T(t-t_n)a_n - T(t-t_0)a_0\| \\
&\qquad + \left\| \int_{t_n}^{t} T(t-s)f(s,y_n(s))\,ds - \int_{t_0}^{t} T(t-s)f(s,y_n(s))\,ds \right\| \\
&\qquad + \left\| \sum_{t_0<t_k<t_n} T(t-t_k)I_k(y_n(t_k)) \right\| \\
&\quad \le Me^{\omega b}|a_n - a_0| + \|T(t-t_n)a_n - T(t-t_0)a_0\| \\
&\qquad + \int_{t_0}^{t_n} \|T(t-s)\|\|f(s,y_n(s))\|\,ds + Me^{\omega b} \sum_{t_0<t_k<t_n} \|I_k(y_n(t_k))\| \\
&\quad \le Me^{\omega b}|a_n - a_0| + \|T(t-t_n)a_n - T(t-t_0)a_0| \\
&\qquad + Me^{\omega b} \int_{t_0}^{t_n} \|f(s,y_n(s))\|\,ds + Me^{\omega b} \sum_{t_0<t_k<t_n} \|I_k(y_n(t_k))\|.
\end{aligned}$$

In addition, $\|y_n\| \le R$ and assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_2)$ yield

$$\begin{aligned}
\|G_{t_0,a_0}(y_n)(t)\| &\le Me^{\omega b}|a_n - a_0| + \|T(t-t_n)a_0 - T(t-t_0)a_0\| \\
&\quad + Me^{\omega b} \int_{t_0}^{t_n} p(s)\psi(R)\,ds + Me^{\omega b} \sum_{t_0<t_k<t_n} c_k\phi_k(R).
\end{aligned}$$

Now $\lim_{n\to\infty} a_n = a_0$ and $\lim_{n\to\infty} t_n = t_0$ imply that $\lim_{n\to\infty} G_{t_0,a_0}(y_n)(t) = 0$. The set $A = \overline{\{G_{t_0,a_0}(y_n)\}}$ is compact and so is $G^{-1}_{t_0,a_0}(A)$ because G is proper. It is clear that $\{y_n\} \subset G^{-1}_{t_0,a_0}(A)$. Without loss of generality, we may assume that $\lim_{n\to\infty} y_n = y_0$, hence $y_0 \in \widetilde{S}(t_0,a_0) \subset U$ but this is a contradiction to the assumption that $y_n \notin U$ for each n.

(b) Π is locally compact. For some $r > 0$, Let $I = [0,1]$ and

$$B = \{x \in S(a) : \|x\|_{\mathrm{PC}} \le r\}$$

and $\{y_n\} \in \Pi(B \times I)$; then there exists $(a_n, \alpha_n) \in B \times I$ such that

$$y_n(t) = \begin{cases} a_n, & \text{for } 0 \le t \le \alpha_n b, \\ z_n(t), & \text{for } \alpha b < t \le b,\ z_n \in S(f_n, \alpha_n b, a_n). \end{cases}$$

Since $S(a)$ is compact, there exist subsequences of $\{a_n\}$ and $\{\alpha_n\}$ which converge to x and α respectively. $\widetilde{S}$ *u.s.c.* implies that for every $\varepsilon > 0$ there exists $n_0 = n(\varepsilon)$ such that $z_n(t) \in \widetilde{S}(t,x) = S(f, \alpha b, x)$, for any $n \ge n_0$. Hence there exists a subsequence of $\{z_n\} \in S(f, \alpha b, x)$. By the compactness of $S(f, \alpha b, x)$, there

exists z such that the subsequence $\{z_n\}$ converges to $z \in S(f, \alpha b, x)$. Therefore Π is locally compact.

Step 2. Π has a closed graph. Let $(x_n, \alpha_n) \longrightarrow (x_*, \alpha), h_n \in \Pi(x_n, \alpha_n)$ and $h_n \to h_*$ as $n \to +\infty$. We shall prove that $h_* \in \Pi(x_*, \alpha)$. $h_n \in \Pi(y_n, \alpha_n)$ means that there exists $z_n \in S(f, \alpha_n b, \phi_n)$ such that for each $t \in J$

$$h_n(t) = \begin{cases} a_n, & \text{for } 0 \le t \le \alpha_n b, \\ z_n(t), & \text{for } \alpha_n b < t \le b. \end{cases}$$

We must prove that there exists $z_* \in S(f, \alpha b, x_*)$ such that for each $t \in J$

$$h_*(t) = \begin{cases} x_*(t), & \text{for } 0 \le t \le \alpha b, \\ z_*(t), & \text{for } \alpha b < t \le b. \end{cases}$$

Clearly $(\alpha_n b, a_n) \to (\alpha, x_*)$ as $n \to \infty$ and we can easily show that there exists a subsequence $\{z_n\}$ converging to some limit z_*. The cases $\alpha = 0$ or $\alpha = 1$ can be treated as in the proof of Theorem 3.7. From the above arguing, we find that $z_* \in S(f, \alpha b, x)$, proving our claim. □

Next, more results regarding the topological structure of the solution sets are derived.

Corollary 3.9. *Let $F\colon J \times E \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ be a Carathéodory multi-map satisfying conditions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$.*

(a) *If F is σ-mLL-selectionable, then $S(a)$ is an R_δ-set.*
(b) *If F is σ-Ca-selectionable, then $S(a)$ is the intersection of R_δ-contractible sets.*

Proof. (a) Since F is σ-Ca-selectionable, there exists a decreasing sequence of multi-maps $F_k\colon [0, b] \times E \to \mathcal{P}(E)$ $(k \in \mathbb{N})$ which have measurable selections and such that

$$F_{k+1}(t, u) \subset F_k(t, x) \quad \text{for almost all } t \in [0, b],\ x \in E$$

and

$$F(t, x) = \bigcap_{k=0}^{\infty} F_k(t, x), \quad t \in [0, b],\ x \in E.$$

Then

$$S(F, a) = \bigcap_{k=0}^{\infty} S(F_k, a).$$

From Theorem 3.7, the set $S(F_k, a)$ is contractible for each $k \in \mathbb{N}$, whence our claim. Part (b) is proved analogously. □

CHAPTER 4

THE PERIODIC PROBLEM

In this section, we present some existence results of mild solutions and study the topological structure of solution sets for a first-order impulsive semi-linear differential inclusions subject to periodic boundary conditions:

$$
(4.1) \qquad \begin{cases} (y' - Ay)(t) \in F(t, y(t)), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = y(b). \end{cases}
$$

We will have to distinguish between the cases when either or not 1 lies in the resolvent of $T(b)$. Accordingly, the problem is either formulated as a fixed point problem for an integral operator or for a Poincaré translation operator. Our first existence result relies on a new nonlinear alternative for compact *u.s.c.* maps. Then, we present some existence results and investigate the topological structure of the solution sets. The particular case $Ay = \lambda y$ for some real parameter $\lambda \neq 0$ is studied in [68], [69] in the finite-dimensional case.

4.1. Existence results: $1 \in \rho(T(b))$

In this subsection, we assume that $1 \in \rho(T(b))$, prove some existence results and describe the structure of the solutions sets. The continuous dependence of solutions on parameters in the both convex and the nonconvex cases are also examined.

Let A be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t \geq 0}$ such that $1 \in \rho(T(b))$ and let $f\colon J \to E$ be a continuous function. The following lemma leads us to define what we mean by a mild solution of problem (4.1).

Lemma 4.1. *If $y \in \mathrm{PC}$ is a mild solution of the problem*

$$\begin{cases} y'(t) - Ay(t) = f(t), & t \in J \setminus \{t_1, \dots, t_m\}, \\ y(t_k^-) - y(t_k) = I_k(y(t_k)), & t \neq t_k,\ k = 1, \dots, m, \\ y(0) = y(b), \end{cases} \tag{4.2}$$

then it is given by

$$\begin{aligned} y(t) = {} & T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k) I_k(y(t_k^-)) + \int_0^b T(b-s) f(s)\, ds\Big) \\ & + \int_0^t T(t-s) f(s))\, ds + \sum_{0<t_k<t} T(t - t_k) I_k(y(t_k^-)), \quad \textit{for } t \in J. \end{aligned} \tag{4.3}$$

Proof. Let y be a solution of problem (4.2) and $L(s) = T(t-s)y(s)$ for fixed $t \in J$. We have

$$\begin{aligned} L'(s) = {} & -T'(t-s)y(s) + T(t-s)y'(s) \\ = {} & -AT(t-s)y(s) + T(t-s)y'(s) \\ = {} & T(t-s)[y'(s) - Ay(s)] = T(t-s)f(s). \end{aligned} \tag{4.4}$$

Let $0 < t < t_1$. Integrating the previous equation, we get for $k = 1$

$$L(t) - L(0) = \int_0^t T(t-s) f(s)\, ds.$$

Hence

$$y(t) = T(t)y(0) + \int_0^t T(t-s) f(s)\, ds.$$

More generally, for $t_k < t < t_{k+1}$, we have

$$\begin{aligned} & \int_0^{t_1} L'(s)\, ds + \int_{t_1}^{t_2} L'(s)\, ds + \ldots + \int_{t_k}^{t} L'(s)\, ds = \int_0^t T(t-s) f(s))\, ds \\ \Leftrightarrow\ & L(t_1^-) - L(0) + L(t_2^-) - L(t_1^+) + \ldots + L(t) - L(t_k^+) = \int_0^t T(t-s) f(s)\, ds. \end{aligned}$$

Therefore

$$y(t) = T(t)y(0) + \sum_{0<t_k<t} [L(t_k^+) - L(t_k^-)] + \int_0^t T(t-s) f(s)\, ds.$$

Since $y(0) = y(b)$ and $1 \in \rho(T(b))$, then $I - T(b)$ is invertible. Hence we obtain after substitution

$$\begin{aligned} y(t) =& T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k)I_k(y(t_k)) + \int_0^b T(b - s)f(s)\,ds\Big) \\ &+ \sum_{0<t_k<t} T(t - t_k)I_k(y(t_k)) + \int_0^t T(t - s)f(s)\,ds, \quad \text{for } t \in J, \end{aligned}$$

proving the lemma. □

Definition 4.2. A function $y \in \mathrm{PC}$ is said to be a mild solution of problem (4.1) if there exists $f \in L^1(J, E)$ such that $f(t) \in F(t, y(t))$ almost everywhere on J, $y(0) = y(b)$ and

$$\begin{aligned} y(t) =& T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k)I_k(y(t_k)) + \int_0^b T(b - s)f(s)\,ds\Big) \\ &+ \int_0^t T(t - s)f(s))\,ds + \sum_{0<t_k<t} T(t - t_k)I_k(y(t_k^-)), \quad \text{for } t \in J. \end{aligned}$$

4.1.1. The convex case: a direct approach. Consider the following assumptions:

($\mathcal{B}_1'$) $F\colon J \times E \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ is an integrably bounded multi-map, i.e. there exists $p \in L^1(J, E)$ such that

$$\|F(t, x)\|_{\mathcal{P}} \le p(t), \quad \text{for a.e. } t \in J \text{ and every } x \in E.$$

($\mathcal{B}_2'$) There exist constants $a_k, b_k > 0$ and $\alpha \in [0, 1)$ such that

$$|I_k(x)| \le a_k|x|^\alpha + b_k, \quad \text{for every } x \in E,\ k = 1, \dots, m.$$

Remark 4.3. One can relax assumption ($\mathcal{B}_1'$) and replace it by the following sublinear growth condition:

($\mathcal{B}_1''$) there exist $p, q \in L^1(J, \mathbb{R}_+)$ and $\beta \in [0, 1 - \alpha)$ such that

$$\|F(t, x)\|_{\mathcal{P}} \le q(t) + p(t)|x|^\beta, \quad \text{for a.e. } t \in J \text{ and every } x \in E$$

Our first existence result is

Theorem 4.4. *Assume $F\colon J \times E \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ is a Carathéodory multi-map satisfying ($\mathcal{B}_1'$)–($\mathcal{B}_2'$) and ($\mathcal{A}_3$)$'$. Then problem* (4.1) *has at least one solution. If further E is a reflexive space, then the solution set is compact in* PC.

Proof. Part 1. Existence of solutions. It is clear that all solutions of problem (4.1) are fixed points of the multi-valued operator $N\colon \mathrm{PC} \to \mathcal{P}(\mathrm{PC})$ defined by

$N(y) := \{h \in \mathrm{PC}\}$ with

$$h(t) = T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k)I(y(t_k^-)) + \int_0^b T(b - s)f(s)\,ds\Big) + \int_0^t T(t - s)f(s))\,ds + \sum_{0<t_k<t} T(t - t_k)I_k(y(t_k^-)), \quad \text{for } t \in J$$

and where $f \in S_{F,y} = \{f \in L^1([0, b], E) : f(t) \in F(t, y(t))$, for almost every $t \in [0, b]\}$. $(\mathcal{B}_1')$ implies that the set $S_{F,y}$ is nonempty. Since, for each $y \in \mathrm{PC}$, the nonlinearity F takes convex values, the selection set $S_{F,y}$ is convex and therefore N has convex values.

Step 1. N is completely continuous.

(a) N sends bounded sets into bounded sets in PC.

Let $q > 0$, $B_q := \{y \in \mathrm{PC} : \|y\|_{\mathrm{PC}} \le q\}$ be a bounded set in PC, and $y \in B_q$. Then for each $h \in N(y)$, there exists $f \in S_{F,y}$ such that

$$h(t) = T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k)I(y(t_k)) + \int_0^b T(b - s)f(s)\,ds\Big) + \int_0^t T(t - s)f(s))\,ds + \sum_{0<t_k<t} T(t - t_k)I_k(y(t_k^-)), \quad \text{for } t \in J.$$

Using Proposition 2.69, we obtain that for each $t \in J$,

$$\|h\|_{\mathrm{PC}} \le e^{2\omega b}\|(I - T(b))^{-1}\|_{B(E)}\Big(\sum_{k=1}^{m}(a_k q^\alpha + b_k) + \|p\|_{L^1}\Big) + e^{\omega b}\|p\|_{L^1} + e^{\omega b}\sum_{k=1}^{m}(a_k q^\alpha + b_k).$$

(b) N maps bounded sets into equicontinuous sets of PC.

Let $\tau_1, \tau_2 \in J\,(0 < \tau_1 < \tau_2)$, $\mathcal{B}_q$ be a bounded set of PC, and $y \in \mathcal{B}_q$. Then for almost every $t \in J$, we have the estimate

$$\begin{aligned}
&|h(\tau_2) - h(\tau_1)| \\
&\quad \le e^{\omega b}\|T(\tau_2) - T(\tau_1)\|_{B(E)}\|(I - T(b))^{-1}\|_{B(E)}\Big(\sum_{k=1}^{m}(a_k q^\alpha + b_k) + \|p\|_{L^1}\Big) \\
&\qquad + \int_{\tau_1}^{\tau_2}\|T(\tau_2 - s)\|_{B(E)}p(s)\,ds + \int_0^{\tau_1}\|T(\tau_1 - s) - T(\tau_2 - s)\|_{B(E)}p(s)\,ds \\
&\qquad + \sum_{\tau_1<t_k<\tau_2}\|T(\tau_1 - t_k) - T(\tau_2 - t_k)\|_{B(E)}(a_k q^\alpha + b_k).
\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$ since $T(t)$ is uniformly continuous. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \dots, m$. It remains to examine the equicontinuity at $t = t_i$. Set

$$\begin{aligned} h_1(t) =& T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k)I(y(t_k)) + \int_0^b T(b-s)f(s)\,ds\Big) \\ &+ \sum_{0<t_k<t} T(t - t_k)I_k(y(t_{k^-})). \end{aligned}$$

First we prove equicontinuity at $t = t_i^-$. Let $\delta_1 > 0$ be such that

$$\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset.$$

We have

$$\begin{aligned} h_1(t_i) =& T(t_i)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k)I(y(t_k)) + \int_0^b T(b-s)f(s)\,ds\Big) \\ &+ \sum_{0<t_k<t_i} T(t_i - t_k)I_k(y(t_k)) \\ =& T(t_i)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k)I(y(t_k)) + \int_0^b T(b-s)f(s)\,ds\Big) \\ &+ \sum_{k=1}^{i-1} T(t_i - t_k)I_k(y(t_k)). \end{aligned}$$

For $0 < h < \delta_1$, we have

$$\begin{aligned} &|h_1(t_i - h) - h_1(t_i)| \\ &\le |(T(t_i - h) - T(t_i))|\|(I - T(b))^{-1}\|_{B(E)}\Big(e^{\omega b}\sum_{k=1}^{m}(a_k q^\alpha + b_k) + e^{\omega b}\|p\|_{L^1}\Big) \\ &\quad + \sum_{k=1}^{i-1} |(T(t_i - h - t_k) - T(t_i - t_k))|[a_k q^\alpha + b_k]. \end{aligned}$$

By the uniform continuity of $T(t)$, the right-hand side tends to zero as $h \to 0$. If $h_2(t) = \int_0^t T(t-s)f(s)\,ds$, then

$$|h_2(t_i - h) - h_2(t_i)| \le \int_0^{t_i - h} |[T(t_i - h - s) - T(t_i - s)]p(s)|\,ds + e^{\omega b}\int_{t_i - h}^{t_i} p(s)\,ds,$$

which also tends to zero as $h \to 0$. Next we prove equicontinuity at $t = t_i^+$. Define

$$\widehat{h}_0(t) = h(t), \quad \text{for } t \in [0, t_1]$$

and

$$\widehat{h}_i(t) = \begin{cases} h(t), & t \in (t_i, t_{i+1}], \\ h(t_i^+), & t = t_i. \end{cases}$$

Fix $\delta_2 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. Then

$$\begin{aligned}\widehat{h}(t_i) =& T(t_i)\Big(\sum_{k=1}^{m} T(b-t_k)I(y(t_k)) + \int_0^b T(b-s)f(s)\,ds\Big) \\ &+ \int_0^{t_i} T(t_i - s)f(s)\,ds + \sum_{k=1}^{i} T(t_i - t_k)I_k(y(t_k)).\end{aligned}$$

For $0 < h < \delta_2$, we have

$$\begin{aligned}|\widehat{h}(t_i + h) - \widehat{h}(t_i)| \le{}& e^{\omega b}|(T(t_i + h) - T(t_i)|\Big(\sum_{k=1}^{m}(a_k q^\alpha + b_k) + \|p\|_{L^1}\Big) \\ &+ \int_0^{t_i} |[T(t_i + h - s) - T(t_i - s)]|p(s)\,ds \\ &+ e^{\omega b}\int_{t_i}^{t_i+h} p(s)\,ds + \sum_{k=1}^{i}[T(t_i + h - t_k) - T(t_i - t_k)](a_k q^\alpha + b_k),\end{aligned}$$

which also tends to zero as $h \to 0$.

(c) As a consequence of parts (a), (b) together with the Arzéla–Ascoli theorem, it remains to show that N maps B_q into a precompact set in E. Let $0 < t \le b$ and let $0 < \varepsilon < t$. For $y \in B_q$, define

$$\begin{aligned}h_\varepsilon(t) =& T(\varepsilon)T(t-\varepsilon)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I(y(t_k)) + \int_0^b T(b-s)f(s)\,ds\Big) \\ &+ T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon)f(s)\,ds + T(\varepsilon)\sum_{0<t_k<t-\varepsilon} T(t-\varepsilon-t_k)I_k(y(t_k)).\end{aligned}$$

The compactness of $\{T(t)\}_{t>0}$ implies that $N_\varepsilon(B_q)$ is a precompact subset in E, where N_ε is the fixed point operator with h replaced by h_ε. Moreover,

$$|h(t) - h_\varepsilon(t)| \le \int_{t-\varepsilon}^{t} \|T(t-s)\|_{B(E)}p(s)\,ds + \sum_{t-\varepsilon<t_k<t} \|T(t-t_k)\|_{B(E)}(a_k q^\alpha + b_k),$$

which tends to 0 as $\varepsilon \to 0$. Therefore, there are precompact sets arbitrarily close to the set $H(t) = \{h(t) : h \in N(y)\}$. This set is then precompact in E.

Step 2. N is *u.s.c.* Let $h_n \to h_*$, $h_n \in N(y_n)$ and $y_n \to y_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) =& T(t)(I-T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_n(t_k)) + \int_0^b T(b-s)f_n(s)\,ds\Big) \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k)) + \int_0^t T(t-s)f_n(s)\,ds \end{aligned}$$

for each $t \in J$. First, we have

$$\begin{aligned} \Big\|&(h_n - T(t)(I-T(b))^{-1})\sum_{k=1}^{m} T(t-t_k)I_k(y_n(t_k))) \\ &- (h_* - T(t)(I-T(b))^{-1})\sum_{k=1}^{m} T(t-t_k)I_k(y_*(t_k)))\Big\|_\infty \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Now, consider the linear continuous operator $\Gamma\colon L^1(J,E) \to \mathrm{PC}(J,E)$ defined by

$$(\Gamma v)(t) = \int_0^t T(t-s)v(s)\,ds + T(t)(I-T(b))^{-1}\int_0^b T(t-s)v(s)\,ds.$$

From the definition of Γ, we know that

$$h_n(t) - T(t)(I-T(b))^{-1})\sum_{k=1}^{m} T(t-t_k)I_k(y_n(t_k) \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$ and $\Gamma \circ S_F$ is a closed graph operator by Lemma 2.10, then there exists $f_* \in S_{F,y_*}$ such that

$$\begin{aligned} h_*(t) =& T(t)(I-T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_*(t_k)) + \int_0^b T(b-s)f_*(s)\,ds\Big) \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k)) + \int_0^t T(t-s)f_*(s)\,ds. \end{aligned}$$

Hence $h_* \in N(y_*)$, proving that N has a closed graph. Finally, Lemma 2.9 implies that N is *u.s.c.*

Step 3. A priori bounds on solutions. Let $y \in \mathrm{PC}$ be such that $y \in N(y)$. Then there exists $f \in S_{F,y}$ such that

$$\begin{aligned} y(t) =& T(t)(I-T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I(y(t_k)) + \int_0^b T(b-s)f(s)\,ds\Big) \\ &+ \int_0^t T(t-s)f(s))\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \quad \text{for } t \in J. \end{aligned}$$

By Proposition 2.69, there exists a constant $\omega \geq 0$ such that

$$|y(t)| \leq e^{2\omega b}\|(I-T(b))^{-1}\|_{B(E)}\bigg(\sum_{k=1}^{m}(a_k|y(t_k)|^{\alpha}+b_k)+\int_0^b |f(s)|\,ds\bigg)$$
$$+e^{\omega b}\int_0^t |f(s)|\,ds + Me^{\omega b}\sum_{k=1}^{m}(a_k|y(t_k^-)|^{\alpha}+b_k),$$

and so

$$\|y\|_{\mathrm{PC}} \leq e^{2\omega b}\|(I-T(b))^{-1}\|_{B(E)}\bigg(\sum_{k=1}^{m}(a_k\|y\|_{\mathrm{PC}}^{\alpha}+b_k)+\|p\|_{L^1}\bigg)$$
$$+e^{\omega b}\|p\|_{L^1}+e^{\omega b}\sum_{k=1}^{m}(a_k\|y\|_{\mathrm{PC}}^{\alpha}+b_k).$$

If $\|y\|_{\mathrm{PC}} > 1$, then since $0 \leq \alpha < 1$, we have

$$\|y\|_{\mathrm{PC}}^{1-\alpha} \leq e^{2\omega b}\|(I-T(b))^{-1}\|_{B(E)}\bigg(\sum_{k=1}^{m}(a_k+b_k)+\|p\|_{L^1}\bigg)$$
$$+e^{\omega b}\|p\|_{L^1}+e^{\omega b}\sum_{k=1}^{m}(a_k+b_k).$$

Hence

$$\|y\|_{\mathrm{PC}} \leq \bigg(e^{2\omega b}\|(I-T(b))^{-1}\|_{B(E)}\bigg(\sum_{k=1}^{m}(a_k+b_k)+\|p\|_{L^1}\bigg)$$
$$+e^{\omega b}\|p\|_{L^1}+e^{\omega b}\sum_{k=1}^{m}(a_k+b_k)\bigg)^{1/(1-\alpha)} := \overline{M}.$$

Finally

$$\|y\|_{\mathrm{PC}} \leq \max(1,\overline{M}) := \widetilde{M}.$$

Let $U := \{y \in \mathrm{PC} : \|y\|_{\mathrm{PC}} < \widetilde{M}+1\}$, and consider the operator $N\colon \overline{U} \to \mathcal{P}_{\mathrm{cv,cp}}(\mathrm{PC})$. From the choice of U, there is no $y \in \partial U$ such that $y \in \gamma N(y)$ for some $\gamma \in (0,1)$. As a consequence of the Leray–Schauder nonlinear alternative (Lemma 2.72), we deduce that N has a fixed point y in U which is a solution of problem (4.1).

Part 2. Compactness of the solution set. Let

$$S_F = \{y \in \mathrm{PC} : y \text{ is a solution of problem (4.1)}\}.$$

From Part 1, $S_F \neq \emptyset$ and there exists $\widetilde{M}$ such that for every $y \in S_F$, $\|y\|_{\mathrm{PC}} \leq \widetilde{M}$. Since N is completely continuous, $N(S_F)$ is relatively compact in PC. Moreover, $S_F \subset \overline{N(S_F)}$. Thus it remains to prove that S_F is closed set in PC. Let

$y_n \in S_F$ be such that y_n converges to some limit y. For every $n \in \mathbb{N}$, there exists $v_n(t) \in F(t, y_n(t))$, for almost every $t \in J$ and such that

$$\begin{aligned}(4.5)\quad y_n(t) =& T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_n(t_k)) + \int_0^b T(b-s)v_n(s)\,ds\Big)\\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k)) + \int_0^t T(t-s)v_n(s)\,ds.\end{aligned}$$

$(\mathcal{B}_1')$ implies that $v_n(t) \in p(t)B(0,1)$, hence $(v_n)_{n\in\mathbb{N}}$ is integrably bounded. Note that this still remains true when $(\mathcal{B}_1)''$ holds for S_F is a bounded set. Since E is reflexive, $(v_n)_{n\in\mathbb{N}}$ has a subsequence, still denoted $(v_n)_{n\in\mathbb{N}}$, which converges weakly to some limit $v \in L^1(J,E)$ (see the proof of Theorem 3.3, Part 1, Step 2). Moreover, the mapping $\Gamma\colon L^1(J,E) \to \mathrm{PC}(J,E)$ defined by

$$\Gamma(g)(t) = \int_0^t T(t-s)g(s)\,ds$$

is a bounded linear operator, hence continuous, these spaces being endowed with their weak topologies (see [26, Theorem 3.10]). Therefore, for almost every $t \in J$, the sequence $y_n(t)$ converges to $y(t)$ and by the continuity of I_k, we obtain at the limit

$$\begin{aligned}y(t) =& T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y(t_k)) + \int_0^b T(b-s)v(s)\,ds\Big)\\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)) + \int_0^t T(t-s)v(s)\,ds.\end{aligned}$$

It remains to prove that $v \in F(t, y(t))$, for almost every $t \in J$. Lemma 2.62 yields the existence of constants $\alpha_i^n \geq 0$, $i = n, \dots, k(n)$ such that $\alpha_i^n = 1 = 0$ for i large enough, $\sum_{i=n}^{\infty} \alpha_i^n = 1$ and the sequence of convex combinations $g_n(\,\cdot\,) = \sum_{i=n}^{\infty} \alpha_i^n v_i(\,\cdot\,)$ converges strongly to v in L^1. Since F takes convex values, using Lemma 2.6, we obtain that

$$\begin{aligned}(4.6)\qquad v(t) &\in \bigcap_{n\geq 1} \overline{\{g_k(t),\, k \geq n\}}, \quad \text{a.e. } t \in J\\ &\subset \bigcap_{n\geq 1} \overline{\mathrm{co}}\{v_k(t),\ k \geq n\}\\ &\subset \bigcap_{n\geq 1} \overline{\mathrm{co}}\Big\{\bigcup_{k\geq n} F(t, y_k(t))\Big\} = \overline{\mathrm{co}}\Big(\limsup_{k\to\infty} F(t, y_k(t))\Big).\end{aligned}$$

Since F is *u.s.c.* with respect to the second argument and has compact values, then by Lemma 2.5, we have

$$\limsup_{n\to\infty} F(t, y_n(t)) = F(t, y(t)), \quad \text{for a.e. } t \in J.$$

This with (4.6) imply that $v(t) \in \overline{\mathrm{co}}\, F(t, y(t))$. Since $F(\,\cdot\,,\,\cdot\,)$ has closed, convex values, we deduce that $v(t) \in F(t, y(t))$, for almost every $t \in J$, as claimed. Hence $y \in S_F$ which yields that S_F is closed, hence compact in PC. □

4.1.2. The convex case: the MNC approach. Since $T(t)$ is strongly continuous, we may assume that the existence $M > 0$ such that

$$\|T(t)\|_{B(E)} \le M, \quad \text{for every } t \in [0, b]. \tag{4.7}$$

Let $F\colon J \times E \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ be a Carathéodory multi-map which satisfies:

($\mathcal{B}_4'$) There exists $\overline{p} \in L^1([0, b], \mathbb{R}_+)$ such that for every bounded D in E,

$$\chi(F(t, D)) \le \overline{p}(t)\chi(D),$$

($\mathcal{B}_5'$) There exist $d_k > 0$ such that for every bounded D in E.

$$\chi(I_k(D)) \le d_k \chi(D), \quad k = 1, \dots, m.$$

Lemma 4.5. *Under conditions* ($\mathcal{B}_1'$) *and* ($\mathcal{B}_4'$), *the operator N is closed, $N(y) \in \mathcal{P}_{\mathrm{cp,cv}}(\mathrm{PC})$, for every $y \in \mathrm{PC}$, and N is u.s.c., where the fixed point operator N is as defined in the proof of Theorem* 4.4.

Proof. Step 1. N is closed.

Let $h_n \to h_*$, $h_n \in N(y_n)$ and $y_n \to y_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that for almost every $t \in J$

$$\begin{aligned} h_n(t) =& T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k) I_k(y_n(t_k)) + \int_0^b T(b - s) f_n(s)\, ds\Big) \\ &+ \sum_{0<t_k<t} T(t - t_k) I_k(y_n(t_k)) + \int_0^t T(t - s) f_n(s)\, ds. \end{aligned} \tag{4.8}$$

Since $\{f_n(t) : n \in \mathbb{N}\} \subseteq F(t, y_n(t))$, assumption ($\mathcal{B}_1'$) implies that $(f_n)_{n\in\mathbb{N}}$ is integrably bounded. In addition, the set $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact for almost every $t \in J$ because assumption ($\mathcal{B}_4'$) both with the convergence of $\{y_n\}_{n\in\mathbb{N}}$ imply that

$$\chi(\{f_n(t) : n \in \mathbb{N}\}) \le \chi(F(t, y_n(t)) \le \overline{p}(t)\chi(\{y_n(t) : n \in \mathbb{N}\}) = 0.$$

Hence the sequence $\{f_n : n \in \mathbb{N}\}$ is semi-compact, hence weakly compact in $L^1([0, b]; E)$ and tends weakly to some limit f_* by Lemma 2.60. Arguing as in

the proof of Theorem 4.4, Part 2, and passing to the limit in (4.8), we obtain that $f_* \in S_{F,y_*}$ and for almost every $t \in J$

$$(4.9)\quad h_*(t) = T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_*(t_k)) + \int_0^b T(b-s)f_*(s)\,ds\Big) + \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k)) + \int_0^t T(t-s)f_*(s)\,ds.$$

As a consequence, $(y_*, h_*) \in Gr\,N$, as claimed.

Step 2. N has compact, convex values.

The convexity of $N(y)$ follows immediately from the convexity of the values of F. To prove the compactness of the values of N, let $N(y) \in \mathcal{P}(E)$ for some $y \in \mathrm{PC}$ and $h_n \in N(y)$. Then there exists $f_n \in S_{F,y}$ satisfying (4.8). Arguing again as in Step 1 and using $(\mathcal{B}_4')$, we can prove that $\{f_n\}$ is semi-compact and converges weakly to some limit $f_* \in F(t, y(t))$, for almost every $t \in [0, b]$; passing to the limit in (4.8), h_n tends to some limit h_* in the closed set $N(y)$ with h_* satisfying (4.9). Therefore the convex set $N(y)$ is sequentially compact, hence compact [108].

Step 3. N is *u.s.c.*

Using Lemma 2.9, it suffices to prove that N is quasicompact. Let K be a compact set in PC and $h_n \in N(y_n)$ where $y_n \in K$. Then there exists $f_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_n(t_k)) + \int_0^b T(b-s)f_n(s)\,ds\Big) + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k)) + \int_0^t T(t-s)f_n(s)\,ds.$$

Since K is compact, we may pass to a subsequence, if necessary, to get that $\{y_n\}$ converges to some limit y_* in PC. Arguing as in Step 1, we can prove the existence of a subsequence of $\{f_n\}$ which converges weakly to some limit f_* and hence h_n converges to h_*, where

$$h_*(t) = T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_*(t_k)) + \int_0^b T(b-s)f_*(s)\,ds\Big) + \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k)) + \int_0^t T(t-s)f_*(s)\,ds.$$

As a consequence, N is *u.s.c.* □

Our second existence result regarding the convex case is

Theorem 4.6. *Assume that F satisfies assumptions $(\mathcal{B}_1')$, $(\mathcal{B}_4')$ and $(\mathcal{B}_5')$. If*

$$q := M(\|M(I-T(b))^{-1}\|_{B(E)}+1)\Big(\sum_{k=1}^{m} d_k + 2\int_0^b \overline{p}(s)\,ds\Big) < 1,$$

then the set of solutions for problem (4.1) *is nonempty and compact.*

Proof. It is clear that all solutions of problem (4.1) are fixed points of the multi-valued operator N defined in Theorem 4.4. By Lemma 4.5, $N(\,\cdot\,) \in \mathcal{P}_{\mathrm{cv,cp}}(\mathrm{PC})$ and it is *u.s.c.* Next, we prove that N is a $\beta-$condensing operator for a suitable MNC β. Given a bounded subset $D \subset \mathcal{K}_0$, let $\mathrm{mod}_C(D)$ the modulus of quasi-equicontinuity of the set of functions D defined by

$$\mathrm{mod}_C(D) = \lim_{\delta\to 0}\sup_{x\in D}\max_{|\tau_2-\tau_1|\le\delta}|x(\tau_1)-x(\tau_2)|.$$

Then $\mathrm{mod}_C(D)$ defines an MNC in PC which satisfies all of the properties in Definition 2.55 except regularity (see, e.g. Example 2.1.2 in [93]). Given the Hausdorff MNC χ, let γ be the real MNC defined on bounded subsets on PC by

$$\gamma(D) = \sup_{t\in[0,b]}\chi(D(t)).$$

Let $D \subset \mathrm{PC}$ be bounded and define the following MNC on bounded subsets of PC by

$$\beta(D) = \max_{D\in\Delta(\mathrm{PC})}(\gamma(D), \mathrm{mod}_C(D)),$$

where $\Delta(\mathrm{PC})$ is the collection of all denumerable bounded subsets of D. Then the MNC β is monotone, regular, and nonsingular (see Example 2.1.4 in [93]). To show that N is $\beta-$condensing, let $B \subset \mathrm{PC}$ be bounded and

$$\beta(B) \le \beta(N(B)). \tag{4.10}$$

We will show that B is relatively compact. Let $\{y_n : n \in \mathbb{N}\} \subset B$ and let $N = L_1 + L_2 \circ \Gamma_1 \circ S_F + \Gamma \circ S_F$ where $L_1\colon \mathrm{PC} \to \mathrm{PC}$ is defined by

$$(L_1 y)(t) = T(t)(I-T(b))^{-1}\sum_{k=1}^{m} T(b-t_k)I_k(y(t_k)) + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)),$$

$L_2\colon \mathbb{R}_+ \to B(E)$ is defined by

$$L_2(y) = T(t)(I-T(b))^{-1}y,$$

$S_F\colon \mathrm{PC} \to L^1(J,E)$ is defined by

$$S_F(y) = \{v \in L^1(J,E) : v(t) \in F(t,y(t)),\ \text{a.e. } t \in [0,b]\},$$

$\Gamma_1\colon L^1(J,E) \to \mathrm{PC}$ is defined by

$$\Gamma_1(f)(t) = \int_0^b T(t-s)f(s)\,ds, \quad t \in [0,b],$$

and

$$\Gamma(f)(t) = \int_0^t T(t-s)f(s)\,ds, \quad t \in [0,b].$$

Then

$$|\Gamma f_1(t) - \Gamma f_2(t)| \le \int_0^t \|T(t-s)\| \cdot |f_1(s) - f_2(s)|\,ds \le Me \int_0^t |f_1(s) - f_2(s)|\,ds.$$

Moreover, each element h_n in $N(y_n)$ can be represented as

$$h_n = L_1(y_n) + T(\,\cdot\,)(I - T(b))^{-1}\Gamma_1(f_n) + \Gamma(f_n), \tag{4.11}$$

with some $f_n \in S_F(y_n)$, and (4.10) yields

$$\beta(\{h_n : n \in \mathbb{N}\}) \ge \beta(\{y_n : n \in \mathbb{N}\}). \tag{4.12}$$

From assumption $(\mathcal{B}_4')$, for almost every $t \in [0,b]$, we have

$$\begin{aligned} \chi(\{f_n(t) : n \in \mathbb{N}\}) &\le \chi(F(t, \{y_n(t)\}_{n=1}^\infty) \le \overline{p}(t)\chi(\{y_n(t)\}_{n=1}^\infty) \\ &\le \overline{p}(t) \sup_{0 \le s \le t} \chi(\{y_n(s)\}_{n=1}^\infty) \le \overline{p}(t)\gamma(\{y_n\}_{n=1}^\infty). \end{aligned} \tag{4.13}$$

Lemmas 2.57 and 2.58 imply that

$$\chi(\{\Gamma(f_n)(t)\}_{n=1}^\infty) \le \gamma(\{y_n\}_{n=1}^\infty) 2M \int_0^t \overline{p}(s)\,ds,$$

$$\begin{aligned} \chi(\{T(\,\cdot\,)(I - T(b))^{-1}\Gamma_1(f_n)(t)\}_{n=1}^\infty) &\le M\|(I - T(b))^{-1}\|_{B(E)}\{\Gamma_1(f_n)(t)\}_{n=1}^\infty \\ &\le 2M^2\|(I - T(b))^{-1}\|_{B(E)}\gamma(\{y_n\}_{n=1}^\infty) \int_0^b \overline{p}(s)\,ds, \end{aligned}$$

and

$$\begin{aligned} &\chi(L_1\{y_n(t)\}_{n=1}^\infty) \\ &\quad \le M^2\|(I - T(b))^{-1}\|_{B(E)} \sum_{k=1}^m \chi(I_k\{y_n(t_k)\}_{n=1}^\infty) + M \sum_{k=1}^m \chi(I_k\{y_n(t_k)\}_{n=1}^\infty) \\ &\quad \le M^2\|(I - T(b))^{-1}\|_{B(E)} \sum_{k=1}^m d_k\chi(\{y_n(t_k)\}_{n=1}^\infty) + M \sum_{k=1}^m d_k\chi(\{y_n(t_k)\}_{n=1}^\infty) \\ &\quad \le (M^2\|(I - T(b))^{-1}\|_{B(E)} + M) \sum_{k=1}^m d_k\gamma(\{y_n\}_{n=1}^\infty). \end{aligned}$$

(4.11) and the lower additivity of γ yield

$$\gamma(\{h_n\}_{n=1}^{\infty}) \le \left(M^2\|(I-T(b))^{-1}\|_{B(E)} + M\right) \cdot \left(\sum_{k=1}^{m} d_k + 2\|\overline{p}\|_{L^1}\right)\gamma(\{y_n\}_{n=1}^{\infty}). \tag{4.14}$$

Therefore

$$\gamma(\{y_n\}_{n=1}^{\infty}) \le \gamma(\{h_n\}_{n=1}^{\infty}) = \sup_{t\in[0,b]} \chi(\{h_n(t)\}_{n=1}^{\infty} \le q\gamma(\{y_n\}_{n=1}^{\infty}). \tag{4.15}$$

Since $0 < q < 1$, we infer that

$$\gamma(\{y_n\}_{n=1}^{\infty}) = 0. \tag{4.16}$$

Back to (4.14), we find that $\gamma(h_n) = 0$. To check that the set $\{h_n\}$ is equicontinuous, we proceed as in the proof of Theorem 4.6, Step 1, part (b) and obtain that $\beta(\{h_n\}_{n=1}^{\infty}) = 0$ which implies, by (4.12), that $\beta(\{y_n\}_{n=1}^{\infty}) = 0$. We have proved that B is relatively compact. Hence $N\colon \overline{U} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathrm{PC})$ is *u.s.c.* and β-condensing, where the open set U is as defined in the proof of Theorem 4.4. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type for condensing maps (Lemma 933), we deduce that N has a fixed point y in U, which is a solution of problem (4.1). Finally, since $\mathrm{Fix}\,(N)$ is bounded, by Lemma 2.77, $\mathrm{Fix}\,(N)$ is further compact. □

4.1.3. The nonconvex case. In this subsection, we first present an existence result for problem (4.1) when the multi-valued nonlinearity is not necessarily convex. In the proof, we will make use of the nonlinear alternative of Leray–Schauder type (Lemma 2.72) combined with a selection theorem due to Bressan–Colombo–Fryszkowski (Lemma 24) for lower semicontinuous multimaps with decomposable values. Finally, using Covitz–Nadler fixed point theorem (Theorem 2.74), we also prove an existence result under Hausdorff Lipschitz conditions.

Theorem 4.7. *Suppose that the hypotheses* $(\mathcal{B}_1')$–$(\mathcal{B}_2')$ *hold together with* $(\mathcal{H}_{\mathrm{lsc}})$. *Then problem* (4.1) *has at least one solution.*

Proof. $(\mathcal{B}_1')$ and $(\mathcal{H}_{lsc})$ imply, by Lemma 2.23, that F is of lower semicontinuous type. From Lemma 24, there is a continuous selection $f\colon \mathrm{PC} \to L^1([0,b],E)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \mathrm{PC}$, where $\mathcal{F}$ is the Nemyts'kiĭ

operator associated with F. Consider the problem

$$\begin{cases} y'(t) - Ay(t) &= f(y)(t), \qquad t \in [0,b],\ t \neq t_k,\ k = 1, \dots, m, \\ \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(0) &= y(b), \end{cases} \tag{4.17}$$

and the operator $G\colon \mathrm{PC} \to \mathrm{PC}$ defined by

$$\begin{aligned} G(y)(t) =& T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I(y(t_k)) + \int_0^b T(b-s)f(y)(s)\,ds\Big) \\ &+ \int_0^t T(t-s)f(y)(s))\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \quad \text{for } t \in [0,b]. \end{aligned}$$

As in Theorem 4.4, we can prove that the single-valued operator G is compact and there exists $M_* > 0$ such that for all possible solutions y, we have $\|y\|_{\mathrm{PC}} \leq M_*$. Now, we only check that G is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in PC, as $n \to +\infty$. Then

$$\begin{aligned} \|G(y_n) - G(y)\|_{\mathrm{PC}} \leq& e^{2\omega b}(\|(I - T(b))^{-1}\|_{B(E)})\|f(y_n) - f(y)\|_{L^1} \\ &+ e^{2\omega b}(\|(I - T(b))^{-1}\|_{B(E)})\sum_{k=1}^{m} |I_k(y_n(t_k^-)) - I_k(y(t_k^-))|, \end{aligned}$$

which, by continuity of f and I_k $(1 \leq k \leq n)$, tends to 0, as $n \to +\infty$. Let

$$U = \{y \in \mathrm{PC} : \|y\|_{\mathrm{PC}} < M_* + 1\}.$$

From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N y$ for in $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of the Leray-Schauder type (Lemma 2.73), we deduce that G has a fixed point $y \in U$ which is a solution of problem (4.17), hence a solution of problem (4.1).

Now, assume that the following hypotheses:

$(\mathcal{C}_1')$ $F\colon J \times E \to \mathcal{P}_{\mathrm{cp}}(E)$; $t \mapsto F(t,x)$ is measurable for each $x \in E$.

$(\mathcal{C}_2')$ There exists a function $l \in L^1(J, \mathbb{R}^+)$ such that

$$H_d(F(t,x), F(t,y)) \leq l(t)|x - y|, \quad \text{for a.e. } t \in J \text{ and all } x, y \in E,$$

with $H_d(0, F(t,0)) \leq l(t)$, for almost every $t \in J$.

$(\mathcal{C}_3')$ There exists constants c_k, such that

$$|I_k(x) - I_k(y)| \leq c_k|x - y|, \quad \text{for } k = 1, \dots, m \text{ and } x, y \in E.$$

Theorem 4.8. *Let assumptions* $(\mathcal{C}_1')$–$(\mathcal{C}_3')$ *be satisfied. If further*

$$Me^{\omega b}\Big(\sum_{k=1}^{k=m} c_k + \|l\|_{L^1}\Big)\Big(1 + Me^{\omega b}\|(I - T(b))^{-1}\|_{B(E)}\Big) < 1,$$

then problem (4.1) *has at least one solution.*

Proof. In order to transform problem (4.1) into a fixed point problem, let the multi-valued operator $N\colon \mathrm{PC} \to \mathcal{P}(\mathrm{PC})$ be as defined in Theorem 4.4. We shall show that N satisfies all the assumptions of Lemma 2.74.

(a) $N(y) \in \mathcal{P}_{\mathrm{cl}}(\mathrm{PC})$ for each $y \in \mathrm{PC}$. The proof is similar to that in Theorem 3.4, Part 1, Step 2 and is omitted.

(b) There exists $\gamma < 1$ such that

$$H_d(N(y), N(\overline{y})) \le \gamma\|y - \overline{y}\|_{\mathrm{PC}}, \quad \text{for all } y, \overline{y} \in \mathrm{PC}.$$

Let $y, \overline{y} \in \mathrm{PC}$ and $h \in N(y)$. Then there exists a measurable selection $g(t) \in F(t, y(t))$ such that for each $t \in J$

$$\begin{aligned} h(t) =& T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k)I(y(t_k)) + \int_0^b T(b - s)g(s)\,ds\Big) \\ &+ \int_0^t T(t - s)g(s)\,ds + \sum_{0<t_k<t} T(t - t_k)I_k(y(t_k^-)). \end{aligned}$$

$(\mathcal{C}_2')$ implies that

$$H_d(F(t, y(t)), F(t, \overline{y}(t))) \le l(t)|y(t) - \overline{y}(t)|, \quad \text{a.e. } t \in J.$$

Hence there is $w \in F(t, \overline{y}(t))$ such that

$$|g(t) - w| \le l(t)|y(t) - \overline{y}(t)|, \quad t \in J.$$

Then consider the mapping $U\colon J \to P(E)$, given by

$$U(t) = \{w \in E : |g(t) - w| \le l(t)|y(t) - \overline{y}(t)|\}, \quad t \in J.$$

$U(t) = \overline{B}(g(t), l(t)|y(t) - \overline{y}(t)|)$ is a closed ball in E. Since g, l, y, $\overline{y}$ are measurable, Theorem III.4.1 in [33] shows that the closed ball U is measurable. In addition $(\mathcal{C})_1'$ and $(\mathcal{C})_2'$ imply that for each $y \in \mathrm{PC}$, $F(t, y(t))$ is measurable. Finally the set $V(t) = U(t) \cap F(t, \overline{y}(t))$ is nonempty since it contains w. Therefore the intersection multi-valued operator V is measurable with nonempty, closed values (see [13], [33], [61]). By Lemma 2.15, there exists a function $\overline{g}(t)$, which is a measurable selection for V. Thus $\overline{g}(t) \in F(t, \overline{y}(t))$ and

$$|g(t) - \overline{g}(t)| \le l(t)|y(t) - \overline{y}(t)|, \quad \text{for a.e. } t \in J.$$

Let us define for almost every $t \in J$

$$\begin{aligned}\overline{h}(t) =& T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k)I(\overline{y}(t_k)) + \int_0^b T(b - s)\overline{g}(s)\,ds\Big)\\ &+ \int_0^t T(t - s)\overline{g}(s)\,ds + \sum_{0<t_k<t} T(t - t_k)I_k(\overline{y}(t_k)).\end{aligned}$$

Then

$$\begin{aligned}|h(t) - \overline{h}(t)| \le{}& M^2 e^{2\omega b}\|(I - T(b))^{-1}\|_{B(E)} \sum_{k=1}^{m} c_k|y(t_k)) - \overline{y}(t_k)|\\ &+ M^2 e^{2\omega b}\|(I - T(b))^{-1}\|_{B(E)} \int_0^b l(s)|y(s) - \overline{y}(s)|\,ds\\ &+ Me^{\omega b}\sum_{k=1}^{m} c_k|y(t_k) - \overline{y}(t_k)| + Me^{\omega b}\int_0^b l(s)|y(s) - \overline{y}(s)\,ds.\end{aligned}$$

Then

$$\|h - \overline{h}\|_\infty \le Me^{\omega b}\Big(\sum_{k=1}^{k=m} c_k + \|l\|_{L^1}\Big)\Big(1 + Me^{\omega b}\|(I - T(b))^{-1}\|_{B(E)}\Big)\|y - \overline{y}\|_{\mathrm{PC}}.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we finally arrive at

$$\begin{aligned}H_d(N(y), N(\overline{y})) \le{}& Me^{\omega b}\Big(\sum_{k=1}^{k=m} c_k + \|l\|_{L^1}\Big)\\ &\cdot\Big(1 + Me^{\omega b}\|(I - T(b))^{-1}\|_{B(E)}\Big)\|y - \overline{y}\|_{\mathrm{PC}}.\end{aligned}$$

So, N is a contraction and thus, by Lemma 2.74, N has a fixed point y, which is a mild solution of (4.1). □

Arguing as in Theorem 4.4, we can also prove the following result the proof of which is omitted.

Theorem 4.9. *Let E be a reflexive Banach space. Suppose that*

$$F\colon J \times E \to \mathcal{P}_{\mathrm{cp}}(E)$$

satisfies all of conditions of Theorem 4.8. *Then the solution set of problem* (4.1) *is nonempty and compact.*

4.1.4. The parameter-dependant case. In this subsection, we consider the following impulsive problem:

$$\begin{cases} (y' - Ay)(t) \in F(t, y(t), \lambda), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ \Delta y_{t=t_k} = I_k(y(t_k^-), \lambda), & k = 1, \dots, m, \\ y(0) = y(b) \end{cases} \tag{4.18}$$

where $F\colon J \times H \times \Lambda \to \mathcal{P}(H)$ is a multi-map with compact values, $I_k(\,\cdot\,,\,\cdot\,)\colon H \times \Lambda \to H$ is a continuous functions, (Λ, d_Λ) is a complete metric space and H is a separable Hilbert space. In the case with no impulses, some existence results and properties of solutions for semi-linear and evolutions differential inclusions with parameters were studied by Hu *et al.* [88], Papageorgiou and Yannakakis [120], and Tolstonogov [135], [Tolstodwa]; see also [14] for a parameter-dependant first-order Cauchy problem. In this section, we consider separately the convex and the nonconvex cases.

4.1.4.1. The convex case. We will assume the following assumptions:

($\widetilde{\mathcal{B}}_1$) The multi-map $F(\,\cdot\,, x, \lambda)\colon [0,b] \to \mathcal{P}_{\mathrm{cp,cv}}(H)$ is measurable for all $x \in E$ and $\lambda \in \Lambda$.

($\widetilde{\mathcal{B}}_2$) The multi-map $F(t,\,\cdot\,,\,\cdot\,)\colon H \times \Lambda \to \mathcal{P}_{\mathrm{cp,cv}}(H)$ is *u.s.c.*, for a.e. $t \in [0,b]$.

($\widetilde{\mathcal{B}}_3$) There exists $\alpha \in [0,1)$ and $p, q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t,x,\lambda)\|_{\mathcal{P}} \le p(t) + q(t)|x|^\alpha, \quad \text{for a.e. } t \in J, \text{ and all } x \in E,\ \lambda \in \Lambda.$$

($\widetilde{\mathcal{B}}_4$) There exist constants $a_k, b_k > 0$ and $\beta \in [0,1)$ such that

$$|I_k(x)| \le a_k|x|^\beta + b_k, \quad \text{for each } x \in E,\ k = 1, \dots, m.$$

($\widetilde{\mathcal{B}}_5$) For every $t > 0$, $T(t)$ is compact.

Theorem 4.10. *Assume that F satisfies ($\widetilde{\mathcal{B}}_1$)$--$($\widetilde{\mathcal{B}}_3$). Then for every fixed $\lambda \in \Lambda$, there exists $y(\,\cdot\,,\lambda) \in \mathrm{PC}$ solution of problem* (4.18).

Proof. For fixed $\lambda \in \Lambda$, let $F_\lambda(t,y) = F(t, y(t), \lambda)$, $(t,y) \in [0,b] \times H$ and let $I_k^\lambda(y) = I_k(y,\lambda)$, $k = 1, \dots, m$. It is clear that $F_\lambda(\,\cdot\,, u)$ is a measurable multi-map for all $u \in E$, $F_\lambda(t,\,\cdot\,)$ is *u.s.c.* and

$$\|F_\lambda(t,x)\|_{\mathcal{P}} \le q(t) + p(t)|x|^\alpha \quad \text{for a.e. } t \in J \text{ and each } x \in H,$$

where $p, q \in L^1(J, \mathbb{R}^+)$ and α are as defined in ($\widetilde{\mathcal{B}}_3$). To transform problem (4.18) into a fixed point problem, consider the operator $N_1\colon \mathrm{PC} \to \mathcal{P}(\mathrm{PC})$ defined by

$N_1(y) := \{h \in \mathrm{PC}\}$ with

$$h(t) = \Big\{T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I(y(t_k),\lambda) + \int_0^b T(b-s)v(s)\,ds\Big) + \int_0^t T(t-s)v(s))\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-),\lambda), \quad \text{for } t \in J\Big\}$$

and where $v \in S_{F_\lambda,y} = \{v \in L^1(J,H) : v(t) \in F(t,y(t),\lambda),\ \text{a.e. } t \in J\}$.

Define the mapping $S\colon \Lambda \to \mathcal{P}_{\mathrm{cp}}(H)$ by

$$S(\lambda) = \{y \in \mathrm{PC} : y \text{ is a solution of problem (4.18)}\}.$$

From Theorem 4.4, $S(\lambda) \neq \emptyset$ so that S is well defined. Next, we prove the upper semi-continuity of solutions with respect to the parameter λ. □

Proposition 4.11. *If hypotheses* $(\widetilde{\mathcal{B}}_1)$–$(\widetilde{\mathcal{B}}_3)$ *hold and* E *is reflexive, then* S *is u.s.c.*

Proof. Step 1. $S(\,\cdot\,) \in \mathcal{P}_{\mathrm{cp}}(H)$. Let $\lambda \in \Lambda$ and $y_n \in S(\lambda)$, $n \in \mathbb{N}$. Then there exists $v_n \in S_{F_\lambda,y_n}$ such that

$$y_n(t) = T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_n(t_k),\lambda) + \int_0^b T(b-s)v_n(s)\,ds\Big) + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k),\lambda) + \int_0^t T(t-s)v_n(s)\,ds, \quad t \in J.$$

From $(\widetilde{\mathcal{B}}_3)$ and the continuity of I_k, $k = 1,\dots,m$, we can prove that there exists $M > 0$ such that $\|y_n\|_{\mathrm{PC}} \le M$, $n \in \mathbb{N}$. As in the proof of Theorem 4.4, Part 2, we can further prove that the set $\{y_n : n \ge 1\}$ is compact in PC; hence there exists a subsequence of $\{y_n\}$ which converges to y in PC. Since $\{v_n\}(\,\cdot\,)$ is integrably bounded, then arguing as in the proof of Theorem 4.4, Step 1, we can find some subsequence which converges weakly to v and then we obtain at the limit:

$$\begin{aligned} y(t) = {} & T(t)(I - T(b))^{-1} \\ & \cdot \Big(\sum_{k=1}^{m} T(b-t_k)I_k(y(t_k^-),\lambda) + \int_0^b T(b-s)v(s)\,ds\Big) \\ & + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-),\lambda) + \int_0^t T(t-s)v(s)\,ds, \quad \text{for } t \in J. \end{aligned} \tag{4.19}$$

Hence $S(\,\cdot\,) \in \mathcal{P}_{\mathrm{cp}}(H)$.

Step 2. $S(\,\cdot\,)$ is quasicompact. Let K be a compact set in Λ. To show that $\overline{S(K)}$ is compact, let $y_n \in S(\lambda_n)$, $\lambda_n \in K$. Then there exists $v_n \in S_{F(\,\cdot\,,\,\cdot\,,\lambda_n,y_n)}$, $n \in \mathbb{N}$ such that, for $t \in J$,

$$\begin{aligned} y_n(t) = {} & T(t)(I - T(b))^{-1} \\ & \cdot \left(\sum_{k=1}^{m} T(b - t_k) I_k(y_n(t_k^-), \lambda_n) + \int_0^b T(b-s) v_n(s)\, ds \right) \\ & + \sum_{0<t_k<t} T(t - t_k) I_k(y_n(t_k^-), \lambda_n) + \int_0^t T(t-s) v_n(s)\, ds. \end{aligned} \tag{4.20}$$

As mentioned in Step 1, $\{y_n : n \geq 1\}$ is compact in PC; then there exists a subsequence of $\{y_n\}$ which converges to y in PC. Since K is compact, there exists a subsequence $\{\lambda_n : n \geq 1\}$ in K such that λ_n converges to $\lambda \in \Lambda$. As we did above, we can easily prove that there exists $v(\,\cdot\,) \in F(\,\cdot\,, y(\,\cdot\,), \lambda)$ such that y satisfies (4.19).

Step 3. $S(\,\cdot\,)$ is closed. For this, let $\lambda_n \in \Lambda$ be such that λ_n converges to λ and let $y_n \in S(\lambda_n)$, $n \in \mathbb{N}$ be a sequence converging to some limit y in PC. Then y_n satisfies (4.20) and as we did above, we can use $(\widetilde{\mathcal{B}}_3)$ to show that the set $\{y_n : n \geq 1\}$ is equicontinuous in PC. Hence, by the Arzelá–Ascoli Theorem, we conclude that there exists a subsequence of $\{y_n\}$ converging to some limit y in PC and there exists a subsequence of $\{v_n\}$ which converges to $v(\,\cdot\,) \in F(\,\cdot\,, y(\,\cdot\,), \lambda)$ such that y satisfies (4.19). Therefore $S(\,\cdot\,)$ has a closed graph, hence *u.s.c.* by Lemma 2.9. □

4.1.4.2. The nonconvex case. The following hypotheses will be assumed in this sub-section:

- $(\widetilde{\mathcal{H}}_1)$ The function $F\colon J \times H \times \Lambda \to \mathcal{P}_{\mathrm{cp}}(H)$ is such that for all $(x, \lambda) \in H \times \Lambda$, the map $t \mapsto F(t, x, \lambda)$ is measurable.
- $(\widetilde{\mathcal{H}}_2)$ For every compact $B \subset \Lambda$, there exists a function $p_B \in L^1(J, H)$ such that
$$H_d(F(t, x, \lambda), F(t, y, \lambda)) \leq p_B(t)|x - y|, \quad \text{for each } x, y \in H.$$
- $(\widetilde{\mathcal{H}}_3)$ There exist $\overline{c_k} > 0$, $k = 1, \ldots, m$, such that
$$|I_k(x, \lambda) - I_k(y, \lambda)| \leq \overline{c_k}|x - y|, \quad \text{for each } x, y \in H,\ \lambda \in \Lambda.$$

Theorem 4.12. *Assume that F satisfies $(\widetilde{\mathcal{H}}_1)$–$(\widetilde{\mathcal{H}}_3)$ and that for every compact subset $B \subset \Lambda$, we have*

$$M^2 \|(I - T(t))^{-1}\|_{B(H)} \left(\|p_B\|_{L^1} + \sum_{k=1}^{m} \overline{c_k} \right) < 1.$$

Then for every $\lambda \in \Lambda$, there exists at least one solution of problem (4.18). *Assume further that $F(\,\cdot\,,\,\cdot\,,\,\cdot\,) \in \mathcal{P}_{\mathrm{cp,cv}}(H)$ and $(\widetilde{\mathcal{B}}_5)$ holds together with*

- $(\widetilde{\mathcal{H}}_4)$ *$\lambda \mapsto F(t,x,\lambda)$ is lower semi-continuous, for almost every $t \in J$ and each $x \in H$.*
- $(\widetilde{\mathcal{H}}_5)$ *For every compact $B \subset \Lambda$, there exist a function $\overline{p}_B \in L^1(J,H)$ and a continuous nondecreasing function $\psi\colon [0,\infty) \to [0,\infty)$ such that*

$$\|F(t,x,\lambda)\|_{\mathcal{P}} \le \overline{p}_B(t)\psi(|x|), \quad \textit{for a.e. } t \in J \textit{ and each } x \in H.$$

Then $S(\,\cdot\,)$ is l.s.c. from Λ into $\mathcal{P}_{\mathrm{cp}}(\mathrm{PC})$.

Proof. Arguing as in Theorem 4.8, we can prove that $S(\lambda) \neq \emptyset$, for every $\lambda \in \Lambda$, and that under $(\widetilde{\mathcal{H}}_1)$–$(\widetilde{\mathcal{H}}_3)$, we have $S(\,\cdot\,) \in \mathcal{P}_{\mathrm{cp}}(\mathrm{PC})$.

To show that $S(\,\cdot\,)$ is *l.s.c.*, let $\lambda_n \to \lambda$ in Λ. We need to show that

$$S(\lambda) \subset \varliminf_{n\to\infty} S(\lambda_n) = \left\{ y \in \mathrm{PC} : y = \lim_{n\to\infty} y_n,\ y_n \in S(\lambda_n),\ n > 0 \right\}.$$

Let $y \in S(\lambda)$; then there exists $v \in S_{F(\,\cdot\,,y(\,\cdot\,),\lambda)}$ such that y satisfies (4.19). Let

$$\begin{aligned} h(t,\lambda_n) &= \mathrm{Proj}(v(t), F(t,y(t),\lambda_n)), \\ g(t,u,\lambda_{n*}) &= \mathrm{Proj}(h(t,\lambda_n), F(t,u,\lambda_n)) \end{aligned}$$

where

$$\begin{aligned} |v(t) - h(t,\lambda_n)| &= d(v(t), F(t,y(t),\lambda_n)), \\ |h(t,\lambda_n) - g(t,u,\lambda_n)| &= d(h(t,\lambda_n), F(t,u,\lambda_n)). \end{aligned}$$

It is clear that $h(\,\cdot\,,\lambda_n)$ and $g(\,\cdot\,,u,\lambda_n)$ are measurable functions. Furthermore, the mapping $u \to g(t,u,\lambda_n)$ is continuous (see [10, Theorem 3.33] or [86, Proposition 3]). Consider the following impulsive problem

$$\begin{cases} y'(t) - Ay(t) = g(t,y,\lambda_n), & \text{a.e. } t \in [0,b], \\ \Delta y|_{t=t_k} = I_k(y(t_k),\lambda_n), & k = 1,\dots,m, \\ y(0) = y(b). \end{cases} \tag{4.21}$$

Since λ_n converges to λ, the set $B = \{\lambda_n : n \ge 1\}$ is compact in Λ. Hence from $(\widetilde{\mathcal{H}}_5)$, there exists $\overline{p}_B \in L^1(J,H)$ such that

$$|g(t,y,\lambda_*)| \le \overline{p}_B(t)\psi(|y|), \quad \text{for all } n \in \mathbb{N},\ y \in H.$$

By the Leray–Schauder nonlinear alternative (Lemma 2.73), we prove that for every $n \in \mathbb{N}$, problem (4.21) has at least one solution denoted by y_n, that is

$$\begin{aligned} y_n(t) =& T(t)(I-T(b))^{-1} \\ &\cdot \left(\sum_{k=1}^{m} T(b-t_k) I_k(y_n(t_k^-), \lambda), \lambda) + \int_0^b T(b-s) g(s, y_n(s), \lambda_n)\, ds \right) \\ &+ \sum_{0<t_k<t} T(t-t_k) I_k(y_n(t_k^-), \lambda_n) + \int_0^t T(t-s) g(s, y_n(s), \lambda_n)\, ds, \quad t \in J. \end{aligned}$$

Next we prove that (y_n) converges to y. We have the following estimates:

$$\begin{aligned} &|y_n(t) - y(t)| \\ &\quad \le [M^2 \|(I-T(t))^{-1}\|_{B(H)} + M] \int_0^b |h(s,\lambda_n) - v(s)|\, ds \\ &\qquad + [M^2 \|(I-T(t))^{-1}\|_{B(H)} + M] \int_0^b |g(s, y_n(s), \lambda_n) - h(s, \lambda_n)|\, ds \\ &\qquad + [M^2 \|(I-T(t))^{-1}\|_{B(H)} + M] \sum_{0<t_k<t}^{m} |I_k(y_n(t_k), \lambda_n) - I_k(y(t_k), \lambda)| \\ &\quad \le [M^2 \|(I-T(t))^{-1}\|_{B(H)} + M] \int_0^b d(v(s), F(s, y(s), \lambda_n))\, ds \\ &\qquad + [M^2 \|(I-T(t))^{-1}\|_{B(H)} + M] \int_0^b d(h(s,\lambda_n), F(s, y(s), \lambda_n)|\, ds \\ &\qquad + [M^2 \|(I-T(t))^{-1}\|_{B(H)} + M] \sum_{k=1}^{m} \overline{c_k} |y_n(t_k) - y(t_k)|. \end{aligned}$$

Then

$$\begin{aligned} &\sup_{s\in[0,t]} |y_n(s) - y(s)| \\ &\qquad \le \frac{[M^2 \|(I-T(t))^{-1}\|_{B(H)} + M] \int_0^b |d(v(s), F(s, y(s), \lambda_n))|\, ds}{1 - [M^2 \|(I-T(t))^{-1}\|_{B(H)} + M] \left(\|p_B\|_{L^1} + \sum_{k=1}^{m} \overline{c_k} \right)}. \end{aligned}$$

Using $(\widetilde{\mathcal{H}}_4)$ and the fact that the multi-map $\lambda \to F(t, y(t), \lambda)$ is *l.s.c.*, we deduce that the mapping $\lambda \to d(v(s), F(t, y(t), \lambda)$ is *u.s.c.* Since $v(t) \in F(t, y(t), \lambda)$, by Fatou's Lemma, and the convergence of λ_n to λ, we obtain that

$$\int_0^b |d(v(s), F(s, y_s, \lambda_n))|\, ds \to 0, \quad \text{as } n \to \infty.$$

Hence $\|y_n - y\|_{\mathrm{PC}} \to 0$, as $n \to +\infty$. Noting that $y_n \in S(\lambda_n)$, $n \in \mathbb{N}$, we infer that

$$S(\lambda) \subset \varliminf_{n\to\infty} S(\lambda_n),$$

as claimed. □

4.1.5. Filippov's Theorem. The family of all nonempty closed and decomposable subsets of $L^1(J,E)$ is denoted by $\mathcal{D}$. Let $S \neq \emptyset$ be a nonempty set. The following result is due to Colombo *et al.* (see [35]).

Lemma 4.13. *Consider a l.s.c. multi-map $G\colon S \to \mathcal{D}$ and assume that $\phi, \psi\colon S \to L^1(J,\mathbb{R}^n)$ are continuous maps such that for every $s \in S$, the set*

$$H(s) = \overline{\{u \in G(s) : |u(t) - \phi(s)(t)| < \psi(s)(t),\ \text{a.e. } t \in J\}}$$

is nonempty. Then the map $H\colon S \to \mathcal{D}$ is l.s.c. and admits a continuous selection.

Next, we prove a Filippov type result for problem (4.1).

Theorem 4.14. *Further to assumptions $(\widetilde{\mathcal{H}}_2)$ and $(\widetilde{\mathcal{H}}_3)$, assume that $F\colon J \times E \to \mathcal{P}_{\mathrm{cp}}(E)$ satisfies the following condition:*

$(\mathcal{H}_2')$ *There exist a continuous mapping $g(\,\cdot\,)\colon \mathrm{PC} \to L^1(J,E)$ and $x \in \mathrm{PC}$ a mild solution of the problem:*

$$\begin{cases} x'(t) - Ax(t) = g(x)(t), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k^-)), & k = 1, \dots, m, \\ x(0) = x(b). \end{cases} \tag{4.22}$$

and there exists a function $r \in L^1(J,\mathbb{R}^+)$ with

$$d(g(x)(t), F(t,x(t))) < r(t), \quad \text{a.e. } t \in J.$$

If

$$R\left(2\|l\|_1 + \sum_{k=1}^m c_k\right) < 1,$$

where

$$R := Me^{\omega b}(M^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1),$$

then problem (4.1) has at least one solution y satisfying the estimates

$$\|y - x\|_{\mathrm{PC}} \le \frac{2R(\widetilde{H} + \|g(x)\|_1)}{1 - R\sum_{k=1}^m c_k}$$

and

$$|y'(t) - Ay(t) - g(x)(t)| \le 2\widetilde{H}\|r\|_1 p(t) + |l(t)|, \quad \text{a.e. } t \in J,$$

where

$$\widetilde{H} = \frac{1 - R\Big(\|l\|_1 + \sum_{k=1}^{m} c_k\Big)}{1 - R\Big(2\|l\|_1 + \sum_{k=1}^{m} c_k\Big)}.$$

Remark 4.15. Assumption $(\mathcal{H}_2')$ is satisfied for instance if g is an L^1-selection of the multi-map F.

Proof of Theorem 4.14.

Step 1. Let $y_0 := x$ and $f_0 := g$, that is

$$\begin{aligned} y_0(t) =& T(t)(I - T(b))^{-1} \\ &\cdot \Big(\sum_{k=1}^{m} T(b - t_k) I_k(y_0(t_k)) + \int_0^b T(b - s) f_0(y_0)(s)\, ds\Big) \\ &+ \sum_{0<t_k<t} T(t - t_k) I_k(y_0(t_k)) + \int_0^t T(t - s) f_0(y_0)(s)\, ds. \end{aligned}$$

Let $G_1\colon \mathrm{PC} \to \mathcal{P}(L^1(J, E))$ be defined by

$$G_1(y) = \{v \in L^1(J, E) : v(t) \in F(t, y(t)),\ \text{a.e. } t \in J\}$$

and $\widetilde{G}_1\colon \mathrm{PC} \to \mathcal{P}(L^1(J, E))$ be given by

$$\widetilde{G}_1(y) = \overline{\{v \in G_1(y) : |v(t) - g(y_0)(t)| < p(t)|y(t) - y_0(t)| + p(t)\}}.$$

Since, from assumption $(\mathcal{H}_2')$, the mapping $t \to F(t, y(t))$ is, for fixed $y \in \mathrm{PC}$, a measurable multi-map, then by Corollary 2.17, there exists a measurable selection $v_1(t) \in F(t, y(t))$ for almost every $t \in J$ such that, using $(\mathcal{A}_3)$,

$$\begin{aligned} |v_1(t) - g(y_0)(t)| &\le d(g(y_0)(t), F(t, y(t))) \\ &\le d(g(y_0)(t), F(t, y_0(t)) + H_d(F(t, y_0(t)), F(t, y(t))) \\ &< r(t) + l(t)|y_0(t) - y(t)|. \end{aligned}$$

Thus $\widetilde{G}_1(y) \neq \emptyset$ and so $\widetilde{G}_1$ is well defined. By Lemma 2.23, F is of lower semi-continuous type. Then G_1 is *l.s.c.* and has decomposable values. Then so is $\widetilde{G}_1$. By Lemma 24, there exists a continuous selection $f_1\colon \mathrm{PC} \to L^1(J, E)$ such that $f_1(y) \in \widetilde{G}_1(y)$ for all $y \in \mathrm{PC}$. Consider the problem

$$\begin{cases} y'(t) - Ay(t) = f_1(y)(t), & t \in J,\ t \neq t_k,\ k = 1, \dots, m, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = y(b). \end{cases} \tag{4.23}$$

Define the operator $\overline{N}_1\colon \mathrm{PC} \to \mathrm{PC}$ by

$$\begin{aligned}\overline{N}_1(y)(t) =& T(t)(I-T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I(y(t_k)) + \int_0^b T(b-s)f_1(y)(s)\,ds\Big)\\ &+ \int_0^t T(t-s)f_1(y)(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \quad \text{for } t\in[0,b].\end{aligned}$$

As in Theorem 4.4, we can prove that $\overline{N}_1$ is completely continuous. To establish a priori estimates on all possible solutions, let $y \in \mathrm{PC}$ be such that $y = \lambda\overline{N}_1(y)$ for some $\lambda \in (0,1)$. By Proposition 2.67, there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\begin{aligned}|y(t)| \leq& M^2e^{2\omega b}\|(I-T(b))^{-1}\|_{B(E)}\Big(\sum_{k=1}^{m}(c_k|y(t_k)| + |I_k(0)|) + \int_0^b |f_1(y)(s)|\,ds\Big)\\ &+ Me^{\omega b}\int_0^t |f_1(y)(s)|\,ds + Me^{\omega b}\sum_{k=1}^{m}[c_k|y(t_k^-)| + |I_k(0)|],\end{aligned}$$

and so

$$\|y\|_{\mathrm{PC}} \leq Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\Big(\sum_{k=1}^{m}(c_k\|y\|_{\mathrm{PC}} + |I_k(0)|) + \|l\|_1\Big).$$

Hence

$$\|y\|_{\mathrm{PC}} \leq \frac{Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\Big(\|l\|_1 + \sum_{k=1}^{m} I_k(0)|\Big)}{1 - Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\sum_{k=1}^{m} c_k} := \overline{M}.$$

Let $U := \{y \in \mathrm{PC} : \|y\|_{\mathrm{PC}} < \overline{M} + 1\}$, and consider the operator $\overline{N}_1\colon \overline{U} \to \mathrm{PC}$. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda\overline{N}_1(y)$ for some $\lambda \in (0,1)$. As a consequence of the Leray–Schauder nonlinear alternative (Lemma 2.73), we deduce that $\overline{N}_1$ has a fixed point y in U which is a solution of problem (6.10). This solution, denoted y_1, satisfies for $t \in J$

$$\begin{aligned}y_1(t) =& T(t)(I-T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_1(t_k)) + \int_0^b T(b-s)f_1(y_1)(s)\,ds\Big)\\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y_1(t_k)) + \int_0^t T(t-s)f_1(y_1)(s)\,ds.\end{aligned}$$

Writing

$$|f_1(y_1) - f_0(y_0)| \leq |f_1(y_1) - f_1(y_0)| + |f_1(y_0) - f_0(y_0)|,$$

and using the fact that

$$f_1(y_1)(t) \in F(t, y_1(t)), f_1(y_0)(t) \in F(t, y_0(t)),$$

we obtain, for almost every $t \in J$, the following estimates

$$\begin{aligned}
&|y_1(t) - y_0(t)| \\
&\quad \le Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\int_0^b |f_1(y_1)(s) - f_0(y_0)(s)|\,ds \\
&\qquad + Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\sum_{k=1}^m |I_k(y_1(t_k)) - I_k(y_0(t_k))| \\
&\quad \le Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\int_0^b l(s)|y_1(s) - y_0(s)|\,ds \\
&\qquad + Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\|r\|_1 \\
&\qquad + Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\sum_{k=1}^m c_k|y_1(t_k)) - y_0(t_k)|.
\end{aligned}$$

Passing to the supremum over J, we get the bound:

$$\|y_1 - y_0\|_{\mathrm{PC}} \le \frac{Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\|r\|_1}{1 - Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\Big(\|l\|_1 + \sum_{k=1}^m c_k\Big)}.$$

Step 2. Define the set-valued map $G_2\colon \mathrm{PC} \to \mathcal{P}(L^1(J,E)$ by $G_2 = G_1$ and

$$\widetilde{G}_2(y) = \overline{\{v \in G_2(y) : |v(t) - f_1(y_1)(t)| < l(t)|y(t) - y_1(t)| + l(t)|y_1(t) - y_0(t)|\}}.$$

Since, for fixed $y \in \mathrm{PC}$, the map $t \to F(t, y(t))$ is measurable, then there exists, by Corollary 2.17, a function $v_2 \in \widetilde{G}_2$ which is a measurable selection of $F(\,\cdot\,, y(\,\cdot\,))$ such that

$$\begin{aligned}
|v_2(t) - f_1(y_1)(t)| &\le d(f_1(y_1)(t), F(t, y(t))) \le H_d(F(t, y_1(t)), F(t, y(t))) \\
&\le l(t)|y_1(t) - y(t)| < l(t)|y_1(t) - y(t)| + l(t)|y_1(t) - y_0(t)|.
\end{aligned}$$

Thus $\widetilde{G}_2(y) \ne \emptyset$. Arguing as in Step 1, we can prove that $\widetilde{G}_2$ has at least a continuous selection which we denote by f_2. Hence, there exists $y_2 \in \mathrm{PC}$ solution of the problem

$$\begin{cases}
y'(t) - Ay(t) = f_2(y)(t), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\
y(t_k^+) - y(t_k) = I_k(y(t_k^-)), & k = 1, \dots, m, \\
y(0) = y(b),
\end{cases} \tag{4.24}$$

that is

$$\begin{aligned} y_2(t) =& T(t)(I-T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_2(t_k)) + \int_0^b T(b-s)f_2(y_2)(s)\,ds\Big) \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y_2(t_k)) + \int_0^t T(t-s)f_2(y_2)(s)\,ds, \quad t\in J. \end{aligned}$$

$f_2 \in \widetilde{G}_2$ implies that $|f_2(y_2)(s) - f_1(y_1)(s)| \le l(s)|y_2(s) - y_(s)| + l(s)|y_1 - y_0|$. Then we have

$$\begin{aligned} &|y_2(t) - y_1(t)| \\ &\quad\le Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\int_0^b |f_2(y_2)(s) - f_1(y_1)(s)|\,ds \\ &\qquad + Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\sum_{k=1}^{m} c_k|y_2(t_k) - y_1(t_k)| \\ &\quad\le Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\int_0^b l(s)|y_2(s) - y_1(s)|\,ds \\ &\qquad + Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\int_0^b l(s)|y_1(s) - y_0(s)|\,ds \\ &\qquad + Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)\sum_{k=1}^{m} c_k|y_2(t_k) - y_1(t_k)|. \end{aligned}$$

Setting $R := Me^{\omega b}(M^{\omega b}\|(I-T(b))^{-1}\|_{B(E)} + 1)$, we deduce that

$$\|y_2 - y_1\|_{\mathrm{PC}} \le \frac{R\|l\|_1\|y_1 - y_0\|_{\mathrm{PC}}}{1 - R\Big(\|l\|_1 + \sum_{k=1}^{m} c_k\Big)} \le \frac{R^2\|l\|_1\|r\|_1}{\Big(1 - R(\|l\|_1 + \sum_{k=1}^{m} c_k)\Big)^2}.$$

Step 3. Define the set-valued map $G_3\colon \mathrm{PC} \to \mathcal{P}(L^1(J,E)$ by $G_3 = G_2 = G_1$ and let

$$\widetilde{G}_3(y) = \overline{\{v \in G_3(y) : |v(t) - f_2(y_2)(t)| < l(t)|y(t) - y_2(t)| + l(t)|y_2(t) - y_1(t)|\}}.$$

Arguing as we did for $\widetilde{G}_2$, we can prove that $\widetilde{G}_3$ is a multi-map of *l.s.c.* type with nonempty decomposable values. Thus there exists a continuous selection $f_3(y) \in \widetilde{G}_3(y)$ for all $y \in \mathrm{PC}$. Then we can prove existence of a solution y_3 such that

$$\begin{aligned} y_3(t) =& T(t)(I-T(b))^{-1}\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_3(t_k)) + \int_0^b T(b-s)f_3(y_3)(s)\,ds\Big) \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(y_3(t_k)) + \int_0^t T(t-s)f_3(y_3)(s)\,ds, \quad t\in J, \end{aligned}$$

is a solution of the problem

$$
(4.25)\qquad \begin{cases} y'(t) - Ay(t) = f_3(y)(t), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ y(t_k^+) - y(t_k) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ \qquad\qquad y(0) = y(b). \end{cases}
$$

Since $f_3(y) \in \widetilde{G}_3(y)$, we obtain the estimates

$$
\begin{aligned}
&|y_3(t) - y_2(t)| \\
&\quad \le Me^{\omega b}(M^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1)\left| \int_0^t |f_3(y_3(s)) - f_2(y_2(s))| \right| ds \\
&\qquad + Me^{\omega b}(M^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1) \sum_{k=1}^m c_k |y_3(t_k) - y_2(t_k)| \\
&\quad \le Me^{\omega b}(M^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1) \int_0^t |l(s)(y_3(s)) - (y_2(s))|\, ds \\
&\qquad + Me^{\omega b}(M^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1) \int_0^t |l(s)(y_2(s)) - (y_1(s))|\, ds \\
&\qquad + Me^{\omega b}(M^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1) \sum_{k=1}^m c_k |y_3(t_k) - y_2(t_k)|.
\end{aligned}
$$

From the estimates of $|y_2(t) - y_1(t)|$ and $|y_1(t) - y_0(t)|$ above, we get the bound

$$
\|y_3 - y_2\|_{\mathrm{PC}} \le \frac{R^3 \|l\|^2 \|r\|}{\left(1 - R\left(\|l\|_1 + \sum_{k=1}^m\right)\right)^3}.
$$

Step 4. Repeating the process for $n = 4, 5, \dots$, we finally arrive at

$$
(4.26)\qquad \|y_n - y_{n-1}\|_{\mathrm{PC}} \le \frac{R^n \|l\|^{n-1} \|r\|}{\left(1 - R\left(\|l\|_1 + \sum_{k=1}^m\right)\right)^n}.
$$

By induction, suppose that (4.26) holds for some n and let $\widetilde{G}_{n+1}$ be a multi-map defined by

$$
\begin{aligned}
\widetilde{G}_{n+1}(y) = \overline{\{v \in G_{n+1}(y) :} \\
\overline{|v(t) - f_n(y_n)(t)| < p(t)|y(t) - y_n(t)| + p(t)|y_n(t) - y_{n-1}(t)|\}}.
\end{aligned}
$$

Since $\widetilde{G}_{n+1}$ is a *l.s.c.* type multi-map, there exists a continuous selection $f_{n+1}(y)$ in $\widetilde{G}_{n+1}(y)$ which allows us to define

$$\begin{aligned}(4.27)\quad y_{n+1}(t) =& T(t)(I-T(b))^{-1}\\ &\cdot\Big(\sum_{k=1}^{m} T(b-t_k)I_k(y_{n+1}(t_k)) + \int_0^b T(b-s)f_{n+1}(y_{n+1})(s)\,ds\Big)\\ &+\sum_{0<t_k<t} T(t-t_k)I_k(y_{n+1}(t_k)) + \int_0^t T(t-s)f_{n+1}(y_{n+1})(s)\,ds,\end{aligned}$$

for $t\in J$. Then

$$\begin{aligned}&|y_{n+1}(t)\ -y_n(t)|\\ &\quad\le Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)}+1)\int_0^b |f_{n+1}(y_{n+1})(s)-f_n(y_n)(s)|\,ds\\ &\qquad + Me^{\omega b}(Me^{\omega b}\|(I-T(b))^{-1}\|_{B(E)}+1)\sum_{k=1}^{m} c_k|y_n(t_k)-y_{n+1}(t_k)|\\ &\quad\le Me^{\omega b}(M^{\omega b}\|(I-T(b))^{-1}\|_{B(E)}+1)\int_0^b |l(s)(y_{n+1}(s))-(y_n(s))|\,ds\\ &\qquad + Me^{\omega b}(M^{\omega b}\|(I-T(b))^{-1}\|_{B(E)}+1)\int_0^b |l(s)(y_n(s))-(y_{n-1}(s))|\,ds\\ &\qquad + Me^{\omega b}(M^{\omega b}\|(I-T(b))^{-1}\|_{B(E)}+1)\sum_{k=1}^{m} c_k|y_{n+1}(t_k)-y_n(t_k)|.\end{aligned}$$

Hence

$$\|y_{n+1}-y_n\|_{\mathrm{PC}} \le \frac{R\|l\|_1\|y_{n+1}-y_n\|_{\mathrm{PC}}}{1-R\Big(\|l\|_1+\sum_{k=1}^{m} c_k\Big)}.$$

Using (4.26), we obtain that

$$\begin{aligned}(4.28)\quad \|y_{n+1}-y_n\|_{\mathrm{PC}} &\le \frac{R^n\|l\|_1^{n+1}\|r\|_1}{\Big(1-R\Big(\|l\|_1+\sum_{k=1}^{m} c_k\Big)\Big)^n}\\ &= \Big(R\|l\|_1\Big/\Big(1-R\Big(\|l\|_1+\sum_{k=1}^{m} c_k\Big)\Big)\Big)^n\|l\|_1\|r\|_1.\end{aligned}$$

Thus (4.26) holds for all $n\in\mathbb{N}$, and so $\{y_n\}$ is a Cauchy sequence in PC, converging uniformly to a function $y\in\mathrm{PC}$, as $n\to+\infty$. Moreover, from the definition of $\widetilde{G}_n(y)$, $n\in\mathbb{N}$, we have

$$|f_{n+1}(y_{n+1})(t)-f_n(y_n)(t)| \le l(t)|y_n(t)-y_{n-1}(t)| + l(t)|y_{n+1}(t)-y_n(t)|$$

for almost every $t \in J$. Therefore, for almost every $t \in J$, $\{f_n(y_n)(t) \colon n \in \mathbb{N}\}$ is also a Cauchy sequence in E and thus converges almost everywhere to some measurable function $f(t)$ in E. Moreover, since $f_0 = g$, we have successively the estimates:

$$\begin{aligned}
|f_n(y_n)(t)| \leq{}& |f_n(y_n)(t) - f_{n-1}(y_{n-1})(t)| \\
&+ |f_{n-1}(y_{n-1})(t) - f_{n-2}(y_{n-2})(t)| + \dots \\
&+ |f_2(y_2)(t) - f_1(y_1)(t)| + |f_1(y_1)(t) - f_0(y_0)(t)| + |f_0(y_0)(t)| \\
\leq{}& 2\sum_{k=1}^{n} l(t)|y_k(t) - y_{k-1}(t)| + |f_0(y_0)(t)| \\
\leq{}& 2l(t)\sum_{k=1}^{\infty} |y_k(t) - y_{k-1}(t)| + |g(x)(t)| \\
\leq{}& 2l(t)\sum_{k=1}^{\infty} \Bigg(R\|l\|_1 \Big/ \Big(1 - R(\|l\|_1 + \sum_{k=0}^{m} c_k)\Big)\Bigg)^k + |g(x)(t)|.
\end{aligned}$$

Hence

$$|f_n(y_n)(t)| \leq 2l(t)\widetilde{H} + |g(x)(t)|, \quad \text{a.e. } t \in J, \tag{4.29}$$

where

$$\widetilde{H} := \frac{1 - R\Big(\|l\|_1 + \sum_{k=1}^{m} c_k\Big)}{1 - R\Big(2\|l\|_1 + \sum_{k=1}^{m} c_k\Big)}.$$

From (4.29) and the Lebesgue Dominated Convergence Theorem, we conclude that $f_n(y_n)$ converges to $f(y)$ in $L^1(J, E)$. Passing to the limit in (4.27), we obtain a solution of problem (4.1), namely

$$\begin{aligned}
y(t) ={}& T(t)(I - T(b))^{-1}\Big(\sum_{k=1}^{m} T(b - t_k)I_k(y(t_k)) + \int_0^b T(b - s)f(y)(s)\,ds\Big) \\
&+ \sum_{0<t_k<t} T(t - t_k)I_k(y(t_k)) + \int_0^t T(t - s)f(y)(s)\,ds, \quad \text{for } t \in J.
\end{aligned}$$

Next, we give estimates for

$$|y'(t) - Ay(t) - g(x)(t)| \quad \text{and} \quad |x(t) - y(t)|, \quad t \in J.$$

We have

$$\begin{aligned}|y'(t) - Ay(t) - g(x)(t)| &= |f(y)(t) - f_0(x)(t)| \\ &\le |f(y)(t) - f_n(y_n)(t)| + |f_n(y_n)(t) - f_0(x)(t)| \\ &\le |f(y)(t) - f_n(y_n)(t)| + \sum_{k=1}^{n} |f_k(y_k)(t) - f_{k-1}(y_{k-1})(t)| \\ &\le |f(y)(t) - f_n(y_n)(t)| + 2\sum_{k=1}^{n} l(t)|y_k(t) - y_{k-1}(t)| + |l(t)|.\end{aligned}$$

Using (4.28) and passing to the limit as $n \to +\infty$, we get

$$\begin{aligned}|y'(t) - Ay(t) - g(x)(t)| &\le 2l(t)\sum_{k=1}^{\infty} |y_k(t) - y_{k-1}(t)| + |l(t)| \\ &\le 2l(t)\sum_{k=0}^{\infty}\left(R\|l\|_1 \Big/ \left(1 - R(\|l\|_1 + \sum_{k=1}^{m} c_k)\right)\right)^k \|l\|_1\|r\|_1 + |l(t)|,\end{aligned}$$

hence

$$|y'(t) - Ay(t) - g(x)(t)| \le (2\|l\|_1\|r\|_1\widetilde{H} + 1)l(t), \quad \text{a.e. } t \in J.$$

Similarly,

$$\begin{aligned}&|x(t) - y(t)| \\ &\le Me^{\omega b}(Me^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1)\int_0^b |f(y)(s) - f_0(y_0)(s)|\,ds \\ &\quad + Me^{\omega b}(Me^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1)\sum_{k=1}^{m} c_k|y(t_k) - x(t_k)| \\ &\le Me^{\omega b}(Me^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1)\int_0^b |f(y)(s) - f_n(y_n)(s)|\,ds \\ &\quad + Me^{\omega b}(Me^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1)\int_0^b |f_n(y_n)(s) - g(x)(s)|\,ds \\ &\quad + Me^{\omega b}(Me^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1)\sum_{k=1}^{m} c_k|x(t_k) - y(t_k)|.\end{aligned}$$

As $n \to \infty$, we finally arrive at

$$\|x - y\|_{\mathrm{PC}} \le \frac{2R(\widetilde{H}\|p\|_1 + |g(x)|)}{1 - Me^{\omega b}\left(Me^{\omega b}\|(I - T(b))^{-1}\|_{B(E)} + 1)\sum_{k=1}^{m} c_k\right)},$$

ending the proof of the theorem. □

4.2. Existence of solutions: $1 \notin \rho(T(b))$

In the particular case where $Ay = \lambda y$, some basic results in the theory of periodic boundary value problems for first order impulsive differential equations and inclusions may be found in [68], [69], [112], [113], [114] and references therein. Now we give an existence result when $1 \notin \rho(T(b))$ using the topological degree theory combined with the Poincaré operator properties. Consider the problem

$$\begin{cases} (y' - Ay)(t) \in \varphi(t, y(t)), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = y(b) \in E, \end{cases} \tag{4.30}$$

where $\varphi\colon J \times E \to \mathcal{P}(E)$ is a multi-map.

4.2.1. A nonlinear alternative. Set

$$\overline{B}(r,0) = \{x \in X : |x| \le r\}, \quad S(r) = \partial\overline{B}(r,0), \quad \text{and} \quad X^* = X \setminus \{0\},$$

where $\overline{B}(r,0)$ is the closed ball in X with center x and radius r while $\partial\overline{B}(r,0)$ stands for the boundary of $\overline{B}(r,0)$ in X. For any ANR-space X, let

$$J(\overline{B}(r,0), X) = \{F : \overline{B}(r,0) \to \mathcal{P}(X) : F \text{ is } u.s.c. \text{ and has } R_\delta\text{-values}\}.$$

Moreover, for any continuous function $f\colon X \to X$ where $X \in$ ANR, define the sets

$$\begin{aligned} J_f(\overline{B}(r,0), X)) &= \{\phi : \overline{B}(r,0) \to \mathcal{P}(X),\ \varphi = f \circ F \\ &\qquad \text{with } F \in J(\overline{B}(r,0), X), \phi(S(r)) \subset E^*\}, \\ CJ(\overline{B}(r,0), X) &= \bigcup\{J_f(\overline{B}(r,0), X)),\ f\colon X \to X\}, \\ CJ_C(\overline{B}(r,0), X) &= \{\Phi\colon \overline{B}(r,0) \to \mathcal{P}(X) : \Phi = f \circ F \text{ with } F \in J(\overline{B}(r,0), X), \\ &\qquad \operatorname{Fix}(\Phi \cap S(r)) = \emptyset \text{ and } \overline{\Phi(\overline{B}(r,0))} \text{ is compact}\}. \end{aligned}$$

In what follows, for given $\Phi \in CJ(\overline{B}(r,0), X)$, we shall associate the vector field $\phi\colon \overline{B}(r,0) \to X$, $\phi = j - \Phi$ defined by

$$\phi(x) = j(x) - \Phi(x), \quad \text{for all } x \in \overline{B}(r,0),$$

where $j\colon \overline{B}(r,0) \to X$, $j(x) = x$ is the inclusion map. Notice that if $\Phi \in CJ_C(\overline{B}(r,0), X)$, then $\phi \in CJ(\overline{B}(r,0), X)$ (Proposition 28.1 in [61]), hence $\phi(S(r)) \subset X^*$. Let

$$\begin{aligned} CJ_{CV}(\overline{B}(r,0), X) = \{&\phi \in CJ(\overline{B}(r,0), X) : \phi \text{ is a compact field} \\ &\text{associated with some } \Phi \in CJ_C(\overline{B}(r,0), X)\}. \end{aligned}$$

It is well known (see, e.g. [61]) that for multi-maps in this class, one can define a notion of topological degree. To this end, we need an appropriate concept of homotopy in $CJ_{CV}(\overline{B}(r,0),X)$.

Lemma 4.16 ([61]). *There exists a map $Deg\colon CJ_{CV}(K(r),X)\to\mathbb{Z}$, called the topological degree function, which satisfies the following properties:*

- ($\mathcal{D}_1$) *If $\varphi\in CJ_{CV}(\overline{B}(r,0),X)$ is of the form $\varphi=f\circ F$ where F is single-valued and continuous, then $\mathrm{Deg}(\varphi)=\deg(\varphi)$, where $\deg(\varphi)$ stands for the Leray–Schauder topological degree of the single valued continuous map $\varphi\colon\overline{B}(r,0)\to X$.*
- ($\mathcal{D}_2$) *If $\mathrm{Deg}(\varphi)\neq 0$, then there exists $u\in\overline{B}(r,0)$ such that $0\in\varphi(u)$.*
- ($\mathcal{D}_3$) *If $\varphi\in CJ_{CV}(\overline{B}(r,0),X)$ and $\{u\in\overline{B}(r,0)\colon 0\in\varphi(u)\}\subset\int\overline{B}(r_0,0)$ for some $0<r_0<r$, then the restriction φ_0 of φ to $\overline{B}(r_0,0)$ is in $CJ_{CV}(\overline{B}(r,0),X)$ and $\mathrm{Deg}(\varphi_0)=Deg(\varphi)$.*
- ($\mathcal{D}_4$) *If ϕ_1,ϕ_2 are homotopic in $JC_{CV}(\overline{B}(r,0),X)$ then $\mathrm{Deg}(\phi_1)=\mathrm{Deg}(\phi_2)$.*
- ($\mathcal{D}_5$) *Let Φ_1, $\Phi_2\in JC_{CV}(\overline{B}(r,0),X)$ and assume that:*

$$x\in\{\lambda\varphi_1(u)+(1-\lambda)\varphi_2(u)\mid \textit{for every } (u,\lambda)\in S(r)\times[0,1]\}$$

then $\mathrm{Deg}(\Phi_1)=\mathrm{Deg}(\Phi_2)$.

The following result is due to Górniewicz.

Proposition 4.17 ([61, Proposition 28.9]). *Let $X\in\mathrm{AR}$ and $\Phi\colon X\to\mathcal{P}(X)$ a compact u.s.c. map of the form:*

$$\Phi=f\circ F\colon X\xrightarrow{F}\mathcal{P}(Y)\xrightarrow{f}X,$$

where F is u.s.c. with R_δ-values, $Y\in\mathrm{ANR}$ and f is continuous. Then

$$\mathrm{Fix}(\Phi)\neq\emptyset.$$

In what follows, our main ingredient tool will be the following nonlinear alternative. The finite-dimensional version is given in [61], Proposition 26.8.

Proposition 4.18 (Nonlinear Alternative). *Let $\phi\colon X\to\mathcal{P}(X)$ be a multi-map associated with Φ and let $\overline{M}>0$ be such that $\phi\in CJ_{CV}(K(\overline{M}),X)$. Then*

- (a) *either there exist $\lambda\in(0,1)$ and $x\in S(\overline{M})$ such that $x\in\lambda\Phi(x)$,*
- (b) *or $Fix(\Phi)\neq\emptyset$ and hence $0\in\phi$.*

Proof. Since $\varphi\in CJ_{CV}(\overline{B}(r,0),X)$, then $\Phi\in CJ_C(\overline{B}(r,0),X)$. Consider the following $\widehat{\Phi}$ multi-map defined by

$$\widehat{\Phi}=f\circ F\circ r\colon X\xrightarrow{r}K(\overline{M})\xrightarrow{F}\mathcal{P}(X)\xrightarrow{f}X,$$

with the radial retraction

$$r(x) = \begin{cases} x, & |x| \le \overline{M}, \\ \dfrac{\overline{M}x}{|x|}, & |x| > \overline{M}. \end{cases}$$

We know that $X \in \mathrm{AR}$ and $F \circ r$ is *u.s.c.* with R_δ values. Furthermore $X \in \mathrm{ANR}$, f is continuous function and

$$\widehat{\Phi}(X) = (f \circ F \circ r)(X) = (f \circ F)(r(X)) = (f \circ F)(K(\overline{M})) = \Phi(K(\overline{M})).$$

This implies that $\widehat{\Phi}$ is compact. From Proposition 4.17, there exists $x \in X$ such that $x \in \widehat{\Phi}(x)$. Assume that $|x| \ge \overline{M}$; then $x \in (f \circ F)(\overline{M}x/|x|)$ which implies that

$$\frac{\overline{M}x}{|x|} \in \frac{\overline{M}}{|x|}\Phi\left(\frac{\overline{M}x}{|x|}\right),$$

leading to a contradiction with $\mathrm{Fix}\,(\lambda\Phi \cap S(r)) = \emptyset$. Then $\mathrm{Fix}\,(\Phi) \ne \emptyset$ and $0 \in \varphi(x)$. □

4.2.2. A Poincaré translation operator. By a Poincaré operator for a differential system, we mean the translation operator (or the Poincaré–Andronov, or Levinson operator, or simply the T-operator [100]) along the trajectories of the associated differential system, and the first return map defined on the cross section of the torus by means of the flow generated by the vector field. Both of these operators are single-valued when the uniqueness of solutions of initial value problems is assumed. In the absence of uniqueness, it is often possible to approximate the right-hand sides of the given system by locally lipschitzian ones (hence implying uniqueness), and then apply a standard limiting argument which may be rather complicated for discontinuous right-hand sides. However, set-valued analysis allows us to handle effectively such problems. For further details, we refer to the monographs [3], [61]. Consider the following impulsive problem

$$\begin{cases} (y' - Ay) \in G(t, y(t)), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(0) = x, \end{cases} \tag{4.31}$$

where $G\colon J \times X \to \mathcal{P}(X)$, is a Carathéodory multifunction map and define a multi-map $S_G\colon X \to \mathcal{P}(\mathrm{PC})$ by

$$S_G(x) = \{y : y \text{ is a solution of problem (4.31)}\},$$

where $x \in X$. For some positive real number b, consider the operator P_b defined by $P_b = \Psi_b \circ S_G$ and called the Poincaré translation map associated with the

impulsive Cauchy problem (4.31) where

$$P_b\colon X \xrightarrow{S_G} \mathcal{P}(\mathrm{PC}) \xrightarrow{\Psi_b} \mathcal{P}(X)$$

and

$$\Psi_b(y) = y(0) - y(b).$$

The following lemma is easily proved.

Lemma 4.19. *Let $G\colon J \times X \to \mathcal{P}_{\mathrm{cv,cp}}(X)$ be a Carathéodory multi-map. Then the periodic problem* (4.31) *has a solution if and only if for some $x \in X$, $0 \in P_b(x)$, where P_b is the Poincaré map associated with* (4.31).

4.2.3. The MNC approach. Let X be ANR; we are in position to state our main existence result. Throughout this subsection, $\{T(t)\}_{t\geq 0}$ is assumed uniformly continuous.

Theorem 4.19. *Let $G\colon J \times X \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ be a Carathéodory multi-map with the upper-Scorza–Dragoni property. Assume that*

($\mathcal{R}_1$) *There exist a function $p_* \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\rho\colon [0,\infty) \to [0,\infty)$ such that*

$$\|G(t,y)\|_{\mathcal{P}} \leq p_*(t)\rho(|y|), \quad \textit{for each } (t,y) \in J \times X,$$

with

$$\int_0^b p_*(s)\,ds < \int_1^\infty \frac{du}{\rho(u)}.$$

($\mathcal{R}_2$) *There exist constants $\overline{c}_k > 0$ and continuous functions $\psi_k\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|I_k(x)| \leq c_k\psi_k(|x|) \quad \textit{for each } x \in X,\ k = 1, \dots, m.$$

($\mathcal{R}_3$) *There exists $\overline{p} \in L^1([0,b], \mathbb{R}^+)$ such that for every bounded subset D in E*

$$\chi(F(t,D)) \leq \overline{p}(t)\chi(D)$$

and there exist $L_k > 0$, $k = 0, \dots, m$ such that

$$q_k := 2Me^{\omega t_{k+1}} \sup_{t\in[t_k,t_{k+1}]} \int_{t_k}^{t_{k+1}} e^{-L_k(t-s)}\overline{p}(s)\,ds < 1,\ k = 0, \dots, m.$$

Here χ is the Hausdorff MNC. *Then problem* (4.30) *has at least one solution.*

Proof. (a) From Theorem 3.3, we know that problem (4.31) has at least one solution, and the solution set $S_G(x)$ is nonempty and compact, for each $x \in X$.

From Lemma 2.29, G is σ-Ca-selectionable and so, by Theorem 3.8, for every $x \in X$, $S_G(x)$ is an R_δ−set. In addition, the mapping $\Psi\colon \mathrm{PC} \to X$ defined by

$$y \mapsto \Psi(y) = y(0) - y(\,\cdot\,)$$

is continuous. Indeed, let $\{y_n\}$ be a sequence such that $y_n \to y$ in PC. Then,

$$|\Psi(y_n)(t) - \Psi(y)(t)| \le 2\|y_n - y\|_{\mathrm{PC}} \to 0, \quad \text{as } n \to \infty.$$

(b) Using the conditions $(\mathcal{R}_1)$–$(\mathcal{R}_2)$, we can prove, as mentioned at the end of the proof of Theorem 3.3, that there exists $M_* > 0$ independent of x such that, for every y solution of problem (4.31), we have $\|y\|_{\mathrm{PC}} \le M_*$. Let

$$K = \{x \in X : |x| \le 2M_* + 1\}.$$

We have to show that $P_b \in CJ_{CV}(K, X)$. Let $x \in P_t(x) = \lambda(\Psi_t \circ S_G)(x)$ for some $\lambda \in (0,1)$. Then, there exists $y \in \mathrm{PC}$ such that $y \in S_G(x)$. This yields $y(0) = x$ and $x = \lambda(x - y(t))$, $x \in S(2M_* + 1)$. For $t \in J$, we have the estimates

$$|x| \le |y(0)| + |y(t)| \le 2\|y\|_{\mathrm{PC}} \le 2M_*$$

which is contradiction to $|x| = 2M_* + 1$.

(c) Making use of Lemma 2.9, we will show that S_G is *u.s.c.* by proving that the graph of S_G

$$\Gamma_G := \{(x, y) : y \in S_G(x)\}$$

is closed. Let $(x_n, y_n) \in \Gamma_G$, i.e. $y_n \in S_G(x_n)$ and let $(x_n, y_n) \to (x, y)$, as $n \to \infty$. Since $y_n \in S_G(x_n)$, there exists $v_n \in S_{G,y_n}$ such that

$$y_n(t) = T(t)x_n + \int_0^t T(t-s)v_n(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)), \quad t \in J. \tag{4.32}$$

Since (x_n, y_n) converges to (x, y), there exists $M_1 > 0$ such that

$$|x_n| \le M_1 \quad \text{for every } n \in \mathbb{N}.$$

Moreover,

$$\|y_n\|_{\mathrm{PC}} \le M_*, \quad \text{for every } n \in \mathbb{N}.$$

So

$$|v_n(t)| \le \overline{p}(t)\rho(M_*), \quad t \in J.$$

Hence, $v_n(t) \in p_*(t)\rho(M_*)\overline{B}(0,1) := H(t)$ for almost every $t \in J$ and $x_n \in M_1 B(0,1)$. The mapping $H\colon J \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ is a multi-map which is integrably bounded. Since $\{v_n(\,\cdot\,) : n \ge 1\} \in H(\,\cdot\,)$ is bounded, we may pass to a subsequence if necessary to obtain that v_n converges weakly to v in $L^1_w(J, X)$. Arguing

as in the proof of Lemma 4.5, we deduce that for each $t \in J$, the sequence $\{y_n\}$ converges to y with

$$y(t) = T(t)x + \int_0^t T(t-s)v(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), \quad t \in J. \tag{4.33}$$

Thus, $y \in S_G(x)$.

(d) It remains to show that $S_G(\,\cdot\,)$ maps compact sets into relatively compact sets of PC. Let K be a compact set in X and let $\{y_n\} \subset S_G(K)$. Then there exists a sequence $\{x_n\} \subset K$ satisfying (4.32). Since $\{x_n\}$ is a compact sequence, there exists a subsequence of $\{x_n\}$ converging to some limit x. From the boundedness of the sequence $\{y_n\}$, $n \in \mathbb{N}$ and arguing as in the proof of Theorem 3.3, we can show that $\{y_n : n \in \mathbb{N}\}$ is equicontinuous in PC. The Arzéla–Ascoli Theorem then implies that there exists a subsequence of $\{y_n\}$ converging to some limit y in PC. By a similar argument to the one above, we can prove that y satisfies (4.33) and $y \in S_G(x)$. Then $P_b \in CJ_{CV}(K, \mathrm{PC})$. By the nonlinear alternative (Proposition 4.18), we conclude that $\mathrm{Fix}\,(j - P_b) \neq \emptyset$, i.e. $0 \in P_b$, ending the proof of the theorem. $\square$

CHAPTER 5

IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS

5.1. Introduction

It is well-known that systems with delay are of great importance in applications. They are described by functional differential equations and inclusions, also called differential equations and inclusions with deviating argument. Among functional differential equations, one may distinguish some special classes of equations, retarded functional differential equations, advanced functional differential equations and inclusions. In particular, retarded functional differential equations and inclusions describe systems or processes whose rate of change of state is determined by their past and present states. Such equations are frequently encountered as mathematical models of most dynamical processes in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory, etc. Especially, since the 1960's, many books have been published on delay differential equations; see, for examples, the books by Burton [28], [29], Èl'sgol'ts [45], Èl'sgol'ts and Norkin [46], Gopalsmy [58], Azbelez *et al.* [15], Hale [78], Hale and Lunel [79], Kolmanovskii and Myshkis [98], Kolmanovskiĭ and Nosov [99], Krasovskii [100], Yoshizawa [141]. Functional differential equations with impulsive effects and fixed moments have been recently addressed by Yujun and Erxin [144] and Yujun [143]. Some existence results on impulsive functional differential equations with finite or infinite delay may be found in [19], [20], [30], [32], [41], [114], [116] and references therein.

Given a real separable Banach space E with norm $|\cdot|$, we will consider in this section the impulsive problem for the following first-order semi-linear differential inclusion:

$$\begin{cases} (y' - Ay)(t) \in F(t, y_t), & \text{a.e. } t \in J', \\ \Delta y_{t=t_k} = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(t) = \phi(t), & t \in [-r, 0], \end{cases} \tag{5.1}$$

where $0 < r < \infty$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $J = [0, b]$, and $J' = J \setminus \{t_1, \ldots, t_m\}$. $F\colon J \times \mathcal{D} \to \mathcal{P}(E)$ is a multifunction and $\phi \in \mathcal{D}$ where

$$\mathcal{D} = \{y\colon [-r, 0] \to E : y \text{ is continuous everywhere except for a finite number of points } \overline{t} \text{ at which } y(\overline{t}^-) \text{ and } y(\overline{t}^+) \text{ exist and satisfy } y(\overline{t}^-) = y(\overline{t})\}.$$

Moreover, if

$$\Omega = \{y\colon [-r, b] \to E,\ y \in PC \cap \mathcal{D}\},$$

then Ω is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y\|_{PC}, \|y\|_{\mathcal{D}}\},$$

where $\|y\|_{\mathcal{D}} = \sup_{t\in[-r,0]} |y(t)|$. For any function y defined on $[-r, b]$ and any $t \in J$, y_t refers to the element of $\mathcal{D}$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0];$$

the function y_t represents the history of the state from time $t - r$ up to the present time t. We can investigate the following questions: existence problems, topological characterization of solution of problem (5.1) as well as Aronszajn-type results. We obtain similar results; hereafter, we only present the results without proofs. For the details, we refer to [41].

5.2. Existence results

The nonlinearity $F\colon J \times \mathcal{D} \to \mathcal{P}_{cp}(E)$ is a Carathéodory multimap which satisfies some of the following assumptions:

($\mathcal{AF}_1$) There exist a function $p \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\rho\colon [0, \infty) \to [0, \infty)$ such that

$$\|F(t, z)\| \le p(t)\rho(\|z\|_{\mathcal{D}}), \quad \text{for a.e. } t \in J \text{ and each } z \in \mathcal{D},$$

and

$$\int_0^b p(s)\, ds < \int_1^\infty \frac{du}{\rho(u)}.$$

($\mathcal{AF}_4$) There exists $\overline{p} \in L^1([0, b], \mathbb{R}^+)$ such that for every bounded subset D in $\mathcal{D}$

$$\chi(F(t, D)) \le \overline{p}(t) \sup\{\chi(D(\theta)) : \theta \in [-r, 0]\}$$

and there exist $L_k > 0$, $k = 0, \ldots, m$ such that

$$q_k := 2Me^{\omega t_{k+1}} \sup_{t\in[t_k, t_{k+1}]} \int_{t_k}^{t_{k+1}} e^{-L_k(t-s)} p(s)\, ds < 1, \quad k = 0, \ldots, m.$$

Here χ is the Hausdorff MNC and $D(\theta) := \{y(\theta),\ y \in \mathcal{D}\}$.

Then we have a first existence result in the convex case:

Theorem 5.1 (The convex case)**.** *Assume that* $F\colon J \times \mathcal{D} \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ *satisfies either* $(\mathcal{AF}_1)$, $(\mathcal{A}_2)$ *and* $(\mathcal{A}_3)$ *or* $(\mathcal{AF}_1)$, $(\mathcal{A}_2)$ *and* $(\mathcal{AF}_4)$. *Then the set of solutions for problem* (5.1) *in nonempty and compact.*

To deal with the nonconvex case, consider the following hypotheses:

$(\mathcal{BF}_1)$ For fixed y, the multi-map $t \mapsto F(t, y)$ is measurable.

$(\mathcal{BF}_2)$ There exists $p \in L^1([0, b], \mathbb{R}^+)$ such that

$$H_d(F(t, z_1), F(t, z_2)) \le p(t)\|z_1 - z_2\|_{\mathcal{D}}, \quad \text{for a.e. } t \in J \text{ and } z_1, z_2 \in \mathcal{D}$$
$$\text{and } F(t, 0) \subset p(t)B(0, 1), \quad \text{for a.e. } t \in J.$$

Then we have

Theorem 5.2 (The nonconvex case)**.** *Under assumptions* $(\mathcal{BF}_1)$–$(\mathcal{BF}_2)$, *the operator solution* $S_{[-r,b]}(\,\cdot\,)$ *of problem* (5.1) *has nonempty closed values and a closed graph.*

Proof. Let $S_{[-r,b]}\colon \mathcal{D} \to \mathcal{P}(E)$ be the operator solution of problem (5.1) defined by

$$S_{[-r,b]}(\phi) = \{y \in \Omega : y \text{ solution of problem (5.1)}\}.$$

With assumptions $(\mathcal{CF}_1)$–$(\mathcal{CF}_2)$, we may use the Covitz–Nadler fixed point theorem (Lemma 2.74), as in the proof of Theorem 3.4, to prove that

$$S_{[-r,b]}(\phi) \ne \emptyset \quad \text{for every } \phi \in \mathcal{D}.$$

Thus we only show the closeness of both the values and the graph of $S_{[-r,b]}(\,\cdot\,)$.

Step 1. $S_{[-r,b]}(\,\cdot\,) \in \mathcal{P}_{\mathrm{cl}}(E)$. For this, let $\phi \in \mathcal{D}$ and let $y_n \in S_{[-r,b]}(\phi)$, $n \in \mathbb{N}$ be a sequence which converges to some limit y_* in Ω. Then

$$y_n(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v_n(s)\,ds & \\ \qquad + \displaystyle\sum_{0<t_k<t} T(t-t_k)I_k((y_n(t_k)), & t \in [0, b], \end{cases}$$

where $v_n \in \{v \in L^1([0, b], E)\colon v(\,\cdot\,) \in F(\,\cdot\,, (y_n\,\cdot\,))\}$. Let v_n be fixed. By Lemma 2.16, for every ε, there exists $w_\varepsilon(\,\cdot\,) \in F(\,\cdot\,, (y_*)_\cdot\,)$ such that

$$|v_n(t) - w_\varepsilon(t)| \le d(v_n(t), F(t, (y_*)_t) + \varepsilon$$

Let $\varepsilon = 1/m$, $m \in \mathbb{N}$, since $F(\,\cdot\,, \cdot,)$ has compact values, then there exists a subsequence $w_m(\,\cdot\,)$ such that

$$w_m(\,\cdot\,) \to w_n(\,\cdot\,), \quad \text{as } m \to \infty, \quad \text{and} \quad w_n(t) \in F(t, (y_*)_t) \quad \text{a.e. } t \in [0, b].$$

Also using the compactness of $F(t,(y_*)_t)$, there exists a subsequence $w_n(\,\cdot\,)$ such that

$$w_n(\,\cdot\,) \to w(\,\cdot\,), \quad \text{as } n \to \infty$$

and

$$\begin{aligned}|v_n(t) - w(t)| &\le |v_n(t) - w_n(t)| + |w_n(t) - w(t)| \\ &\le d(v_n(t), F(t,(y_*)_t) + |v_n(t) - w(t)| \\ &\le p(t)\|y_n - y_*\|_\Omega + |w_n(t) - w(t)|.\end{aligned}$$

This implies that $v_n(\,\cdot\,)$ converges to $w(\,\cdot\,) \in F(\,\cdot\,,(y_*)_\cdot\,)$.

Now, we prove that $w \in L^1(J,E))$. Using the fact that $F(\,\cdot\,,\,\cdot\,)$ is p-Lipschitz together with Lemma 2.16, we get

$$|v_n(t)| \le d(v_n(t), F(t,0)) + 2p(t) \le 2p(t) + p(t)|y_n(t)|, \quad \text{a.e. } t \in [0,b].$$

Thus $|v_n(t)| \le (2 + M_*)p(t)$, for almost every $t \in [0,b]$.

$$|y_n(t) - z(t)| \le Me^{\omega b}\int_0^b |v_n(s) - v(s)|\,ds + Me^{\omega b}\sum_{k=1}^m |I_k((y_n(t_k)) - I_k((y_*(t_k))|,$$

where

$$z(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v(s)\,ds & \\ \qquad + \displaystyle\sum_{0<t_k<t} T(t-t_k)I_k((y_*(t_k)), & t \in [0,b]. \end{cases}$$

Using the continuity of I_k and the Lebesgue dominated convergence theorem, we conclude that $y_* = z$.

Step 2. $S_{[-r,b]}$ has a closed graph. Let $\phi_n \to \phi_*$, $y_n \in S_{[-r,b]}(\phi_n)$ and $y_n \to y_*$. $y_n \in S_{[-r,b]}(\phi_n)$ means that there exists $g_n \in L^1$ such that, for each $t \in [-r,0]$, $y_n(t) = \phi_n(t)$ and for $t \in [0,b]$,

$$y_n(t) = T(t)\phi_n(0) + \int_0^t T(t-s)g_n(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)).$$

We prove that $y_* \in S_{[-r,b]}(\phi_*)$, i.e. there exists $g_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$y_*(t) = T(t)\phi_*(0) + \int_0^t T(t-s)g_*(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)).$$

Using the fact that F has compact values together with $(\mathcal{B}F_1)$–$(\mathcal{B}F_2)$, we may pass to a subsequence, if necessary, to get that $\{g_n\}$ converges to some limit g_*

in $L^1(J,E)$. Since the functions I_k, $k=1,\dots,m$ are continuous, we obtain the estimates

$$\begin{aligned}
&\Big|y_*(t) - T(t)\phi_*(0) - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)) - \int_0^t T(t-s)g_*(s)\,ds\Big| \\
&\quad\le \Big|\Big(y_n(t) - T(t)\phi_n(0) - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)) - \int_0^t T(t-s)g_n(s)\,ds\Big) \\
&\qquad - \Big(y_*(t) - T(t)\phi_*(0) - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)) - \int_0^t T(t-s)g_*(s)\,ds\Big)\Big| \\
&\quad\le \|y_n - y_*\|_\Omega + Me^{\omega b}\sum_{k=1}^m |I_k(y_n(t_k)) - I_k(y_*(t_k))| \\
&\qquad + \|T(t)\|_{B(E)}\cdot|\phi_n(0)-\phi_*(0)| + Me^{\omega b}\int_0^b |g_n(s)-g_*(s)|\,ds.
\end{aligned}$$

The right-hand side terms tend to 0, as $n\to+\infty$, proving our claim. □

5.3. Structure of the solution set

Regarding the topological properties of solution sets, we can prove similar results to those in Section 5.2 (see the proofs of Theorems 3.6 and 3.7). Consider the first-order impulsive single-valued problem:

$$\text{(5.2)}\qquad \begin{cases} y'(t) - Ay(t) = f(t,y_t), & \text{a.e. } t\in J=[t_0,b]\setminus\{t_1,\dots,t_m\},\\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k=1,\dots,m,\\ y(t)=\phi(t), & t\in[-r,t_0],\end{cases}$$

where $f\colon J\times\mathcal{D}\to E$ is a given function and $\phi\in\mathcal{D}$. Denote by $S(f,\phi)$ the set of all solutions of problem (5.2) and assume that $f\colon J\times\mathcal{D}\to E$ is an L^1-Carathéodory function that satisfies

($\mathcal{C}F_2$) There exist a function $p\in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\rho\colon[0,\infty)\to[0,\infty)$ such that

$$|f(t,x)|\le p(t)\rho(\|x\|_{\mathcal{D}}),\quad \text{for a.e. } t\in J \text{ and each } x\in\mathcal{D}$$

and

$$\int_0^b p(s)\,ds < \int_{Me^{\omega b}\|\phi\|_{\mathcal{D}}}^\infty \frac{du}{\rho(u)}.$$

We have

Theorem 5.3. *Assume that assumption* $(\mathcal{C}F_1)$ *hold together with either* $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ *or* $(\mathcal{A}_2)$, $(\mathcal{A}_4)$. *Then the set* $S(f,\phi)$ *is* R_δ.

Proof. Let $F\colon \Omega \to \Omega$ be defined by:

$$L(y)(t) = \begin{cases} \phi(t), & t \in [-r, t_0], \\ T(t-t_0)\phi(t_0) + \displaystyle\int_{t_0}^{t} T(t-s) f(s, y_s)\, ds \\ \qquad\qquad + \displaystyle\sum_{t_0 < t_k < t} T(t-t_k) I_k(y(t_k)), & t \in (t_0, b]. \end{cases}$$

Thus $\operatorname{Fix}(L) = S(f,\phi)$. From Theorem 5.1, we know that $S(f,\phi) \neq \emptyset$ and there exists $\overline{M} > 0$ such that

$$\|y\|_\Omega \le \overline{M}, \quad \text{for every } y \in S(f,\phi).$$

Define

$$\widetilde{f}(t, y_t) = \begin{cases} f(t, y_t), & \text{if } \|y_t\|_{\mathcal{D}} \le \overline{M}, \\ f\left(t, \dfrac{\overline{M} y_t}{\|y_t\|_{\mathcal{D}}}\right), & \text{if } \|y_t\|_{\mathcal{D}} \ge \overline{M}. \end{cases}$$

Since f is L^1-Carathéodory, the function $\widetilde{f}$ is Carathéodory and is integrably bounded by $(\mathcal{D}F_2)$. So there exists $h \in L^1(J, \mathbb{R}^+)$ such that

$$|\widetilde{f}(t,x)| \le h(t), \quad \text{for a.e. } t \text{ and all } x \in \mathcal{D}. \tag{5.4}$$

Consider the modified problem

$$\begin{cases} y'(t) - Ay(t) = \widetilde{f}(t, y_t), & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, \dots, m, \\ y(t) = \phi(t), & t \in [-r, t_0]. \end{cases}$$

We can easily prove that $S(f,\phi) = S(\widetilde{f},\phi) = \operatorname{Fix}(\widetilde{F})$, where $\widetilde{F}\colon \Omega \to \Omega$ is defined by

$$\widetilde{F}(y)(t) = \begin{cases} \phi(t), & t \in [-r, t_0], \\ T(t-t_0)\phi(t_0) + \displaystyle\int_{t_0}^{t} T(t-s) \widetilde{f}(s, y_s)\, ds \\ \qquad\qquad + \displaystyle\sum_{0 < t_k < t} T(t-t_k) I_k(y(t_k^-)), & t \in [t_0, b]. \end{cases}$$

By the inequality (5.3) and the continuity of I_k, we deduce that

$$\|\widetilde{L}(y)\|_\Omega \le M e^{\omega b} \|\phi\|_{\mathcal{D}} + M e^{\omega b} \|h\|_{L^1} + M e^{\omega b} \sum_{k=1}^{m} c_k \phi_k(\overline{M}) := R.$$

Then $\widetilde{L}$ is uniformly bounded. As in Theorem 5.1, we can prove that $\widetilde{L}\colon \Omega \to \Omega$ is compact which allows us to define the compact perturbation of the identity

$\widetilde{G}(y) = y - \widetilde{L}(y)$ which is a proper map. From the compactness of $\widetilde{F}$, we can easily prove that all conditions of Theorems 2.47 and 2.46 are met. Therefore the solution set $S(\widetilde{f}, \phi) = \widetilde{G}^{-1}(0)$ is an R_δ set. $\square$

CHAPTER 6

IMPULSIVE DIFFERENTIAL INCLUSIONS ON THE HALF-LINE

Many properties of solutions for differential equations, such as stability or oscillation, require global properties of solutions. This is the main motivation to search for sufficient conditions that ensure global existence of solutions for impulsive differential equations and inclusions. In this direction, some questions have been addressed by Graef and Ouahab [68], [70], Guo [72], [73], Guo and Liu [74], Henderson and Ouahab [80]–[82], Marino *et al* [107], Ouahab [114], Stamov and Stamova [130], Weng [138], and Yan [139], [140]. Consider the following problem:

$$\begin{cases} y'(t) \in F(t, y(t)), & \text{a.e. }, t \in J := [0, \infty) \setminus \{t_1, t_2, \dots\}, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, 2 \dots \\ y(0) = a, \end{cases} \tag{6.1}$$

where $F\colon J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multi-map, $\mathcal{P}(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$, $0 = t_0 < t_1 < \dots < t_k < \dots$, $\lim_{k\to\infty} t_k = \infty$. $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$ where $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ stand for the right and the left limits of y at $t = t_k$, respectively and $a \in \mathbb{R}^n$. Throughout this section, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ for $k = 1, 2, \dots$ and $(\mathbb{R}^n, \|\cdot\|)$ is the n-euclidian Banach space.

6.1. Existence results and compactness of solution sets

Consider the Banach space $\mathrm{PC}_b = \{y \in \mathrm{PC}(\mathbb{R}^+, \mathbb{R}^n) : y \text{ is bounded}\}$, where

$$\begin{aligned} \mathrm{PC}(\mathbb{R}^+, \mathbb{R}^n) = \{&y\colon [0, \infty) \to \mathbb{R}^n,\ y_k \in C((t_k, t_{k+1}), \mathbb{R}^n),\ k = 0, \dots, m, \\ &y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k) = y(t_k^-),\ \text{for } k = 1, 2, \dots\} \end{aligned}$$

and $y_k := y|_{(t_k,t_{k+1}]}$. Endowed with the norm

$$\|y\|_b = \sup\{\|y(t)\| : t \in [0,\infty)\},$$

PC_b is a Banach space. We let $\mathrm{PC}_l = \{y \in \mathrm{PC}_b(\mathbb{R}^+,\mathbb{R}^n) : \lim_{t\to+\infty} \text{ exists}\}$.

Next we define what we mean by a solution of problem (6.1).

Definition 6.1. A function $y \in \mathrm{PC}$ is said to be a solution of problem (6.1) if there exists $v \in L^1(J,\mathbb{R}^n)$ such that $v(t) \in F(t,y(t))$, almost everywhere on J, $y(0) = a$ and

$$y(t) = a + \int_0^t v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad \text{a.e. } t \in J.$$

6.1.1. The convex u.s.c. case. In this subsection, we present a global existence result and prove the compactness of solution sets for problem (6.1) by using a nonlinear alternative for multi-maps combined with a compactness argument. The nonlinearity is *u.s.c.* with respect to the second variable and satisfies a Nagumo growth condition.

Theorem 6.2.

(a) *Assume that the impulsive functions $I_k \in C(\mathbb{R}^n,\mathbb{R}^n)$ satisfy*

($\mathcal{HF}_1$) *there exist $a_k, b_k > 0$ such that*

$$\|I_k(x)\| \le a_k\|x\| + d_k, \quad \textit{for every } x \in \mathbb{R}^n,\ k = 1,2,\ldots$$

with

$$\sum_{k=1}^{\infty} a_k < 1 \quad \textit{and} \quad \sum_{k=1}^{\infty} b_k < \infty.$$

(b) *The Carathéodory multi-map $F\colon J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ has compact, convex values and satisfies*

($\mathcal{HF}_2$) *there exist a continuous nondecreasing function $\psi\colon [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1(J,\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} \le p(t)\psi(\|x\|), \quad \textit{for a.e. } t \in J \textit{ and each } x \in \mathbb{R}^n,$$

with

$$\int_0^\infty m(s)\,ds < \int_c^\infty \frac{du}{\psi(u)}$$

where

$$m(s) = \frac{p(s)}{1 - \sum_{k=1}^{\infty} a_k} \quad \textit{and} \quad c = \frac{\|a\| + \sum_{k=1}^{\infty} b_k}{1 - \sum_{k=1}^{\infty} a_k}.$$

Then problem (6.1) *has at least one solution. Moreover, the solution set* $S_F(a)$ *is compact and the multi-map* $S_F\colon a \to S_F(a)$ *is u.s.c.*

The following compactness criterion on unbounded domains is a simple extension of Corduneanu's compactness criterion in $C_b(\mathbb{R}^+, \mathbb{R}^n)$ (see [36, p. 62]).

Lemma 6.3. *Let* $M \subset \mathrm{PC}_\ell = \{x \in \mathrm{PC}(\mathbb{R}^+, \mathbb{R}^n) : \lim_{t\to\infty} x(t)$ *exists*$\}$. *Then* M *is relatively compact if it satisfies the following conditions:*

(a) M *is uniformly bounded in* $\mathrm{PC}_\ell(\mathbb{R}^+, \mathbb{R}^n)$.
(b) *The functions belonging to* M *are almost equicontinuous on* $\mathbb{R}^+$, *i.e. equicontinuous on every compact interval of* $\mathbb{R}^+$.
(c) *The functions from* M *are equiconvergent, that is, given* $\varepsilon > 0$, *there corresponds* $T(\varepsilon) > 0$ *such that* $|x(t) - x(\infty)| < \varepsilon$ *for any* $t \geq T(\varepsilon)$ *and* $x \in M$.

Proof of Theorem 6.2.

Step 1. Existence of solutions. Consider the operator $N\colon \mathrm{PC} \to \mathcal{P}(\mathrm{PC})$ defined for $y \in \mathrm{PC}$ by

$$N(y) = \left\{ h \in \mathrm{PC}_b : \ h(t) = a + \int_0^t v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k)), \text{ a.e. } t \in J \right\}, \tag{6.2}$$

where $v \in S_{F,y} = \{v \in L^1(J, \mathbb{R}^n) \colon v(t) \in F(t, y(t)), \text{ a.e. } t \in J\}$. Fixed points of the operator N are solutions of problem (6.1). We shall show that N satisfies the assumptions of Lemma 2.72. Finally notice that since $S_{F,y}$ is convex (because F has convex values), then N takes convex values.

Claim 1. $N(\mathrm{PC}_b) \subset \mathrm{PC}_\ell$. Indeed, if $y \in \mathrm{PC}_b$ and $h \in N(y)$ then there exists $v \in S_{F,y}$ such that

$$h(t) = a + \int_0^t v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad \text{a.e. } t \in J. \tag{6.3}$$

Since $v \in L^1(J)$ and

$$\sum_{0<t_k<\infty} I_k(y(t_k)) \leq \|y\|_{\mathrm{PC}_b} + \sum_{k=1}^{\infty} d_k,$$

then clearly $h \in \mathrm{PC}_\ell$ and $h(\infty) = a + \int_0^{+\infty} v(s)\,ds + \sum_{0<t_k<\infty} I_k(y(t_k))$. Moreover,

$$\begin{aligned} \|h(t)\| &\leq \|a\| + \int_0^t \|F(s, y(s))\|_{\mathcal{P}}\,ds + \sum_{0<t_k<t} \|I_k(y(t_k))\| \\ &\leq \|a\| + \int_0^t p(s)\psi(\|y(s)\|)\,ds + \sum_{0<t_k<t} (a_k\|y(t_k)\| + b_k). \end{aligned}$$

Hence

$$\|h\|_{\mathrm{PC}_b} \le \|a\| + \psi(|y|_{\mathrm{PC}_b}) \int_0^\infty p(s)\,ds + \|y\|_{\mathrm{PC}_b} + \sum_{k=1}^\infty d_k.$$

This shows that N sends bounded sets into bounded sets in PC_ℓ.

Claim 2. *N sends bounded sets in PC_b into almost equicontinuous sets of PC_ℓ.* Let $r > 0$, $B_r := \{y \in \mathrm{PC}_b : \|y\|_{\mathrm{PC}_b} \le r\}$ be a bounded set in PC_b, $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, and $y \in B_r$. For each $h \in N(y)$, we have

$$\begin{aligned}\|h(\tau_2) - h(\tau_1)\| &\le \int_{\tau_1}^{\tau_2} \|v(s)\|\,ds + \sum_{\tau_1 < t_k < \tau_2} \|I_k(y(t_k))\| \\ &\le \psi(r) \int_{\tau_1}^{\tau_2} p(s)\,ds + \sum_{\tau_1 < t_k < \tau_2} (a_k r + b_k).\end{aligned}$$

Since $\sum_{k=1}^\infty a_k < \infty$, $\sum_{k=1}^\infty b_k < \infty$ and $p \in L^1(J, \mathbb{R}^+)$, the right-hand term tends to zero as $|\tau_1 - \tau_2| \to 0$, proving equicontinuity for the case where $t \ne t_i^-$, $i = 1, 2, \dots$ To prove equicontinuity at $t = t_i$ for some $i \in \mathbb{N}^*$, fix $\varepsilon_0 > 0$ such that $\{t_j : j \ne i\} \cap [t_i - \varepsilon_0, t_i + \varepsilon_0] = \emptyset$. Then for each $0 < \varepsilon < \varepsilon_0$, we have

$$\|h(t_i) - h(t_i - \varepsilon)\| \le \int_{t_i - \varepsilon}^{t_i} \|v(s)\|\,ds \le \psi(r) \int_{t_i - \varepsilon}^{t_i} p(s)\,ds.$$

Since $p \in L^1(J, \mathbb{R}^+)$, the right-hand term tends to 0 as $\varepsilon \to 0$. The equicontinuity at t_i^+ $(i = 1, \dots)$ is proved in a similar way.

Claim 3. We now show that the set $N(\overline{B}(0, r))$ is equiconvergent at ∞, i.e. for every $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that $\|h(t) - h(+\infty)\| \le \varepsilon$ for every $t \ge T$ and each $h \in N(\overline{B}(0, r))$. Letting $h \in N(y)$ for some $y \in \overline{B}(0, r)$, there exists $v \in S_{F,y}$ such that h satisfies (6.3). Then

$$\begin{aligned}\|h(t) - h(+\infty)\| &\le \int_t^{+\infty} \|v(s)\|\,ds + \sum_{t \le t_k < \infty} \|I_k(y(t_k))\| \\ &\le \psi(r) \int_t^\infty p(s)\,ds + \sum_{t \le t_k < \infty} (a_k r + b_k).\end{aligned}$$

Since $\sum_{k=1}^\infty a_k < \infty$, $\sum_{k=1}^\infty b_k < \infty$ and $p \in L^1(J, \mathbb{R}^+)$, there exist k_0 and $T(\varepsilon) > 0$ such that

$$\sum_{k=k_0}^\infty (a_k r + b_k) \le \frac{\varepsilon}{2} \quad \text{and} \quad \int_t^\infty p(s) < \frac{\varepsilon}{2\psi(r)}, \quad \text{for all } t \ge T(\varepsilon).$$

Hence

$$\|h(t) - h(+\infty)\| \le \varepsilon, \quad \text{for all } t \ge \max(k_0, T(\varepsilon)).$$

Then $N(B(0,r))$ is equiconvergent. With Lemma 6.3 and Claims 1–3, we conclude that N is completely continuous.

Claim 4. *N is u.s.c.* To this end, we show that N has a closed graph. Let $h_n \in N(y_n)$ such that $h_n \to h$ and $y_n \to y$, as $n \to +\infty$. Then there exists $M > 0$ such that $\|y_n\| \le M$. We shall prove that $h \in N(y)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$h_n(t) = a + \int_0^t v_n(s)\,ds + \sum_{0<t_k<t} I_k(y_n(t_k)).$$

$(\mathcal{HF}_2)$ implies that $v_n(t) \in p(t)\psi(M)B(0,1)$. Then $(v_n)_{n\in\mathbb{N}}$ is integrably bounded in $L^1(J,\mathbb{R}^n)$ hence semi-compact. By Lemma 2.60, there exists a subsequence, still denoted $(v_n)_{n\in\mathbb{N}}$, which converges weakly to some limit $v \in L^1(J,\mathbb{R}^n)$. Moreover, the mapping $\Gamma\colon L^1(J,\mathbb{R}^n) \to \mathrm{PC}_b(J,\mathbb{R}^n)$ defined by

$$\Gamma(g)(t) = \int_0^t g(s)\,ds$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies (see [44] or [26, Theorem 3.10]). Therefore for almost every $t \in J$, the sequence $y_n(t)$ converges to $y(t)$ and by continuity of I_k, we arrive at

$$h(t) = a + \int_0^t v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k)).$$

It remains to prove that $v \in F(t,y(t))$, for almost every $t \in J$. Mazur's Lemma 2.62 yields the existence of $\alpha_i^n \ge 0$, $i = n,\dots,k(n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n(\,\cdot\,) = \sum_{i=1}^{k(n)} \alpha_i^n v_i(\,\cdot\,)$ converges strongly to v in L^1. Using Lemma 2.6, we obtain that

$$\begin{aligned} v(t) &\in \bigcap_{n\ge1} \overline{\{g_k(t),\ k \ge n\}}, \quad \text{a.e. } t \in J \\ &\subset \bigcap_{n\ge1} \overline{\mathrm{co}}\{v_k(t),\ k \ge n\} \\ &\subset \bigcap_{n\ge1} \overline{\mathrm{co}}\{\bigcup_{k\ge n} F(t,y_k(t))\} = \overline{\mathrm{co}}(\limsup_{k\to\infty} F(t,y_k(t))). \end{aligned} \tag{6.4}$$

However, the fact that the multi-valued $x \to F(\,\cdot\,,x)$ is *u.s.c.* and has compact values together with Lemma 2.5 imply that

$$\limsup_{n\to\infty} F(t,y_n(t)) = F(t,y(t)), \quad \text{a.e. } t \in J.$$

This with (6.4) yield that $v(t) \in \overline{\mathrm{co}}\, F(t, y(t))$. Finally $F(\,\cdot\,,\,\cdot\,)$ has closed, convex values, hence $v(t) \in F(t, y(t))$, for almost every $t \in J$. Thus $h \in N(y)$, proving that N has a closed graph. Finally, with Lemma 2.9 and the compactness of N, we conclude that N is *u.s.c.*

Claim 5. *A priori bounds on solutions.* Let $y \in \mathrm{PC}_b$ be such that $y \in N(y)$. Then there exists $v \in S_{F,y}$ such that

$$y(t) = a + \int_0^t v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad \text{a.e. } t \in J.$$

Arguing as in Claim 1, we obtain the estimates:

$$\|y(t)\| \le \|a\| + \int_0^t p(s)\psi(\|y(s)\|)\,ds + \sum_{0<t_k<t} (a_k\|y(t_k)\| + b_k), \quad \text{a.e. } t \in J.$$

Letting $\alpha(t) = \sup\{\|y(s)\| : s \in [0,t]\}$ and using the increasing character of ψ, we get

$$\alpha(t) \le \|a\| + \int_0^t p(s)\psi(\alpha(s))\,ds + \sum_{0<t_k<t} (a_k\alpha(t) + b_k).$$

Hence

$$\alpha(t) \le \frac{1}{1-\sum_{k=1}^{\infty} a_k}\left(\|a\| + \int_0^t p(s)\psi(\alpha(s))\,ds + \sum_{k=1}^{\infty} b_k\right).$$

Denoting by $\beta(t)$ the right-hand side, we have

$$\|y(t)\| \le \alpha(t) \le \beta(t), \quad t \in J$$

as well as

$$\beta(0) = \frac{\|a\| + \sum\limits_{k=1}^{\infty} d_k}{1 - \sum\limits_{k=1}^{\infty} a_k} \quad \text{and} \quad \beta'(t) = \frac{p(t)\psi(\alpha(t))}{1 - \sum\limits_{k=1}^{\infty} a_k} \le \frac{p(t)\psi(\beta(t))}{1 - \sum\limits_{k=1}^{\infty} a_k}.$$

$(\mathcal{HF}_1)$ implies that for $t \in J$

$$\Gamma(z(t)) = \int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)} \le \frac{1}{1 - \sum\limits_{k=1}^{\infty} a_k} \int_0^{\infty} p(s)\,ds < \int_{\beta(0)}^{\infty} \frac{ds}{\psi(s)} = \Gamma(+\infty).$$

Thus

$$\beta(t) \le \Gamma^{-1}\left(\frac{\|p\|_{L^1}}{1 - \sum\limits_{k=1}^{\infty} a_k}\right), \quad \text{for every } t \in J,$$

where $\Gamma(z) = \int_{\beta(0)}^{z} du/\psi(u)$. As a consequence

$$\|y\|_{\mathrm{PC}_b} \leq \Gamma^{-1}\left(\frac{\|p\|_{L^1}}{1-\sum_{k=1}^{\infty} a_k}\right) := \widetilde{M}.$$

Finally, let

$$U := \{y \in \mathrm{PC}_b : \|y\|_{\mathrm{PC}_b} < \widetilde{M} + 1\}$$

and consider the operator $N\colon \overline{U} \to \mathcal{P}_{\mathrm{cv,cp}}(\mathrm{PC}_b)$. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the multivalued version of the nonlinear alternative of Leray–Schauder (Lemma 2.72), N has a fixed point y in U which is a solution of problem (6.1).

Step 2. Compactness of the solution set. For each $a \in \mathbb{R}^n$, let

$$S_F(a) = \{y \in \mathrm{PC}_b : y \text{ is a solution of problem (6.1)}\}.$$

From Step 1, there exists $\widetilde{M}$ such that for every $y \in S_F(a)$, $\|y\|_{\mathrm{PC}_b} \leq \widetilde{M}$. Since N is completely continuous, $N(S_F(a))$ is relatively compact in PC_b. Let $y \in S_F(a)$; then $y \in N(y)$ hence $S_F(a) \subset \overline{N(S_F(a))}$. It remains to prove that $S_F(a)$ is a closed subset in PC_b. Let $\{y_n : n \in \mathbb{N}\} \subset S_F(a)$ be such that $(y_n)_{n\in\mathbb{N}}$ converges to y. For every $n \in \mathbb{N}$, there exists v_n such that $v_n(t) \in F(t, y_n(t))$, for almost every $t \in J$ and

$$y_n(t) = a + \int_0^t v_n(s)\,ds + \sum_{0<t_k<t} I_k(y_n(t_k)). \tag{6.5}$$

Arguing as in Step 1, Claim 4, we can prove that there exists v such that $v(t) \in F(t, y(t))$ and

$$y(t) = a + \int_0^t v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad \text{a.e. } t \in J.$$

Therefore $y \in S_F(a)$ which yields that $S_F(a)$ is closed, hence compact subset in PC_b. Finally, we prove that $S_F(\,\cdot\,)$ is *u.s.c.* by proving that the graph of S_F

$$\Gamma_{S_F} := \{(a, y) : y \in S_F(a)\}$$

is closed. Let $(a_n, y_n) \in \Gamma_{S_F}$ be such that $(a_n, y_n) \to (a, y)$ as $n \to \infty$. Since $y_n \in S_F(a_n)$, there exists $v_n \in L^1(J, \mathbb{R}^n)$ such that

$$y_n(t) = a_n + \int_0^t v_n(s)\,ds + \sum_{0<t_k<t} I_k(y_n(t_k)), \quad t \in J.$$

Arguing as in Claim 4, we can prove that there exists $v \in S_{F,y}$ such that

$$y(t) = y(t) = a + \int_0^t v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad \text{a.e. } t \in J.$$

Thus, $y \in S_F(a)$. Now, we show that S_F maps bounded sets into relatively compact sets of PC. Let B be a bounded set in $\mathbb{R}^n$ and let $(y_n) \subset S_F(B)$. Then there exists $\{a_n\} \subset B$ such that

$$y_n(t) = a_n + \int_0^t v_n(s)\,ds + \sum_{0<t_k<t} I_k(y_n(t_k)), \quad t \in J,$$

where $v_n \in S_{F,y_n}$, $n \in \mathbb{N}$. Since $\{a_n\}$ is a bounded sequence, there exists a subsequence of $\{a_n\}$ converging to a. As in Step 1, Claims 2–3, we can show that $\{y_n : n \in \mathbb{N}\}$ is equicontinuous on every compact subset of J and equiconvergent at ∞. As a consequence of Lemma 6.3, we conclude that there exists a subsequence of $\{y_n\}$ converging to y in PC. By a similar argument of Claim 4, we can prove that

$$y(t) = a + \int_0^t v(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad \text{a.e. } t \in J,$$

where $v \in S_{F,y}$. Thus, $y \in \overline{S(B)}$. This implies that $S_F(\,\cdot\,)$ is *u.s.c.*, ending the proof of Theorem 6.2. □

6.1.2. The nonconvex Lipschitz case. In this subsubsection, we prove existence of solutions for problem (6.1) under Hausdorff–Lipschitz conditions. Our main tool will be the nonlinear alternative of Frigon for multi-valued contractions (Lemma 2.79).

Theorem 6.4. *Suppose the multi-map $F\colon J \times \mathbb{R}^n \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R}^n)$ is such that $t \to F(t,\,\cdot\,)$ is a measurable and*

($\mathcal{HF}_3$) *for each $k = 1, 2, \ldots$, there exist $l_k \in L^1([0, t_k], \mathbb{R}^+)$ such that*

$$H_d(F(t,x), F(t,y)) \le l_k(t)\|x - y\|, \quad \textit{for } x, y \in \mathbb{R}^n \textit{ and a.e. } t \in \widetilde{J_k}$$

and

$$F(t,0) \subset l_k(t)\overline{B}(0,1), \quad \textit{for a.e. } t \in \widetilde{J_k}.$$

($\mathcal{HF}_4$) $\sum_{k=1}^{\infty} \|I_k(0)\| < \infty$ *and there exist constants $\overline{c_k} \ge 0$ such that*

$$\sum_{k=1}^{\infty} \overline{c_k} < 1 \quad \textit{and} \quad \|I_k(x) - I_k(y)\| \le \overline{c_k}\|x - y\|,$$

for each $x, y \in \mathbb{R}^n$.

Then problem (6.1) *has at least one solution.*

Here and hereafter $\widetilde{J_k} = [0, t_k] \setminus \{t_j, 0 < j < k\}$.

Proof. We begin by defining a family of semi-norms on PC, thus rendering PC a Fréchet space. Let τ be a sufficiently large real parameter, say

$$\frac{1}{\tau} + \sum_{k=1}^{\infty} \overline{c_k} < 1.$$

For each $n \in \mathbb{N}$, define in PC the semi-norms

$$\|y\|_n = \sup\{e^{-\tau L_n(t)}\|y(t)\| : 0 \le t \le t_n\}$$

where

$$L_n(t) = \int_0^t l_n(s)\, ds.$$

Thus $\mathrm{PC} = \bigcap_{n\ge 1} \mathrm{PC}_n$ where $\mathrm{PC}_n = \{y : \widetilde{J_n} \to \mathbb{R}^n$ such that y is continuous everywhere except for some t_k at which $y(t_k^-)$ and $y(t_k^+)$ exist and $y(t_k^-) = y(t_k)$ $(k = 1, 2, \dots, n-1)$. Then PC is a Fréchet space with the family of semi-norms $\{\|\cdot\|_n\}$. In order to transform problem (6.1) into a fixed point problem, define the operator $N\colon \mathrm{PC} \to \mathcal{P}(\mathrm{PC})$ by

$$N(y) = \left\{ h \in \mathrm{PC} : h(t) = a + \int_0^t v(s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)),\ \text{a.e. } t \in J \right\}$$

where $v \in S_{F,y} = \{v \in L^1(J, \mathbb{R}^n) : v(t) \in F(t, y(t)),\ t \in J\}$. Clearly, the fixed points of the operator N are solutions of problem (6.1). We first show that $N\colon \overline{U} \to \mathcal{P}_{\mathrm{cl}}(\mathrm{PC})$ is an admissible multi-valued contraction, where $U \subset \mathrm{PC}$ is some open subset.

Step 1. We claim that there exists $\gamma < 1$ such that

$$H_d(N(y), N(\overline{y})) \le \gamma \|y - \overline{y}\|_n, \quad \text{for each } y, \overline{y} \in \mathrm{PC}_n.$$

Let $y, \overline{y} \in \mathrm{PC}_n$ and $h \in N(y)$. Then there exists $v \in S_{F,y}$ such that

$$h(t) = a + \int_0^t v(s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad \text{a.e. } t \in \widetilde{J_n}.$$

$(\mathcal{HF}_3)$ implies that

$$H_d(F(t, y(t)), F(t, \overline{y}(t)) \le l_n(t)\|y(t) - \overline{y}(t)\|, \quad \text{a.e. } t \in \widetilde{J_n}.$$

Hence, there is some $w \in F(t, \overline{y}(t)$ such that

$$\|v(t) - w\| \le l_n(t)\|y(t) - \overline{y}(t)\|, \quad t \in \widetilde{J_n}.$$

Consider the multi-map $U_n\colon \widetilde{J_n} \to \mathcal{P}(\mathbb{R}^n)$ defined by

$$U_n(t) = \{w \in F(t, y(t)) : |v(t) - w| \le l_n(t)\|y(t) - \overline{y}(t)\|, \text{ a.e. } t \in \widetilde{J_n}\}.$$

Then $U_n(t)$ is a nonempty set and Theorem III.4.1 in [33] tells us that U_n is measurable. Moreover, the multi-valued intersection operator $V_n(\,\cdot\,) = U_n(\,\cdot\,) \cap F(\,\cdot\,, \overline{y}(\,\cdot\,))$ is measurable. Therefore, by Lemma 2.15, there exists a function $t \mapsto \overline{v}_n(t)$, which is a measurable selection for V_n, that is $\overline{v}_n(t) \in F(t, \overline{y}(t))$ and

$$\|v(t) - \overline{v}_n(t)\| \le l_n(t)\|y(t) - \overline{y}(t)\|, \quad \text{a.e. } t \in \widetilde{J_n}.$$

Define the function $\overline{h}$ by

$$\overline{h}(t) = a + \int_0^t \overline{v}_n(s)\,ds + \sum_{0<t_k<t} I_k(\overline{y}(t_k^-)).$$

Using $(\mathcal{HF}_4)$, we have, for $t \in \widetilde{J_n}$,

$$\begin{aligned}
\|h(t) - \overline{h}(t)\| &\le \int_0^t \|v(s) - \overline{v}_n(s)\|\,ds + \sum_{0<t_k<t} \|I_k(y(t_k^-)) - I_k(\overline{y}(t_k^-))\| \\
&\le \int_0^t l_n(s)\|y(s) - \overline{y}(s)\|\,ds + \sum_{0<t_k<t} \overline{c_k}\|y(t_k) - \overline{y}(t_k)\| \\
&\le \int_0^t l_n(s)e^{\tau L_n(s)}e^{-\tau L_n(s)}\|y(s) - \overline{y}(s)\|\,ds \\
&\quad + \sum_{0<t_k<t} \overline{c_k}e^{\tau L_n(t)}e^{-\tau L_n(t)}\|y(t_k) - \overline{y}(t_k)\| \\
&\le \int_0^t l_n(s)e^{\tau L_n(s)}\,ds\|y - \overline{y}\|_n + \sum_{0<t_k<t} \overline{c_k}e^{\tau L(t)}\|y - \overline{y}\|_n \\
&\le \int_0^t \frac{1}{\tau}(e^{\tau L_n(s)})'\,ds\|y - \overline{y}\|_n + \sum_{k=1}^n \overline{c_k}e^{\tau L_n(t)}\|y - \overline{y}\|_n \\
&\le e^{\tau L_n(t)}\left(\frac{1}{\tau} + \sum_{k=1}^n \overline{c_k}\right)\|y - \overline{y}\|_n.
\end{aligned}$$

Thus

$$e^{-\tau L_n(t)}\|h(t) - \overline{h}(t)\| \le \left(\frac{1}{\tau} + \sum_{k=1}^n \overline{c_k}\right)\|y - \overline{y}\|_n.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we finally arrive at the estimate

$$H_{d_n}(N(y), N(\overline{y})) \le \left(\frac{1}{\tau} + \sum_{k=1}^n \overline{c_k}\right)\|y - \overline{y}\|_n.$$

In addition, since F is compact valued, we can prove that N has compact values too. Let $x \in \overline{U}$ and $\varepsilon > 0$. If $x \notin N(x)$, then $d_n(x, N(x)) \neq 0$. Since $N(x)$ is compact, there exists $y \in N(x)$ such that $d_n(x, N(x)) = \|x - y\|_n$ and we have

$$\|x - y\|_n \leq d_n(x, N(x)) + \varepsilon.$$

If $x \in N(x)$, then we may take $y = x$. Therefore N is an admissible operator contraction.

Step 2. A priori estimates. Given $t \in \widetilde{J_n}$, let $y \in \lambda N(y)$ for some $\lambda \in (0, 1]$. Then there exists $v \in S_{F,y}$ such that

$$\begin{aligned} |y(t)| &\leq \|a\| + \int_0^t |v(s)|\, ds + \sum_{0<t_k<t} \|I_k(y(t_k))\| \\ &\leq \|a\| + \int_0^t l_n(s)(1 + \|y(s)\|)\, ds + \sum_{k=1}^n \overline{c_k}\|y(t_k^-)\| + \sum_{k=1}^n \|I_k(0)\|. \end{aligned}$$

Consider the function μ defined on $\widetilde{J_n}$ by

$$\mu(t) = \sup\{\|y(s)\| : 0 \leq s \leq t\}.$$

By the previous inequality, we have for $t \in \widetilde{J_n}$

$$\mu(t) \leq \frac{1}{1 - \sum_{k=1}^n \overline{c_k}} \left(\|a\| + \sum_{k=1}^n \|I_k(0)\| + \int_0^t l_n(s)(1 + \mu(s))\, ds \right).$$

Let us denote the right-hand side of the above inequality as $\beta(t)$. Then we have

$$\beta(0) = \frac{\|a\| + \sum_{k=1}^\infty \|I_k(0)\|}{1 - \sum_{k=1}^n \overline{c_k}} = c, \qquad \mu(t) \leq \beta(t), \quad t \in \widetilde{J_n}$$

and

$$\beta'(t) = \frac{l_n(t)(1 + \mu(t))}{1 - \sum_{k=1}^\infty \overline{c_k}} \leq \frac{l_n(t)(1 + \beta(t))}{1 - \sum_{k=1}^\infty \overline{c_k}}, \quad t \in \widetilde{J_n}.$$

Integrating over $t \in \widetilde{J_n}$ yields

$$\int_{\beta(0)}^{\beta(t)} \frac{ds}{1 + s} \leq \frac{1}{1 - \sum_{k=1}^\infty c_k} \int_0^{t_n} l_n(s)\, ds =: M_n.$$

Hence $\beta(t) \le K_n := (1+\beta(0))e^{M_n}$ and as a consequence

$$\|y(t)\| \le \mu(t) \le \beta(t) \le K_n, \quad t \in \widetilde{J_n}.$$

Therefore $\|y\|_n \le K_n$, for all $n \in \mathbb{N}$. Let

$$U = \{y \in \mathrm{PC} : \|y\|_n < K_n + 1, \text{ for all } n \in \mathbb{N}\}.$$

Clearly, U is an open subset of PC and there is no $y \in \partial U$ such that $y \in \lambda N(y)$ and $\lambda \in (0,1)$. By Lemma 2.79 and Steps 1, 2, N has at least one fixed point y, solution of problem (6.1). □

6.1.3. The nonconvex l.s.c. case. Our third existence result for problem (6.1) deals with the case where the nonlinearity is lower semi-continuous with respect to the second argument and does not necessarily take convex values. In the proof, we will make use of the nonlinear alternative of Leray–Schauder type (Lemma 2.72) combined with a selection theorem for lower semi-continuous multi-maps with decomposable values.

Theorem 6.5. *Suppose that*

($\widetilde{\mathcal{HF}_5}$) *there exist $a_k, b_k > 0$ such that*

$$\|I_k(x)\| \le a_k\|x\| + b_k, \quad \textit{for every } x \in \mathbb{R}^n,\ k = 1, 2, \ldots$$

with

$$\sum_{k=1}^{\infty} a_k < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} b_k < \infty.$$

($\widetilde{\mathcal{HF}_6}$) *$F\colon [0,\infty) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a nonempty compact valued multi-map satisfying ($\mathcal{H}_{\mathrm{lsc}}$) and there exist $p \in L^1_{\mathrm{loc}}([0,\infty), \mathbb{R}^n)$ and a continuous nondecreasing function $\psi\colon [0,\infty) \to [0,\infty)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} \le p(t)\psi(\|x\|), \quad \textit{for a.e. } t \in J \textit{ and each } x \in \mathbb{R}^n$$

with

$$\int_{\|a\|}^{\infty} \frac{du}{\psi(u)} = \infty.$$

Then problem (6.1) *has at least one solution.*

The following result is known as Gronwall–Bihari Theorem.

Lemma 6.6 ([17])**.** *Let $u, g\colon I \to \mathbb{R}$ be positive real continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $h\colon \mathbb{R} \to (0,+\infty)$ such that*

$$u(t) \le c + \int_a^t g(s)h(u(s))\,ds, \quad \textit{for all } t \in I.$$

Then

$$u(t) \le H^{-1}\Big(\int_a^t g(s)\,ds\Big), \quad \textit{for all } t \in I$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_a^b g(s)\,ds.$$

Here H^{-1} *refers to the inverse of the function* $H(u) = \int_c^u dy/h(y)$ *for* $u \ge c$.

Proof of Theorem 6.5. Let $F\colon \widetilde{J_m} \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, where $\widetilde{J_m} = [0, t_m]$. $(\widetilde{\mathcal{HF}_5})$ implies by Lemma 2.23, that F is of *l.s.c.* type. From Lemma 24, there is a continuous selection $f_m\colon \mathrm{PC}(\widetilde{J_m}, \mathbb{R}^n) \to L^1(\widetilde{J_m}, \mathbb{R}^n)$ such that $f_m(y) \in \mathcal{F}_m(y)$ for every $y \in \mathrm{PC}(\widetilde{J_m}, \mathbb{R}^n)$ where $\mathcal{F}_m$ is the Nemyts'kiĭ operator associated with F on $\widetilde{J_m}$:

$$\mathcal{F}_m(y) := \{v \in L^1(J_m, \mathbb{R}^n) : v(t) \in F(t, y(t)),\ \text{a.e. } t \in \widetilde{J_m}\}.$$

Let $f\colon \mathrm{PC} \to L^1_{\mathrm{loc}}([0,\infty), \mathbb{R}^n)$ be defined by

$$f(y)(t) = f_m(y)(t), \quad \text{a.e. } t \in \widetilde{J_m}.$$

Then $\mathrm{PC} = \bigcap_{m \ge 1} \mathrm{PC}_m$ is a Fréchet space with family of semi-norms $\{\|\cdot\|_m\}$ where

$$\|y\|_m = \sup\{|y(t)| : t \in \widetilde{J_m}\}.$$

Consider the problem

$$\begin{cases} y'(t) = f(y)(t), & \text{a.e. } t \in J, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, \dots, \\ y(0) = a \end{cases} \tag{6.6}$$

and the operator $L\colon \mathrm{PC} \to \mathrm{PC}$ defined by

$$L(y)(t) = a + \int_0^t f(y)(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad \text{a.e. } t \in J.$$

Clearly, the fixed points of the operator L are mild solutions of problem (6.1).

Step 1. A priori estimates.] Let y be a possible solution of problem (6.1). For $t \in [0, t_1]$, we have

$$y(t) = a + \int_0^t f(y)(s)\,ds.$$

Then

$$\|y(t)\| \le \|a\| + \int_0^t p(s)\psi(\|y(s)\|)\,ds.$$

By Lemma 6.6, we have

$$\|y(t)\| \le \Gamma_1^{-1}\bigg(\int_0^t p(s)\,ds\bigg), \quad t \in [0,t_1], \text{ where } \Gamma_1(z) = \int_{\|a\|}^z \frac{du}{\psi(u)}.$$

For $t \in (t_1, t_2]$, we have

$$y(t) = a + \int_0^t f(y)(s)\,ds + I_1(y(t_1)).$$

Then

$$\begin{aligned}\|y(t)\| &\le \|a\| + \|I_1(y(t_1))\| + \int_0^t p(s)\psi(\|y(s)\|)\,ds \\ &\le \|a\| + K_1 + \int_0^t p(s)\psi(\|y(s)v)\,ds,\end{aligned}$$

where

$$K_1 = \sup\{\|I_1(z)\| : z \in \overline{B}(0, M_0)\} \quad \text{and} \quad M_0 = \Gamma_1^{-1}\bigg(\int_0^{t_1} p(s)\,ds\bigg).$$

By Lemma 6.6, we get

$$\|y(t)\| \le \Gamma_2^{-1}\bigg(\int_0^t p(s)\,ds\bigg), \quad t \in (t_1, t_2], \text{ where } \Gamma_2(z) = \int_{\|a\|+K_1}^z \frac{du}{\psi(u)}.$$

We continue this process until we get, for every $t \in (t_{m-1}, t_m]$, the estimate

$$\|y(t)\| \le \Gamma_m^{-1}\bigg(\int_0^t p(s)\,ds\bigg), \quad t \in (t_{m-1}, t_m],$$

where

$$\begin{aligned}\Gamma_m(z) &= \int_{\|a\|+K_{m-1}}^z \frac{du}{\psi(u)}, \\ K_{m-1} &= \sup\{\|I_{m-1}(z)\| : z \in \overline{B}(0, M_{m-2})\}, \\ M_{m-2} &= \Gamma_{m-1}^{-1}\bigg(\int_0^{t_{m-1}} p(s)\,ds\bigg).\end{aligned}$$

Let

$$C = \bigg\{y \in \mathrm{PC} : \|y(t)\| \le \Gamma_m^{-1}\bigg(\int_0^t p(s)\,ds\bigg),\ t \in (t_{m-1}, t_m],\ m = 1, 2, \dots\bigg\}.$$

It is clear that C is a convex closed and bounded set in PC.

Step 2. $L(C) \subset C$. Let $y \in C$, then we have for $t \in [0, t_1]$

$$\begin{aligned}
\|L(y)(t)\| &\leq \|a\| + \int_0^t \|f(y)(s)\|\, ds \leq \|a\| + \int_0^t p(s)\psi(\|y(s)\|)\, ds \\
&\leq \|a\| + \int_0^t p(s)\psi\bigg(\Gamma_1^{-1}\bigg(\int_0^s p(r)dr\bigg)\bigg)\, ds \\
&= \|a\| + \int_0^t p(s)(\Gamma_1^{-1})'\bigg(\int_0^s p(r)dr\bigg)\, ds \\
&= \|a\| + \int_0^t \bigg(\Gamma_1^{-1}\bigg(\int_0^s p(r)dr\bigg)\bigg)'\, ds.
\end{aligned}$$

We have used the fact that $\Gamma_1^{-1}(0) = \|a\|$ and

$$\psi(z) = \frac{1}{(\Gamma_1)'(z)} = (\Gamma_1^{-1})'(\Gamma_1)(z)).$$

Lemma 6.6 yields

$$\|L(y)(t)\| \leq \Gamma_1^{-1}\bigg(\int_0^t p(s)\, ds\bigg), \quad \text{a.e. } t \in [0, t_1]. \tag{6.7}$$

Also for $t \in (t_1, t_2]$, we have

$$\begin{aligned}
\|L(y)(t)\| &\leq \|a\| + \|I_1(y(t_1))\| + \int_0^t p(s)\psi(\|y(s)\|)\, ds \\
&\leq \|a\| + K_1 + \int_0^t p(s)\psi(\|y(s)\|)\, ds.
\end{aligned}$$

Arguing as above, we obtain

$$\|L(y)(t)\| \leq \Gamma_2^{-1}\bigg(\int_0^t p(s)\, ds\bigg), \quad \text{a.e. } t \in (t_1, t_2]. \tag{6.8}$$

We continue this process until we arrive at the estimate

$$\|L(y)(t)\| \leq \Gamma_m^{-1}\bigg(\int_0^t p(s)\, ds\bigg), \quad \text{a.e. } t \in (t_{m-1}, t_m], \tag{6.9}$$

proving that $L(C) \subset C$; this implies that $L(C)$ is a bounded set in the Fréchet space PC. As in Claims 2-3, Step 1 of the proof of Theorem 6.2, we can prove that for every $m \in \mathbb{N}$, the operator $L\colon \mathrm{PC}_m \to \mathrm{PC}_m$ is completely continuous, then $L\colon \mathrm{PC} \to \mathrm{PC}$ is continuous and $L(C)$ is relatively compact. By Lemma 2.80, we conclude that L has at least one fixed point, solution of problem (6.6), hence a solution of problem (6.1). $\square$

6.2. Topological structure of solution sets: the projective limit approach

In order to define an inverse system for problem (6.1), consider, for every $m \in \mathbb{N}$, the sequence of problems

$$\begin{cases} y'(t) \in F(t, y(t)), & \text{a.e. } t \in \widetilde{J_m}, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, \dots, m-1, \\ y(0) = a, \end{cases} \tag{6.10}$$

where $\widetilde{J_m} = [0, t_m] \setminus \{t_j, 0 < j < m\}$. Under the assumptions of Theorem 6.2 (see Step 1, Claim 5 of the proof) and Theorem 6.4 (see Step 2 of the proof), we can prove the existence of a constant $M_m > 0$ such that for all y solution of problem (6.10), we have $\|y\|_\infty \le M_m$. Consider the problem

$$\begin{cases} y'(t) \in G_m(t, y(t)), & \text{a.e. } t \in \widetilde{J_m}, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, \dots, m-1, \\ y(0) = a, \end{cases} \tag{6.11}$$

where $G_m \colon [0, t_m] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is the multi-map defined by

$$G_m(t, y) = \begin{cases} F(t, y), & \|y\| \le M_m, \\ F\left(t, \dfrac{yM_m}{\|y\|}\right), & \|y\| \ge M_m. \end{cases} \tag{6.12}$$

6.2.1. The nonconvex case. We need the following result owed to Bressan–Cellina–Fryszkowski.

Lemma 6.7 (See [25, Theorem 2])**.** *Let E be a Banach space, $X = L^1(\Omega, E)$, for some measure space Ω, and $N \colon E \to \mathcal{P}(X)$ a contraction map with decomposable values. Then* $\operatorname{Fix}(N)$ *is an absolute retract.*

Regarding the geometric structure of the solution set of problem (6.1), we state and prove our main result.

Theorem 6.8. *Under assumptions of Theorem* 6.4*, the solution set for problem* (6.1) *can be obtained as a limit of inverse system of* AR*-spaces, for every $a \in \mathbb{R}^n$.*

Proof. Step 1. A fixed point formulation. Since F is H_d-Lipschitz, the multi-map G_m defined by (6.12) is H_d-Lipschitz too. Now F has at most linear growth, then there exists $l \in L^1_{\mathrm{loc}}(J, \mathbb{R}^+)$ such that for a.e. $t \in \widetilde{J_m}$ and $y \in \mathbb{R}^n$, we have the estimates:

$$\|G_m(t, y)\|_{\mathcal{P}} \le l(t)(1 + M_m), \tag{6.13}$$

$$\|G_m(t, y)\|_{\mathcal{P}} \le l(t)(1 + \|y\|). \tag{6.14}$$

In addition, if $y \in S_F$ then $\|y\| \leq M_m$ implies that $F(t, y(t)) = G_m(t, y(t))$ hence $y \in S_G$. Conversely, let $y \in S_G$; (6.14) shows that G as also a linear growth and thus Gronwal's Lemma can be also applied and yields that $\|y\| \leq M_m$; hence again $F(t, y(t)) = G(t, y(t))$ and $y \in S_F$. This proves that F and G yield the same solution set. For $m \in \mathbb{N}^*$, define

$$\mathrm{PAC}_m := \{y : \widetilde{J_m} \to \mathbb{R}^n : y \in \mathrm{AC}((t_k, t_{k+1}), \mathbb{R}^n),\ k = 0, \dots m-1\}$$

and

$$\Omega_m = \{y \in \mathrm{PAC}_m : y(0) = a\}. \tag{6.15}$$

Then $(\Omega_m, \|\cdot\|_{\Omega_m})$ is a Banach space with norm

$$\|y\|_{\Omega_m} = \|y\|_\infty + \|y'\|_{L^1}.$$

Note that $\mathrm{PAC} = \bigcap_{m\in\mathbb{N}} \mathrm{PAC}_m$ is the appropriate space for solutions. Consider the multi-valued Nemyts'kiĭ operator associated with G

$$\overline{K}_m : \Omega_m \to \mathcal{P}(L^1(\widetilde{J_m}, \mathbb{R}^n))$$

defined by

$$\overline{K}_m(y) = \{v \in L^1(\widetilde{J_m}, \mathbb{R}^n) : v(t) \in G_m(t, y(t)),\ \text{a.e. } t \in \widetilde{J_m}\}$$

and

$$\overline{L}_m \colon L^1(\widetilde{J_m}, \mathbb{R}^n) \to \Omega_m$$

defined by

$$\overline{L}_m(v)(t) = \begin{cases} L_0(v)(t), & \text{if } t \in [0, t_1], \\ L_1(v)(t), & \text{if } t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ L_{m-1}(v)(t), & \text{if } t \in (t_{m-1}, t_m], \end{cases} \tag{6.16}$$

where

$$\begin{aligned}
L_0(v)(t) &= a + \int_0^t v(s)\,ds, && t \in [0, t_1], \\
L_1(v)(t) &= L_0(v)(t_1) + I_1(L_0(v)(t_1)) + \int_{t_1}^t v(s)\,ds, && t \in (t_1, t_2], \\
L_2(v)(t) &= L_1(v)(t_2) + I_2(L_1(v)(t_2)) + \int_{t_2}^t v(s)\,ds, && t \in (t_2, t_3], \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
L_{m-1}(v)(t) &= L_{m-2}(v)(t_{m-1}) \\
&\quad + I_{m-1}(L_{m-2}(v)(t_{m-1})) + \int_{t_{m-1}}^t v(s)\,ds, && t \in (t_{m-1}, t_m].
\end{aligned}$$

From (6.16), we can easily check that

$$\overline{L}_m(v)(t) = a + \sum_{0<t_k<t} I_k(L_{k-1}(v)(t_k)) + \int_0^t v(s)\,ds, \quad \text{a.e. } t \in \widetilde{J_m}.$$

If $S_{G_m}(a)$ is the solution set of problem (6.11), then $S_{G_m}(a) = \operatorname{Fix}(\overline{L}_m \circ \overline{K}_m)$, where

$$\overline{L}_m \circ \overline{K}_m : \Omega_m \to \mathcal{P}(\Omega_m).$$

Step 2. $\overline{L}_m$ is a homeomorphism.

• $\overline{L}_m$ is injective. Let $v_1, v_2 \in L^1(\widetilde{J_m}, \mathbb{R}^n)$ be such that $\overline{L}_m(v_1) = \overline{L}_m(v_2)$. Then

$$\begin{aligned}
&\int_0^t v_1(s)\,ds = \int_0^t v_2(s)\,ds, && t \in [0, t_1],\\
&L_0(v_1)(t_1) + I_1(L_0(v_1)(t_1)) + \int_{t_1}^t v_1(s)\,ds\\
&\qquad = L_0(v_2)(t_1) + I_1(L_0(v_2)(t_1)) + \int_{t_1}^t v_2(s)\,ds, && t \in (t_1, t_2],\\
&L_1(v_1)(t_2) + I_2(L_1(v_1)(t_2)) + \int_{t_2}^t v_1(s)\,ds\\
&\qquad = L_1(v_2)(t_2) + I_2(L_1(v_2)(t_2)) + \int_{t_2}^t v_2(s)\,ds, && t \in (t_2, t_3],\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&L_{m-1}(v_1)(t_{m_1}) + I_{m-1}(L_{m-1}(v_1)(t_{m-1})) + \int_{t_{m-1}}^t v_1(s)\,ds\\
&\qquad = L_{m-1}(v_2)(t_{m_1})\\
&\qquad\quad + I_{m-1}(L_{m-1}(v_2)(t_{m-1})) + \int_{t_{m-1}}^t v_2(s)\,ds, && t \in (t_{m-1}, t_m].
\end{aligned}$$

This implies that $v_1 = v_2$.

• $\overline{L}_m$ is surjective. Let $y \in \Omega_m$, then $y(t) = a + \int_0^t y'(s)\,ds$, $t \in [0, t_1]$ and $y' \in L^1([0, t_1], \mathbb{R}^n)$, i.e.

$$y(t) = L_0(y')(t), \quad t \in [0, t_1].$$

Since $y \in \mathrm{AC}((t_1, t_2), \mathbb{R}^n)$, then

$$y(t) = L_0(y')(t_1) + I_1(L_0(y'(t_1))) + \int_{t_1}^t y'(s)\,ds, \quad t \in (t_1, t_2].$$

We continue the process until we arrive at the formula

$$y(t) = L_{m-2}(y')(t_{m-1}) + I_{m-1}(L_{m-2}(y'(t_{m-1})) + \int_{t_{m-1}}^t y'(s)\,ds, \quad t \in (t_{m-1}, t_m].$$

Hence there exists $y' \in L^1(\widetilde{J_m}, \mathbb{R}^n)$ such that $\overline{L}_m(y') = y$.

• $\overline{L}_m$ is continuous. Let $v_n \in L^1(\widetilde{J_m}, \mathbb{R}^n)$ be such that v_n converges to v in $L^1(\widetilde{J_m}, \mathbb{R}^n)$, as $n \to +\infty$. We show that $\overline{L}_m(v_n)$ converges to $\overline{L}_m(v)$. We have

$$\|\overline{L}_m(v_n)(t) - \overline{L}_m(v)(t)\| \leq \sum_{k=1}^{m-1} \|I_k(L_{k-1}(v_n)(t_k)) - I_k(L_{k-1}(v)(t_k))\| + (2m-1)\int_0^{t_m} \|v_n(s) - v(s)\|\, ds.$$

Then

$$\|\overline{L}_m(v_n) - \overline{L}_m(v)\|_{\Omega_m} \leq \sum_{k=1}^{m-1} \|I_k(L_{k-1}(v_n)(t_k)) - I_k(L_{k-1}(v)(t_k))\| + (2m-1)\|v_n - v\|_{L^1}.$$

Since

$$\|L_0(v_n) - L_0(v)\|_\infty \leq \|v_n - v\|_{L^1} \to 0, \quad \text{as } n \to \infty$$

and using the fact that I_1 is a continuous function, we get

$$\|I_1(L_0(v_n)(t_1)) - I_1(L_0(v)(t_1)\| \to 0, \quad \text{as } n \to \infty.$$

More generally, for each $k \in \{2, \dots, m-1\}$, we can prove that

$$\|I_k(L_{k-1}(v_n)(t_k)) - I_k(L_{k-1}(v)(t_k))\| \to 0, \quad \text{as } n \to \infty.$$

Then

$$\|\overline{L}_m(v_n) - \overline{L}_m(v)\|_{\Omega_m} \to 0, \quad \text{as } n \to \infty.$$

To prove that ${\overline{L}_m}^{-1}$ is continuous, let $y_n \in \Omega_m$ be a sequence converging to y, as $n \to +\infty$. Since ${\overline{L}_m}^{-1}(y) = y'$, we have

$$\|\overline{L}_m^{-1}(y_n) - \overline{L}_m^{-1}(y)\|_{L^1} \leq \|y_n' - y'\|_{L^1} \to 0, \quad \text{as } n \to \infty.$$

Hence $\overline{L}_m$ is a homeomorphism.

Step 3. The set $\mathcal{G}_m = \{y' : y \in S_{G_m}(a)\}$ is an absolute retract. Let

$$\widetilde{N}_m = \overline{K}_m \circ \overline{L}_m \colon L^1(\widetilde{J_m}, \mathbb{R}^n) \to \mathcal{P}(L^1(\widetilde{J_m}, \mathbb{R}^n))$$

be defined by

$$(\widetilde{N}_m)(v) = \{u \in L^1(\widetilde{J_m}, \mathbb{R}^n) : u(t) \in F(t, \overline{L}_m(v)(t)), \text{ a.e. } t \in \widetilde{J_m}\}.$$

then $\mathcal{G}_m = \operatorname{Fix}(\widetilde{N}_m)$. Moreover, since G is H_d-Lipschitz, then $\overline{K}_m$ is H_d-Lipschitz. To prove that $\widetilde{N}_m$ is a contraction, let $v_1, v_2 \in L^1(\widetilde{J_m}, \mathbb{R}^n)$. Then there exist $h_i \in (\widetilde{N}_m)(v_i)$, $i = 1, 2$ such that

$$h_i(t) \in G(t, \overline{L}_m(v_i)(t)), \quad \text{a.e. } t \in \widetilde{J_m}.$$

For almost every $t \in \widetilde{J_m}$ we have

$$H_d(G(t, \overline{L}_m(v_1)(t)), G(t, \overline{L}_m(v_2)(t)) \leq l_m(t)\|\overline{L}_m(v_1)(t) - \overline{L}_m(v_2)(t)\|.$$

Hence, there exists some $w \in G(t, \overline{L}_m(v_2)(t))$ such that

$$\|h_1(t) - w\| \leq l_m(t)\|\overline{L}_m(v_1)(t) - \overline{L}_m(v_2)(t)\|, \quad \text{a.e. } t \in \widetilde{J_m}.$$

Consider the multi-map $U\colon \widetilde{J_m} \to \mathcal{P}(\mathbb{R}^n)$ defined by

$$U(t) = \{w \in \mathbb{R}^n : \|h_1(t) - w\| \leq l_m(t)\|\overline{L}_m(v_1)(t) - \overline{L}_m(v_2)(t)\|\}.$$

Since the operator V defined by $V(t) = U(t) \cap G(t, \overline{L}_m(v_2)(t))$, is a measurable multi-map, there exists a function $t \to h_2(t)$ which is a measurable selection for V (see the proof of Theorem 6.4). Hence $h_2(t) \in G(t, \overline{L}_m(v_2)(t))$ and

$$\|h_1(t) - h_2(t)\| \leq l_m(t)\|\overline{L}_m(v_1)(t) - \overline{L}_m(v_2)(t)\|, \quad \text{a.e. } t \in \widetilde{J_m}. \tag{6.17}$$

Form the definition of $\overline{L}_m$, we can prove that there exists $D_m > 0$ such that

$$\|\overline{L}_m(v_1)(t) - \overline{L}_m(v_2)(t)\| \leq D_m \int_0^t \|v_1(s) - v_2(s)\|\, ds. \tag{6.18}$$

Let $\tau_m > \max(D_m, 1)$, $L_m(t) = \int_0^t l_m(s)\, ds$, and let L^1_m be the weighted space of Lebesgue measurable functions such that

$$\int_0^{t_m} \|v(t)\| e^{-\tau_m L_m(t)}\, dt < \infty.$$

Endowed with norm

$$\|v\|_m = \int_0^{t_m} \|v(t)\| e^{-\tau_m L_m(t)} dt, \quad \text{for } v \in L^1_m,$$

this is a Banach space. From the inequalities (6.17) and (6.18), we obtain the following estimates after an integration by parts:

$$\begin{aligned}
\|h_1 - h_2\|_m &= \int_0^{t_m} e^{-\tau_m L_m(t)} \|h_1(t) - h_2(t)\| dt \\
&\leq \int_0^{t_m} l_m(t) e^{-\tau_m L_m(t)} D_m dt \int_0^t \|v_1(s) - v_2(s)\|\, ds \\
&= \frac{-D_m}{\tau_m} \int_0^{t_m} [e^{-\tau_m L_m(t)}]' dt \int_0^t \|v_1(s) - v_2(s)\|\, ds \\
&= \frac{-D_m}{\tau_m} \Big\{ e^{-\tau_m L_m(t_m)} \int_0^{t_m} \|v_1(t) - v_2(t)\|\, dt \\
&\quad - \int_0^{t_m} e^{-\tau_m L_m(t)} \|v_1(t) - v_2(t)\|\, dt \Big\}.
\end{aligned}$$

Hence

$$\|h_1 - h_2\|_m \leq \frac{D_m}{\tau_m}\|v_1 - v_2\|_m.$$

By an analogous relation obtained by interchanging the roles of h_1 and h_2, we finally obtain that for every $v_1, v_2 \in L^1(\widetilde{J_m}, \mathbb{R}^n)$

$$H_d(\widetilde{N}_m(v_1), \widetilde{N}_m(v_2)) \leq \frac{D_m}{\tau_m}\|v_1 - v_2\|_m.$$

The multi-map $t \to G(t, \cdot)$ is measurable and $x \to G(\cdot, x)$ is H_d-continuous. In addition $G(\cdot, \cdot)$ has compact values; hence $G(\cdot, \cdot)$ is measurable, continuous. Since the measurable multifunction G is integrably bounded, Lemma 2.23 implies that the Nemyts'kiĭ operator $\overline{K}_m$ has decomposable values; then $\widetilde{N}_m$ has decomposable values. Since $\widetilde{N}_m$ is contractive and $\mathrm{Fix}\,(\widetilde{N}_m) = \{y' : y \in S_{G_m}(a)\} = \mathcal{G}_m$, by Lemma 25, we conclude that the set $\mathcal{G}_m$ is an absolute retract.

Step 4. Conclusion. Using the fact that $\overline{L}_m$ is a homeomorphism and homeomorphic spaces have the same AR topological structure (Lemma 2.36), we deduce that the set $\mathrm{Fix}\,(\overline{L}_m \circ \overline{K}_m) = S_{G_m}(a)$ is also an absolute retract, hence acyclic. It remains to show that $\{\overline{L}_m \circ \overline{K}_m\}$ is a map of the inverse system $\{\Omega_m, \pi^p_m, \mathbb{N}\}$ where $\pi^p_m(y) = y|_{[0,t_m]}$, for every $y \in \Omega_p$ and $p \geq m$. We have

$$(\overline{L}_m \circ \overline{K}_m)\pi^p_m(y)(t) = \Big\{a + \sum_{0<t_k<t} I_k((L_{k-1}v)(t_k)) + \int_0^t v(s)\,ds : \\ v \in L^1(\widetilde{J_m}) \text{ and } v(t) \in G(t, y(t)), \text{ a.e. } t \in \widetilde{J_m}\Big\},$$

$$\pi^p_m(\overline{L}_m \circ \overline{K}_m)(y)(t) = \Big\{a + \sum_{0<t_k<t} I_k((L_{k-1}v)(t_k)) + \int_0^t v(s)\,ds : \\ v \in L^1(\widetilde{J_p}) \text{ and } v(t) \in G(t, y(t)), \text{ a.e. } t \in \widetilde{J_p}\Big\}.$$

Since every Fréchet space is a limit of some inverse system of Banach spaces, PAC is a limit of the Banach spaces PAC_m; hence the maps $\{\overline{L}_m \circ \overline{K}_m\}$ induce the limit one $N\colon \Omega \to \Omega$ with $N(y)|_{[0,t_m]} = \{(\overline{L}_m \circ \overline{K}_m)(y|_{[0,t_m]})\}$ where

$$\Omega = \{y \in \mathrm{PAC} : y(0) = a\}. \tag{6.19}$$

Since $\mathrm{Fix}\,(N) = S_F(a)$, we conclude the proof. □

6.2.2. The convex case. In this section, we prove that the solution set for problem (6.1) is an R_δ-set in case where F has convex compact values.

Theorem 6.9. *Under the assumptions of Theorem* 6.2*, the solution set is homeomorphic to the intersection of absolute retracts. If further F is upper-Scorza–Dragoni map, then the solution set is R_δ.*

Proof. Step 1. The solution set is homeomorphic to an intersection of AR spaces. From assumptions and using the fact that F is a Carathéodory function with convex and compact values, we can apply the results of Benassi and Gavioli (see [18, Theorem 5.1]) to get a sequence of multi-maps $F_k\colon J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ such that, for every $k \geq 1$,

(a) F_k has nonempty, closed and connected values with the same growth as F,
(b) $F_k(\,\cdot\,, y)$ is measurable, for every $y \in \mathbb{R}^n$,
(c) $F_k(t, \cdot\,)$ is locally Lipschitzian, for almost every $t \in J$,
(d) $F(t,y) \subset F_{k+1}(t,y) \subset F_k(t,y)$, for almost every $t \in J$ and all $y \in \mathbb{R}^n$,
(e) $\lim_{k\to\infty} H_d(F(t,y), F_k(t,y)) = 0$, for almost every $t \in J$ and all $y \in \mathbb{R}^n$.

For every $k \geq 1$, consider the sequence of impulsive problems

$$\begin{cases} y'(t) \in F_k(t, y(t)), & \text{a.e. } t \in \widetilde{J_m}, \\ \Delta y|_{t=t_i} = I_i(y(t_i^-)), & i = 1, \dots, m-1, \\ y(0) = a \end{cases} \tag{6.20}$$

and denote by $S_m^k(a)$ the solution set of problem (6.20). Since for almost every $t \in \widetilde{J_m}$, $F_k(t, \cdot\,)$ is Lipschitzian on bounded subsets of $\mathbb{R}^n$, using [25, Theorem 2], we can prove that $S_m^k(a)$ is an absolute retract. Consider the multi-valued operators N_m and N_m^k defined by

$$N_m(y) := \left\{ h \in \Omega_m : h(t) = a + \int_0^t v(s)\,ds + \sum_{0<t_i<t} I_k(y(t_i)),\ \text{a.e. } t \in \widetilde{J_m} \right\}$$
$$\text{where } v \in S_{F,y} = \{v \in L^1([0,t_m], \mathbb{R}^n) : v(t) \in F(t, y(t)),\ \text{a.e. } t \in \widetilde{J_m}\},$$

$$N_m^k(y) := \left\{ h \in \Omega_m : h(t) = a + \int_0^t v_k(s)\,ds + \sum_{0<t_i<t} I_k(y(t_i)),\ \text{a.e. } t \in \widetilde{J_m} \right\}$$
$$\text{where } v_k \in S_{F_k,y} = \{v_k \in L^1(\widetilde{J_m}, \mathbb{R}^n) : v_k(t) \in F_k(t, y(t)),\ \text{a.e. } t \in \widetilde{J_m}\}.$$

Note that $\operatorname{Fix}(N_m^k) = S_m^k(a) \in \mathrm{AR}$ and $S_m(a) = \bigcap_{k=1}^\infty S_m^k(a)$, where S_m is the solution set for problem (6.10). Moreover, the map N_m is map of the inverse system $(\Omega_m, \pi_m^p, \mathbb{N})$ where Ω_m is defined in (6.15). It is clear that $\operatorname{Fix}(N) = S_F(a)$. Since PAC is a limit of the Banach spaces PAC_m, then N_m induce the

limit map $N\colon \Omega \to \Omega$ where Ω is defined in (6.19) and N is defined by

$$N(y) := \left\{ h \in \Omega : h(t) = a + \int_0^t v(s)\,ds + \sum_{0<t_i<t} I_k(y(t_i)),\ \text{a.e. } t \in J \right\}$$
$$\text{where } v \in S_{F,y} = \{v \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n) : v(t) \in F(t,y(t)) \text{ a.e. } t \in J\}.$$

From Proposition 2.53, we have $S_F(a) = \varprojlim S_m(a)$. Let

$$Z_m = \left\{ (y_i) \in \prod_{i=1}^{\infty} S_i(a) : y_i = y_m\{[0,t_i],\ \text{for } i \le m\} \right\},$$
$$\overline{Z}_k^m = \left\{ (y_i) \in \prod_{i=1}^{\infty} S_i^k(a) : y_i = y_m\{[0,t_i],\ \text{for } i \le m\} \right\} \cong \prod_{i=m}^{\infty} S_i^k(a) \in \mathrm{AR}.$$

Therefore $S_F(a) = \varprojlim S_m(a) = \bigcap_{m=1}^{\infty} Z_m = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{m} \overline{Z}_k^m = \bigcap_{m=1}^{\infty} \overline{Z}_m^m$, proving our claim.

Step 2. The solution set is R_δ. By Theorem 6.2, the solution set is nonempty. Moreover $S_F(a) = \cap_{m=1}^{\infty} S_m(a)$, where $S_m(a)$ is a solution set of problem (6.10). Since F is upper-Scorza–Dragoni map, then the function G defined by (6.12) is also upper-Scorza-Dragoni map. This implies that G is σ-Ca-selectionable and m-LL-selectionable. Then there exists a sequence of multi-maps G_m such that $G_m(\,\cdot\,,\,\cdot\,)$ is Carathéodory and $G(t,x) = \bigcap_{m=1}^{\infty} G_m(t,x)$. Using the same method as in [43], Theorems 6.11 and 6.18 and making use of the Aronszajn–Browder–Gupta results (see Theorems 2.46 and 2.47), we can prove that the solution set $S_{G_m}(a)$ of problem (6.11) is an R_δ-set. Hence the set $S_m(a)$ is R_δ. Moreover, it can be easily seen that $\{\overline{L}_m \circ \overline{K}_m\}$ is a map of the inverse system $\{\mathrm{PC}_m, \pi_m^p, \mathbb{N}\}$, where $\pi_m^p(y) = y|_{[0,t_m]}$, for every $y \in \Omega_p$ and $p \ge m$ Indeed, since the Fréchet space PC is a limit of PC_m, the mappings $\{\overline{L}_m \circ \overline{K}_m\}_{m=1}^{\infty}$ induce the limit mapping $N\colon \mathrm{PC} \to \mathrm{PC}$ defined by

$$N(y) := \left\{ h \in \mathrm{PC} : h(t) = a + \int_0^t f(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)),\ \text{for a.e. } t \in J \right\}$$
$$\text{where } f \in S_{F,y} = \{f \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n) : f(t) \in F(t,y(t)),\ \text{a.e. } t \in J\}.$$

Note that the fixed point set of the mapping N is the solution set of problem (6.1). Applying Proposition 2.53, the solution set of problem (6.1) is an R_δ-set.□

6.2.3. The terminal problem. Our final existence theorem is concerned with the terminal problem, also called target problem, namely problem (6.1) in which the initial condition is replaced by a limit condition at positive infinity. We also prove compactness and acyclicity of the solution sets extending similar

results obtained in [5] for differential inclusions with impulses. Consider the following problem

$$\begin{cases} y'(t) \in F(t, y(t)), & \text{a.e. } t \in J, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, 2, \dots, \\ \lim\limits_{t\to\infty} y(t) = y_\infty \in \mathbb{R}^n. \end{cases} \tag{6.21}$$

For the existence problem, our arguments are based on the Covitz and Nadler fixed point theorem for multi-valued contraction. We have:

Theorem 6.10. *Assume that $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ and the multi-map*

$$F\colon J \times \mathbb{R}^n \mapsto \mathcal{P}_{\mathrm{cp}}(\mathbb{R}^n)$$

is such that $t \to F(t, \cdot)$ is a measurable,

($\mathcal{HF}_7$) *there exists $l \in L^1(J, \mathbb{R}^+)$ such that $F(t, 0) \subset l(t)\overline{B}(0, 1)$, for almost every $t \in J$, and*

$$H_d(F(t, x), F(t, y)) \le l(t)\|x - y\|, \quad \textit{for every } x, y \in \mathbb{R}^n \textit{ and a.e. } t \in J.$$

If

$$\|l\|_{L^1} + \sum_{k=1}^{\infty} c_k < 1, \tag{6.22}$$

then the boundary value problem (6.21) *has at least one solution. If further $F\colon \mathbb{R}^+ \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a Carathéodory multi-map with compact convex values, then the solution set is a contractible compact set, hence acyclic.*

Proof. Part 1. Existence of solution.

Consider the operator $N\colon \mathrm{PC}_b \to \mathcal{P}(\mathrm{PC}_b)$ defined by $N(y) = \{h \in \mathrm{PC}_b\}$ such that for $t \in J$

$$h(t) = y_\infty - \int_t^\infty v(s)\, ds - \sum_{t_k \ge t} I_k(y(t_k)),$$

where $v \in S_{F,y} = \{v \in L^1(J, \mathbb{R}^n) : v(t) \in F(t, y(t))$, for almsot every $t \in J\}$. Clearly, the fixed points of the operator N are solutions of problem (6.21).

Step 1. $N(y) \in \mathcal{P}_{cl}(\mathrm{PC})$, for each $y \in \mathrm{PC}_b$. Indeed, let $(y_n)_{n\ge 0} \in N(y)$ be such that $y_n \to \widetilde{y}$ in PC_b. Then there exists $v_n \in S_{F,y}$ such that for almost every $t \in J$

$$y_n(t) = y_\infty - \int_t^\infty v_n(s)\, ds - \sum_{t_k \ge t} I_k(y(t_k)). \tag{6.23}$$

Since F has compact values, then there exists a subsequence $(v_{n_m}(\,\cdot\,))$ which converges to $v(\,\cdot\,)$ in $\mathbb{R}^n$. From $(\widetilde{\mathcal{HT}}_3)$, we have

$$\|v_{n_m}(t)\| \le l(t)\|y\|_{\mathrm{PC}} + l(t), \quad \text{a.e. } t \in J.$$

The Lebesgue dominated convergence theorem then implies that $\int_t^\infty v_n(s)\,ds$ converges to $\int_t^\infty v(s)\,ds$. Passing to the limit in (6.23) yiel ds that $\widetilde{y} \in N(y)$, as claimed.

Step 2. N is a contraction. Let $y, \overline{y} \in \mathrm{PC}_b$ and $h \in N(y)$. Then there exists $v(t) \in F(t, \overline{y}(t))$ such that for almost every $t \in J$

$$h(t) = y_\infty - \int_t^\infty v(s))\,ds - \sum_{t_k \ge t} I_k(y(t_k)).$$

$(\mathcal{HF}_7)$ implies that there is $u \in F(t, \overline{y}(t))$ such that

$$|v(t) - u| \le l(t)\|y(t) - \overline{y}(t)\|, \quad \text{a.e. } t \in J.$$

Consider the multi-map defined by $V(t) = U(t) \cap F(t, \overline{y}(t))$ where $U\colon [0,\infty) \to \mathcal{P}_{\mathrm{cl}}(\mathbb{R}^n)$ is given by

$$U(t) = \{u \in \mathbb{R}^n : \|v(t) - u\| \le l(t)\|y(t) - \overline{y}(t)\|\}.$$

Arguing as in the proof of Theorem 6.4, we can find a function $\overline{v}(t)$ which is a measurable selection for V. Thus, $\overline{v}(t) \in F(t, \overline{y}(t))$ and

$$\|v(t) - \overline{v}(t)\| \le l(t)\|y(t) - \overline{y}(t)\|, \quad \text{a.e. } t \in J.$$

Let us define for almost every $t \in J$

$$\overline{h}(t) = y_\infty - \int_t^\infty \overline{v}(s)\,ds - \sum_{t_k \ge t} I_k(y(t_k)).$$

Then $\overline{h} \in N(\overline{y})$ and

$$\begin{aligned}\|h(t) - \overline{h}(t)\| &\le \int_0^\infty l(s)\|\overline{y}(s) - y(s)\|\,ds + \sum_{k=1}^\infty c_k\|\overline{y}(t_k) - y(t_k)\| \\ &\le \int_0^\infty l(s)\|\overline{y} - y\|_{\mathrm{PC}_b}\,ds + \sum_{k=1}^\infty c_k\|\overline{y} - y\|_{\mathrm{PC}_b}.\end{aligned}$$

Hence

$$\|h - \overline{h}\|_{\mathrm{PC}_b} \le \left(\|l\|_{L^1} + \sum_{k=1}^\infty c_k\right)\|y - \overline{y}\|_{\mathrm{PC}_b}.$$

Since $y \in \mathrm{PC}_b$ and $h \in N(y)$ are arbitrary, it follows that

$$\sup_{h \in N(y)} d(h, N(\overline{y})) \leq \left(\|l\|_{L^1} + \sum_{k=1}^{\infty} c_k \right) \|y - \overline{y}\|_{\mathrm{PC}_b}.$$

By an analogous relation obtained by interchanging the roles of y and $\overline{y}$, we obtain that

$$H_d(N(y), N(\overline{y})) \leq \left(\|l\|_{L^1} + \sum_{k=1}^{\infty} c_k \right) \|y - \overline{y}\|_{\mathrm{PC}_b}.$$

By (6.22), N is a contraction on PC_b and thus Lemma 2.74 implies that N has a fixed point in PC_b, solution of problem (6.21). For each $y_\infty \in \mathbb{R}^n$, let $S(y_\infty)$ be the solution set of problem (6.21).

Part 2. Structure of the solution sets.

First, we prove that for each $y_\infty \in \mathbb{R}^n$, $S(y_\infty)$ is acyclic. Let $y \in S(y_\infty)$, then there exists $v \in L^1(J, \mathbb{R}^+)$ such that $v(t) \in F(t, y(t))$ for almost every $t \in J$ and

$$y(t) = y_\infty - \int_t^\infty v(s)\, ds - \sum_{t_k \geq t} I_k(y(t_k)).$$

Consequently

$$\|y(t)| \leq |y_\infty\| + \int_0^\infty l(s)\|y(s)\|\, ds + \|l\|_{L^1} + \sum_{k=1}^{\infty} c_k \|y(t_k)\| + \sum_{k=1}^{\infty} \|I_k(0)\|,$$

for almost every $t \in J$. Then

$$\|y\|_{\mathrm{PC}_b} \leq |y_\infty| + \int_0^\infty l(s)\|y\|_{\mathrm{PC}_b}\, ds + \|l\|_{L^1} + \sum_{k=1}^{\infty} c_k \|y\|_{\mathrm{PC}_b} + \sum_{k=1}^{\infty} |I_k(0)|.$$

Hence

$$\|y\|_{\mathrm{PC}_b} \leq M := \frac{1}{1 - \|l\|_{L^1} - \sum_{k=1}^{\infty} c_k} \left(\|y_\infty\| + \|l\|_{L^1} + \sum_{k=1}^{\infty} \|I_k(0)\| \right). \tag{6.24}$$

Step 1. Definition of a homotopy. Consider the modified problem:

$$\begin{cases} y'(t) \in G(t, y(t)), & \text{a.e. } t \in J, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), & k = 1, 2, \dots, \\ \lim\limits_{t \to \infty} y(t) = y_\infty \in \mathbb{R}^n, \end{cases} \tag{6.25}$$

where $G\colon \mathbb{R}^+ \times \mathbb{R}^n \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ is the multi-map defined by

$$G(t, y)) = \begin{cases} F(t, y), & \|y\| \leq M_*, \\ F\left(t, \dfrac{yM_*}{\|y\|}\right), & \|y\| > M_*, \end{cases}$$

where $M_* > M$. Now, we show that $S(y_\infty) = G(y_\infty)$ where $G(y_\infty)$ is the solution set of problem (6.25). Let $y \in S(y_\infty)$ then $y(t_k^+) - y(t_k^-) = I_k(y(t_k))$ for every $k \in \{1, 2, \dots\}$, $\lim_{t\to\infty} y(t) = y_\infty$ and $y'(t) \in F(t, y(t))$, for almost every $t \in J$. Thus

$$\|y(t)\| \le M, \text{ a.e. } t \in J \;\Rightarrow\; F(t, y(t)) = G(t, y(t)), \text{ a.e. } t \in J.$$

This implies that $y \in G(y_\infty)$, hence $S(y_\infty) \subset G(y_\infty)$. Conversely, let $y \in G(y_\infty)$, then

$$y(t) = y_\infty - \int_t^\infty v(s)\,ds - \sum_{t_k \ge t} I_k(y(t_k)), \quad \text{a.e. } t \in J,$$

where $v \in S_{G,y}$. Set $W = \{t \in [0, \infty) : \|y(t)\| > M_*\}$, we show that $\operatorname{meas}(W) = 0$ where meas is the Lebesgue measure. Assume that $\operatorname{meas}(W) \ne 0$. Then

$$y(t) = y_\infty - \int_W^\infty v(s)\,ds - \int_{[t,\infty)\setminus W}^\infty v(s)\,ds - \sum_{t_k \ge t} I_k(y(t_k)).$$

As a consequence

$$\|y(t)\| \le |y_\infty| + \int_W |v(s)|\,ds + \int_{[0,\infty)\setminus W} \|v(s)\|\,ds + \sum_{k=1}^\infty c_k \|y(t_k)\| + \sum_{k=1}^\infty \|I_k(0)\|.$$

From the definition of G, $(\mathcal{HF}_4)$ and $(\widetilde{\mathcal{HT}}_3)$, we obtain the successive estimates

$$\begin{aligned}
\|y(t)\| &\le \|y_\infty\| + \int_W \|v(s)\|\,ds + \int_{[0,\infty)\setminus W} \|v(s)\|\,ds + \sum_{k=1}^\infty c_k \|y(t_k)\| + \sum_{k=1}^\infty \|I_k(0)\| \\
&\le \|y_\infty\| + \int_W l(s)\,ds + \int_W M_* l(s)\,ds + \int_{[0,\infty)\setminus W} l(s)\|y(s)\|\,ds \\
&\quad + \int_{[0,\infty)\setminus W} l(s)\,ds + \sum_{k=1}^\infty c_k \|y(t_k)\| + \sum_{k=1}^\infty \|I_k(0)\| \\
&\le \|y_\infty\| + \int_0^\infty l(s)\,ds + \int_0^\infty l(s)\|y(s)\|\,ds + \sum_{k=1}^\infty c_k \|y(t_k)\| + \sum_{k=1}^\infty \|I_k(0)\|.
\end{aligned}$$

Then

$$\|y\|_{\mathrm{PC}_b} \le \|y_\infty\| + \|l\|_{L^1} + \|l\|_{L^1} \|y\|_{\mathrm{PC}_b} + \sum_{k=1}^\infty c_k \|y\|_{\mathrm{PC}_b} + \sum_{k=1}^\infty \|I_k(0)\|.$$

Therefore

$$\|y\|_{\mathrm{PC}_b} \le \frac{1}{1 - \|l\|_{L^1} - \sum_{k=1}^\infty c_k} \left(\|y_\infty\| + \|l\|_{L^1} + \sum_{k=1}^\infty \|I_k(0)\| \right) = M.$$

This is a contradiction with $M < M_*$; then $\operatorname{meas}(W) = 0$, this implies that

$$G(t, y(t)) = F(t, y(t)), \quad \text{a.e. } t \in J.$$

So, $S(y_\infty) = G(y_\infty)$. Since $G(\,\cdot\,,\,\cdot\,) \in \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ because $F(\,\cdot\,,\,\cdot\,) \in \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ and is *u.s.c.*, then there exists a Carathéodory Lipschitzian selection (see [12]) $g\colon \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$g(t, x) \in G(t, x), \quad \text{for a.e. } t \in J \text{ and } x \in \mathbb{R}^n.$$

Since $\|l\|_{L^1} + \sum_{k=1}^{\infty} c_k < 1$, then we can choose a Lipschitzian function $\gamma \in L^1(J, \mathbb{R}^+)$ such that

$$\int_0^\infty \gamma(s)\, ds + \sum_{k=1}^{\infty} c_k < 1$$

(see [12]) and

$$\|g(t, x) - g(t, y)\| \le \gamma(t)\|x - y\|, \quad \text{for a.e. } t \in J \text{ and each } x, y \in \mathbb{R}^n.$$

Consider the single-valued problem

$$\begin{cases} x'(t) = g(t, x(t)), & \text{a.e. } t \in J, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), & k = 1, 2, \dots, \\ \lim\limits_{t\to\infty} x(t) = y_\infty \in \mathbb{R}^n. \end{cases} \tag{6.26}$$

Since g is a Carathéodory Lipschitz function, we can prove that problem (6.26) has a unique global solution denoted by $\overline{x}$ (see, e.g. [114], Theorems 3.3 and 3.5). Furthermore $\overline{x} \in \mathrm{PC}$ and $\lim_{t\to\infty} \overline{x}(t) = y_\infty$. Since $g(t, x) \in G(t, x)$, then

$$\|\overline{x}\|_{\mathrm{PC}_b} \le M_*$$

which implies that $\overline{x} \in \mathrm{PC}_b$. Define the homotopy $h\colon S(y_\infty) \times [0, 1] \to S(y_\infty)$ by

$$h(y, \alpha)(t) = \begin{cases} y(t), & \text{for } t \ge \dfrac{1}{\alpha} - \alpha,\ \alpha \ne 0, \\ \overline{z}(t), & \text{for } 0 \le t < \dfrac{1}{\alpha} - \alpha,\ \alpha \ne 0, \\ \overline{x}(t), & \text{for } \alpha = 0, \end{cases}$$

where, for each $\alpha \in (0, 1)$, $\overline{z}$ is the unique solution of the backward Cauchy problem:

$$\begin{cases} z'(t) =, g(t, z(t)), & \text{a.e. } t \in \left[0, \dfrac{1}{\alpha} - \alpha\right] \setminus \{t_1, t_2, \dots\}, \\ \Delta z|_{t=t_k} = I_k(z(t_k^-)), & k = 1, 2, \dots, \\ z\left(\dfrac{1}{\alpha} - \alpha\right) = \overline{x}\left(\dfrac{1}{\alpha} - \alpha\right). \end{cases} \tag{6.27}$$

Step 2. h is a continuous function. Let $(y_n, \alpha_n) \in S(y_\infty) \times [0,1]$ be such that $(y_n, \alpha_n) \to (y, \alpha)$, as $n \to +\infty$. We shall prove that $h(y_n, \alpha_n) \to h(y, \alpha)$. We have

$$h(y_n, \alpha_n)(t) = \begin{cases} y_n(t), & \text{for } t \geq \dfrac{1}{\alpha_n} - \alpha_n,\ \alpha_n \neq 0, \\ \overline{z}(t), & \text{for } 0 \leq t < \dfrac{1}{\alpha_n} - \alpha_n,\ \alpha_n \neq 0, \\ \overline{x}(t), & \text{for } \alpha_n = 0. \end{cases}$$

We distinguish between different cases:

• If $\lim_{n\to\infty} \alpha_n = 0$, then $\|h(y_n, \alpha_n)(t) - h(y, 0)(t)\| = \|\overline{z}(t) - \overline{x}(t)\|$, for n large enough.

Indeed when $\lim_{n\to\infty} \alpha_n = 0$,

$$\forall\, t \in (0, \infty),\ \exists\, n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow \frac{1}{\alpha_n} - \alpha_n > t$$

and thus $h(y_n, \alpha_n) = \overline{x}(t)$. Moreover,

$$\lim_{n\to+\infty} \overline{z}\left(\frac{1}{\alpha_n} - \alpha_n\right) = \lim_{n\to+\infty} \overline{x}\left(\frac{1}{\alpha_n} - \alpha_n\right) = y_\infty.$$

Hence $\lim_{n\to\infty} |h(y, \alpha_n)(t) - h(y, 0)(t)| = 0$.

If $t = 0$, we have $\lim_{n\to\infty} |h(y, \alpha_n)(0) - h(y, 0)(0)| = 0$.

• Let $\alpha_n \neq 0$ and $0 < \lim_{n\to\infty} \alpha_n = \alpha \leq 1$. Since $\overline{z}_n$ is a solution of problem (6.27), then for every $n \geq 1$ and for almost every $t \in [0, 1/\alpha_n - \alpha_n]$, we have

$$\overline{z}(t) = \overline{x}\left(\frac{1}{\alpha_n} - \alpha_n\right) - \int_t^{1/\alpha_n - \alpha_n} g(s, \overline{z}(s))\, ds - \sum_{t < t_k < 1/\alpha_n - \alpha_n} I_k(\overline{z}(t_k)).$$

Without loss of generality, we may assume $\alpha_n \geq \alpha$; hence $1/\alpha_n - \alpha_n \leq 1/\alpha - \alpha$. Consider the function

$$\widetilde{z}_n(t) = \begin{cases} \overline{z}(t), & \text{for } t \in \left[0, \dfrac{1}{\alpha_n} - \alpha_n\right], \\ \overline{x}\left(\dfrac{1}{\alpha_n} - \alpha_n\right), & \text{for } \dfrac{1}{\alpha_n} - \alpha_n \leq t \leq \dfrac{1}{\alpha} - \alpha. \end{cases}$$

Since g and I_k for $k = 1, \ldots$ are Lipschitzian functions and $\overline{x}$ is bounded, then there exists a positive real number M independent of $n \in \mathbb{N}$ such that

$$\sup\left\{\|\widetilde{z}_n(t)\| : t \in \left[0, \frac{1}{\alpha} - \alpha\right]\right\} \leq M, \quad \text{for every } n \in \mathbb{N}.$$

Moreover, the set $\{\widetilde{z}_n(t) : t \in [0, 1/\alpha - \alpha]\}$ is equicontinuous in $\mathrm{PC}_b([0, 1/\alpha - \alpha], \mathbb{R}^n)$. If $\tau_1, \tau_2 \in [0, 1/\alpha_n - \alpha_n]$, then

$$\|z_n(\tau_1) - z_n(\tau_2)\| \leq \int_{\tau_1}^{\tau_1} \|g(s, \overline{z}(s))\|\, ds + \sum_{\tau_1 < t_k < \tau_2} \|I_k(\overline{z}(t_k)\|.$$

Using the fact that $\overline{x}$ is bounded, we can easily prove that there exist $K_* > 0$ such that $\|\overline{z}\|_{\mathrm{PC}} \le K_*$ (for the proof, we refer the interested reader to [114, Theorem 3.3]). Then

$$\|\widetilde{z}_n(\tau_1) - \widetilde{z}_n(\tau_2)\| \le \int_{\tau_1}^{\tau_2} K_*\gamma(s)\,ds + \int_{\tau_1}^{\tau_2} l(s)\,ds + \sum_{\tau_1<t_k<\tau_2} \sup\{\|I_k(x)\| : x \in \overline{B}(0,K_*)\}.$$

If $\tau_1, \tau_2 \in [1/\alpha_n - \alpha_n, 1/\alpha - \alpha]$, then

$$\|\widetilde{z}_n(\tau_1) - \widetilde{z}_n(\tau_2)\| = 0.$$

In the case $\tau_1 \in [0, 1/\alpha_n - \alpha_n]$ and $\tau_2 \in [1/\alpha_n - \alpha_n, 1/\alpha - \alpha]$, we get

$$\begin{aligned}\|\widetilde{z}_n(\tau_1) - \widetilde{z}_n(\tau_2)\| &\le \int_{\tau_1}^{1/\alpha_n-\alpha_n} K_*\gamma(s)\,ds \\ &\quad + \int_{\tau_1}^{\tau_2} l(s)\,ds + \sum_{\tau_1<t_k<\tau_2} \sup\{\|I_k(x)\| : 0 \le \|x\| \le K_*\} \\ &\le \int_{\tau_1}^{\tau_2} K_*\gamma(s)\,ds + \int_{\tau_1}^{\tau_2} l(s)\,ds + \sum_{\tau_1<t_k<\tau_2} \sup\{\|I_k(x)\| : 0 \le \|x\| \le K_*\}.\end{aligned}$$

By Ascoli–Arzéla Lemma, there exists $\widetilde{z} \in \mathrm{PC}_b([0, 1/\alpha - \alpha], \mathbb{R}^n)$ such that $(\widetilde{z}_n)_{n\in\mathbb{N}}$ converges to $\widetilde{z}$ and

$$\widetilde{z}(t) = \overline{x}\left(\frac{1}{\alpha} - \alpha\right) - \int_t^{1/\alpha-\alpha} g(s, \widetilde{z}(s))\,ds - \sum_{t<t_k<1/\alpha-\alpha} I_k(\widetilde{z}(t_k)).$$

Hence $\widetilde{z}$ is solution of problem (6.27). By uniqueness, $\widetilde{z} = \overline{z}$. Thus

$$\sup\left\{\|\widetilde{z}_n(t) - \overline{z}(t)\| : t \in \left[0, \frac{1}{\alpha} - \alpha\right]\right\} \to 0, \quad \text{as } n \to \infty.$$

• Finally, let $t \ge 1/\alpha_n - \alpha_n$ and $\alpha_n \ne 0$. Since (y_n) converges to y in PC_b, we have

$$\sup\{\|y_n(t) - y(t)\| : t \in [0,\infty)\} \to 0, \quad \text{as } n \to \infty.$$

To sum up

$$\sup\{\|h(y_n, \alpha_n)(t) - h(y, \alpha)(t)\| : t \in [0,\infty)\} \to 0, \quad \text{as } n \to \infty,$$

proving our claim.

Step 3. $S(y_\infty)$ is closed in PC_b. Let $\{y_n : n \in \mathbb{N}\} \subset S(y_\infty)$ be such that (y_n) converges to y in PC_b, an $n \to +\infty$. Then there exists $v_n \in S_{G,y_n}$ such that

$$y_n(t) = y_\infty - \int_t^\infty v_n(s)\,ds - \sum_{t_k \geq t} I_k(y_n(t_k)), \quad \text{a.e. } t \in J.$$

From (6.24), there exists $M > 0$ such that

$$\sup\{\|y_n(t)\| : t \in \mathbb{R}^+\} \leq M \tag{6.28}$$

and

$$\|v_n(t)\| \leq l(t)M + l(t), \quad \text{a.e. } t \in J.$$

Since $\overline{B}(0,1)$ is compact in $\mathbb{R}^n$, then there exists a subsequence $(v_{n_m}(\,\cdot\,))$ which converges to some limit $v(\,\cdot\,)$. By the Lebesgue dominated convergence theorem, we conclude that $v \in L^1(\mathbb{R}^+, \mathbb{R}^n)$. Using the fact that $G(\,\cdot\,,\,\cdot\,) \in \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R}^n)$ and that $G(t,\,\cdot\,)$ is *u.s.c.*, the map $G(t,\,\cdot\,)$ has a closed graph, hence $v \in S_{G,y}$. Therefore

$$y(t) = y_\infty - \int_t^\infty v(s)\,ds - \sum_{t_k \geq t} I_k(y(t_k)), \quad \text{a.e. } t \in J,$$

proving that $y \in S(y_\infty)$. From Steps 1–3, h is a continuous homotopy with $h(y,0) = \overline{x}$ and $h(y,1) = y$ (see also [56] for the case of differential inclusions). Therefore the set $S(\,\cdot\,)$ is contractible. Let $(y_n)_{n\in\mathbb{N}} \subset S(y_\infty)$. (6.24) shows that $(y_n)_{n\in\mathbb{N}}$ is uniformly bounded. Also, we can easily prove that for every compact interval $J_* \subset [0,\infty)$, $\{y_n :\ n \in \mathbb{N}\}$ is almost equicontinuous. It remains to only prove that $\{y_n : n \in \mathbb{N}\}$ is equiconvergent at ∞, i.e. for every $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that $\|y_n(t) - y_n(\infty)\| \leq \varepsilon$ for every $t \geq T$ and each $n \in \mathbb{N}$. Let $y_n \in \{y_n : n \in \mathbb{N}\}$, then there exists $v_n \in S_{F,y_n}$ such that

$$y_n(t) = y_\infty - \int_t^\infty v_n(s)\,ds - \sum_{t_k \geq t} I_k(y_n(t_k)), \quad \text{a.e. } t \in J.$$

Then

$$\begin{aligned}\|y_n(t) - y_n(\infty)\| &\leq \int_t^\infty \|v_n(s)\|\,ds + \sum_{t \leq t_k < \infty} \|I_k(y_n(t_k))\| \\ &\leq (1+M)\int_t^\infty l(s)\,ds + \sum_{t \leq t_k < \infty} (c_k M + \|I_k(0)\|).\end{aligned}$$

Since $\sum_{k=1}^\infty c_k < \infty$, $\sum_{k=1}^\infty \|I_k(0)\| < \infty$ and $p \in L^1(J, \mathbb{R}^+)$, there exist k_0 and $T(\varepsilon) > 0$ such that

$$\sum_{k=k_0}^\infty (c_k M + \|I_k(0)\|) \leq \frac{\varepsilon}{2}$$

and

$$\int_t^\infty l(s)\,ds < \frac{\varepsilon}{2(M+1)}, \quad \text{for all } t \geq T(\varepsilon).$$

Hence

$$\|y_n(t) - y_n(\infty)\| \leq \varepsilon, \quad \text{a.e. } t \geq \max(k_0, T(\varepsilon)).$$

Then $\{y_n : n \in \mathbb{N}\}$ is equiconvergent. With Lemma 6.3 implies that $S(y_\infty)$ is compact, hence acyclic. The proof of Theorem 6.10 is complete. □

6.3. Using solution sets to prove existence results

Using J-mapping and fixed point index, we are going to prove an existence result of solutions for an impulsive differential inclusion on the half-line. Let X be a metric space.

Definition 6.11. A compact, nonempty subset $A \subset X$ is called ∞-proximally connected in X if, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for any $n = 1, 2, \dots$, and for any map $g\colon \partial\Delta^n \to O_\delta(A)$, there exists an extension $f\colon \Delta^n \to O_\varepsilon(A)$ such that $f|_{\partial\Delta^n} = g$ (f is an extension of g) and where Δ^n is an $n-$dimensional standard simplex; $\partial\Delta^n$ stands for the boundary of Δ^n. Neighbourhoods are taken as subsets of X.

Moreover, one can see that the above notion gives us information about embedding of A into X rather than the structure of A. In spite of this, we have the following result formulated (D.M. Hyman, 1969).

Proposition 6.12 (see [91])**.** *Let X be an* ANR *and A a nonempty compact subset. Then A is R_δ-subset if and only if it is ∞-proximally connected.*

Let use note the following immediate result:

Proposition 6.13. *Let Y be a space, X be a neighbourhood retract of Y and $Z \subset X$ be a compact, ∞-proximally connected set in Y. Then Z is ∞-proximally connected in X.*

Proof. Take an arbitrary $k \in \mathbb{N}$. We will show that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each map $g\colon \partial\Delta^k \to O_\delta(Z) \cap X$, there is an extension $f\colon \Delta^k \to O_\varepsilon(Z) \cap Y$ of f. Let Ω be an open subset of Y and $r\colon \Omega \to X$ be a retraction. Take an arbitrary $\varepsilon > 0$, put $U := r^{-1}(O_\varepsilon(Z) \cap X) \subset \Omega$ and choose $\eta > 0$ such that $O_\eta(Z) \subset U$. There is δ, $0 < \delta < \eta$ such that, for every map $g^*\colon \partial\Delta^k \to O_\delta(Z) \cap X$, there is an $f^*\colon \Delta^k \to O_\eta(Z)$ such that $f^*|_{\partial\Delta^k} = g^*$. The map $f = r \circ f^*\colon \Delta^k \to O_\varepsilon(Z) \cap X$ is then an extension of f. □

Definition 6.14. A multi-map $\phi\colon X \to \mathcal{P}(Y)$ is called J-mapping ($\phi \in J(X, Y)$ for short), provided the set $\phi(x)$ is ∞-proximally connected for every $x \in X$.

Remark 6.15. From Propositions 6.12 and 6.13, we have the following two facts. First, if Y is a neighbourhood retract of the Fréchet space F, then $\phi \in J(X,Y)$ if, $\phi(x)$ is an R_δ-set, for every $x \in X$. Secondly, if $\phi \in J(X,Y)$ and $r\colon Z \to X$, then $\phi \circ r \in J(Z,Y)$. Moreover, if $\phi_i \in J(X_i,Y_i)$ for $i = 1,2$, then $\phi_1 \times \phi_2 \in J(X_1 \times X_2, Y_1 \times Y_2)$.

Assume that X is a retract of a Fréchet space F and D is an open subset of X. Let $\Phi \in J(D,F)$ be locally compact, let $\mathrm{Fix}\,(\Phi)$ be compact, and let the following condition hold:

(A) For each $x \in \mathrm{Fix}\,(\Phi)$, there exists an open neighbourhood U_x in D of x such that $\Phi(U_x) \subset X$.

Definition 6.16. Let X be a retract of the Fréchet space F and D is an open subset of X. A multi-map $\Phi\colon X \to \mathcal{P}(Y)$ is called J_A-mapping if it is a locally compact $J-$map from D to F with compact fixed point set and satisfies (A). It will be denoted by $J_A(D,F)$.

Definition 6.17. We say that $\Phi,\ \Psi \in J_A(D,F)$ are homotopic in $J_A(D,F)$, if there exists a homotopy $H \in J([0,1]\times D,F)$ such that $H(0,\,\cdot\,) = \Phi,\ H(1,\,\cdot\,) = \Psi$, for every $x \in D$ there is an open neighbourhood V_x of x in D such that $H|_{V_x\times[0,1]}$ is compact, and

(A_H) for each $x \in D$ and $t \in [0,1]$ we have

$$x \in H(t,x) \Rightarrow \text{ there exists } U_x \text{ open in } D : H([0,1]\times U_x) \subset X.$$

Remark 6.18. Note that the condition (A_H) is equivalent to the following one:

(B_H) If $\{x_n\}_{n\geq 1} \subset D$ converges to $x \in H(t,x)$ for some $t \in [0,1]$, then

$$H([0,1]\times\{x_n\}) \subset X, \quad \text{for } n \text{ sufficiently large.}$$

First, we need the following auxiliary fixed point result (see Corollary 10.5 in [7]).

Theorem 6.19. *Let X be a retract of the Fréchet space F, and $D \in X$ an open subset. Let H be a homotopy in $J_A(D,F)$ such that $H(x,0) = x_0$ for all $x \in D$. Then there exists $x \in D$ such that $x \in H(x,1)$.*

Although in Theorems 6.2, 6.4, and 6.5, the solvability of given problems is guaranteed as well; the Lipschitzianity is rather restrictive. Therefore, we look for better existence results. We start with a compactness result.

Theorem 6.20. *Let $G\colon J\times\mathbb{R}^n\times\mathbb{R}^m\to\mathcal{P}(\mathbb{R}^n)$ be a Carathéodory map and let $S\subset \mathrm{PC}\cap\bigcup_{k=1}^{\infty}\mathrm{AC}(J_k,\mathbb{R}^n)$. Assume that*

($\mathcal{HJ}_1$) *there exists a subset Q of* PC *such that for some $q\in Q$, the set $T(q)$ of all solutions of the problem*

$$\begin{cases} y'(t)\in G(t,y(t),q(t)), & \text{a.e. } t\in J\setminus\{t_1,t_2,\dots\},\\ \Delta y_{t=t_k}=I_k(y(t_k^-)), & k=1,2\dots,\\ y\in S, \end{cases} \tag{6.29}$$

where $J:=[0,\infty)$, $J_k=(t_k,t_{k+1})$, $I_k\in C(\mathbb{R}^n,\mathbb{R}^n)$, $k=1,2,\dots$ is nonempty.

($\mathcal{HJ}_2$) *$T(Q)$ is bounded in* PC,

($\mathcal{HJ}_3$) *There exists $p\in L^1_{\mathrm{loc}}(J,\mathbb{R}_+)$ such that*

$$\|G(t,y(t),q(t))\|_{\mathcal{P}}\le p(t),\quad \text{a.e. } t\in J,$$

for all $(q,x)\in\mathrm{Gr}(T)$.

Then $T(Q)$ is relatively compact subset in PC. *Moreover, under assumptions ($\mathcal{HJ}_1$)–($\mathcal{HJ}_3$), the multi-map $T\colon Q\to\mathcal{P}(S)$ is u.s.c. with compact values if and only if T has a closed graph.*

Proof. Let $y\in T(Q)$ then there exists $q\in Q$ such that $y\in T(q)$; hence there exists $v\in S_{F,G}$ such that

$$y(t)=y(0)+\int_0^t v(s)\,ds+\sum_{0<t_k<t} I_k(y(t_k)),\quad t\in J.$$

where

$$S_{G,y}=\{v\in L^1_{\mathrm{loc}}(J,\mathbb{R}^n): v(t)\in G(t,y(t),q(t)) \text{ a.e. } t\in J\}.$$

Given $r_1,r_2\in J$ ($r_1<r_2$), there exists $b>0$ such that $r_1,r_2\in[0,b]$. Then

$$\|y(r_2)-y(r_1)\|\le\int_{r_1}^{r_2}\|v(s)\|\,ds+\sum_{r_1\le t_k<r_2}\|I_k(y(t_k))\|.$$

Since $T(Q)$ is bounded then there exists $M_n>0$, $n\in N$ such that

$$\|y\|_n\le M_n,\quad \text{for each } n\in\mathbb{N},$$

where for each $n\in\mathbb{N}$,

$$\|y\|_n=\sup\{\|y(t)\|: 0\le t\le t_n\}$$

stands for the semi-norms in PC_n. Then there exists $n\in\mathbb{N}$ such that $b\le n$, hence

$$\|y(r_2)-y(r_1)\|\le\int_{r_1}^{r_2}p(s)\,ds+\sum_{r_1\le t_k<r_2}\sup_{x\in B(0,M_n)}\|I_k(x)\|.$$

The right-hand side tends to zero as $r_2 - r_1 \to 0$. This implies that for every interval K compact in J, the set $T(Q)$ is equicontinuous. Hence $T(Q)$ is relatively compact. As in Theorem 4.4, we can prove that T has a closed graph, then T is *u.s.c.* □

We can now state our final existence result in this work.

Theorem 6.21. *Consider the boundary value problem*

$$\begin{cases} y'(t) \in F(t, y(t)), & a.e.\ t \in J \setminus \{t_1, \dots\}, \\ \Delta y_{t=t_k} = I_k(y(t_k^-)), & k = 1, 2, \dots, \\ y \in, S, \end{cases} \tag{6.30}$$

where $F\colon J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a Carathéodory map and S is a subset of $\mathrm{PC} \cap \bigcup_{k=1}^{\infty} AC(J_k, \mathbb{R}^n)$. *Let $G\colon J \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathcal{P}(\mathbb{R}^n)$ be a Carathéodory map such that*

$$G(t, c, c, 1) \subset F(t, c) \quad \text{for all } (t, c) \in J \times \mathbb{R}^n.$$

Assume that the following conditions hold

($\mathcal{HJ}_4$) *There exist a retract Q of* PC *and a closed bounded subset S_1 of S such that the associated problem*

$$\begin{cases} y'(t) \in G(t, y(t), q(t), \lambda), & a.e.\ t \in J \setminus \{t_1, \dots\}, \\ \Delta y_{t=t_k} = I_k(y(t_k^-)), & k = 1, 2, \dots, \\ y \in S_1, \end{cases} \tag{6.31}$$

is solvable with R_δ-set of solutions, for each $(\lambda, q) \in [0,1] \times Q$.

($\mathcal{HJ}_5$) *There exists a locally integrable function $p\colon J \to \mathbb{R}_+$ such that*

$$\|G(t, y(t), q(t), \lambda)\| \le p(t), \quad a.e.\ in\ J,$$

for any $(q, \lambda, y) \in \mathrm{Gr}(T)$, where T denotes the set-valued map which assigns to any $(q, \lambda) \in Q \times [0,1]$ the set of solutions of (6.31).

($\mathcal{HJ}_6$) *$T(\{0\} \times Q) \subset Q$. If $Q \ni q_j \to q \in Q$, $q \in T(\lambda, q)$, then there exists $j_0 \in \mathbb{N}$ such that, for every $j \ge j_0$, $\theta \in [0,1]$ and $q \in T(q_j, \theta)$, we have $y \in Q$.*

Then problem (6.30) *has a solution.*

Proof. Let $Q' = \{y \in \mathrm{PC}(J, \mathbb{R}^{n+1}) : y(t) = (\lambda, q(t)),\ q \in Q,\ \lambda \in [0,1]\}$. From Theorem 6.20, the set-valued map $T\colon [0,1] \times Q \to \mathcal{P}(S_1)$ is *u.s.c.* and from Proposition 6.12, we can conclude that T belongs to the $J([0,1] \times Q, \mathrm{PC}(J, \mathbb{R}^n))$. Moreover, it has a relatively compact image. Given $\Phi = T(0, \cdot)$ and $\Psi = T(1, \cdot)$, it is clear that $\Phi, \Psi \in J_A(Q, \mathrm{PC})$. Assumption ($\mathcal{HJ}_2$) implies that T form a homotopy of Φ and Ψ in $J_A(Q, \mathrm{PC})$. By Theorem 6.19, we conclude that $T(1, \cdot)$ has at least one fixed point. □

CHAPTER 7

CONCLUDING REMARKS

In this survey paper, we investigated problem (3.1) and more generally problem (5.1) under various assumptions on the multi-valued right hand-side nonlinearity and we obtained a number of new results regarding existence of mild solutions as well as Filippov type results (see also [34], [49]). The main assumptions on the nonlinearity are the Carathéodory and the Lipschitz conditions with respect to the Hausdorff distance and some Nagumo–Bernstein type growth conditions. We have used fixed point theory for multi-valued analysis together with general properties from functional analysis and measure of noncompacness. Under the same hypotheses, we have also obtained closedness and even compactness of the solution sets.

Concerning initial-valued problems for first-order differential inclusions on bounded intervals, De Blasi and Myjak [39] proved in 1986 that the solution set is R_δ whenever the nonlinearity F is a Carathéodory multi-map which is bounded and compact convex valued. In [60], Górniewicz noticed the result remains true if F has sublinear growth and proved the previous results. The results of Theorem 6.9 were obtained by Andres *et al.* [6] in case of differential inclusions and the solution set was proved to have compact R_δ structure in [4] under convex compact right-hand sides. In this work, we have extended these results to initial problem for impulsive differential inclusions on bounded and differential inclusions on unbounded intervals. Using of the Poincaré operator and a recent nonlinear alternative, we have also discussed the problem with periodic boundary condition and obtained some existence results and theorems about structure of solution sets.

Some existence results for first and second order differential inclusions on the half-line can be found in the literature (see e.g. [5], [8], [56] and the references therein). Also some abstract differential equations and inclusions on Fréchet spaces are considered in [16] where the projective limit approach is employed.

In 1976, Lasry and Robert [105] proved that, if the nonlinearity F is compact, convex valued, *u.s.c.* and bounded, then the set of all solutions for first-order differential inclusions with right-hand side F is a compact and acyclic set. In 1986, Górniewicz [64] discussed the topological structure of the set of solutions (contractibility, acyclic,...) when F is an ML- or σ-selectionable. Later, Himmelberg and van Vlek [85] proved that the solution set is in fact an R_δ-set. In [6], Andres *et al.* obtained some multi-valued generalizations to the Aronszajn–Browder–Gupta theory [9], [27]. In this survey, we have focused on the structure topology of solutions set for impulsive differential equations and inclusions. Using the limit of inverse systems, such results have been extended to problems set on unbounded intervals of the real line. We hope this survey can provide some contribution to the questions of existence and topological structure for impulsive first-order differential inclusions on bounded and unbounded domains.

Acknowledgments. The authors would like to warmly thank Prof. dr hab. A. Fryszkowski and Prof. dr hab. J. Motyl for careful reading of this survey paper; their comments and suggestions have led to a profound improvement of the first draft.

REFERENCES

[1] Z. Agur, L. Cojocaru, G. Mazaur, R.M. Anderson and Y.L. Danon, *Pulse mass measles vaccination across age cohorts*, Proc. Nat. Acad. Sci. USA. **90** (1993), 11698–11702.

[2] R.R. Akhmerov, M.I. Kamenskiĭ, A.S. Potapov, A.E. Rodkina and B.N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*, Nauka, Novosibirsk, 1986 (in Russian); Engl. transl.: Birkhäuser, Basel, 1992.

[3] J. Andres, *On the multi-valued Poincaré operators*, Topol. Methods Nonlinear Anal. **10** (1997), 171–182.

[4] J. Andres, G. Gabor and L. Górniewicz, *Boundary value problems on infinite intervals*, Trans. Amer. Math. Soc. **351** (12) (1999), 4861–4903.

[5] J. Andres, G. Gabor and L. Górniewicz, *Topological structure of solution sets to multi-valued asymptotic problems*, Z. Anal. Anwend. **19** (1) (2000), 35–60.

[6] J. Andres, G. Gabor and L. Górniewicz, *Acyclicity of solution sets to functional inclusions*, Nonlinear Anal. **49** (2002), 671–688.

[7] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Kluwer, Dordrecht, 2003.

[8] J. Andres and M. Pavlačková, *Topological structure of solution sets to asymptotic boundary value problems*, J. Differential Equations **248** (2010), 127–150.

[9] *Le correspondant topologique de l'unicité dans la théorie des équations différentielles*, Ann. Math. **43** (1942), 730–738.

[10] H. Attouch, *Variational Convergence for Functionals and Operators*, Pitman, London, 1984.

[11] J.P. Aubin, *Impulse Differential Inclusions and Hybrid Systems: a Viability Approach*, Lecture Notes, Université Paris–Dauphine, 2002.

[12] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer–Verlag, Berlin, Heidelberg, New York, 1984.

[13] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[14] E.P. Avgerinos and N.S. Papageorgiou, *Topological properties of the solution set of integrodifferential inclusions*, Comment. Math. Univ. Carolinae **36** (3) (1995), 429–442.

[15] N.B. Azbelev, V.P. Maximov and L.F. Rakhmatulina, *Introduction to the Functional Differential Equations*, Nauka, Moscow, 1991. (in Russian)

[16] A. Bąkowska and G. Gabor, *Topological structure of solution sets to differential problems in Fréchet spaces*, Ann. Polon. Math. **95** (1) (2009), 17–36.

[17] D.D. Bainov and P.S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.

[18] C. Benassi and A. Gavioli, *Approximation from the exterior of Carathéodory multifunctions*, Acta. Univ. Palacki. Olomuc. **39** (2000), 17–35.

[19] M. Benchohra, L. Górniewicz, S.K. Ntouyas and A. Ouahab, *Controllability results for impulsive functional differential inclusions*, Reports on Math. Phys. **54** (2004), 211–227.

[20] M. Benchohra, J. Henderson and S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, vol. 2, Hindawi Publishing Corporation, New York, 2006.

[21] M. Benchohra and A. Ouahab, *Some uniqueness results for impulsive semilinear neutral functional differential equations*, Georgian Math. J. **9** (2002), 423–430.

[22] R. Bielawski, L. Górniewicz and S. Plaskacz, *Topological approach to differential inclusions on closed subset of* $\mathbb{R}^n$, Dynam. Report. Expositions Dynam. Systems (N.S.) **1** (1992), 225–250.

[23] K. Borsuk, *Theory of Retracts*, vol. 1967, Polish Scientific Publishers, Warszawa.

[24] A. Bressan and G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math. **90** (1988), 70–85.

[25] A. Bressan, A. Cellina and A. Fryszkowski, *A class of absolute retracts in spaces of integrable functions*, Proc. Amer. Math. Soc. **112** (1991), 413–418.

[26] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitxt, Springer, 2011.

[27] F.E. Browder and G.P. Gupta, *Topological degree and nonlinear mappings of analytic type in Banach spaces*, J. Math. Anal. Appl. **26** (1969), 390–402.

[28] T.A. Burton, *Stability and Periodic Solutions of Ordinary and Functional-Differential Equations*, Mathematics in Science and Engineering, vol. 178, Academic Press, Inc., Orlando, FL, 1985.

[29] T.A. Burton, *Volterra Integral and Differential Equations*, Second edition, Mathematics Science and Engineering, vol. 202, Elsevier B.V., Amsterdam, 2005.

[30] T. Cardinali and P. Rubbioni, *Mild solutions for impulsive semilinear evolution differential inclusions*, J. Appl. Funct. Anal. **1** (3) (2006), 303–325.

[31] T. Cardinali and P. Rubbioni, *On some properties of semilinear functional differential inclusions in abstract spaces*, J. Concrete. Appl. Math. **4** (1) (2006), 183–214.

[32] T. Cardinali and P. Rubbioni, *Impulsive semilinear differential inclusions: topological structure of the solutions set and solutions on noncompact domains*, Nonlinear Anal. **69** (1) (2008), 73–84.

[33] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, vol. 580, Springer–Verlag, Berlin, Heidelberg, New York, 1977.

[34] A. Cernea, *A Filippov type existence theorem for a class of second-order differential inclusions*, J. Ineq. Pure and Appl. Math. **9** (2) (2008), 1–6.

[35] R.M. Colombo, A. Fryszkowski, T. Rzeżuchowski and V. Staicu, *Continuous selection of solutions sets of Lipschitzean differential inclusions*, Funkcial. Ekvac. **34** (1991), 321–330.

[36] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[37] H. Covitz and S.B. Nadler Jr., *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970), 5–11.

[38] K. Czarnowski and T. Pruszko, *On the structure of fixed point sets of compact maps in B_0 spaces with applications to integral and differential equations in unbounded domain*, J. Math. Anal. Appl. **54** (1991), 151–163.

[39] F.S. De Blasi and J. Myjak, *On the solutions sets for differential inclusions*, Bull. Polish Acad. Sci. Math. **12** (1985), 17–23.

[40] K. Deimling, *Multi-valued Differential Equations*, De Gruyter, Berlin, New York, 1992.

[41] S. Djebali, L. Górniewicz and A. Ouahab, *Filippov's theorem and structure of solution sets for first order impulsive semilinear functional differential inclusions*, Topol. Methods Nonlinear Anal. **32** (2008), 261–312.

[42] S. Djebali, L. Górniewicz and A. Ouahab, *First order periodic impulsive semilinear differential inclusions: existence and structure of solution sets*, Math. Comput. Model. **52** (2010), 683–714.

[43] S. Djebali, L. Górniewicz and A. Ouahab, *Topological structure of solution sets for impulsive differential inclusions in Fréchet spaces*, Nonlinear Anal. **74** (2011), 2141–2169.

[44] N. Dunford and J.T. Schwartz, *Linear Operators. Part* I. *General Theory*, Wiley-Interscience, New York, 1967.

[45] L.È. Èl'sgol'ts, *Introduction to the Theory of Differential Equations with Deviating Arguments*, translated from the Russian by Robert J. McLaughlim, Holden–Dely, Inc., San Francisco, California, London, Amsterdam, 1966.

[46] L.È. Èl'sgol'ts and S.B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, translated from the Russian by John L. Casti, Mathematics in Science and Engineering, vol. 105, Academic Press [A Subsidiary of Harcourt Brace Jovanovich, Publishers], New York–London, 1973.

[47] K.J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Grad. Texts in Math., vol. 194, Springer–Verlag, New York, 2000.

[48] R. Engelking, *Outline of General Topology*, North-Holland, PWN, 1968.

[49] H. Frankowska, *A priori estimates for operational differential inclusions*, J. Differential Equations **84** (1990), 100–128.

[50] M. Frigon, *Fixed point results for multi-valued contractions on gauge spaces, set valued mappings with application in nonlinear analysis*, Ser. Math. Anal. Appl. **4** (2002), 175–181.

[51] M. Frigon and A. Granas, *Théorèmes d'existence pour des inclusions différentielles sans convexité*, C. R. Acad. Sci. Paris, Sér. I **310** (1990), 819–822.

[52] M. Frigon, L. Górniewicz and T. Kaczynski, *Differential inclusions and implicit equations on closed subsets of $\mathbb{R}^n$*, World Cong. of Nonlinear Anal., Tampa (1992), 1797–1806.

[53] A. Fryszkowski, *Continuous selections for a class of non-convex multi-maps*, Studia Math. **75** (1983), 163–174.

[54] A. Fryszkowski, *Fixed Point Theory for Decomposable Sets,*, Topological Fixed Point Theory and its Applications, vol. 2, Kluwer Academic Pulishers, Dordrecht, 2004.

[55] G. Gabor, *On the acyclicity of fixed point sets of multi-maps*, Topol. Methods Nonlinear Anal. **14** (1999), 327–343.

[56] G. Gabor, *Some results on existence and structure of solution sets to differential inclusions on the halfline*, Boll. Un. Mat. Ital. B **5** (8) (2002), 431–446.

[57] Yu.E. Gliklih and V.V. Obukhovskiĭ, *Differential equations of the Carathéodory type on Hilbert manifolds*, Trudy Mat. Fak. Voronezh Univ. (N.S.) **1** (1996), 23–28. (in Russian)

[58] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Mathematics and its Applications, vol. 74, Kluwer Academic Publishers Group, Dordrecht, 1992.

[59] C. Gori, V. Obukhovskiĭ, M. Ragni and P. Rubbioni, *Existence and continuous dependence results for semilinear functional differential inclusions with infinite delay*, Nonlinear Anal. **51** (2002), 765–782.

[60] L. Górniewicz, *Topological structure of solution sets: current results*, Arch. Math. **36** (5) (2000), 343–382.

[61] L. Górniewicz, *Topological Fixed Point Theory of Multi-valued Mappings*, Mathematics and its Applications, vol. 495, Kluwer Academic Publishers, Dordrecht, 1999.

[62] L. Górniewicz, *Homological methods in fixed point theory of multi-valued maps*, Dissert. Math. **129** (1976), 1–71.

[63] L. Górniewicz, *Topological approach to differential inclusions*, Topological Methods in Differential Equations and Inclusions (M. Frigon and A. Granas, eds.), Kluwer Acad. Publ. Ser. C: Math. and Phys. Sci., vol. 47, 1995, pp. 129–190.

[64] L. Górniewicz, *On the solution sets of differential inclusions*, J. Math. Anal. Appl. **113** (1986), 235–244.

[65] L. Górniewicz, P. Nistri and V. Obukhovskiĭ, *Differential inclusions on proximate retracts of Hilbert spaces*, Internat. J. Nonlinear Differential Equations Theory Meth. Appl. **3** (1997), 13–26.

[66] L. Górniewicz, S.K. Ntouyas and D. O'Regan, *Existence results for first and second order semilinear impulsive differential inclusions*, Topol. Methods Nonlinear Anal. **26** (2005), 135–162.

[67] J.R. Graef, J. Henderson and A. Ouahab, *Differential inclusions with nonlocal conditions: existence results and topological properties of solution sets*, Topol. Methods Nonlinear Anal. **37** (2011), 117–145.

[68] J.R. Graef and A. Ouahab, *First order impulsive differential inclusions with periodic condition*, Electron. J. Qual. Theory Differential Equations **31** (2008), 1–40.

[69] J.R. Graef and A. Ouahab, *Structure of solutions sets and a continuous version of Filippov's Theorem for first order impulsive differential inclusions with periodic conditions*, Electron. J. Qual. Theory Differential Equations **24** (2009), 1–23.

[70] J.R. Graef and A. Ouahab, *Global existence and uniqueness results for impulsive functional differential equations with variable times and multiple delays*, Dynam. Contin. Discrete Impuls. Systems Ser. A Math. Anal. **16** (1) (2009), 27–40.

[71] A. Granas and J. Dugundji, *Fixed Point Theory* (2003), Springer-Verlag, New York.

[72] D. Guo, *Boundary value problem for impulsive integro-differential equations on unbounded domains in a Banach space*, Appl. Math. Comput. **99** (1999), 1-15.

[73] G. Guo, *A class of second-order implusive integro-differential equations on unbounded domain in a Banach space*, Appl. Math. Comput. **125** (2002), 59–77.

[74] D. Guo and X.Z. Liu, *Impulsive integro-differential equations on unbounded domain in Banach space*, Nonlinear Stud. **3** (1996), 49–57.

[75] H. Guo and L. Chen, *The effects of impulsive harvest on a predator-prey system with distributed time delay*, Comm. Nonlinear Sci. Numer Simulat. **14** (2009), 2301–2309.

[76] G. Haddad and J.M. Lasry, *Periodic solutions of functional differential inclusions and fixed points of σ-selectionable correspondences*, J. Math. Anal. Appl. **96** (1983), 295–312.

[77] A. Halanay and D. Wexler, *Teoria Calitativa a Systeme cu Impulduri* (1968), Editura Republicii Socialiste Romania, Bucharest.

[78] J.K. Hale, *Theory of Functional Differential Equations*, Springer–Verlag, New York, Heidelberg, 1977.

[79] J.K. Hale and S.M.V. Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, vol. 99, Springer–Verlag, New York, 1993.

[80] J. Henderson and A. Ouahab, *Global existence results for impulsive functional differential inclusions with multiple delay in Fréchet spaces*, PanAmerican Math. J. **15** (2005), 73–89.

[81] J. Henderson and A. Ouahab, *Extrapolation spaces and controllability of impulsive semilinear functional differential inclusions with infinite delay in Fréchet spaces*, Appl. Anal. **85** (2006), 1255–1270.

[82] J. Henderson and A. Ouahab, *Local and gobal existence and uniqueness results for second and higher order impulsive functional differential equations with infinite delay*, Australian J. Appl. Math. **4** (6) (2007), 1–26.

[83] J. Henderson and A. Ouahab, *On the solutions sets for first order impulsive neutral functional differential inclusions in Banach spaces*, Dynam. Contin. Discrete Impuls. Systems Ser. A Math. Anal. **18** (2011), 53–114.

[84] R. Hille and R.S. Philips, *Functional Analysis and Semi-groups*, Amer. Math. Soc., Providence, RI, 1957.

[85] C. Himmelberg and F. van Vleck, *A note on the solution sets of differential inclusions*, Rocky Mountain J. Math. **12** (1982), 621–625.

[86] Sh. Hu and N.S. Papageorgiou, *Handbook of Multi-valued Analysis, Volume*I: *Theory*, Kluwer, Dordrecht, 1997.

[87] Sh. Hu and N.S. Papageorgiou, *Handbook of Multi-valued Analysis, Volume*II: *Applications* (2000), Kluwer, Dordrecht, The Netherlands.

[88] Sh. Hu, N.S. Papageorgiou and V. Lakshmikantham, *On the properties of the solutions set of semilinear evolution inclusions*, Nonlinear Anal. **24** (1995), 1683–1712.

[89] J. Hui and J.L.S. Chen, *Impulsive vaccination of SIR epidemic models with nonlinear incidence rates*, Discrete Contin. Dynam. Systems **4** (2004), 595–605.

[90] M. Hukuhara, *Sur les systèmes des équations différentielles ordinaires*, Japan J. Math. **5** (1928), 345–350.

[91] D.M. Hyman, *On decreasing sequences of compact absolute retracts*, Fund. Math. **64** (1969), 91–97.

[92] J. Jiao, X. Meng and L. Chen, *A new stage structured predator-prey Gomportz model with time delay and impulsive porturbations*, Appl. Math. Comput. **196** (2008), 705–719.

[93] M. Kamenskiĭ, V. Obukhovskiĭ and P. Zecca, *Condensing Multi-valued Maps and Semilinear Differential Inclusions in Banach Spaces* (2001), Walter de Gruyter & Co., Berlin.

[94] L.V. Kantorovich and G.P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1964.

[95] M. Kečkemétyová, *On the structure of the set of solutions of nonlinear boundary value problems for ODEs on unbounded intervals*, Math. Slovaca **53** (3) (2006), 333–347.

[96] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.

[97] H. Kneser, *Uber die Lösungen eine system gewöhnlicher differential Gleichungen,das der Lipschitzschen Bedingung nicht genügt*, Akad. Wiss. Phys. Math. Kl. **4** (1923), 171–174.

[98] V.B. Kolmanovskiĭ and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[99] V.B. Kolmanovskiĭ and A.R. Nosov, *Stability of Functional Dfferential Equations*, Mathematics in Science and Engineering, vol. 180, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1986.

[100] M.A. Krasnosel'skiĭ, *Translation Operator along the Trajectories of Differential Equations*, Nauka, Moscow, 1966. (in Russian)

[101] E. Kruger-Thiemr, *Formal theory of drug dosage regiments* 1, J. Theoret. Biol. **13** (1966), 212–235.

[102] E. Kruger-Thiemr, *Formal theory of drug dosage regiments* 2, J. Theoret. Biol. **23** (1969), 169–190.

[103] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[104] A. Lasota and Z. Opial, *An application of the Kakutani–Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.

[105] J.M. Lasry and R. Robert, *Analyse non linéaire multivoque*, Publ. Centre de Recherche de Mathématique de la Décision, vol. 7611, Université de Dauphine, Paris, pp. 1–190.

[106] P.H. Leslie, *Some further notes on the use of matrices in population mathematics*, Biometrika **35** (1948), 213–245.

[107] G. Marino, P. Pielramala and L. Muglia, *Impulsive neutral integrodifferential equations on unbounded intervals*, Mediterr. J. Math. **1** (2004), 3–42.

[108] R.E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Math., vol. 183, Springer–Verlag, New York, 1998.

[109] X. Meng, J. Jiao and L. Chen, *The dynamics of an age structured predator-prey model with disturbing pulse and time delays*, Nonlinear Anal. **9** (2008), 547–561.

[110] V.D. Milman and A.A. Myshkis, *On the stability of motion in the presence of impulses*, Siberian Math. J. **1** (1960), 233–237. (in Russian)

[111] J. Musielak, *Introduction to Functional Analysis*, PWN, Warszawa, 1976. (in Polish)

[112] J.J. Nieto, *Impulsive resonance periodic problems of first order*, Appl. Math. Lett. **15** (2002), 489–493.

[113] J.J. Nieto, *Periodic boundary value problems for first-order impulsive ordinary differential equations*, Nonlinear Anal. **51** (2002), 1223–1232.

[114] A. Ouahab, *Existence and uniqueness results for impulsive functional differential equations with scalar multiple delay and infinite delay*, Nonlinear Anal. **67** (2006), 1027–1041.

[115] A. Ouahab, *Existence and uniqueness results for impulsive functional differential equations with scalar multiple delay and infinite delay*, Nonlinear Anal. **67** (2007), 1027–1041.

[116] A. Ouahab, *Some Contributions in Impulsives Differential Equations and Inclusions with Fixed and Variable Times*, PhD Dissertation, University of Sidi-Bel-Abbès, Algeria, 2006.

[117] S.G. Pandit and S.G. Deo, *Differential Systems Involving Impulses*, Lecture Notes in Mathematics, vol. 954, Springer–Verlag, 1982.

[118] N.S. Papageorgiou, *A continuous version of the relaxation theorem for nonlinear evolution inclusions*, Kodai Math. J. **18** (1995), 169–186.

[119] N.S. Papageorgiou and S.Th. Kyritsi-Yiallourou, *Handbook of Applied Analysis*, Adv. in Mech. and Math., vol. 19, Springer, 2009.

[120] N.S. Papageorgiou and N. Yannakakis, *Nonlinear parametric evolution inclusions*, Math. Nachr. **233/234** (2002), 201–219.

[121] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer–Verlag, New York, 1983.

[122] G. Peano, *Démonstration de l'intégrabilité des équations différentielles ordinaires*, Mat. Ann. **37** (1890), 182–238.

[123] N.A. Perestyuk, V.A. Plotnikov, A.M. Samoĭlenko and N.V. Skripnik, *Differential Equations with Impulse Effects Multivalued Right-hand Sides with Discontinuities*, Walter de Gruyter & Co., Berlin, 2011.

[124] T. Pruszko, *Some applications of the topological degree theory to multi-valued boundary value problems*, Dissertationes Math. **230** (1984), 1–50.

[125] B. Ricceri, *Une propriété topologique de l'ensemble des points fixe d'une contraction multivoque à valeurs convexes*, Atti Accad. Naz. Linei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. (1987), 283–286.

[126] Y.V. Rogovchenko, *Impulsive evolution systems: Main results and new trends*, Dynam. Contin. Discrete Impuls. Systems **3** (1997), 57–88.

[127] A.M. Samoĭlenko and N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[128] G.V. Smirnov, *Introduction to the Theory of Differential Inclusions*, Graduate Studies in Mathematics, vol. 41, American Mathematical Society, Providence, 2002.

[129] X. Song and H. Guo, *Extinction and permanence of a kind of pest-predator models impulsive effect and infinite delay*, J. Korean Math. Soc. **44** (2007), 3270–342.

[130] G.T. Stamov and I.M. Stamova, *Second method of Lyapunov and existence of integral manifolds for impulsive differential equations*, SUT J. Math. **32** (1996), 101–107.

[131] I. Stamova, *Stability Analysis of Impulsive Functional Differential Equations*, Walter de Gruyter & Co., Berlin, 2009.

[132] S. Sufla, *Sets of fixed points nonlinear mappings in function spaces*, Funkcial. Ekvac. **22** (1979), 121–126.

[133] J.M.A. Toledano, T.D. Benavides and G.L. Azedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser, Basel, 1997.

[134] A.A. Tolstonogov, *Differential Inclusions in Banach Spaces*, Kluwer, Dordrecht, 2000.

[135] A.A. Tolstonogov, *Approximation of attainable sets of an evolution inclusion of subdifferential type*, Sibirsk. Mat. Zh. **44** (2003), 883–904.

[136] A.A. Tolstonogov, *Properties of attainable sets of evolution inclusions and control systems of subdifferential type*, Sibirsk. Mat. Zh. **45** (2004), 920–945.

[137] Q. Wang, Z. Wang, M. Ding and H. Zhang, *Periodicity and attractivity of a ratio-dependent Leslie system with impulses*, J. Math. Anal. Appl. **376** (2011), 212–220.

[138] P. Weng, *Global existence in integrable space for impulsive FDE with P-Delay*, Dynam. Discrete Impuls. Systems **9** (2002), 321–337.

[139] B. Yan, *On L^p_{loc}-solutions of nonlinear impulsive Volterra integral equations in Banach spaces*, SUT J. Math. **33** (1997), 121–137.

[140] B. Yan, *The existence of positive solutions of nonlinear impulsive Fredholm integral equations in Banach spaces*, Dynam. Contin. Discrete Impuls. Systems **6** (1999), 289–300.

[141] T. Yoshizawa, *Stability Theorey by Liapunov's Second Method*, The Mathematical Society of Japan, Tokyo, 1966.

[142] K. Yosida, *Functional Analysis*, 6th ed., Springer–Verlag, Berlin, 1980.

[143] D. Yujun, *Periodic boundary value problems for functional differential equations with impulses*, J. Math. Anal. Appl. **210** (1997), 170–181.

[144] D. Yujun and Z. Erxin, *An application of coincidence degree continuation theorem in existence of solutions of impulsive differential equations*, J. Math. Anal. Appl. **197** (1996), 875–889.

[145] G. Vidossich, *On the structure of the set of solutions of nonlinear equations*, J. Math. Anal. Appl. **34** (1971), 602–617.

[146] G. Vidossich, *A fixed point theorem for function spaces*, J. Math. Anal. Appl. **36** (1971), 581–587.

[147] D. Wagner, *Survey of measurable selection theorems*, SIAM J. Control Optim. **15** (1977), 859–903.

[148] Q.J. Zhu, *On the solution set of differential inclusions in Banach space*, J. Differential Equations **93** (2) (1991), 213–237.