Juliusz Schauder Center for Nonlinear Studies
Nicolaus Copernicus University

FIXED POINTS OF MULTIVALUED WEIGHTED MAPS

Robert Skiba

Faculty of Mathematics and Computer Science
Nicolaus Copernicus University

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Recenzenci rozprawy doktorskiej:

prof. dr hab. Wojciech Kryszewski
dr hab. Sławomir Nowak, prof. Uniwersytetu Warszawskiego

Centrum Badań Nieliniowych im. Juliusza Schaudera
Uniwersytet Mikołaja Kopernika
ul. Chopina 12/18, 87-100 Toruń

Redakcja: tel. +48 (56) 611 34 28, faks: +48 (56) 622 89 79

e-mail: tmna@ncu.pl
http://www.cbn.ncu.pl

Skad komputerowy w TEX-u: Jolanta Szelatyńska

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PREFACE

The present work is the revised version of a part of Ph.D. thesis of the autor written at the Faculty of Mathematics and Computers Science of the Nicolaus Copernicus University in Toruń, Poland.

The paper, divided into 5 chapters, is devoted to the study of properties of two special classes of multivalued maps. The first, which we will call weighted carriers, was discovered by Gabriele Darbo in 1950. The second class was introduced by him in 1957 (and independently by R. Jerrard with a slightly different formulation) under the name of weighted maps.

The main goal is to give a self-contained presentation of the theory of multivalued weighted maps. To be precise, we present the most important results and methods which concern the fixed point theory of multivalued weighted mappings.

The first chapter is a brief exposition of some basic facts on the theory of retracts, the Čech homology theory, general properties of multivalued maps, algebraic limits and the Lefschetz number.

In the second chapter we study general properties of multivalued weighted maps. In particular, we briefly describe the construction of Darbo homology theory, which will be frequently used in this work. We also present how the cross product can be defined in terms of Darbo homology. Next, we apply the Darbo homology to the theory of Lefschetz number and establish the Lefschetz fixed point theorem for weighted maps. Moreover, we construct a topological degree theory for such a class of maps. Additionally, the topological essentiality is introduced and studied. The last part of this chapter contains a detailed investigation of the problem of the existence of extensions of weighted maps with values belonging to the so-called w-LC^n-space.

The third chapter is devoted to a detailed description of the basic properties of weighted carriers.

In the fourth chapter we study sets having various w-UV-properties. We shall compare w-UV-notions with acyclicity with respect to the Čech homology. In particular, we give a necessary and sufficient condition for a given compact subset

A of an ANR X to be k-acyclic in the sense of Čech homology with the coefficients in the field of rational numbers $\mathbb{Q}$. Furthermore, we deal with the existence of graph-approximations. It should be noted that in general an upper semicontinuous multivalued map has no single-valued continuous approximation and therefore we study the finite-valued version of this problem. More precisely, using the approximation techniques developed in [3], [10] and [28], we show that a weighted carrier defined on a compact ANR with values having w-UV^{ω}-property can be approximated in graph by weighted maps. In particular, we prove that any upper semicontinuous multivalued map with compact and acyclic values (with respect to the Čech homology with rational coefficients) from a compact ANR to an ANR admits arbitrarily close weighted graph-approximations. The above graph-approximation result allows us to define the induced homomorphism (in the Darbo homology) for weighted carriers. Next, some generalizations of the Lefschetz fixed point theorem for weighted carriers from a paper of G. Conti and J. Pejsachowicz [10] are presented. In the last section of this chapter, using the method developed in the work of R. Bader and W. Kryszewski [4], we construct the topological degree for compositions of weighted carriers.

In the fifth chapter we show that the Nielsen fixed point theory cannot be extended to the multivalued weighted case.

The main results of this paper are the following:

- The basic properties of weighted carriers;
- The description of w-UV-notions in terms of homology;
- The results concerning approximability of weighted carriers;
- The Lefschetz fixed point theorems;
- The topological degree.

Some applications of the results of this work in the theory of differential equations and inclusions will be given in a forthcoming paper of the present author.

The author wishes to express his thanks to Professor L. Górniewicz for encouragement to take up the effort of writing this work. I am greatly indebted to Professor W. Kryszewski and Professor J. Pejsachowicz for valuable questions, suggestions and discussions. Many thanks to Doctor B. Klemp-Dyczek who verified my English and suggested further improvements.

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CHAPTER 1

TOPOLOGICAL BACKGROUND

1.1. Preliminaries

In what follows, by a space we understand a metric space. If (X, d_X) is a metric space, $\varepsilon > 0$ and $A \subset X$, then by the ε-neighbourhood of A in X we mean the set $O_\varepsilon(A) := \{x \in X \mid d_X(x, A) < \varepsilon\}$, where $d_X(x, A) = \inf_{a\in A} d_X(x, a)$ is the distance of a point $x \in X$ from the set A. Moreover, $D(A, x) := \{x \in X \mid d_X(x, A) \leqslant \varepsilon\}$. In particular, $B(x, \varepsilon) := O_\varepsilon(x) = \{y \in X \mid d_X(y, x) < \varepsilon\}$ (resp. $D(x, \varepsilon) = \{y \in X \mid d_X(y, x) \leqslant \varepsilon\}$) is the open ball (closed disk) of radius ε centered at $x \in X$.

Given a space X, by a piece of X we shall mean any open and closed subset of X. Throughout this paper, $\#X$ denotes a power of a set X. A pair of spaces is understood to be a pair (X, A) where X is a space and A is a subset of X.

If $A \subset X$, then $\overline{A}$, $\operatorname{int} A$ and ∂A denote the closure, the interior and the boundary of A, respectively. By D^{n+1} we shall understand the unit closed disk in $\mathbb{R}^{n+1}$ and $\partial D^{n+1} = S^n$. By S^n_+ and S^n_- we mean the closed northern and southern hemispheres of S^n, respectively, $n \geqslant 1$. Then $S^{n-1} = S^n_+ \cap S^n_-$. Recall that we can think of S^n as the one point compactyfication of $\mathbb{R}^n$, in other words, $S^n = \mathbb{R}^n \cup \{\infty\}$.

By $\check{H}_*(X, G)$ we denote the Čech homology (graded) of a space X with coefficients in a group G ([17]). A space X will be called positively acyclic (resp. k-acyclic, $k \geqslant 1$) if $\check{H}_n(X, \mathbb{Q}) = 0$ for $n \geqslant 1$ (resp. $\check{H}_i(X, \mathbb{Q}) = 0$ for $1 \leqslant i \leqslant k$).

Given a space X, by $\dim X$ we shall denote the covering dimension of X. For more information on the covering dimension we refer to [18]. If $A \subset X$ and $\mathbf{U}$ is a collection of sets in X, then $\operatorname{st}(A, \mathbf{U}) := \bigcup\{U \in \mathbf{U} \mid U \cap A \neq \emptyset\}$ is the star of A with respect to $\mathbf{U}$.

Throughout this paper, on the Cartesian product $X \times Y$ of two metric spaces (X, d_X) and (Y, d_Y) we shall consider the following metric:

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

1.2. ARs and ANRs

First, we are going to recall the notion of an n-simplex in a vector space E.

By an n-simplex in E we shall understand a geometrically independent subset of E having precisely $n+1$ points ([1]). Simplexes will be denoted by Greek letters, i.e. σ, μ, τ, etc. (sometimes such a simplex σ will be denoted by the symbol $[p_0,\dots,p_n]$, where $p_i \in E$ for $0 \leqslant i \leqslant n$). If σ and τ are simplexes and $\sigma \subset \tau$ then σ is called a face of τ. A geometric simplex $|\sigma|$ is the convex hull of a simplex σ. The union of all proper faces of σ is called the boundary of σ and it is denoted by $\partial\sigma$. By the geometric boundary of σ we understand a set $|\partial\sigma| := \cup\{|\tau| \mid \tau \subset \sigma,\ \tau \neq \sigma\}$ and by $\langle\sigma\rangle$ we mean the following set $\langle\sigma\rangle := |\sigma| \setminus |\partial\sigma|$. Here and in what follows, we shall denote by Δ_n the n-dimensional standard simplex.

An abstract simplicial complex is a collection $\mathbb{S}$ of non-empty and finite sets such that if $s \in \mathbb{S}$ and $\emptyset \neq s' \subset s$ then $s' \in \mathbb{S}$. The elements of the set $S = \bigcup_{s\in\mathbb{S}} s$ are called the vertices of the abstract simplicial complex. Consider E as the free real module with basis S. Observe now that any $s \in \mathbb{S}$ is geometrically independent in E and, consequently, any element $s \in \mathbb{S}$ is an n-simplex, where $n := (\#s)-1$. So, from this moment on, we will denote the elements of $\mathbb{S}$ by Greek letters. Moreover, for every $\sigma_1, \sigma_2 \in \mathbb{S}$ we have $|\sigma_1| \cap |\sigma_2| = |\sigma_1 \cap \sigma_2|$. Denote by $|\mathbb{S}|$ the set $|\mathbb{S}| := \bigcup_{\sigma\in\mathbb{S}} |\sigma|$. Then the set $|\mathbb{S}|$ together with the Whitehead topology is called a geometric realization of $\mathbb{S}$ (see [6]). If v is a vertex of $\mathbb{S}$, then $\mathrm{st}(v, |\mathbb{S}|) := \bigcup\{\langle\tau\rangle \mid v \in |\tau|, \tau \in \mathbb{S}\}$ is the star of v. For each vertex v of $\mathbb{S}$ every $\mathrm{st}(v, |\mathbb{S}|)$ is open in $|\mathbb{S}|$.

Now, we recall the notion of a retract. A subset $A \subset X$ is called the retract of X if there exists a continuous map $r\colon X \to A$ (called a retraction) such that $r(x) = x$ for all $x \in A$. In addition, we shall say that A is a neighbourhood retract of X if there exists an open subset $U \subset X$ such that $A \subset U$ and A is a retract of U. Following K. Borsuk we shall introduce the notion of an absolute retract (AR) and the notion of absolute neighbourhood retract (ANR). We shall also use the notion of an embedding. Namely, by an embedding of a space X into Y we shall understand any homeomorphism $h\colon X \to Y$ from X onto its image such that $h(X)$ is a closed subset of Y.

Definition 1.2.1. We shall say that $X \in \mathrm{AR}$ ($X \in \mathrm{ANR}$) if and only if for any space Y and for any embedding $h\colon X \to Y$ the set $h(X)$ is a retract of Y ($h(X)$ is a neighbourhood retract of Y).

([1]) Let E be a vector space. Then:

(a) the vectors $e_1,\dots,e_n$ are said to be geometrically independent if for all elements $t_1,\dots,t_n \in \mathbb{R}$ with $\sum_{i=1}^n t_i = 0$ and $\sum_{i=1}^n t_i \cdot e_i = 0 \in E$ we have $t_1 = t_2 = \ldots = t_n = 0$;

(b) a subset $s \subset E$ is geometrically independent if and only if every its finite subset is.

A single-valued continuous map $f: X \to Y$ is said to be an r-map if there exists a single-valued continuous map $g: Y \to X$ such that $f \circ g = \mathrm{id}_Y$, where $\mathrm{id}_Y: Y \to Y$ is the identity map. If there exists an r-map $f: X \to Y$ then the space Y is called an r-image of the space X.

Theorem 1.2.2 ([23]). *$X \in \mathrm{AR}$ if and only if X is an r-image of some normed space E.*

Theorem 1.2.3 ([23]). *$X \in \mathrm{ANR}$ if and only if X is an r-image of some open subset U of a normed space E.*

Theorem 1.2.4 ([7]). *Suppose that the space X is the union of two closed subsets X_1 and X_2 and let $X_0 = X_1 \cap X_2$. Then:*

(a) *If $X_0, X_1, X_2 \in \mathrm{AR}$, then $X \in \mathrm{AR}$.*
(b) *If $X_0, X_1, X_2 \in \mathrm{ANR}$, then $X \in \mathrm{ANR}$.*
(c) *If $X, X_0 \in \mathrm{AR}$, then $X_1, X_2 \in \mathrm{AR}$.*
(d) *If $X, X_0 \in \mathrm{ANR}$, then $X_1, X_2 \in \mathrm{ANR}$.*

Lemma 1.2.5. *Let $Y \in \mathrm{ANR}$, X be an arbitrary space and $A \subset X$ be a closed subset. Assume that $f, g: X \to Y$ are such that there is a homotopy $h: A \times [0,1] \to Y$ with $h(x,0) = f(x), h(x,1) = g(x)$ for every $x \in A$. Then there exists a neighbourhood U of A in X and a homotopy $H: U \times [0,1] \to Y$ such that $H|A \times [0,1] = h$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for every $x \in U$.*

Proof. Let $B = (A \times [0,1]) \cup (X \times \{0\}) \cup (X \times \{1\})$ and let $k: B \to Y$ be defined by

$$k(x,t) = \begin{cases} h(x,t) & \text{if } x \in A, \\ f(x) & \text{if } t = 0, \\ g(x) & \text{if } t = 1. \end{cases}$$

Since Y is an ANR, k has an extension $g: N \to Y$ over a neighbourhood N of B in $X \times [0,1]$. Because of the compactness of $[0,1]$, there is an open neighbourhood U of A such that $U \times [0,1] \subset N$. Finally, define $H: U \times [0,1] \to Y$ by $H := g|U \times [0,1]$, which completes the proof. □

Lemma 1.2.6 ([7]). *Let Q^ω be the Hilbert cube and let $A \subset Q^\omega$ be a compact subset. Then there exists a sequence $\{Z_i\}_{i=1}^\infty$ of compact ANRs such that*

$$Z_1 = Q^\omega, \qquad Z_{i+1} \subset \operatorname{int} Z_i, \qquad A = \bigcap_{i=1}^\infty Z_i.$$

In the sequel we shall make repeatedly use of the following result due to Girolo.

Lemma 1.2.7 (Girolo, [21]). *If K is a compact subset of an open set U in a normed space E, then there exists a compact ANR X such that $K \subset X \subset U$.*

From the above lemma we deduce that a family of all compact ANRs contained in U is cofinal in the family of all compact subsets of U.

Finally, let us recall the following important embedding theorem.

Theorem 1.2.8 (Arens–Eells Embedding Theorem)**.** *Let X be a metric sapce. Then there exists a normed space E and an isometry $\Theta: X \to E$ such that $\Theta(X)$ is a closed subset of E.*

1.3. Multivalued mappings — general properties

By a map we shall mean a single-valued continuous transformation of spaces and by a multivalued map φ of a space X into a space Y we mean a correspondence which associates to every $x \in X$ a non-empty and compact subset $\varphi(x) \subset Y$, and we write $\varphi: X \multimap Y$. In the sequel, the symbol $f: X \to Y$ is reserved for single-valued mappings. Moreover, we associate with φ the graph Γ_φ of φ by putting:

$$\Gamma_\varphi := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}.$$

Definition 1.3.1. A multivalued map $\varphi: X \multimap Y$ is *upper semicontinuous* (u.s.c.) if for any open subset U of Y the set $\varphi^{-1}(U) := \{x \in X \mid \varphi(x) \subset U\}$ is open in X.

Proposition 1.3.2. *If $\varphi: X \multimap Y$ is* u.s.c. *then the graph Γ_φ is a closed subset of $X \times Y$.*

Proposition 1.3.3. *Let $\varphi: X \multimap Y$ be a multivalued map. The map φ is upper semicontinuous and has compact values if and only if, given a sequence $(x_n, y_n) \in \Gamma_\varphi$, if $x_n \to x_0$, then there exists a subsequence y_{n_k} such that $y_{n_k} \to y_0 \in \varphi(x_0)$.*

Definition 1.3.4. A multivalued map $\varphi: X \multimap Y$ is *lower semicontinuous* (l.s.c.) if for any open subset U of Y the set $\varphi_+^{-1}(U) := \{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ is open in X.

Proposition 1.3.5. *Let $\varphi: X \multimap Y$ be a multivalued map. The map φ is lower semicontinuous at $x_0 \in X$ if and only if, for any $y_0 \in \varphi(x_0)$ and a sequence $x_n \to x_0$, there exists a sequence $y_n \to y_0$ such that $y_n \in \varphi(x_n)$ for all $n \in \mathbb{N}$.*

Definition 1.3.6. We say that $\varphi: X \multimap Y$ is compact if the closure $\overline{\varphi(X)}$ of $\varphi(X)$ is compact.

Definition 1.3.7. We say that $\varphi: X \multimap Y$ is *continuous provided* if is both upper semicontinuous and lower semicontinuous.

Given pairs (X, A), (Y, B), by a map $\varphi: (X, A) \multimap (Y, B)$ we shall mean a multivalued map $\varphi: X \multimap Y$ satisfying the condition $\varphi(A) \subset B$. If $\varphi: X \multimap Y$

and $A \subset X$, then the composition of the inclusion $i: A \to X$ and $\varphi: X \multimap Y$ is denoted by $\varphi|A: A \multimap Y$.

If $\varphi: (X, A) \multimap (Y, B)$ is a multivalued map, then $\varphi_X: X \multimap Y$ and $\varphi_A: A \multimap B$ denote the evident multivalued maps determined by φ.

Remark 1.3.8. The basic terminology concerning multivalued maps used throughout this work is taken from the book by Górniewicz [23]. Moreover, for more information about multivalued maps we refer the reader to [1] and [23].

1.4. Direct and inverse limits

In this section we recall the concepts of direct limit and inverse limit which will be needed in the sequel.

Definition 1.4.1 ([32])**.** Let $\{M_i\}_{i \in I}$ be a family of R-modules indexed by the directed set I. Assume that for $i \leqslant j$ we are given a homomorphism $\pi_{j,i}: M_i \to M_j$ such that

(a) $\pi_{i,i} = \mathrm{id}$,
(b) $\pi_{j,i} = \pi_{j,k} \circ \pi_{k,i}$ for $i \leqslant k \leqslant j$.

Then the family $S := \{M_i, \pi_{j,i} \mid i, j \in I\}$ is called a direct system of R-modules. Given such a direct system S, the direct limit of this system is an R-module M_∞ together with a family of homomorphisms $\pi_i: M_i \to M_\infty$ indexed by I such that

(c) $\pi_j \circ \pi_{j,i} = \pi_i$ for $i \leqslant j$

and such that this collection is universal with respect to the following property. For any R-module N and any family of homomorphisms $f_i: M_\infty \to N$ satisfying

(d) $f_j \circ \pi_{j,i} = f_i$ for $i \leqslant j$

there is a unique homomorphism $f: M_\infty \to N$ such that

(e) $f_i = f \circ \pi_i$ for all $i \in I$.

Proposition 1.4.2. *The direct limit of $S = \{M_i, \pi_{j,i} \mid i, j \in I\}$ always exists.*

Throughout this book, an R-module M_∞ will be denoted by the symbol

$$\varinjlim_{i \in I} M_i.$$

Moreover, the unique homomorphism f will be denoted by the symbol

$$\varinjlim_{i \in I} f_i.$$

Remark 1.4.3. If we reverse the arrows in the above definition, then we get the notion of an inverse system and limit. Moreover, the analogue of Proposition 1.4.2 for inverse systems is also true.

1.5. The Čech homology functor

By $\check{H}_*(X,G)$ we denote the Čech homology (graded) of a space X with coefficients in a group G ([17]). A space X will be called positively acyclic (resp. k-acyclic, $k \geqslant 1$) if $\check{H}_n(X,\mathbb{Q}) = 0$ for $n \geqslant 1$ (resp. $\check{H}_i(X,\mathbb{Q}) = 0$ for $1 \leqslant i \leqslant k$). The following nontrivial theorem will be useful for our present purposes.

Lemma 1.5.1 ([31]). *Let X be a compact space and let $X_1 \supset X_2 \supset \dots$ be a descending sequence of compact spaces with $X = \bigcap_{i=1}^{\infty} X_i$. Then*

$$\check{H}_*(X,\mathbb{Q}) = \varprojlim \check{H}_*(X_i,\mathbb{Q}).$$

Lemma 1.5.2 ([31]). *Let $(X_1,A_1) \supset (X_2,A_2) \supset \dots$ be a descending sequence of compact pairs with $(X,A) = \bigcap_{i=1}^{\infty}(X_i,A_i)$. In addition, fix n and let $\{z_1,\dots,z_s\}$ be a linearly independent set in $\check{H}_n(X,A;\mathbb{Q})$. Then there exists an index k_n such that:*

(a) *$\{j_k(z_1),\dots,j_k(z_s)\}$ is linearly independent in $\check{H}_n(X_k,A_k)$ for all $k \geqslant k_n$, where $j_k\colon \check{H}_n(X,A;\mathbb{Q}) \to \check{H}_n(X_k,A_k;\mathbb{Q})$ are induced by the inclusion $(X,A) \to (X_k,A_k)$,*
(b) *in particular, for all $k \geqslant k_n$, $j_k\colon \check{H}_n(X,A;\mathbb{Q}) \to \check{H}_n(X_k,A_k;\mathbb{Q})$ is a monomorphism on the space $E_s := \langle z_1,\dots,z_s\rangle$ generated by $\{z_1,\dots,z_s\}$.*

Theorem 1.5.3 ([57]). *There exists a transformation T from the arbitrary homology theory with compact supports over a coefficient group G to the Čech homology over the same coefficient group G such that*

(a) *to each metric space X it assigns a homomorphism*

$$T(X)\colon H(X,G) \to \check{H}(X,G),$$

(b) *for any single-valued map $f\colon X \to Y$ the following diagram*

$$\begin{tikzcd} H(X,G) \arrow[r, "f_*"] \arrow[d, "T(X)"'] & H(Y,G) \arrow[d, "T(Y)"] \\ \check{H}(X,G) \arrow[r, "f_*"'] & \check{H}(Y,G) \end{tikzcd}$$

commutes. Moreover, if X is a metric absolute neighbourhood retract, then $T(X)$: $H(X,G) \to \check{H}(X,G)$ is an isomorphism.

1.6. The Lefschetz number

In this section all the vector spaces are taken over $\mathbb{Q}$ and all maps between such spaces are linear. First we shall recall the notion of an ordinary trace. Let

$L\colon E \to E$ be an endomorphism of a finite-dimensional vector space E and let $e_1, \dots, e_n$ be a basis for E. Then for every e_i we can write

$$L(e_i) = \sum_{j=1}^{n} a_{ij} e_j.$$

Hence we have the matrix $A = [a_{ij}]_{i,j=1}^{n}$ of L. The trace of A is given by the formula:

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}.$$

By a trace of an endomorphism of a finite-dimensional vector space $L\colon E \to E$, written $\operatorname{tr}(L)$, we shall understand the trace of the matrix of f with respect to some basis for E. The above definition is correct, i.e. it does not depend on the choice of the basis for E. Now we shall collect the important and well-known properties of the defined trace $\operatorname{tr}(L)$.

Proposition 1.6.1. *Assume that in the category of finite-dimensional vector spaces the following diagram commutes*

$$\begin{tikzcd} E' \arrow[r, "u"] \arrow[d, "L'"'] & E'' \arrow[d, "L''"] \arrow[dl, "v"'] \\ E' \arrow[r, "u"'] & E'' \end{tikzcd}$$

Then $\operatorname{tr}(L') = \operatorname{tr}(L'')$, *or equivalently,* $\operatorname{tr}(vu) = \operatorname{tr}(uv)$.

Proposition 1.6.2. *Given a commutative diagram of finite-dimensional vector spaces with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r] \arrow[d, "L'"'] & E \arrow[r] \arrow[d, "L"] & E'' \arrow[r] \arrow[d, "L''"] & 0 \\ 0 \arrow[r] & E' \arrow[r] & E \arrow[r] & E'' \arrow[r] & 0 \end{tikzcd}$$

we have $\operatorname{tr}(L) = \operatorname{tr}(L') + \operatorname{tr}(L'')$.

Definition 1.6.3. A graded vector space $E = \{E_n\}$ is of finite type provided $E_n = 0$ for almost all $n \in \mathbb{N}$ and $\dim E_n < \infty$ for all $n \in \mathbb{N}$.

Definition 1.6.4. Let $E = \{E_n\}_{n \geq 0}$ be a graded vector space of finite type and let $L = \{L_n\}_{n \geq 0}$ be an endomorphism of degree zero (i.e. $L_n\colon E_n \to E_n$) of E. Then the Lefschetz number $\lambda(L)$ of L is defined by

$$\lambda(L) = \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}(L_n).$$

It is well-known that one can generalize the Lefschetz number. To begin with, we have to generalize the notion of the trace. Let $L\colon E \to E$ be an endomorphism of an arbitrary vector space E. By $L^{(n)}: E \to E$ we denote the n-th iterate of L. Let us note that the kernels

$$\operatorname{Ker} L \subset \operatorname{Ker} L^{(2)} \subset \ldots \subset \operatorname{Ker} L^{(n)} \subset \ldots$$

form an increasing sequence of subspaces of E. Next, let us define the set $\mathcal{N}(L)$ by the formula:

$$\mathcal{N}(L) = \{x \in E \mid L^{(n)}(x) = 0 \text{ for some } n\}.$$

It is clear that

$$\mathcal{N}(L) = \bigcup_{n \geq 1} \operatorname{Ker} L^{(n)}.$$

Let us observe that L maps $\mathcal{N}(L)$ into itself and, consequently, we get the induced endomorphism $\widetilde{L}: \widetilde{E} \to \widetilde{E}$, where $\widetilde{E} = E/\mathcal{N}(L)$ is the factor space. It is easy to see that $\widetilde{L}: \widetilde{E} \to \widetilde{E}$ is a monomorphism. Now we are able to define the Leray trace.

Definition 1.6.5. Let $L: E \to E$ be an endomorphism of a vector space E. Assume that $\dim \widetilde{E} < \infty$. Then we put $\operatorname{Tr}(L) = \operatorname{tr}(\widetilde{L})$ and $\operatorname{Tr}(L)$ is called the *generalized trace* of L.

Note that Propositions 1.6.1 and 1.6.2 also hold for the generalized trace. Moreover, one can show the following:

Proposition 1.6.6. *Let $L: E \to E$ be an endomorphism. If $\dim E < \infty$, then $\operatorname{Tr}(L) = \operatorname{tr}(L)$.*

We are now ready to define the generalized Lefschetz number. Let $L = \{L_n\}_{n \geq 0}$ be an endomorphism of degree zero of a graded vector space $E = \{E_n\}_{n \geq 0}$.

Definition 1.6.7. We shall say that L is a Leray endomorphism provided that the graded vector space $\widetilde{E} = \{\widetilde{E}_n\}_{n \geq 0}$ is of finite type. If L is the Leray endomorphism, then we can define the generalized Lefschetz number $\Lambda(L)$ of L by putting:

$$\Lambda(L) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Tr}(L_n).$$

Now from Proposition 1.6.6 we have the following:

Proposition 1.6.8. *Let $L: E \to E$ be an endomorphism of degree zero. If E is a graded vector space of finite type then*

$$\Lambda(L) = \lambda(L).$$

The next two statements follow immediately from Propositions 1.6.1 and 1.6.2, respectively.

Proposition 1.6.9. *Assume that in the category of graded vector spaces the following diagram commutes:*

$$\begin{tikzcd} E' \arrow[r, "u"] \arrow[d, "L'"'] & E'' \arrow[d, "L''"] \arrow[dl, "v"'] \\ E' \arrow[r, "u"'] & E'' \end{tikzcd}$$

Then if one of the maps L', L is a Leray endomorphism, then so is the other and in that case

$$\Lambda(v \circ u) = \Lambda(L') = \Lambda(L'') = \Lambda(u \circ v).$$

Proposition 1.6.10. *Let*

$$\begin{tikzcd} \cdots \arrow[r] & E'_n \arrow[r] \arrow[d, "L'_n"'] & E_n \arrow[r] \arrow[d, "L_n"'] & E''_n \arrow[r] \arrow[d, "L''_n"] & E'_{n-1} \arrow[r] \arrow[d, "L'_{n-1}"] & \cdots \\ \cdots \arrow[r] & E'_n \arrow[r] & E_n \arrow[r] & E''_n \arrow[r] & E'_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

be a commutative diagram of vector spaces in which the rows are exact. If two of the following endomorphisms

$$L = \{L_n\}_{n\geq 0}, \quad L' = \{L'_n\}_{n\geq 0}, \quad L'' = \{L''_n\}_{n\geq 0}$$

are Leray endomorphisms, then so is the third, and in that case we have:

$$\Lambda(L'') + \Lambda(L') = \Lambda(L).$$

Finally, we recall the notion of a weakly nilpotent endomorphism.

Definition 1.6.11. A linear map $L\colon E \to E$ of a vector space E into itself is called *weakly nilpotent* provided for every $x \in X$ there exists a natural number $n = n_x$ such that $L^{n_x}(x) = 0$.

From the above definition we deduce that $L\colon E \to E$ is weakly nilpotent if and only if $\mathcal{N}(L) = E$.

Proposition 1.6.12. *If $L\colon E \to E$ is weakly nilpotent then* $\mathrm{Tr}(L)$ *is well-defined and* $\mathrm{Tr}(L) = 0$.

We say that an endomorphism $L = \{L_n\}_{n\geq 0}\colon E \to E$ is weakly nilpotent if and only if $L_n\colon E_n \to E_n$ is weakly nilpotent for every n, where $E = \{E_n\}_{n\geq 0}$ is a graded vector space. From Proposition 1.6.12 we obtain the following proposition.

Proposition 1.6.13. *Any weakly nilpotent endomorphism* $L: E \to E$ *of a graded vector space is a Leray endomorphism and* $\Lambda(L) = 0$.

Remark 1.6.14. Notice that all the proofs of the results presented in this section can be found in [23] or [31].

CHAPTER 2

w-MAPS

2.1. Definition and examples

The aim of this short preliminary section is to recall the notion of weighted maps. Next we present several examples of such maps.

Definition 2.1.1. A *weighted map* from X to Y with coefficients in $\mathbb{Q}$ (or simply a *w-map*) is a pair $\psi = (\sigma_\psi, w_\psi)$ satisfying the following conditions:

(a) $\sigma_\psi\colon X \multimap Y$ is a multivalued upper semicontinuous map such that $\sigma_\psi(x)$ is a finite subset of Y for all $x \in X$;

(b) $w_\psi\colon X \times Y \to \mathbb{Q}$ is a function with the following properties:

  (b1) $w_\psi(x, y) = 0$ for any $y \notin \sigma_\psi(x)$;

  (b2) for any open subset U of Y and $x \in X$ such that $\sigma_\psi(x) \cap \partial U = \emptyset$ there exists an open neighbourhood V of the point x such that:

$$\sum_{y\in U} w_\psi(x, y) = \sum_{y\in U} w_\psi(z, y),$$

  for every $z \in V$.

For simplicity of notation, we denote a multivalued weighted mapping from X to Y by $\psi\colon X \multimap Y$. Thus, by $\psi(x)$ we shall mean $\sigma_\psi(x)$ for all $x \in X$. The map σ_ψ from the above definition will be called the support of ψ and w_ψ the weight of ψ. The class of weighted maps was introduced in 1957 by G. Darbo and independently by R. Jerrard. Let us notice that our definition of weighted map is a slight modification of the one introduced by G. Darbo (and R. Jerrard), but all the results of [36], [39], [53]–[55] are also true for weighted maps defined above. Moreover, the above definition seems to be more convenient in our considerations.

Remark 2.1.2. Notice that in Definition 2.1.1 one can also consider any commutative ring R with identity instead of $\mathbb{Q}$. To simplify our exposition, we restrict ourselves to the case when $R = \mathbb{Q}$.

Now, we give some examples of weighted maps.

Example 2.1.3. Each continuous map $f\colon X \to Y$ can be considered as a weighted one by assigning the coefficient 1 to each $f(x)$.

Example 2.1.4. Let $\psi\colon X \multimap Y$ be a continuous map such that for all $x \in X$, $\psi(x)$ consists of 1 or exactly n points (with $n \geqslant 2$ fixed). A weight $w_\psi\colon X \times Y \to \mathbb{Q}$ can be defined by

$$w_\psi(x,y) = \begin{cases} 0 & \text{if } y \notin \psi(x), \\ n & \text{if } \{y\} = \psi(x), \\ 1 & \text{otherwise.} \end{cases}$$

It is not difficult to verify that $\psi = (\psi, w_\psi)$ is a weighted map.

Example 2.1.5. Let $f\colon X \to \mathrm{SP}^n Y$ be a continuous single-valued map and let $\Pi\colon \mathrm{SP}^n Y \multimap Y$ be a multivalued map which is defined by

$$\Pi(x_1^{k_1} \ldots x_s^{k_s}) = \{x_1, \ldots, x_s\},$$

where $\mathrm{SP}^n Y$ denotes the n-th symmetric product of Y and $x_1^{k_1} \ldots x_s^{k_s}$ denotes an equivalence class in $\mathrm{SP}^n Y$ ([2]). Then f induces a w-map $\varphi = (\sigma_\varphi, w_\varphi)$, where $\sigma_\varphi\colon X \multimap Y$ and $w_\varphi\colon X \times Y \to \mathbb{Q}$ are defined by

$$\sigma_\varphi(x) = \Pi \circ f(x)$$

and

$$w_\varphi(x,y) = \begin{cases} k_i & \text{if } y \in \sigma_\varphi(x), \\ 0 & \text{if } y \notin \sigma_\varphi(x). \end{cases}$$

Thus if $f\colon X \to \mathrm{SP}^n X$ is a single-valued map, then $\Pi \circ f\colon X \multimap X$ is a weighted map and the fixed point theorems for maps into symmetric products ([51]) are direct consequences of the corresponding fixed point theorems obtained for weighted maps.

Example 2.1.6. Further examples can be found in [53].

Example 2.1.7. Let $\varphi\colon [0,1] \multimap [0,1]$ be defined by

$$\varphi(x) = \begin{cases} \{1\} & \text{if } 0 \leqslant x < 1/2, \\ \{0,1\} & \text{if } x = 1/2, \\ \{0\} & \text{if } 1/2 < x \leqslant 1. \end{cases}$$

Of course, φ is upper semicontinuous without fixed points. One can show that if $w_\varphi\colon [0,1] \times [0,1] \to \mathbb{Q}$ is an arbitrary weight of φ, then $w_\varphi(x,y) = 0$ for all $x, y \in [0,1]$. For such a map φ there exists only trivial weight.

([2]) Let X be a metric space and let $n \geqslant 2$ be an integer. The n-th symmetric group S_n acts on the n-th cartesian product X^n by the formula $(s, (x_1, \ldots, x_n)) \mapsto (x_{s(1)}, \ldots, x_{s(n)})$ where $s \in S_n$. The n-th symmetric product $\mathrm{SP}^n X$ of X is the orbit space X^n / S_n. One can prove that $\mathrm{SP}^n X$ is also a metric space. For more details concerning the definition of the metric in $\mathrm{SP}^n X$, we refer for example to [51].

2.2. Elementary properties

In this section, some basic properties of multivalued weighted maps are presented.

Proposition 2.2.1. *If $\psi, \varphi \colon X \multimap Y$ are two w-maps, then*

$$\psi \cup \varphi = (\sigma_{\psi\cup\varphi}, w_{\psi\cup\varphi})$$

is also a w-map, where $\sigma_{\psi\cup\varphi}\colon X \multimap Y$ and $w_{\psi\cup\varphi}\colon X \times Y \to \mathbb{Q}$ are defined by

$$\begin{aligned} \sigma_{\psi\cup\varphi}(x) &= \sigma_\psi(x) \cup \sigma_\varphi(x), \\ w_{\psi\cup\varphi}(x,y) &= w_\psi(x,y) + w_\varphi(x,y), \end{aligned}$$

for all $x \in X$ and $y \in Y$.

Proposition 2.2.2. *If $\psi\colon X \multimap Y$ is a w-map and $\alpha \in \mathbb{Q}$, then $\alpha \cdot \psi = (\sigma_{\alpha\cdot\psi}, w_{\alpha\cdot\psi})$ is also a w-map, where $\sigma_{\alpha\cdot\psi}\colon X \multimap Y$ and $w_{\alpha\cdot\psi}\colon X \times Y \to \mathbb{Q}$ are defined as follows: $\sigma_{\alpha\cdot\psi}(x) = \sigma_\psi(x)$ and $w_{\alpha\cdot\psi}(x,y) = \alpha \cdot w_\psi(x,y)$ for all $x \in X$ and $y \in Y$.*

The proofs of Propositions 2.2.1 and 2.2.2 are straightforward.

Proposition 2.2.3. *If $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$ are w-maps, then $\psi \circ \varphi\colon X \multimap Z$ is a w-map, where its support $\sigma_{\varphi\circ\psi}$ is the composition of σ_φ and σ_ψ and a weight $w_{\psi\circ\varphi}\colon X \times Z \to \mathbb{Q}$ is defined by the formula:*

$$w_{\psi\circ\varphi}(x,z) = \sum_{y\in Y} w_\varphi(x,y) \cdot w_\psi(y,z),$$

for every $x \in X$ and $z \in Z$.

Proof. It is easy to see that the first condition of Definition 2.1.1 and the condition (b1) hold true for $\psi \circ \varphi$. Now, we shall prove that the condition (b2) of Definition 2.1.1 is also satisfied for $\psi \circ \varphi$. For this purpose, let us fix $x_0 \in X$ and let U be an open subset of Z such that $\psi \circ \varphi(x_0) \cap \partial U = \emptyset$. We have to show that there exists an open neighbourhood W_{x_0} of x_0 in X such that

$$\sum_{z\in U} w_{\psi\circ\varphi}(x_0, z) = \sum_{z\in U} w_{\psi\circ\varphi}(x', z), \tag{2.1}$$

for all $x' \in W_{x_0}$. We have

$$\begin{aligned} \sum_{z\in U} w_{\psi\circ\varphi}(x_0,z) &= \sum_{z\in U}\Bigg(\sum_{y\in Y} w_\varphi(x_0,y)\cdot w_\psi(y,z_0)\Bigg) \\ &= \sum_{y\in Y}\Bigg(\sum_{z\in U} w_\varphi(x_0,y)\cdot w_\psi(y,z)\Bigg) = \sum_{y\in Y} w_\varphi(x_0,y)\cdot\Bigg(\sum_{z\in U} w_\psi(y,z)\Bigg). \end{aligned}$$

Let

$$\varphi(x_0) = \{y_0^1, \dots, y_0^k\} \quad \text{and} \quad a_y := \sum_{z\in U} w_\psi(y,z).$$

Next, we obtain

$$\begin{aligned}\sum_{y\in Y} w_\varphi(x_o,y)\cdot\Big(\sum_{z\in U} w_\psi(y,z)\Big) &= \sum_{y\in Y} w_\varphi(x_0,y)\cdot a_y\\ &= \sum_{y\in\varphi(x_0)} w_\varphi(x_0,y)\cdot a_y = \sum_{i=1}^{k} w_\varphi(x_0,y_0^i)\cdot a_{y_0^i}.\end{aligned}$$

Hence, we get

$$\sum_{z\in U} w_{\psi\circ\varphi}(x_0,z) = \sum_{i=1}^{k} w_\varphi(x_0,y_0^i)\cdot a_{y_0^i}. \tag{2.2}$$

Now it is enough to show that the right hand sides of (2.1) and (2.2) are equal for some open neighbourhood of x_0 in X. Since ψ is a w-map, it follows that there exist open subsets $V_{y_0^i}$ of Y, $1\leqslant i\leqslant k$, such that

$$\begin{aligned} y_0^i &\in V_{y_0^i} && \text{for } i=1,\dots,k;\\ V_{y_0^i}\cap V_{y_0^j} &= \emptyset && \text{for } i\neq j;\\ a_{y_0^i} = \sum_{z\in U} w_\psi(y_0^i,z) &= \sum_{z\in U} w_\psi(y,z) = a_y && \text{for } y\in V_{y_0^i},\ i=1,\dots,k. \end{aligned} \tag{2.3}$$

Moreover, since φ is also a w-map, we deduce that there exist an open neighbourhoods $W^i_{x_0}$ of x_0 in X, $1\leqslant i\leqslant k$, such that

$$w_\varphi(x_0,y_0^i) = \sum_{y\in V_{y_0^i}} w_\varphi(x',y) = \sum_{y\in\varphi(x')\cap V_{y_0^i}} w_\varphi(x',y), \tag{2.4}$$

for $x'\in W^i_{x_0}$. From the upper semicontinuity of φ we infer that there exists an open neighbourhood $\widetilde{W}_{x_0}$ of x_0 in X such that

$$\varphi(\widetilde{W}_{x_0}) \subset \bigcup_{i=1}^{k} V_{y_0^i}. \tag{2.5}$$

Finally, we let

$$W_0 := \widetilde{W}_{x_0}\cap W^1_{x_0}\cap\ldots\cap W^k_{x_0}.$$

Then for any $x'\in W_{x_0}$ we obtain

$$\begin{aligned}\sum_{z\in U} w_{\psi\circ\varphi}(x',z) &= \sum_{z\in U}\Big(\sum_{y\in Y} w_\varphi(x',y)\cdot w_\psi(y,z)\Big)\\ &= \sum_{y\in Y}\sum_{z\in U} w_\varphi(x',y)\cdot w_\psi(y,z) = \sum_{y\in Y} w_\varphi(x',y)\cdot\Big(\sum_{z\in U} w_\psi(y,z)\Big)\\ &= \sum_{y\in Y} w_\varphi(x',y)\cdot a_y = \sum_{y\in\varphi(x')} (x',y)\cdot a_y.\end{aligned}$$

Consequently, in view of (2.5), we deduce that

$$\sum_{y\in\varphi(x')} w_\varphi(x',y)\cdot a_y = \sum_{y\in\varphi(x')\cap(\bigcup_{i=1}^k V_{y_0^i})} w_\varphi(x',y) = \sum_{i=1}^k \sum_{y\in\varphi(x')\cap V_{y_0^i}} w_\varphi(x',y)\cdot a_y.$$

Let us observe that, by (2.3), we have $a_y = a_{y_0}$ for $y \in V_{y_0^i}$. Therefore, we obtain

$$\begin{aligned}\sum_{i=1}^k \sum_{y\in\varphi(x')\cap V_{y_0^i}} w_\varphi(x',y)a_y &= \sum_{i=1}^k \sum_{y\in\varphi(x')\cap V_{y_0^i}} w_\varphi(x',y)a_{y_0^i}\\ &= \sum_{i=1}^k a_{y_0^i}\Bigg(\sum_{y\in\varphi(x')\cap V_{y_0^i}} w_\varphi(x',y)\Bigg) = \sum_{i=1}^k a_{y_0^i} w_\varphi(x_0,y_0^i),\end{aligned}$$

where the last equality follows from (2.4). The proof is complete. $\square$

Proposition 2.2.4. *Let $\varphi_i = (\sigma_{\varphi_i}, w_{\varphi_i})\colon X_i \multimap Y_i$, $1 \leqslant i \leqslant 2$, be two weighted maps. Then $\varphi_1\times\varphi_2 = (\sigma_{\varphi_1\times\varphi_2}, w_{\varphi_1\times\varphi_2})$ is a weighted map, where*

$$\begin{gathered}\sigma_{\varphi_1\times\varphi_2}\colon X_1\times X_2 \multimap Y_1\times Y_2,\\ w_{\varphi_1\times\varphi_2}\colon (X_1\times X_2)\times(Y_1\times Y_2)\to\mathbb{Q}\end{gathered}$$

are defined as follows

$$\begin{aligned}\sigma_{\varphi_1\times\varphi_2}(x_1,x_2) &= \sigma_{\varphi_1}(x_1)\times\sigma_{\varphi_2}(x_2),\\ w_{\varphi_1\times\varphi_2}((x_1,x_2),(y_1,y_2)) &= w_{\varphi_1}(x_1,y_1)\cdot w_{\varphi_2}(x_2,y_2),\end{aligned}$$

for every $x_1\in X_1$, $x_2\in X_2$, $y_1\in Y_1$, $y_2\in Y_2$.

Proof. The proof can be found in [39] or [23]. $\square$

Now we shall give the definition of homotopy in the category of weighted maps.

Definition 2.2.5. Given two w-maps ψ and φ from X to Y, we say that ψ is *w-homotopic* to φ ($\psi \sim_w \varphi$) if there exists a w-map $\theta\colon X\times[0,1]\multimap Y$ such that

$$\begin{aligned}w_\theta((x,0),y) &= w_\psi(x,y) \quad\text{and}\quad w_\theta((x,1),y) = w_\varphi(x,y),\\ \sigma_\theta(x,0) &= \sigma_\psi(x) \qquad\quad\text{and}\qquad\quad \sigma_\theta(x,1) = \sigma_\varphi(x),\end{aligned}$$

for any $x\in X$, $y\in Y$.

Definition 2.2.6. Let $\varphi\colon X \multimap Y$ be a w-map and let X be a connected space. Then the sum
$$\sum_{y\in Y} w_\varphi(x,y)$$
is called the *weighted index* of φ, where $x \in X$. We shall denote it by $I_w(\varphi)$.

The above definition is correct because the sum $\sum_y w_\varphi(x,y)$ does not depend on the choice of $x \in X$ provided the space X is connected (see [36] or Lemma 3.1.14 below).

In the following proposition we shall list some important properties of the weighted index.

Proposition 2.2.7. *The above index has the following properties:*

(a) *If $\varphi, \psi\colon X \multimap Y$ are w-homotopic, then $I_w(\varphi) = I_w(\psi)$.*

(b) *If $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$ are two w-maps, then*
$$I_w(\psi \circ \varphi) = I_w(\psi) \cdot I_w(\varphi).$$

(c) *If $f\colon X \to Y$ is a continuous map, then $I_w(f) = 1$.*

Proof. The proof can be found in [23]. □

Proposition 2.2.8 ([36])**.** *Let $\varphi\colon X \multimap Y$ be a weighted map such that $\varphi(X) \subset \bigcup_{i=1}^s V_i$, where V_i, $i = 1,\dots,s$, are open subsets of Y with $V_i \cap V_j = \emptyset$ for $i \neq j$. Assume also that the following condition is satisfied: $\varphi(x) \cap V_i \neq \emptyset$ for all $x \in X$ and $i = 1,\dots,s$. Then there exist w-maps $\varphi_i\colon X \multimap Y$ with $\varphi_i(X) \subset V_i$, $1 \leqslant i \leqslant s$, such that $\varphi = \bigcup_{i=1}^s \varphi_i$.*

Lemma 2.2.9. *Let $\psi, \varphi\colon X \multimap Y$ be two w-maps such that*
$$w_\psi(x,y) = w_\psi(x,y),$$
for each $x \in X, y \in Y$. Then there exists a weighted map $\theta\colon X \times [0,1] \multimap Y$ such that
$$\begin{aligned} \theta(x,0) &= \varphi(x), \quad & w_\theta((x,0),y) &= w_\varphi(x,y),\\ \theta(x,1) &= \psi(x), \quad & w_\theta((x,1),y) &= w_\psi(x,y), \end{aligned}$$
for $x \in X$, $y \in Y$.

Proof. It is enough to define a w-map $\theta\colon X \times [0,1] \multimap Y$ as follows:
$$\theta(x,t) = \begin{cases} \varphi(x) & \text{if } t \in [0,1/3),\\ \varphi(x) \cup \psi(x) & \text{if } t \in [1/3,2/3],\\ \psi(x) & \text{if } t \in (2/3,1], \end{cases}$$
and
$$w_\theta((x,t),y) = w_\varphi(x,y), \quad \text{for all } x \in X,\ y \in Y,\ t \in [0,1].$$ □

Lemma 2.2.10. *Let Y be a path-connected space. Then, for any w-map $\varphi\colon \{0,1\} \multimap Y$ satisfying the condition $\sum_{y\in Y} w_\varphi(0,y) = \sum_{y\in Y} w_\varphi(1,y)$, there exists a weighted map $\widetilde{\varphi}\colon [0,1] \multimap Y$ such that*

$$\widetilde{\varphi} = \bigcup_{i=1}^{s} \lambda_i f_i \quad \textit{and} \quad \widetilde{\varphi}|\{0,1\} = \varphi,$$

where $f_i\colon [0,1] \to Y$ is single-valued continuous function and $\lambda_i \in \mathbb{Q}$ for $i = 1,\dots,s$.

Proof. The standard proof may be found in [10]. However, let us provide a simple direct proof. Let $\varphi(0) = \{x_1,\dots,x_n\}$ and $\varphi(1) = \{y_1,\dots,y_m\}$. The proof will be divided into two steps.

Step 1. We assume that $n \geqslant m$. Let

$$\begin{aligned} \alpha_i &:= w_\varphi(0,x_i) \quad \text{for } 1 \leqslant i \leqslant n, \\ \beta_j &:= w_\varphi(1,y_j) \quad \text{for } 1 \leqslant j \leqslant m. \end{aligned}$$

First, we shall consider the case $m = 1$. Then by the connectedness of Y there exist continuous functions $h_i\colon [0,1] \to Y$ such that

$$h_i(0) = x_i \quad \text{and} \quad h_i(1) = y_1,$$

for $i = 1,\dots,n$. Consequently, it is enough to define $\widetilde{\varphi}\colon [0,1] \multimap Y$ as follows

$$\widetilde{\varphi} := \bigcup_{i=1}^{n} \alpha_i h_i.$$

Let $m > 1$. We put

$$\begin{aligned} \gamma_{x_1} &= \alpha_1, \\ \gamma_{y_i} &= \beta_i - \gamma_{x_i}, \quad \gamma_{x_{i+1}} = \alpha_{i+1} - \gamma_{y_i}, \quad \text{for } i = 1,\dots,m-1, \end{aligned}$$

if $n > m$, then we put

$$\gamma_{x_{m+l}} = \alpha_{m+l}, \quad \text{for } l = 1,\dots,n-m.$$

Since Y is path-connected, there exist continuous functions

$$h_{x_1},\dots,h_{x_n}\colon [0,1] \to Y \quad \text{and} \quad h_{y_1},\dots,h_{y_{m-1}}\colon [0,1] \to Y$$

such that

$$\begin{aligned} h_{x_i}(0) &= x_i, & h_{x_i}(1) &= y_i, & &\text{for } i = 1,\dots,m-1, \\ h_{y_i}(0) &= y_i, & h_{y_i}(1) &= x_{i+1}, & &\text{for } i = 1,\dots,m-1, \\ h_{x_{m+l}}(0) &= x_{m+l}, & h_{x_{m+l}}(1) &= y_m, & &\text{for } l = 0,\dots,n-m. \end{aligned}$$

Now it is enough to define $\widetilde{\varphi}\colon [0,1] \multimap Y$ as follows

$$\widetilde{\varphi} = \left(\bigcup_{i=1}^{n} \gamma_{x_i} h_{x_i} \right) \cup \left(\bigcup_{j=1}^{m-1} \gamma_{y_j} h_{y_j}^{-1} \right),$$

where $h_{y_j}^{-1}(t) := h_{y_j}(1-t)$ for all $t \in [0,1]$ and $j = 1, \ldots, m-1$.

Step 2. We assume that $m \geqslant n$. Let us define a w-map $\psi\colon \{0,1\} \multimap Y$ by $\psi(t) = \varphi(1-t)$ for $t \in \{0,1\}$. Then, by Step 1, there exists a w-map $\widetilde{\psi}\colon [0,1] \multimap Y$ such that

$$\widetilde{\psi} = \bigcup_{i=1}^{s} \lambda_i f_i \quad \text{and} \quad \widetilde{\psi}|\{0,1\} = \psi,$$

where $\lambda_i \in \mathbb{Q}$ and $f_i\colon [0,1] \to Y$ are continuous functions for $i = 1, \ldots, s$. Consequently, a w-map $\widetilde{\varphi}\colon [0,1] \multimap Y$ defined by the formula

$$\widetilde{\varphi} = \bigcup_{i=1}^{s} \lambda_i f_i^{-1}$$

is the desired extension of $\varphi\colon \{0,1\} \multimap Y$, where $f_i^{-1}(t) := f_i(1-t)$ for $t \in [0,1]$ and $i = 1, \ldots, s$. This completes the proof of the lemma. $\square$

Lemma 2.2.11 (Gluing Lemma)**.** *Assume that a space X is a union of two closed subsets $X = A_1 \cup A_2$ and $A_1 \cap A_2 \neq \emptyset$. If there are two weighted maps $\varphi_1\colon A_1 \multimap Y, \varphi_2\colon A_2 \multimap Y$ such that*

$$\begin{aligned} \sigma_{\varphi_1}(x) &= \sigma_{\varphi_2}(x) && \text{for all } x \in A_1 \cap A_2, \\ w_{\varphi_1}(x,y) &= w_{\varphi_2}(x,y) && \text{for all } x \in A_1 \cap A_2,\ y \in Y, \end{aligned}$$

then a pair $\varphi = (\sigma_\varphi, w_\varphi)$ given by

$$\sigma_\varphi(x) := \begin{cases} \sigma_{\varphi_1}(x) & \text{if } x \in A_1, \\ \sigma_{\varphi_2}(x) & \text{if } x \in A_2, \end{cases}$$

and

$$w_\varphi(x,y) := \begin{cases} w_{\varphi_1}(x,y) & \text{if } x \in A_1,\ y \in Y, \\ w_{\varphi_2}(x,y) & \text{if } x \in A_2,\ y \in Y, \end{cases}$$

is a weighted map.

The proof of Lemma 2.2.11 is straightforward.

Now, we shall consider some algebraic properties of w-maps. They will play a crucial role in the topological essentiality.

Definition 2.2.12. Let E be a normed space and let $\psi, \varphi\colon X \multimap E$ be two w-maps. By $\psi + \varphi\colon X \multimap E$ we shall understand a pair $\psi + \varphi = (\sigma_{\psi+\varphi}, w_{\psi+\varphi})$, where

$$\sigma_{\psi+\varphi}\colon X \multimap E \quad \text{and} \quad w_{\psi+\varphi}\colon X \times E \to \mathbb{Q}$$

are defined as follows:

$$\begin{aligned} \sigma_{\psi+\varphi}(x) &= \{u+v \mid u \in \psi(x) \text{ and } v \in \varphi(x)\}; \\ w_{\psi+\varphi}(x,u) &= \sum_{e\in E} w_\psi(x, u-e)\cdot w_\varphi(x,e). \end{aligned}$$

Proposition 2.2.13 (see [61]). *The above pair $\psi+\varphi = (\sigma_{\psi+\varphi}, w_{\psi+\varphi})$ is a weighted map.*

Definition 2.2.14. Let E be a normed space and let $\psi, \varphi\colon X \multimap E$ be two w-maps. By $\psi-\varphi\colon X \multimap E$ we shall understand a pair $\psi-\varphi = (\sigma_{\psi-\varphi}, w_{\psi-\varphi})$, where

$$\sigma_{\psi-\varphi}\colon X \multimap E \quad \text{and} \quad w_{\psi-\varphi}\colon X\times E \to \Omega$$

are defined as follows:

$$\begin{aligned} \sigma_{\psi-\varphi}(x) &= \{u-v \mid u \in \psi(x) \text{ and } v \in \varphi(x)\}; \\ w_{\psi-\varphi}(x,u) &= \sum_{e\in E} w_\psi(x, u+e)\cdot w_\varphi(x,e). \end{aligned}$$

Proposition 2.2.15 ([61]). *A pair $\psi-\varphi = (\sigma_{\psi-\varphi}, w_{\psi-\varphi})$ is a weighted map.*

Definition 2.2.16. Let E be a normed space and $\varphi\colon X \multimap E$ and let $s\colon X \to \mathbb{R}$ be a continuous function. By $s\varphi\colon X \multimap E$ we shall understand a pair $s\varphi = (\sigma_{s\varphi}, w_{s\varphi})$, where

$$\sigma_{s\varphi}\colon X \multimap E \quad \text{and} \quad w_{s\varphi}\colon X\times E \to \mathbb{Q}$$

are defined as follows:

$$\sigma_{s\varphi}(x) = \{s(x)u \mid u \in \varphi(x)\}$$

and

$$w_{s\varphi}(x,u) = \begin{cases} w_\varphi\left(x, \dfrac{u}{s(x)}\right) & \text{if } s(x) \neq 0, \\ \displaystyle\sum_{e\in\mathbb{E}} w_\varphi(x,e) & \text{if } s(x) = 0,\ u = 0, \\ 0 & \text{if } s(x) = 0,\ u \neq 0. \end{cases}$$

Proposition 2.2.17 (see [61]). *A pair $s\varphi = (\sigma_{s\varphi}, w_{s\varphi})$ defined above is a weighted map.*

Proposition 2.2.18 ([61]). *Let $\varphi, \psi\colon X \multimap E$ be two w-maps and let $s\colon X \to \mathbb{R}$ be a continuous map, where X is a connected space. Then*

(a) $I_w(\varphi+\psi) = I_w(\varphi)\cdot I_w(\psi)$;
(b) $I_w(\varphi-\psi) = I_w(\varphi)\cdot I_w(\psi)$;
(c) $I_w(s\cdot\varphi) = I_w(\varphi)$.

Remark 2.2.19. Let $\varphi, \psi\colon X \multimap E$ be two w-maps. It is easy to see that the equation:

$$w_{\varphi+\psi}(x,u) = w_\varphi(x,u) + w_\psi(x,u)$$

is not true in general.

2.3. Darbo homology functor

2.3.1. Basic constructions. By $\mathcal{W}$ we shall denote the category of metric spaces and weighted maps with coefficients in $\mathbb{Q}$. In particular, by $\mathcal{W}(X,Y)$ we shall understand the class of all w-maps from X to Y. Let us define an equivalence relation $\sim$ on $\mathcal{W}(X,Y)$ as follows: $\psi \sim \varphi$ if and only if $w_\psi = w_\varphi$. The class of equivalence classes will be denoted by $(X,Y) := \mathcal{W}(X,Y)/\sim$.

Darbo constructed a homology theory for weighted maps by adapting the usual construction of the singular homology functor. In what follows we briefly describe his construction. Let Δ_k be the geometrical k-simplex. For any $0 \leqslant i \leqslant k$ consider the map $d_k^i\colon \Delta_{k-1} \to \Delta_k$ given by the inclusion of Δ_{k-1} as the face opposite to the i-th vertex of Δ_k. Given a space X we shall consider the graded vector space $\mathbb{C}(X,\mathbb{Q}) = \{\mathbb{C}_k(X,\mathbb{Q})\}_{k\geqslant 0}$, where $\mathbb{C}_k(X,\mathbb{Q}) := (\Delta_k, X)$. So, we can define a boundary operator ∂ in $\mathbb{C}(X,\mathbb{Q})$ as follows:

$$\partial_k s = \bigcup_{i=0}^{k} (-1)^i s \circ d_k^i \in \mathbb{C}_{k-1}(X,\mathbb{Q}),$$

for any $s \in \mathbb{C}_k(X,\mathbb{Q})$ and $k > 0$; if $k = 0$, define $\partial_0 s = 0$. One can easily prove that $\partial_k \partial_{k+1} = 0$, for all $k \geqslant 0$.

The graded vector space

$$\mathbb{H}_*(X,\mathbb{Q}) = \{\mathbb{H}_k(X,\mathbb{Q})\}_{k\geqslant 0}$$

of the complex $(\mathbb{C}(X,\mathbb{Q}),\partial)$ will be called the *Darbo homology* of the space X over $\mathbb{Q}$. Any weighted map $\varphi\colon X \multimap Y$ induces in a functorial way a linear map $\varphi_*\colon \mathbb{H}_*(X,\mathbb{Q}) \to \mathbb{H}_*(Y,\mathbb{Q})$ (of degree zero). It is well-known how to define

$$\mathbb{H}_*(X,A;\mathbb{Q}) \quad \text{and} \quad \varphi_*\colon \mathbb{H}_*(X,A;\mathbb{Q}) \to \mathbb{H}_*(Y,B;\mathbb{Q})$$

for a pair (X,A) and a w-map $\varphi\colon (X,A) \multimap (Y,B)$.

Let us note that two w-homotopic w-maps induce the same linear map in Darbo homology. With this $\mathbb{H}_*$ becomes additive functor from $\mathcal{W}$ to the category of graded vector spaces which is invariant under the w-homotopy. Darbo (and Jerrard) showed that the functor $\mathbb{H}_*$ satisfies the Eilenberg–Steenrod axioms for a homology theory with compact carriers. For more details concerning the notion of Darbo homology [12], [36] are recomended.

Theorem 2.3.1 ([64]). *On the category of all pairs of absolute neighbourhood retracts and single-valued continuous maps the singular homology functor and the Darbo homology functor are naturally isomorphic.*

Let X_1, X_2 be two subspaces of a space X with $X = X_1 \cup X_2$. We denote this situation by $(X; X_1, X_2)$ and call it a triad. Now we want to relate $\mathbb{H}_*(X_1, \mathbb{Q})$, $\mathbb{H}_*(X_2, \mathbb{Q})$, $\mathbb{H}_*(X_1 \cap X_2, \mathbb{Q})$ and $\mathbb{H}_*(X_1 \cup X_2, \mathbb{Q})$.

Definition 2.3.2. A triad $(X; X_1, X_2)$ is called *excisive* if the inclusion $j\colon (X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2) = (X, X_2)$ induces isomorphisms

$$j_*\colon \mathbb{H}_n(X_1, X_1 \cap X_2; \mathbb{Q}) \longrightarrow \mathbb{H}_n(X, X_2; \mathbb{Q}) \quad \text{for all } n.$$

Proposition 2.3.3. *Let Y and Z be two subspaces of a space X. If $X = \operatorname{int} Y \cup \operatorname{int} Z$, then the triad $(X; Y, Z)$ is excisive.*

Proof. The proof follows exactly the same lines as it was done in the case of singular homology in [66] (see also [15]), therefore we leave it to the reader as an exercise. □

Proposition 2.3.4. (S^n, S^n_+, S^n_-) (3) *is an excisive triad.*

Proof. This fact was proved in terms of singular homology, for example, in [32]. In view of Theorem 2.3.1, Proposition 2.3.4 follows from the corresponding statement in [32]. □

Proposition 2.3.5. *Let A be a closed subset of $S^n \subset S^{n+1}$ and let $0 < \varepsilon < \sqrt{2}$. In addition, let $O^+_\varepsilon(A) := O_\varepsilon(A) \cap S^{n+1}_+$ and $O^-_\varepsilon(A) := O_\varepsilon(A) \cap S^{n+1}_-$, where $O_\varepsilon(A) := \{x \in S^{n+1} \mid \operatorname{dist}(x, A) < \varepsilon\}$. Then*

$$(O_\varepsilon(A) \setminus A; O^+_\varepsilon(A) \setminus A, O^-_\varepsilon(A) \setminus A)$$

is an excisive triad.

Proof. Let us observe that if $A = S^n$, then the assertion follows immediately from Proposition 2.3.3. So we can assume that $A \neq S^n$. Let

$$\widetilde{O}^+_\varepsilon := (O^+_\varepsilon(A) \setminus A) \cup \left\{ (x_1, \dots, x_{n+2}) \in O^-_\varepsilon(A) \;\middle|\; \frac{(x_1, \dots, x_{n+1})}{||(x_1, \dots, x_{n+1})||} \notin A \right\},$$
$$\widetilde{O}^-_\varepsilon := (O^-_\varepsilon(A) \setminus A) \cup \left\{ (x_1, \dots, x_{n+2}) \in O^+_\varepsilon(A) \;\middle|\; \frac{(x_1, \dots, x_{n+1})}{||(x_1, \dots, x_{n+1})||} \notin A \right\}.$$

Then, by Proposition 2.3.3, $(O_\varepsilon(A) \setminus A; \widetilde{O}^+_\varepsilon, \widetilde{O}^-_\varepsilon)$ is an excisive triad. Now the assertion of our proposition follows from the fact that $O^+_\varepsilon(A) \setminus A$ (resp. $O^-_\varepsilon(A) \setminus A$) is a deformation retract of $\widetilde{O}^+_\varepsilon$ (resp. $\widetilde{O}^-_\varepsilon$), which completes the proof. □

In a similar way, one can also prove the following result.

(3) Definition of S^n_+ and S^n_- is given in Preliminaries

Proposition 2.3.6. *Let A, $O_\varepsilon^+(A)$ and $O_\varepsilon^-(A)$ be as above. In addition, let*

$$\mathbb{R}^n_+ := \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n \geqslant 0\}, \quad \mathbb{R}^n_- := \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n \leqslant 0\}.$$

Then

(a) $(O_\varepsilon(A); O_\varepsilon^+(A), O_\varepsilon^-(A))$,
(b) $(S^n \setminus A; S^n_+ \setminus A, S^n_- \setminus A)$,
(c) $(\mathbb{R}^n; \mathbb{R}^n_+, \mathbb{R}^n_-)$,
(d) $(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n_+ \setminus \{0\}, \mathbb{R}^n_- \setminus \{0\})$

are the excisive triads.

Theorem 2.3.7 (Mayer–Vietories exact sequence)**.** *Let $(A; A_1, A_2) \subset (X; X_1, X_2)$ be the pair of excisive triads. Then there exists an exact sequence*

$$\begin{aligned}\cdots \to \mathbb{H}_{n+1}(X_1 \cup X_2, A_1 \cup A_2; \mathbb{Q}) &\xrightarrow{\partial_*} \mathbb{H}_n(X_1 \cap X_2, A_1 \cap A_2; \mathbb{Q}) \xrightarrow{(j_{1*}, -j_{2*})} \\ &\to \mathbb{H}_n(X_1, A_1; \mathbb{Q}) \oplus \mathbb{H}_n(X_2, A_2; \mathbb{Q}) \xrightarrow{i_{1*}+i_{2*}} \\ &\to \mathbb{H}_n(X_1 \cup X_2, A_1 \cup A_2; \mathbb{Q}) \xrightarrow{\partial_*} \mathbb{H}_{n-1}(X_1 \cap X_2, A_1 \cap A_2; \mathbb{Q}) \to \cdots\end{aligned}$$

where i_{1*}, i_{2*}, j_{1*} and j_{2*} are induced by inclusions.

For the proof and more details we refer the reader to [15]. The above exact sequence is called the Mayer–Vietoris sequence of the pair of excisive triads and ∂_* is called the *Mayer–Vietoris homomorphism.* If $A = \emptyset$, then Theorem 2.3.7 reduces to the following theorem:

Theorem 2.3.8 (Mayer–Vietories exact sequence)**.** *Let $(X; X_1, X_2)$ be an excisive triad. Then there exists an exact sequence*

$$\begin{aligned}\cdots \to \mathbb{H}_{n+1}(X_1 \cup X_2, \mathbb{Q}) &\xrightarrow{\partial_*} \mathbb{H}_n(X_1 \cap X_2, \mathbb{Q}) \xrightarrow{(j_{1*}, -j_{2*})} \\ &\to \mathbb{H}_n(X_1, \mathbb{Q}) \oplus \mathbb{H}_n(X_2, \mathbb{Q}) \xrightarrow{i_{1*}+i_{2*}} \\ &\to \mathbb{H}_n(X_1 \cup X_2, \mathbb{Q}) \xrightarrow{\partial_*} \mathbb{H}_{n-1}(X_1 \cap X_2, \mathbb{Q}) \to \cdots\end{aligned}$$

where i_{1*}, i_{2*}, j_{1*} and j_{2*} are induced by inclusions.

Moreover, Mayer–Vietoris sequences are functorial, namely:

Theorem 2.3.9 (cf. [15])**.** *A weighted map $\varphi\colon (X; X_1, X_2) \multimap (Y; Y_1, Y_2)$ of excisive triads, i.e. a weighted map $\varphi\colon X \multimap Y$ with $\varphi(X_i) \subset Y_i$, induces a homomorphism of the corresponding* (*absolute or relative*) *Mayer–Vietoris sequences.*

Definition 2.3.10. A non-empty space X is called *acyclic* (with respect to the Darboo homology) provided

(a) $\mathbb{H}_n(X, \mathbb{Q}) = 0$ for all $n \geqslant 1$, and
(b) $\mathbb{H}_0(X, \mathbb{Q}) \approx \mathbb{Q}$.

Definition 2.3.11. A weighted map $\varphi\colon X \multimap X$ is called a *Lefschetz map* provided that $\varphi_*\colon \mathbb{H}_*(X,\mathbb{Q}) \to \mathbb{H}_*(X,\mathbb{Q})$ is a Leray endomorphism; for such a φ we define the Lefschetz number $\Lambda(\varphi)$ of φ by $\Lambda(\varphi) = \Lambda(\varphi_*)$.

Proposition 2.3.12. *Let $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap X$ be two weighted maps. If one of ψ,φ is a Lefschetz map, then so is the other, and $\Lambda(\varphi\circ\psi) = \Lambda(\psi\circ\varphi)$.*

In what follows, we shall also make use of the following lemma.

Lemma 2.3.13. *Let $\varphi\colon X \multimap Y$ be a weighted map and let Y be a path-connected space. If $X \subset Y$, then $\varphi_{*0}([\sigma]) = I_w(\varphi)[j_X \circ \sigma]$ for all $[\sigma] \in \mathbb{H}_0(X,\mathbb{Q})$, where $j_X\colon X \to Y$ is the inclusion.*

Proof. This follows directly from the construction of the Darbo homology functor and, therefore, we leave the details to the reader. □

The following corollary follows from Definition 2.3.11 and Lemma 2.3.13.

Corollary 2.3.14. *Let X be an acyclic* ANR *and let $\varphi\colon X \multimap X$ be a compact weighted map. Then $\Lambda(\varphi) = I_w(\varphi)$.*

2.3.2. The homology cross products. The purpose of this section is to describe the notion of the cross product in Darbo homology. The construction of the cross product follows the same lines as in the case of a singular homology (cf. [15], [32]).

Given (X,A) and (Y,B), we write

$$(X,A)\times(Y,B) = (X\times Y, A\times Y \cup X\times B).$$

Define $\eta_n^i\colon \Delta_n \to \Delta_{n-1}$ as follows

$$\eta_n^i(e_k) = \begin{cases} e_k & \text{if } 0 \le k \le i, \\ e_{k-1} & \text{if } i < k \le n, \end{cases}$$

for $0 \le i \le n-1$. Let $I = \{i_1,\dots,i_p : i_k < i_{k+1},\ 1 \le k < p\}$ be a subset of $\{0,\dots,n-1\}$. In addition, let $\eta_I\colon \Delta_n \to \Delta_{n-p}$ be given by

$$\eta_I = \eta^{i_1}_{n-p+1} \circ \dots \circ \eta^{i_p}_n.$$

If, in particular, $I = \emptyset$ then we let $\eta_I = \mathrm{id}$.

Let θ be the collection of all subsets consisting of p elements of $\{0,\dots,n-1\}$. For $I \in \theta$ we write $J = \{0,\dots,n-1\}\setminus I$ and for $I \in \theta$ denote by $\varepsilon(I)$ the cardinal number of the set $\{(i,j) : i \in I,\ j \in J,\ i > j\}$.

Definition 2.3.15. Let $p+q = n$, $n \geqslant 0$. Then

$$\nabla_n\colon (\mathbb{C}(X,\mathbb{Q}) \otimes \mathbb{C}(Y,\mathbb{Q}))_n \to \mathbb{C}_n(X\times Y,\mathbb{Q})$$

is defined by

$$\nabla_n(\sigma\otimes\tau) = \sum_{I\in\theta} (-1)^{\varepsilon(I)} (\sigma\circ\eta_J, \tau\circ\eta_I),$$

where $\sigma \in \mathbb{C}_p(X,\mathbb{Q}), \tau \in \mathbb{C}_q(Y,\mathbb{Q})$. The family

$$\nabla = \{\nabla_n\}_{n\geqslant 0}\colon \mathbb{C}(X,\mathbb{Q}) \otimes \mathbb{C}(Y,\mathbb{Q}) \to \mathbb{C}(X\times Y,\mathbb{Q})$$

is called the Eilenberg–MacLane map ([4]).

Remark 2.3.16. Recall that the above definition applies for singular complexes as well.

Lemma 2.3.17. *The Eilenberg–Maclane map ∇ is natural with respect to weighted maps, i.e. the following diagram commutes*

$$\begin{tikzcd}
\mathbb{C}(X,\mathbb{Q}) \otimes \mathbb{C}(Y,\mathbb{Q}) \arrow[r, "\nabla"] \arrow[d, "\varphi_\# \otimes \psi_\#"'] & \mathbb{C}(X\times Y,\mathbb{Q}) \arrow[d, "(\varphi\times\psi)_\#"] \\
\mathbb{C}(X',\mathbb{Q}) \otimes \mathbb{C}(Y',\mathbb{Q}) \arrow[r, "\nabla"'] & \mathbb{C}(X'\times Y',\mathbb{Q})
\end{tikzcd}$$

where $\varphi\colon X \multimap X'$ and $\psi\colon Y \multimap Y'$ are w-maps.

Proof. This lemma follows immediately form the definitions. □

Remark 2.3.18. Recall that if (C,∂) and (C',∂') are algebraic chain complexes and

$$[C\otimes C']_n = \bigoplus_{p=0}^{n} C_p \otimes C'_{n-p}$$

then $D_n\colon [C\otimes C']_n \to [C\otimes C']_{n-1}$ is defined by

$$D_n(z\otimes z') = \partial z \otimes z' + (-1)^p z \otimes \partial' z'$$

for $z \in C_p, z' \in C'_{n-p}$ nad extended by linearity.

Theorem 2.3.19 ([16])**.** *Let $\nabla_n\colon [S(\Delta_p)\otimes S(\Delta_q)]_n \to S_n(\Delta_p\times\Delta_q)$ be the Eilenberg–MacLane map. Then the diagram*

$$\begin{tikzcd}
S_n(\Delta_p\times\Delta_q,\mathbb{Q}) \arrow[r, "\partial_n"] & S_{n-1}(\Delta_p\times\Delta_q,\mathbb{Q}) \\
{[S(\Delta_p,\mathbb{Q})\otimes S(\Delta_q,\mathbb{Q})]_n} \arrow[u, "\nabla_n"] \arrow[r, "D_n"'] & {[S(\Delta_p,\mathbb{Q})\otimes S(\Delta_q,\mathbb{Q})]_{n-1}} \arrow[u, "\nabla_n"']
\end{tikzcd}$$

is commutative.

Remark 2.3.20. If $\varphi\colon X \multimap Y$ is a weighted map and $\sigma \in \mathbb{C}_n(X,\mathbb{Q})$, then $\varphi\circ\sigma \in \mathbb{C}_n(Y,\mathbb{Q})$. Extending by linearity gives a chain homomorphism $\varphi_\#\colon \mathbb{C}_n(X,\mathbb{Q}) \to \mathbb{C}_n(Y,\mathbb{Q})$, for every $n \geqslant 0$.

([4]) It should be noted that ∇ is sometimes called a shuffle homomorphism (cf. [32]).

Theorem 2.3.21. *Let $\nabla\colon \mathbb{C}(X,\mathbb{Q})\otimes\mathbb{C}(Y,\mathbb{Q})\to\mathbb{C}(X\times Y,\mathbb{Q})$ be the Eilenberg–MacLane map. Then the following diagram*

$$\begin{array}{ccc} \mathbb{C}_n(X\times Y,\mathbb{Q}) & \xrightarrow{\partial_n} & \mathbb{C}_{n-1}(X\times Y,\mathbb{Q}) \\ \nabla_n\uparrow & & \uparrow\nabla_{n-1} \\ [\mathbb{C}(X,\mathbb{Q})\otimes\mathbb{C}(Y,\mathbb{Q})]_n & \xrightarrow[D_n]{} & [\mathbb{C}(X,\mathbb{Q})\otimes\mathbb{C}(Y,\mathbb{Q})]_{n-1} \end{array}$$

is commutative.

Proof. Let $\sigma\otimes\tau\in\mathbb{C}_p(X,\mathbb{Q})\otimes\mathbb{C}_q(Y,\mathbb{Q})$. Let us observe that

$$\sigma\otimes\tau=(\sigma\otimes\tau)\circ(\lambda_p\otimes\lambda_q),$$

where $\lambda_p\colon\Delta_p\to\Delta_p$ and $\lambda_q\colon\Delta_q\to\Delta_q$ are the identity maps. Then we have

$$\begin{aligned}
\nabla_{n-1}D_n(\sigma\otimes\tau) &= \nabla_{n-1}D_n((\sigma\otimes\tau)\circ(\lambda_p\otimes\lambda_q))\\
&= \nabla_{n-1}D_n((\sigma_\#\otimes\tau_\#)(\lambda_p\otimes\lambda_q))\\
&= \nabla_{n-1}(\sigma_\#\otimes\tau_\#)(D_n(\lambda_p\otimes\lambda_q))\\
&\overset{2.3.17}{=} (\sigma\times\tau)_\#\nabla_{n-1}(D_n(\lambda_p\otimes\lambda_q))\\
&\overset{2.3.19}{=} (\sigma\times\tau)_\#\partial_n(\nabla_n(\lambda_p\otimes\lambda_q))\\
&= \partial_n(\sigma\times\tau)_\#\nabla_n(\lambda_p\otimes\lambda_q)\\
&\overset{2.3.17}{=} \partial_n\nabla_n((\sigma_\#\otimes\tau_\#)(\lambda_p\otimes\lambda_q))\\
&= \partial_n\nabla_n((\sigma\otimes\tau)\circ(\lambda_p\otimes\lambda_q))=\partial_n\nabla_n(\sigma\otimes\tau),
\end{aligned}$$

which completes the proof. $\square$

From the above considerations we conclude the following corollary.

Corollary 2.3.22. *The following diagram commutes*

$$\begin{array}{ccc} H_n(\mathbb{C}(X,\mathbb{Q})\otimes\mathbb{C}(Y,\mathbb{Q})) & \xrightarrow{\nabla_{*n}} & H_n(\mathbb{C}(X\times Y,\mathbb{Q})) \\ {\scriptstyle(\varphi\otimes\psi)_{*n}}\downarrow & & \downarrow{\scriptstyle(\varphi\times\psi)_{*n}} \\ H_n(\mathbb{C}(X',\mathbb{Q})\otimes\mathbb{C}(Y',\mathbb{Q})) & \xrightarrow[\nabla_{*n}]{} & H_n(\mathbb{C}(X'\times Y',\mathbb{Q})) \end{array}$$

where $\varphi\colon X\multimap X'$ and $\psi\colon Y\multimap Y'$ are weighted maps.

Definition 2.3.23 (cf. [15])**.** Let $[\sigma]\in\mathbb{H}_p(X,A;\mathbb{Q})$ and $[\tau]\in\mathbb{H}_q(Y,B;\mathbb{Q})$. Then the cross product $\times\colon\mathbb{H}_p(X,A;\mathbb{Q})\otimes\mathbb{H}_q(Y,B;\mathbb{Q})\to\mathbb{H}_{p+q}((X,A)\times(Y,B);\mathbb{Q})$ is defined by

$$\times([\sigma]\otimes[\tau]):=[\nabla(\sigma\otimes\tau)]\in\mathbb{H}_{p+q}((X,A)\times(Y,B);\mathbb{Q}).$$

The image under $\times$ of $[\sigma]\otimes[\tau]$ will be denoted by $[\sigma]\times[\tau]$.

From Corollary 2.3.22 we get the following proposition.

Proposition 2.3.24 (Naturality). *Let $\varphi\colon (X,A) \multimap (X',A')$ and $\psi\colon (Y,B) \multimap (Y',B')$ be two weighted maps. Then*

$$(\varphi \times \psi)_*(a \times b) = \varphi_*(a) \times \psi_*(b)$$

for any $a \in \mathbb{H}_n(X,A;\mathbb{Q})$ and $b \in \mathbb{H}_n(Y,B;\mathbb{Q})$.

Remark 2.3.25. It is also important to notice that if $\xi \in \mathbb{H}_p(X,A;\mathbb{Q})$, $\eta \in \mathbb{H}_q(Y,B;\mathbb{Q})$ and $\gamma \in \mathbb{H}_r(Z,C;\mathbb{Q})$, then

$$(\xi \times \eta) \times \gamma = \xi \times (\eta \times \gamma).$$

The proof of the above equality proceeds exactly the same lines as the proof of it in the case of the singular homology. For the detailed treatment of the fact that the cross product is associative we refer the reader to [15], [32] and [62].

2.4. The w-homotopy functor

A w-map $\varphi\colon (X,x_0) \multimap (Y,y_0)$ between pointed spaces will be called a pointed w-map if $\varphi(x_0) = y_0$. Let $\mathcal{W}_0$ be the category of pointed spaces and pointed w-maps with the weighted index equal to 0. Given two weighted maps φ_0 and φ_1 from (X,x_0) to (Y,y_0), we say that φ_0 is w-homotopic to φ_1 relative to x_0 (written $\varphi_0 \sim_w \varphi_1$ rel x_0) if there exists a weighted map $\theta\colon X \times [0,1] \multimap Y$ satisfying two conditions of Definition 2.2.5 and $\theta(x_0,t) = y_0$ for any $t \in [0,1]$. This θ is called the pointed w-homotopy between φ_0 and φ_1. It is easy to see that the pointed w-homotopy is an equivalence relation on $\mathcal{W}_0$. For a space X with a basepoint $x_0 \in X$, define $\pi_n^w(X,x_0)$ to be the set of the pointed classes of w-maps $\varphi\colon (S^n,s_0) \multimap (X,x_0)$ having the weighted index $I_w(\varphi) = 0$, where s_0 is a base point of the n-sphere S^n. Notice that $\pi_n^w(X,x_0)$ admits a natural structure of $\mathbb{Q}$-module under the following operations:

$$[\varphi] + [\psi] := [\varphi \cup \psi], \qquad \lambda[\varphi] = [\lambda\varphi],$$

where $[\varphi],[\psi] \in \pi_n^w(X,x_0)$, $\lambda \in \mathbb{Q}$. For any pointed space X, and $n \geqslant 0$, the $\mathbb{Q}$-module $\pi_n^w(X,x_0)$ is called the n-th w-homotopy $\mathbb{Q}$-module of X. It is easy to see that in the definition of $\pi_n^w(X,x_0)$ we can replace the unit sphere S^n by $\partial\Delta_{n+1}$. Notice that the concept of w-homotopy was systematically studied in [40], [53] and [55].

The Hurewicz map $h_n\colon \pi_n^w(X,x_0) \to \widetilde{\mathbb{H}}_n(X,\mathbb{Q})$ is defined in the usual way. Namely, $h_n(\alpha) = \alpha_*(1_n)$, where $\widetilde{\mathbb{H}}$ denotes the reduced (Darbo) homology and 1_n is a generator of $\widetilde{\mathbb{H}}_n(S^n,\mathbb{Q})$. In the sequel we shall use the following result:

Theorem 2.4.1 ([55]). *If X is an absolute neighbourhood retract, then the Hurewicz map $h_n\colon \pi_n^w(X,x_0) \to \widetilde{\mathbb{H}}_n(X,\mathbb{Q})$ is an isomorphism for every $n \geqslant 0$*

and any $x_0 \in X$. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_n^w(X,x_0) & \xrightarrow{\varphi_{*n}} & \pi_n^w(Y,y_0) \\ {\scriptstyle h_n}\big\downarrow & & \big\downarrow{\scriptstyle h_n} \\ \widetilde{\mathbb{H}}_n(X,\mathbb{Q}) & \xrightarrow[\varphi_{*n}]{} & \widetilde{\mathbb{H}}_n(Y,\mathbb{Q}) \end{array}$$

for any weighted map $\varphi\colon X \multimap Y$ and $n \geqslant 0$.

Lemma 2.4.2. *Let $\varphi\colon S^n \multimap Y$ be a weighted map. In addition, let us assume that there exists a point $x_0 \in S^n$ such that $\varphi(x_0)$ consists of one point. If φ can be extended over D^{n+1}, then φ is w-homotopic to $I_w(\varphi)k$ relative to x_0, where $k\colon S^n \to Y$ is the constant map at $\varphi(x_0)$ [5].*

Proof. Let $\widetilde{\varphi}\colon D^{n+1} \multimap Y$ be an extension of φ and let $c\colon S^n \to D^{n+1}$ be defined by $c(x) = x_0$ for all $x \in S^n$. Since the inclusion $i\colon S^n \to D^{n+1}$ and $c\colon S^n \to D^{n+1}$ are w-homotopic relative to x_0, it follows that $\widetilde{\varphi} \circ i$ and $\widetilde{\varphi} \circ c$ are also w-homotopic relative to x_0. Let $k\colon S^n \to Y$ be defined to be $\widetilde{\varphi} \circ c$. Consequently, $\varphi \sim_w I_w(\varphi)k$, because $\widetilde{\varphi} \circ i = \varphi$ and $\widetilde{\varphi} \circ c = I_w(\varphi)k$, which completes the proof. □

Proposition 2.4.3. *Let X be an* ANR, *let A be a closed* ANR *subspace of X and let Y be an arbitrary metric space. If $\varphi\colon A \times [0,1] \multimap Y$ is a weighted map such that $\varphi_0\colon A \multimap Y$ is extendable to a w-map $\widetilde{\varphi_0}\colon X \multimap Y$, then there is a w-map $\overline{\varphi}\colon X \times [0,1] \multimap Y$ such that*

(a) $\overline{\varphi}|X \times \{0\} = \widetilde{\varphi_0}$,

(b) $\overline{\varphi_t}|A = \varphi_t$, *for every $t \in [0,1]$,*

where $\varphi_t(x) := \varphi(t,x)$ and $\overline{\varphi_t}(x) := \overline{\varphi}(t,x)$ for all $t \in [0,1]$ and $x \in A$.

The above proposition is in fact a special case of more general fact, which is proved in Chapter 3 (see Proposition 3.2.11).

2.5. The Lefschetz fixed point theory for w-maps

To begin with, we shall recall a few notions and their properties. Let $\psi\colon X \multimap X$ be a w-map. If the induced homomorphism $\psi_*\colon \mathbb{H}_*(X,\mathbb{Q}) \to \mathbb{H}_*(X,\mathbb{Q})$ is a Leray endomorphism, then ψ is called a Lefschetz map and for such a w-map ψ we can define the Lefschetz number $\Lambda(\psi)$ of ψ by putting:

$$\Lambda(\psi) = \Lambda(\psi_*).$$

Clearly, if ψ and φ are w-homotopic then $\Lambda(\psi) = \Lambda(\varphi)$. Let us note that if φ has a trivial weight then φ is a Lefschetz w-map and $\Lambda(\varphi) = 0$.

[5] By the constant map at $y_0 \in Y$ we shall understand the function $k\colon X \to Y$ with $k(x) = y_0$ for all $x \in X$.

In 1961 G. Darbo extended the Lefschetz fixed point theorem to w-maps from a compact ANR to itself ([14]). We refer also to [56] for more information about his proof. Moreover, this theorem has been also obtained independently by R. Jerrard ([36]), but only for polyhedra. In 1967 A. Granas gave a generalization of the Lefschetz fixed point theorem for single-valued maps to the case of absolute neighbourhood retracts. Using the method due to A. Granas one can show the following theorem.

Theorem 2.5.1 (Lefschetz Fixed Point Theorem, [60]). *Let X be an* ANR *and let $\varphi: X \multimap X$ be a compact w-map ($\varphi \in K_w(X)$). Then*

(a) *φ is a Lefschetz map;*
(b) *$\Lambda(\varphi) \neq 0$ implies that φ has a fixed point.*

Let us notice that if X is an ANR, then $\mathbb{H}_*(X, \mathbb{Q}) = \{\mathbb{H}_n(X, \mathbb{Q})\}_{n \geq 0}$ does not need to be a graded vector space of finite type. Therefore, for a given φ the ordinary Lefschetz number cannot be well-defined. So, in the proof of the above theorem we have to consider a concept of the generalized Lefschetz number.

Now, following G. Fournier and L. Górniewicz ([22], [19], [23]) we show how the above theorem can be extended to a class of non-compact mappings. For this purpose, we recall the necessary notions and facts. Let (X, A) be a pair of spaces. Given a weighted map $\varphi: (X, A) \multimap (X, A)$ we denote by $\varphi_X: X \multimap X$ and $\varphi_A: A \multimap A$ the respective contractions of φ. Let us consider the graded vector space $\mathbb{H}_*(X, A; \mathbb{Q}) = \{\mathbb{H}_n(X, A; \mathbb{Q})\}_{n \geqslant 0}$. A weighted map $\varphi: (X, A) \multimap (X, A)$ is called a Lefschetz map provided $\varphi_*: \mathbb{H}_*(X, A; \mathbb{Q}) \to \mathbb{H}_*(X, A; \mathbb{Q})$ is a Leray endomorphism. For a weighted map φ we can define the Lefschetz number $\Lambda(\varphi)$ of φ by putting $\Lambda(\varphi) = \Lambda(\varphi_*)$. The following proposition expresses a basic property of the generalized Lefschetz number:

Proposition 2.5.2 ([60]). *Let $\varphi: (X, A) \multimap (X, A)$ be a weighted map. If two of the following weighted maps φ, φ_A, φ_X are the Lefschetz maps, then so is the third one, and in this case we have:*

$$\Lambda(\varphi) = \Lambda(\varphi_X) - \Lambda(\varphi_A).$$

Definition 2.5.3. A weighted map $\varphi: X \multimap X$ is said to be a *compact absorbing contraction* if there exists an open subset U of X such that

(a) $\varphi(U) \subset U$ and the map $\widetilde{\varphi}: U \multimap U$, $\widetilde{\varphi}(x) = \varphi(x)$, is compact,
(b) for every $x \in X$ there exists a natural number n_x such that $\varphi^{n_x}(x) \subset U$.

The set of all compact absorbing contractions will be denoted by $\mathrm{CAC}_w(X)$. Evidently, any compact weighted map $\varphi: X \multimap X$ is a compact absorbing contraction (it is enough to take $U = X$). The main property of the compact absorbing contraction is given in the following:

Proposition 2.5.4 ([60]). *If $\varphi\colon (X,U) \multimap (X,U)$ is a weighted map such that U satisfies two conditions of Definition* 2.5.3*, then φ is a Lefschetz map and $\Lambda(\varphi) = 0$.*

After these preliminaries we are able to formulate the following theorem (see [23]).

Theorem 2.5.5 (Lefschetz Fixed Point Theorem, [60]). *Let $X \in \mathrm{ANR}$ and $\varphi \in \mathrm{CAC}_w(X)$. Then*

(a) *φ is a Lefschetz map;*
(b) *$\Lambda(\varphi) \neq 0$ implies that φ has a fixed point.*

Proof. We choose an open subset $U \subset X$ according to Definition 2.5.3. Let $\overline{\varphi}\colon U \multimap U$ be defined by $\overline{\varphi}(x) = \varphi(x)$, for all $x \in U$. In addition, we consider the map $\widetilde{\varphi}\colon (X,U) \multimap (X,U)$, $\widetilde{\varphi}(x) = \varphi(x)$, for all $x \in X$. From Proposition 2.5.4 we deduce that $\widetilde{\varphi}$ is a Lefschetz map and $\Lambda(\widetilde{\varphi}) = 0$. Since $\overline{\varphi}$ is a compact weighted map and $U \in \mathrm{ANR}$, we conclude from Theorem 2.5.1 that $\overline{\varphi}$ is a Lefschetz map. Consequently, by applying Proposition 2.5.2, we deduce that φ is a Lefschetz map and $\Lambda(\overline{\varphi}) = \Lambda(\varphi)$. Now, if we assume that $\Lambda(\varphi) \neq 0$, then $\Lambda(\overline{\varphi}) \neq 0$ and, consequently, Theorem 2.5.1 implies that $\overline{\varphi}$ has a fixed point. Hence φ has a fixed point, which completes the proof. $\square$

As an immediate consequence of the above theorem we obtain the following corollary.

Corollary 2.5.6. *Let X be an acyclic* ANR *(i.e. $\mathbb{H}_0(X,\mathbb{Q}) \approx \mathbb{Q}$ and $\mathbb{H}_n(X,\mathbb{Q}) = 0$ for every $n \geqslant 1$) or, in particular, a convex subset of a normed space and let $\varphi\colon X \multimap X$ be a w-map with $I_w(\varphi) \neq 0$. Then*

(a) *if $\varphi \in K_w(X)$, then φ has a fixed point;*
(b) *if $\varphi \in \mathrm{CAC}_w(X)$, then φ has a fixed point.*

2.6. Topological degree for w-maps

2.6.1. Topological degree in $\mathbb{R}^n$. The aim of this section is to define the topological degree for w-maps. Topological degree for weighted maps was studied in various forms (see [11], [41], [50]). Our presentation here follows the lines of Ph.D. thesis of S. Jodko-Narkiewicz, but some properties of the topological degree for weighted maps presented in this work, as far as the author knows, are proved for the first time. We first recall the definition and properties of the topological degree. It will be defined by means of the Darbo homology functor. Let U be a bounded open subset of $\mathbb{R}^n$. Furthermore, we set

$$\mathbb{A}(U,\mathbb{R}^n) = \{\varphi\colon \overline{U} \multimap \mathbb{R}^n \mid \varphi \text{ is a weighted map and } 0 \notin \varphi(\partial U)\}$$

where $\overline{U}$ denotes the closure of U and ∂U is the boundary of U. Now, we shall define a map $\deg\colon \mathbb{A}(U,\mathbb{R}^n) \to \mathbb{Q}$. Let us recall that we can think of S^n as the one point compactyfication of $\mathbb{R}^n$, in other words, $S^n = \mathbb{R}^n \cup \{\infty\}$. Take any $\varphi \in \mathbb{A}(U, \mathbb{R}^n)$. Then we have

$$S^n \xrightarrow{k} (S^n, S^n \setminus \varphi_+^{-1}(0)) \xleftarrow{j} (U, U \setminus \varphi_+^{-1}(0)) \xrightarrow{\varphi} (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}),$$

where k, j are the inclusions. Now, we can apply the n-dimensional Darbo homology functor with rational coefficients (from now on up to the end of this chapter, we will omit coefficients from the notations) to the above diagram and we get

$$\begin{tikzcd} \mathbb{Q} = \mathbb{H}_n(S^n) \arrow[r, "k_{*n}"] & \mathbb{H}_n(S^n, S^n \setminus \varphi_+^{-1}(0)) & \mathbb{H}_n(U, U \setminus \varphi_+^{-1}(0)) \arrow[l, "j_{*n}"'] \arrow[d, "\varphi_{*n}"] \\ & & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = \mathbb{Q} \end{tikzcd}$$

where, by means of the excision axiom, j_{*n} is an isomorphism. We define

$$O_\varphi := j_{*n}^{-1} \circ k_{*n}(\mu_n) \in \mathbb{H}_n(S^n, S^n \setminus \varphi_+^{-1}(0)), \tag{2.6}$$

where μ_n is defined in Remark 2.6.1 below.

Remark 2.6.1. Choose, once and for all, a generator $\alpha_1 \in \mathbb{H}_1(\mathbb{R}, \mathbb{R} \setminus \{0\}) \approx \mathbb{Q}$ and assume inductively that a generator $\alpha_{n-1} \in \mathbb{H}_{n-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \setminus \{0\})$ has been constructed. Then we define a generator $\alpha_n \in \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ by setting $\alpha_1 \times \alpha_{n-1}$. Consider also the composition

$$\mathbb{H}_n(S^n) \xrightarrow{k_*} \mathbb{H}_n(S^n, S^n \setminus 0) \xleftarrow{j_*} \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

and define

$$\mu_n = k_*^{-1} \circ j_*(\alpha_n) \in \mathbb{H}_n(S^n) \approx \mathbb{Q}.$$

Since the composition is an isomorphism, μ_n is a generator of $\mathbb{H}_n(S^n)$.

Moreover, taking into account Remark 2.3.25, we obtain the following fact.

Lemma 2.6.2. *Let $\mu_n \in \mathbb{H}_n(S^n)$, $\mu_m \in \mathbb{H}_m(S^m)$, $\mu_{n+m} \in \mathbb{H}_{n+m}(S^{n+m})$. In addition, let*

$$\alpha_m \in \mathbb{H}_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}), \quad \alpha_n \in \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}),$$
$$\alpha_{n+m} \in \mathbb{H}_{n+m}(\mathbb{R}^{n+m}, \mathbb{R}^{n+m} \setminus \{0\}).$$

Then

$$\mu_n \times \mu_m = \mu_{n+m}, \qquad \alpha_n \times \alpha_m = \alpha_{n+m},$$

where $\times$ stands for the cross product.

Definition 2.6.3. Let $\varphi \in \mathbb{A}(U, \mathbb{R}^n)$. We define the *topological degree* $\deg(\varphi, U, \mathbb{R}^n)$ of φ by

$$\varphi_{*n}(O_\varphi) = \deg(\varphi, U, \mathbb{R}^n) \cdot \alpha_n. \tag{2.7}$$

Lemma 2.6.4. *Given $\varphi \in \mathbb{A}(U, \mathbb{R}^n)$, let V be an open subset of U and let K be a compact set such that $\varphi_+^{-1}(0) \subset K \subset V$. Then*

$$(\varphi|V)_{*n} \circ (j'_{*n})^{-1} \circ k'_{*n}(\mu_n) = \varphi_{*n}(O_\varphi),$$

where $k'_{*n}\colon \mathbb{H}_n(S^n) \to \mathbb{H}_n(S^n, S^n \setminus K)$ and $j'_{*n}\colon \mathbb{H}_n(V, V \setminus K) \to \mathbb{H}_n(S^n, S^n \setminus K)$ are homomorphisms induced by inclusions.

Proof. The lemma follows easily from the following commutative diagram:

$$\begin{tikzcd}
\mathbb{H}_n(S^n) \arrow[r, "k_{*n}"] \arrow[d, "\mathrm{id}"'] & \mathbb{H}_n(S^n, S^n \setminus F) & \mathbb{H}_n(U, U \setminus F) \arrow[l, "j_{*n}"'] \arrow[r, "\varphi_{*n}"] & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\
\mathbb{H}_n(S^n) \arrow[r, "k'_{*n}"'] & \mathbb{H}_n(S^n, S^n \setminus K) \arrow[u] & \mathbb{H}_n(V, V \setminus K) \arrow[l, "j'_{*n}"] \arrow[u] \arrow[r, "(\varphi|V)_{*n}"'] & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

where $F := \varphi_+^{-1}(0)$. □

Remark 2.6.5. The above lemma implies that we may replace a set $\varphi_+^{-1}(0)$ in the definition of the topological degree of φ by any larger compact set K contained in U.

Below we shall list some properties of the topological degree defined above.

Proposition 2.6.6 (Existence)**.** *Let $\varphi \in \mathbb{A}(U, \mathbb{R}^n)$. If $\deg(\varphi, U, \mathbb{R}^n) \neq 0$, then $0 \in \varphi(x)$ for some $x \in U$.*

Proof. Suppose that $0 \notin \varphi(x)$ for all $x \in U$. Hence $\varphi_+^{-1}(0) = \emptyset$ and therefore $\mathbb{H}_n(U, U \setminus \varphi_+^{-1}(0)) = 0$, and we conclude that $\deg(\varphi, U, \mathbb{R}^n) = 0$. □

Proposition 2.6.7 (Excision)**.** *Let $\varphi \in \mathbb{A}(U, \mathbb{R}^n)$ and let V be an open subset of U containing $\varphi_+^{-1}(0)$, then*

$$\deg(\varphi, U, \mathbb{R}^n) = \deg(\varphi|\overline{V}, V, \mathbb{R}^n).$$

Proof. The proof follows immediately from the commutative diagram:

$$\begin{tikzcd}
\mathbb{H}_n(S^n) \arrow[r, "k_{*n}"] \arrow[dr, "k_{*n}"'] & \mathbb{H}_n(S^n, S^n \setminus F) & \mathbb{H}_n(U, U \setminus F) \arrow[l, "j_{*n}"'] \arrow[r, "\varphi_{*n}"] & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\
& \mathbb{H}_n(S^n, S^n \setminus F) \arrow[u] & \mathbb{H}_n(V, V \setminus F) \arrow[l, "j'_{*n}"] \arrow[u] \arrow[ur, "(\varphi|V)_{*n}"'] &
\end{tikzcd}$$

where $F := \varphi_+^{-1}(0)$. □

Lemma 2.6.8. *Let U_1 and U_2 be disjoint open subsets of S^n and let F be a closed subset of S^n such that $F \subset U_1 \cup U_2$. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathbb{H}_n(S^n, S^n \setminus F) \arrow[r, "(i_{1*}, i_{1*})"] & \mathbb{H}_n(S^n, S^n \setminus F_1) \oplus \mathbb{H}_n(S^n, S^n \setminus F_2) \\
\mathbb{H}_n(U_1 \cup U_2, U_1 \cup U_2 \setminus F) \arrow[u, "j_*"] & \mathbb{H}_n(U_1, U_1 \setminus F_1) \oplus \mathbb{H}_n(U_2, U_2 \setminus F_2) \arrow[l, "h_{1*}+h_{2*}"] \arrow[u, "j_{1*} \oplus j_{2*}"']
\end{tikzcd}$$

where all the homomorphisms in the above diagram are induced by inclusions and $F_i := F \cap U_i$ for $i = 1, 2$.

Proof. The above fact was given in terms of singular homology in [9]. Consequently, taking into account Theorem 2.3.1, we conclude that the above lemma is also true. □

Proposition 2.6.9 (Additivity). *Let $\varphi \in \mathbb{A}(U, \mathbb{R})$ and let U_1 and U_2 be two disjoint open subsets of U such that $\varphi_+^{-1}(0) \subset U_1 \cup U_2$. Then*

$$\deg(\varphi, U, \mathbb{R}^n) = \deg(\varphi_1, U_1, \mathbb{R}^n) + \deg(\varphi_2, U_2, \mathbb{R}^n),$$

where φ_i denotes the restriction of φ to U_i.

Proof. By Proposition 2.6.6, we may replace U by $U_1 \cup U_2$ in the definition of $\deg(\varphi, U, \mathbb{R}^n)$. Let $F = \varphi_+^{-1}(0)$ and $F_i = F \cap U_i$. Now, it is enough to prove that the following diagram

$$\begin{tikzcd}
\mathbb{H}_n(S^n) \arrow[d, "k_*"] \arrow[dr, "(k_{1*}, k_{2*})"] & \\
\mathbb{H}_n(S^n, S^n \setminus F) \arrow[r, "(i_{1*}, i_{2*})"] & \mathbb{H}_n(S^n, S^n \setminus F_1) \oplus \mathbb{H}_n(S^n, S^n \setminus F_2) \\
\mathbb{H}_n(S^n, S^n \setminus F) \arrow[u, "\simeq", "j_*"'] \arrow[d, "\varphi_*"] & \mathbb{H}_n(U_1, U_1 \setminus F_1) \oplus \mathbb{H}_n(U_2, U_2 \setminus F_2) \arrow[l, "h_{1*}+h_{2*}"'] \arrow[u, "\simeq", "j_{1*} \oplus j_{2*}"'] \arrow[dl, "\varphi_{1*}+\varphi_{2*}"] \\
\mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) &
\end{tikzcd}$$

commutes, in which, except for the weighted map φ and its restrictions, all the homomorphisms are induced by inclusions. For this purpose, let us observe that the middle square commutes by Lemma 2.6.8. Furthermore, the commutativity of the top and bottom triangles follows from the definitions and the proof is complete. □

Proposition 2.6.10 (Unity)**.** *If U contains the origin and $i\colon U \hookrightarrow \mathbb{R}^n$ is the inclusion of U in $\mathbb{R}^n$, then* $\deg(i, U, \mathbb{R}^n) = 1$.

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{H}_n(S^n) \arrow[r, "k_{*n}"] \arrow[dr, "k_{*n}"'] & \mathbb{H}_n(S^n, S^n \setminus \{0\}) & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[l, "j_{*n}"'] \\
 & \mathbb{H}_n(S^n, S^n \setminus \{0\}) \arrow[u, "\mathrm{id}_{*n}"'] & \mathbb{H}_n(U, U \setminus \{0\}) \arrow[l, "j'_{*n}"] \arrow[u, "i_{*n}"']
\end{tikzcd}$$

Now, taking into account Remark 2.6.1, (2.6) and (2.7), we obtain the desired conclusion. $\square$

Proposition 2.6.11 (Linearity)**.** *If $\varphi = \alpha\varphi_1 \cup \beta\varphi_2$, then*

$$\deg(\varphi, U, \mathbb{R}^n) = \alpha \cdot \deg(\varphi_1, U, \mathbb{R}^n) + \beta \cdot \deg(\varphi_2, U, \mathbb{R}^n),$$

where $\alpha, \beta \in \mathbb{Q}$.

Proof. Let $F := \varphi_+^{-1}(0)$, $K_1 := (\varphi_1)_+^{-1}(0)$ and $K_2 := (\varphi_2)_+^{-1}(0)$. Then $F = K_1 \cup K_2$. Consider the following diagrams:

$$\begin{tikzcd}
\mathbb{H}_n(S^n) \arrow[r, "k_{*n}"] \arrow[d, "\mathrm{id}"'] & \mathbb{H}_n(S^n, S^n \setminus F) & \mathbb{H}_n(U, U \setminus F) \arrow[l, "j_{*n}"'] \arrow[r, "(\varphi_1)_{*n}"] & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\
\mathbb{H}_n(S^n) \arrow[r, "k'_{*n}"'] & \mathbb{H}_n(S^n, S^n \setminus K_1) \arrow[u] & \mathbb{H}_n(U, U \setminus K_1) \arrow[l, "j'_{*n}"] \arrow[r, "(\varphi'_1)_{*n}"'] & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{H}_n(S^n) \arrow[r, "k_{*n}"] \arrow[d, "\mathrm{id}"'] & \mathbb{H}_n(S^n, S^n \setminus F) & \mathbb{H}_n(U, U \setminus F) \arrow[l, "j_{*n}"'] \arrow[r, "(\varphi_2)_{*n}"] & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\
\mathbb{H}_n(S^n) \arrow[r, "\widetilde{k}_{*n}"'] & \mathbb{H}_n(S^n, S^n \setminus K_2) \arrow[u] & \mathbb{H}_n(U, U \setminus K_2) \arrow[l, "\widetilde{j}_{*n}"] \arrow[r, "(\widetilde{\varphi}_2)_{*n}"'] & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

$$\mathbb{H}_n(S^n) \xrightarrow{k_{*n}} \mathbb{H}_n(S^n, S^n \setminus F) \xleftarrow{j_{*n}} \mathbb{H}_n(U, U \setminus F) \xrightarrow{\varphi_{*n}} \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

Moreover, $\varphi_{*n} = (\alpha\varphi_1 \cup \beta\varphi_2)_{*n} = \alpha(\varphi_1)_{*n} + \beta(\varphi_2)_{*n}$. Consequently, taking into account the above commutative diagrams and Definition 2.6.3, we obtain

$$\begin{aligned}
\deg(\varphi, U, \mathbb{R}^n) \cdot \alpha_n &= \varphi_{*n}(O_\varphi) = \alpha(\varphi_1)_{*n}(O_\varphi) + \beta(\varphi_2)_{*n}(O_\varphi) \\
&= \alpha(\varphi'_1)_{*n}(O_{\varphi_1}) + \beta(\widetilde{\varphi}_2)_{*n}(O_{\varphi_2}) \\
&= \alpha \cdot \deg(\varphi_1, U, \mathbb{R}^n) \cdot \alpha_n + \beta \cdot \deg(\varphi_2, U, \mathbb{R}^n) \cdot \alpha_n \\
&= (\alpha \cdot \deg(\varphi_1, U, \mathbb{R}^n) + \beta \cdot \deg(\varphi_2, U, \mathbb{R}^n)) \cdot \alpha_n,
\end{aligned}$$

which proves that

$$\deg(\varphi, U, \mathbb{R}^n) = \alpha \cdot \deg(\varphi_1, U, \mathbb{R}^n) + \beta \cdot \deg(\varphi_2, U, \mathbb{R}^n)$$

as required. $\square$

Proposition 2.6.12 (w-Homotopy invariance)**.** *Let $\Upsilon\colon \overline{U}\times[0,1] \multimap \mathbb{R}^n$ be a w-map and*

$$K := \{x \in \overline{U} \mid 0 \in \Upsilon(x,t) \text{ for some } t \in [0,1]\}.$$

If $K \cap \partial U = \emptyset$, then $\deg(\Upsilon_0, U, \mathbb{R}^n) = \deg(\Upsilon_1, U, \mathbb{R}^n)$, where $\Upsilon_0(x) = \Upsilon(x,0)$ and $\Upsilon_1(x) = \Upsilon(x,1)$, for all $x \in \overline{U}$.

Proof. First, let us observe that $(\Upsilon_t)_+^{-1}(0) \subset K$ for every $t \in [0,1]$. Consider now the following diagram:

$$\mathbb{H}_n(S^n) \xrightarrow{k_{*n}} \mathbb{H}_n(S^n, S^n\setminus K) \xleftarrow{j_{*n}} \mathbb{H}_n(U, U\setminus K) \xrightarrow{(\Upsilon_t)_{*n}} \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$$

for any $t \in [0,1]$. Then, by Remark 2.6.5 and (2.7), we obtain

$$(\Upsilon_t)_{*n} \circ (j_{*n})^{-1} \circ k_{*n}(\mu_n) = \deg(\Upsilon_t, U, \mathbb{R}^n)\cdot \alpha_n, \tag{2.8}$$

for all $t \in [0,1]$. From the w-homotopy invariance of the Darbo homology functor it follows that

$$(\Upsilon_0)_* = (\Upsilon_1)_*. \tag{2.9}$$

Consequently, taking into account (2.8) and (2.9), we get

$$\deg(\Upsilon_0, U, \mathbb{R}^n) = \deg(\Upsilon_1, U, \mathbb{R}^n),$$

which completes the proof. $\square$

Before proceeding further, we need to prove the following lemma.

Lemma 2.6.13. *Let $\varphi \in \mathbb{A}(U,\mathbb{R}^n)$ and $\psi \in \mathbb{A}(V,\mathbb{R}^m)$. Then*

$$O_\varphi \times O_\psi = O_{\varphi\times\psi} \in \mathbb{H}_n(S^n, S^n \setminus (\varphi\times\psi)_+^{-1}(0)).$$

Following [9] we define a map $\pi\colon S^p\times S^q \to S^{p+q}$, $p \geqslant 0, q\geqslant 0$, which will be needed in the proof of the above lemma. Define an equivalence relation $\sim$ on $[0,1]^{p+q} = [0,1]^p \times [0,1]^q$ by setting $(x_1,y_1)\sim(x_2,y_2)$ if and only if one of the following conditions holds

(1) $x_1, x_2 \in \partial I^p$ and $y_1 = y_2$,
(2) $x_1 = x_2$ and $y_1, y_2 \in \partial I^q$.

It is well-known that the quotient space $([0,1]^p\times[0,1]^q)/\sim$ is homeomorphic to $S^p\times S^q$ and that the quotient space $[0,1]^{p+q}/\partial[0,1]^{p+q}$ is homeomorphic to S^{p+q}. Consequently, we can define

$$\pi\colon ([0,1]^p\times[0,1]^q)/\sim \longrightarrow [0,1]^{p+q}/\partial[0,1]^{p+q}$$

by $\pi([x]_\sim) = [x]_{\partial[0,1]^{p+q}}$ for any $[x]_\sim \in [0,1]^p\times[0,1]^q/\sim$. Thus, π induces the following homomorphism:

$$\pi_*\colon \mathbb{H}_{p+q}(S^p\times S^q) \to \mathbb{H}_{p+q}(S^{p+q}).$$

Lemma 2.6.14. *Let $\mu_p \in \mathbb{H}_p(S^p)$, $\mu_q \in \mathbb{H}_q(S^q)$ and let π_* be as above. Then*

$$\pi_*(\mu_p \times \mu_q) = \mu_{p+q}.$$

Proof. The proof of the above statement is the same as for singular homology given in [9]. $\square$

Now we are ready to give a proof of Lemma 2.6.13.

Proof of Lemma 2.6.13 (due to R. Brown). Let $F := \varphi_+^{-1}(0)$, $G := \psi_+^{-1}(0)$. Consider the following diagram

$$\begin{tikzcd}
\mathbb{H}_m(S^m) \otimes \mathbb{H}_n(S^n) \arrow[r, "\times"] \arrow[d, "k'_* \otimes k''_*"'] & \mathbb{H}_{m+n}(S^m \times S^n) \arrow[d, "(k' \times k'')_*"] \\
\mathbb{H}_m(S^m, S^m \setminus F) \otimes \mathbb{H}_n(S^n, S^n \setminus G) \arrow[r, "\times"] \arrow[d, "{j'_*}^{-1} \otimes {j''_*}^{-1}"'] & \mathbb{H}_{m+n}(S^m \times S^n, S^m \times S^n \setminus F \times G) \arrow[d, "(j' \times j'')_*^{-1}"] \\
\mathbb{H}_m(U, U \setminus F) \otimes \mathbb{H}_n(V, V \setminus G) \arrow[r, "\times"'] & \mathbb{H}_{m+n}(U \times V, U \times V \setminus F \times G)
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\pi_*"] & \mathbb{H}_{m+n}(S^{m+n}) \arrow[d, "k_*"] \\
{} \arrow[r, "\pi_*"] & \mathbb{H}_{m+n}(S^{m+n}, S^{m+n} \setminus F \times G) \arrow[d, "j_*^{-1}"] \\
{} \arrow[r, "\mathrm{id}"] & \mathbb{H}_{m+n}(U \times V, U \times V \setminus F \times G)
\end{tikzcd}$$

in which, except for the cross product and π_*, all the homomorphisms are induced by inclusions. Commutativity of the above diagram can be established in the same way as in [9]. Consequently, taking into account Lemmas 2.6.2 and 2.6.14 and commutativity of the above diagram, we deduce that $O_\varphi \times O_\psi = O_{\varphi\times\psi}$, which completes the proof. $\square$

Proposition 2.6.15 (Multiplicativity)**.** *Let $\varphi \in \mathbb{A}(U, \mathbb{R}^n)$ and $\psi \in \mathbb{A}(V, \mathbb{R}^m)$. Then $\varphi \times \psi \in \mathbb{A}(U \times V, \mathbb{R}^{n+m})$ and*

$$\deg(\varphi \times \psi, U \times V, \mathbb{R}^{n+m}) = \deg(\varphi, U, \mathbb{R}^n) \cdot \deg(\psi, V, \mathbb{R}^m).$$

Proof. First, let us observe that $(\varphi \times \psi)_+^{-1}(0) = \varphi_+^{-1}(0) \times \psi_+^{-1}(0)$. Thus $\varphi \times \psi \in \mathbb{A}(U \times V, \mathbb{R}^{n+m})$. Moreover, we have

$$\begin{aligned}
\deg(\varphi \times \psi, U \times V, \mathbb{R}^n \times \mathbb{R}^m)\alpha_{n+m} &= (\varphi \times \psi)_*(O_{\varphi\times\psi}) \\
&\overset{2.6.13}{=} (\varphi \times \psi)_{*n}(O_\varphi \times O_\psi) \overset{2.3.24}{=} \varphi_{*n}(O_\varphi) \times \psi_{*n}(O_\psi) \\
&= (\deg(\varphi, U, \mathbb{R}^n)\alpha_n) \times (\deg(\psi, V, \mathbb{R}^m)\alpha_m) \\
&= \deg(\varphi, U, \mathbb{R}^n) \cdot \deg(\psi, V, \mathbb{R}^m)\alpha_n \times \alpha_m.
\end{aligned}$$

Since, by Lemma 2.6.2, $\alpha_{n+m} = \alpha_n \times \alpha_m$, it follows that

$$\deg(\varphi \times \psi, U \times V, \mathbb{R}^{n+m}) = \deg(\varphi, U, \mathbb{R}^n) \cdot \deg(\psi, V, \mathbb{R}^m),$$

which completes the proof. $\square$

Proposition 2.6.16. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism and let $\varphi \in \mathbb{A}(U, \mathbb{R}^n)$. Then*

$$\deg(T \circ \varphi \circ T^{-1}|T(\overline{U}), T(U), \mathbb{R}^n) = \deg(\varphi, U, \mathbb{R}^n).$$

Proof. Let $K := \varphi_+^{-1}(0)$. Then $(T \circ \varphi \circ T^{-1})_+^{-1}(0) = T(K)$. Consider now the following commutative diagram:

$$\begin{tikzcd} \mathbb{H}_n(S^n) \arrow[r, "i_{*n}"] \arrow[d, "\cong"', "\widehat{T}_{*n}"] & \mathbb{H}_n(S^n, S^n \setminus K) \arrow[d, "\cong"', "\widetilde{T}_{*n}"] & \mathbb{H}_n(U, U \setminus K) \arrow[l, "j_{*n}"', "\cong"] \arrow[d, "\cong"', "(T|U)_{*n}"] \\ \mathbb{H}_n(S^n) \arrow[r, "i_{*n}"'] & \mathbb{H}_n(S^n, S^n \setminus T(K)) & \mathbb{H}_n(T(U), T(U) \setminus T(K)) \arrow[l, "j_{*n}", "\cong"'] \end{tikzcd} \tag{2.10}$$

where $\widehat{T}\colon S^n = \mathbb{R}^n \cup \{\infty\} \to S^n = \mathbb{R}^n \cup \{\infty\}$ is given by

$$\widehat{T}(x) = \begin{cases} T(x) & \text{for } x \in \mathbb{R}^n, \\ \infty & \text{for } x = \infty, \end{cases}$$

and $\widetilde{T}$ is induced by $\widehat{T}$. Moreover, the following diagram also commutes:

$$\begin{tikzcd} \mathbb{H}_n(S^n) \arrow[r, "i_{*n}"] \arrow[d, "\cong"', "\widehat{T}_{*n}"] & \mathbb{H}_n(S^n, S^n \setminus \{0\}) \arrow[d, "\cong"', "T'_{*n}"] & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[l, "\cong", "j_{*n}"'] \arrow[d, "\cong"', "T_{*n}"] \\ \mathbb{H}_n(S^n) \arrow[r, "i_{*n}"'] & \mathbb{H}_n(S^n, S^n \setminus \{0\}) & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[l, "\cong", "j_{*n}"'] \end{tikzcd} \tag{2.11}$$

where $T'\colon (S^n, S^n \setminus \{0\}) \to (S^n, S^n \setminus \{0\})$ is induced by $\widehat{T}$.

We shall consider two possible cases: $T_{*n}(\alpha_n) = \alpha_n$ or $T_{*n}(\alpha_n) = -\alpha_n$, where α_n is a generator of $\mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. First, suppose that $T_{*n}(\alpha_n) = \alpha_n$. Then, by Remark 2.6.1 and (2.11), we get $\widehat{T}_*(\alpha_n) = \alpha_n$. Consequently, from (2.10) we obtain

$$(T|U)_{*n}(O_\varphi) = O_{T\circ\varphi\circ T^{-1}}.$$

Thus

$$\begin{aligned} T_{*n} \circ \varphi_{*n} \circ (T^{-1}|T(U))_{*n}(O_{T\circ\varphi\circ T^{-1}}) &= T_{*n} \circ \varphi_{*n}(O_\varphi) \\ &= T_{*n}(\deg(\varphi, U, \mathbb{R}^n)\alpha_n) = \deg(\varphi, U, \mathbb{R}^n) T_{*n}(\alpha_n) = \deg(\varphi, U, \mathbb{R}^n)\alpha_n. \end{aligned}$$

But

$$(T \circ \varphi \circ T^{-1}|T(U))_{*n} = T_{*n} \circ \varphi_{*n} \circ (T^{-1}|T(U))_{*n}$$

and therefore

$$\deg(T \circ \varphi \circ T^{-1}|T(\overline{U}), T(U), \mathbb{R}^n) = \deg(\varphi, U, \mathbb{R}^n), \tag{2.12}$$

which completes the proof of the case $T_{*n}(\alpha_n) = \alpha_n$.

If $T_*(\alpha_n) = -\alpha_n$, then, taking into account Remark 2.6.1 and (2.11), we deduce that $\widehat{T}_*(\alpha_n) = -\alpha_n$ and hence, by (2.10), we have $(T|U)_{*n}(O_\varphi) = -O_{T\circ\varphi\circ T^{-1}}$. Thus,

$$\begin{aligned} T_{*n} \circ \varphi_{*n} \circ (T^{-1}|T(U))_{*n}(O_{T\circ\varphi\circ T^{-1}}) &= T_{*n} \circ \varphi_{*n}(-O_\varphi) \\ &= -T_{*n}(\deg(\varphi, U, \mathbb{R}^n)\alpha_n) = -\deg(\varphi, U, \mathbb{R}^n) T_{*n}(\alpha_n) \\ &= -\deg(\varphi, U, \mathbb{R}^n) \cdot (-\alpha_n) = \deg(\varphi, U, \mathbb{R}^n) \cdot \alpha_n. \end{aligned}$$

So in this case the equality (2.12) is also true, which completes the proof of the proposition. □

Lemma 2.6.17. *Let $\varphi \in \mathbb{A}(U, \mathbb{R}^n)$. If $\varphi(\overline{U}) \subset \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$, then*

$$\deg(\varphi, U, \mathbb{R}^n) = \deg(\pi_{n-1} \circ \varphi \circ i_{n-1} | \overline{V}, V, \mathbb{R}^{n-1}),$$

where $\pi_{n-1} \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection onto the first $n-1$ coordinates, $i_{n-1} \colon \mathbb{R}^{n-1} \to \mathbb{R}^n$ is the inclusion and $V := \pi_{n-1}(U \cap (\mathbb{R}^{n-1} \times \{0\}))$.

Proof. Let $Z := \varphi_+^{-1}(0) \subset U$. If $Z = \emptyset$, then

$$\deg(\varphi, U, \mathbb{R}^n) = 0 = \deg(\pi_{n-1} \circ \varphi \circ i_{n-1} | \overline{V}, V, \mathbb{R}^{n-1}).$$

So we can assume that $Z \neq \emptyset$. Since Z is compact, there exists $\varepsilon > 0$ such that $O_\varepsilon(Z) \subset U$. By Propositions 2.3.5 and 2.3.6 and Theorem 2.3.9, the following diagram is commutative

$$\begin{tikzcd} \mathbb{H}_n(S^n) \arrow[r, "k_*"] \arrow[d, "\partial_*", "\cong"'] & \mathbb{H}_n(S^n, S^n \setminus Z) \arrow[d, "\partial_*"] & \mathbb{H}_n(O_\varepsilon(Z), O_\varepsilon(Z) \setminus Z) \arrow[l, "\cong", "j_*"'] \arrow[r, "\widetilde{\varphi}_*"] \arrow[d, "\partial_*"] & \mathbb{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \arrow[d, "\partial_*", "\cong"'] \\ \mathbb{H}_{n-1}(S^{n-1}) \arrow[r, "k'_*"'] & \mathbb{H}_{n-1}(S^{n-1}, S^{n-1} \setminus Z) & \mathbb{H}_{n-1}(\widetilde{O}_\varepsilon(Z), \widetilde{O}_\varepsilon(A) \setminus Z) \arrow[l, "\cong", "j'_*"'] \arrow[r, "\varphi'_*"'] & \mathbb{H}_{n-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \setminus \{0\}) \end{tikzcd}$$

where $\widetilde{\varphi}$ and φ' are induced by φ, ∂_* is the Mayer–Vietoris homomorphism and $\widetilde{O}_\varepsilon(Z) := O_\varepsilon(Z) \cap \mathbb{R}^{n-1}$. Now the assertion of our lemma follows immediately from the above diagram. □

Proposition 2.6.18. *Let $\varphi \in \mathbb{A}(U, \mathbb{R}^n)$. If $\varphi(\overline{U}) \subset \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$, $n > m$, then*

$$\deg(\varphi, U, \mathbb{R}^n) = \deg(\pi_m \circ \varphi \circ i_m | \overline{U_m}, U_m, \mathbb{R}^m),$$

where $\pi_m\colon \mathbb{R}^n \to \mathbb{R}^m$ *is the projection onto the first* m *coordinates,* $i_m\colon \mathbb{R}^m \to \mathbb{R}^n$ *is the inclusion and* $U_m := \pi_m(U \cap (\mathbb{R}^m \times \{0\}))$.

Proof. Let $k := n - m$ Then, by applying k times Lemma 2.6.17, we obtain the desired conclusion. $\square$

We now extend the topological degree to any n-dimensional normed linear space E^n. Let U be an open and bounded subset of E^n and let $T\colon \mathbb{R}^n \to E^n$ be any linear isomorphism. Furthermore, we put

$$\mathbb{A}(U, E^n) = \{\varphi\colon \overline{U} \multimap E^n \mid \varphi \text{ is a weighted map and } 0 \notin \varphi(\partial U)\}.$$

Definition 2.6.19. Let $\varphi \in \mathbb{A}(U, E^n)$. We define the topological degree of $\varphi\colon \overline{U} \multimap E^n$ by

$$\deg(\varphi, U, E^n) := \deg(T^{-1} \circ \varphi \circ T|T^{-1}(\overline{U}), T^{-1}(U), \mathbb{R}^n). \tag{2.13}$$

Lemma 2.6.20. *The above definition is independent of the choice of the isomorphism* $T\colon \mathbb{R}^n \to E^n$.

Proof. Let $T_1\colon \mathbb{R}^n \to E^n$ and $T_1^{-1}\colon \mathbb{R}^n \to E^n$ be two linear isomorphisms. We will show that

$$\deg(T_1^{-1} \circ \varphi \circ T_1|T_1^{-1}(\overline{U}), T_1^{-1}(U), \mathbb{R}^n) = \deg(T_2^{-1} \circ \varphi \circ T_2|T_2^{-1}(\overline{U}), T_2^{-1}(U), \mathbb{R}^n).$$

Let $T_3\colon \mathbb{R}^n \to \mathbb{R}^n$ be given by $T_3 := T_2^{-1} \circ T_1$. Then Proposition 2.6.16 implies that

$$\begin{aligned}\deg(T_1^{-1} \circ \varphi \circ T_1|T_1^{-1}(\overline{U}), T_1^{-1}(U), \mathbb{R}^n)&\\ &= \deg(T_3 \circ (T_1^{-1} \circ \varphi \circ T_1) \circ T_3^{-1}|T_3(T_1^{-1}(\overline{U})), T_3(T_1^{-1}(U)), \mathbb{R}^n)\\ &= \deg(T_2^{-1} \circ \varphi \circ T_2|T_2^{-1}(\overline{U}), T_2^{-1}(U), \mathbb{R}^n),\end{aligned}$$

which completes the proof. $\square$

Theorem 2.6.21. *Let* $\varphi \in \mathbb{A}(U, E^n)$. *The topological degree defined in* (2.13) *satisfies the following properties:*

(a) (Existence) *If* $\deg(\varphi, U, E^n) \neq 0$, *then* $0 \in \varphi(x)$ *for some* $x \in U$.

(b) (Additivity) *Let* U_1 *and* U_2 *be two disjoint open subsets of* U *such that* $\varphi_+^{-1}(0) \subset U_1 \cup U_2$. *Then*

$$\deg(\varphi, U, E^n) = \deg(\varphi_1, U_1, E^n) + \deg(\varphi_2, U_2, E^n),$$

where φ_i *denotes the restriction of* φ *to* U_i.

(c) (Contraction) *Let* $\varphi\colon \overline{U} \multimap E'$ *be a weighted map such that* $\varphi_+^{-1}(0) \cap \partial U = \emptyset$, *where* E' *is some linear subspace of* E^n. *Then*

$$\deg(i \circ \varphi, U, E^n) = \deg(\varphi|\overline{U \cap E'}, U \cap E', E'),$$

where $i\colon E' \hookrightarrow E^n$ *is the inclusion.*

(d) (Linearity) *If* $\varphi = \alpha\varphi_1 \cup \beta\varphi_2$*, then*

$$\deg(\varphi, U, E^n) = \alpha \cdot \deg(\varphi_1, U, E^n) + \beta \cdot \deg(\varphi_2, U, E^n).$$

(e) (Unity) *If* $i\colon \overline{U} \hookrightarrow E^n$ *is the inclusion and* $0 \in U$*, then* $\deg(i, U, E^n) = 1$.

(f) (Multiplicativity) *Let* $\varphi_1 \in \mathbb{A}(U_1, E^n)$ *and* $\varphi_2 \in \mathbb{A}(U_2, E^m)$*. Then* $\varphi_1 \times \varphi_2 \in \mathbb{A}(U_1 \times U_2, E^n \times E^m)$ *and*

$$\deg(\varphi_1 \times \varphi_2, U_1 \times U_2, E^n \times E^m) = \deg(\varphi_1, U_1, E^n) \cdot \deg(\varphi_2, U_2, E^m).$$

(g) (w-Homotopy invariance) *Let* $\Upsilon\colon \overline{U} \times [0,1] \multimap E^n$ *be a w-map and let*

$$K := \{x \in \overline{U} : 0 \in \Upsilon(x,t) \text{ for some } t \in [0,1]\}.$$

If $K \cap \partial U = \emptyset$*, then* $\deg(\Upsilon_0, U, E^n) = \deg(\Upsilon_1, U, E^n)$.

Proof. Properties (a)–(g) for this topological degree follow immediately from those of the topological degree in $\mathbb{R}^n$. □

2.6.2. Topological degree in normed spaces. Throughout this section, E will always denote a normed space with the norm $||\cdot||$. By U we denote an open and bounded subset of a normed space E.

Now, we define the topological degree for weighted maps in normed spaces. We begin with the Schauder approximation theorem.

Theorem 2.6.22. *Let X be a space, U an open subset of a normed space E and $f\colon X \to U$ a compact map. Then, for each sufficiently small $\varepsilon > 0$, there exists a finite-dimensional subspace E_ε of E and a compact map $f_\varepsilon\colon X \to E_\varepsilon$ such that*

(a) $\overline{f_\varepsilon(X)} \subset E_\varepsilon \cap U$,
(b) $||f(x) - f_\varepsilon(x)|| < \varepsilon$ *for all* $x \in X$,
(c) f_ε *is homotopic to* f.

We also state a simple lemma that will be frequently used.

Lemma 2.6.23. *Let $\varphi\colon \overline{U} \multimap E$ be a compact weighted map such that $x \notin \varphi(x)$ for all $x \in \partial U$, then*

$$\varepsilon_0(\varphi, U) := \inf\{||x-y|| \mid x \in \partial U,\ y \in \varphi(x)\} > 0.$$

Definition 2.6.24. Let $\varphi\colon \overline{U} \multimap E$ be a weighted map. Then a weighted map $\Phi\colon \overline{U} \multimap E$ defined by the formula:

$$\Phi(x) = x - \varphi(x) \quad \text{for every } x \in \overline{U}$$

is called a *weighted vector field associated with* φ, and if φ is compact then Φ is called a *compact weighted vector field.*

By $\mathbb{A}_c(U, E)$ we shall denote the set of all compact weighted vector fields $\Phi\colon \overline{U} \multimap E$ satisfying the following condition: $0 \notin \Phi(x)$, for all $x \in \partial U$.

Let $\Phi = i - \varphi \in \mathbb{A}_c(U,E)$, $K := \overline{\varphi(\overline{U})}$ and $\varepsilon < (1/2)\varepsilon_0(\varphi,U)$. By using the Schauder approximation theorem, we get an ε-approximation $\pi_\varepsilon\colon K \to E_\varepsilon$ of the inclusion $j\colon K \to E$. Let $U_\varepsilon = U \cap E_\varepsilon$ and define $\varphi_\varepsilon\colon \overline{U_\varepsilon} \multimap E_\varepsilon$ to be the restriction of $\pi_\varepsilon \circ \varphi$ to $\overline{U_\varepsilon}$. Now, let us observe that $0 \notin (i_\varepsilon - \varphi_\varepsilon)(x)$ for any $x \in \partial_{E_\varepsilon} U_\varepsilon$ (6), where $i_\varepsilon\colon \overline{U_\varepsilon} \to E_\varepsilon$ is the inclusion. Indeed, suppose on the contrary that $0 \in (i_\varepsilon - \varphi_\varepsilon)(x_0)$ for some $x_0 \in \partial_{E_\varepsilon} U_\varepsilon$. Then $x_0 \in \varphi_\varepsilon(x_0)$ and hence $x_0 = \pi_\varepsilon(y_0)$ for some $y_0 \in \varphi(x_0)$. Consequently, we obtain

$$||y_0 - x_0|| = ||y_0 - \pi_\varepsilon(y_0)|| < \varepsilon < (1/2)\varepsilon_0(\varphi,U). \tag{2.14}$$

On the other hand, since $\partial_{E_\varepsilon} U_\varepsilon \subset \partial U$, it follows that $x_0 \in \partial U$, and then

$$||y_0 - x_0|| \geqslant \varepsilon_0(\varphi,U). \tag{2.15}$$

Now, taking into account (2.14) and (2.15), we obtain a contradiction, so this finishes the proof that $(i_\varepsilon - \varphi_\varepsilon)_+^{-1}(0) \cap \partial_{E_\varepsilon} U_\varepsilon = \emptyset$. Consequently, we have proved that

$$i_\varepsilon - \varphi_\varepsilon \in \mathbb{A}(U_\varepsilon, E_\varepsilon).$$

Therefore we can formulate the following definition.

Definition 2.6.25. Let $\Phi \in \mathbb{A}_c(U,E)$. We define the topological degree of $\Phi = i - \varphi$ as follows:

$$\deg(\Phi,U,E) := \deg(i_\varepsilon - \varphi_\varepsilon, U_\varepsilon, E_\varepsilon), \tag{2.16}$$

where $i_\varepsilon - \varphi_\varepsilon$ is obtained by the above procedure and $\deg(i_\varepsilon - \varphi_\varepsilon, U_\varepsilon, E_\varepsilon)$ is defined by (2.7).

Now we will show that the above definition is correct.

Lemma 2.6.26. *Definition* 2.6.25 *does not depend on the choice of ε, K and $\pi_\varepsilon\colon K \to E_\varepsilon$, provided $\varepsilon < (1/2)\varepsilon_0(\varphi,U)$.*

Proof. Let $\Phi = i - \varphi \in \mathbb{A}_c(U,E)$. Take $\varepsilon, K, \pi_\varepsilon\colon K \to E_\varepsilon$ and $\varepsilon', K', \pi'_{\varepsilon'}\colon K' \to E_{\varepsilon'}$ such that

$$\begin{aligned} \varepsilon &< (1/2)\varepsilon_0(\varphi,U), & \overline{\varphi(\overline{U})} &\subset K, & \|\pi_\varepsilon(y) - y\| &< \varepsilon,\\ \varepsilon' &< (1/2)\varepsilon_0(\varphi,U), & \overline{\varphi(\overline{U})} &\subset K', & \|\pi'_{\varepsilon'}(y') - y'\| &< \varepsilon', \end{aligned}$$

for $y \in K$, $y' \in K'$. We shall show that

$$\deg(i_\varepsilon - \varphi_\varepsilon, U_\varepsilon, E_\varepsilon) = \deg(i_{\varepsilon'} - \varphi_{\varepsilon'}, U_{\varepsilon'}, E_{\varepsilon'}).$$

For this purpose, it is enough to consider two cases

(6) The symbol $\partial_{E_\varepsilon} U_\varepsilon$ denotes the boundary of U_ε with respect to E_ε.

Case 1. We assume that $K \subset K'$. Let E_0 be a finite-dimensional subspace of a normed space E such that $E_\varepsilon \subset E_0$ and $E_{\varepsilon'} \subset E_0$. Then, by Theorem 2.6.21 (the contraction property of topological degree), we obtain

$$\deg(i_\varepsilon - \pi_\varepsilon \circ \varphi|\overline{U \cap E_\varepsilon}, U \cap E_\varepsilon, E_\varepsilon) = \deg(i_0 - j^0_\varepsilon \circ \pi_\varepsilon \circ \varphi|\overline{U \cap E_0}, U \cap E_0, E_0),$$

$$\begin{aligned}\deg(&i_{\varepsilon'} - \pi'_{\varepsilon'} \circ \varphi|\overline{U \cap E_{\varepsilon'}}, U \cap E_{\varepsilon'}, E_{\varepsilon'}) \\ &= \deg(i_0 - j^0_{\varepsilon'} \circ \pi'_{\varepsilon'} \circ \varphi|\overline{U \cap E_0}, U \cap E_0, E_0),\end{aligned}$$

where $j^0_\varepsilon : E_\varepsilon \to E_0$, $j^0_{\varepsilon'} : E_{\varepsilon'} \to E_0$ and $i_0 : \overline{U \cap E_0} \to E_0$ are the inclusions. Now we shall prove that

$$\begin{aligned}\deg(&i_0 - j^0_\varepsilon \circ \pi_\varepsilon \circ \varphi|\overline{U \cap E_0}, U \cap E_0, E_0) \\ &= \deg(i_0 - j^0_{\varepsilon'} \circ \pi'_{\varepsilon'} \circ \varphi|\overline{U \cap E_0}, U \cap E_0, E_0).\end{aligned} \tag{2.17}$$

For this purpose, it is enough to show that there exists a w-homotopy

$$\Upsilon : \overline{U \cap E_0} \times [0,1] \multimap E_0$$

such that

(1) $\Upsilon(x,0) = (j^0_\varepsilon \circ \pi_\varepsilon \circ \varphi|\overline{U \cap E_0})(x)$,
(2) $\Upsilon(x,1) = (j^0_{\varepsilon'} \circ \pi'_{\varepsilon'} \circ \varphi|\overline{U \cap E_0})(x)$,
(3) $\{x \in \overline{U \cap E_0} \mid x \in \Upsilon(x,t) \text{ for some } t \in [0,1]\} \cap \partial_{E_0}(U \cap E_0) = \emptyset$.

Consider the following diagram:

$$\begin{array}{ccccc} \overline{U \cap E_0} \times [0,1] & \xrightarrow{\overline{\varphi} \times \mathrm{id}} & K \times [0,1] & \xrightarrow{\Delta \times \mathrm{id}} & K \times K \times [0,1] \\ & & & & \Big\downarrow{\scriptstyle f_1 \times f_2 \times \mathrm{id}} \\ & & & & E_0 \times E_0 \times [0,1] \\ & & & & \Big\downarrow{\scriptstyle \lambda} \\ & & & & E_0 \end{array}$$

where $\overline{\varphi} : \overline{U \cap E_0} \multimap K$, $\Delta : K \to K \times K$, $\lambda : E_0 \times E_0 \times [0,1] \to E_0$, are defined by:

$$\overline{\varphi}(x) = \varphi(x), \quad \Delta(x) = (x,x), \quad \lambda(x,y,t) = (1-t)x + ty,$$

and

$$f_1(x) = j^0_\varepsilon \circ \pi_\varepsilon, \quad f_2(x) = j^0_{\varepsilon'} \circ \pi'_{\varepsilon'} \circ j',$$

where $j' : K \hookrightarrow K'$ is the inclusion. The above diagram allows us to define a weighted map $\Upsilon : \overline{U \cap E_0} \times [0,1] \multimap E_0$ as follows

$$\Upsilon(x,t) = \lambda \circ (f_1 \times f_2 \times \mathrm{id}) \circ (\Delta \times \mathrm{id}) \circ (\overline{\varphi} \times \mathrm{id})(x,t),$$

for $x \in \overline{U \cap E_0}$ and $t \in [0,1]$. It is easy to see that

(1) $\Upsilon(x,0) = j^0_\varepsilon \circ \pi_\varepsilon \circ \overline{\varphi}(x) = (j^0_\varepsilon \circ \pi_\varepsilon \circ \varphi|\overline{U \cap E_0})(x)$,
(2) $\Upsilon(x,1) = j^0_{\varepsilon'} \circ \pi'_{\varepsilon'} \circ j' \circ \overline{\varphi}(x) = (j^0_{\varepsilon'} \circ \pi'_{\varepsilon'} \circ \varphi|\overline{U \cap E_0})(x)$.

Now we shall show that

$$\{x \in \overline{U \cap E_0} \mid x \in \Upsilon'(x,t) \text{ for some } t \in [0,1]\} \cap \partial_{E_0}(U \cap E_0) = \emptyset.$$

Indeed, suppose on the contrary that there exists a point $x_0 \in \partial_{E_0}(U \cap E_0)$ and $t_0 \in [0,1]$ such that $x_0 \in \Upsilon(x_0, t_0)$. Then there exists $y_0 \in \varphi(x_0)$ such that

$$x_0 = (1-t_0)f_1(y_0) + t_0 f_2(y_0).$$

Hence,

$$\|x_0 - y_0\| \leqslant (1-t_0)\|f_1(y_0) - y_0\| + t_0\|f_2(y_0) - y_0\|. \tag{2.18}$$

However,

$$\|\pi_\varepsilon(y_0) - y_0\| < \varepsilon < \frac{1}{2}\varepsilon_0(\varphi, U) \quad \text{and} \quad \|\pi'_{\varepsilon'}(y_0) - y_0\| < \varepsilon' < \frac{1}{2}\varepsilon_0(\varphi, U),$$

so

$$\|f_1(y_0) - y_0\| < \frac{1}{2}\varepsilon_0(\varphi, U) \quad \text{and} \quad \|f_2(y_0) - y_0\| < \frac{1}{2}\varepsilon_0(\varphi, U). \tag{2.19}$$

Moreover,

$$\|x_0 - y_0\| \geqslant \varepsilon_0(\varphi, U), \tag{2.20}$$

since $x_0 \in \partial U$ and $y_0 \in \varphi(x_0)$. Now, taking into account (2.18)–(2.20), we get

$$\varepsilon_0(\varphi, U) < \varepsilon_0(\varphi, U),$$

a contradiction. Consequently, by the w-homotopy invariance of the topological degree of weighted maps (see Theorem 2.6.21), we get (2.17). Finally, note that the same proof works in the case when $K' \subset K$.

Case 2. In this case we assume that $K \not\subset K'$ and $K' \not\subset K$. Let $K'' := K \cup K'$ and let $\varepsilon'' < \varepsilon_0(\varphi U)$. In adddition, let $\pi''_\varepsilon \colon K'' \to E''_\varepsilon$ be any continuous function such that

$$\|\pi''_{\varepsilon''}(y) - y\| < \varepsilon'',$$

for all $y \in K''$. Now it is enough to apply Case 1 to the following two situations:

$$\varepsilon, K, \pi_\varepsilon \colon K \to E_\varepsilon \quad \text{and} \quad \varepsilon'', K'', \pi''_\varepsilon \colon K'' \to E''_\varepsilon,$$
$$\varepsilon'', K'', \pi''_\varepsilon \colon K'' \to E''_\varepsilon \quad \text{and} \quad \varepsilon', K', \pi'_{\varepsilon'} \colon K' \to E_{\varepsilon'}.$$

Indeed, by Case 1, we obtain

$$\begin{multline} \deg(i_\varepsilon - \pi_\varepsilon \circ \varphi|\overline{U \cap E_\varepsilon}, U \cap E_\varepsilon, E_\varepsilon) \\ = \deg(i''_{\varepsilon''} - \pi''_{\varepsilon''} \circ \varphi|\overline{U \cap E''_{\varepsilon''}}, U \cap E''_{\varepsilon''}, E''_{\varepsilon''}), \end{multline} \tag{2.21}$$

$$\begin{multline} \deg(i''_{\varepsilon''} - \pi''_{\varepsilon''} \circ \varphi|\overline{U \cap E''_{\varepsilon''}}, U \cap E''_{\varepsilon''}, E''_{\varepsilon''}) \\ = \deg(i_{\varepsilon'} - \pi'_{\varepsilon'} \circ \varphi|\overline{U \cap E_{\varepsilon'}}, U \cap E_{\varepsilon'}, E_{\varepsilon'}). \end{multline} \tag{2.22}$$

Hence, taking into account (2.21) and (2.22), we get

$$\deg(i_\varepsilon - \pi_\varepsilon \circ \varphi|\overline{U \cap E_\varepsilon}, U \cap E_\varepsilon, E_\varepsilon) = \deg(i_{\varepsilon'} - \pi'_{\varepsilon'} \circ \varphi|\overline{U \cap E_{\varepsilon'}}, U \cap E_{\varepsilon'}, E_{\varepsilon'}),$$

which completes the proof of Case 2. □

We shall collect below the most important properties of the topological degree.

Theorem 2.6.27. *Let $\Phi \in \mathbb{A}_c(U,E)$. The topological degree defined in* (2.16) *satisfies the following properties:*

(a) (Existence) *If* $\deg(\Phi, U, E) \neq 0$, *then* $0 \in \Phi(x)$ *for some* $x \in U$.
(b) (Additivity) *Let U_1 and U_2 be two disjoint open subsets of U such that $\Phi_+^{-1}(0) \subset U_1 \cup U_2$. Then*

$$\deg(\Phi, U, E) = \deg(\Phi_1, U_1, E) + \deg(\Phi_2, U_2, E),$$

where Φ_i denotes the restriction of Φ to U_i.
(c) (Contraction) *Let $\Phi\colon \overline{U} \multimap E'$ be a weighted map such that $\Phi_+^{-1}(0) \cap \partial U = \emptyset$, where E' is some linear subspace of E. Then*

$$\deg(j \circ \Phi, U, E) = \deg(\Phi|\overline{U \cap E'}, U \cap E', E'),$$

where $j\colon E' \hookrightarrow E$ is the inclusion.
(d) (Linearity) *If $\Phi = \alpha\Phi_1 \cup \beta\Phi_2$, then*

$$\deg(\Phi, U, E) = \alpha \cdot \deg(\Phi_1, U, E) + \beta \cdot \deg(\Phi_2, U, E).$$

(e) (Unity) *If $\varphi\colon \overline{U} \to E$ is the constant map sending $\overline{U}$ to 0 and $0 \in U$, then* $\deg(i - \varphi, U, E) = 1$.
(f) (Multiplicativity) *Let $\Phi_1 \in \mathbb{A}_c(U_1, E_1)$ and $\Phi_2 \in \mathbb{A}_c(U_2, E_2)$. Then $\Phi_1 \times \Phi_2 \in \mathbb{A}_c(U_1 \times U_2, E_1 \times E_2)$ and*

$$\deg(\Phi_1 \times \Phi_2, U_1 \times U_2, E_1 \times E_2) = \deg(\Phi_1, U_1, E_1) \cdot \deg(\Phi_2, U_2, E_2).$$

(g) (w-Homotopy invariance) *Let $\gamma\colon \overline{U} \times [0,1] \multimap E$ be a compact w-map and let*

$$K := \{x \in \overline{U} \mid x \in \gamma(x,t) \text{ for some } t \in [0,1]\}.$$

If $K \cap \partial U = \emptyset$, then $\deg(i - \gamma_0, U, E) = \deg(i - \gamma_1, U, E)$.

Proof. (a) Existence. Let $\deg(\Phi, U, E) \neq 0$. Let $\varepsilon_n = 1/n$ for $n > m_0 := 2/\varepsilon_0(\varphi, U)$. Then, by (2.16), we have

$$\deg(\Phi, U, E) = \deg(i_{\varepsilon_n} - \varphi_{\varepsilon_n}, U_{\varepsilon_n}, E_{\varepsilon_n}).$$

Hence,

$$\deg(i_{\varepsilon_n} - \varphi_{\varepsilon_n}, U_{\varepsilon_n}, E_{\varepsilon_n}) \neq 0, \quad \text{for all } n > m_0.$$

Consequently, there exists an $x_n \in U$ with $x_n \in \varphi_{\varepsilon_n}(x_n) = \pi_{\varepsilon_n} \circ \varphi(x_n)$, for any $n > m_0$, and hence $x_n \in O_{\varepsilon_n}(\varphi(x_n))$. Now $\bigcup_{n=m_0}^{\infty} \varphi(x_n)$ is relatively compact, therefore there is a subsequence $\{y_{k_n}\}_{n=0}^{\infty}$ convergent to y and such that

$$y_{k_n} \in \varphi(x_{k_n}), \quad ||x_{k_n} - y_{k_n}|| < \varepsilon_{k_n},$$

therefore $x_{k_n} \to y$ as $n \to \infty$. Consequently, from the upper semicontinuity of φ we have $y \in \varphi(y)$ and hence $0 \in \Phi(y)$, as required.

(b) Additivity. Let $\varepsilon < (1/2)\min\{\varepsilon_0(\varphi|\overline{U_1}, U_1), \varepsilon_0(\varphi|\overline{U_2}, u_2), \varepsilon_0(\varphi, U)\}$. Then

$$\deg(\Phi, U, E) = \deg(i_\varepsilon - \varphi_\varepsilon, U_\varepsilon, E_\varepsilon). \tag{2.23}$$

But $U_\varepsilon = U \cap E_\varepsilon = (U_1 \cup U_2) \cap E_\varepsilon = (U_1 \cap E_\varepsilon) \cup (U_2 \cap E_\varepsilon) = U_{1\varepsilon} \cup U_{2\varepsilon}$. Consequently, the additivity property of the topological degree (see Theorem 2.6.21) implies that

$$\begin{aligned} \deg(i_\varepsilon - \varphi_\varepsilon, U_\varepsilon, E_\varepsilon) \\ = \deg((i_\varepsilon - \varphi_\varepsilon)|(\overline{U_{1\varepsilon}}), U_{1\varepsilon}, E_\varepsilon) + \deg((i_\varepsilon - \varphi_\varepsilon)|(\overline{U_{2\varepsilon}}), U_{2\varepsilon}, E_\varepsilon). \end{aligned} \tag{2.24}$$

On the other hand, in view of (2.16), we have

$$\deg(\Phi_1, U_1, E) = \deg((i_\varepsilon - \varphi_\varepsilon)|(\overline{U_{1\varepsilon}}), U_{1\varepsilon}, E_\varepsilon), \tag{2.25}$$

$$\deg(\Phi_2, U_2, E) = \deg((i_\varepsilon - \varphi_\varepsilon)|(\overline{U_{2\varepsilon}}), U_{2\varepsilon}, E_\varepsilon). \tag{2.26}$$

Hence, taking into account (2.23)–(2.26), we obtain

$$\deg(\Phi, U, E) = \deg(\Phi_1, U_1, E) + \deg(\Phi_2, U_2, E)$$

as required.

(c) Contraction. Let $\Phi = i - \varphi$ and $K := \overline{\varphi(\overline{U})} \subset E'$. To begin with, observe that $\varepsilon_0(\varphi, U) \leqslant \varepsilon_0(\varphi|\overline{U \cap E'}, U \cap E')$ (since $\partial_{E'}(U \cap E') \subset (\partial U) \cap E'$). Let $\pi_\varepsilon \colon K \to E'_\varepsilon$ be an ε-approximation of the inclusion $i_K \colon K \to E'$, where $\varepsilon < \varepsilon_0(\varphi, U)/2$. Consequently, by (2.16), we have

$$\begin{aligned} \deg(j \circ \Phi, U, E) &= \deg(i - \pi_\varepsilon \circ \varphi, U'_\varepsilon, E'_\varepsilon), \\ \deg(\Phi|\overline{U \cap E'}, U \cap E', E') &= \deg(i - \pi_\varepsilon \circ \varphi, U'_\varepsilon, E'_\varepsilon), \end{aligned}$$

thus

$$\deg(j \circ \Phi, U, E) = \deg(\Phi|\overline{U \cap E'}, U \cap E', E').$$

(d) Linearity. Let $\Phi = (\alpha + \beta)i - (\alpha\varphi_1 \cup \beta\varphi_2)$ and let $K := \overline{(\varphi_1 \cup \varphi_2)(\overline{U})}$. Observe that $\varepsilon_0(\alpha\varphi_1 \cup \beta\varphi_2, U) \leqslant \varepsilon_0(\alpha\varphi_1, U)$ and $\varepsilon_0(\alpha\varphi_1 \cup \beta\varphi_2, U) \leqslant \varepsilon_0(\beta\varphi_2, U)$.

Let $\varepsilon < (1/2)\varepsilon_0(\alpha\varphi_1 \cup \beta\varphi_2, U)$. In addition, let $\pi_\varepsilon : K \to E_\varepsilon$ be a continuous function such that $||y - \pi_\varepsilon(y)|| < \varepsilon$ for all $y \in K$. Then

$$\begin{aligned}
&\deg(\alpha\Phi_1 \cup \beta\Phi_2, U, E) \\
&\qquad = \deg((\alpha+\beta)i_\varepsilon - (\alpha(\pi_\varepsilon \circ \varphi_1) \cup \beta(\pi_\varepsilon \circ \varphi_2)), U_\varepsilon, E_\varepsilon) \\
&\qquad = \deg(\alpha(i_\varepsilon - \pi_\varepsilon \circ \varphi_1) \cup \beta(i_\varepsilon - \pi_\varepsilon \circ \varphi_2), U_\varepsilon, E_\varepsilon) \\
&\qquad \overset{(*)}{=} \alpha \cdot \deg(i_\varepsilon - \pi_\varepsilon \circ \varphi_1, U_\varepsilon, E_\varepsilon) + \beta \cdot \deg(i_\varepsilon - \pi_\varepsilon \circ \varphi_2 U_\varepsilon, E_\varepsilon) \\
&\qquad = \alpha \cdot \deg(\Phi_1, U, E) + \beta \cdot \deg(\Phi_2, U, E),
\end{aligned}$$

where the equality $(*)$ follows from the linearity property of topological degree for weighted maps in finite-dimensional normed spaces (see Theorem 2.6.21).

(e) Unity. Let $K := \{0\}$ and let E_0 be a finite-dimensional subspace of E. We define a function $\pi_0 : K \to E_0$ by $\pi_0(x) = 0$ for $x \in K$. Then

$$\deg(\Phi, U, E) = \deg(i - \varphi, U, E) = \deg(i_0 - \pi_0 \circ \varphi, U_0, E_0) = 1,$$

where the last equality follows from the unity property of the topological degree for w-maps in a finite-dimensional normed space (see Theorem 2.6.21).

(f) Multiplicativity. Let $\Phi_1 = i_1 - \varphi_1$, $\Phi_2 = i_2 - \varphi_2$ and $K_1 = \overline{\varphi_1(\overline{U_1})}$, $K_2 = \overline{\varphi_2(\overline{U_2})}$. Let us observe that

$$\varepsilon_0(\varphi_1, U_1) \leqslant \varepsilon_0(\varphi_1 \times \varphi_2, U_1 \times U_2) \quad \text{and} \quad \varepsilon_0(\varphi_2, U_2) \leqslant \varepsilon_0(\varphi_1 \times \varphi_2, U_1 \times U_2) \quad (^7).$$

Take $\varepsilon < (1/2)\min\{\varepsilon_0(\varphi_1, U_1), \varepsilon_0(\varphi_2, U_2)\}$. Let $\pi_\varepsilon^1 : K_1 \to E_\varepsilon^1$ and $\pi_\varepsilon^2 : K_2 \to E_\varepsilon^2$ be two continuous functions such that

$$\begin{aligned}
||e_1 - \pi_\varepsilon^1(e_1)||_{E_1} &< \varepsilon \quad \text{for all } e_1 \in K_1, \\
||e_2 - \pi_\varepsilon^2(e_2)||_{E_2} &< \varepsilon \quad \text{for all } e_2 \in K_2.
\end{aligned}$$

Then the function $\pi_\varepsilon^1 \times \pi_\varepsilon^2 : K_1 \times K_2 \to E_\varepsilon^1 \times E_\varepsilon^2$ satisfies the following condition

$$||(e_1, e_1) - \pi_\varepsilon^1 \times \pi_\varepsilon^2(e_1, e_2)||_{E_1 \times E_2} < \varepsilon,$$

for all $(e_1, e_2) \in K_1 \times K_2$. Consequently, taking into account Definition 2.6.25, we obtain

$$\begin{aligned}
(2.27)\quad &\deg(\Phi_1 \times \Phi_2, U_1 \times U_2, E_1 \times E_2) \\
&\qquad = \deg(i_{1\varepsilon} \times i_{2\varepsilon} - (\pi_\varepsilon^1 \times \pi_\varepsilon^2) \circ (\varphi_1 \times \varphi_2), U_{1\varepsilon} \times U_{2\varepsilon}, E_\varepsilon^1 \times E_\varepsilon^2).
\end{aligned}$$

On the other hand, from the multiplicativity of the topological degree of weighted maps in finite-dimensional normed spaces we deduce that

$$\begin{aligned}
(2.28)\quad &\deg(i_{1\varepsilon} \times i_{2\varepsilon} - (\pi_\varepsilon^1 \times \pi_\varepsilon^2) \circ (\varphi_1 \times \varphi_2), U_{1\varepsilon} \times U_{2\varepsilon}, E_\varepsilon^1 \times E_\varepsilon^2) \\
&\qquad = \deg(i_{1\varepsilon} - \pi_\varepsilon^1 \circ \varphi_1, U_{1\varepsilon}, E_\varepsilon^1) \cdot \deg(i_{2\varepsilon} - \pi_\varepsilon^2 \circ \varphi_2, U_{2\varepsilon}, E_\varepsilon^2).
\end{aligned}$$

(^{7}) Recall that on the Cartesian product $E \times F$ of two normed spaces $(E, ||\cdot||_E)$, $(F, ||\cdot||_F)$ we consider the following norm $||(e,f)|| := \max\{||e||_E, ||f||_F\}$, for every $e \in E$, $f \in F$.

Moreover, by Definition 2.6.25,

$$\deg(\Phi_1, U_1, E_1) = \deg(i_{1\varepsilon} - \pi_\varepsilon^1 \circ \varphi_1, U_{1\varepsilon}, E_\varepsilon^1), \tag{2.29}$$

$$\deg(\Phi_2, U_2, E_2) = \deg(i_{2\varepsilon} - \pi_\varepsilon^2 \circ \varphi_2, U_{2\varepsilon}, E_\varepsilon^2). \tag{2.30}$$

Hence, by (2.27)–(2.30), one obtains

$$\deg(\Phi_1 \times \Phi_2, U_1 \times U_2, E_1 \times E_2) = \deg(\Phi_1, U_1, E_1) \cdot \deg(\Phi_2, U_2, E_2)$$

as required.

(g) w-Homotopy invariance. Let $\gamma: \overline{U} \times [0,1] \multimap E$ be a compact w-map and let $K := \{x \in \overline{U} \mid x \in \gamma(x,t) \text{ for some } t \in [0,1]\}$. Assume that $K \cap \partial U = \emptyset$. Then

$$\varepsilon_0(\gamma) := \inf\{||x - y|| \mid x \in \partial U,\ y \in \gamma(x,t) \text{ for some } t \in [0,1]\} > 0$$

and $\varepsilon_0(\gamma_t, U) \geqslant \varepsilon_0(\gamma)$ for all $t \in [0,1]$. Let $K := \overline{\gamma(\overline{U} \times [0,1])}$ and $\varepsilon < (1/2)\varepsilon_0(\gamma)$. In addition, let $\pi_\varepsilon: K \to E_\varepsilon$ be an ε-approximation of the inclusion $j: K \to E$. Define $\widetilde{\gamma}: \overline{U_\varepsilon} \times [0,1] \multimap E_\varepsilon$ by $\widetilde{\gamma}(x,t) = i_\varepsilon(x) - \pi_\varepsilon \circ (\gamma|\overline{U_\varepsilon} \times [0,1])(x,t)$, for all $x \in \overline{U_\varepsilon}$, $t \in [0,1]$. Arguing as at the beginning of this section, one shows that

$$\{x \in \overline{U_\varepsilon} \mid x \in \widetilde{\gamma}(x,t) \text{ for some } t \in [0,1]\} \cap \partial_{E_\varepsilon} U_\varepsilon = \emptyset.$$

Consequently, by the w-homotopy property of the topological degree of weighted maps in E_ε (see Theorem 2.6.21), we obtain

$$\begin{aligned}\deg(i_\varepsilon - \pi_\varepsilon \circ \gamma_0|\overline{U_\varepsilon}, U_\varepsilon, E_\varepsilon) &= \deg(\widetilde{\gamma}_0, U_\varepsilon, E_\varepsilon) \\ &= \deg(\widetilde{\gamma}_1, U_\varepsilon, E_\varepsilon) = \deg(i_\varepsilon - \pi_\varepsilon \circ \gamma_1|\overline{U_\varepsilon}, U_\varepsilon, E_\varepsilon).\end{aligned} \tag{2.31}$$

On the other hand, by Definition 2.6.25, one has

$$\deg(i - \gamma_0, U, E) = \deg(i_\varepsilon - \pi_\varepsilon \circ \gamma_0|\overline{U_\varepsilon}, U_\varepsilon, E_\varepsilon), \tag{2.32}$$

$$\deg(i - \gamma_1, U, E) = \deg(i_\varepsilon - \pi_\varepsilon \circ \gamma_1|\overline{U_\varepsilon}, U_\varepsilon, E_\varepsilon). \tag{2.33}$$

Thus, by (2.31)–(2.33), we get

$$\deg(i - \gamma_0, U, E) = \deg(i - \gamma_1, U, E)$$

as required. The proof of Theorem 2.6.27 is complete. □

2.7. Topological essentiality

In this section we define a notion of topological essentiality for weighted mappings. Topological essentiality can be defined on a larger class of mappings than the topological degree but yields less information.

The notion of topological essentiality (sometimes called topological transversality) in the single-valued case was introduced by A. Granas and later studied by many authors (cf. [5], [20], [29], [23]).

The results presented in this section are taken from the paper [61].

In what follows, E and F are two real normed spaces and U is an open bounded subset of E. In this section all w-maps $\varphi\colon X \multimap Y$ are assumed to have a weight $w_\varphi\colon X \times Y \to \mathbb{Q}$ such that the sum $\sum_{y\in Y} w_\varphi(x,y)$ does not depend on $x \in X$ and $\sum_{y\in Y} w_\varphi(x,y) \neq 0$. We let:

$$\begin{aligned} W_{\partial U}(U,F) &= \{\varphi\colon \overline{U} \multimap F \mid \varphi \text{ is a } w\text{-map and } 0 \notin \varphi(\partial U)\},\\ W_C(U,F) &= \{\varphi\colon \overline{U} \multimap F \mid \varphi \text{ is a } w\text{-map and compact}\},\\ W_0(U,F) &= \{\varphi\colon \overline{U} \multimap F \mid \varphi \in W_C(U,F) \text{ and } \varphi(x) = \{0\} \text{ for every } x \in \partial U\}. \end{aligned}$$

Definition 2.7.1. A weighted map $\varphi \in W_{\partial U}(U,F)$ is called *essential* (with respect to $W_0(U,F)$) provided for any $\psi \in W_0(U,F)$ there exists a point $x \in U$ such that $\varphi(x) \cap \psi(x) \neq \emptyset$.

Let us observe that if $E = F$ then the notion of essentiality can be reinterpreted as $\mathbb{Z}_2$ topological degree. We give now some examples of essential w-maps.

Example 2.7.2. Let $\varphi \in W_{\partial B}(B,\mathbb{R})$, where B is an open ball of radius $r > 0$ centered at $0 \in E$. If there exist $x_0, x_1 \in \partial B$ such that

$$u < 0 \quad \text{for every } u \in \varphi(x_0), \qquad v > 0 \quad \text{for every } v \in \varphi(x_1),$$

then φ is an essential w-map.

To see this, we shall need the following lemma:

Lemma 2.7.3 ([10])**.** *Let $\varphi\colon [a,b] \multimap \mathbb{R}$ be a w-map from the interval $[a,b]$ to $\mathbb{R}$ with $I_w(\varphi) \neq 0$. Suppose that $\varphi(a) \subset \mathbb{R}^-$ and $\varphi(b) \subset \mathbb{R}^+$. Then $0 \in \varphi(x)$ for some $x \in [a,b]$.*

Let $\psi \in W_0(B,\mathbb{R})$ and suppose on the contrary that $\varphi(x) \cap \psi(x) = \emptyset$ for every $x \in \overline{B}$. We define $\eta\colon \overline{B} \multimap \mathbb{R}$ by the formula:

$$\eta(x) = \varphi(x) - \psi(x).$$

In view of Propositions 2.2.15 and 2.2.18, we deduce that η is a w-map with $I_w(\eta) \neq 0$. Moreover, $0 \notin \eta(x)$ for every $x \in \overline{B}$. Let $\gamma\colon [0,1] \to \overline{B}$ be a path between x_0 and x_1 and let $\widetilde{\eta} = \eta \circ \gamma$. It is easy to see that $\widetilde{\eta}$ has the following properties:

(a) $\widetilde{\eta}$ is a w-map with $I_w(\widetilde{\eta}) \neq 0$,
(b) $\widetilde{\eta}(0) \subset \mathbb{R}^-$ and $\widetilde{\eta}(1) \subset \mathbb{R}^+$,
(c) $0 \notin \widetilde{\eta}(x)$ for every $x \in [0,1]$.

But this contradicts Lemma 2.7.3. This ends the proof of essentiality of φ.

Example 2.7.4 (Essentiality of homeomorphism)**.** Let U be an open and bounded subset of E such that $\overline{U} \in \mathrm{AR}$ and let $f\colon \overline{U} \to f(\overline{U})$ be a homeomorphism such that $f(\overline{U})$ is a closed subset of F. In addition, assume that $f(U)$ is an open subset of F and $0 \in f(U)$. Then f is an essential w-map.

Indeed, let $\psi \in W_0(U,F)$. Since $f(\overline{U}) \in \mathrm{AR}$, there exists a retraction $r\colon F \to f(\overline{U})$. Let us denote by $g\colon f(\overline{U}) \to \overline{U}$ an inverse function of f. Consider

$$K = \{x \in \overline{U} \mid f(x) \in (t \cdot \psi(x)) \text{ for some } t \in [0,1]\}.$$

It is easy to see that K is closed in $\overline{U}$ and nonempty (since $0 \in f(U)$). Let $s\colon F \to [0,1]$ be an Urysohn function such that $s(y) = 1$ for $y \in f(B)$ and $s(y) = 0$ for $y \in F \setminus f(U)$. The definition of s is correct, because $f(B)$ is closed in F and $f(B) \subset f(U)$. Define $\eta\colon F \multimap F$ by the formula:

$$\eta(y) = s(y)\psi(g(r(y))),$$

for every $y \in F$. It is easy to see that η is a compact w-map. Moreover, $I_w(\eta) \neq 0$. Indeed,

$$I_w(\eta) = I_w(s \circ \psi \circ g \circ r) = I_w(s)I_w(\psi)I_w(g)I_w(r) = 1 \cdot I_w(\psi) \cdot 1 \cdot 1 \neq 0.$$

Hence, in view of Corollary 2.5.6, we have a fixed point: $y \in \eta(y)$. If $y \in F \setminus f(U)$, then $s(y) = 0$ and $y = 0$, but $0 \in f(U)$, so we get a contradiction. Therefore we deduce that $y \in f(U)$. It follows that there exists a point $x \in U$ such that $f(x) = y$. Consequently, $f(x) \in s(f(x))\psi(x)$. Then $x \in B$ and hence $f(x) \in \psi(x)$. This completes the proof of essentiality of f. Let us list several properties of the topological essentiality.

Proposition 2.7.5 (Existence)**.** *If $\varphi \in W_{\partial U}(U,F)$ is essential, then there exists $x \in U$ such that $0 \in \varphi(x)$.*

Proposition 2.7.6 (Compact perturbation)**.** *If $\varphi \in W_{\partial U}(U,F)$ is essential and $\eta \in W_0(U,F)$, then $\varphi + \eta \in W_{\partial U}(U,F)$ is an essential w-map.*

Proposition 2.7.7 (Coincidence)**.** *Assume that $\varphi \in W_{\partial U}(U,F)$ is an essential w-map and $\eta \in W_C(U,F)$. Let*

$$B = \{x \in \overline{U} \mid \varphi(x) \cap (t\eta(x)) \neq \emptyset \text{ for some } t \in [0,1]\}.$$

If $B \subset U$, then φ and η have a coincidence.

Proposition 2.7.8 (Normalization)**.** *Assume that $0 \notin \partial U$ and $\overline{U} \in \mathrm{AR}$. Then the inclusion map is an essential w-map if and only if $0 \in U$.*

Proposition 2.7.9 (Localization)**.** *Let $\varphi \in W_{\partial U}(U,F)$ be an essential w-map. Assume that V is an open subset of U such that $\varphi_+^{-1}(\{0\}) \subset V$ and $\overline{V} \in \mathrm{AR}$. Then the restriction $\varphi|\overline{V}$ of φ to $\overline{V}$ is an essential w-map.*

Proposition 2.7.10 (w-homotopy)**.** *Let $\varphi \in W_{\partial U}(U, F)$ be an essential w-map. If $H: \overline{U} \times [0,1] \multimap F$ is a compact w-map such that*

(a) *$H(x, 0) = \{0\}$ for every $x \in \partial U$,*
(b) *$\{x \in \overline{U} \mid \varphi(x) \cap H(x,t) \neq \emptyset$ for some $t \in [0,1]\} \subset U$*

then the map $\varphi(\,\cdot\,) - H(\,\cdot\,, 1)$ is an essential w-map.

The topological essentiality has many applications in fixed point theory, analysis and other fields. In this book we give a few examples.

Proposition 2.7.11. *Let $\varphi \in W_{\partial U}(U, F)$ be an essential w-map and proper (i.e. for any compact $K \subset F$, $\varphi_+^{-1}(K)$ is compact). If D is a connected component of $F \setminus \varphi(\partial U)$ which contains 0, then $D \subset \varphi(\overline{U})$.*

Proof. The set $\varphi(\partial U)$ is a closed subset of F, because any proper map is closed. Let $v \in D$. We shall show that $v \in \varphi(\overline{U})$. Since $F \setminus \varphi(\partial U)$ is open, its components are open, and for open set in F connectedness is the same as path-connectedness. So, let $\sigma: [0,1] \to D$ be a continuous curve with $\sigma(0) = 0$ and $\sigma(1) = v$. Define w-homotopy $\eta: \overline{U} \times [0,1] \multimap F$ by the formula:

$$\eta(x,t) = \sigma(t),$$

for every $(x,t) \in \overline{U} \times [0,1]$. Now, we can apply the w-homotopy property to deduce that

$$\varphi(\,\cdot\,) - \eta(\,\cdot\,, 1): \overline{U} \times [0,1] \multimap F$$

is an essential w-map. Notice that

$$\varphi(x) - \eta(x, 1) = \varphi(x) - \{v\},$$

for every $x \in \overline{U}$. Thus, from the existence property we deduce that $v \in \varphi(U)$. This completes the proof. $\square$

Proposition 2.7.12. *Let $\varphi \in W_{\partial U}(U, F)$ be an essential weighted map and let $\psi \in W_C(U, F)$. If $\varphi(x) \cap \psi(x) = \emptyset$ for every $x \in \partial U$, then at least one of the following conditions holds:*

(a) *there exists $x \in U$ such that $\varphi(x) \cap \psi(x) \neq \emptyset$;*
(b) *there exists $\lambda \in (0,1)$ and $x \in \partial U$ such that $\varphi(x) \cap (\lambda\psi(x)) \neq \emptyset$.*

To see this, it is enough to apply the w-homotopy property for φ and ψ, where $H(x,t) = t \cdot \psi(x)$ for $x \in \overline{U}$ and $t \in [0,1]$. Let us observe that if $E = F$, then from Proposition 2.7.12 and the normalization property it follows the following result.

Proposition 2.7.13 (Nonlinear alternative)**.** *Let $\psi \in W_C(U, F)$ and $0 \in U$, then at least one of the following conditions is satisfied:*

(a) $\mathrm{Fix}(\psi) \neq \emptyset$,
(b) *there exists $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda\psi(x)$.*

2.8. Extension theorems

The notion of n-locally connected space was introduced by S. Lefschetz (see [47]). In this chapter we shall consider this notion, but with a slightly different formulation and under the name of locally nw-connected spaces.

The aim of this chapter is twofold. First, we show a useful characterization of locally nw-connected spaces (see Theorem 2.8.19 below). Next, we apply this theorem to prove the Lefschetz-type result for such spaces.

Remark 2.8.1. In this section all weighted maps $\varphi: X \multimap Y$ are assumed to have a weight $w_\varphi: X \times Y \to \mathbb{Q}$ such that the sum $\sum_{y \in Y} w_\varphi(x, y)$ does not depend on $x \in X$. Moreover, in this section by a space we shall mean a Hausdorff topological space. For a given Hausdorff topological space X, by τ_X we shall denote a topology on X.

We begin with the following definition.

Definition 2.8.2. Let G be an open subset of a space X, and let $\mathbf{U} = \{U_\mu\}$, $\mu \in M$, be an open covering of G. We say that $\mathbf{U}$ is *canonical with respect to* X provided that the following two conditions are satisfied:

(a) $\mathbf{U}$ is locally finite, i.e. for each $g \in G$ there is a neighbourhood V of g such that $V \cap U_\mu \neq \emptyset$ for a finite number of $\mu \in M$ at most.

(b) For each point $x \in X \setminus G$ and each neighbourhood $V_x \subset X$ of x in X there is a neighbourhood W_x of x in X such that $U_\mu \cap W_x \neq \emptyset$ implies $U_\mu \subset V_x$.

Theorem 2.8.3 ([7]). *If the space X is metric, then for each open subset G of X there exists a canonical covering of G with respect to X.*

Remark 2.8.4. Let X be a metric space and let $\mathbf{U} = \{U_\mu\}$ be an open covering of X. Then to a given covering $\mathbf{U}$ we can associate an abstract simplicial complex $N(\mathbf{U})$ whose vertices v_{U_μ} correspond to the open sets U_μ, and a set of $k+1$ vertices constitute a k-simplex if and only if the $k+1$ corresponding U_μ's have a nonempty intersection. This simplicial complex $N(\mathbf{U})$ is called the nerve of the open covering $\mathbf{U}$. The geometric realization of $N(\mathbf{U})$ (with the weak topology) will be denoted by $|N(\mathbf{U})|$.

Definition 2.8.5. Let X and A be two spaces. A space X is said to be *locally w-homotopically trivial over a space A at a point $x_0 \in X$* provided that for each neighbourhood U_{x_0} of x_0 in X there is a neighbourhood U'_{x_0} of x_0 contained in U_{x_0} such that every weighted map $\varphi: A \multimap U'_{x_0}$ is w-homotopic in U_{x_0} to a constant map with the weighted index equal to $I_w(\varphi)$.

Definition 2.8.6. Let $n \geqslant 0$. A space X is said to be *locally nw-connected* if it is locally w-homotopically trivial over S^n.

Remark 2.8.7. If a space X is kw-connected for each $k = 0, \ldots, n$, then we shall write $X \in w\text{-LC}^n$.

Definition 2.8.8. A space X is said to be *locally w-contractible at a point* $x_0 \in X$ provided that each neighbourhood U of x_0 contains a neighbourhood U_0 of x_0 which is w-contractible to a point in U. A space X is said to be *locally w-contractible* if it is locally w-contractible at each of its points.

Remark 2.8.9. Let X and Y be two spaces and let $y_0 \in Y$. The constant map at y_0 is the function $c\colon X \to Y$ with $c(x) = y_0$ for all $x \in X$.

Definition 2.8.10. A set $X_0 \subset X$ is said to be *w-contractible to a point* $x_0 \in X$ if the inclusion $i\colon X_0 \to X$ is w-homotopic to the constant map at x_0.

Remark 2.8.11. If a space X is locally contractible, then it is locally w-contractible.

Proposition 2.8.12. *If a space X is locally w-contractible, then $X \in w\text{-}\mathrm{LC}^n$ for each $n \geqslant 0$.*

Proof. Let $n \geqslant 0$ and let $x_0 \in X$ be a fixed point. In addition, let $U_{x_0} \subset X$ be an open neighbourhood of a point x_0. Since a space X is locally w-contractible, it follows that there exists an open neighbourhood V_{x_0} of a point x_0 with $i\colon V_{x_0} \hookrightarrow U_{x_0}$ such that the inclusion $V_{x_0} \hookrightarrow U_{x_0}$ is w-homotopic to the constant map $c\colon V_{x_0} \to U_{x_0}$ at x_0. Let $\varphi\colon S^n \multimap V_{x_0}$ be a weighted map. Hence $i \circ \varphi \sim_w c \circ \varphi$ and $c \circ \varphi = I_w(\varphi) \cdot c_0$, where $c_0\colon S^n \to U_{x_0}$ is the constant map at x_0, which completes the proof. $\square$

Let X be a metric space and let A be a closed subset of X. Assume, furthermore, that $\dim(X \setminus A) \leqslant n+1$. Then there exists a canonical covering $\mathbf{U} = \{U_\mu\}_{\mu\in\Lambda}$ of the set $X \setminus A$ such that each point $x \in X \setminus A$ belongs to $n+2$ sets U_μ at most. Let $N(\mathbf{U})$ be the nerve of this covering. Notice that for each $\sigma \in N(\mathbf{U})$ we have $\dim \sigma \leqslant n+1$. Consider the following set:

$$\mathcal{Z} := A \cup |N(\mathbf{U})|. \tag{2.34}$$

Now we are going to define, for each $x \in \mathcal{Z}$, a collection $N(x)$ of subsets of $\mathcal{Z}$ as follows:

(1) If $x \in \operatorname{int} A$, then $N(x) := \{O_x \cap A \mid O_x \in \tau_X \text{ and } x \in O_x\}$.
(2) If $x \in |N(\mathbf{U})|$, then $N(x) := \{O_x \mid O_x \in \tau_{|N(\mathbf{U})|} \text{ and } x \in O_x\}$.
(3) If $x \in \partial A$, then

$$N(x) := \{(O_x \cap A) \cup N[O'_x] \mid O_x, O'_x \in \tau_X \text{ and } x \in O_x \cap O'_x\},$$

where $N[O'_x] := \bigcup\{\mathrm{st}(v_{U_\mu}, |N(\mathbf{U})|) \mid U_\mu \subset O'_x\}$.

It can be shown that the family $\{N(x)\}_{x\in\mathcal{Z}}$ induces a unique topology on $\mathcal{Z}$ such that $N(x)$ is an open neighbourhood basis at x, for each $x \in \mathcal{Z}$.

Lemma 2.8.13. *Let $N[O'_x]$ ba as above and let*

$$\widetilde{N}[O_x] := \bigcup\{|\tau| \mid \tau \in N(\mathbf{U}) \text{ and } |\tau| \subset N[O_x]\}.$$

Then there exists an open neighbourhood of a point x in X such that $N[O'_x] \subset \widetilde{N}[O_x]$.

Proof. Since $\mathbf{U}$ is a canonical covering, we deduce that for each $x \in \partial A$ and O_x there exists an open neighbourhood O'_x of x such that if $U_\mu \cap O'_x$, then $U_\mu \subset O_x$.

Now, we shall show that $N[O'_x] \subset \widetilde{N}[O_x]$. For this purpose, take any point $y \in N[O'_x]$. Then there exists $U_{\mu_0} \in \mathbf{U}$ such that $U_{\mu_0} \subset O'_x$ and $y \in \operatorname{st}(v_{U_{\mu_0}}, |N(\mathbf{U})|)$. Hence, there exists a simplex $\tau \in N(\mathbf{U})$ with $y \in \langle\tau\rangle \subset \operatorname{st}(v_{U_{\mu_0}}, |N(\mathbf{U})|)$. Let $v_{U_{\mu_0}}, \dots, v_{U_{\mu_s}}$ be the vertices of simplex τ. Since $U_{\mu_i} \cap U_{\mu_0} \neq \emptyset$, for $1 \leqslant i \leqslant s$, it follows that $U_{\mu_i} \subset O_x$. Consequently, we obtain $\operatorname{st}(v_{\mu_i}, |N(U)|) \subset N[O_x]$, for $i = 1, \dots, s$, and hence

$$|\tau| \subset \bigcup_{i=0}^{s} \operatorname{st}(v_{U_{\mu_0}}, |N(\mathbf{U})|).$$

Thus $y \in |\tau| \subset \widetilde{N}[O_x]$, which completes the proof of the lemma. □

Theorem 2.8.14. *Let A be a closed subset of a metric space X with $\dim(X \setminus A) \leqslant n+1$ and let $Y \in w\text{-LC}^n$. Then for every weighted map $\varphi\colon A \multimap Y$ there exists a neighbourhood U of A in $\mathcal{Z}$ such that φ can be extended to a weighted map $\widetilde{\varphi}\colon U \multimap Y$.*

Proof. The main idea of the proof is based on [7]. Let $\dim(X \setminus A) = n_0$ and let $N(\mathbf{U})$ be a simplicial complex according to a canonical covering $\mathbf{U}$ of the set $X \setminus A$. Let us fix $U_\mu \in \mathbf{U}$ and let $\varepsilon_\mu \colon = \sup_{x \in U_\mu} d(x, A)$. Hence $O^X_{2\varepsilon_\mu}(U_\mu) \cap A \neq \emptyset$ ([8]). Thus we can assign to each set U_μ a point $a_{U_\mu} \in O^X_{2\varepsilon_\mu}(U_\mu) \cap A$. Now we proceed inductively over skeleta of $N(\mathbf{U})$. Let $\varphi_0 = (\sigma_{\varphi_0}, w_{\varphi_0})\colon A \cup |N(\mathbf{U})^{(0)}| \multimap Y$ be defined by the formula

$$\sigma_{\varphi_0}(x) = \begin{cases} \sigma_\varphi(x) & \text{if } x \in A, \\ \sigma_{\varphi_2}(x) & \text{if } x = v_{U_\mu} \in |N(\mathbf{U})^{(0)}|, \end{cases}$$

and

$$w_{\varphi_0}(x, y) = \begin{cases} w_\varphi(x, y) & \text{if } x \in A, \\ w_\varphi(a_\mu, y) & \text{if } x = v_{U_\mu}. \end{cases}$$

Now, we shall show that φ_0 is a weighted map. Let us observe that it suffices to consider the case $x \in \partial A$. For this purpose, let $x_0 \in \partial A$ be a fixed point and let U be an open subset of Y. From the upper semicontinuity of $\sigma_\varphi\colon A \multimap Y$ we deduce that there exists $\varepsilon > 0$ such that $\sigma_\varphi(O^X_\varepsilon(x_0) \cap A) \subset U$. Let W_{x_0} be an

([8]) In what follows, given a subset B of a space X and $\varepsilon > 0$, we denote by $O^X_\varepsilon(B)$ the ε-neighbourhood of B in X

open neighbourhood of x_0 in X satisfying the condition (b) of Definition 2.8.2 for x_0 and $V_{x_0} = O^X_{\varepsilon/3}(x_0)$. Consequently, we obtain

$$\sigma_{\varphi_0}((O^X_\varepsilon(x_0) \cap A) \cup (N[W_{x_0}] \cap |N(\mathbf{U})^{(0)}|)) \subset U,$$

which proves the upper semicontinuity of σ_{φ_0} at x_0. Now we are going to show that w_{φ_0} satisfies the condition (b2) of Definition 2.1.1. To see this, let us take any open subset V of Y such that $\sigma_{\varphi_0}(x_0) \cap \partial V = \emptyset$. Then there exists an open neighbourhood V_{x_0} of x_0 in X such that

$$\sum_{y \in V} w_{\varphi_0}(x_0, y) = \sum_{y \in V} w_{\varphi_0}(x', y) \tag{2.35}$$

for $x' \in V_{x_0} \cap A$. Since U is a canonical covering, so there exists an open neighbourhood W_{x_0} of x_0 in $\mathcal{Z}$ such that if $U_\mu \subset W_{x_0}$, then $\mathrm{st}(U_\mu, \mathbf{U}) \subset V_{x_0}$. Let $B(x_0, 3\varepsilon_0) \subset W_{x_0}$ be an open ball. Now let us take any $U_\mu \subset B(x_0, \varepsilon_0)$. Then $\varepsilon_\mu = \sup_{x \in U_\mu} d(x, A) \leqslant \varepsilon_0$ and $O^X_{2\varepsilon_\mu}(U_\mu) \subset O^X_{2\varepsilon_\mu}(B(x_0, \varepsilon_0)) \subset B(x_0, 3\varepsilon_0)$. Hence $O^X_{2\varepsilon_\mu}(U_\mu) \cap A \subset B(x_0, 3\varepsilon_0) \cap A \subset V_{x_0} \cap A$, for all $U_\mu \subset B(x_0, \varepsilon_0)$. Moreover, if $v_{U_\mu} \in N[B(x_0, \varepsilon_0)]$, then $a_{U_\mu} \in O^X_{2\varepsilon_\mu}(U_\mu) \cap A \subset V_{x_0} \cap A$. Therefore for each $v_{U_\mu} \in N[B(x_0, \varepsilon_0)] \cap |N(U)^{(0)}|$, in view of (2.35), we have

$$\sum_{y \in V} w_{\varphi_0}(x_0, y) = \sum_{y \in V} w_{\varphi_0}(a_{U_\mu}, y),$$

which completes the proof that φ_0 is a weighted map.

Let $W_0 = \mathcal{Z}$ (let us observe that if $n_0 = 0$, then the proof is complete; thus we can assume that $n_0 > 0$). Assume now inductively that for some index k (with $0 \leqslant k < n_0$) a neighbourhood W_k of A in $\mathcal{Z}$ and a weighted map $\varphi_k \colon A \cup (|N(\mathbf{U})^{(k)}| \cap W_k) \multimap Y$ are already constructed.

Now we shall show that there exists an open neighbourhood W_{k+1} of A in $\mathcal{Z}$ and a weighted map $\varphi_{k+1} \colon A \cup (|N(\mathbf{U})^{(k+1)}| \cap W_{k+1}) \multimap Y$ such that $W_{k+1} \subset W_k$ and $\varphi_{k+1}(x) = \varphi_k(x)$ for any $x \in A \cup (|N(\mathbf{U})^{(k)}| \cap W_{k+1})$. For this purpose, we shall need the following lemma.

Lemma 2.8.15. *Let $x_0 \in \partial A$ be a fixed point and let $\varphi_k \colon A \cup (|N(\mathbf{U})^{(k)}| \cap W_k) \multimap Y$ be as above. In addition, let $\varphi_k(x_0) = \{y_1, \dots, y_s\}$. Then for each $\varepsilon > 0$ there is $0 < \delta < \varepsilon$ and an open neighbourhood $O^{\mathcal{Z}}_{x_0} \subset W_k$ of x_0 in $\mathcal{Z}$ such that for each $(k+1)$-dimensional geometric simplex $|\sigma|$ lying in $O^{\mathcal{Z}}_{x_0}$ the following three conditions are satisfied:*

(a) $\varphi_k(|\partial\sigma|) \subset \bigcup_{i=1}^s B(y_i, \delta)$.

(b) $\sum_{y \in B(y_i, \varepsilon)} w_{\varphi_k}(x, y) = \sum_{y \in B(y_i, \varepsilon)} w_{\varphi_k}(x_0, y)$ *for any* $x \in |\partial\sigma|$ *and* $i = 1, \dots, s$.

(c) *If* $\varphi_k(|\partial\sigma|) \subset \bigcup_{j \in I} B(y_j, \delta)$ *for some subset I of $\{1, \dots, s\}$, then $\varphi_k||\partial\sigma|$ admits an extension φ_σ over $|\sigma|$ such that* $\varphi_\sigma(|\sigma|) \subset \bigcup_{j \in I} B(y_j, \varepsilon)$.

Proof. Let us fix $\varepsilon > 0$. We can assume that $B(y_i, \varepsilon) \cap B(y_j, \varepsilon) = \emptyset$ for $i \neq j$. Since $Y \in w\text{-LC}^n$, it follows that there is $\delta < \varepsilon$ such that for any $1 \leqslant i \leqslant s$, any $(k+1)$-dimensional simplex $|\sigma|$, and any weighted map $\psi\colon |\partial\sigma| \multimap B(y_i, \delta)$ there exists a weighted map $\widetilde{\psi}\colon |\sigma| \multimap B(y_i, \varepsilon)$ with $\widetilde{\psi}(x) = \psi(x)$ for all $x \in |\partial\sigma|$. Moreover, since φ_k is a w-map, we deduce that there exists an open neighbourhood $O_{x_0}^{\mathcal{Z}} \subset W_k$ of x_0 in $\mathcal{Z}$ such that

$$\varphi_k(x) \subset \bigcup_{i=1}^{s} B(y_i, \delta);$$
$$\sum_{y \in B(y_i,\delta)} w_{\varphi_k}(x, y) = \sum_{y \in B(y_i,\delta)} w_{\varphi_k}(x_0, y),$$

for all $x \in (A \cup |N(\mathbf{U})^{(k)}|) \cap O_{x_0}^{\mathcal{Z}}$ and $1 \leqslant i \leqslant s$. Let $|\sigma| \subset O_{x_0}^{\mathcal{Z}}$ be a $(k+1)$-dimensional simplex and let $\varphi_k(|\partial\sigma|) \subset \bigcup_{l=1}^{s'} B(y_{i_l}, \delta)$ for some subset $I = \{i_1, \ldots, i_{s'}\}$ of $\{1, \ldots, s\}$. In addition, let $\alpha\colon |\partial\sigma| \multimap Y$ be a weighted map defined by $\alpha(x) = \{y_1, \ldots, y_s\}$ for all $x \in |\partial\sigma|$, where $w_\alpha\colon |\partial\sigma| \times Y \to \mathbb{Q}$ is defined by the formula $w_\alpha(x, y) = 0$ for all $x \in |\partial\sigma|, y \in Y$. Then, by Proposition 2.4.3, a weighted map $\varphi_{\partial\sigma}^{\alpha} := (\varphi_k||\partial\sigma|) \cup \alpha$ has the following decomposition

$$\varphi_{\partial\sigma}^{\alpha} = \varphi^1 \cup \ldots \cup \varphi^{s'},$$

where any w-map φ^l satisfies the following condition: $\varphi^l(|\partial\sigma|) \subset B(y_{i_l}, \delta)$, for $1 \leqslant l \leqslant s'$. Consequently, any w-map $\varphi^l\colon |\partial\sigma| \multimap Y$ admits an extension $\widetilde{\varphi}^l\colon |\sigma| \multimap Y$ over $|\sigma|$ such that $\widetilde{\varphi}^l(|\sigma|) \subset B(y_{i_l}, \varepsilon)$. Let $\varphi_\sigma^\alpha\colon |\sigma| \multimap Y$ be defined by

$$\varphi_\sigma^\alpha = \widetilde{\varphi}^1 \cup \ldots \cup \widetilde{\varphi}^{s'}.$$

Since, by Lemma 2.2.9, a weighted map $\varphi_\sigma^\alpha||\partial\sigma|$ is w-homotopic to $\varphi_k||\partial\sigma|$, we deduce from Proposition 2.4.3 that a weighted map $\varphi_k||\partial\sigma|$ can be extended to a weighted map $\varphi_\sigma\colon |\sigma| \multimap Y$ with $\varphi_\sigma(|\sigma|) \subset \bigcup_{l=1}^{s'} B(y_{i_l}, \varepsilon)$ and this completes the proof of the lemma. □

Proof of Theorem 2.8.14 (continuation). From Lemma 2.8.15 it follows that for each $x \in \partial A$ there exists an open subset $W_x := (O_x \cap A) \cup N[O'_x]$ of $\mathcal{Z}$ such that for each $(k+1)$-dimensional simplex $|\sigma|$ contained in W_x there exists an extension $\widetilde{\varphi}_\sigma\colon |\sigma| \multimap Y$ of $\varphi_k||\partial\sigma|$. Let

$$\widetilde{W}_{k+1} := \bigg(\bigcup_{x \in \partial A} \widetilde{N}[O'_x]\bigg) \cup A.$$

Notice that $\widetilde{W}_{k+1}$ does not need to be open in $\mathcal{Z}$. But taking into account Lemma 2.8.13, we deduce that for each $x \in \partial A$ there exists an open neighbourhood O''_x of x in X such that $N[O''_x] \subset \widetilde{N}[O'_x]$. Consequently, the set

$$W_{k+1} := \bigg(\bigcup_{x \in \partial A} N[O''_x]\bigg) \cup A.$$

is open in $\mathcal{Z}$. So for each $(k+1)$-dimensional simplex $|\sigma|$ contained in W_{k+1} we can define the following number as follows

$$\varepsilon_\sigma := \inf\{\varepsilon \mid \varphi_k||\partial\sigma| \text{ can be extended to } \widetilde{\varphi}_\sigma \text{ over } |\sigma| \text{ with values in } O_\varepsilon(\varphi_k(|\partial\sigma|))\}.$$

Now we shall prove the following lemma.

Lemma 2.8.16. *Let $x_0 \in \partial A$ be a fixed point and let $\varphi_k\colon A \cup (|N(\mathbf{U}^{(k)})| \cap W_k) \multimap Y$ be a weighted map. Then for each $\varepsilon > 0$ there exists an open neighbourhood $O^{\mathcal{Z}}_{x_0} \subset W_k$ of x_0 in $\mathcal{Z}$ such that for any $(k+1)$-dimensional simplex $|\sigma|$ contained in $O^{\mathcal{Z}}_{x_0} \cap W_{k+1}$ the number ε_σ satisfies the following condition: $\varepsilon_\sigma < \varepsilon/2$.*

Proof. Let $\varphi_k(x_0) = \{y_1, \dots, y_s\}$ and let $\varepsilon > 0$. We can assume that $B(y_j, \varepsilon) \cap B(y_i, \varepsilon) = \emptyset$ for $i \neq j$. Let $\delta < \varepsilon/8$ and $O^{\mathcal{Z}}_{x_0} \subset W_k$ satisfy the assertion of Lemma 2.8.15. Let us take any $(k+1)$-dimensional simplex $|\sigma|$ contained in $O^{\mathcal{Z}}_{x_0} \cap W_{k+1}$. Then for a weighted map $\varphi_k||\partial\sigma|\colon |\partial\sigma| \multimap Y$ there exists a subset I_σ of $\{1, \dots, s\}$ such that

$$\begin{aligned}(\varphi_k||\partial\sigma|)(|\partial\sigma|) \cap B(y_{i_l}, \delta) &\neq \emptyset \quad \text{for } i_l \in I_\sigma,\\ (\varphi_k||\partial\sigma|)(|\partial\sigma|) \cap B(y_{i_l}, \delta) &= \emptyset \quad \text{for } i_l \notin I_\sigma.\end{aligned}$$

Hence, in view of Lemma 2.8.15, there exists an extension $\widetilde{\varphi}_\sigma\colon |\sigma| \multimap Y$ of $\varphi_k||\partial\sigma|$ such that

$$\widetilde{\varphi}(|\sigma|) \subset O_{\varepsilon/8}(\{y_{i_1}, \dots, y_{i_{s'}}\})$$

where $s' := \#I_\sigma$. Since for each $i_l \in I_\sigma$ the following condition holds

$$y_l \in O_\delta((\varphi_k||\partial\sigma|))(|\partial\sigma|),$$

it follows that

$$O_{\varepsilon/8}(\{y_{i_1}, \dots, y_{i_{s'}}\}) \subset O_{\varepsilon/8+\delta}((\varphi_k||\partial\sigma|)(|\partial\sigma|)) \subset O_{\varepsilon/4}((\varphi_k||\partial\sigma|)(|\partial\sigma|)).$$

Consequently, we obtain $\widetilde{\varphi}_\sigma(|\sigma|) \subset O_{\varepsilon/4}((\varphi_k||\partial\sigma|)(|\partial\sigma|))$. Thus, $\varepsilon_\sigma < \varepsilon/2$, which completes the proof of the lemma. □

The End of the Proof of Theorem 2.8.14. From the above consideration it follows that we can assign to each $(k+1)$-dimensional simplex $|\tau| \subset W_{k+1}$ an extension $\widetilde{\varphi}_\tau\colon |\tau| \multimap Y$ of $\varphi_k||\partial\tau|$ such that

$$\widetilde{\varphi}_\tau \subset O_{2\varepsilon_\tau}((\varphi_k||\partial\tau|)(|\partial\tau|)). \tag{2.36}$$

Consequently, we can define

$$\begin{aligned}&\sigma_{\varphi_{k+1}}\colon A \cup (|N(\mathbf{U})^{(k+1)}| \cap W_{k+1}) \multimap Y,\\ &w_{\varphi_{k+1}}\colon A \cup (|N(\mathbf{U})^{(k+1)}| \cap W_{k+1}) \times Y \to \mathbb{Q},\end{aligned}$$

by the formulas:

$$\sigma_{\varphi_{k+1}}(x) = \begin{cases} \sigma_{\varphi_k}(x) & \text{if } x \in A, \\ \sigma_{\widetilde{\varphi}_\tau}(x) & \text{if } x \in |\tau| \subset |N(\mathbf{U})^{(k+1)}| \cap W_{k+1}, \end{cases}$$

$$w_{\varphi_{k+1}}(x) = \begin{cases} w_{\varphi_k}(x,y) & \text{if } x \in A,\ y \in Y \\ w_{\widetilde{\varphi}_\tau}(x,y) & \text{if } x \in |\tau| \subset |N(\mathbf{U})^{(k+1)}| \cap W_{k+1},\ y \in Y. \end{cases}$$

Now we shall show that $\varphi_{k+1} = (\sigma_{\varphi_{k+1}}, w_{\varphi_{k+1}})$ is a weighted map. First, we prove that $\sigma_{\varphi_{k+1}}$ is upper semicontinuous. Of course, the upper semicontinuity of $\sigma_{\varphi_{k+1}}$ at every point x of the set $(A \cup (|N(\mathbf{U})^{(k+1)}| \cap W_{k+1})) \setminus \partial A$ is evident, so it remains to check this at the points of ∂A. For this purpose, let us fix a point $x_0 \in \partial A$. The upper semicontinuity of $\sigma_{\varphi_{k+1}}$ at x_0 will be proved if we show that for any $\varepsilon > 0$ there exists an open neighbourhood $O^{\mathcal{Z}}_{x_0} \subset W_{k+1}$ of x_0 in $\mathcal{Z}$ such that $\sigma_{\varphi_{k+1}}(x) \subset O_\varepsilon(\sigma_{\varphi_{k+1}}(x_0))$ for all $x \in O^{\mathcal{Z}}_{x_0}$. Let us observe that an upper semicontinuity of σ_{φ_k} at a point x_0 implies the existence of an open neighbourhood $\widetilde{O}^{\mathcal{Z}}_{x_0} = \mathcal{O} \cap A \cup N[\mathcal{O}'_{x_0}] \subset W_k$ of x_0 such that

$$\sigma_{\varphi_k}(x) \subset O_{\varepsilon/16}(\sigma_{\varphi_k}(x_0)) \text{ for } x \in (\mathcal{O}_{x_0} \cap A) \cup (N[\mathcal{O}'_{x_0}]) \cap |N(\mathbf{U})^{(k)}|). \tag{2.37}$$

Moreover, by Lemma 2.8.16, we can assume that $\varepsilon_\tau < \varepsilon/8$ for each $(k+1)$-dimensional simplex $|\tau|$ contained in $N[\mathcal{O}'_{x_0}] \cap |N(\mathbf{U})^{(k+1)}|$. Consequently, if $|\tau| \subset N[\mathcal{O}'_{x_0}] \cap |N(\mathbf{U})^{(k+1)}| \cap W_{k+1}$, then by (2.36) and (2.37) we have

$$\begin{aligned} \sigma_{\varphi_{k+1}}(\tau) &\subset O_{2\varepsilon_\tau}((\sigma_{\varphi_k||\partial\tau|}))(|\partial\tau|) \subset O_{\varepsilon/4}(O_{\varepsilon/16}(\sigma_{\varphi_k}(x_0))) \\ &\subset O_{\varepsilon/2}(\sigma_{\varphi_k}(x_0)) = O_{\varepsilon/2}(\sigma_{\varphi_{k+1}}(x_0)). \end{aligned}$$

From Lemma 2.8.15 it follows that there exists an open neighbourhood $\mathcal{O}''_{x_0}$ of x_0 in X such that $N[\mathcal{O}''_{x_0}] \subset \widetilde{N}[\mathcal{O}'_{x_0}]$ and hence for each $x \in (\mathcal{O}_{x_0} \cap A) \cup (N[\mathcal{O}''_{x_0}] \cap |N(\mathbf{U})^{(k+1)}| \cap W_{k+1})$ we obtain

$$\sigma_{\varphi_{k+1}}(x) \subset O_{\varepsilon/2}(\sigma_{\varphi_{k+1}}(x_0)),$$

which proves that $\sigma_{\varphi_{k+1}}$ is upper semicontinuous at x_0.

Now, we shall show that $w_{\varphi_{k+1}}$ satisfies the condition (b2) of Definition 2.1.1. Notice that the condition (b1) of Definition 2.1.1 is obviously satisfied by $w_{\varphi_{k+1}}$. Moreover, to check the condition (b2) of Definition 2.1.1 it suffices to restrict our considerations to the case of $x \in \partial A$. For this purpose, let us fix a point $x_0 \in \partial A$ and let V be an open subset of Y such that $\sigma_{\varphi_{k+1}}(x_0) \cap \partial V = \emptyset$. In addition, let $\sigma_{\varphi_{k+1}}(x_0) = \{y_1, \ldots, y_s\}$ and $\sigma_{\varphi_{k+1}}(x_0) \cap V = \{y_{i_1}, \ldots, y_{i_{s'}}\}$. Choose $\varepsilon > 0$ such that

(i) $B(y_i, \varepsilon) \cap B(y_j, \varepsilon) = \emptyset$ for $i \neq j$,
(ii) $B(y_{i_l}, \varepsilon) \subset V$ for $l = 1, \ldots, s'$,
(iii) $B(y_j, \varepsilon) \cap V = \emptyset$ for $y_j \notin V$.

Since σ_{φ_k} is a weighted map, there is an open neighbourhood $\mathcal{O}^{\mathcal{Z}}_{x_0} = (\mathcal{O}_{x_0} \cap A) \cup N[\mathcal{O}'_{x_0}] \subset W_k$ of x_0 in $\mathcal{Z}$ such that

$$\sum_{y \in \bigcup\limits_{i=1}^{s'} B(y_{i_l},\varepsilon)} w_{\varphi_k}(x,y) = \sum_{y \in \bigcup\limits_{i=1}^{s'} B(y_{i_l},\varepsilon)} w_{\varphi_k}(x_0,y) = \sum_{y \in \bigcup\limits_{i=1}^{s'} B(y_{i_l},\varepsilon)} w_{\varphi_{k+1}}(x_0,y) \tag{2.38}$$

for all $x \in \mathcal{O}^{\mathcal{Z}}_{x_0} \cap (|N(\mathbf{U})^{(k)}| \cup A)$. Moreover, there exists an open neighbourhood $\widetilde{\mathcal{O}}^{\mathcal{Z}}_{x_0} := (\mathcal{O}_{x_0} \cap A) \cup N[\widetilde{\mathcal{O}}'_{x_0}] \subset W_{k+1}$ of x_0 in $\mathcal{Z}$ such that

$$\sigma_{\varphi_{k+1}}(x) \subset \bigcup_{i=1}^{s} B(y_i,\varepsilon),$$

for all $x \in \widetilde{\mathcal{O}}^{\mathcal{Z}}_{x_0} \cap (|N(\mathbf{U})^{(k+1)}| \cup A)$. Let

$$\widehat{\mathcal{O}}^{\mathcal{Z}}_{x_0} := (\mathcal{O}_{x_0} \cap \mathcal{O}_{x_0} \cap A) \cup N[\mathcal{O}'_{x_0} \cap \widetilde{\mathcal{O}}'_{x_0}] \subset \mathcal{O}^{\mathcal{Z}}_{x_0} \cap \widetilde{\mathcal{O}}^{\mathcal{Z}}_{x_0} \subset W_{k+1}.$$

Let $|\tau|$ be a $(k+1)$-dimensional simplex contained in $N[\mathcal{O}'_{x_0} \cap \widetilde{\mathcal{O}}'_{x_0}]$. Then for each $x \in |\tau|$ we have

$$\begin{aligned}
\sum_{y \in V} w_{\varphi_{k+1}}(x,y) &= \sum_{y \in \bigcup\limits_{i=1}^{s'} K(y_{j_i},\varepsilon)} w_{\varphi_{k+1}}(x,y) = I_w\left(\varphi_{k+1}|\left(|\sigma|, \bigcup_{i=1}^{s'} K(y_{j_i},\varepsilon)\right)\right) \\
&= \sum_{y \in \bigcup\limits_{i=1}^{s'} K(y_{j_i},\varepsilon)} w_{\varphi_{k+1}}(x',y) = \sum_{y \in \bigcup\limits_{i=1}^{s'} K(y_{j_i},\varepsilon)} w_{\varphi_k}(x',y) \\
&\overset{(2.38)}{=} \sum_{y \in \bigcup\limits_{i=1}^{s'} K(y_{j_i},\varepsilon)} w_{\varphi_{k+1}}(x_0,y) = \sum_{y \in V} w_{\varphi_{k+1}}(x_0,y),
\end{aligned}$$

for all $x' \in |\partial\tau|$. From Lemma 2.8.13 it follows that there exists an open neighbourhood $\mathcal{O}''_{x_0} \subset \mathcal{O}'_{x_0} \cap \widetilde{\mathcal{O}}'_{x_0}$ of x_0 in X such that $N[\mathcal{O}''_{x_0}] \subset \widetilde{N}[\mathcal{O}'_{x_0} \cap \widetilde{\mathcal{O}}'_{x_0}]$. Consequently, from the above considerations we obtain the following equation

$$\sum_{y \in V} w_{\varphi_{k+1}}(x,y) = \sum_{y \in V} w_{\varphi_{k+1}}(x_0,y),$$

for all $x \in (\mathcal{O}_{x_0} \cap \widetilde{\mathcal{O}}_{x_0} \cap A) \cup (N[\mathcal{O}''_{x_0}] \cap |N(\mathbf{U})^{(k+1)}|) \subset W_{k+1}$, which completes the proof of the fact that $w_{\varphi_{k+1}}$ satisfies the condition (b2) of Definition 2.1.1.

Consequently, after n_0 steps, we arrive at the desired weighted map $\widetilde{\varphi} := \varphi_{n_0}$ of the neighbourhood $U := W_{n_0}$ of A in $\mathcal{Z}$. The proof of the theorem is complete. □

The following lemma is crucial for our further considerations.

Lemma 2.8.17 ([7]). *Let A be a closed subset of a metric space X and let $\mathcal{Z}$ be as in* (2.34)*. Then there exists a continuous map $\chi\colon X \to \mathcal{Z}$ satisfying the condition: $\chi(x) = x$ for every point $x \in A$.*

From Theorem 2.8.14 and Lemma 2.8.17 we immediately obtain the following theorem.

Theorem 2.8.18. *Let A be a closed subset of a metric space X and let $\dim(X \setminus A) \leqslant n+1$. Assume further that $Y \in w\text{-LC}^n$. Then any weighted map $\varphi\colon A \multimap Y$ can be extended to a weighted map $\widetilde{\varphi}\colon U \multimap Y$, where U is an open neighbourhood of A in X.*

Theorem 2.8.19. *For any metric space Y and $n \geqslant 0$ the following conditions are equivalent:*

(a) $Y \in w\text{-LC}^n$.
(b) *If A is a closed subset of a metric space X and $\dim(X \setminus A) \leqslant n+1$, then for every weighted map $\varphi\colon A \multimap Y$ there exists a neighbourhood U of A in X such that φ can be extended to a weighted map $\widetilde{\varphi}\colon U \multimap Y$.*
(c) *If V_y is a neighbourhood of a point $y \in Y$, then there exists a neighbourhood $V_0 \subset V_y$ such that, for any metric space X and for any closed subset of A in X satisfying the condition $\dim(X \setminus A) \leqslant n+1$, any weighted map $\varphi\colon A \multimap V_0$ can be extended to a weighted map $\widetilde{\varphi}\colon X \multimap V_y$.*
(d) *If V_y is a neighbourhood of a point $y \in Y$, then there exists a neighbourhood $V_0 \subset V_y$ such that for any metric space X with $\dim(X) \leqslant n$ and for any weighted map $\varphi\colon X \multimap V_0$ there exists a weighted map $\phi\colon X \times [0,1] \multimap V_y$ with the following properties: $\phi(x,0) = \varphi(x), \phi(x,1) = I_w(\varphi)y$, for any $x \in X$.*

Proof. The implication (a)$\Rightarrow$(b) has been proved in Theorem 2.8.18. Moreover, the proof of the implications (b)$\Rightarrow$(c), (c)$\Rightarrow$(d), (d)$\Rightarrow$(a) goes similar to the proof of the corresponding implications in the proof of Theorem 9.1 in [7] and therefore it is left as an exercise to the reader. $\square$

Corollary 2.8.20. *Let X be a metric space with $\dim(X) \leqslant n$. Then $X \in w\text{-LC}^n$ if and only if X is locally w-contractible.*

Proof. The implication $\Leftarrow$ follows from Proposition 2.8.12. Conversely, let us fix $x_0 \in X$ and let $U_{x_0} \subset X$ be an open neighbourhood of x_0. In addition, let $V_0 \subset U_{x_0}$ satisfy condition (d) in Theorem 2.8.19. Since $\dim(V_0) \leqslant \dim(X) \leqslant n$, it follows by condition (d) in Theorem 2.8.19 that the inclusion $i\colon V_0 \to U_0$ is w-homotopic to the constant map $k\colon V_0 \to U_0$ at x_0, which completes the proof of the corollary. $\square$

Now, we are going to prove the following version of the Lefschetz fixed point theorem for weighted maps. From now on, only metric spaces are considered.

Lemma 2.8.21. *Let X be a compact space and let Y be an* ANR. *In addition, let $\alpha: Y \multimap X$ be a weighted map, let $j: X \to Y$ be an embedding. Then a weighted map $\varphi: X \multimap X$ given by $\varphi(x) = \alpha \circ j(x)$, for all $x \in X$, admits the following properties:*

(a) *φ is a Lefschetz map,*
(b) *if $\Lambda(\varphi) \neq 0$, then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

Proof. From Theorem 2.5.1 it follows that a weighted map $j \circ \alpha: Y \multimap Y$ is a Lefschetz map. Consequently, by Proposition 2.3.12, $\alpha \circ j$ is a Lefschetz map and $\Lambda(\alpha \circ j) = \Lambda(j \circ \alpha)$. Let $\Lambda(\varphi) \neq 0$. Then $\Lambda(j \circ \alpha) \neq 0$. Now, from Theorem 2.5.1 we deduce that there exists a point $y_0 \in Y$ such that $y_0 \in j \circ \alpha(y_0)$. Thus $j^{-1}(y_0) \in j^{-1} \circ j \circ \alpha(y_0) = \alpha(y_0) = (\alpha \circ j)(j^{-1}(y_0))$, which completes the proof. □

Theorem 2.8.22 ([31])**.** *For every $n \geqslant 1$ there exists a compact $(n+1)$-dimensional absolute retract $X^{(n)}$ such that every separable metric space of dimension $\leqslant n$ has an embedding into $X^{(n)}$.*

Theorem 2.8.23. *Let K be a compact space with* $\dim K \leqslant n$ *and assume that $\varphi: K \multimap K$ can be factored as $K \overset{\alpha}{\multimap} X \overset{f}{\to} K$, where X is a w-*LC^n*-space, φ is a weighted map and f is a single-valued continuous map. Then*

(a) *φ is a Lefschetz map,*
(b) *if $\Lambda(\varphi) \neq 0$, then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

Proof. Our proof of the above theorem is based upon ideas found in [31]. From Theorem 2.8.22 it follows that there is an embedding $j: K \hookrightarrow \mathbf{Y}^{(n)}$ of K into a compact absolute retract $Y^{(n)}$ with $\dim(Y^{(n)}) = n+1$. Since $\dim(Y^{(n)} \setminus j(K)) \leqslant n+1$ (see [18]), it follows from Theorem 2.8.19 that there is an extension $\widetilde{\alpha}: U \multimap X$ of the weighted map $\alpha \circ j^{-1}: j(K) \multimap X$ over an open neighbourhood U of $j(K)$ in $Y^{(n)}$. Consider the commutative diagram

$$\begin{tikzcd} U \arrow[d, multimap, "\widetilde{\alpha}"'] & j(K) \arrow[l, hook', "i"'] \\ X \arrow[dr, "f"] & K \arrow[u, "j"'] \arrow[l, multimap, "\alpha"'] \arrow[d, multimap, "\varphi"] \\ & K \end{tikzcd}$$

where $i: j(K) \to U$ is the inclusion. Consequently, $\varphi(x) = (f \circ \widetilde{\alpha}) \circ (i \circ j)(x)$. Now, since U is an ANR, our assertion follows at once from Lemma 2.8.21. □

CHAPTER 3

WEIGHTED CARRIERS

In this chapter, we give a definition and several examples of weighted carriers. Next, we prove some properties of such mappings that will be used in the sequel.

3.1. Definition and examples

Given any multivalued map $\Phi\colon X \multimap Y$ we put

$$D(\Phi) = \{(V, x) \mid V \text{ is an open subset of Y and } \Phi(x) \cap \partial V = \emptyset\}.$$

Definition 3.1.1. A multivalued u.s.c. map $\Phi\colon X \multimap Y$ with compact values is said to be a *weighted carrier* if there exists a function $I_{w\mathrm{loc}}\colon D(\Phi) \to \mathbb{Q}$ satisfying the following conditions:

(a) (Existence) If $I_{w\mathrm{loc}}(\Phi, V, x) \neq 0$, then $\Phi(x) \cap V \neq \emptyset$.

(b) (Local invariance) For every $(V, x) \in D(\Phi)$ there exists an open neighbourhood U_x of a point x such that for each $x' \in U_x$ we have

$$I_{w\mathrm{loc}}(\Phi, V, x) = I_{w\mathrm{loc}}(\Phi, V, x').$$

(c) (Additivity) If $\Phi(x) \cap V \subset \bigcup_{i=1}^{k} V_i$, where V_i, $1 \leqslant i \leqslant k$, are open disjoint subsets of V, then

$$I_{w\mathrm{loc}}(\Phi, V, x) = \sum_{i=1}^{k} I_{w\mathrm{loc}}(\Phi, V_i, x).$$

A function $I_{w\mathrm{loc}}\colon D(\Phi) \to \mathbb{Q}$ satisfying the above conditions will be called the *local weighted index* of Φ.

Moreover, notice that for a weighted carrier Ψ, the set $\Psi(x)$ does not need to have a finite number of connected components.

Remark 3.1.2. Let us notice that Definition 3.1.1 is equivalent to that of [10], but our definition of weighted carriers will turn out to be much more useful in our considerations.

Remark 3.1.3. The additivity property in the case of $k = 1$ will be called the *excision property.*

Below, we shall present a number of examples of weighted carriers.

Example 3.1.4. It is easy to see that if $\Phi: X \multimap Y$ is an upper semicontinuous map with connected values, then Φ is a w-carrier. Indeed, it is enough to define a function $I_{w\mathrm{loc}}: D(\Phi) \to \mathbb{Q}$ as follows

$$I_{w\mathrm{loc}}(\Phi, U, x) := \begin{cases} 1 & \text{if } \Phi(x) \cap U \neq \emptyset, \\ 0 & \text{if } \Phi(x) \cap U = \emptyset, \end{cases}$$

for any $(U, x) \in D(\Phi)$.

Example 3.1.5. If $\varphi: X \multimap Y$ is a weighted map, then $I_{w\mathrm{loc}}: D(\varphi) \to \mathbb{Q}$ is defined by $I_{w\mathrm{loc}}(\varphi, U, x) := \sum_{y \in U} w_\varphi(x, y)$ for any $(U, x) \in D(\varphi)$.

Example 3.1.6. Let X be a compact ANR and let $f: X \times [0,1] \to X$ be a continuous function with the Lefschetz number $\lambda(f_0) \neq 0$ of f_0, where $f_0(x) = f(x, 0)$ for all $x \in X$. Then a multivalued (u.s.c.) map $\Phi: [0,1] \multimap X$ defined by $\Phi(t) = \{x \mid f_t(x) := f(x,t) = x\}$ for all $t \in [0,1]$ is a weighted carrier, because a function $I_{w\mathrm{loc}}: D(\Phi) \to \mathbb{Q}$ given by

$$I_{w\mathrm{loc}}(\Phi, U, t) := \mathrm{ind}(f_t, U, X)$$

satisfies all the conditions of Definition 3.1.1, where $\mathrm{ind}(f_t, U, X)$ denotes the fixed point index for single-valued maps (for more information on the fixed point index for single-valued maps, see [31]).

Example 3.1.7. Let M and N be two topological manifolds of the same dimension and let $f: M \to N$ be a proper map (i.e. for any compact $K \subset N$, $f^{-1}(K)$ is compact) from M onto N. Then the multivalued map $\Phi_f: N \multimap M$ given by $\Phi_f(n) = f^{-1}(n)$, for all $n \in N$, is a weighted carrier. For more details concerning this example we recommend [10].

Example 3.1.8. Let $\Psi: X \multimap Y$ be a continuous map such that $\Psi(x)$ has 1 or n acyclic components for all $x \in X$, where $n \geqslant 2$ is fixed. Then $I_{w\mathrm{loc}}: D(\Psi) \to \mathbb{Q}$ is defined as follows

$$I_{w\mathrm{loc}}(\Psi, U, x) := \begin{cases} 1 & \text{if } \Psi(x) \subset U, \\ k_x & \text{if } \Psi(x) \cap U \neq \emptyset, \\ 0 & \text{if } \Psi(x) \cap U = \emptyset, \end{cases}$$

where k_x denotes a number of connected components of $\Psi(x)$ contained in U.

Example 3.1.9. Let $\Psi\colon X \multimap X$ be an upper semicontinuous multivalued map and let $\Psi(x)$ consist of a finite number of $\mathbb{Q}$-acyclic components, for any $x \in X$. If Ψ is an m-multivalued map (see Remark 3.1.11 below), then Ψ is a weighted carrier. Note that the converse is not true, i.e. a weighted carrier need not be an m-multivalued map (see Example 3.1.10 below).

Example 3.1.10. Let $Y = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$ be a subset of $\mathbb{R}$ and let $\Psi\colon X \multimap Y$ be defined by $\Psi(x) = Y$ for any $x \in X$. In addition, let $\mathcal{B}$ be a collection of subsets of Y such that

$$B \in \mathcal{B} \iff B = Y \text{ or } B = \{1/i\} \text{ or } B = Y \setminus \{1, 1/2, \dots, 1/i\} \quad \text{for some } i \in \mathbb{N}.$$

Now, we shall define a nontrivial function $I_{w\mathrm{loc}}\colon D(\Psi) \to \mathbb{Q}$. For this purpose, let us observe that $(V, x) \in D(\Psi)$ if and only if it can be represented as a finite union of elements of $\mathcal{B}$. Consequently, it is enough to define a function $I_{w\mathrm{loc}}$ on all pairs (B, x) with $B \in \mathcal{B}$ and $x \in X$. Let

$$B_0 = Y,\ B_1 = Y \setminus \{1\},\ B_2 = Y \setminus \{1, 1/2\}, \dots, B_n = Y \setminus \{1, 1/2, \dots, 1/n\}, \dots .$$

Then, we put

$$\begin{aligned}
I_{w\mathrm{loc}}(\Psi, B_0, x) &:= 1 && \text{for } x \in X,\\
I_{w\mathrm{loc}}(\Psi, B_1, x) &:= 0 && \text{for } x \in X,\\
I_{w\mathrm{loc}}(\Psi, B_2, x) &:= 1 && \text{for } x \in X,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
I_{w\mathrm{loc}}(\Psi, B_n, x) &:= (1/2)((-1)^n + 1) && \text{for } x \in X,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

and

$$\begin{aligned}
I_{w\mathrm{loc}}(\Psi, \{1/(2k+1)\}, x) &= 1 && \text{for } x \in X,\ k \geqslant 0,\\
I_{w\mathrm{loc}}(\Psi, \{1/(2k)\}, x) &= -1 && \text{for } x \in X,\ k \geqslant 1.
\end{aligned}$$

One can easily show that the above function $I_{w\mathrm{loc}}$ satisfies the conditions of Definition 3.1.1. Of course, Ψ is not an m-multivalued map. Using this example one can construct another example of a weighted carrier, which is not an m-multivalued map.

Remark 3.1.11. Let $\Psi\colon X \multimap Y$ be an upper semicontinuous multivalued map with compact values. Two points $(x_1, y_1), (x_2, y_2) \in \Gamma_\Psi$ are equivalent ($(x_1, y_1) \sim (x_2, y_2)$) if and only if $x_1 = x_2$ and y_1, y_2 are in the same connected component of $\Psi(x_1) = \Psi(x_2)$. This defines a new set $\widetilde{\Gamma}_\Psi = \Gamma_\Psi/_\sim$ with elements denoted by $(x, C(x))$, where $C(x)$ denotes a connected component of $\Psi(x)$ as a subset of Y.

In what follows, a map $m\colon \widetilde{\Gamma}_\Psi \to \mathbb{Q}$ is called the multiplicity function for Ψ. Note that in the above definition $\mathbb{Q}$ can be replaced by an arbitrary ring without zero divisors.

Let Ψ and m be as above. Then a map Ψ is called an m-multivalued map if the following two conditions are satisfied:

(a) $\Psi(x)$ consists of finitely many connected components for each $x \in X$,
(b) for all $x_0 \in K$ with $\Psi(x_0) = C_1(x_0) \cup \ldots \cup C_s(x_0)$, $s = s(x_0)$, and disjoint open neighbourhoods U_i of $C_i(x_0)$ in Y there exists a neighbourhood U of x_0 such that:
$$\Psi(U) \subset \bigcup_{i=1}^{s} U_i$$
and
$$m(x_0, C_i(x_0)) = \sum_{C(x) \subset U_i} m(x, C(x)) \quad \text{for all } x \in U,\ i = 1, \ldots, s.$$

Example 3.1.12. Further examples can be found in [10].

Remark 3.1.13. From now on, all multivalued weighted carriers $\Psi\colon X \multimap Y$ are assumed to have a local weighted index $I_{w\mathrm{loc}}\colon D(\Psi) \to \mathbb{Q}$ such that $I_{w\mathrm{loc}}(\Psi, Y, x)$ does not depend on $x \in X$ (for instance, if X is connected, then the number $I_w(\Psi, Y, x)$ does not depend on the choice of $x \in X$, see Lemma 3.1.14 below).

Lemma 3.1.14. *Let $\Psi\colon X \multimap Y$ be a weighted carrier and let X be a connected space. Then, for every $x, x' \in X$,*
$$I_{w\mathrm{loc}}(\Psi, Y, x) = I_{w\mathrm{loc}}(\Psi, Y, x').$$

Proof. Assume to the contrary that there are two points $x_0, x_0' \in X$ such that
$$I_{w\mathrm{loc}}(\Psi, Y, x_0) \neq I_{w\mathrm{loc}}(\Psi, Y, x_0').$$
Let
$$\begin{aligned} X_1 &:= \{x \in X \mid I_{w\mathrm{loc}}(\Psi, Y, x) = I_{w\mathrm{loc}}(\Psi, Y, x_0')\}, \\ X_1 &:= \{x \in X \mid I_{w\mathrm{loc}}(\Psi, Y, x) \neq I_{w\mathrm{loc}}(\Psi, Y, x_0')\}. \end{aligned}$$
From the local invariance property of $I_{w\mathrm{loc}}$ it follows that X_1 and X_2 are open. Moreover, $X_1 \cap X_2 = \emptyset$, $X_1 \neq \emptyset$, $X_2 \neq \emptyset$ and $X = X_1 \cup X_2$, so we get a contradiction. $\square$

The assumption in Remark 3.1.13 allows us to give the following definition.

Definition 3.1.15. Let $\Psi: X \multimap Y$ be a weighted carrier and let $x_0 \in X$. Then the number

$$I_w(\Psi) := I_{w\mathrm{loc}}(\Psi, Y, x_0)$$

is said to be the *weighted index* of Ψ.

3.2. Basic properties

Lemma 3.2.1. *Let $\Psi: X \multimap Y$ be a weighted carrier and let U be an open subset of Y. In addition, let X_0 be a connected subset of X such that $\Psi(x) \cap \partial U = \emptyset$ for each $x \in X_0$. Then*

$$I_{w\mathrm{loc}}(\Psi, U, x) = I_{w\mathrm{loc}}(\Psi, U, y)$$

for any $x, y \in X_0$.

Proof. Let us define a map $I: X_0 \to \mathbb{Q}$ by $I(y) := I_{w\mathrm{loc}}(\Psi, U, y)$, where the set $\mathbb{Q}$ is endowed with the discrete topology. Then from the local invariance of $I_{w\mathrm{loc}}$ we infer that the above function I is locally constant. Therefore, by the connectedness of X_0, I is constant, which completes the proof. □

Definition 3.2.2. Let U be an open subset of Y and let $\Psi: X \multimap Y$ be a weighted carrier. Let C be a connected subset of X satisfying the following condition: $\Psi(x) \cap \partial U = \emptyset$. Define $I_{w\mathrm{loc}}(\Psi|(C, U))$ to be $I_{w\mathrm{loc}}(\Psi|(C, U)) := I_{w\mathrm{loc}}(\Psi, U, c_0)$, where $c_0 \in C$ is an arbitrary fixed point.

Let $\Psi: Y \multimap Z$ and $\Phi: X \multimap Y$ be two weighted carriers. Assume also that for any $x \in X$, the set $\Phi(x)$ consists of finitely many connected components $C_1^x, \ldots, C_{s_x}^x$. Now, let us fix a point $x \in X$. Since C_i^x, $i = 1, \ldots, s_x$, are compact disjoint subsets of $\Phi(x)$, there exist open subsets V_i^x of Z such that

$$C_i^x \subset V_i^x \quad \text{and} \quad V_i^x \cap V_j^x = \emptyset, \tag{3.1}$$

for $i \neq j$ and $i, j = 1, \ldots, s_x$. Let U be an open subset of Z such that $\Psi \circ \Phi(x) \cap \partial U = \emptyset$.

Definition 3.2.3. Under the above assumptions we let

$$I_{w\mathrm{loc}}(\Psi \circ \Phi, U, x) = \sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi, V_i^x, x) \cdot I_{w\mathrm{loc}}(\Psi|(C_i^x, U)),$$

where $I_{w\mathrm{loc}}(\Psi|(C_i^x, U))$ is defined as in Definition 3.2.2.

Let us observe that from the localization property of $I_{w\mathrm{loc}}$ for Φ it follows that $I_{w\mathrm{loc}}(\Phi, V_i^x, x)$ does not depend on the choice of V_i^x, and hence the above definition is correct.

Proposition 3.2.4. *Let $\Psi: Y \multimap Z$ and $\Phi: X \multimap Y$ be as above. Then a function $I_{w\mathrm{loc}}: D(\Psi \circ \Phi) \to \mathbb{Q}$ defined as in Definition* 3.2.3 *satisfies the existence, local invariance and additivity properties* (*and hence $\Psi \circ \Phi$ is a weighted carrier*).

Proof. Let us fix $x \in X$. Let $\Phi(x) = C_1^x \cup \ldots \cup C_{s_x}^x$, where C_i^x are components of $\Phi(x)$. Moreover, let U be an open subset of Z such that $\Psi \circ \Phi(x) \cap \partial U = \emptyset$.

(Existence) Let $I_{w\mathrm{loc}}(\Psi \circ \Phi, U, x) \neq 0$. Then there exists $1 \leqslant i_0 \leqslant s_x$ such that

$$I_{w\mathrm{loc}}(\Phi, V_{i_0}^x, x) \cdot I_{w\mathrm{loc}}(\Psi|(C_{i_0}^x, U)) \neq 0.$$

Since $I_{w\mathrm{loc}}(\Psi|(C_{i_0}^x, U)) = I_{w\mathrm{loc}}(\Psi, U, c_{i_0})$ for any point $c_{i_0} \in C_{i_0}^x$, it follows that $I_{w\mathrm{loc}}(\Psi, U, c_{i_0}) \neq 0$. Consequently, $\Psi(c_{i_0}) \cap U \neq \emptyset$ and hence $\Psi(\Phi(x)) \cap U \neq \emptyset$, because $c_{i_0} \in C_{i_0}^x \subset \Phi(x)$.

(Local invariance) We first shall show that for any C_i^x, $i = 1, \ldots, s_x$, there exists an open neighbourhood W_i^x of C_i^x in Y such that

$$I_{w\mathrm{loc}}(\Psi|(C_i^x, U)) = I_{w\mathrm{loc}}(\Psi, U, y) \quad \text{for all } y \in W_i^x \tag{3.2}$$

and $W_i^x \cap W_j^x = \emptyset$ for $i \neq j$. For this purpose, we fix C_j^x. By the local invariance of $I_{w\mathrm{loc}}$ for Ψ we infer that for any $y \in C_j^x$ there exists an open neighbourhood O'_y of y such that for each $y' \in O'_y$ the following equalities hold

$$I_{w\mathrm{loc}}(\Psi, U, y') = I_{w\mathrm{loc}}(\Psi, U, y) = I_{w\mathrm{loc}}(\Psi|(C_j^x, U)).$$

Since Ψ is u.s.c. and $\Psi(y) \cap \partial U = \emptyset$ for $y \in C_j^x$, it follows that for any $y \in C_j^x$ there exists an open neighbourhood O''_y of y such that $\Psi(y') \cap \partial U = \emptyset$ for each $y' \in O''_y$. Let $O_y := O'_y \cap O''_y$ for $y \in C_j^x$. Moreover, let $\widetilde{W}_j^x := \bigcup_{y \in C_j^x} O_y$. Then

$$I_{w\mathrm{loc}}(\Psi, U, y) = I_{w\mathrm{loc}}(\Psi|(C_j^x, U)) \quad \text{for } y \in \widetilde{W}_j^x.$$

It is easy to see that there exist open sets $\widehat{W}_i^x$, $i = 1, \ldots, s_x$, such that

$$C_i^x \subset \widehat{W}_i^x \quad \text{and} \quad \widehat{W}_i^x \cap \widehat{W}_j^x = \emptyset \quad \text{for } i \neq j.$$

Obviously, if we put $W_i^x := \widetilde{W}_i^x \cap \widehat{W}_i^x$, then $W_i^x \cap W_j^x = \emptyset$ for $i \neq j$ and $i, j = 1, \ldots, s_x$; which completes the proof of (3.2). Now let us put $I_{w\mathrm{loc}}(\Psi|(W_i^x, U)) := I_{w\mathrm{loc}}(\Psi, U, y)$, where $y \in W_i^x$ is an arbitrary fixed point. Hence

$$I_{w\mathrm{loc}}(\Psi|(W_i^x, U)) = I_{w\mathrm{loc}}(\Psi|(C_i^x, U)) \tag{3.3}$$

for all $1 \leqslant i \leqslant s_x$. Consequently, from the local invariance of $I_{w\mathrm{loc}}$ for Φ we infer that for each $1 \leqslant i \leqslant s_x$ there exists an open neighbourhood O_i^x of the point x such that

$$I_{w\mathrm{loc}}(\Phi, W_i^x, x) = I_{w\mathrm{loc}}(\Phi, W_i^x, x') \tag{3.4}$$

for all $x' \in O_i^x$. Since Φ is u.s.c. we can deduce that there exists an open neighbourhood $\widetilde{O}_x$ of the point x such that $\Phi(\widetilde{O}_x) \subset \bigcup_{i=1}^{s_x} W_i^x$. Let $O_x :=$

$\widetilde{O}_x \cap (\bigcap_{i=1}^{s_x} O_i^x)$. Since the sets W_i^x, $1 \leqslant i \leqslant s_x$, satisfy the condition (3.1), we have

$$I_{w\mathrm{loc}}(\Psi \circ \Phi, U, x) = \sum_{j=1}^{s_x} I_{w\mathrm{loc}}(\Phi, W_j^x, x) \cdot I_{w\mathrm{loc}}(\Psi|(C_j^x, U)).$$

Now we shall show that for any $x' \in O_x$ the following equality holds

$$I_{w\mathrm{loc}}(\Psi \circ \Phi, U, x) = I_{w\mathrm{loc}}(\Psi \circ \Phi, U, x'),$$

where

$$I_{w\mathrm{loc}}(\Psi \circ \Phi, U, x') = \sum_{j=1}^{s_{x'}} I_{w\mathrm{loc}}(\Phi, V_j^{x'}, x') \cdot I_{w\mathrm{loc}}(\Psi|(C_j^{x'}, U)),$$

$C_j^{x'}$ are components of $\Phi(x')$, $1 \leqslant j \leqslant s_{x'}$, and $V_j^{x'}$ are open subsets of Y such that

$$C_j^{x'} \subset V_j^{x'} \quad \text{and} \quad V_i^{x'} \cap V_j^{x'} = \emptyset \quad \text{for } i \neq j.$$

For this purpose, let us fix $x' \in O_x$. Let $I_i^{x'} := \{1 \leqslant k \leqslant s_{x'} \mid C_k^{x'} \subset W_i^x\}$ for $1 \leqslant i \leqslant s_x$ ([11]). Then

$$\begin{aligned}(3.5) \qquad & \sum_{j=1}^{s_{x'}} I_{w\mathrm{loc}}(\Phi, V_j^{x'}, x'), \cdot I_{w\mathrm{loc}}(\Psi|(C_j^{x'}, U)) \\ &= \sum_{i=1}^{s_x} \sum_{j \in I_i^{x'}} I_{w\mathrm{loc}}(\Phi, V_j^{x'}, x') \cdot I_{w\mathrm{loc}}(\Psi|(C_j^{x'}, U)) \\ &= \sum_{i=1}^{s_x} \sum_{j \in I_i^{x'}} I_{w\mathrm{loc}}(\Phi, V_j^{x'}, x') \cdot I_{w\mathrm{loc}}(\Psi|(C_i^{x}, U))\end{aligned}$$

where the last equality follows from the fact that for any $j \in I_i^{x'}$ and any $y \in C_j^{x'} \subset W_i^x$ we have

$$I_{w\mathrm{loc}}(\Psi|(C_j^{x'}, U)) = I_{w\mathrm{loc}}(\Psi, U, y) = I_{w\mathrm{loc}}(\Psi|(W_i^x, U)) \overset{(3.3)}{=} I_{w\mathrm{loc}}(\Psi|(C_i^x, U)).$$

Consequently, we have

$$(3.5) = \sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Psi|(C_i^x, U)) \cdot \Big(\sum_{j \in I_i^{x'}} I_{w\mathrm{loc}}(\Phi, V_j^{x'}, x') \Big).$$

([11]) Let us note that the set $I_i^{x'}$ defined above can be empty, but it holds only in the case $I_{w\mathrm{loc}}(\Phi, W_i^x, x) = 0$. Thus, if $I_i^{x'} = \emptyset$, then we put

$$\sum_{j \in I_i^{x'}} I_{w\mathrm{loc}}(\Phi, V_j^{x'}, x') \cdot I_{w\mathrm{loc}}(\Psi|(C_j^{x'}, U)) = 0.$$

Now, let us observe that, if we show that

$$\sum_{j\in I_i^{x'}} I_{w\mathrm{loc}}(\Phi, V_j^{x'}, x') = I_{w\mathrm{loc}}(\Phi, W_i^x, x), \tag{3.6}$$

then the proof of the local invariance of $I_{w\mathrm{loc}}$ will be completed, because

$$\sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Psi|(C_i^x, U)) \cdot I_{w\mathrm{loc}}(\Phi, W_i^x, x) = I_{w\mathrm{loc}}(\Psi\circ\Phi, U, x).$$

Now let us fix $i \in \{1,\ldots,s_x\}$. Since

$$\Phi(x')\cap V_j^{x'} = C_j^{x'} \subset W_i^x \quad \text{and} \quad \Phi(x')\cap V_j^{x'} \subset V_j^{x'} \quad \text{for any } j\in I_i^{x'},$$

we deduce from the excision property of $I_{w\mathrm{loc}}$ for Φ that

$$I_{w\mathrm{loc}}(\Phi, V_j^{x'}, x') = I_{w\mathrm{loc}}(\Phi, V_j^{x'}\cap W_i^x, x')$$

for all $j\in I_i^{x'}$. Hence

$$\begin{aligned}\sum_{j\in I_i^{x'}} I_{w\mathrm{loc}}(\Phi, V_j^{x'}, x') &= \sum_{j\in I_i^{x'}} I_{w\mathrm{loc}}(\Phi, V_j^{x'}\cap W_i^x, x')\\ &\overset{(*)}{=} I_{w\mathrm{loc}}\bigg(\Phi, \bigg(\bigcup_{j\in I_i^{x'}} V_j^{x'}\cap W_i^x\bigg), x'\bigg)\\ &= I_{w\mathrm{loc}}\bigg(\Phi, \bigg(\bigcup_{j\in I_i^{x'}} V_j^{x'}\bigg)\cap W_i^x, x'\bigg),\end{aligned}$$

where the equality $(*)$ holds true by the additivity property of $I_{w\mathrm{loc}}$ for Φ. Consequently, applying the excision property of $I_{w\mathrm{loc}}$, we obtain

$$I_{w\mathrm{loc}}\bigg(\Phi, \bigg(\bigcup_{j\in I_i^{x'}} V_j^{x'}\bigg)\cap W_i^x, x'\bigg) = I_{w\mathrm{loc}}(\Phi, W_i^x, x') \overset{(3.4)}{=} I_{w\mathrm{loc}}(\Phi, W_i^x, x),$$

which completes the proof of (3.6).

(Additivity) Let $\Psi\circ\Phi(x)\cap U \subset \bigcup_{j=1}^k U_j$, $U_m\cap U_n=\emptyset$ for $m\neq n$, $U_j\subset U$ for $1\leqslant j\leqslant k$. We first show that

$$I_{w\mathrm{loc}}(\Psi|(C_i^x, U)) = \sum_{j=1}^k I_{w\mathrm{loc}}(\Psi|(C_i^x, U_j)) \tag{3.7}$$

for $1\leqslant i\leqslant s_x$. For this purpose, let us fix $1\leqslant i_0\leqslant s_x$ and $c_{i_0}^x\in C_{i_0}^x$. Since $\Psi(c_{i_0}^x)\cap U\subset\bigcup_{j=1}^k U_j$, we deduce from the additivity property of $I_{w\mathrm{loc}}$ for Ψ that

$$I_{w\mathrm{loc}}(\Psi|(C_i^x, U)) = I_{w\mathrm{loc}}(\Psi, U, c_{i_0}^x) = \sum_{j=1}^k I_{w\mathrm{loc}}(\Psi, U_j, c_{i_0}^x),$$

and, taking into account the following equality

$$I_{w\mathrm{loc}}(\Psi|(C_i^x, U_j)) = I_{w\mathrm{loc}}(\Psi, U_j, c_{i_0}^x),$$

we obtain (3.7). Consequently,

$$\begin{aligned}
I_{w\mathrm{loc}}(\Psi \circ \Phi, U, x) &= \sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi, V_i^x, x) \cdot I_{w\mathrm{loc}}(\Psi|(C_i^x, U)) \\
&= \sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi, V_i^x, x) \cdot \left(\sum_{j=1}^{k} I_{w\mathrm{loc}}(\Psi|(C_i^x, U_j) \right) \\
&= \sum_{j=1}^{k} \sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi, V_i^x, x) \cdot I_{w\mathrm{loc}}(\Psi|(C_i^x, U_j)) \\
&= \sum_{j=1}^{k} I_{w\mathrm{loc}}(\Psi \circ \Phi, U_j, x),
\end{aligned}$$

which completes the proof of the additivity property of $I_{w\mathrm{loc}}$ for $\Psi \circ \Phi$. □

As an easy consequence of Proposition 3.2.4 we obtain the following corollary:

Corollary 3.2.5. *Let $f\colon Y \to Z$ be a single-valued map and let $\Psi\colon X \multimap Y$ be a weighted carrier. Then*

$$I_{w\mathrm{loc}}(\Psi \circ f, U, x) = I_{w\mathrm{loc}}(\Psi, U, f(x)).$$

Lemma 3.2.6. *Let $\Psi\colon X \multimap Y$ be an upper semicontinuous multivalued map and let $f\colon Y \to Z$ be a continuous function. If U is an open subset of Y such that $f \circ \Psi \cap \partial U = \emptyset$, then $\Psi(x) \cap \partial f^{-1}(U) = \emptyset$.*

Proof. Let $f \circ \Psi(x) \cap U = \emptyset$. Suppose, contrary to our claim, that $\Psi(x) \cap \partial f^{-1}(U) \neq \emptyset$. Now, observe that

$$\begin{aligned}
\partial f^{-1}(U) &= \overline{f^{-1}(U)} \cap (Y \setminus f^{-1}(U)) \subset f^{-1}(\overline{U}) \cap (Y \setminus f^{-1}(U)) \\
&= f^{-1}(\overline{U}) \cap f^{-1}(Z \setminus U) = f^{-1}(\overline{U} \cap (Z \setminus U)) = f^{-1}(\partial U).
\end{aligned} \tag{3.8}$$

Take a point $y_0 \in \Psi(x) \cap \partial f^{-1}(U)$. Then, by (3.8), we obtain $y_0 \in f^{-1}(\partial U)$. Consequently, $f(y_0) \in f \circ \Psi(x) \cap \partial U$, which contradicts the fact that $f \circ \Psi \cap \partial U = \emptyset$. The proof of the lemma is complete. □

Moreover, we have the following corollary:

Corollary 3.2.7. *Let $f\colon Y \to Z$ be a continuous function and let $\Psi\colon X \multimap Y$ be a weighted carrier. Assume additionally that $\Psi(x)$ consists of finitely many connected components for each $x \in X$. Then*

$$I_{w\mathrm{loc}}(f \circ \Psi, U, x) = I_{w\mathrm{loc}}(\Psi, f^{-1}(U), x).$$

Proof. Let $x \in X$ be a fixed point and let U be an open subset of Z such that $f \circ \Psi(x) \cap \partial U = \emptyset$. Hence, by Lemma 3.2.6, we infer that $(\Psi, f^{-1}(U), x) \in D(\Psi)$. Let $\Psi(x) = C_1 \cup \ldots \cup C_{s_x}$, where C_i, $1 \leqslant i \leqslant s_x$, are connected components of $\Psi(x)$. Now, we choose open sets $V_1^x, \ldots, V_{s_x}^x$ in Y such that

$$C_i \subset V_i^x \quad \text{and} \quad V_i^x \cap V_j^x = \emptyset$$

for $i \neq j$. Since $\Psi(x) \cap \partial f^{-1}(U) = \emptyset$, it follows that

$$\Psi(x) \cap f^{-1}(U) = C_{k_1} \cup \ldots \cup C_{k_s}.$$

Without loss of generality we may assume that $V_j^x \subset f^{-1}(U)$ provided $C_j \subset f^{-1}(U) \cap \Psi(x)$. Then

$$\begin{aligned} I_{w\mathrm{loc}}(f \circ \Psi, U, x) &\overset{(1)}{=} \sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Psi, V_i^x, x) I_{w\mathrm{loc}}(f|(C_i, U)) \\ &\overset{(2)}{=} \sum_{C_i \subset f^{-1}(U) \cap \Psi(x)} I_{w\mathrm{loc}}(\Psi, V_i^x, x) I_{w\mathrm{loc}}(f|(C_i, U)), \end{aligned}$$

where the equality (1) follows from the definition of $I_{w\mathrm{loc}}$ for $f \circ \Psi$, and the last equality uses the fact that $I_{w\mathrm{loc}}(f|(C_i, U)) = 0$ for $C_i \subset Y \setminus f^{-1}(U)$. Since $I_{w\mathrm{loc}}(f|(C_i, U)) = 1$ for $C_i \subset f^{-1}(U) \cap \Psi(x)$, we have

$$\begin{aligned} &\sum_{C_i \subset f^{-1}(U) \cap \Psi(x)} I_{w\mathrm{loc}}(\Psi, V_i^x, x) I_{w\mathrm{loc}}(f|(C_i, U)) \\ &\qquad = \sum_{C_i \subset f^{-1}(U) \cap \Psi(x)} I_{w\mathrm{loc}}(\Psi, V_i^x, x) \cdot 1 \\ &\qquad \overset{(3)}{=} I_{w\mathrm{loc}}\Bigg(\Psi, \bigcup_{\{i | C_i \subset f^{-1}(U) \cap \Psi(x)\}} V_i^x, x\Bigg) \overset{(4)}{=} I_{w\mathrm{loc}}(\Psi, f^{-1}(U), x), \end{aligned}$$

where (3) follows from the additivity property of $I_{w\mathrm{loc}}$ and (4) follows from the excision property of $I_{w\mathrm{loc}}$, since

$$\Psi(x) \cap f^{-1}(U) \subset \bigcup_{\{i | C_i \subset f^{-1}(U) \cap \Psi(x)\}} V_i^x \subset f^{-1}(U),$$

and the result follows. $\square$

Corollary 3.2.7 leads to the following definition.

Definition 3.2.8. Let $\Psi\colon X \multimap Y$ be a weighted carrier and let $f\colon Y \to Z$ be a single-valued map. Then $I_{w\mathrm{loc}}\colon D(f \circ \Psi) \to \mathbb{Q}$ is defined by

$$I_{w\mathrm{loc}}(f \circ \Psi, U, x) := I_{w\mathrm{loc}}(\Psi, f^{-1}(U), x), \tag{3.9}$$

for any $(U, x) \in D(f \circ \Psi)$.

By Lemma 3.2.6, if $(U, x) \in D(f \circ \Psi)$, then $(f^{-1}(U), x) \in D(\Psi)$, which implies that the above definition is correct.

Proposition 3.2.9. *Let $\Psi\colon X \multimap Y$ and $f\colon Y \multimap Z$ be as above. Then a function $I_{w\mathrm{loc}}\colon D(f\circ\Psi) \to \mathbb{Q}$ defined as in* (3.9) *satisfies the existence, local invariance and additivity properties.*

Proof. It is easy to see that $f\circ\Psi$ is an upper semicontinuous map. Let us fix a point $x \in X$. Let U be an open subset of Z such that $f\circ\Psi(x)\cap\partial U=\emptyset$.

(Existence) Assume that $I_{w\mathrm{loc}}(f\circ\Psi,U,x)\neq 0$. Then $I_{w\mathrm{loc}}(\Psi,f^{-1}(U),x)\neq 0$. Consequently, from the existence property of $I_{w\mathrm{loc}}$ for Ψ it follows that $\Psi(x)\cap f^{-1}(U)\neq\emptyset$. Thus $f\circ\Psi(x)\cap U\neq\emptyset$, since

$$\emptyset\neq f(\Psi(x)\cap f^{-1}(U))\subset f\circ\Psi(x)\cap f\circ f^{-1}(U)\subset f\circ\Psi(x)\cap U.$$

(Local invariance) Since $f\circ\Psi$ is upper semicontinuous, it follows that there exists an open neighbourhood V_x of the point x such that $f\circ\Psi(y)\cap\partial U=\emptyset$ for all $y\in V_x$. Moreover, from the local invariance of $I_{w\mathrm{loc}}$ for Ψ we deduce that there exists an open neighbourhood W_x of x such that

$$I_{w\mathrm{loc}}(\Psi,f^{-1}(U),x)=I_{w\mathrm{loc}}(\Psi,f^{-1}(U),z), \tag{3.10}$$

for all $z\in W_x$. Let $O_x:=V_x\cap W_x$. Then $(U,z)\in D(f\circ\Psi)$ for all $z\in O_x$ and

$$\begin{aligned} I_{w\mathrm{loc}}(f\circ\Psi,U,x) &= I_{w\mathrm{loc}}(\Psi,f^{-1}(U),x)\\ &\overset{(3.10)}{=} I_{w\mathrm{loc}}(\Psi,f^{-1}(U),z)=I_{w\mathrm{loc}}(f\circ\Psi,U,z).\end{aligned}$$

(Additivity) Let $f\circ\Psi(x)\cap U\subset\bigcup_{i=1}^k V_i\subset U$ and $V_i\cap V_j=\emptyset$ for $i\neq j$. Observe that

$$\begin{aligned}\Psi(x)\cap f^{-1}(U) &\subset f^{-1}(f\circ\Psi(x))\cap f^{-1}(U)=f^{-1}(f\circ\Psi(x)\cap U)\\ &\subset f^{-1}\left(\bigcup_{i=1}^k V_i\right)=\bigcup_{i=1}^k f^{-1}(V_i).\end{aligned}$$

Then, by the additivity property of $I_{w\mathrm{loc}}$ for Ψ, we obtain

$$\begin{aligned} I_{w\mathrm{loc}}(f\circ\Psi,U,x) &= I_{w\mathrm{loc}}(\Psi,f^{-1}(U),x)\\ &=\sum_{i=1}^k I_{w\mathrm{loc}}(\Psi,f^{-1}(V_i),x)=\sum_{i=1}^k I_{w\mathrm{loc}}(f\circ\Psi,V_i,x),\end{aligned}$$

which completes the proof. □

Lemma 3.2.10 (Gluing lemma for weighted carriers)**.** *Assume that a space X is a union of two closed subsets $X=A_1\cup A_2$ and $A_1\cap A_2\neq\emptyset$. Let $\Psi_1\colon A_1\multimap Y$ and $\Psi_2\colon A_2\multimap Y$ be two weighted carriers such that*

$$\Psi_1(a)=\Psi_2(a)\quad\textit{and}\quad I_{w\mathrm{loc}}(\Psi_1,U,a)=I_{w\mathrm{loc}}(\Psi_2,U,a),$$

for all $a \in A_1 \cap A_2$ *and for all* $(U,a) \in D(\Psi_1)$ ([12]). *Then* $\Psi\colon X \multimap Y$ *given by*

$$\Psi(x) = \begin{cases} \Psi_1(x) & \text{for } x \in A_1, \\ \Psi_2(x) & \text{for } x \in A_2, \end{cases}$$

and

$$I_{w\mathrm{loc}}(\Psi,U,x) = \begin{cases} I_{w\mathrm{loc}}(\Psi_1,U,x) & \text{for } x \in A_1, \\ I_{w\mathrm{loc}}(\Psi_2,U,x) & \text{for } x \in A_2. \end{cases}$$

is a weighted carrier.

Proof. The proof is straightforward. □

Proposition 3.2.11. *Let* X *be an* ANR, *let* A *be a closed* ANR *subspace of* X *and let* Y *be an arbitrary metric space. If* $\Psi\colon A \times [0,1] \multimap Y$ *is a weighted carrier such that* $\Psi_0\colon A \multimap Y$ *is extendable to a w-carrier* $\widetilde{\Psi_0}\colon X \multimap Y$, *then there is a w-carrier* $\overline{\Psi}\colon X \times [0,1] \multimap Y$ *such that*

(a) $\overline{\Psi}|X \times \{0\} = \widetilde{\Psi_0}$,
(b) $\overline{\Psi_t}|A = \Psi_t$, *for every* $t \in [0,1]$,

where $\Psi_t(x) := \Psi(t,x)$ *and* $\overline{\Psi_t}(x) := \overline{\Psi}(t,x)$ *for all* $t \in [0,1]$ *and* $x \in A$.

Proof. The proof proceeds along the same lines as in the case of single-valued maps in [35]. For the sake of completeness we give the details. Indeed, since $X \times \{0\} \cup A \times [0,1]$ is a compact ANR in $X \times [0,1]$ (see Theorem 1.2.4), it follows there exists an open neighbourhood V of $X \times \{0\} \cup A \times [0,1]$ in $X \times [0,1]$ and a retraction $r\colon V \to X \times \{0\} \cup A \times [0,1]$. Let $\widehat{\Psi}\colon X \times \{0\} \cup A \times [0,1] \multimap Y$ be given by

$$\widehat{\Psi}(x,t) = \begin{cases} \Psi(x,t) & \text{for } x \in A, t \in [0,1], \\ \widetilde{\Psi}_0(x) & \text{for } x \in X. \end{cases}$$

By Lemma 3.2.10, $\widehat{\Psi}$ is a weighted carrier. Let $\widehat{\Psi}'\colon V \multimap Y$ be a weighted carrier given by the formula $\widehat{\Psi}'(x,t) = \widehat{\Psi} \circ r(x,t)$ and let $u\colon X \to [0,1]$ be an Urysohn function such that

$$u(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus V. \end{cases}$$

Finally, define $\overline{\Psi}\colon X \times [0,1] \multimap Y$ by

$$\overline{\Psi}(x,t) = \widehat{\Psi}'(x,u(x)t),$$

for all $(x,t) \in X \times [0,1]$. It is easy to see that $\overline{\Psi}$ is the desired extension of Ψ, which completes the proof. □

Now we are able to prove:

([12]) Observe that if $a \in A_1 \cap A_2$, then $(U,a) \in D(\Psi_1)$ if and only if $(U,a) \in D(\Psi_2)$.

Corollary 3.2.12. *Let $X, A \subset X$ and Y be as in Proposition* 3.2.11 *and let $V \subset U$ be subsets of Y. In addition, let $\varphi: A \multimap V$ be a w-map. Then φ can be extended to a w-map $\widetilde{\varphi}: X \multimap U$ if and only if $\varphi \cup (-I_w(\varphi))y_0: A \multimap V$ can be extended to a w-map $\overline{\varphi}: X \multimap U$, where $y_0 \in Y$ is any fixed point.*

Proof. The implication $\Rightarrow$ is obvious.

$\Leftarrow$ Let $\varphi \cup (-I_w(\varphi))y_0: A \multimap V$ be a weighted map and let $\overline{\varphi}: X \multimap U$ be an extension of $\varphi \cup (-I_w(\varphi))y_0$ over X. Then a weighted map $\overline{\varphi} \cup I_w(\varphi)y_0: X \multimap U$ satisfies

$$w_{\overline{\varphi} \cup I_w(\varphi)y_0}(x, y) = w_{i \circ \varphi}(x, y) \quad \text{for all } x \in A,\ y \in U,$$

where $i: V \hookrightarrow U$ is the inclusion. Hence, in view of Lemma 2.2.9, a w-map $\overline{\varphi} \cup I_w(\varphi)y_0|A: A \multimap U$ is w-homotopic to $i \circ \varphi: A \multimap U$. Consequently, by Proposition 3.2.11, it follows that there exists a w-map $\widetilde{\varphi}: X \multimap U$ with $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in A$, which completes the proof. □

At the end of Chapter 3, we prove some properties of I_w for weighted carriers.

Proposition 3.2.13. *Let $\Psi: Y \multimap Z$ and $\Phi: X \multimap Y$ be as in Definition* 3.2.3. *Then*

$$I_w(\Psi \circ \Phi) = I_w(\Psi) \cdot I_w(\Phi).$$

Proof. Let us fix a point $x \in X$. Then, we have

$$\begin{aligned}
I_{w\mathrm{loc}}(\Psi \circ \Phi, Z, x) &= \sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi, V_i^x, x) \cdot I_{w\mathrm{loc}}(\Psi|(C_i^x, Z)) \\
&= \sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi, V_i^x, x) \cdot I_{w\mathrm{loc}}(\Psi, Z, c_i^x) = \sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi, V_i^x, x) \cdot I_w(\Psi) \\
&= I_w(\Psi) \cdot \left(\sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi, V_i^x, x) \right) = I_w(\Psi) \cdot \left(I_{w\mathrm{loc}}\left(\Phi, \bigcup_{i=1}^{s_x} V_i^x, x \right) \right) \\
&= I_w(\Psi) \cdot (I_{w\mathrm{loc}}(\Phi, Y, x)) = I_w(\Psi) \cdot I_w(\Phi),
\end{aligned}$$

where $c_i^x \in C_i^x$. □

Corollary 3.2.14. *Let $f: Y \to Z$ be a continuous function and $\Psi: X \multimap Y$ a weighted carrier. Then*

$$I_w(f \circ \Psi) = I_w(\Psi).$$

Proof. This corollary follows immediately from Definitions 3.1.15 and 3.2.8. □

Proposition 3.2.15. *Let $f: X_1 \to X_2$ and $g: X_3 \to X_4$ be two continuous functions. In addition, let $\Psi: X_2 \multimap X_3$ be a weighted carrier. Then*

$$I_w(g \circ \Psi \circ f) = I_w(\Psi).$$

Proof. First, observe that Proposition 3.2.9 together with Proposition 3.2.4 implies that $g \circ \Psi \circ f$ is a weighted carrier. Finally, the assertion follows from Proposition 3.2.13, Corollary 3.2.14 and the fact that $I_\omega(f) = 1$. □

Definition 3.2.16. We say that two weighted carriers $\Psi, \Phi: X \multimap Y$ are *homotopic* if there exists a weighted carrier $\Upsilon: X \times [0,1] \multimap Y$ such that

$$\begin{aligned} &\Upsilon(x,0) = \Psi(x), \quad \Upsilon(x,1) = \Phi(x) && \text{for all } x \in X;\\ &I_{w\mathrm{loc}}(\Upsilon, U, (x,0)) = I_{w\mathrm{loc}}(\Psi, U, x) && \text{for all } (U,x) \in D(\Psi);\\ &I_{w\mathrm{loc}}(\Upsilon, U, (x,1)) = I_{w\mathrm{loc}}(\Phi, U, x) && \text{for all } (U,x) \in D(\Phi). \end{aligned}$$

Proposition 3.2.17. *If two weighted carriers $\Psi, \Phi: X \multimap Y$ are homotopic, then*

$$I_w(\Psi) = I_w(\Phi).$$

Proof. Let $\Upsilon: X \times [0,1] \multimap Y$ be a homotopy between Ψ and Φ and let $x_0 \in X$ be a fixed point. Furthermore, define a multivalued map $\sigma: [0,1] \multimap Y$ by

$$\sigma(t) = \Upsilon(x_0, t)$$

for all $t \in [0,1]$. It is easy to see that σ is a weighted carrier. Since $[0,1]$ is connected, it follows from Lemma 3.1.14 that

$$I_{w\mathrm{loc}}(\sigma, Y, 0) = I_{w\mathrm{loc}}(\sigma, Y, 1). \tag{3.11}$$

On the other hand, we have

$$I_w(\Psi) = I_{w\mathrm{loc}}(\Psi, Y, x_0) = I_{w\mathrm{loc}}(\Upsilon, Y, (x_0, 0)) = I_{w\mathrm{loc}}(\sigma, Y, 0), \tag{3.12}$$

$$I_w(\Phi) = I_{w\mathrm{loc}}(\Phi, Y, x_0) = I_{w\mathrm{loc}}(\Upsilon, Y, (x_0, 1)) = I_{w\mathrm{loc}}(\sigma, Y, 1). \tag{3.13}$$

Consequently, the conclusion follows from (3.11)–(3.13). $\square$

CHAPTER 4

APPROXIMATION METHODS

In this chapter we are going to study some approximation techniques allowing to investigate fixed points of weighted carriers. The use of the approximation technique in the fixed point theory of weighted carriers goes back to J. Pejsachowicz [54] and G. Conti, J. Pejsachowicz [10].

4.1. Graph-approximations

Definition 4.1.1 ([54]). Let $\Psi: X \multimap Y$ be a weighted carrier and $X_0 \subset X$, and let $\varepsilon > 0$. A weighted map $\psi: X_0 \multimap Y$ is said to be an ε-approximation of $\Psi: X \multimap Y$ (written $\psi \in a_w(\Psi, \varepsilon)$) if

(a) $\psi(x) \subset O_\varepsilon(\Psi(O_\varepsilon(x)))$ for all $x \in X_0$,
(b) $I_{w\mathrm{loc}}(\psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, x)$ for any piece C of $O_\varepsilon(\Psi(O_\varepsilon(x)))$ ([13]) and $x \in X_0$.

Remark 4.1.2. From Lemma 4.1.3 below it follows that the above definition is correct, i.e. $(C, x) \in D(\Psi)$ and $(C, x) \in D(\psi)$. Moreover, we have $I_w(\Psi) = I_w(\varphi)$ since

$$I_w(\Psi) = I_{w\mathrm{loc}}(\Psi, O_\varepsilon(\Psi(O_\varepsilon(x))), x) = I_{w\mathrm{loc}}(\varphi, O_\varepsilon(\Psi(O_\varepsilon(x))), x) = I_w(\varphi).$$

Lemma 4.1.3. *Let U be an open subset of X and let C be a piece of U. If K is a subset of U, then $K \cap \partial C = \emptyset$ (where ∂C denotes the boundary of C with respect to X).*

Proof. It is enough to show that $\partial C \cap U = \emptyset$. For this purpose, let us observe that C and $U \setminus C$ are open in X. Consequently, $\partial C \cap (U \setminus C) = \emptyset$ and $\partial C \cap C = \emptyset$; and hence $\partial C \cap U = \emptyset$. This completes the proof. $\square$

Moreover, we have the following result:

([13]) Given a space Z, by a piece of Z we mean any open and closed subset of Z.

Proposition 4.1.4. *Let $\Psi: X \multimap Y$ be a w-carrier and let $\varphi: X \multimap Y$ be a w-map. In addition, let $0 < \varepsilon_1 < \varepsilon_2$. If φ is an ε_1-approximation of Ψ, then φ is also an ε_2-approximation of Ψ.*

Proof. The first condition of Definition 4.1.1 is obviously satisfied, only the second one needs to be proved. For this purpose, let us fix $x \in X$ and let C be a piece of $O_{\varepsilon_2}(\Psi(O_{\varepsilon_2}(x)))$. Then $\widetilde{C} := C \cap O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x)))$ is a piece of $O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x)))$. Since φ is an ε_1-approximation of Ψ, it follows that

$$I_{w\mathrm{loc}}(\varphi, \widetilde{C}, x) = I_{w\mathrm{loc}}(\Psi, \widetilde{C}, x). \tag{4.1}$$

Consequently, by the excision property of $I_{w\mathrm{loc}}$, we obtain

$$I_{w\mathrm{loc}}(\varphi, \widetilde{C}, x) = I_{w\mathrm{loc}}(\varphi, C, x), \tag{4.2}$$

$$I_{w\mathrm{loc}}(\Psi, \widetilde{C}, x) = I_{w\mathrm{loc}}(\Psi, C, x). \tag{4.3}$$

Now, taking into account (4.1)–(4.3), we have

$$I_{w\mathrm{loc}}(\varphi, C, x) = I_{w\mathrm{loc}}(\Psi, C, x),$$

which completes the proof. □

Remark 4.1.5. Let $\Psi: X \multimap Y$ be a w-carrier and let $X_0 \subset X$. If a weighted map $\varphi: X_0 \multimap Y$ is a δ-approximation of $\Psi|X_0$, then φ is also a δ-approximation of Ψ.

The following lemma will be used repeatedly throughout this paper.

Lemma 4.1.6 ([28])**.** *Let $\psi: X \multimap Y$ and $\varphi: Y \multimap Z$ be two upper semicontinuous multivalued maps. If X is a compact space, then for every $\varepsilon > 0$ there is $\delta > 0$ such that $O_\delta(\varphi)O_\delta(\psi)(x) \subset O_\varepsilon(\varphi \circ \psi(O_\varepsilon(x)))$ for any $x \in X$, where $O_\delta(\varphi)O_\delta(\psi)(x) := O_\delta(\varphi(O_\delta(O_\delta(\psi(O_\delta(x))))))$.*

Now we use the above lemma to obtain the following proposition which will be needed in the sequel.

Proposition 4.1.7. *Let X be a compact space, $\varphi: X \multimap Y$ a weighted map and $\Phi: Y \multimap Z$ a weighted carrier. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\psi: Y \multimap Z$ is a δ-approximation of Φ, then $\psi \circ \varphi$ is an ε-approximation of $\Phi \circ \varphi$.*

Proof. Let $\varepsilon > 0$. From Lemma 4.1.6 it follows that there exists $\delta > 0$ such that

$$O_\delta(\Phi)O_\delta(\varphi)(x) \subset O_\varepsilon(\Phi \circ \varphi(O_\varepsilon(x))),$$

for all $x \in X$. Let $\psi: Y \multimap Z$ be a δ-approximation of $\Phi: Y \multimap Z$. Let us fix $x \in X$. Then

$$\psi(\varphi(x)) \subset O_\delta(\Phi(O_\delta(\varphi(x)))) \subset O_\delta(\Phi)O_\delta(\varphi)(x) \subset O_\varepsilon(\Phi \circ \varphi(O_\varepsilon(x))).$$

What is left is to show that

$$I_{w\mathrm{loc}}(\psi\circ\varphi, C, x) = I_{w\mathrm{loc}}(\Phi\circ\varphi, C, x)$$

for any piece C of $O_\varepsilon(\Phi\circ\varphi(O_\varepsilon(x)))$. Let $\varphi(x) = \{y_1, \ldots, y_{n_x}\}$. Now let us observe (see Definition 3.2.3) that

$$\begin{aligned} I_{w\mathrm{loc}}(\psi\circ\varphi, C, x) &= \sum_{i=1}^{n_x} I_{w\mathrm{loc}}(\varphi, V_i^x, x)\cdot I_{w\mathrm{loc}}(\psi, C, y_i),\\ I_{w\mathrm{loc}}(\Phi\circ\varphi, C, x) &= \sum_{i=1}^{n_x} I_{w\mathrm{loc}}(\varphi, V_i^x, x)\cdot I_{w\mathrm{loc}}(\Phi, C, y_i), \end{aligned}$$

where $V_1^x, \ldots, V_{n_x}^x$ satisfy the following conditions

$$y_i \in V_i^x \quad \text{and} \quad V_i^x \cap V_j^x = \emptyset \quad \text{for } i \neq j.$$

Consequently, it is enough to show that the following equality holds

$$I_{w\mathrm{loc}}(\psi, C, y_i) = I_{w\mathrm{loc}}(\Phi, C, y_i).$$

For this purpose, let us observe that

$$\begin{aligned} I_{w\mathrm{loc}}(\psi, C, y_i) &= I_{w\mathrm{loc}}(\psi, C\cap O_\delta(\Phi(O_\delta(y_i))), y_i)\\ &= I_{w\mathrm{loc}}(\Phi, C\cap O_\delta(\Phi(O_\delta(y_i))), y_i) = I_{w\mathrm{loc}}(\Phi, C, y_i), \end{aligned} \tag{4.4}$$

where the first equality and the last one follow from the excision property of $I_{w\mathrm{loc}}$, because

$$\begin{aligned} \psi(y_i)\cap C &\subset C\cap O_\delta(\Phi(O_\delta(y_i))) \subset C,\\ \Phi(y_i)\cap C &\subset C\cap O_\delta(\Phi(O_\delta(y_i))) \subset C. \end{aligned}$$

Moreover, since $C\cap O_\delta(\Phi(O_\delta(y_i)))$ is a piece of $O_\delta(\Phi(O_\delta(y_i)))$, we deduce that the second equality in (4.4) follows from the fact that ψ is a δ-approximation of Φ. This completes the proof. □

Corollary 4.1.8. *Let $X_0 \subset X$ be a compact subset and let $\Psi\colon X \multimap Y$ be a weighted carrier. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi\colon X \multimap Y$ is a δ-approximation of Ψ, then $\varphi|X_0\colon X_0 \multimap Y$ is an ε-approximation of $\Psi|X_0\colon X_0 \multimap Y$.*

Before proceeding further, we prove some necessary lemmas.

Lemma 4.1.9. *Let X be a locally connected space and let $\Psi\colon X \multimap Y$ be a weighted carrier. Then for every $\varepsilon > 0$ and $x \in X$ there exists $\delta_x > 0$ such that for any $y \in B(x, \delta_x)$ and any piece C of $O_\varepsilon(\Psi(x))$ the following equation holds:*

$$I_{w\mathrm{loc}}(\Psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, y).$$

Proof. Let us fix $\varepsilon > 0$ and $x \in X$. Since Ψ is a weighted carrier, it follows that there exists $\eta_x > 0$ such that

$$\Psi(B(x,\eta_x)) \subset O_\varepsilon(\Psi(x)), \tag{4.5}$$

for all $y \in B(x,\eta_x)$. Additionally, since X is locally connected, it follows that there exists a connected neighbourhood ([14]) V_x of x and $\delta_x > 0$ such that $B(x,\delta_x) \subset V_x \subset B(x,\eta_x)$. Now, let C be a piece of $O_\varepsilon(\Psi(x))$. Then for all $y \in B(x,\eta_x)$ we have $\Psi(y) \cap \partial C = \emptyset$ (where ∂C denotes the boundary of C with respect to Y), by (4.5) and Lemma 4.1.3. Consequently, in view of Lemma 3.2.1, we obtain

$$I_{w\mathrm{loc}}(\Psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, y)$$

for all $y \in V_x$; and hence for all $y \in B(x,\delta_x)$. This completes the proof. □

Lemma 4.1.10. *Let X be a compact locally connected space and let $\Psi\colon X \multimap Y$ be a weighted carrier. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if two points $x, y \in X$ satisfy the following condition $d_X(x,y) < \delta$, then there exists a point $z \in X$ such that*

$$\Psi(x) \subset O_\varepsilon(\Psi(z)) \quad \textit{and} \quad \Psi(y) \subset O_\varepsilon(\Psi(z)), \tag{4.6}$$

$$z \in O_\varepsilon(x) \quad \textit{and} \quad z \in O_\varepsilon(y), \tag{4.7}$$

$$I_{w\mathrm{loc}}(\Psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, z) = I_{w\mathrm{loc}}(\Psi, C, y), \tag{4.8}$$

for any piece C of $O_\varepsilon(\Psi(z))$.

Proof. Let us fix $\varepsilon > 0$. Since Ψ is an upper semicontinuous multivalued map with compact values, it follows that for any $x \in X$ there exists $0 < \delta'_x < \varepsilon$ such that $\Psi(B(x,\delta'_x)) \subset O_\varepsilon(\Psi(x))$. Moreover, in view of Lemma 4.1.9, there exists $\delta''_x > 0$ such that for any piece C of $O_\varepsilon(\Psi(x))$ and any $y \in B(x,\delta''_x)$ we have the following equality

$$I_{w\mathrm{loc}}(\Psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, y). \tag{4.9}$$

Let $\delta_x := (1/2)\min\{\delta'_x, \delta''_x\}$ and let $\{B(x,\delta_x)\}_{x\in X}$ be the open covering of X. Since X is compact, there exists a finite subcovering $B(x_1,\delta_{x_1}), \ldots, B(x_k,\delta_{x_k})$ of this covering. Let us put $\delta := (1/2)\min\{\delta_{x_1}, \ldots, \delta_{x_k}\}$. Now we shall show that such a δ satisfies the conclusion of Lemma 4.1.10. Indeed, let us take two points x and y with $d_X(x,y) < \delta$. Then for a point x there exists $1 \leqslant i_0 \leqslant k$ such that $x \in B(x_{i_0}, \delta_{x_{i_0}})$. Let $z := x_{i_0}$. Then $\Psi(x) \subset O_\varepsilon(\Psi(z))$. Since $d_X(y,z) \leqslant d_X(y,x) + d_X(x,z) < \delta + \delta_z < \varepsilon/2 + \varepsilon/2 = \varepsilon$, we deduce that $\Psi(y) \subset O_\varepsilon(\Psi(z))$ and $y \in O_\varepsilon(z)$; and hence (4.6) and (4.7) are satisfied. Finally, (4.8) follows from (4.9) and the fact that $d_X(x,z) < \delta''_z$ and $d_X(y,z) < \delta''_z$, which completes the proof. □

([14]) Recall that by neighbourhood of x in X we mean always a set containing x in its interior; thus a neighbourhood does not need to be open.

Now we are able to prove the following corollary.

Corollary 4.1.11. *Let X be a compact locally connected space and let $\Psi\colon X \multimap Y$ be a weighted carrier. Then for every $\varepsilon > 0$ there exists $\delta_\Psi > 0$ such that for every $x \in X$ and every piece C of $O_\varepsilon(\Psi(O_\varepsilon(x)))$ we have*

$$I_{w\mathrm{loc}}(\Psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, y) \quad \text{for any } y \in B(x, \delta_\Psi).$$

Proof. Let us fix $\varepsilon > 0$ and let $\delta > 0$ be as in Lemma 4.1.10 according to Ψ and ε. We shall show that such a δ satisfies the conclusion of the above corollary. For this purpose, let us choose a point y such that $d_X(x,y) < \delta$. Then, by Lemma 4.1.10, we deduce that there exists a point z such that

$$\begin{gathered} O_\varepsilon(\Psi(z)) \subset O_\varepsilon(\Psi(O_\varepsilon(x))), \\ \Psi(x) \subset O_\varepsilon(\Psi(z)) \quad \text{and} \quad \Psi(y) \subset O_\varepsilon(\Psi(z)). \end{gathered}$$

Let C be a piece of $O_\varepsilon(\Psi(O_\varepsilon(x)))$. Since $C \cap O_\varepsilon(\Psi(z))$ is open and closed in $O_\varepsilon(\Psi(z))$, it follows by Lemma 4.1.10 and the excision property of $I_{w\mathrm{loc}}$ that

$$\begin{aligned} I_{w\mathrm{loc}}(\Psi, C, x) &= I_{w\mathrm{loc}}(\Psi, C \cap O_\varepsilon(\Psi(z)), x) \overset{4.1.10}{=} I_{w\mathrm{loc}}(\Psi, C \cap O_\varepsilon(\Psi(z)), z) \\ &\overset{4.1.10}{=} I_{w\mathrm{loc}}(\Psi, C \cap O_\varepsilon(\Psi(z)), y) = I_{w\mathrm{loc}}(\Psi, C, y), \end{aligned}$$

which completes the proof. □

Lemma 4.1.12. *Let $\Psi\colon X \multimap Y$ and $\varphi\colon X \multimap Y$ be two upper semicontinuous multivalued map with compact values. In addition, let $\delta > 0$. Then the following conditions are equivalent:*

(a) $\varphi(x) \subset O_\delta(\Psi(O_\delta(x)))$ *for all $x \in X$.*
(b) $\Gamma_\varphi \subset O_\delta(\Gamma_\Psi)$ *(recall that in $X \times Y$ we consider the max-metric).*

Proof. Assume that $\varphi(x) \subset O_\delta(\Psi(O_\delta(x)))$ for all $x \in X$. Take $(x,y) \in \Gamma_\varphi$. Then $y \in O_\delta(\Psi(O_\delta(x)))$; hence, there exists $\widetilde{y} \in Y$ such that $y \in O_\delta(\widetilde{y})$ and $\widetilde{y} \in \Psi(O_\delta(x))$. But $\widetilde{y} \in \Psi(O_\delta(x))$ if and only if there exists $\widetilde{x} \in O_\delta(x)$ such that $\widetilde{y} \in \Psi(\widetilde{x})$. Thus $(x,y) \in O_\delta(\widetilde{x}) \times O_\delta(\widetilde{y}) \subset O_\delta(\Gamma_\Psi)$, so this finishes the proof that $\Gamma_\varphi \subset O_\delta(\Gamma_\Psi)$. Conversely, suppose $\Gamma_\varphi \subset O_\delta(\Gamma_\Psi)$. Take $(x,y) \in \Gamma_\varphi$. Then $y \in \varphi(x)$. Moreover, there exists $(\widetilde{x}, \widetilde{y}) \in \Gamma_\Psi$ such that $(x,y) \in O_\delta((\widetilde{x}, \widetilde{y})) = O_\delta(\widetilde{x}) \times O_\delta(\widetilde{y})$. Thus $y \in O_\delta(\widetilde{y}) \subset O_\delta(\Psi(\widetilde{x})) \subset O_\delta(\Psi(O_\delta(x)))$, which completes the proof that $\varphi(x) \subset O_\delta(\Psi(O_\delta(x)))$. □

Proposition 4.1.13. *Let C be a compact subset of $X \subset Y$ and let $\varphi\colon X \multimap Y$ be an upper semicontinuous multivalued map with compact values such that $\mathrm{Fix}(\varphi) \cap C = \emptyset$. Then there exists $\delta > 0$ such that $x \notin D(\varphi(D(x,\delta)),\delta)$ for all $x \in C$.*

Proof. Suppose, to the contrary, that such $\delta > 0$ does not exist. Then we obtain a sequence $\{x_n\} \subset C$ such that $x_n \in D(\varphi(D(x_n, 1/n)), 1/n)$. Consequently, we get a sequence $\{y_n\}$ with

$$y_n \in \varphi(D(x_n, 1/n)), \quad d(x_n, y_n) < 2/n, \tag{4.10}$$

for all $n \in \mathbb{N}$. Since $y_n \in \varphi(D(x_n, 1/n))$, it follows that there exists a sequence $\widetilde{x}_n \in D(x_n, 1/n)$ such that $y_n \in \varphi(\widetilde{x}_n)$. The compactness of C implies that, passing to subsequence if necessary, $x_n \xrightarrow{n\to\infty} x_0 \in C$. Thus

$$\widetilde{x}_n \xrightarrow{n\to\infty} x_0, \quad y_n \xrightarrow{n\to\infty} x_0. \tag{4.11}$$

By the upper semicontinuity of φ and (4.11), we obtain $x_0 \in \varphi(x_0)$. Consequently, $\mathrm{Fix}(\varphi) \cap C \neq \emptyset$, a contradiction. □

Corollary 4.1.14. *Let C be a compact subset of $X \subset Y$ and let $\varphi: X \multimap Y$ be an upper semicontinuous multivalued map with compact values such that $\mathrm{Fix}(\varphi) \cap C = \emptyset$. Then there exists $\delta > 0$ such that if a multivalued map $\psi: X \multimap Y$ satisfies the condition $\Gamma_\psi \subset O_\delta(\Gamma_\varphi)$, then $\mathrm{Fix}(\psi) \cap C = \emptyset$.*

Proof. This result follows immediately from Lemma 4.1.12 and Proposition 4.1.13. □

Proposition 4.1.15. *Let E^n be a finite-dimensional normed space and let U be an open and bounded subset of E^n. In addition, let $\varphi: \overline{U} \multimap E^n$ be an upper semicontinuous multivalued map with compact values such that*

$$\{x \in \overline{U} \mid 0 \in \varphi(x)\} \cap \partial U = \emptyset.$$

Then there exists $\delta > 0$ such that if a multivalued map $\psi: \overline{U} \multimap E^n$ satisfies the condition $\Gamma_\psi \subset O_\delta(\Gamma_\varphi)$, then $\{x \in \overline{U} \mid 0 \in \psi(x)\} \cap \partial U = \emptyset$.

Proof. The proof is quite similar to that of Proposition 4.1.13 and is left to the reader. □

Proposition 4.1.16. *Let X be a compact space and let $\Psi: X \multimap Y$ be a weighted carrier. In addition, let $f: Y \to Z$ be a continuous function. Then for any $\varepsilon > 0$ there exists $0 < \delta \leqslant \varepsilon$ such that $f \circ \varphi \in a_w(f \circ \Psi, \varepsilon)$ provided $\varphi \in a_w(\Psi, \delta)$.*

Proof. Let $\varepsilon > 0$. Then, by Lemma 4.1.6, there exists $\delta > 0$ such that

$$O_\delta(f) O_\delta(\Psi)(x) \subset O_\varepsilon(f \circ \Psi(O_\varepsilon(x))), \tag{4.12}$$

for all $x \in X$. Let $\varphi \in a_w(\Psi, \delta)$. Hence, by (4.12), one has

$$f \circ \varphi(x) \subset O_\varepsilon(f \circ \Psi(O_\varepsilon(x))),$$

for all $x \in X$. It now remains to show that for any piece $C \subset O_\varepsilon(f \circ \Psi(O_\varepsilon(x)))$ the following equality holds

$$I_{w\mathrm{loc}}(f \circ \Psi, C, x) = I_{w\mathrm{loc}}(f \circ \varphi, C, x). \tag{4.13}$$

For this purpose, it is enough to show that

$$I_{w\mathrm{loc}}(\Psi, f^{-1}(C), x) = I_{w\mathrm{loc}}(\varphi, f^{-1}(C), x) \tag{4.14}$$

since

$$\begin{aligned} I_{w\mathrm{loc}}(f\circ\Psi, C, x) &\overset{3.2.8}{=} I_{w\mathrm{loc}}(\Psi, f^{-1}(C), x),\\ I_{w\mathrm{loc}}(f\circ\varphi, C, x) &\overset{3.2.7}{=} I_{w\mathrm{loc}}(\varphi, f^{-1}(C), x). \end{aligned}$$

Let us fix a point $x \in X$ and let C be a piece of $O_\varepsilon(f\circ\Psi(O_\varepsilon(x)))$. Now, we shall prove that $f^{-1}(C)\cap O_\delta(\Psi(O_\delta(x)))$ is open and closed in $O_\delta(\Psi(O_\delta(x)))$. Since

$$f(O_\delta(\Psi(O_\delta(x)))) \subset O_\varepsilon(f\circ\Psi(O_\varepsilon(x))),$$

it follows that one can define the following continuous function

$$\overline{f}\colon O_\delta(\Psi(O_\delta(x))) \to O_\varepsilon(f\circ\Psi(O_\varepsilon(x))),$$

by $\overline{f}(x) = f(x)$ for all $x \in O_\delta(\Psi(O_\delta(x)))$. It is easy to see that

$$f^{-1}(C)\cap O_\delta(\Psi(O_\delta(x))) = (\overline{f})^{-1}(C). \tag{4.15}$$

Moreover, observe that $(\overline{f})^{-1}(C)$ is open and closed in $O_\delta(\Psi(O_\delta(x)))$, thus, by (4.15), $f^{-1}(C)\cap O_\delta(\Psi(O_\delta(x)))$ is open and closed in $O_\delta(\Psi(O_\delta(x)))$ as required. Returning now to the proof of (4.13), we deduce from the excision property of $I_{w\mathrm{loc}}$ for Ψ that

$$I_{w\mathrm{loc}}(\Psi, f^{-1}(C), x) = I_{w\mathrm{loc}}(\Psi, f^{-1}(C)\cap O_\delta(\Psi(O_\delta(x))), x), \tag{4.16}$$

$$I_{w\mathrm{loc}}(\varphi, f^{-1}(C), x) = I_{w\mathrm{loc}}(\varphi, f^{-1}(C)\cap O_\delta(\Psi(O_\delta(x))), x) \tag{4.17}$$

since

$$\begin{aligned} \Psi(x)\cap f^{-1}(C) &\subset f^{-1}(C)\cap O_\delta(\Psi(O_\delta(x))) \subset f^{-1}(C),\\ \varphi(x)\cap f^{-1}(C) &\subset f^{-1}(C)\cap O_\delta(\Psi(O_\delta(x))) \subset f^{-1}(C). \end{aligned}$$

Since $f^{-1}(C)\cap O_\delta(\Psi(O_\delta(x)))$ is a piece of $O_\delta(\Psi(O_\delta(x)))$ and $\varphi \in a_w(\Psi,\delta)$, we deduce that

$$\begin{aligned} &I_{w\mathrm{loc}}(\Psi, f^{-1}(C)\cap O_\delta(\Psi(O_\delta(x))), x)\\ &\qquad = I_{w\mathrm{loc}}(\varphi, f^{-1}(C)\cap O_\delta(\Psi(O_\delta(x))), x). \end{aligned} \tag{4.18}$$

Consequently, taking into account (4.15)–(4.18) we obtain (4.13), which completes the proof. $\square$

4.2. w-UV-sets

Following [43], we propose the following definitions, which will play a crucial role in the sequel.

Definition 4.2.1. Let $V \subset U$ be subsets of a space Y. We say that the inclusion $V \hookrightarrow U$ is *w-homotopy 0-trivial* if for any connected component C of V and for any weighted map $\varphi\colon \partial\Delta_1 \multimap C$ satisfying the following condition

$$\sum_{y \in C} w_\varphi(0, y) = \sum_{y \in C} w_\varphi(1, y)$$

there exists a weighted map $\widetilde{\varphi}\colon \Delta_1 \multimap U$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for every $x \in \partial\Delta_1$ (15).

Definition 4.2.2. Let $V \subset U$ be subsets of a space Y and let $n \geqslant 1$ be an integer. The inclusion $V \hookrightarrow U$ is said to be *w-homotopy n-trivial* if it is w-homotopy 0-trivial and for any integer $1 < k \leqslant n+1$ and for every weighted map $\varphi\colon \partial\Delta_k \multimap V$ there exists a w-map $\widetilde{\varphi}\colon \Delta_k \multimap U$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for every $x \in \partial\Delta_k$.

It is easy to see that we can replace in the above definition Δ_{k+1} by the unit closed disk D^{k+1} in $\mathbb{R}^{k+1}$ and $\partial\Delta_{k+1}$ by the unit sphere S^k, for $k \geq 0$.

Definition 4.2.3. Let A be a compact subset of a space X. We say that the inclusion $A \hookrightarrow X$ has:

(a) *w-UV^n-property* ($n \geqslant 0$) if for every $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that the inclusion $O_\delta(A) \to O_\varepsilon(A)$ is w-homotopy n-trivial;
(b) *w-UV^ω-property* if it has w-UV^n-property for each $n \geqslant 0$.

Now, we are going to show some facts concerning the above notions. In particular, we will prove that the class of sets satisfying some w-UV-properties is quite large.

Proposition 4.2.4. *Let X be a locally connected space (16), let A be a compact subset of X and let $n \geqslant 1$. If for any $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon$, such that:*

(a) *$O_\delta(A) \hookrightarrow O_\varepsilon(A)$ is w-homotopy 0-trivial,*
(b) *for each positive integer $1 \leqslant k \leqslant n$ and $x_0 \in O_\delta(A)$, the inclusion $O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces the trivial homomorphism*

$$\pi_k^w(O_\delta(A), x_0) \to \pi_k^w(O_\varepsilon(A), x_0),$$

(15) By a 1-dimensional simplex Δ_1 we mean a line segment $[0, 1]$.

(16) If a space X is locally connected and V is an open subset of X, then V is locally connected. Hence any connected component C of V is open in X. This observation will be of use in the proof of Proposition 4.2.4 and later.

then the inclusion $A \hookrightarrow X$ has a w-UV^n-property.

Proof. The proof will be divided into a number of steps. (We proceed by proving successively more general cases.)

Step 1. Fix $\varepsilon > 0$ and let $\delta > 0$ be such that the induced homomorphism

$$\pi_k^w(O_\delta(A), x_0) \to \pi_k^w(O_\varepsilon(A), x_0) \tag{4.19}$$

is trivial for $1 \leqslant k \leqslant n$ and for all $x_0 \in O_\delta(A)$. We divide Step 1 into a sequence of cases.

Case A. Let $\varphi: S^n \multimap O_\delta(A)$ be a w-map with $I_w(\varphi) = 0$ and $\varphi(s_0) = x_0$, where $s_0 \in S^n$ is a fixed point. Since the homomorphism (4.19) is trivial, it follows that a w-map $i \circ \varphi: S^n \multimap O_\varepsilon(A)$ is w-homotopic to the constant map at x_0 (with the weighted index equal to 0), where $i: O_\delta(A) \to O_\varepsilon(A)$ is the inclusion. Hence, in view of Proposition 3.2.11, we conclude that $i \circ \varphi$ can be extended to a w-map $\widetilde{\varphi}: D^{n+1} \multimap O_\varepsilon(A)$.

Case B. Let $\varphi: S^n \multimap O_\delta(A)$ be a w-map with $I_w(\varphi) \neq 0$ and $\varphi(s_0) = x_0$ (s_0 as in Case A). Let $\psi: S^n \multimap O_\delta(A)$ be given by $\psi = \varphi \cup (-I_w(\varphi))y_0$, where y_0 is an arbitrary fixed point of A. Since $I_w(\psi) = 0$, we conclude, by Case A, that there exists a weighted map $\widetilde{\psi}: D^{n+1} \multimap O_\varepsilon(A)$ such that $\widetilde{\psi}(x) = \psi(x)$ for each $x \in S^n$. Therefore, by Corollary 3.2.12, we obtain a w-map $\widetilde{\varphi}: D^{n+1} \multimap O_\varepsilon(A)$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in S^n$.

Case C. Let $\varphi: S^n \multimap O_\delta(A)$ be a w-map and let us assume that $\#\varphi(s_0) > 1$. Assume also that there exists a w-map $\alpha: [0,1] \multimap O_\delta(A)$ such that

$$\alpha(0) = \varphi(s_0), \quad \alpha(1) = x_0,$$
$$w_\alpha(0, y) = w_\varphi(s_0, y) \quad \text{for all } y \in O_\delta(A).$$

Now let us define $\Upsilon: (S^n \times \{0\}) \cup (\{s_0\} \times [0,1]) \multimap O_\delta(A)$ by

$$\Upsilon(x,t) = \begin{cases} \varphi(x) & \text{if } t = 0, \\ \alpha(t) & \text{if } x = s_0. \end{cases}$$

Then, in view of Proposition 3.2.11, there exists a w-map $\widetilde{\Upsilon}: S^n \times [0,1] \multimap O_\delta(A)$ such that $\widetilde{\Upsilon}|(S^n \times \{0\}) \cup (\{s_0\} \times [0,1]) = \Upsilon$. Now, applying Case A or Case B to $\widetilde{\Upsilon}(\,\cdot\,,1): S^n \multimap O_\delta(A)$ ([17]), we obtain an extension $\overline{\Upsilon}: D^{n+1} \multimap O_\varepsilon(A)$ of $\widetilde{\Upsilon}(\,\cdot\,,1)$. Let

$$\Upsilon_0: (S^n \times \{0\}) \cup (\{s_0\} \times [0,1]) \cup (D^{n+1} \times \{1\}) \multimap O_\varepsilon(A)$$

be defined as follows

$$\Upsilon_0(x,t) = \begin{cases} \varphi(x) & \text{if } t = 0, \\ \alpha(t) & \text{if } x = s_0, \\ \overline{\Upsilon}(x) & \text{if } t = 1. \end{cases}$$

([17]) If $I_w(\widetilde{\Upsilon}(\,\cdot\,,1)) = 0$, then we apply Case A, otherwise we apply Case B.

Since $(S^n\times\{0\})\cup(\{s_0\}\times[0,1])\cup(D^{n+1}\times\{1\})$ is an ANR and is closed in $D^{n+1}\times[0,1]$, we infer from Proposition 3.2.11 that we can extend Υ_0 to a weighted map $\widehat{\Upsilon}\colon D^{n+1}\times[0,1]\multimap O_\varepsilon(A)$. Finally, let us observe that $\widehat{\Upsilon}(\,\cdot\,,0)\colon D^{n+1}\multimap O_\varepsilon(A)$ satisfies the following condition $\widehat{\Upsilon}(x,0)=\varphi(x)$, for every $x\in S^n$.

Step 2. Let $\varepsilon>0$. Then under the assumptions of Proposition 4.2.4 it follows that there exists $\delta<\varepsilon$ such that

(a) $O_\delta(A)\hookrightarrow O_\varepsilon(A)$ is w-homotopy 0-trivial,
(b) for each positive integer $1\leqslant k\leqslant n$ and $x_0\in O_\delta(A)$ the inclusion $O_\delta(A)\hookrightarrow O_\varepsilon(A)$ induces the trivial homomorphism
$$\pi_k^w(O_\delta(A),x_0)\to\pi_k^w(O_\varepsilon(A),x_0).$$

Moreover, for δ there exists $\eta<\delta$ such that

(c) $O_\eta(A)\hookrightarrow O_\delta(A)$ is w-homotopy 0-trivial.

Let us fix $1\leqslant k\leqslant n$ and let s_0 be the base point of S^k. Now we shall show that for any w-map $\varphi\colon S^k\multimap O_\eta(A)$ there exists a w-map $\widetilde{\varphi}\colon D^{k+1}\multimap O_\varepsilon(A)$ with $\widetilde{\varphi}(x)=\varphi(x)$ for all $x\in S^k$. To see this, let us fix a w-map $\varphi\colon S^k\multimap O_\eta(A)$. Let us observe that if for a given w-map $\varphi\colon S^k\multimap O_\eta(A)$ there exists a w-map $\alpha\colon[0,1]\multimap O_\delta(A)$ such that

$$\alpha(0)=\varphi(s_0),\quad w_\alpha(0,y)=w_\varphi(s_0,y)\quad\text{for all } y\in O_\eta(A),\ \#\alpha(1)=1, \tag{4.20}$$

then by Step 1 we infer that there exists a w-map $\widetilde{\varphi}\colon D^{k+1}\multimap O_\varepsilon(A)$ with $\widetilde{\varphi}(x)=\varphi(x)$ for all $x\in S^k$. Therefore we can assume that for $\varphi\colon S^k\multimap O_\eta(A)$ there is no $\alpha\colon[0,1]\multimap O_\delta(A)$ satisfying (4.20). Let $O_\eta(A)=\bigcup_{j\in I}O_j^A$, where O_j^A is the connected component of $O_\eta(A)$. Since $O_\eta(A)$ is locally connected, it follows that the connected components of $O_\eta(A)$ are open in $O_\eta(A)$. Hence, by the compactness of $\varphi(S^k)$, we obtain
$$\#I':=\{j\in I\mid\varphi(S^k)\cap O_j^A\neq\emptyset\}<\infty.$$

Obviously, $\varphi(S^k)\subset\bigcup_{j_m\in I'}O_{j_m}^A$. Let us choose a point y_{j_m} in each component $O_{j_m}^A$ and let us define a w-map $\alpha\colon S^k\multimap O_\eta(A)$ as follows
$$\alpha(x)=\{y_{j_1},\dots,y_{j_s}\},\quad w_\alpha(x,y)=0,$$
for all $x\in S^k$, $y\in O_\eta(A)$, where $s:=\#I'$. Let $\varphi^\alpha\colon S^k\multimap O_\eta(A)$ be defined by $\varphi^\alpha:=\varphi\cup\alpha$. Then, by Proposition 2.2.8, a w-map φ^α has the following decomposition $\varphi^\alpha=\varphi_1^\alpha\cup\ldots\cup\varphi_s^\alpha$, where any w-map φ_m^α satisfies the following condition $\varphi_m^\alpha(S^k)\subset O_{j_m}^A$. Let $\beta_m\colon\{0,1\}\multimap O_{j_m}^A$, $m=1,\dots,s$, be defined as follows
$$\begin{gathered}\beta_m(0)=\varphi_m^\alpha(s_0),\quad\beta_m(1)=y_{j_m},\\ w_{\beta_m}(0,y):=w_{\varphi_m^\alpha}(s_0,y),\quad\text{for all } y\in O_{j_m}^A,\\ w_{\beta_m}(1,y_{j_m}):=I_w(\varphi_m^\alpha),\quad w_{\beta_m}(1,y):=0\quad\text{for } y\neq y_{j_m},\end{gathered}$$

where $1 \leqslant m \leqslant s$. Since the inclusion $O_\eta(A) \hookrightarrow O_\delta(A)$ is w-homotopy 0-trivial, it follows that for any β_m there exists a w-map $\widetilde{\beta_m}\colon [0,1] \multimap O_\delta(A)$ with $\widetilde{\beta_m}|\{0,1\} = \beta_m$. Hence, by Step 1, for any w-map $\varphi_m^\alpha\colon S^k \multimap O_\eta(A) \subset O_\delta(A)$ there exists a w-map $\widetilde{\varphi}_m^\alpha\colon D^{k+1} \multimap O_\varepsilon(A)$ such that $\widetilde{\varphi}_m^\alpha(x) = \varphi_m^\alpha(x)$ for all $x \in S^k$. Consequently, a w-map $\widetilde{\varphi}\colon D^{k+1} \multimap O_\varepsilon(A)$ given by

$$\widetilde{\varphi}^\alpha = \widetilde{\varphi}_1^\alpha \cup \ldots \cup \widetilde{\varphi}_m^\alpha$$

is an extension of $\varphi^\alpha\colon S^k \multimap O_\eta(A)$. Since w-maps φ and φ^α satisfy the condition:

$$w_\varphi(x,y) = w_{\varphi^\alpha}(x,y),$$

for all $x \in S^k$ and $y \in O_\varepsilon(A)$, Lemma 2.2.9 implies that φ is w-homotopic to φ^α; and hence, by Proposition 3.2.11, we conclude that there exists a w-map $\widetilde{\varphi}\colon D^{k+1} \multimap O_\varepsilon(A)$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in S^k$. □

Now we will prove that the converse of the last statement is also true.

Proposition 4.2.5. *Let X be a space and let A be a compact subset of X. If the inclusion $A \hookrightarrow X$ has a w-UV^n-property ($n \geqslant 1$), then for each $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon$, such that the homomorphism*

$$h_k\colon \pi_k^w(O_\delta(A), x_0) \to \pi_k^w(O_\varepsilon(A), x_0)$$

induced by the inclusion $i\colon O_\delta(A) \to O_\varepsilon(A)$ is trivial for $1 \leqslant k \leqslant n$ and for all $x_0 \in O_\delta(A)$.

Proof. Let us fix $\varepsilon > 0$. Let $\delta > 0$ be such that for any $1 \leqslant k \leqslant n$ and any w-map $\varphi\colon S^k \multimap O_\delta(A)$ there exists a w-map $\widetilde{\varphi}\colon D^{k+1} \multimap O_\varepsilon(A)$ with $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in S^k$. Now we are going to show that the induced homomorphism $h_k\colon \pi_k^w(O_\delta(A), x_0) \to \pi_k^w(O_\varepsilon(A), x_0)$ is trivial for any $1 \leqslant k \leqslant n$ and for each $x_0 \in O_\delta(A)$. To see this, let us fix $1 \leqslant k \leqslant n$ and $x_0 \in O_\delta(A)$. Let $\varphi\colon (S^k, s_0) \multimap (O_\delta(A), x_0)$ be a pointed w-map with $I_w(\varphi) = 0$. Hence, by the definition of δ, we infer that there exists a w-map $\widetilde{\varphi}\colon D^{k+1} \multimap O_\varepsilon(A)$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for $x \in S^k$. Then, in view of Lemma 2.4.2, $i \circ \varphi$ is w-homotopic to the constant map at x_0 (with the weighted index equal to 0) relative to s_0, which proves that the homomorphism h_k is trivial. □

As an immediate consequence of the above propositions we obtain:

Corollary 4.2.6. *Let X be a locally connected space and let A be a compact subset of X. Then the inclusion $A \hookrightarrow X$ has a w-UV^n-property ($n \geqslant 1$) if and only if for any $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon$, such that*

(a) *$O_\delta(A) \hookrightarrow O_\varepsilon(A)$ is w-homotopy 0-trivial,*
(b) *for each positive integer $1 \leqslant k \leqslant n$ and $x_0 \in O_\delta(A)$ the inclusion $O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces the trivial homomorphism*

$$\pi_k^w(O_\delta(A), x_0) \to \pi_k^w(O_\varepsilon(A), x_0).$$

Proposition 4.2.7. *Let X be a locally path-connected space and let $A \subset X$ be a compact subspace. Then for any open subsets U and V satisfying the condition $A \subset V \subset U \subset X$ the inclusion $V \hookrightarrow U$ is w-homotopy 0-trivial.*

Proof. Let $V \subset U$ be open subsets of X and let C be a connected component of V. Due to our assumptions C is locally path-connected and connected. Hence C is path-connected. Let $\varphi\colon \partial\Delta_1 \multimap C$ be a weighted map with $\sum_{y \in C} w_\varphi(0, y) = \sum_{y \in C} w_\varphi(1, y)$. Then, in view of Lemma 2.2.10, there exists a weighted map $\widetilde{\varphi}\colon \Delta_1 \multimap C$ with $\widetilde{\varphi}|\partial\Delta_1 = \varphi$. This completes the proof. $\square$

Consequently, combining Corollary 4.2.6 with Proposition 4.2.7, one obtains the following corollary.

Corollary 4.2.8. *Let A be a compact subset of a locally path-connected space X, $n \geqslant 1$. Then A has a w-UV^n-property if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that the inclusion $O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces the trivial homomorphism $h_k\colon \pi_k^w(O_\delta(A), x_0) \to \pi_k^w(O_\varepsilon(A), x_0)$ for any $1 \leqslant k \leqslant n$ and for all $x_0 \in O_\delta(A)$.*

Taking into account Corollary 4.2.8 and Theorems 1.5.3 and 2.4.1 we get the following theorem.

Theorem 4.2.9. *Let X be an* ANR *and let A be a compact subset of X, $k \geqslant 1$. Then the inclusion $j\colon A \hookrightarrow X$ has a w-UV^k-property if and only if for each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that the induced homomorphism $j_i\colon \check{H}_i(O_\delta(A), \mathbb{Q}) \to \check{H}_i(O_\varepsilon(A), \mathbb{Q})$ is trivial for each $1 \leqslant i \leqslant k$.*

The following lemma is crucial for our considerations.

Lemma 4.2.10. *Let X, Y be* ANR*s and let $X_0 \subset X$ and $Y_0 \subset Y$ be compact subsets. In addition, assume that the inclusion $X_0 \hookrightarrow X$ has a w-UV^n-property, where $n \geqslant 1$. If Y_0 is homeomorphic to X_0, then the inclusion $Y_0 \hookrightarrow Y$ has also a w-UV^n-property.*

The proof of the above lemma is the same as in [4]. The only difference is using the w-homotopy functor instead of the homotopy functor.

Proposition 4.2.11. *Let X be the Hilbert cube and let $A \subset X$ be a compact subset. Assume, furthermore, that A is k-acyclic, $k \geqslant 1$. Then for each $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that the inclusion $j\colon O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces a trivial homomorphism $j_{*l}\colon \check{H}_l(O_\delta(A), \mathbb{Q}) \to \check{H}_l(O_\varepsilon(A), \mathbb{Q})$ for each $1 \leqslant l \leqslant k$.*

Proof. Our proof is based upon ideas found in [31]. Since X is the Hilbert cube, there exists, in view of lemma 1.2.6, a sequence $\{Z_i\}_{i=1}^\infty$ of compact ANR-spaces such that $Z_{i+1} \subset Z_i$, for $i \geqslant 1$, and $\bigcap_{i=1}^\infty Z_i = A$. Let us fix Z_{i_0}. Now consider the diagram

$$\begin{tikzcd}
 & \check{H}_l(Z_s,\mathbb{Q}) \arrow[d, "p_l^s"] & \\
\check{H}_l(A,\mathbb{Q}) \arrow[r, "j_l"] & \check{H}_l(Z_{i_0},\mathbb{Q}) \arrow[r, "w_l"] \arrow[d, "\lambda_l^s"] & \check{H}_l(Z_{i_0},A;\mathbb{Q}) \arrow[dl, "\mu_l^s"] \\
 & \check{H}_l(Z_{i_0},Z_s;\mathbb{Q}), &
\end{tikzcd}$$

where all homomorphisms are induced by inclusions, the triangle is commutative; and both the horizontal and vertical lines are exact, $s \geqslant i_0, l \geqslant 1$. Let us observe that if $\check{H}_l(A,\mathbb{Q}) = 0$ for some $l \geqslant 1$, then from the above diagram we deduce that $\ker w_l = 0$. Additionally, $\dim \operatorname{Im} w_l < \infty$, because compact ANRs have the Čech homology of finite type. Now we shall show that there exists an index N_{i_0} such that the homomorphisms $p_l^s\colon \check{H}_l(Z_s,\mathbb{Q}) \to \check{H}_l(Z_{i_0},\mathbb{Q})$ are trivial for $s \geqslant N_{i_0}$ and $1 \leqslant l \leqslant k$. Let $1 \leqslant l_0 \leqslant k$ be fixed and let $z_1^{l_0},\dots,z_{s_{l_0}}^{l_0}$ be a basis for $w_{l_0}(\check{H}_{l_0}(Z_{i_0},\mathbb{Q})) \subset \check{H}_{l_0}(Z_{i_0},A;\mathbb{Q})$. Now, by applying Lemma 1.5.2 to

$$(Z_{i_0},Z_s) \supset (Z_{i_0},Z_{s+1}) \supset (Z_{i_0},Z_{s+2}) \supset \dots$$

and $\mu_{l_0}^s\colon \check{H}_{l_0}(Z_{i_0},A;\mathbb{Q}) \to \check{H}_{l_0}(Z_{i_0},Z_s;\mathbb{Q})$ for $s \geqslant i_0$, we obtain $N_{l_0}^{i_0} \geqslant i_0$ such that the homomorphism $\mu_{l_0}^s|\langle z_1^{l_0},\dots,z_{s_{l_0}}^{l_0}\rangle\colon \langle z_1^{l_0},\dots,z_{s_{l_0}}^{l_0}\rangle \to \check{H}_{l_0}(Z_{i_0},Z_s;\mathbb{Q})$ is a monomorphism for $s \geqslant N_{l_0}^{i_0}$. Moreover, since $\ker w_{l_0} = 0$ and $\lambda_{l_0}^s = \mu_{l_0}^s \circ w_{l_0}$, we deduce that the homomorphism $\lambda_{l_0}^s\colon \check{H}_{l_0}(Z_{i_0},\mathbb{Q}) \to \check{H}_{l_0}(Z_{i_0},Z_s;\mathbb{Q})$ is a monomorphism for all $s \geqslant N_{l_0}^{i_0}$. Thus, from the exactness of the vertical sequence in the above diagram, we infer that $\operatorname{Im} p_{l_0}^s = 0$ for $s \geqslant N_{l_0}^{i_0}$. Let $N_{i_0} := max\{N_1^{i_0},\dots,N_k^{i_0}\}$. Then for $1 \leqslant l \leqslant k$ and $s \geqslant N_{i_0}$ the homomorphism

$$p_l^s\colon \check{H}_l(Z_s,\mathbb{Q}) \to \check{H}_l(Z_{i_0},\mathbb{Q}) \tag{4.21}$$

is trivial. Let $\varepsilon > 0$. Now, let us observe that there exists i_0 such that $Z_s \subset O_\varepsilon(A)$ for $s \geqslant i_0$, because $A = \bigcap_{i=1}^\infty Z_i$ and $Z_{i+1} \subset Z_i$. Let us fix $s \geqslant N_{i_0}$ ($N_{i_0} \geqslant i_0$). Since Z_s is a compact ANR, there exists an open subset $U \subset X$ with $Z_s \subset U \subset O_\varepsilon(A)$ and a retraction $r_s\colon U \to Z_s$. Let $f\colon U \to O_\varepsilon(A)$ be factored as

$$U \xrightarrow{r_s} Z_s \xrightarrow{j_s} Z_{i_0} \xrightarrow{i_s} O_\varepsilon(A),$$

where j_s and i_s are the inclusions. Then, by the compactness of A and Lemma 1.2.5, we infer that there exists $\delta < \varepsilon$ with $O_\delta(A) \subset U$ and such that $f|O_\delta(A)$ is homotopic to the inclusion $j\colon O_\delta(A) \to O_\varepsilon(A)$. Hence

$$(f|O_\delta(A))_{*l} = j_{*l}\colon \check{H}_l(O_\delta(A),\mathbb{Q}) \to \check{H}_l(O_\varepsilon(A),\mathbb{Q}).$$

But $(f|O_\delta(A))_{*l} = (i_s)_{*l} \circ (j_s)_{*l} \circ (r_s|O_\delta(A))_{*l}$ and $(j_s)_{*l} = p_l^s$, so, in view of (4.21), the homomorphism $(f|O_\delta(A))_{*l}$ is trivial for $1 \leqslant l \leqslant k$. Consequently, $j_{*l}\colon \check{H}_l(O_\delta(A), \mathbb{Q}) \to \check{H}_l(O_\varepsilon(A), \mathbb{Q})$ is the trivial homomorphism for $1 \leqslant l \leqslant k$, which completes the proof. $\square$

Corollary 4.2.12. *Let X be an* ANR *and let $A \subset X$ be a compact subset. Assume, furthermore, that A is k-acyclic, $k \geqslant 1$. Then for each $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that the inclusion $j\colon O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces a trivial homomorphism $j_{*l}\colon \check{H}_l(O_\delta(A), \mathbb{Q}) \to \check{H}_l(O_\varepsilon(A), \mathbb{Q})$ for $1 \leqslant l \leqslant k$.*

Proof. Since any compact metric space admits an embedding into the Hilbert cube Q^ω, it follows that there exists a compact subset B of Q^ω which is homeomorphic to A. Moreover, since A is k-acyclic and since A is homeomorphic to B, we deduce that B is also k-acyclic. Now, in view of Proposition 4.2.11 and Theorem 4.2.9, we infer that the inclusion $B \hookrightarrow Q^\omega$ has a w-UV^k-property. Hence, by Lemma 4.2.10, the inclusion $A \hookrightarrow X$ has a w-UV^k-property. Consequently, by Theorem 4.2.9, the assertion follows. $\square$

Proposition 4.2.13. *Let X and $A \subset X$ be as in Proposition 4.2.11. If for each $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that the inclusion $j\colon O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces a trivial homomorphism $j_{*l}\colon \check{H}_l(O_\delta(A), \mathbb{Q}) \to \check{H}_l(O_\varepsilon(A), \mathbb{Q})$ for any $1 \leqslant l \leqslant k$, then A is k-acyclic.*

Proof. Let $\{Z_n\}_{n=1}^\infty$ be a sequence as in Lemma 1.2.6 corresponding to A. Then, under our assumptions, there exist two sequences $\{\varepsilon_m\}_{m=1}^\infty$ and $\{i_m\}_{m=1}^\infty$ such that

(1) $Z_{i_1} = Q^\omega$, $\varepsilon_1 = 1$, $i_1 = 1$, $\varepsilon_{m+1} < \varepsilon_m$;
(2) $Z_{i_{m+1}} \subset O_{\varepsilon_m}(A) \subset Z_{i_m}$ for $m \geqslant 1$;
(3) the inclusion $j\colon O_{\varepsilon_{m+1}}(A) \to O_{\varepsilon_m}(A)$ induces the trivial homomorphism $j_{*l}\colon \check{H}_l(O_{\varepsilon_{m+1}}(A), \mathbb{Q}) \to \check{H}_l(O_{\varepsilon_m}(A), \mathbb{Q})$ for any $m \geqslant 1$, $1 \leqslant l \leqslant k$.

Since the inclusion $J_{i_m}\colon Z_{i_{m+2}} \hookrightarrow Z_{i_m}$ can be factored as

$$Z_{i_{m+2}} \hookrightarrow O_{\varepsilon_{m+1}}(A) \hookrightarrow O_{\varepsilon_m}(A) \hookrightarrow Z_{i_m},$$

the induced homomorphism $(J_{i_m})_{*l}\colon \check{H}_l(Z_{i_{m+2}}, \mathbb{Q}) \to \check{H}_l(Z_{i_m}, \mathbb{Q})$ is trivial for any $m \geqslant 1$ and $1 \leqslant l \leqslant k$. Therefore

$$\varprojlim_m \check{H}_l(Z_{i_{(2m-1)}}, \mathbb{Q}) = 0, \tag{4.22}$$

and since $A = \bigcap_{m=1}^\infty Z_{i_{(2m-1)}}$, so by Lemma 1.5.1 and (4.22) we infer that $\check{H}_l(A, \mathbb{Q}) = 0$ for $1 \leqslant l \leqslant k$, which completes the proof. $\square$

Corollary 4.2.14. *Let X be an* ANR *and let $A \subset X$ be a compact subset. If for each $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that the inclusion $j\colon O_\delta(A) \hookrightarrow O_\varepsilon(A)$*

induces a trivial homomorphism $j_{*l}\colon \check{H}_l(O_\delta(A),\mathbb{Q}) \to \check{H}_l(O_\varepsilon(A),\mathbb{Q})$ *for any* $1 \leqslant l \leqslant k$, *then* A *is* k*-acyclic.*

Proof. By the same argument as in the proof of Corollary 4.2.12, there exists a compact subset B of the Hilbert cube Q^ω which is homeomorphic to A. Under our assumptions Theorem 4.2.9 implies that the inclusion $A \hookrightarrow X$ has a w-UV^k-property. Hence, by Lemma 4.2.10, it follows that the inclusion $B \hookrightarrow Q^\omega$ has a w-UV^k-property. Consequently, by Theorem 4.2.9 and Proposition 4.2.13, we infer that B is k-acyclic. Since A is homeomorphic to B, we deduce that A is also k-acyclic. This completes the proof. □

Finally, from Corollaries 4.2.12 and 4.2.14 we obtain the following theorem.

Theorem 4.2.15. *Let* X *be an* ANR *and let* $A \subset X$ *be a compact subset,* $k \geqslant 1$. *Then* A *is* k*-acyclic if and only if for each* $\varepsilon > 0$ *there exists* $\delta < \varepsilon$ *such that the inclusion* $j\colon O_\delta(A) \hookrightarrow O_\varepsilon(A)$ *induces a trivial homomorphism* $j_{*l}\colon \check{H}_l(O_\delta(A),\mathbb{Q}) \to \check{H}_l(O_\varepsilon(A),\mathbb{Q})$ *for all* $1 \leqslant l \leqslant k$.

Now let us observe that Theorem 4.2.9 together with Theorem 4.2.15 implies the following theorem.

Theorem 4.2.16. *Let* X *nad* $A \subset X$ *be as in Theorem* 4.2.9 *and* $k \geqslant 1$. *Then* A *is* k*-acyclic if and only if the inclusion* $A \hookrightarrow X$ *has a* w*-*UV^k*-property.*

Since a subset A of a space X is positively acyclic if and only if it is k-acyclic for all $k \geqslant 1$, we obtain, by Theorem 4.2.16, the main result of this section.

Corollary 4.2.17. *Let* X *be an* ANR *and let* A *be a compact subset of* X. *Then* A *is positively acyclic if and only if the inclusion* $j\colon A \hookrightarrow X$ *has a* w*-*UV^ω*-property.*

We shall conclude this section by introducing the following notion, which will be used in what follows.

Definition 4.2.18. Let $0 \leqslant n < \infty$ or $n = \omega$. A weighted carrier $\Psi\colon X \multimap Y$ is said to be a *w-UV^n-valued carrier* if, for each $x \in X$, the inclusion $\Psi(x) \hookrightarrow Y$ has w-UV^n-property.

4.3. Existence of approximations

In this section, we improve an approximability theorem for weighted carriers defined on compact polyhedra due to G. Conti and J. Pejsachowicz (see Theorem 4.1 in [10]). Next, following [3] and [27], we extend the above theorem to the case of compact ANRs.

We start with the following lemma that is crucial in what follows.

Lemma 4.3.1. *Let X be a compact space and let Y be a space, $n \geqslant 0$. If $\Psi: X \multimap Y$ is a w-UV^n-valued carrier, then for each $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon$, such that for each $x \in X$ two properties hold:*

(a) *for any connected component C of $O_\delta(\Psi(O_\delta(x)))$ and for every weighted map $\varphi: \partial\Delta_1 \multimap C$ with $\sum_{y\in C} w_\varphi(0,y) = \sum_{y\in C} w_\varphi(1,y)$ there exists a weighted map $\widetilde{\varphi}: \Delta_1 \multimap O_\varepsilon(\Psi(O_\varepsilon(x)))$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in \partial\Delta_1$;*

(b) *if $n > 0$, then for each k, $1 < k \leqslant n+1$, and any weighted map $\varphi: \partial\Delta_k \multimap O_\delta(\Psi(O_\delta(x)))$ there exists a w-map $\widetilde{\varphi}: \Delta_k \multimap O_\varepsilon(\Psi(O_\varepsilon(x)))$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in \partial\Delta_k$.*

The proof of the above lemma is similar in spirit to that of [27, Lemma 5.8], so the details are left to the reader.

We shall establish the first approximation result of this section.

Theorem 4.3.2. *Let X be a compact polyhedron* ([18]) *and let A be a subpolyhedron of X. Let Y be a locally connected space. If $\dim(X \setminus A) \leqslant n+1$ and $\Psi: X \multimap Y$ is a w-UV^n-valued carrier, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi_0: A \multimap Y$ is a δ-approximation of $\Psi: X \multimap Y$, then there exists a w-map $\varphi: X \multimap Y$ being an ε-approximation of Ψ with $\varphi|A = \varphi_0$.*

Proof. The main idea of our proof follows from [10], [27]. Let us fix $\varepsilon > 0$ and let $\dim(X \setminus A) = n_0$. By using Lemma 4.3.1, we can construct a sequence $\{\varepsilon_i\}_{i=0}^{n_0}$ ([19]) such that

(1) $\varepsilon_{n_0} := \varepsilon$,

(2) $4\varepsilon_i < \varepsilon_{i+1}$ for $0 \leqslant i \leqslant n_0 - 1$,

(3) for any $x \in X$, any connected component C of $O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x)))$, and any weighted map $\varphi: \partial\Delta_1 \multimap C$ with $\sum_{y\in C} w_\varphi(0,y) = \sum_{y\in C} w_\varphi(1,y)$ there exists a weighted map $\widetilde{\varphi}: \Delta_1 \multimap O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(x)))$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in \partial\Delta_1$,

(4) for any $x \in X$, any positive k, $1 \leqslant k \leqslant n_0 - 1$, and any w-map
$$\varphi: \partial\Delta_{k+1} \multimap O_{2\varepsilon_k}(\Psi(O_{2\varepsilon_k}(x)))$$
there exists a w-map $\widetilde{\varphi}: \Delta_{k+1} \multimap O_{\varepsilon_{k+1}/2}(\Psi(O_{\varepsilon_{k+1}/2}(x)))$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in \partial\Delta_{k+1}$.

Let $\delta := \varepsilon_0$ and let $\varphi_0: A \multimap Y$ be a δ-approximation of $\Psi: X \multimap Y$. Let (K, L) be a triangulation of (X, A) finer than the covering $\{O_{\varepsilon_0/2}(x)\}_{x\in X}$ of X, i.e. $|K| = X$, $|L| = A$ and let L be a subcomplex of K. We shall prove now that $\varphi: A \multimap Y$ can be extended to an ε-approximation of $\Psi: X \multimap Y$. For this

([18]) Recall that by a polyhedron we shall mean the space $|K|$ of a simplicial complex K with the Whitehead topology (see also Preliminaries).

([19]) During the construction we can assume that $n_0 \geqslant 1$, because otherwise $n_0 = 0$ and then we put $\{\varepsilon_i\}_{i=0}^{n_0} := \varepsilon$.

purpose, choose for each simplex σ of $K\setminus L$ a point x_σ such that $|\sigma|\subset O_{\varepsilon_0/2}(x_\sigma)$. Let us notice that if σ is a vertex v of $K\setminus L$, then we can take $x_\sigma = v$. Let v be a vertex of K such that $v\notin L$. Since Y is locally connected, it follows that the open set $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v)))$ is also locally connected and hence the connected components of $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v)))$ are open in $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v)))$. Consequently, by the compactness of $\Psi(v)$, we infer that it meets only a finite number of connected components of $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v)))$, say $C_1^v,\ldots,C_{r_v}^v$. Let us choose a point y_i^v in each C_i^v. We define a weighted map $\varphi^0\colon |K^{(0)}|\cup|L|\multimap Y$ ([20]) by the formula

$$\varphi^0(x)=\begin{cases}\varphi_0(x) & \text{if } x\in|L|,\\ \displaystyle\sum_{i=1}^{r_v} I_{w\mathrm{loc}}(\Psi,C_i^v,v)y_i^v & \text{if } x=v\in|K^{(0)}|\setminus|L|.\end{cases}$$

Obviously, φ^0 is an ε_0-approximation of $\Psi\colon|K|\multimap Y$. Now we extend φ^0 to $|K^{(1)}|\cup|L|$. For this purpose, let us fix a 1-dimensional simplex $\sigma=[v_0,v_1]$ such that $\sigma\notin L$. Since $|\sigma|\subset O_{\varepsilon_0/2}(x_\sigma)$, we have

$$O_{\varepsilon_0}(v_i)\subset O_{\varepsilon_0}(|\sigma|)\subset O_{\varepsilon_0}(O_{\varepsilon_0/2}(x_\sigma))\subset O_{2\varepsilon_0}(x_\sigma),\quad \text{for } i=0,1.$$

Moreover, since φ^0 is an ε_0-approximation of $\Psi\colon|K|\multimap Y$, we infer that

$$\varphi^0(v_i)\subset O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_i)))\subset O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x_\sigma))),\quad \text{for } i=0,1.$$

Now we shall show that for each piece C of $O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x_\sigma)))$ the following condition holds:

$$I_{w\mathrm{loc}}(\varphi^0,C,v_0)=I_{w\mathrm{loc}}(\varphi^0,C,v_1).$$

Indeed, let us fix a piece C of $O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x_\sigma)))$. Let

$$C_i := C\cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_i))),$$

for $i=0,1$. Then C_i is a piece of $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_i)))$, for $i=0,1$. Since φ^0 is an ε_0-approximation of $\Psi\colon|K|\multimap Y$, we obtain

$$I_{w\mathrm{loc}}(\varphi^0,C,v_i)=I_{w\mathrm{loc}}(\varphi^0,C_i,v_i)=I_{w\mathrm{loc}}(\Psi,C_i,v_i)=I_{w\mathrm{loc}}(\Psi,C,v_i),\tag{4.23}$$

for $i=0,1$; where the first equality and the last one above follow from the excision property of $I_{w\mathrm{loc}}$. Consequently, since $|\sigma|$ is connected and

$$\Psi(|\sigma|)\subset\Psi(O_{\varepsilon_0/2}(x_\delta))\subset O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x_\delta))),$$

we deduce from Lemmas 3.2.1 and 4.1.3 that

$$I_{w\mathrm{loc}}(\Psi,C,v_0)=I_{w\mathrm{loc}}(\Psi,C,v_1).$$

Hence, taking into account (4.23) and (4.24), we obtain

$$I_{w\mathrm{loc}}(\varphi^0,C,v_0)=I_{w\mathrm{loc}}(\varphi^0,C,v_1).$$

([20]) Given a simplicial complex we shall denote by $K^{(i)}$ the simplex of K consisting of all simplexes $\sigma\in K$ with $\dim(\sigma)\leqslant i$.

Thus, by the definition of ε_0, we can extend $\varphi^0||\partial\sigma|\colon|\partial\sigma| \multimap O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x_\sigma)))$ to

$$\varphi_\sigma\colon|\sigma| \multimap O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(x_\sigma))).$$

Now we are going to show that φ_σ is an ε_1-approximation of $\Psi\colon|K| \multimap Y$. First, let us observe that for each $x \in |\sigma|$ we have $x_\sigma \in O_{\varepsilon_0/2}(x)$, since $|\sigma| \subset O_{\varepsilon_0/2}(x_\sigma)$. Thus

$$O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(x_\sigma))) \subset O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(O_{\varepsilon_0/2}(x)))) \subset O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x))).$$

This shows that $\varphi_\sigma(x) \subset O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x)))$, for each $x \in |\sigma|$. So, it is enough to show that if C is any piece of $O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x)))$, then

$$I_{w\mathrm{loc}}(\varphi_\sigma, C, x) = I_{w\mathrm{loc}}(\Psi, C, x).$$

For this purpose, let us fix $x \in |\sigma|$ and let C be a piece of $O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x)))$. Since

$$\begin{gathered}\varphi_\sigma(v_0) \subset \varphi_\sigma(|\sigma|) \subset O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(x_\sigma))) \subset O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x))),\\ \Psi(v_0) \subset \Psi(|\sigma|) \subset O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(x_\sigma))) \subset O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x))),\end{gathered}$$

it follows, in view of Lemma 4.1.3 and Lemma 3.2.1, that

$$I_{w\mathrm{loc}}(\varphi_\sigma, C, v_0) = I_{w\mathrm{loc}}(\varphi_\sigma, C, x), \tag{4.25}$$

$$I_{w\mathrm{loc}}(\Psi, C, v_0) = I_{w\mathrm{loc}}(\Psi, C, x). \tag{4.26}$$

Since $\varphi_\sigma||\partial\sigma|$ is an ε_0-approximation of $\Psi\colon|K| \multimap Y$, we conclude that

$$\varphi_\sigma(v_0) \subset O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0))) \subset O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x))).$$

Then, by the excision property of $I_{w\mathrm{loc}}$, we have

$$I_{w\mathrm{loc}}(\varphi_\sigma, C, v_0) = I_{w\mathrm{loc}}(\varphi_\sigma, C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0))), v_0), \tag{4.27}$$

$$I_{w\mathrm{loc}}(\Psi, C, v_0) = I_{w\mathrm{loc}}(\Psi, C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0))), v_0). \tag{4.28}$$

Now let us observe that $C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0)))$ is a piece of $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0)))$. Hence, taking into account the fact that $\varphi_\sigma||\partial\sigma|$ is an ε_0-approximation of $\Psi\colon|K| \multimap Y$, we obtain

$$I_{w\mathrm{loc}}(\varphi_\sigma, C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0))), v_0) = I_{w\mathrm{loc}}(\Psi, C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0))), v_0). \tag{4.29}$$

Consequently, from (4.25)–(4.29) we get

$$I_{w\mathrm{loc}}(\varphi_\sigma, C, x) = I_{w\mathrm{loc}}(\Psi, C, x),$$

which proves that φ_σ satisfies the second condition of Definition 4.1.1. Now using the gluing lemma we obtain a weighted map $\varphi^1\colon|K^{(1)}| \cup |L| \multimap Y$ being an ε_1-approximation of $\Psi\colon|K| \multimap Y$ with $\varphi^1||K^{(0)}| \cup |L| = \varphi^0$. Suppose now that $\varphi^r\colon|K^{(r)}| \cup |L| \multimap Y$ is an ε_r-approximation of $\Psi\colon|K| \multimap Y$, $r < n_0$. Let τ be an

$(r+1)$-dimensional simplex such that $\tau \notin L$. Then $|\tau| \subset O_{\varepsilon_0/2}(x_\tau) \subset O_{\varepsilon_r/2}(x_\tau)$ and $\varphi^r(x) \subset O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(x)))$, for all $x \in |\partial\tau|$. Consequently,

$$\varphi^r(|\partial\tau|) \subset O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(|\partial\tau|))) \subset O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(O_{\varepsilon_r}(x_\tau)))) \subset O_{2\varepsilon_r}(\Psi(O_{2\varepsilon_r}(x_\tau))).$$

Thus, by the definition of ε_r, a w-map $\varphi^r||\partial\tau|\colon |\partial\tau| \multimap O_{2\varepsilon_r}(\Psi(O_{2\varepsilon_r}(x_\tau)))$ admits an extension to

$$\varphi_\tau^{r+1}\colon |\tau| \multimap O_{\varepsilon_{r+1}/2}(\Psi(O_{\varepsilon_{r+1}/2}(x_\tau))).$$

Let us observe now that for each $x \in |\tau|$ we have

$$O_{\varepsilon_{r+1}/2}(\Psi(O_{\varepsilon_{r+1}/2}(x_\tau))) \subset O_{\varepsilon_{r+1}}(\Psi(O_{\varepsilon_{r+1}}(x))),$$

because $|\tau| \subset O_{\varepsilon_0/2}(x_\tau) \subset O_{\varepsilon_{r+1}/2}(x_\tau)$ and hence

$$\varphi_\tau^{r+1}(x) \subset O_{\varepsilon_{r+1}}(\Psi(O_{\varepsilon_{r+1}}(x))),$$

for each $x \in |\tau|$. This implies that φ_τ^{r+1} satisfies the first condition of Definition 4.1.1. Let us fix $x_0 \in |\partial\tau|$. Now we shall prove that φ_τ^{r+1} also satisfies the second condition of Definition 4.1.1. For this end, let us fix $x \in |\tau|$ and let C be a piece of $O_{\varepsilon_{r+1}}(\Psi(O_{\varepsilon_{r+1}}(x)))$. Then by Lemma 4.1.3 and Lemma 3.2.1 we get

$$I_{w\mathrm{loc}}(\varphi_\tau^{r+1}, C, x_0) = I_{w\mathrm{loc}}(\varphi_\tau^{r+1}, C, x), \tag{4.30}$$
$$I_{w\mathrm{loc}}(\Psi, C, x_0) = I_{w\mathrm{loc}}(\Psi, C, x). \tag{4.31}$$

Moreover,

$$I_{w\mathrm{loc}}(\varphi_\tau^{r+1}, C \cap O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(x_0))), x_0) = I_{w\mathrm{loc}}(\Psi, C \cap O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(x_0))), x_0), \tag{4.32}$$

because $\varphi_\tau^{r+1}||\partial\tau|$ is an ε_r-approximation of Ψ. Next, by the excision property of $I_{w\mathrm{loc}}$, we infer that

$$I_{w\mathrm{loc}}(\varphi_\tau^{r+1}, C \cap O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(x_0))), x_0) = I_{w\mathrm{loc}}(\varphi_\tau^{r+1}, C, x_0), \tag{4.33}$$
$$I_{w\mathrm{loc}}(\Psi, C \cap O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(x_0))), x_0) = I_{w\mathrm{loc}}(\Psi, C, x_0). \tag{4.34}$$

Therefore, taking into account (4.30)–(4.34), we obtain

$$I_{w\mathrm{loc}}(\varphi_\tau^{r+1}||\partial\tau|, C, x) = I_{w\mathrm{loc}}(\Psi, C, x),$$

which ends the proof that φ_τ^{r+1} is an ε_{r+1}-approximation of Ψ. Now, using the gluing lemma, we obtain a weighted map $\varphi^{r+1}\colon |K^{(r+1)}| \cup |L| \multimap Y$ being an ε_{r+1}-approximation of $\Psi\colon |K| \multimap Y$ with $\varphi^{r+1}||K^{(r)}| \cup |L| = \varphi^r$. This completes the inductive step. Consequently, after n_0+1 steps, we arrive at an ε-approximation $\varphi := \varphi^{n_0}$ of Ψ. The theorem is proved. □

Let us notice that the following theorem was proved in [10].

Theorem 4.3.3. *Let $X_0 \subset X$ be a finite polyhedral pair, let Y be a metric* ANR *and let $\Phi: X \multimap Y$ be a weighted carrier with positively acyclic values. Given any $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-approximation $\varphi: X_0 \multimap Y$ of $\Phi|X_0: X_0 \multimap Y$ can be extended to an ε-approximation $\widetilde{\varphi}: X \multimap Y$ of Φ.*

It should be noted that Theorem 4.3.2 was proved under the weaker assumptions than Theorem 4.3.3 but with a slight different conclusion, which will be much more convenient in applications.

The following three lemmas will be used in the proof of the main result of this section.

Lemma 4.3.4 ([23])**.** *Let K be a compact subset of X and let U be an open neighbourhood of K in X. Then for any retraction $r: U \to X$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $O_\delta(K) \subset U$ and $d_X(r(x), x) < \varepsilon$ for each $x \in O_\delta(K)$.*

Lemma 4.3.5 ([49])**.** *Let (X, A) be a pair of compact* ANR*s and let $\eta > 0$. Then there is a finite polyhedral pair (P, P_0) and maps of pairs $p: (P, P_0) \to (X, A)$ and $q: (X, A) \to (P, P_0)$ such that $d_X(p \circ q(x), x) < \eta$ for each $x \in X$.*

Lemma 4.3.6 ([8])**.** *Let X be a compact* ANR *and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that if $f_0, f_1: X \to X$ are δ-close* ([21])*, then there exists a map $h: X \times [0, 1] \to X$ such that*

(a) *$h(x, 0) = f_0(x)$ for all $x \in X$,*
(b) *$h(x, 1) = f_1(x)$ for all $x \in X$,*
(c) $\operatorname{diam}(h(\{x\} \times [0, 1])) < \varepsilon$ *for any $x \in X$, where* $\operatorname{diam}(h(\{x\} \times [0, 1])) := \sup\{d_X(h(x, t_1), h(x, t_2)), t_1, t_2 \in [0, 1]\}$.

We shall now prove the following lemma that will play a central role in the sequel.

Lemma 4.3.7. *Let (X, A) be a pair of compact* ANR*s and let Y be a metric space. In addition, let $\Psi: X \multimap Y$ be a weighted carrier and let $\varepsilon > 0$. Then there exists $\gamma > 0$ such that if a weighted map $\psi_0: (X \times \{0\}) \cup (A \times [0, 1]) \multimap Y$ has the property that the weighted maps $\psi_0(\,\cdot\,, 0): X \multimap Y$ and $\psi_0(\,\cdot\,, t): A \multimap Y$, for each $t \in [0, 1]$, are γ-approximations of Ψ, then there exists a weighted map $\psi: X \times [0, 1] \multimap Y$ such that for each $t \in [0, 1]$ the weighted map $\psi(\,\cdot\,, t): X \multimap Y$ is an ε-approximation of Ψ and $\psi|(X \times \{0\}) \cup (A \times [0, 1]) = \psi_0$.*

Proof. The basic idea of the proof follows from [3]. Let $M := (X \times \{0\}) \cup (A \times [0, 1])$. Since $X \times \{0\}$, $A \times [0, 1]$, $(X \times \{0\}) \cap (A \times [0, 1]) = A \times \{0\}$ are ANRs, we infer from Theorem 1.2.4 that M is also an ANR. Hence there exists an open neighbourhood $U \subset X \times [0, 1]$ of M and a retraction $r: U \to M$. Let δ_Ψ

([21]) Let $f, g: X \to X$ be two mappings and let d_X be a metric in X, $\varepsilon > 0$. We shall say that f and g are ε-close provided for every $x \in X$ we have $d_X(f(x), g(x)) < \varepsilon$.

be as in Corollary 4.1.11 for Ψ and $\varepsilon/2$. From Lemma 4.3.4 it follows that there exists $0 < \gamma < \min\{\varepsilon/2, \delta_\Psi\}$ such that

$$O_\gamma(M) \subset U \text{ and } d_{X\times[0,1]}(r(z), z) < \min\{\varepsilon/2, \delta_\Psi\} \tag{4.35}$$

for every $z \in O_\gamma(M)$. Now take a w-map ψ_0 as in the formulation of Lemma 4.3.7 according to γ. Define a w-map $\overline{\psi}\colon O_\gamma(M) \multimap Y$ by the formula: $\psi_0 \circ r$. Let us observe that for each $(x,t) \in O_\gamma(M)$ we have

$$\overline{\psi}(x,t) \subset O_\varepsilon(\Psi(O_\varepsilon(x))). \tag{4.36}$$

Indeed, let $(x',t') := r(x,t)$. Then, by (4.35), we get

$$d_X(x',x) \leqslant d_{X\times[0,1]}(r(x,t),(x,t)) < \min\{\varepsilon/2, \delta_\Psi\}. \tag{4.37}$$

Therefore

$$\overline{\psi}(x,t) = \psi_0(x',t') \subset O_\gamma(\Psi(O_\gamma(x'))) \subset O_\varepsilon(\Psi(O_\varepsilon(x))),$$

which verifies (4.36). Let V be an open neighbourhood of A in X such that $V\times[0,1] \subset O_\gamma(M)$. Since A and $X\setminus V$ are disjoint subsets of X, there exists an Urysohn function, i.e. there is a map $u\colon X \to [0,1]$ such that $u(x) = 1$, for every $x \in A$ and $u(x) = 0$, for every $X \setminus V$. Define a w-map $\psi\colon X\times[0,1] \multimap Y$ by

$$\psi(x,t) = \overline{\psi}(x,u(x)t).$$

Now, let us observe that from (4.36) we get

$$\psi(x,t) \subset O_\varepsilon(\Psi(O_\varepsilon(x)))$$

for all $(x,t) \in X\times[0,1]$. Therefore the proof will be completed, if we show that for each $t \in [0,1]$ a w-map $\psi(\,\cdot\,,t)\colon X \multimap Y$ satisfies the second condition of Definition 4.1.1. To this end, we need to consider 3 cases.

Case 1. Let $x_0 \in A$ and let C be a piece of $O_\varepsilon(\Psi(O_\varepsilon(x_0)))$. In addition, let us fix $t_0 \in [0,1]$. Then we have

$$\psi(x_0,t_0) = \overline{\psi}(x_0,u(x_0)t_0) = \overline{\psi}(x_0,t_0) = \psi_0\circ r(x_0,t_0) = \psi_0(x_0,t_0).$$

Hence

$$I_{w\mathrm{loc}}(\psi(\,\cdot\,,t_0),C,x_0) = I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,t_0),C,x_0).$$

Since $O_\gamma(\Psi(O_\gamma(x_0))) \subset O_\varepsilon(\Psi(O_\varepsilon(x_0)))$ and $C \subset O_\varepsilon(\Psi(O_\varepsilon(x_0)))$, we infer that $C\cap O_\gamma(\Psi(O_\gamma(x_0)))$ is a piece of $O_\gamma(\Psi(O_\gamma(x_0)))$. Therefore

$$\begin{aligned} I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,t_0),C,x_0) &= I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,t_0),C\cap O_\gamma(\Psi(O_\gamma(x_0))),x_0) \\ &= I_{w\mathrm{loc}}(\Psi,C\cap O_\gamma(\Psi(O_\gamma(x_0))),x_0), \end{aligned}$$

where the first equality follows from the excision property of $I_{w\text{loc}}$ for $\psi_0(\,\cdot\,,t_0)$, but the second one follows from the fact that $\psi_0(\,\cdot\,,t_0)\colon A \multimap Y$ is a γ-approximation of Ψ. Using once again the excision property of $I_{w\text{loc}}$, we get

$$I_{w\text{loc}}(\Psi, C\cap O_\gamma(\Psi(O_\gamma(x_0))), x_0) = I_{w\text{loc}}(\Psi, C, x_0),$$

which proves that

$$I_{w\text{loc}}(\psi(\,\cdot\,,t_0), C, x_0) = I_{w\text{loc}}(\Psi, C, x_0).$$

The proof of Case 1 is complete.

Case 2. Let $x_0 \in X\setminus V$ and let $C\subset O_\varepsilon(\Psi(O_\varepsilon(x_0))$ be as above. In addition, let us fix $t_0\in[0,1]$. Then

$$\psi(x_0,t_0) = \overline{\psi}(x_0, u(x_0)t_0) = \overline{\psi}(x_0,0) = \psi_0\circ r(x_0,0) = \psi_0(x_0,0).$$

Hence

$$I_{w\text{loc}}(\psi(\,\cdot\,,t_0), C, x_0) = I_{w\text{loc}}(\psi_0(\,\cdot\,,0), C, x_0).$$

Since $\psi_0(\,\cdot\,,0)$ is a γ-approximation of Ψ we have

$$\begin{aligned}
&(4.38) && I_{w\text{loc}}(\psi_0(\,\cdot\,,0), C, x_0) = I_{w\text{loc}}(\psi_0(\,\cdot\,,0), C\cap O_\gamma(\Psi(O_\gamma(x_0))), x_0)\\
& && \qquad = I_{w\text{loc}}(\Psi, C\cap O_\gamma(\Psi(O_\gamma(x_0))), x_0)\\
&(4.39) && \qquad = I_{w\text{loc}}(\Psi, C, x_0)
\end{aligned}$$

where the equalities in (4.38) and (4.39) follow from the excision property of $I_{w\text{loc}}$. Therefore

$$I_{w\text{loc}}(\psi(\,\cdot\,,t_0), C, x_0) = I_{w\text{loc}}(\Psi, C, x_0),$$

which completes the proof of Case 2.

Case 3. (In this case Corollary 4.1.11 plays a crucial role) Let $x_0\in V\setminus A$ and let $C\subset O_\varepsilon(\Psi(O_\varepsilon(x_0)))$ be a piece of $O_\varepsilon(\Psi(O_\varepsilon(x_0)))$. In addition, let us fix $t_0\in[0,1]$. Let $(x',t') := r(x_0,t_0)$. Since

$$\psi(x_0,t_0) = \psi_0\circ r(x_0,t_0) = \psi_0(x',t'),$$

we have

$$I_{w\text{loc}}(\psi(\,\cdot\,,t_0), C, x_0) = I_{w\text{loc}}(\psi_0\circ r, C, (x_0,t_0)) = I_{w\text{loc}}(\psi_0(\,\cdot\,,t'), C, x'). \tag{4.40}$$

Moreover,

$$\psi_0(x',t') \subset O_\gamma(\Psi(O_\gamma(x'))) \subset O_\varepsilon(\Psi(O_\varepsilon(x_0))),$$

because $\psi_0(\,\cdot\,,t')$ is a γ-approximation of Ψ and $d_X(x_0,x') < \varepsilon/2$ by (4.37). Consequently, the same reasoning as in Case 1 establishes that

$$I_{w\text{loc}}(\psi_0(\,\cdot\,,t'), C, x') = I_{w\text{loc}}(\Psi, C, x'). \tag{4.41}$$

Since $d_X(x_0,x') < \delta_\Psi$, Corollary 4.1.11 implies that

$$I_{w\text{loc}}(\Psi, C, x') = I_{w\text{loc}}(\Psi, C, x_0). \tag{4.42}$$

Therefore, taking into account (4.40)–(4.42), we get

$$I_{w\mathrm{loc}}(\psi(\,\cdot\,,t_0),C,x_0) = I_{w\mathrm{loc}}(\Psi,C,x_0),$$

which completes the proof of Case 3, and hence the lemma follows. $\square$

We now prove the main result of this section.

Theorem 4.3.8. *Let X be a compact* ANR*, let $A \subset X$ be a closed* ANR*, and let Y be a locally connected space. In addition, let $\Psi\colon X \multimap Y$ be a w-UV^ω-valued carrier. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\psi_0\colon A \multimap Y$ is a δ-approximation of $\Psi\colon X \multimap Y$, then there exists a weighted map $\psi\colon X \multimap Y$ being an ε-approximation of Ψ with $\psi|A = \psi_0$.*

Proof. The proof is based on [3]. Let $\varepsilon > 0$ be fixed. Let γ be as in Lemma 4.3.7 according to X, A, Ψ, and ε. Let δ_Ψ be as in Corollary 4.1.11 according to Ψ and ε. In addition, let $\eta > 0$ be as in Lemma 4.3.6 for X and $\min\{\gamma/2,\delta_\Psi\}$. We can assume that $\eta \leqslant \min\{\gamma/2,\delta_\Psi\}$. Then for a pair of compact ANRs (X,A) and η there is, by Lemma 4.3.5, a polyhedral pair (P,P_0) and maps of pairs

$$p\colon (P,P_0) \to (X,A) \quad \text{and} \quad q\colon (X,A) \to (P,P_0) \tag{4.43}$$

such that for each $x \in X$ we have $d_X(p\circ q(x),x) < \eta$. Since P is compact space, we infer that there exists $0 < \mu \leqslant \gamma$ such that if $x,y \in P$, then $d_X(p(x),q(x)) < \gamma/2$ provided that $d_P(x,y) < \mu$. Thus $p(O_\mu(q(x))) \subset O_\gamma(x)$ for each $x \in X$. Let $\Phi\colon P \multimap Y$ be a weighted carrier given by $\Phi = \Psi \circ p$. It is easy to see that Φ is a w-UV^ω-valued carrier. In view of Theorem 4.3.2, there exists $\nu > 0$ such that if $\theta_0\colon P_0 \multimap Y$ is a ν-approximation of $\Phi\colon P \multimap Y$, then there exists a μ-approximation $\theta\colon P \multimap Y$ of Φ with $\theta|P_0 = \theta_0$. Next, in view of Proposition 4.1.7, there exists $0 < \delta \leqslant \gamma/2$ such that if $\psi_0\colon A \multimap Y$ is a δ-approximation of $\Psi\colon X \multimap Y$, then $\psi_0 \circ p|P_0\colon P_0 \multimap Y$ is a ν-approximation of $\Phi\colon P \multimap Y$. Now, let ψ_0 be a δ-approximation of Ψ. We shall prove that there exists a weighted map $\psi\colon X \multimap Y$ being a δ-approximation of Ψ with $\psi|A = \psi_0$. Let us observe that by the choice of δ the composition $\psi_0 \circ p|P_0$ is a ν-approximation of $\Phi\colon P \multimap Y$. Therefore, from Theorem 4.3.2 it follows that a weighted map $\psi_0 \circ p|P_0\colon P_0 \multimap Y$ admits an extension $\widetilde{\psi}\colon P \multimap Y$ being a μ-approximation of Φ. Let us define now a weighted map $\overline{\psi}\colon X \multimap Y$ by $\overline{\psi} = \widetilde{\psi} \circ q$. We shall show now that $\overline{\psi}$ is a γ-approximation of $\Psi\colon X \multimap Y$. Since $\widetilde{\psi}$ is a μ-approximation of Φ, we have

$$\overline{\psi}(x) = \widetilde{\psi}(q(x)) \subset O_\mu(\Phi(O_\mu(q(x))))$$

for each $x \in X$. Hence, taking into account a definition of Φ and (4.43), we get

$$\begin{aligned} O_\mu(\Phi(O_\mu(q(x)))) &= O_\mu(\Psi \circ p(O_\mu(q(x)))) \\ &\subset O_\mu(\Psi(O_\gamma(x))) \subset O_\gamma(\Psi(O_\gamma(x))), \end{aligned}$$

for each $x \in X$. Now, we are going to show that for any $x \in X$ and for any piece C of $O_\gamma(\Psi(O_\gamma(x)))$ the following condition is satisfied:

$$I_{w\mathrm{loc}}(\overline{\psi}, C, x) = I_{w\mathrm{loc}}(\Psi, C, x).$$

To this end, let us fix $x \in X$ and $C \subset O_\gamma(\Psi(O_\gamma(x)))$. Let us observe that

$$\begin{aligned} &I_{w\mathrm{loc}}(\overline{\psi}, C, x) \\ &\qquad = I_{w\mathrm{loc}}(\overline{\psi}, C \cap O_\mu(\Psi(O_\mu(q(x)))), x) && (\text{excision of } I_{w\mathrm{loc}}) \\ &\qquad = I_{w\mathrm{loc}}(\widetilde{\psi} \circ q, C \cap O_\mu(\Psi(O_\mu(q(x)))), x) && (\overline{\psi} = \widetilde{\psi} \circ q) \\ &\qquad = I_{w\mathrm{loc}}(\widetilde{\psi}, C \cap O_\mu(\Psi(O_\mu(q(x)))), q(x)) && (\text{Corollary 3.2.5}) \\ &\qquad = I_{w\mathrm{loc}}(\Phi, C \cap O_\mu(\Psi(O_\mu(q(x)))), q(x)), \end{aligned}$$

where the last equality follows from the fact that $\widetilde{\psi}$ is a μ-approximation of Φ. Consequently, we get

$$\begin{aligned} &I_{w\mathrm{loc}}(\Phi, C \cap O_\mu(\Psi(O_\mu(q(x)))), q(x)) \\ &\qquad\qquad = I_{w\mathrm{loc}}(\Phi, C, q(x)) && (\text{excision of } I_{w\mathrm{loc}}) \\ &\qquad\qquad = I_{w\mathrm{loc}}(\Psi \circ p, C, q(x)) && (\Phi = \Psi \circ p) \\ &\qquad\qquad = I_{w\mathrm{loc}}(\Psi, C, p \circ q(x)) && (\text{Corollary 3.2.5}). \end{aligned}$$

Since $d_X(p \circ q(x), x) < \eta \leqslant \delta_\Psi$, due to Corollary 4.1.11, we get

$$I_{w\mathrm{loc}}(\Psi, C, p \circ q(x)) = I_{w\mathrm{loc}}(\Psi, C, x).$$

Summing up, we have showed that

$$I_{w\mathrm{loc}}(\overline{\psi}, C, x) = I_{w\mathrm{loc}}(\Psi, C, x),$$

which proves that $\overline{\psi}$ is a γ-approximation of Ψ. Now we shall use Lemma 4.3.7 to modify the weighted map $\overline{\psi}$ because $\overline{\psi}$ is not the required approximation of Ψ yet. For this purpose, let us recall that for each $a \in A$ we have

$$\overline{\psi}(a) = \widetilde{\psi} \circ q(a) = \psi_0(p \circ q(a)).$$

Moreover, $\mathrm{id}_A\colon A \to A$ and $p \circ q\colon A \to A$ are η-close. Therefore, in view of Lemma 4.3.6, there exists a map $h\colon A \times [0,1] \to A$ such that

$$h(\,\cdot\,, 0) = p \circ q|A \quad \text{and} \quad h(\,\cdot\,, 1) = \mathrm{id}_A, \tag{4.44}$$

$$\mathrm{diam}(h(\{a\} \times [0,1])) < \min\{\gamma/2, \delta_\Psi\}. \tag{4.45}$$

Let $\phi_0\colon (X \times \{0\}) \cup (A \times [0,1]) \multimap Y$ be given by

$$\phi_0(x,t) = \begin{cases} \overline{\psi}(x) & \text{if } (x,t) \in X \times \{0\}, \\ \psi_0 \circ h(x,t) & \text{if } (x,t) \in A \times [0,1]. \end{cases}$$

Since

$$\psi_0 \circ h(a, 0) = (\psi_0 \circ p) \circ q(a) = \overline{\psi}(a),$$

for all $a \in A$, in view of Corollary 3.2.5, we get

$$\begin{aligned} I_{w\text{loc}}(\overline{\psi}, U, a) &= I_{w\text{loc}}(\psi_0 \circ (p \circ q), U, a) = I_{w\text{loc}}(\psi_0, U, p \circ q(a)) \\ &= I_{w\text{loc}}(\psi_0, U, h(a,0)) = I_{w\text{loc}}(\psi_0 \circ h, U, (a,0)), \end{aligned}$$

where $a \in A$ and U is an open subset of Y such that $\overline{\psi}(a) \cap \partial U = \emptyset$. Hence, by the gluing lemma, ϕ_0 is a weighted map. We shall show now that for all $t \in [0,1]$ a w-map $\phi_0(\,\cdot\,, t)$ is a γ-approximation of Ψ (the case $t = 0$ has already been proved). Let us fix $t \in (0,1]$ and $a \in A$. Then

$$\phi_0(a,t) = \psi_0 \circ h(a,t) \subset O_\delta(\Psi(O_\delta(h(a,t)))),$$

because ψ_0 is a δ-approximation of Ψ. Moreover, since $h(a,t) \in O_{\gamma/2}(a)$, we have $O_\delta(h(a,t)) \subset O_{\gamma/2+\delta}(a)$. Thus

$$\phi_0(a,t) \subset O_\gamma(\Psi(O_\gamma(a))), \tag{4.46}$$

since $\delta \leqslant \gamma/2$. Let C be a piece of $O_\gamma(\Psi(O_\gamma(a)))$. Now we are going to prove that

$$I_{w\text{loc}}(\phi_0(\,\cdot\,,t), C, a) = I_{w\text{loc}}(\Psi, C, a).$$

First, note that

$$\begin{aligned} I_{w\text{loc}}(\phi_0(\,\cdot\,,t), C, a) &= I_{w\text{loc}}(\psi_0 \circ h(\,\cdot\,,t), C, a) \quad && (\phi_0(\,\cdot\,,t) = \psi_0 \circ h(\,\cdot\,,t)) \\ &= I_{w\text{loc}}(\psi_0, C, h(a,t)). && (\text{Corollary 3.2.5}) \end{aligned}$$

Additionally,

$$\begin{aligned} \psi_0 \circ h(a,t) &\subset O_\delta(\Psi(O_\delta(h(a,t)))) \subset O_\delta(\Psi(O_\delta(O_{\gamma/2}(a)))) \\ &\subset O_{\gamma/2}(\Psi(O_\gamma(a))) \subset O_\gamma(\Psi(O_\gamma(a))), \end{aligned}$$

and hence $C \cap O_\delta(\Psi(O_\delta(h(a,t))))$ is a piece of $O_\delta(\Psi(O_\delta(h(a,t))))$. Consequently, by the excision property of $I_{w\text{loc}}$, we get

$$I_{w\text{loc}}(\psi_0, C, h(a,t)) = I_{w\text{loc}}(\psi_0, C \cap O_\delta(\Psi(O_\delta(h(a,t)))), h(a,t))$$

and

$$\begin{aligned} &I_{w\text{loc}}(\psi_0, C \cap O_\delta(\Psi(O_\delta(h(a,t)))), h(a,t)) \\ &\qquad\qquad = I_{w\text{loc}}(\Psi, C \cap O_\delta(\Psi(O_\delta(h(a,t)))), h(a,t)), \end{aligned}$$

because ψ_0 is a δ-approximation of Ψ. Thus

$$I_{w\text{loc}}(\Psi, C \cap O_\delta(\Psi(O_\delta(h(a,t)))), h(a,t)) = I_{w\text{loc}}(\Psi, C, h(a,t)) = I_{w\text{loc}}(\Psi, C, a),$$

where the first equality follows form the excision property of $I_{w\text{loc}}$, and the second one holds by Corollary 4.1.11, because, by (4.45), $d_X(h(a,t), a) < \delta_\Psi$ (let us recall that δ_Ψ was defined at the beginning of our proof). Consequently, we have showed that

$$I_{w\text{loc}}(\phi_0(\,\cdot\,,t), C, a) = I_{w\text{loc}}(\Psi, C, a). \tag{4.47}$$

From (4.46) and (4.47) we infer that the assumptions of Lemma 4.3.7 are satisfied, and hence ϕ_0 admits an extension $\phi: X \times [0,1] \multimap Y$ such that for each $t \in [0,1]$ the w-map $\phi(\,\cdot\,,t): X \multimap Y$ is an ε-approximation of Ψ. Finally, to complete the proof, we define $\psi: X \multimap Y$ by putting $\psi := \phi(\,\cdot\,,1)$. □

In particular, we obtain the following corollary (see Corollary 4.2.17).

Corollary 4.3.9. *Let X be a compact* ANR*, let $A \subset X$ be a closed* ANR*, and let Y be an* ANR*. In addition, let $\Psi: X \multimap Y$ be an upper semicontinuous multivalued map with acyclic values* (*with respect to the Čech homology with coeffcients in $\mathbb{Q}$*)*. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi_0: A \multimap Y$ is a δ-approximation of $\Psi: X \multimap Y$, then there exists a weighted map $\varphi: X \multimap Y$ being an ε-approximation of Ψ with $\varphi|A = \varphi_0$.*

Moreover, from the above considerations we obtain the following corollaries.

Corollary 4.3.10. *Let $\Psi: X \multimap Y$ be a w-UV^ω-valued carrier, let X be a compact* ANR *and let Y be a locally connected space. Then for each $\varepsilon > 0$ there exists an ε-approximation $\varphi: X \multimap Y$ of Ψ.*

Proof. It is enough to take $A = \emptyset$ in Theorem 4.3.8. □

Corollary 4.3.11. *Let $\Theta: X \times [0,1] \multimap Y$ be a w-UV^ω-valued carrier and let X, Y be as above. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi_i: X \multimap Y$ is a δ-approximation of Θ, then there exists an ε-approximation $\psi: X \times [0,1] \multimap Y$ of Θ such that $\psi|X \times \{i\} = \varphi_i$ for $i = 0, 1$.*

Proof. Let us take $A = X \times \{0\} \cup X \times \{1\}$ and let $\psi_0: A \multimap Y$ be defined as follows

$$\psi_0(x,t) = \begin{cases} \varphi_0(x) & \text{if } (x,t) \in X \times \{0\}, \\ \varphi_1(x) & \text{if } (x,t) \in X \times \{1\}. \end{cases}$$

This completes the proof if we invoke Theorem 4.3.8. □

Corollary 4.3.12. *Let X be a compact* ANR *and let Y be an* ANR*. In addition, let $\Theta: X \times [0,1] \multimap Y$ be a weighted carrier with positively acyclic values* (*with respect to the Čech homology with coefficients in $\mathbb{Q}$*)*. Then there is $\delta > 0$ such that if $\varphi_i: X \multimap Y$ is a δ-approximation of Θ, for $i = 0, 1$, then $\varphi_0, \varphi_1: X \multimap Y$ are w-homotopic.*

Corollary 4.3.13. *Let X and Y be as in Corollary* 4.3.11*. Let $\Psi: X \multimap Y$ be a w-UV^ω-valued carrier. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that for any two δ-approximations $\varphi_0, \varphi_1: X \multimap Y$ of Ψ there exists a w-homotopy $\psi: X \times [0,1] \multimap Y$ such that*

(a) *$\psi(\,\cdot\,,0) = \varphi_0$ and $\psi(\,\cdot\,,1) = \varphi_1$,*
(b) *$\psi(\,\cdot\,,t)$ is an ε-approximation of Ψ for any $t \in [0,1]$.*

Proof. Let $\Theta: X \times [0,1] \multimap Y$ be a w-UV^ω-valued carrier defined by $\Theta(x,t) = \Psi(x)$ for all $x \in X$ and $t \in [0,1]$. Then, in view of Corollary 4.3.11, there exists the required weighted homotopy ψ. □

In particular, we get:

Corollary 4.3.14. *Let X, Y and $\Psi: X \multimap Y$ be as above. Then there is $\delta > 0$ such that any two δ-approximations $\psi_0, \psi_1: X \multimap Y$ of Ψ are w-homotopic.*

Corollary 4.3.15. *Let X be a compact* ANR *and let Y be an* ANR. *In addition, let $\Psi: X \multimap Y$ be a weighted carrier with positively acyclic values* (*with respect to the Čech homology with coefficients in $\mathbb{Q}$*). *Then there is $\delta_\Psi > 0$ such that any two δ_Ψ-approximations $\psi_0, \psi_1: X \multimap Y$ of Ψ are w-homotopic.*

We shall end this section by proving approximation results for w-carriers defined on pairs of compact ANR's.

We say that $\Psi: (X,A) \multimap (Y,B)$ is a w-carrier if $\Psi_X: X \multimap Y$ is a w-carrier and $\Psi(A) \subset B$. It is easy to see that if $\Psi: (X,A) \multimap (Y,B)$ is a w-carrier, then $\Psi_A: A \multimap B$ is also a w-carrier.

Definition 4.3.16. Let $\Psi: (X,A) \multimap (Y,B)$ be a w-carrier and let $\varepsilon > 0$. We say that a w-map $\varphi: (X,A) \multimap (Y,B)$ is an *ε-approximation of* Ψ if $\varphi_A: A \multimap B$ is an ε-approximation of Ψ_A and $\varphi_X: X \multimap Y$ is an ε-approximation of $\Psi_X: X \multimap Y$.

Definition 4.3.17. We say that $\Psi: (X,A) \multimap (Y,B)$ is a *w-UV^ω-valued carrier* if $\Psi_A: A \multimap B$ and $\Psi_X: X \multimap Y$ are w-UV^ω-valued carriers.

Theorem 4.3.18. *Let (X,A) be pair of compact* ANR′*s, let (Y,B) be a pair of locally connected spaces and let $\Phi: (X,A) \multimap (Y,B)$ be a w-UV^ω-valued carrier. Then for each $\varepsilon > 0$ there is a w-map $\varphi: (X,A) \multimap (Y,B)$ such that φ is an ε-approximation of Φ.*

Proof. The proof is similar to that of [3, Theorem 3.1(i)], but for the sake of completeness we give details. Let us take $\varepsilon > 0$ and let $0 < \delta < \varepsilon$ be as in Theorem 4.3.8. Since $\Phi_A: A \multimap B$ is a w-UV^ω-valued carrier, we conclude, using Corollary 4.3.10, that there is a δ-approximation $\varphi_0: A \multimap B$ of Φ_A. Consequently, in view of Theorem 4.3.8, there exists a weighted map $\varphi: X \multimap Y$ such that φ is an ε-approximation of $\Phi_X: X \multimap Y$ and $\varphi|A = \varphi_0$. Hence we obtain a weighted map $\varphi: (X,A) \multimap (Y,B)$ being an ε-approximation of $\Phi: (X,A) \multimap (Y,B)$. This completes the proof. □

Similarly to [3, Theorem 3.1(ii)], we get the following theorem.

Theorem 4.3.19. *Let (X,A), (Y,B) and $\Phi: (X,A) \multimap (Y,B)$ be as in the formulation of Theorem* 4.3.18. *Then for each $\varepsilon > 0$ there is $\delta > 0$ such that if $\varphi, \psi: (X,A) \multimap (Y,B)$ are δ-approximations of Φ, then there exists a w-homotopy*

$\theta\colon (X \times [0,1], A \times [0,1]) \multimap (Y,B)$ such that $\theta(\,\cdot\,,0) = \varphi(\,\cdot\,)$, $\theta(\,\cdot\,,1) = \psi(\,\cdot\,)$ and $\theta(\,\cdot\,,t)\colon (X,A) \multimap (Y,B)$ is an ε-approximation of Φ for each $t \in [0,1]$.

Proof. Let $\varepsilon > 0$ and let $\overline{\Phi}\colon (X \times [0,1], A \times [0,1]) \multimap (Y,B)$ be defined by $\overline{\Phi}(x,t) := \Phi(x)$ for all $t \in [0,1]$, $x \in X$. It is easy to see that $\overline{\Phi}$ is a w-UV^ω-valued carrier. Additionally, let us define $M := (X\times\{0\})\cup(A\times[0,1])\cup(X\times\{1\})$. Using the same arguments as in the proof of Lemma 4.3.7, we see that M is an absolute neighbourhood retract. Let $0 < \gamma < \varepsilon$ be as in the formulation of Theorem 4.3.8 for $X \times [0,1]$, M, $\overline{\Phi}_{X\times[0,1]}\colon X \times [0,1] \multimap Y$ and ε. Moreover, Corollary 4.3.13 provides $0 < \delta < \gamma$ according to $\Phi_A\colon A \multimap B$ and γ. Let $\varphi,\psi\colon (X,A) \multimap (Y,B)$ be given δ-approximations of Φ. Then, by definition of δ, there is a w-homotopy $\overline{\theta}\colon A \times [0,1] \multimap B$ such that

(a) $\overline{\theta}(\,\cdot\,,t)$ is a γ-approximation of Φ_A for all $t \in [0,1]$,
(b) $\overline{\theta}(\,\cdot\,,0) = \varphi_A(\,\cdot\,)$ and $\overline{\theta}(\,\cdot\,,1) = \psi_A(\,\cdot\,)$.

Now let us define a w-map $\widetilde{\theta}\colon M \multimap Y$ as follows

$$\widetilde{\theta}(x,t) = \begin{cases} \varphi_X(x) & \text{if } (x,t) \in X \times \{0\}, \\ \overline{\theta}(x,t) & \text{if } (x,t) \in A \times [0,1], \\ \psi_X(x) & \text{if } (x,t) \in X \times \{1\}. \end{cases}$$

Since $\widetilde{\theta}$ is a γ-approximation of $\overline{\Phi}_{X\times[0,1]}\colon X \times [0,1] \multimap Y$, we conclude from Theorem 4.3.8 that there exists an extension $\widehat{\theta}\colon X\times[0,1] \multimap Y$ of $\widetilde{\theta}$ over $X\times[0,1]$ such that $\widehat{\theta}$ is an ε-approximation of $\overline{\Phi}_{X\times[0,1]}\colon X \times [0,1] \multimap Y$, which implies that there is a weighted map $\theta\colon (X \times [0,1], A \times [0,1]) \multimap (Y,B)$ satisfying all requirements of the assertion. This completes the proof. $\square$

Corollary 4.3.20. *Let (X,A), (Y,B) and $\Phi\colon (X,A) \multimap (Y,B)$ be as above. Then there is $\delta > 0$ such that if $\varphi,\psi\colon (X,A) \multimap (Y,B)$ are δ-approximations of Φ, then there is a w-homotopy $\theta\colon (X \times [0,1], A \times [0,1]) \multimap (Y,B)$ such that $\theta(\,\cdot\,,0) = \varphi(\,\cdot\,)$ and $\theta(\,\cdot\,,1) = \psi(\,\cdot\,)$.*

4.4. Bijection theorem

In this section we will present a weighted version of Theorem 4.5 given in [27]. More precisely, we obtain a bijection between homotopy classes of weighted carriers and homotopy classes of weighted maps.

Given two spaces X, Y, we put

$$\mathbb{A}_0(X,Y) = \{\Psi\colon X \multimap Y \mid \Psi \text{ is a } w\text{-carrier and for every } \varepsilon > 0, a_w(\Psi,\varepsilon) \neq \emptyset\}.$$

Definition 4.4.1. We say that $\Psi \in \mathbb{A}(X,Y)$ if

(a) $\Psi \in \mathbb{A}_0(X,Y)$;
(b) for any $\delta > 0$, there is $\varepsilon > 0$ such that any two weighted maps $\varphi,\psi \in a_w(\Psi,\varepsilon)$ are joined by a w-homotopy $\gamma\colon X \times [0,1] \multimap Y$ such that $\gamma(\,\cdot\,,t) \in a_w(\Psi,\delta)$ for all $t \in [0,1]$.

Moreover, we put

$$\mathbb{A}_C(X_0, X) = \{\Psi \in \mathbb{A}(X_0, X) \mid \mathrm{Fix}(\Psi) \cap C = \emptyset\},$$

where $X_0 \subset X$ and C is a closed subset of X_0. In what follows, by $\mathbb{W}_C(X_0, X)$ we shall denote the set of all weighted maps $\varphi\colon X_0 \multimap X$ such that $\mathrm{Fix}(\varphi) \cap C = \emptyset$.

Now we will give the definition of homotopy in $\mathbb{A}_C(X_0, X)$.

Definition 4.4.2. Two weighted carriers $\Psi, \Phi \in \mathbb{A}_C(X_0, X)$ are called *homotopic* (in $\mathbb{A}_C(X_0, X)$) if there exists a mapping $\Upsilon \in \mathbb{A}_0(X_0 \times [0,1], X)$ such that

$$\Upsilon(x, 0) = \Psi(x), \quad \Upsilon(x, 1) = \Phi(x) \quad \text{for all } x \in X_0$$

and $x \notin \Upsilon(x, t)$ for all $x \in C$, $t \in [0,1]$. If Ψ and Φ are homotopic, then we write $\Psi \sim_C \Phi$.

Proposition 4.4.3. *If C is a closed subset of a compact space X_0, then the relation $\sim_C$ is an equivalence.*

To prove the above proposition, we need the following lemma.

Lemma 4.4.4. *Let X be a compact space and let $\Upsilon \in \mathbb{A}_0(X \times [0,1], Y)$. Then for each $\varepsilon > 0$ and $t \in [0,1]$ there is $\delta > 0$ such that if $\varphi\colon X \times [0,1] \multimap Y$ is a δ-approximation of Υ, then φ_t is an ε-approximation of Υ_t.*

Proof. Let $\varepsilon > 0$ and fix $t \in [0,1]$. In addition, let $\lambda_t\colon X \to X \times [0,1]$ be given by $\lambda_t(x) = (x, t)$. Obviously, $\Upsilon_t = \Upsilon \circ \lambda_t$ and $\varphi_t = \varphi \circ \lambda_t$, so the conclusion follows from Proposition 4.1.7. □

Proof of Proposition 4.4.3. Obviously $\sim_C$ is reflexive and symmetric. To get the transitivity we need Lemma 4.4.4. The proof of this fact uses the same idea as in the proof of Proposition 4.4 in [27], therefore, we omit it. □

We denote the homotopy class of $\Psi \in \mathbb{A}_C(X_0, X)$ by $[\Psi]_C$ and the set of all homotopy classes by $[\mathbb{A}_C(X_0, X)]$.

We say that two weighted maps in $\mathbb{W}_C(X_0, X)$ are homotopic if there exists a weighted and fixed point free on C homotopy joining these weighted maps. We denote the homotopy class of $\varphi \in \mathbb{W}_C(X_0, X)$ by $[\varphi]_C$ and the set of all homotopy classes by $[\mathbb{W}_C(X_0, X)]$.

Remark 4.4.5. Let X be an ANR and let $X_0 \subset X$ be a compact ANR. Then, in view of Corollary 4.3.13 and Corollary 4.2.17, we have $\mathbb{W}_C(X_0, X) \subset \mathbb{A}_C(X_0, X)$, where C is a closed subset of X_0.

Now, we are in position to prove the main result of this section.

Theorem 4.4.6. *Let (X, X_0) be a pair of compact* ANR*'s. If C is a closed subset of X_0, then there is a bijection $F\colon [\mathbb{A}_C(X_0, X)] \to [\mathbb{W}_C(X_0, X)]$.*

Proof. The proof of the above theorem is essentially based on the ideas from [27]. Let $[\Phi]_C \in [\mathbb{A}_C(X_0, X)]$. Then, in view of Corollary 4.1.14, there exists $\delta(\Phi) > 0$ such that any $\delta(\Phi, C)$-approximations of Φ is fixed point free on C. Since $\Phi \in \mathbb{A}_C(X_0, X)$, there exists $\varepsilon(\Phi, C) < \delta(\Phi, C)$ such that for any $0 < \varepsilon \leqslant \varepsilon(\Phi, C)$, any two ε-approximation of Φ are joined by a w-homotopy being a $\delta(\Phi, C)$-approximations of Φ. Consequently, the above considerations allow us to define a function F as follows:

$$F([\Phi]_C) = [\varphi]_C,$$

where $\varphi\colon X_0 \multimap X$ is an arbitrary $\varepsilon(\Phi, C)$-approximation of Φ. One can show that the above definition is correct (The proof uses the arguments similar to those in the proof of Theorem 4.5 in [27], therefore, we omit it).

Now we have to show that F is a bijection. Observe that the surjectivity of F follows from Remark 4.4.5. To prove the injectivity of F it is enough to show that for any $\Phi \in \mathbb{A}_C(X_0, X)$ there exists $\varepsilon > 0$ such that if φ is an ε-approximation of Φ, then $\varphi \in [\Phi]_C$. For this purpose, let us fix $\Phi \in \mathbb{A}_C(X_0, X)$. In addition, let $\delta > 0$ be as in Proposition 4.1.13 for Φ and let $0 < \varepsilon < \delta$ be as in Definition 4.4.1 for Φ and δ.

Before moving further, we need to prove the following lemma.

Lemma 4.4.7. *Let X_0 and X be as above and let $\Phi\colon X_0 \multimap X$ be a weighted carrier. In addition, let $\delta > 0$. Then $\Psi_\Phi\colon X_0 \multimap X$ given by*

$$\Psi_\Phi(x) = D(\Phi(D(x, \delta)), \delta)$$

is a weighted carrier.

Proof. First, we shall show that Ψ_Φ is upper semicontinuous. For this purpose, it is enough to show that the graph Γ_{Ψ_Φ} is a closed subset of $X_0 \times X$. Let $\{(x_n, y_n)\} \subset \Gamma_{\Psi_\Phi}$ be a sequence such that $\{(x_n, y_n)\} \to (x, y) \in X_0 \times X$. Since $y_n \in \Psi_\Phi(x_n) = D(\Phi(D(x_n, \delta)), \delta)$, it follows that there exists a sequence $\{\widetilde{y}_n\}$ such that $\widetilde{y}_n \in \Phi(D(x_n, \delta))$ and $d_X(y_n, \widetilde{y}_n) \leqslant \delta + 1/n$, for all $n \in \mathbb{N}$. Moreover, there exists a sequence $\tilde{\mathrm{x}}_n \in D(x_n, \delta)$ with $\widetilde{y}_n \in \Phi(\tilde{\mathrm{x}}_n)$. We can assume, without loss of generality, that $\widetilde{x}_n \to \widetilde{x}_0 \in D(x_0, \delta)$ and $\widetilde{y}_n \to \widetilde{y}_0$. Since Φ is u.s.c. we conclude that

$$\widetilde{y}_0 \in \Phi(\tilde{\mathrm{x}}_0) \subset \Phi(D(x_0, \delta)).$$

Thus

$$y_0 \in D(\widetilde{y}_0, \delta) \subset D(\Phi(\widetilde{x}_0), \delta) \subset D(\Phi(D(x_0, \delta)), \delta),$$

and hence $(x_0, y_0) \in \Gamma_{\Psi_\Phi}$.

The remaining part of the proof will be devoted to the construction of a local weighted carrier for Ψ_Φ. To this end, let us fix a point $x_0 \in X$ and let V be an open subset of X such that $\Psi_\Phi(x_0) \cap \partial V = \emptyset$. Then we put

$$I_{w\mathrm{loc}}(\Psi_\Phi, x_0, V) := I_{w\mathrm{loc}}(\Phi, x_0, V). \tag{4.48}$$

First, observe, first, that the above definition is correct since $\Phi(x) \subset \Psi_\Phi(x)$ for any $x \in X$. Moreover, it is easy to see that the local weighted index given by

(4.48) has the following properties: existence and additivity. In order to state the local invariance property of $I_{w\mathrm{loc}}$ for Ψ_Φ, take a point $x_0 \in X$ and an open subset V of X with $\Psi_\Phi(x_0) \cap \partial V = \emptyset$. Since $\Psi_\Phi(x_0) \cap V$ and $\Psi_\Phi(x_0) \cap (X \setminus \overline{V})$ are compact, there exist open sets U_1 and U_2 such that

$$\Psi_\Phi(x_0) \cap V \subset U_1 \subset \overline{U}_1 \subset V \quad \text{and} \quad \Psi_\Phi(x_0) \cap (X \setminus \overline{V}) \subset U_2 \subset \overline{U}_2 \subset X \setminus \overline{V}.$$

The upper semicontinuity of Φ_Ψ implies that there exists an open neighbourhood O'_{x_0} of x_0 such that $\Psi_\Phi(x) \subset U_1 \cup U_2$ for all $x \in O'_{x_0}$. Moreover, there exists an open neighbourhood O'_{x_0} of a point x_0 such that

$$I_{w\mathrm{loc}}(\Phi, x_0, V) = I_{w\mathrm{loc}}(\Phi, x, V)$$

for any $x \in O''_{x_0}$. Let $O_{x_0} := O'_{x_0} \cap O''_{x_0}$. Consequently, $(\Psi_\Phi, V, x) \in D(\Phi_\Psi)$ and

$$I_{w\mathrm{loc}}(\Psi_\Phi, x_0, V) = I_{w\mathrm{loc}}(\Phi, x_0, V) = I_{w\mathrm{loc}}(\Phi, x, V) = I_{w\mathrm{loc}}(\Psi_\Phi, x, V),$$

for any $x \in O_{x_0}$, which completes the proof of Lemma 4.4.7. $\square$

Let φ be an ε-approximation of Φ and let $\Psi_\Phi\colon X_0 \multimap X$ be given by $\Psi_\Phi(x) = D(\Phi(D(x,\varepsilon)),\varepsilon)$ for any $x \in X_0$. In addition, define $\chi\colon X_0 \times [0,1] \multimap X$ as follows

$$\chi(x,t) = \begin{cases} \Phi(x) & \text{if } t \in [0,1/3), \\ \Psi_\Phi(x) & \text{if } t \in [1/3,2/3], \\ \varphi(x) & \text{if } t \in (2/3,1]. \end{cases}$$

Following the same lines as in the proof of Lemma 4.4.7, one can easily prove that χ is upper semicontinuous. Define a local weighted carrier $I_{w\mathrm{loc}}\colon D(\chi) \to \mathbb{Q}$ for χ by

$$I_{w\mathrm{loc}}(\chi, V, (x,t)) = \begin{cases} I_{w\mathrm{loc}}(\Phi, V, x) & \text{if } t \in [0,1/3), \\ I_{w\mathrm{loc}}(\Psi_\Phi, V, x) & \text{if } t \in [1/3,2/3], \\ I_{w\mathrm{loc}}(\varphi, V, x) & \text{if } t \in (2/3,1], \end{cases} \tag{4.49}$$

for all $(\chi, V, (x,t)) \in D(\chi)$, where $\Psi_\Phi(x) := D(\Phi(D(x,\varepsilon)),\varepsilon)$ for any $x \in X_0$. Now observe that such a local weighted carrier has the following properties: existence and additivity. In order to prove the local invariance property of $I_{w\mathrm{loc}}$ we need to consider only two cases.

Case 1. Fix $x_0 \in X_0$ and let $t_0 = 1/3$. In addition, let V be an open subset of X such that $\chi(x_0, t_0) \cap \partial V = \emptyset$. Since Ψ_Φ is a weighted carrier, there exists an open neighbourhood O_{x_0} of x_0 such that

$$I_{w\mathrm{loc}}(\Psi_\Phi, V, x_0) = I_{w\mathrm{loc}}(\Psi_\Phi, V, x), \tag{4.50}$$

for any $x \in O_{x_0}$. Moreover, $(\Phi, V, x) \in D(\Psi_\Phi)$ for all $x \in O_{x_0}$. Consequently, by (4.48), one has

$$I_{w\mathrm{loc}}(\Psi_\Phi, V, x) = I_{w\mathrm{loc}}(\Phi, V, x), \tag{4.51}$$

for any $x \in O_{x_0}$. Thus, taking into account (4.49)–(4.51), one obtains

$$I_{w\mathrm{loc}}(\chi, V, (x_0, t_0)) = I_{w\mathrm{loc}}(\chi, V, (x, t)),$$

for all $(x, t) \in O_{x_0} \times (0, 2/3)$.

Case 2. Fix $x_0 \in X_0$ and let $t_0 = 2/3$. Let V be an open subset of X such that $\chi(x_0, t_0) \cap \partial V = \emptyset$. Since Ψ_Φ is a weighted carrier, there exists an open neighbourhood O_{x_0} of x_0 such that

$$I_{w\mathrm{loc}}(\Psi_\Phi, V, x_0) = I_{w\mathrm{loc}}(\Psi_\Phi, V, x) \quad \text{and} \quad \Psi_\Phi(x) \cap \partial V = \emptyset, \tag{4.52}$$

for all $x \in O_{x_0}$. Since φ is an ε-approximation of Φ, for all $x \in X_0$ and any piece C of $O_\varepsilon(\Psi(O_\varepsilon(x)))$, one has

$$I_{w\mathrm{loc}}(\varphi, C, x) = I_{w\mathrm{loc}}(\Phi, C, x). \tag{4.53}$$

Observe that $O_\varepsilon(\Phi(O_\varepsilon(x))) \cap \partial V = \emptyset$, for any $x \in O_{x_0}$, since $O_\varepsilon(\Phi(O_\varepsilon(x))) \subset \Psi_\Phi(x)$ and $\Psi_\Phi(x) \cap \partial V = \emptyset$. Consequently, any connected component C of $O_\varepsilon(\Phi(O_\varepsilon(x)))$, for all $x \in O_{x_0}$, is contained either in V or in $X \setminus \overline{V}$. Since X is locally connected, we infer that any connected component C of $O_\varepsilon(\Phi(O_\varepsilon(x)))$ is open in $O_\varepsilon(\Phi(O_\varepsilon(x)))$. We have

$$V \cap O_\varepsilon(\Phi(O_\varepsilon(x))) = \bigcup C_i^x,$$

where all the C_i^x's are connected components of $O_\varepsilon(\Phi(O_\varepsilon(x)))$ contained in V. Therefore,

$$\begin{aligned} I_{w\mathrm{loc}}(\Phi, V, x) &= I_{w\mathrm{loc}}(\Phi, V \cap O_\varepsilon(\Phi(O_\varepsilon(x))), x) \\ &= I_{w\mathrm{loc}}\Big(\Phi, \bigcup C_i^x, x\Big) = \sum_i I_{w\mathrm{loc}}(\Phi, C_i^x, x). \end{aligned} \tag{4.54}$$

Note that the compactness of $\Phi(x)$ and the existence property of $I_{w\mathrm{loc}}$ for Φ guarantees that the summation in (4.54) is finite. Hence, in view of (4.53), we obtain

$$\begin{aligned} \sum_i I_{w\mathrm{loc}}(\Phi, C_i^x, x) &= \sum_i I_{w\mathrm{loc}}(\varphi, C_i^x, x) = I_{w\mathrm{loc}}(\varphi, \bigcup C_i^x, x) \\ &= I_{w\mathrm{loc}}(\varphi, V \cap O_\varepsilon(\Phi(O_\varepsilon(x))), x) = I_{w\mathrm{loc}}(\varphi, V, x), \end{aligned}$$

for $x \in O_{x_0}$, where the last equality follows from the excision property of $I_{w\mathrm{loc}}$ for φ since $\varphi(x) \subset O_\varepsilon(\Phi(O_\varepsilon(x))) \subset V \cup X \setminus \overline{V}$. Consequently,

$$I_{w\mathrm{loc}}(\Phi, V, x) = I_{w\mathrm{loc}}(\varphi, V, x),$$

for $x \in O_{x_0}$. But

$$I_{w\mathrm{loc}}(\Psi_\Phi, V, x) = I_{w\mathrm{loc}}(\Phi, V, x),$$

for all $x \in X_0$, so

$$I_{w\mathrm{loc}}(\varphi, V, x) = I_{w\mathrm{loc}}(\Psi_\Phi, V, x), \tag{4.55}$$

for all $x \in O_{x_0}$. Thus, taking into account (4.49), (4.52) and (4.55), we obtain

$$I_{w\mathrm{loc}}(\chi, V, (x_0, 2/3)) = I_{w\mathrm{loc}}(\chi, V, (x, t))$$

for all $(x,t) \in O_{x_0} \times (1/3, 1)$, which completes the proof that the local invariance property of $I_{w\mathrm{loc}}$ holds for χ.

Now we shall show that $\chi \in \mathbb{A}_0(X_0 \times [0,1], X)$. Let $\eta > 0$ (of course, we can assume that $\eta < \varepsilon$). Let $\phi\colon X_0 \multimap X$ be an η-approximation of Φ. Then, by the definition of ε, there is a w-homotopy $\Upsilon\colon X_0 \times [1/3, 2/3] \multimap X$ such that

$$\Upsilon(\,\cdot\,, 1/3) = \phi(\,\cdot\,), \quad \Upsilon(\,\cdot\,, 2/3) = \varphi(\,\cdot\,), \quad \Upsilon(\,\cdot\,, t) \in a_w(\Phi, \delta) \quad \text{for } t \in [1/3, 2/3].$$

Define $\vartheta\colon X_0 \times [0,1] \multimap X$ by

$$\vartheta(x,t) = \begin{cases} \phi(x) & \text{if } t \in [0, 1/3), \\ \Upsilon(x,t) & \text{if } t \in [1/3, 2/3], \\ \varphi(x) & \text{if } t \in (2/3, 1]. \end{cases}$$

It is easy to see that ϑ is an η-approximation of χ. Moreover, from the construction of χ it follows that $x \notin \chi(x,t)$ for $x \in C, t \in [0,1]$ and, consequently, $\chi \in \mathbb{A}_C(X_0 \times [0,1], X)$. This completes the proof. □

Remark 4.4.8. Let us note that Theorem 4.4.6 allows us to construct the fixed point index for weighted carriers defined on a compact ANR. The details of the construction will appear in the forthcoming work of the present author.

4.4.1. Induced homomorphisms. In this section, our aim is to define the homomorphism induced in the Darbo homology for weighted carriers having positively acyclic values (with respect to the Čech homology).

We start with the following remark.

Remark 4.4.9. From now on, by a space we shall understand a metric ANR.

If X, Y are spaces, then by $\mathbb{M}(X,Y)$ we denote the class of all weighted carriers $\Phi\colon X \multimap Y$ with positively acyclic values (with respect to the Čech homology with coefficients in the field of rational numbers $\mathbb{Q}$) and such that $I_w(\Phi) \neq 0$.

Definition 4.4.10. Let X be a compact ANR and let $\Phi \in \mathbb{M}(X,Y)$. We define the induced homomorphism $\Phi_*\colon \mathbb{H}_*(X,\mathbb{Q}) \to \mathbb{H}_*(Y,\mathbb{Q})$ by

$$\Phi_* := \varphi_*,$$

where $\varphi \in a_w(\Phi, \delta_\Phi)$ (δ_Φ is given as in Corollary 4.3.15).

Notice that Corollary 4.3.15 guarantees that the above definition is correct.

Proposition 4.4.11. *Let X be a compact* ANR. *If two weighted carriers $\Phi, \Psi \in \mathbb{M}(X, Y)$ are joined by a weighted carrier $\Upsilon \in \mathbb{M}(X \times [0,1], Y)$, then $\Phi_* = \Psi_*$.*

Proof. Our proposition follows immediately from Definition 4.4.10, Corollary 4.3.12 and the w-homotopy invariance of the Darbo homology functor. □

Lemma 4.4.12. *Let $\Psi \in \mathbb{M}(Y, Z)$ and let $f\colon X \to Y$ be a continuous function, where X and Y are compact. Then $(\Psi \circ f)_* = \Psi_* \circ f_*$.*

Proof. By Proposition 4.1.7, there exists $\delta > 0$ such that if $\varphi \in a_w(\Psi, \delta)$, then $\varphi \circ f \in a_w(\Psi \circ f, \delta_{\Psi \circ f})$ ($\delta_{\Psi \circ f}$ is given as in Corollary 4.3.15). Let $\gamma := \min\{\delta, \delta_{\Psi \circ f}\}$. Then for every $\varphi_f \in a_w(\Psi \circ f, \delta_{\Psi \circ f})$ and $\varphi \in a_w(\Psi, \gamma)$ there is a w-map $\chi\colon X \times [0,1] \multimap Z$ such that $\chi(x, 0) = \varphi_f(x)$ and $\chi(x, 1) = \varphi \circ f(x)$ for any $x \in X$. Consequently, taking into account the w-homotopy invariance of the Darbo homology functor, we obtain

$$(\Psi \circ f)_* = (\varphi_f)_* = (\varphi \circ f)_* = \Psi_* \circ f_*,$$

which completes the proof. □

Lemma 4.4.13. *Let $\Psi \in \mathbb{M}(X, Y)$ and let $f\colon Y \to Z$ be a continuous function, where X and Y are compact. If $f \circ \Psi \in \mathbb{M}(X, Z)$, then $(f \circ \Psi)_* = f_* \circ \Psi_*$.*

Proof. Let $\varepsilon := \min\{\delta_\Psi, \delta_{f \circ \Psi}\}$. By Proposition 4.1.16, there exists $\delta \leqslant \varepsilon$ such that if $\varphi \in a_w(\Psi, \delta)$, then $f \circ \Psi \in a_w(f \circ \Psi, \varepsilon)$. Let $\varphi \in a_w(\Psi, \delta)$. Then

$$f_* \circ \Psi_* = f_* \circ \varphi_* = (f \circ \varphi)_* = (f \circ \Psi)_*,$$

which completes the proof. □

If E is a normed space, then by $\mathcal{U}(E)$ we denote the class of all open subsets of E. Let U be an open subset of a normed space E and let $\mathcal{C}(U)$ be the family of all compact ANRs contained in U, directed by the inclusion, i.e. $A \leqslant B$ if and only if $A \subset B$ for $A, B \in \mathcal{C}(U)$. Let us consider the following direct system

$$S_U = \{\mathbb{H}_*(\mathcal{D}, \mathbb{Q}), (i_{\mathcal{D}_2, \mathcal{D}_1})_*\colon \mathbb{H}_*(\mathcal{D}_1, \mathbb{Q}) \to \mathbb{H}_*(\mathcal{D}_2, \mathbb{Q}) \mid \mathcal{D}, \mathcal{D}_1, \mathcal{D}_2 \in \mathcal{C}(U)\}, \tag{4.56}$$

where the homomorphism $(i_{\mathcal{D}_2, \mathcal{D}_1})_*\colon \mathbb{H}_*(\mathcal{D}_1, \mathbb{Q}) \to \mathbb{H}_*(\mathcal{D}_2, \mathbb{Q})$ is induced by the inclusion $i_{\mathcal{D}_2, \mathcal{D}_1}\colon \mathcal{D}_1 \hookrightarrow \mathcal{D}_2$. Since the Darbo homology functor satisfies the axiom of compact carriers and since $\mathcal{C}(U)$ is cofinal in the family of all compact subsets of U, it follows that

$$\mathbb{H}_*(U, \mathbb{Q}) = \varinjlim_{\mathcal{D} \in \mathcal{C}(U)} \mathbb{H}_*(\mathcal{D}, \mathbb{Q}).$$

Given $\Psi \in \mathbb{M}(U, Y)$, we define the family of induced homomorphisms

$$\{(\Psi|\mathcal{D})_*\colon \mathbb{H}_*(\mathcal{D}, \mathbb{Q}) \to \mathbb{H}_*(Y, \mathbb{Q}) \mid \mathcal{D} \in \mathcal{C}(U)\}. \tag{4.57}$$

Lemma 4.4.14. *Under the above assumptions we have*

$$(\Psi|\mathcal{D}_1)_* = (\Psi|\mathcal{D}_2 \circ i_{\mathcal{D}_2,\mathcal{D}_1})_* = (\Psi|\mathcal{D}_2)_* \circ (i_{\mathcal{D}_2,\mathcal{D}_1})_*.$$

Proof. The conclusion of the lemma follows from Lemma 4.4.12. □

From the universal property of the direct limit of (4.56) (see Definition1.4.2) we get for the family (4.57) the unique homomorphism

$$\Psi_*^\infty := \varinjlim_{\mathcal{D}\in\mathcal{C}(U)} (\Psi|\mathcal{D})_*$$

satisfying the following condition:

$$\Psi_*^\infty \circ (i_{\mathcal{D}})_* = (\Psi|\mathcal{D})_*, \tag{4.58}$$

for any homomorphism $(i_{\mathcal{D}})_*\colon \mathbb{H}_*(\mathcal{D},\mathbb{Q}) \to \mathbb{H}_*(U,\mathbb{Q})$ induced by the inclusion $i_{\mathcal{D}}\colon \mathcal{D} \hookrightarrow U$, where $\mathcal{D} \in \mathcal{C}(U)$. Consequently, the above considerations allow us to give the following definition.

Definition 4.4.15. Let U be an open subset of a normed space E and let $\Psi \in \mathbb{M}(U,Y)$. Then the induced homomorphism $\Psi_*\colon \mathbb{H}_*(U,\mathbb{Q}) \to \mathbb{H}_*(Y,\mathbb{Q})$ is defined as follows

$$\Psi_* := \Psi_*^\infty.$$

Remark 4.4.16. Let $\Psi \in \mathbb{M}(U,Y)$. If some homomorphism $h_*\colon \mathbb{H}_*(U,\mathbb{Q}) \to \mathbb{H}_*(Y,\mathbb{Q})$ satisfies the following condition $h_* \circ (i_{\mathcal{D}})_* = (\Psi|\mathcal{D})_*$ for any homomorphism $(i_{\mathcal{D}})_*\colon \mathbb{H}_*(\mathcal{D},\mathbb{Q}) \to \mathbb{H}_*(U,\mathbb{Q})$ induced by the inclusion $i_{\mathcal{D}}\colon \mathcal{D} \hookrightarrow U$ and any $\mathcal{D} \in \mathcal{C}(U)$, then from the universal property of the direct limit (see Definition 1.4.2) we get $h_* = \Psi_*^\infty$.

Proposition 4.4.17. *Let U be an open subset of a normed space E and let $\Phi, \Psi \in \mathbb{M}(U,Y)$. If there exists a weighted carrier $\Upsilon \in \mathbb{M}(U \times [0,1], Y)$ such that $\Upsilon(\,\cdot\,,0) = \Phi(\,\cdot\,)$ and $\Upsilon(\,\cdot\,,1) = \Psi(\,\cdot\,)$, then $\Phi_* = \Psi_*$.*

Proof. Let $\mathcal{D} \in \mathcal{C}(U)$. Then, by Proposition 4.4.11, we obtain $(\Phi|\mathcal{D})_* = (\Psi|\mathcal{D})_*$. Consequently,

$$(\Phi)_*^\infty \circ (i_{\mathcal{D}})_* = (\Phi|\mathcal{D})_* = (\Psi|\mathcal{D})_* = (\Psi)_*^\infty \circ (i_{\mathcal{D}})_*$$

(see also (4.58)) for any homomorphism $(i_{\mathcal{D}})_*\colon \mathbb{H}_*(\mathcal{D},\mathbb{Q}) \to \mathbb{H}_*(U,\mathbb{Q})$ induced by the inclusion $i_{\mathcal{D}}\colon \mathcal{D} \hookrightarrow U$ and any $\mathcal{D} \in \mathcal{C}(U)$. Thus, by Remark 4.4.16 and Definition 4.4.15, we obtain $\Phi_* = \Psi_*$, which completes the proof. □

Proposition 4.4.18. *Let $\Psi \in \mathbb{M}(V, Y)$ and let $f: U \to V$ be a continuous function, where U and V are two open subsets of some normed spaces E_1 and E_2, respectively. Then $(\Psi \circ f)_* = \Psi_* \circ f_*$.*

Proof. First observe that $\Psi \circ f \in \mathbb{M}(U, Y)$. Let

$$f_*: \mathbb{H}_*(U, \mathbb{Q}) \to \mathbb{H}_*(V, \mathbb{Q}), \quad \Psi_*: \mathbb{H}_*(V, \mathbb{Q}) = \varinjlim_{\mathcal{D} \in \mathcal{C}(\mathbb{V})} \mathbb{H}_*(\mathcal{D}, \mathbb{Q}) \to \mathbb{H}_*(Y, \mathbb{Q}),$$
$$(\Psi \circ f)_*: \mathbb{H}_*(U, \mathbb{Q}) = \varinjlim_{\mathcal{D}' \in \mathcal{C}(U)} \mathbb{H}_*(\mathcal{D}', \mathbb{Q}) \to \mathbb{H}_*(Y, \mathbb{Q})$$

be induced homomorphisms. Now the proof will be divided into two steps.

Step 1. Let $Z \subset U$ be a compact ANR. We shall show that

$$\Psi_* \circ (f|Z)_* = (\Psi \circ f|Z)_*. \tag{4.59}$$

Since $f(Z)$ is a compact subset of V, there is, by Lemma 1.2.7, a compact ANR X such that $f(Z) \subset X \subset V$. It is clear that

$$(j_X \circ f|(Z, X))_* = (j_X)_* \circ (f|(Z, X))_*,$$

where $j_X: X \to V$ is the inclusion and $f|(Z, X)$ is the contraction of f to the pair (Z, X). Thus

$$\Psi_* \circ (f|Z)_* = \Psi_* \circ (j_X \circ f|(Z, X))_* = \Psi_* \circ (j_X)_* \circ (f|(Z, X))_*. \tag{4.60}$$

From Definition 4.4.15 and (4.58) it follows that

$$\Psi_* \circ (j_X)_* \circ (f|(Z, X))_* = (\Psi|X)_* \circ (f|(Z, X))_*. \tag{4.61}$$

Moreover, by Lemma 4.4.12, we have

$$(\Psi|X)_* \circ (f|(Z, X))_* = (\Psi|X \circ f|(Z, X))_* = (\Psi \circ f|Z)_*. \tag{4.62}$$

Consequently, taking into account (4.60)–(4.62), we obtain

$$\Psi_* \circ (f|Z)_* = (\Psi \circ f|Z)_*,$$

as required.

Step 2. Now we shall show that for any compact ANR $Z \subset U$ and for any inclusion $j_Z: Z \to U$ the following equality holds

$$(\Psi \circ f)_* \circ (j_Z)_* = \Psi_* \circ f_* \circ (j_Z)_*. \tag{4.63}$$

By Step 1, one has

$$\Psi_* \circ f_* \circ (j_Z)_* = \Psi_* \circ (f \circ j_Z)_* = (\Psi \circ f|Z)_*, \tag{4.64}$$

for any compact ANR $Z \subset U$. On the other hand, let us recall (see Definition 4.4.15 and (4.58) for $\Psi \circ f$) that the homomorphism $(\Psi \circ f)_*$ induced by $\Psi \circ f$ has the following property

$$(\Psi \circ f)_* \circ (j_Z)_* = (\Psi \circ f \circ j_Z)_* = (\Psi \circ f|Z)_*, \tag{4.65}$$

for any compact ANR $Z \subset U$. Consequently, in view of (4.64) and (4.65), we get (4.63). Hence, by Remark 4.4.16 (for $\Psi \circ f$), we deduce that

$$(\Psi \circ f)_* = \Psi_* \circ f_*,$$

which completes the proof. □

Proposition 4.4.19. *Let $\Psi \in \mathbb{M}(U, X)$ and let $f\colon X \to Y$ be a continuous function, where U is an open subset of a normed space E. If $f \circ \Psi \in \mathbb{M}(U, Y)$, then*

$$(f \circ \Psi)_* = f_* \circ \Psi_*.$$

Proof. Let $Z \subset U$ be a compact ANR and let $j_Z\colon Z \to U$ be the inclusion. Observe that (in view of Definition 4.4.15 and (4.58) for Ψ and $\Psi \circ f$) Ψ_* and $(f \circ \Psi)_*$ satisfy the following equalities

$$\Psi_* \circ (j_Z)_* = (\Psi|Z)_*, \tag{4.66}$$

$$(f \circ \Psi)_* \circ (j_Z)_* = ((f \circ \Psi)|Z)_*. \tag{4.67}$$

In view of (4.66), one has

$$f_* \circ \Psi_* \circ (j_Z)_* = f_* \circ (\Psi|Z)_*.$$

Since $\Psi|Z \in \mathbb{M}(Z, X)$ and $f \circ \Psi|Z \in \mathbb{M}(Z, Y)$, it follows from Lemma 4.4.13 that

$$(f \circ \Psi|Z)_* = f_* \circ (\Psi|Z)_*. \tag{4.68}$$

Thus, taking into account (4.66)–(4.68), we obtain

$$(f \circ \Psi)_* \circ (j_Z)_* = f_* \circ \Psi_* \circ (j_Z)_*,$$

for any compact ANR $Z \subset U$. Consequently, by Remark 4.4.16 (for $f \circ \Psi$), one has

$$(f \circ \Psi)_* = f_* \circ \Psi_*$$

as required. □

4.5. Fixed point theorems for w-carriers

In this section we would like to show how the technique of weighted approximations can be used to give a generalization of the Lefschetz fixed point theorem for weighted carriers obtained in [10].

In what follows, we use the following notations:

$\mathcal{A}$ — the class of all ANR's;

$\mathcal{K}$ — the class of all compact ANR's;

$\mathcal{U}$ — the class of all open subsets of normed spaces.

Let $\mathcal{P}$ be a distinguished class of spaces. Following [4], we write $\Phi \in \mathbb{M}(\mathcal{P})(X,Y)$ if there exists a factorization

$$D_\Phi\colon X = X_0 \overset{\Phi_1}{\multimap} X_1 \overset{\Phi_2}{\multimap} \cdots \overset{\Phi_n}{\multimap} X_n = Y \tag{4.69}$$

($n = n(\Phi)$ depends on Φ), where $\Phi = \Phi_n \circ \Phi_{n-1} \circ \ldots \circ \Phi_1$, $\Phi_i \in \mathbb{M}(X_{i-1}, X_i)$, $1 \leqslant i \leqslant n$, and $X_i \in \mathcal{P}$, for $i = 0, \ldots, n-1$, and $X_n \in \mathcal{A}$. In this case we say that D_Φ is a decomposition (in $\mathbb{M}(\mathcal{P})$) of Φ.

Definition 4.5.1. If $\Phi, \Psi \in \mathbb{M}(\mathcal{P})(X,Y)$ have decompositions D_Φ (see (4.69)) and

$$D_\Psi\colon X = X_0 \overset{\Psi_1}{\multimap} X_1 \overset{\Psi_2}{\multimap} \cdots \overset{\Psi_m}{\multimap} X_m = Y$$

then we say that the compositions D_Φ and D_Ψ are homotopic in $\mathbb{M}(\mathcal{P})$ if $n = m$, $X_i = X_i'$, and there is a map $\chi_i \in \mathbb{M}(X_{i-1} \times [0,1], X_i)$ with $\chi_i(\,\cdot\,,0) = \Phi_i$, $\chi_i(\,\cdot\,,1) = \Psi_i$, $1 \leqslant i \leqslant n$. The multivalued map $\chi\colon X \times [0,1] \multimap Y$ given by

$$\chi(x,t) := \chi_n \circ \overline{\chi}_{n-1} \circ \ldots \circ \overline{\chi}_1(x,t),$$

where $\overline{\chi}_i \in \mathbb{M}(X_{i-1} \times [0,1], X_i \times [0,1])$ is given by $\overline{\chi}_i(x,t) = \chi_i(x,t) \times \{t\}$ for $x \in X_{i-1}$, $t \in [0,1]$, $1 \leqslant i \leqslant n-1$, is called a homotopy (observe that $\chi(\,\cdot\,,0) = \Phi(\,\cdot\,)$, $\chi(\,\cdot\,,1) = \Psi(\,\cdot\,)$).

Definition 4.5.2. Let $\Psi \in \mathbb{M}(\mathcal{P})(X,Y)$ have two decompositions

$$D\colon X = X_0 \overset{\Psi_1}{\multimap} X_1 \overset{\Psi_2}{\multimap} \cdots \overset{\Psi_n}{\multimap} X_n = Y,$$
$$D'\colon X = X_0' \overset{\Psi_1'}{\multimap} X_1' \overset{\Psi_2'}{\multimap} \cdots \overset{\Psi_m'}{\multimap} X_m' = Y.$$

We say that D' dominates over D (written $D' > D$), if $n = m$ and, for each $1 \leqslant i \leqslant n$, there is $h_i\colon X_i \to X_i'$ with $h_0 = h_n = \mathrm{id}$ such that a diagram

$$\begin{array}{ccc} X_{i-1} & \overset{\Psi_i}{\multimap} & X_i \\ {\scriptstyle h_{i-1}}\downarrow & & \downarrow{\scriptstyle h_i} \\ X_{i-1}' & \underset{\Psi_i'}{\multimap} & X_i' \end{array}$$

commutes (i.e. $\Psi_i' \circ h_{i-1} = h_i \circ \Psi_i$) for $1 \leqslant i \leqslant n$. If $D > D'$ and $D' > D$ then we say that D and D' are equivalent.

Lemma 4.5.3. *Let $\Psi \in \mathbb{M}(\mathcal{K})(X,X)$ have a decomposition*

$$X = X_0 \xrightarrow{\Psi_1}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\ X_n = X.$$

Then $(\Psi_n)_* \circ \ldots \circ (\Psi_1)_*$ is a Leray endomorphism.

Proof. It follows from the fact that if a space X is a compact ANR, then $\mathbb{H}_*(X,\mathbb{Q})$ is of finite type (see Definition 1.6.7). □

Lemma 4.5.4. *Let $\Psi \in \mathbb{M}(\mathcal{K})(X,Y)$ have a decomposition*

$$X = X_0 \xrightarrow{\Psi_1}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\ X_n = Y.$$

In addition, let Y be a path-connected space and $X \subset Y$. Then

$$(\Psi_n)_{*0} \circ \ldots \circ (\Psi_1)_{*0}([\sigma]) = I_w(\Psi_n) \cdot I_w(\Psi_{n-1}) \cdot \ldots \cdot I_w(\Psi_1)[j_X \circ \sigma],$$

for all $[\sigma] \in \mathbb{H}_0(X,\mathbb{Q})$, where $j_X \colon X \to Y$ is the inclusion.

Proof. By Definition 4.4.10, one has $(\Psi_i)_* := (\varphi_i)_*$, for $1 \leqslant i \leqslant n$, where $\varphi_i \in a_w(\Psi_i.\delta_{\Psi_i})$. Then

$$\begin{aligned}
&(4.70) \qquad (\Psi_n)_{*0} \circ \ldots \circ (\Psi_1)_{*0}([\sigma]) = (\varphi_n)_{*0} \circ \ldots \circ (\varphi_1)_{*0}([\sigma])\\
&(4.71) \qquad = (\varphi_n \circ \ldots \circ \varphi_1)_{*0}([\sigma])\\
&(4.72) \qquad \overset{(*)}{=} I_w(\varphi_n \circ \ldots \circ \varphi_1)[j_X \circ \sigma]\\
&(4.73) \qquad \overset{2.2.7}{=} I_w(\varphi_n) \cdot \ldots \cdot I_w(\varphi_1)[j_X \circ \sigma],
\end{aligned}$$

where the equality $(*)$ follows from Lemma 2.3.13. On the other hand, by Remark 4.1.2, we have

$$I_w(\Psi_i) = I_w(\varphi_i), \tag{4.74}$$

for $i = 1, \ldots, n$. Consequently, the assertion follows by combining (4.70)–(4.73) with (4.74). □

Definition 4.5.5. Let $\Psi \in \mathbb{M}(\mathcal{K})(X,X)$ have a decomposition

$$D_\Psi \colon X = X_0 \xrightarrow{\Psi_1}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\ X_n = X.$$

Then we define the Lefschetz number of D_Ψ by

$$\lambda(D_\Psi) := \lambda((\Psi_n)_* \circ \ldots \circ (\Psi_1)_*).$$

Lemma 4.5.6. *Let $\Psi \in \mathbb{M}(\mathcal{K})(X,X)$ and D_Ψ be as in Definition* 4.5.5. *Then for every $\varepsilon > 0$ there exists $\gamma > 0$ such that*

$$(\varphi_n \circ \ldots \circ \varphi_1)(x) \subset O_\varepsilon(\Psi(O_\varepsilon(x))) \quad \text{for each } x \in X,$$
$$I_w(\varphi_n \circ \ldots \circ \varphi_1) \neq 0,$$

provided that $\varphi_i \in a_w(\Psi_i, \gamma)$, for any $1 \leqslant i \leqslant n$.

Proof. The first assertion follows from Lemma 4.1.6, while the second assertion follows from the following facts

$$\begin{aligned} I_w(\Psi_i) &= I_w(\varphi_i) \neq 0 \quad \text{for } 1 \leqslant i \leqslant n, && \text{(Remark 4.1.2)} \\ I_w(\varphi_n \circ \ldots \circ \varphi_1) &= I_w(\varphi_n) \cdot \ldots \cdot I_w(\varphi_1). && \text{(Proposition 2.2.7)} \end{aligned}$$ □

Moreover, from Lemma 4.5.4 and Definition 4.5.5 it follows the following corollary.

Corollary 4.5.7. *Let $\Psi \in \mathbb{M}(\mathcal{K})(X,X)$ and D_Ψ be as in Definition* 4.5.5. *In addition, let X be an acyclic space* (*with respect to the Darbo homology with rational coefficients*). *Then*

$$\lambda(D_\Psi) = I_w(\Psi_n) \cdot I_w(\Psi_{n-1}) \cdot \ldots \cdot I_w(\Psi_1).$$

Now we are able to prove the following theorem.

Theorem 4.5.8. *Let $\Psi \in \mathbb{M}(\mathcal{K})(X,X)$. If*

$$D_\Psi \colon X = X_0 \xrightarrow{\Psi_1}\!\!\circ\, X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\, X_n = X$$

is a decomposition of Ψ, then $\lambda(D_\Psi) \neq 0$ implies that Ψ has a fixed point.

Proof. Suppose that $x \notin \Psi(x)$ for all $x \in X$. Then from Corollary 4.1.14 it follows that there exists $\delta_0 > 0$ such that if $\varphi \colon X \multimap X$ is a weighted map with $\Gamma_\varphi \subset O_{\delta_0}(\Gamma_\Psi)$, then $\mathrm{Fix}(\varphi) = \emptyset$. Let $\gamma > 0$ be given as in Lemma 4.5.6 according to D_Ψ and δ_0. Moreover, in view of Definition 4.4.10, one has

$$\Psi_{n*} \circ \ldots \circ \Psi_{1*} = \varphi_{n*} \circ \ldots \circ \varphi_{1*} = (\varphi_n \circ \ldots \circ \varphi_1)_*,$$

where $\varphi_i \in a_w(\Psi_i, \gamma_0)$, $1 \leqslant i \leqslant n$, and $\gamma_0 := \min\{\gamma, \delta_0, \delta_{\Psi_1}, \delta_{\Psi_2}, \ldots, \delta_{\Psi_n}\}$. Then

$$0 \neq \lambda(D_\Psi) = \lambda((\varphi_n \circ \ldots \circ \varphi_1)_*) \tag{4.75}$$
$$\mathrm{Fix}(\varphi_n \circ \ldots \circ \varphi_1) = \emptyset. \tag{4.76}$$

Now, taking into account (4.75) and (4.76), we obtain a contradiction with Theorem 2.5.1, which completes the proof. □

Lemma 4.5.9. *If $\Psi \in \mathbb{M}(\mathcal{K})(X,X)$ has two decompositions*

$$D_\Psi\colon X = X_0 \xrightarrow{\Psi_1}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\ X_n = X,$$
$$D'_\Psi\colon X = X'_0 \xrightarrow{\Psi'_1}\!\!\circ\ X'_1 \xrightarrow{\Psi'_2}\!\!\circ \cdots \xrightarrow{\Psi'_n}\!\!\circ\ X'_n = X$$

such that $D_\Psi < D'_\Psi$, then $\lambda(D_\Psi) = \lambda(D'_\Psi)$.

Proof. Without loss of generality we may assume that $n = 3$. Since $D_\Psi < D'_\Psi$, there is a continuous map $h_i\colon X_i \to X'_i$ with $h_0 = h_3 = id_X$, $1 \leqslant i \leqslant 3$, such that the following diagram

$$\begin{tikzcd} X_{i-1} \arrow[r, "\Psi_i"] \arrow[d, "h_{i-1}"'] & X_i \arrow[d, "h_i"] \\ X'_{i-1} \arrow[r, "\Psi'_i"'] & X'_i \end{tikzcd} \tag{4.77}$$

commutes, for $1 \leqslant i \leqslant 3$. By (4.77), Lemma 4.4.12 and Lemma 4.4.13, we obtain

$$\begin{aligned}(\Psi'_3)_* \circ (\Psi'_2)_* \circ (\Psi'_1)_* &= (\Psi'_3)_* \circ (\Psi'_2)_* \circ (h_1 \circ \Psi_1)_* \\ &= (\Psi'_3)_* \circ (\Psi'_2)_* \circ (h_1)_* \circ (\Psi_1)_* = (\Psi'_3)_* \circ (\Psi'_2 \circ h_1)_* \circ (\Psi_1)_* \\ &= (\Psi'_3)_* \circ (h_2 \circ \Psi_2)_* \circ (\Psi_1)_* = (\Psi'_3)_* \circ (h_2)_* \circ (\Psi_2)_* \circ (\Psi_1)_* \\ &= (\Psi'_3 \circ h_2)_* \circ (\Psi_2)_* \circ (\Psi_1)_* = (\Psi_3)_* \circ (\Psi_2)_* \circ (\Psi_1)_*.\end{aligned}$$

Consequently, we get

$$\lambda(D_\Psi) = \lambda((\Psi_3)_* \circ (\Psi_2)_* \circ (\Psi_1)_*) = \lambda((\Psi'_3)_* \circ (\Psi'_2)_* \circ (\Psi'_1)_*) = \lambda(D'_\Psi),$$

which completes the proof. □

From Lemma 4.5.9 it follows immediately the following corollary.

Corollary 4.5.10. *Let $\Psi \in \mathbb{M}(\mathcal{K})(X,X)$. If Ψ has a decomposition*

$$\begin{aligned} D_\Psi\colon X = X_0 \xrightarrow{\Psi_1}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \\ \xrightarrow{\Psi_{k-1}}\!\!\circ\ X_{k-1} \xrightarrow{h_k} X_k \xrightarrow{\Psi_{k+1}}\!\!\circ\ X_{k+1} \xrightarrow{\Psi_{k+2}}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\ X_n = X,\end{aligned}$$

where $h_k\colon X_{k-1} \to X_k$ is a continuous function, then

$$\widetilde{D}_\Psi\colon X = X_0 \xrightarrow{\Psi_1}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_{k-1}}\!\!\circ\ X_{k-1} \xrightarrow{\Psi_{k+1}\circ h_k}\!\!\circ\ X_{k+1} \xrightarrow{\Psi_{k+2}}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\ X_n = X$$

is a decomposition of Ψ and $\lambda(D_\Psi) = \lambda(\widetilde{D}_\Psi)$.

We can give the following definition.

Definition 4.5.11. Let $\Psi \in \mathbb{M}(\mathcal{U})(X,X)$ be a compact multivalued map. In addition, let Ψ have a decomposition

$$D_\Psi \colon X = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = X.$$

Then we define the Lefschetz number of D_Ψ by

$$\Lambda(D_\Psi) := \Lambda((\Psi_n)_* \circ \ldots \circ (\Psi_1)_*).$$

The above definition is correct since the following lemma holds.

Lemma 4.5.12. *Let Ψ and D_Ψ be as in Definition* 4.5.11*. Then $(\Psi_n)_* \circ \ldots \circ (\Psi_1)_*$ is a Leray endomorphism.*

Proof. Let $K := \overline{\Psi(X)}$. Then, by Lemma 1.2.7, there exists a compact ANR Z such that $K \subset Z \subset X$. We have the following diagram:

$$\begin{tikzcd}
\mathbb{H}_*(Z,\mathbb{Q}) \arrow[r, "(j_Z)_*"] \arrow[d, "(\Psi'_n)_*\circ\ldots\circ(\Psi_1|Z)_*"'] & \mathbb{H}_*(X,\mathbb{Q}) \arrow[d, "(\Psi_n)_*\circ\ldots\circ(\Psi_1)_*"] \arrow[dl, "(\Psi'_n)_*\circ\ldots\circ(\Psi_1)_*"'] \\
\mathbb{H}_*(Z,\mathbb{Q}) \arrow[r, "(j_Z)_*"'] & \mathbb{H}_*(X,\mathbb{Q})
\end{tikzcd}$$

where $j_Z \colon Z \to X$ is the inclusion and $\Psi'_n := \Psi_n|(X_{n-1}, Z)$. Commutativity of the lower triangle follows from the fact that $(j_Z)_* \circ (\Psi'_n)_* = (\Psi_n)_*$. Moreover, commutativity of the upper triangle follows form Definition 4.4.15 and the fact that $(\Psi_1)_* \circ (j_Z)_* = (\Psi_1|Z)_*$. Consequently, by Proposition 1.6.9, it follows that $(\Psi'_n)_* \circ \ldots \circ (\Psi_1)_*$ is a Leray endomorphism since $(\Psi'_n)_* \circ \ldots \circ (\Psi_1|Z)_*$ is a Leray endomorphism (by Lemma 4.5.3), which completes the proof. $\square$

In the proof of the next lemma we shall make use of the following remark.

Remark 4.5.13. Let $\Psi \in \mathbb{M}(\mathcal{U})(U,Y)$ have a decomposition

$$D_\Psi \colon U = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = Y.$$

Let

$$S_0 = \{\mathbb{H}_0(Z,\mathbb{Q}), (i_{Z_2,Z_1})_* \colon \mathbb{H}_0(Z_1,\mathbb{Q}) \to \mathbb{H}_0(Z_2,\mathbb{Q}) \mid Z, Z_1, Z_2 \in \mathcal{C}(U)\}.$$

be a direct system. Consider also a family of homomorphisms

$$\{(\Psi_n)_{*0} \circ \ldots \circ (\Psi_1|Z)_{*0} \colon \mathbb{H}_0(Z,\mathbb{Q}) \to \mathbb{H}_0(Y,\mathbb{Q})\}_{Z\in\mathcal{C}(X)}.$$

If some homomorphism $h \colon \mathbb{H}_0(U,\mathbb{Q}) \to \mathbb{H}_0(Y,\mathbb{Q})$ satisfies the following condition

$$h \circ (i_{\mathcal{D}})_{*0} = (\Psi_n)_{*0} \circ \ldots \circ (\Psi_1|\mathcal{D})_{*0}$$

for any homomorphism $(i_{\mathcal{D}})_{*0} \colon \mathbb{H}_0(\mathcal{D}, \mathbb{Q}) \to \mathbb{H}_0(U, \mathbb{Q})$ induced by the inclusion $i_{\mathcal{D}} \colon \mathcal{D} \hookrightarrow U$ and any $\mathcal{D} \in \mathcal{C}(U)$, then from the universal property of the direct limit (see Definition 1.4.2) one obtains

$$h = (\Psi_n)_{*0} \circ \ldots \circ (\Psi_1)_{*0}.$$

Lemma 4.5.14. *Let $\Psi \in \mathbb{M}(\mathcal{U})(X, Y)$ have a decomposition*

$$X = X_0 \overset{\Psi_1}{\multimap} X_1 \overset{\Psi_2}{\multimap} \cdots \overset{\Psi_n}{\multimap} X_n = Y.$$

In addition, let Y be a path-connected space and $X \subset Y$. Then

$$(\Psi_n)_{*0} \circ \ldots \circ (\Psi_1)_{*0}([\sigma]) = I_w(\Psi_n) \cdot I_w(\Psi_{n-1}) \cdot \ldots \cdot I_w(\Psi_1)[j_X \circ \sigma],$$

for all $[\sigma] \in \mathbb{H}_0(X, \mathbb{Q})$, where $j_X \colon X \to Y$ is the inclusion.

Proof. Let $h \colon \mathbb{H}_0(X, \mathbb{Q}) \to \mathbb{H}_0(Y, \mathbb{Q})$ be a homomorphism defined by

$$h([\sigma]) = I_w(\Psi_n) \cdot \ldots \cdot I_w(\Psi_1)[j_X \circ \sigma],$$

for all $[\sigma] \in \mathbb{H}_0(X, \mathbb{Q})$. In addition, let $Z \subset X$ be a compact ANR and let $j_Z \colon Z \to X$ be the inclusion. Then we have the following commutative diagrams

$$\begin{tikzcd} \mathbb{H}_0(Z,\mathbb{Q}) \arrow[r, "(j_Z)_{*0}"] \arrow[dr, "\widetilde{D}_*"'] & \mathbb{H}_0(X,\mathbb{Q}) \arrow[d, "D_{\Psi *}"] \\ & \mathbb{H}_0(X,\mathbb{Q}) \end{tikzcd} \tag{4.78}$$

$$\begin{tikzcd} \mathbb{H}_0(Z,\mathbb{Q}) \arrow[r, "(j_Z)_{*0}"] \arrow[dr, "\widetilde{D}_*"'] & \mathbb{H}_0(X,\mathbb{Q}) \arrow[d, "h"] \\ & \mathbb{H}_0(X,\mathbb{Q}) \end{tikzcd} \tag{4.79}$$

where $D_{\Psi *} = (\Psi_n)_{*0} \circ \ldots \circ (\Psi_1)_{*0}$ and $\widetilde{D}_* = (\Psi_n)_{*0} \circ \ldots \circ (\Psi_1|Z)_{*0}$. The commutativity of the diagram (4.78) follows from the following equality

$$(\Psi_1)_{*0} \circ (j_Z)_{*0} = (\Psi_1|Z)_{*0},$$

which holds by Definition 4.4.15 and (4.58). Moreover, by Lemma 4.5.4, we have

$$\begin{aligned} (\Psi_n)_{*0} \circ \ldots \circ (\Psi_1|Z)_{*0}([\tau]) &= I_w(\Psi_n) \cdot \ldots \cdot I_w(\Psi_1|Z)[j_Z \circ \tau] \\ &= I_w(\Psi_n) \cdot \ldots \cdot I_w(\Psi_1)[j_Z \circ \tau], \end{aligned}$$

for all $[\tau] \in \mathbb{H}_0(Z, \mathbb{Q})$. Since

$$h \circ (j_Z)_{*0}([\tau]) = h([j_Z \circ \tau]) = I_w(\Psi_n) \cdot \ldots \cdot I_w(\Psi_1)[j_Z \circ \tau],$$

it follows that

$$\widetilde{D}_*[\tau] = h \circ (j_Z)_{*0}([\tau]),$$

for all $[\tau] \in \mathbb{H}_0(Z, \mathbb{Q})$, this proves the commutativity of the diagram (4.79).

Consider now a family of homomorphisms

$$\{(\Psi_n)_{*0} \circ \ldots \circ (\Psi_1|Z)_{*0} \colon \mathbb{H}_0(Z, \mathbb{Q}) \to \mathbb{H}_0(Y, \mathbb{Q}) \mid Z \in \mathcal{C}(X)\}$$

and a direct system

$$S_0 = \{\mathbb{H}_0(Z, \mathbb{Q}), (i_{Z_2,Z_1})_* \colon \mathbb{H}_0(Z_1, \mathbb{Q}) \to \mathbb{H}_0(Z_2, \mathbb{Q}) \mid Z, Z_1, Z_2 \in \mathcal{C}(X)\}.$$

Consequently, taking into account the above considerations and Remark 4.5.13, we obtain the desired conclusion. □

Corollary 4.5.15. *Let $\Psi \in \mathbb{M}(\mathcal{U})(X, X)$ and D_Ψ be as in Definition* 4.5.11. *In addition, let X be an acyclic space* (*with respect to the Darbo homology with rational coefficients*). *Then*

$$\Lambda(D_\Psi) = I_w(\Psi_n) \cdot I_w(\Psi_{n-1}) \cdot \ldots \cdot I_w(\Psi_1).$$

Proof. Follows immediately from Definition 4.5.11 and Lemma 4.5.14. □

Theorem 4.5.16. *Let $\Psi \in \mathbb{M}(\mathcal{U})(X, X)$ be a compact multlivalued map. If*

$$D_\Psi \colon X = X_0 \overset{\Psi_1}{\multimap} X_1 \overset{\Psi_2}{\multimap} \cdots \overset{\Psi_n}{\multimap} X_n = X$$

is a decomposition of Ψ, then

(a) *D_Ψ is a Lefschetz map;*
(b) *$\Lambda(D_\Psi) \neq 0$ implies that Ψ has a fixed point.*

Proof. From Lemma 4.5.12 it follows that D_Ψ is a Lefschetz map. Moreover, arguing as in the proof of Lemma 4.5.12, there exists a compact ANR Z such that $\overline{\Psi(X)} \subset Z$. Now consider the following diagram:

$$\begin{tikzcd}
\mathbb{H}_*(Z, \mathbb{Q}) \arrow[r, "(j_Z)_*"] \arrow[d, "(\Psi'_n)_* \circ \ldots \circ (\Psi_1|Z)_*"'] & \mathbb{H}_*(X, \mathbb{Q}) \arrow[d, "(\Psi_n)_* \circ \ldots \circ (\Psi_1)_*"] \arrow[dl, "(\Psi'_n)_* \circ \ldots \circ (\Psi_1)_*"'] \\
\mathbb{H}_*(Z, \mathbb{Q}) \arrow[r, "(j_Z)_*"'] & \mathbb{H}_*(X, \mathbb{Q})
\end{tikzcd}$$

where $j_Z \colon Z \to X$ is the inclusion and $\Psi'_n := \Psi_n|(X_{n-1}, Z)$. Then, by Propositions 1.6.8 and 1.6.9, we have

$$\lambda((\Psi'_n)_* \circ \ldots \circ (\Psi_1|Z)_*) = \Lambda(D_\Psi).$$

Now, if we assume that $\Lambda(D_\Psi) \neq 0$, then $\lambda((\Psi'_n)_* \circ \ldots \circ (\Psi_1|Z)_*) \neq 0$. Thus, from Theorem 4.5.8 we deduce that $\mathrm{Fix}((\Psi'_n)_* \circ (\Psi_{n-1})_* \circ \ldots \circ (\Psi_1|Z)_*) \neq \emptyset$, and hence, $\mathrm{Fix}(\Psi) \neq \emptyset$, which completes the proof. □

Corollary 4.5.17. *Let $\Psi \in \mathbb{M}(\mathcal{U})(X,X)$ be a compact multivalued map. If X is an acyclic* ANR *or $X \in$* AR*, then Ψ has a fixed point.*

Proof. Let

$$D_\Psi \colon X = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = X$$

be a decomposition of Ψ. Then, in view of Corollary 4.5.15, we have

$$\Lambda(D_\Psi) = I_w(\Psi_n) \cdot \ldots \cdot I_w(\Psi_1).$$

Since $\Psi \in \mathbb{M}(\mathcal{U})(X,X)$, it follows that $I_w(\Psi_n) \cdot \ldots \cdot I_w(\Psi_1) \neq 0$. Thus the conclusion follows from Theorem 4.5.16. $\square$

Lemma 4.5.18. *If a compact multivalued map $\Psi \in \mathbb{M}(\mathcal{U})(X,X)$ has two decompositions*

$$D_\Psi \colon X = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = X,$$
$$D'_\Psi \colon X = X'_0 \xrightarrow{\Psi'_1} X'_1 \xrightarrow{\Psi'_2} \cdots \xrightarrow{\Psi'_n} X'_n = X$$

such that $D_\Psi < D'_\Psi$, then $\Lambda(D_\Psi) = \Lambda(D'_\Psi)$.

Proof. As in the proof of Lemma 4.5.9 we may assume that $n = 3$. Moreover, since $D_\Psi < D'_\Psi$, there is a continuous map $h_i \colon X_i \to X'_i$ with $h_0 = h_3 = \mathrm{id}_X$, $1 \leqslant i \leqslant 3$, such that the following diagram

(4.80)
$$\begin{tikzcd} X_{i-1} \arrow[r, "\Psi_i"] \arrow[d, "h_{i-1}"'] & X_i \arrow[d, "h_i"] \\ X'_{i-1} \arrow[r, "\Psi'_i"'] & X'_i \end{tikzcd}$$

commutes for $1 \leqslant i \leqslant 3$. Now, using (4.80), Proposition 4.4.18 and Proposition 4.4.19 (see the proof of Lemma 4.5.9), one can deduce that

$$(\Psi'_3)_* \circ (\Psi'_2)_* \circ (\Psi'_1)_* = (\Psi_3)_* \circ (\Psi_2)_* \circ (\Psi_1)_*.$$

Hence $\Lambda(D_\Psi) = \Lambda(D_{\Psi'})$. $\square$

Corollary 4.5.19. *Let $\Psi \in \mathbb{M}(\mathcal{U})(X,X)$ be a compact multlivalued map. If Ψ has a decomposition*

$$D_\Psi \colon X \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{k-1}} X_{k-1} \xrightarrow{h_k} X_k \xrightarrow{\Psi_{k+1}} X_{k+1} \xrightarrow{\Psi_{k+2}} \cdots \xrightarrow{\Psi_n} X,$$

where $h_k \colon X_{k-1} \to X_k$ is a continuous function, then

$$\widetilde{D}_\Psi \colon X \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{k-1}} X_{k-1} \xrightarrow{\Psi_{k+1}\circ h_k} X_{k+1} \xrightarrow{\Psi_{k+2}} \cdots \xrightarrow{\Psi_n} X$$

is a decomposition of Ψ *and* $\Lambda(D_\Psi) = \Lambda(\widetilde{D}_\Psi)$.

Proof. This corollary follows immediately from Lemma 4.5.18. $\square$

Let $\Psi \in \mathbb{M}(\mathcal{A})(X,X)$ be a compact multivalued map and let Ψ have the following decomposition

$$D_\Psi\colon X = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = X.$$

Observe that the Arens–Eells theorem implies that there are normed spaces E_i and embeddings $s_i\colon X_i \to E_i$ (i.e. homeomorphisms onto closed subsets). Moreover, there are open sets $U_i \subset E_i$ and maps $r_i\colon U_i \to X_i$ such that $r_i \circ s_i = id_{X_i}$ (since $X_i \in \mathcal{A}$, for $1 \leqslant i \leqslant n$), where $E_0 = E_n$, $s_0 = s_n$, $U_0 = U_n$, $r_0 = r_n$. Let $\overline{\Psi}_i\colon U_{i-1} \multimap U_i$ be given by $\overline{\Psi}_i = s_i \circ \Psi_i \circ r_{i-1}$, $1 \leqslant i \leqslant n$. It is easy to see that $\overline{\Psi}_i \in \mathbb{M}(U_{i-1}, U_i)$. Consequently, we deduce that $\overline{\Psi} \in \mathbb{M}(\mathcal{U})(U,U)$ with the following decomposition:

$$D_{\overline{\Psi}}\colon U = U_0 \xrightarrow{\overline{\Psi}_1} U_1 \xrightarrow{\overline{\Psi}_2} \cdots \xrightarrow{\overline{\Psi}_n} U_n = U. \tag{4.81}$$

Remark 4.5.20. It is easy to see that $r_0 \circ \overline{\Psi} \circ s_0 = \Psi$, where $\Psi \in \mathbb{M}(\mathcal{A})(X,X)$ and $\overline{\Psi}$ is given by (4.81).

Definition 4.5.21. We define the Lefschetz number of D_Ψ by the formula

$$\Lambda(D_\Psi) := \Lambda(D_{\overline{\Psi}}),$$

where $D_{\overline{\Psi}}$ is given by (4.81).

Lemma 4.5.22. *The above definition is correct.*

Proof. Let us take E'_i, s'_i, U'_i, $r'_i\colon U'_i \to X_i$ and E''_i, s''_i, U''_i, $r''_i\colon U''_i \to X_i$ as above. In addition, let $\Psi'_i\colon U'_{i-1} \multimap U'_i$ and $\Psi''_i\colon U''_{i-1} \multimap U''_i$ be given by $\Psi'_i = s'_i \circ \Psi_i \circ r'_{i-1}$ and $\Psi''_i = s''_i \circ \Psi_i \circ r''_{i-1}$, $1 \leqslant i \leqslant n$, where $E'_0 = E'_n$, $s'_0 = s'_n$, $U'_0 = U'_n$, $r'_0 = r'_n$ and $E''_0 = E''_n$, $s''_0 = s''_n$, $U''_0 = U''_n$, $r''_0 = r''_n$. Consider also the following decompositions:

$$D_{\Psi'}\colon U'_0 \xrightarrow{\Psi'_1} U'_1 \xrightarrow{\Psi'_2} \cdots \xrightarrow{\Psi'_n} U'_n,$$
$$D_{\Psi''}\colon U''_0 \xrightarrow{\Psi''_1} U''_1 \xrightarrow{\Psi''_2} \cdots \xrightarrow{\Psi''_n} U''_n.$$

We shall show that $\Lambda(D_{\Psi'}) = \Lambda(D_{\Psi''})$. For this purpose, let

$$D_{\widetilde{\Psi}'}\colon U''_0 \xrightarrow{\widetilde{\Psi}'_1} U'_1 \xrightarrow{\Psi'_2} U'_2 \xrightarrow{\Psi'_3} U'_3 \longrightarrow \cdots \xrightarrow{\Psi'_{n-1}} U'_{n-1} \xrightarrow{\widetilde{\Psi}'_n} U''_n,$$

where $\widetilde{\Psi}'_1 := s'_1 \circ \Psi_1 \circ r''_0$ and $\widetilde{\Psi}'_n := s''_n \circ \Psi_n \circ r'_{n-1}$.

Now the proof will be divided into two steps.

Step 1. As a first step we will show that

$$\Lambda(D_{\Psi'}) = \Lambda(D_{\widetilde{\Psi}'}). \tag{4.82}$$

By Definition 4.5.11, we have

$$\Lambda(D_{\Psi'}) = \Lambda((\Psi'_n)_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (\Psi'_1)_*),$$
$$\Lambda(D_{\widetilde{\Psi}'}) = \Lambda((\widetilde{\Psi}'_n)_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (\widetilde{\Psi}'_1)_*).$$

Let us observe that

$$\begin{aligned}
&(\Psi'_n)_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (\Psi'_1)_* \\
&= (s'_n \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_* \\
&= (s'_n \circ \mathrm{id}_{X_n} \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_* \\
&= (s'_n \circ (r''_n \circ s''_n) \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_* \\
&= ((s'_n \circ r''_n) \circ s''_n \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_* \\
&= (s'_n \circ r''_n)_* \circ (s''_n \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_*,
\end{aligned}$$

where the last equality follows from the fact that

$$((s'_n \circ r''_n) \circ s''_n \circ \Psi_n \circ r'_{n-1})_* = (s'_n \circ r''_n)_* \circ (s''_n \circ \Psi_n \circ r'_{n-1})_*$$

(see Proposition 4.4.19). Consequently, we have

$$\begin{aligned}
&\Lambda((\Psi'_n)_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (\Psi'_1)_*) \\
&= \Lambda((s'_n \circ r''_n)_* \circ (s''_n \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_*).
\end{aligned}$$

Moreover, by Proposition 1.6.9, one obtains

$$\begin{aligned}
&\Lambda((s'_n \circ r''_n)_* \circ (s''_n \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_*) \\
&= \Lambda((s''_n \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_* \circ (s'_n \circ r''_n)_*) \\
&= \Lambda((s''_n \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_* \circ (s'_0 \circ r''_n)_*).
\end{aligned}$$

On the other hand, by Proposition 4.4.19, we have

$$\begin{aligned}
(s'_1 \circ \Psi_1 \circ r'_0)_* \circ (s'_0 \circ r''_n)_* &= (s'_1 \circ \Psi_1 \circ r'_0 \circ s'_0 \circ r''_n)_* \\
&= (s'_1 \circ \Psi_1 \circ \mathrm{id}_{X_0} \circ r''_n)_* = (s'_1 \circ \Psi_1 \circ r''_n)_*.
\end{aligned}$$

Consequently, taking into account the above considerations, we obtain

$$\begin{aligned}
&\Lambda(D_{\Psi'}) \\
&\quad= \Lambda((s'_n \circ r''_n)_* \circ (s''_n \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_*) \\
&\quad= \Lambda((s''_n \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r'_0)_* \circ (s'_0 \circ r''_n)_*) \\
&\quad= \Lambda((s''_n \circ \Psi_n \circ r'_{n-1})_* \circ (\Psi'_{n-1})_* \circ \ldots \circ (\Psi'_2)_* \circ (s'_1 \circ \Psi_1 \circ r''_n)_*) = \Lambda(D_{\widetilde{\Psi}'}),
\end{aligned}$$

as required.

Step 2. Since $D_{\widetilde{\Psi}'} > D_{\Psi''}$, it follows from Lemma 4.5.18 that

$$\Lambda(D_{\widetilde{\Psi}'}) = \Lambda(D_{\Psi''}). \tag{4.83}$$

Finally, taking into account (4.82) and (4.83), we get the desired conclusion, which completes the proof of the lemma. □

Theorem 4.5.23. *Let $\Psi \in \mathbb{M}(\mathcal{A})(X,X)$ be a compact multivalued map. If*

$$D_\Psi \colon X = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = X$$

is a decomposition of Ψ, then $\Lambda(D_\Psi) \neq 0$ implies that Ψ has a fixed point.

Proof. Let

$$D_{\overline{\Psi}} \colon U = U_0 \xrightarrow{\overline{\Psi}_1} U_1 \xrightarrow{\overline{\Psi}_2} \cdots \xrightarrow{\overline{\Psi}_n} U_n = U. \tag{4.84}$$

be a decomposition according to (4.81). Then, by Definition 4.5.21, $\Lambda(D_\Psi) = \Lambda(D_{\overline{\Psi}})$. Hence, if $\Lambda(D_\Psi) \neq 0$, then $\Lambda(D_{\overline{\Psi}}) \neq 0$. Consequently, in view of Theorem 4.5.16, we obtain $\mathrm{Fix}(\overline{\Psi}) \neq \emptyset$. Thus, by Remark 4.5.20, $\mathrm{Fix}(\Psi) \neq \emptyset$, which completes the proof. □

Corollary 4.5.24. *If X is an acyclic* ANR *or $X \in$* AR*, then any compact multivalued map $\Psi \in \mathbb{M}(\mathcal{A})(X,X)$ has a fixed point.*

Proof. Let

$$D_\Psi \colon X = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = X$$

be a decomposition of Ψ and let $D_{\overline{\Psi}} \colon U \multimap U$ be a decomposition defined by (4.81). Then, by Definition 4.5.21, one has

$$\Lambda(D_\Psi) = \Lambda((\overline{\Psi}_n)_* \circ \ldots \circ (\overline{\Psi}_1)_*).$$

Since $(\overline{\Psi}_n) \circ \ldots \circ (\overline{\Psi}_1) \in \mathbb{M}(\mathcal{U})(U,U)$, it follows from Corollary 4.5.15 that

$$\Lambda((\overline{\Psi}_n)_* \circ \ldots \circ (\overline{\Psi}_1)_*) = I_w(\overline{\Psi}_n) \cdot \ldots \cdot I_w(\overline{\Psi}_1).$$

Furthermore, by Proposition 3.2.15, we have $I_w(\overline{\Psi}_i) = I_w(\Psi_i)$, for $1 \leqslant i \leqslant n$. Consequently,

$$\Lambda((\overline{\Psi}_n)_* \circ \ldots \circ (\overline{\Psi}_1)_*) = I_w(\Psi_n) \cdot \ldots \cdot I_w(\Psi_1) \neq 0.$$

So the assertion follows from Theorem 4.5.23. □

4.6. Topological degree for compositions of w-carriers

Our principal aim in this section is to define and investigate the topological degree for compositions of weighted carriers having positively acyclic values. For this purpose we use the technique of weighted approximation on the graph. Our approach is based on the paper [4].

Before giving the formal definition of topological degree, we establish the following lemma.

Lemma 4.6.1. *Let X be a compact* ANR *and let Y be an* ANR. *In addition, let*

$$D_\Psi : X = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = Y$$

be a decomposition of $\Psi \in \mathbb{M}(\mathcal{K})(X,Y)$. Then for any $\delta > 0$ there is $\varepsilon((\Psi, D_\Psi), X) > 0$ such that, for each $0 < \varepsilon \leqslant \varepsilon((\Psi, D_\Psi), X)$, if $\varphi_i, \psi_i \in a_w(\Psi_i, \varepsilon)$ for $1 \leqslant i \leqslant n$, then there exists a weighted map $\chi : X \times [0,1] \multimap Y$ such that

$$\begin{aligned} \chi(x,0) &= \varphi_n \circ \ldots \circ \varphi_1(x) && \text{for each } x \in X, \\ \chi(x,1) &= \psi_n \circ \ldots \circ \psi_1(x) && \text{for each } x \in X, \\ \Gamma_{\chi_t} &\subset O_\delta(\Gamma_\Psi) && \text{for each } t \in [0,1]. \end{aligned}$$

Proof. It is sufficient to prove the assertion for $n = 2$ (i.e. $D_\Psi = \Psi_2 \circ \Psi_1$). The more general case can be proved analogously. Let $\delta > 0$. Then, by Lemma 4.1.6, there exists $0 < \gamma < \delta$ such that

$$O_\gamma(\Psi_2) O_\gamma(\Psi_1)(x) \subset O_\delta(\Psi_2 \circ \Psi_1(O_\delta(x))), \tag{4.85}$$

for all $x \in X$. Since $\Psi_1 \in \mathbb{A}(X, X_1)$ and $\Psi_2 \in \mathbb{A}(X_2, Y)$, there is $\xi \leqslant \gamma$ such that, for $\varphi_1', \varphi_1'' \in a_w(\Phi_1, \xi)$ and $\varphi_2', \varphi_2'' \in a_w(\Phi_2, \xi)$, there are w-maps

$$\phi_1 : X \times [0,1] \multimap X_1, \quad \phi_2 : X_1 \times [0,1] \multimap Y$$

such that

$$\phi_1(\,\cdot\,,0) = \varphi_1'(\,\cdot\,), \quad \phi_1(\,\cdot\,,1) = \varphi_1''(\,\cdot\,), \quad \phi_1(\,\cdot\,,t) \in a_w(\Psi_1, \gamma) \tag{4.86}$$

$$\phi_2(\,\cdot\,,0) = \varphi_2'(\,\cdot\,), \quad \phi_2(\,\cdot\,,1) = \varphi_2''(\,\cdot\,), \quad \phi_2(\,\cdot\,,t) \in a_w(\Psi_2, \gamma). \tag{4.87}$$

for all $t \in [0,1]$.

Let $\varepsilon((\Phi, D_\Phi), X) := \xi$. Then, in view of (4.85)–(4.87), we see that

$$\begin{aligned} (\phi_2)_t \circ (\phi_1)_t(x) &\subset (\phi_2)_t(O_\gamma(\Phi_1(O_\gamma(x)))) \\ &\subset O_\gamma(\Phi_2(O_\gamma(O_\gamma(\Phi_1(O_\gamma(x)))))) \subset O_\delta(\Phi_2 \circ \Phi_1(O_\delta(x))), \end{aligned}$$

where $(\phi_1)_t(x) = \phi_1(x,t)$, for all $t \in [0,1]$, $x \in X$, and $(\phi_2)_t(z) = \phi_2(z,t)$, for all $t \in [0,1]$, $z \in X_1$. Let $\overline{\phi}_1 : X \times [0,1] \multimap X_1 \times [0,1]$ be given by

$$\overline{\phi}_1(x,t) = \phi_1(x,t) \times \{t\} \quad \text{for all } t \in [0,1],\ x \in X.$$

Then $\chi := \phi_2 \circ \overline{\phi}_1$ is a desired w-homotopy, which completes the proof. □

Let E^m be a finite-dimensional normed space and let U be an open subset of E^m such that $\overline{U} \in \mathcal{K}$. Then we put

$$\mathbb{M}(\mathcal{K})_{\partial U}(\overline{U}, E^m) := \{\Psi \in \mathbb{M}(\mathcal{K})(\overline{U}, E) \mid \Psi_+^{-1}(0) \cap \partial U = \emptyset\}.$$

Now, from Proposition 4.1.15 and Lemma 4.6.1 we obtain the following corollary.

Corollary 4.6.2. *Let U be an open subset of E^m such that $\overline{U}$ is a compact* ANR. *Let*

$$D_\Psi\colon \overline{U} = X_0 \overset{\Psi_1}{\multimap} X_1 \overset{\Psi_2}{\multimap} \cdots \overset{\Psi_n}{\multimap} X_n = E^m$$

be a decomposition of $\Psi \in \mathbb{M}(\mathcal{K})_{\partial U}(\overline{U}, E^m)$. Then there exists $\varepsilon((\Psi, D_\Psi), U) > 0$ such that, for each $0 < \varepsilon \leqslant \varepsilon((\Psi, D_\Psi), U)$, if $\varphi_i, \psi_i \in a_w(\Psi_i, \varepsilon)$ for $1 \leqslant i \leqslant n$, then there exists a multivalued map $\chi\colon \overline{U} \times [0,1] \multimap \mathbb{R}^n$ such that

$$\begin{aligned}\chi(x,0) &= \varphi_n \circ \ldots \circ \varphi_1(x) \quad \textit{for each } x \in \overline{U},\\ \chi(x,1) &= \psi_n \circ \ldots \circ \psi_1(x) \quad \textit{for each } x \in \overline{U},\end{aligned}$$

$$\{x \in \overline{U} \mid 0 \in \chi(x,t) \textit{ for some } t \in [0,1]\} \cap \partial U = \emptyset.$$

Now we are able to formulate the definition of topological degree.

Definition 4.6.3. If $\Phi, \Psi \in \mathbb{M}(\mathcal{K})_{\partial U}(\overline{U}, E^m)$ have decompositions

$$\begin{aligned}D_\Phi\colon \overline{U} &= X_0 \overset{\Phi_1}{\multimap} X_1 \overset{\Phi_2}{\multimap} \cdots \overset{\Phi_n}{\multimap} X_n = E^m,\\ D_\Psi\colon \overline{U} &= X_0 \overset{\Psi_1}{\multimap} X_1 \overset{\Psi_2}{\multimap} \cdots \overset{\Psi_k}{\multimap} X_k = E^m,\end{aligned}$$

then we say that the compositions D_Φ and D_Ψ are *homotopic* in $\mathbb{M}(\mathcal{K})_{\partial U}$ if $n = k$, $X_i = X_i'$, and there is a map $\chi_i \in \mathbb{M}(X_{i-1} \times [0,1], X_i)$ with $\chi_i(\,\cdot\,,0) = \Phi_i$, $\chi_i(\,\cdot\,,1) = \Psi_i$, $1 \leqslant i \leqslant n$ such that a homotopy $\chi\colon \overline{U} \times [0,1] \multimap E$ given by

$$\chi(x,t) := \chi_n \circ \ldots \circ \overline{\chi}_1(x,t), \tag{4.88}$$

where $\overline{\chi}_i(x,t) = \chi_i(x,t) \times \{t\}$ for $x \in X_{i-1}$, $t \in [0,1]$, $1 \leqslant i \leqslant n-1$, satisfies the following condition

$$\{x \in \overline{U} \mid 0 \in \chi(x,t) \text{ for some } t \in [0,1]\} \cap \partial U = \emptyset.$$

Definition 4.6.4. Let U be an open subset of $\mathbb{R}^n$ such that $\overline{U}$ is a compact ANR. Let $\Psi \in \mathbb{M}(\mathcal{K})_{\partial U}(\overline{U}, \mathbb{R}^n)$ and let

$$D_\Psi\colon \overline{U} = X_0 \overset{\Psi_1}{\multimap} X_1 \overset{\Psi_2}{\multimap} \cdots \overset{\Psi_n}{\multimap} X_n = \mathbb{R}^n$$

be a decomposition of Ψ. We define a *topological degree* of (Ψ, D_Ψ) by the formula

$$\operatorname{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) := \deg(\varphi_n \circ \ldots \circ \varphi_1, U, \mathbb{R}^n),$$

where deg stands for the topological degree for weighted maps (see Chapter 2) and $\varphi_i \in a_w(\Psi_i, \varepsilon)$, $1 \leqslant i \leqslant n$, and $\varepsilon \leqslant \varepsilon((\Psi, D_\Psi), \overline{U})$.

The correctness of this definition follows from Corollary 4.6.2 and the w-homotopy invariance of the topological degree of weighted maps.

Remark 4.6.5. It should be noted that the topological degree of Ψ depends on its decomposition D_Ψ (see Example 4.6.6 below), and therefore we shall use the notation $\operatorname{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n)$ instead of $\operatorname{Deg}(\Psi, U, \mathbb{R}^n)$.

Example 4.6.6. Let $D(0,1) \subset \mathbb{R}^2$ be the closed disk with the centre at $0 \in \mathbb{R}^2$ and radius $r = 1$. Additionally, we shall identify $\mathbb{R}^2$ with the field of complex numbers. Thus, given $z \in \mathbb{R}^2$, we can write

$$z = |z|(\cos\alpha + i\sin\alpha),$$

where $|\cdot|$ denotes a modulus of a complex number. Let $\Psi\colon D(0,1) \multimap D(0,1)$ be defined by

$$\Psi(z) = \{|z|\cdot x \mid x \in S^1\}$$

for all $x \in D(0,1)$. In addition, let $\Phi\colon D(0,1) \multimap D(0,1)$, $f_1, f_2\colon D(0,1) \to \mathbb{R}^2$ be defined by

$$\Phi(|z|(\cos\alpha + \sin\alpha)) = \{|z|(\cos(\alpha+\beta) + i\sin(\alpha+\beta), \beta \in [0, 3\pi/4])\},$$
$$f_1(z) = z^2 \quad \text{and} \quad f_2(z) = z^3.$$

From Example 3.1.4 it follows that Ψ is a weighted carrier. Observe that $\Psi = f_1 \circ \Phi$ and $\Psi = f_2 \circ \Phi$. Let $id\colon D(0,1) \to D(0,1)$ be the identity map. It is easy to see that $\mathrm{id} \in a_w(\Phi, \varepsilon)$ for any $\varepsilon > 0$. Consequently, we have

$$\begin{aligned} \mathrm{Deg}((\Psi, f_1\circ\Phi), B(0,1), \mathbb{R}^2) &= \deg(f_1 \circ \mathrm{id}, B(0,1), \mathbb{R}^2) = 2, \\ \mathrm{Deg}((\Psi, f_2\circ\Phi), B(0,1), \mathbb{R}^2) &= \deg(f_2 \circ \mathrm{id}, B(0,1), \mathbb{R}^2) = 3. \end{aligned}$$

Proposition 4.6.7. *If $\Psi \in \mathbb{M}(\mathcal{K})_{\partial U}(\overline{U}, \mathbb{R}^n)$ has two decompositions*

$$D_\Psi\colon \overline{U} = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_m} X_m = \mathbb{R}^n, \tag{4.89}$$

$$D'_\Psi\colon \overline{U} = X'_0 \xrightarrow{\Psi'_1} X'_1 \xrightarrow{\Psi'_2} \cdots \xrightarrow{\Psi'_m} X'_m = \mathbb{R}^n \tag{4.90}$$

such that $D_\Psi < D'_\Psi$, then

$$\mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) = \mathrm{Deg}((\Psi, D'_\Psi), U, \mathbb{R}^n).$$

Proof. To prove this proposition one applies arguments similar to those used in the proof of Proposition 2.2 in [4]. The only difference is that in the proof of Proposition 4.6.7 one has to use the corresponding graph-approximation results presented in this work (see also the proof of Lemma 4.5.9). □

As an immediate consequence of the above proposition, we obtain

Corollary 4.6.8. *Let $\Psi \in \mathbb{M}(\mathcal{K})_{\partial U}(\overline{U}, \mathbb{R}^n)$. If Ψ has a decomposition*

$$D_\Psi\colon \overline{U} \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{k-1}} X_{k-1} \xrightarrow{h_k} X_k \xrightarrow{\Psi_{k+1}} X_{k+1} \xrightarrow{\Psi_{k+2}} \cdots \xrightarrow{\Psi_m} \mathbb{R}^n,$$

where $h_k\colon X_{k-1} \to X_k$ is a continuous function, then

$$\widetilde{D}_\Psi\colon \overline{U} \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{k-1}} X_{k-1} \xrightarrow{\Psi_{k+1}\circ h_k} X_{k+1} \xrightarrow{\Psi_{k+2}} \cdots \xrightarrow{\Psi_m} \mathbb{R}^n$$

is a decomposition of Ψ *and*

$$\mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) = \mathrm{Deg}((\Psi, \widetilde{D}_\Psi), U, \mathbb{R}^n).$$

Theorem 4.6.9. *Let* $\Psi, \Phi \in \mathbb{M}(\mathcal{K})_{\partial U}(\overline{U}, \mathbb{R}^n)$ *have decompositions*

$$D_\Psi\colon \overline{U} = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_m} X_m = \mathbb{R}^n,$$
$$D_\Phi\colon \overline{U} = Y_0 \xrightarrow{\Phi_1} Y_1 \xrightarrow{\Phi_2} \cdots \xrightarrow{\Phi_m} Y_m = \mathbb{R}^n,$$

respectively.

(a) (Existence) *If* $\mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) \neq 0$*, then* $0 \in \Psi(x)$ *for some* $x \in U$*.*

(b) (Additivity) *Let* U_1 *and* U_2 *be disjoint open subsets of* U *such that* $\Psi_+^{-1}(0) \subset U_1 \cup U_2$ *and let* $\widetilde{\Psi}_i$ *denote the restriction of* Ψ *to* $\overline{U}_i$*, then*

$$\mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) = \mathrm{Deg}((\widetilde{\Psi}_1, D_{\widetilde{\Psi}_1}), U_1, \mathbb{R}^n) + \mathrm{Deg}((\widetilde{\Psi}_2, D_{\widetilde{\Psi}_2}), U_2, \mathbb{R}^n),$$

where

$$D_{\widetilde{\Psi}_1}\colon \overline{U_1} \xrightarrow{\Psi_1|\overline{U_1}} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_m} X_m = \mathbb{R}^n,$$
$$D_{\widetilde{\Psi}_2}\colon \overline{U_2} \xrightarrow{\Psi_1|\overline{U_2}} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_m} X_m = \mathbb{R}^n.$$

(c) (Contraction) *If* $\Psi(\overline{U}) \subset \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ *and* $\overline{U_k} \in \mathrm{ANR}$ (22)*, then*

$$\mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) = \mathrm{Deg}((\pi_k \circ \Psi|\overline{U_k}, D'), U_k, \mathbb{R}^k),$$

where $\pi_k\colon \mathbb{R}^n \to \mathbb{R}^k$ *is the projection onto the first* k *coordinates,* $U_k = \pi_k(U \cap (\mathbb{R}^k \times \{0\}))$ *and*

$$D'\colon \overline{U_k} \xrightarrow{\Psi_1|\overline{U_m}} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{m-1}} X_{m-1} \xrightarrow{\pi_k \circ \Psi_m} \mathbb{R}^k.$$

(d) (Linearity) *Let* $\Upsilon = \alpha\Psi$ *and*

$$D_\Upsilon\colon \overline{U} = X_0 \xrightarrow{\alpha\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_m} X_m = \mathbb{R}^n.$$

Then

$$\mathrm{Deg}((\Upsilon, D_\Upsilon), U, \mathbb{R}^n) = \alpha \cdot \mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n).$$

(e) (Homotopy invariance) *If the decompositions* D_Ψ*,* D_Φ *are homotopic in* $\mathbb{M}(\mathcal{K})_{\partial U}$*, then*

$$\mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) = \mathrm{Deg}((\Phi, D_\Phi), U, \mathbb{R}^n).$$

(22) It is easy to see that if U is an open subset of $\mathbb{R}^n$ such that $\overline{U} \in \mathrm{ANR}$, then the set $\pi_k(U \cap (\mathbb{R}^m \times \{0\}))$ does not to be an ANR.

(f) (Linear invariance) *Let* $T\colon \mathbb{R}^n \to \mathbb{R}^n$ *be a linear isomorphism. Then*

$$\mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) = \mathrm{Deg}((T \circ \Psi \circ T^{-1}|T(\overline{U}), D'_\Psi), T(U), \mathbb{R}^n),$$

where

$$D'_\Psi\colon T^{-1}(\overline{U}) \xrightarrow{\Psi_1 \circ T} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{n-1}} X_{m-1} \xrightarrow{T^{-1} \circ \Psi_m} X_m = \mathbb{R}^n.$$

Proof. (a) Existence. Suppose that $\mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) \neq 0$. In addition, we can assume without loss of generality that Ψ is a composition of two weighted carriers, i.e. $\Psi = \Psi_2 \circ \Psi_1$. Take $\varepsilon_n = 1/n$ for $n > 1/\varepsilon((\Psi, D_\Psi), \overline{U})$. Then, by Lemma 4.1.6, for any ε_n there exists $\delta_n < \varepsilon_n$ such that

$$O_{\delta_n}(\Psi_2) O_{\delta_n}(\Psi_1)(x) \subset O_{\varepsilon_n}(\Psi_2 \circ \Psi_1(O_{\varepsilon_n}(x)))$$

for all $x \in \overline{U}$. Furthermore, we have

$$0 \neq \mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) = \deg(\psi_{\delta_n} \circ \varphi_{\delta_n}, U, \mathbb{R}^n), \tag{4.91}$$

where $\varphi_{\delta_n} \in a_w(\Psi_1, \delta_n), \psi_{\delta_n} \in a_w(\Psi_2, \delta_n)$. Consequently, it follows from the existence property of the topological degree for weighted maps that there exists a sequence $x_n \in U$ such that

$$0 \in \psi_{\delta_n} \circ \varphi_{\delta_n}(x_n) \subset O_{\delta_n}(\Psi_2) O_{\delta_n}(\Psi_1)(x_n) \subset O_{\varepsilon_n}(\Psi_2 \circ \Psi_1(O_{\varepsilon_n}(x_n))).$$

By the compactness of $\overline{U}$, (up to a subsequence) $x_n \to x_0$. Finally, the upper semicontinuity of Ψ implies that $0 \in \Psi(x_0)$ as required.

(b) Additivity. Let

$$\varepsilon = \min\{\varepsilon((\Psi, D_\Psi), \overline{U}), \varepsilon((\widetilde{\Psi}_1, D_{\widetilde{\Psi}_1}), \overline{U_1}), \varepsilon((\widetilde{\Psi}_2, D_{\widetilde{\Psi}_2}), \overline{U_2})\}.$$

So, by Proposition 4.1.7, there exists $\delta < \varepsilon$ such that if $\varphi \in a_w(\Psi_1, \delta)$, then $\varphi|\overline{U_i} \in a_w(\Psi_1|\overline{U_i}, \varepsilon)$ for $i = 1, 2$. Let $\varphi_i \in a_w(\Psi_i, \delta)$ for $i = 1, \dots, m$. Then, we have

$$\begin{aligned}
&\mathrm{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) = \deg(\varphi_m \circ \ldots \circ \varphi_1, U, \mathbb{R}^n) \\
&\overset{(2)}{=} \deg(\varphi_m \circ \ldots \circ \varphi_1|\overline{U_1}, U_1, \mathbb{R}^n) + \deg(\varphi_m \circ \ldots \circ \varphi_1|\overline{U_2}, U_2, \mathbb{R}^n) \\
&\overset{(3)}{=} \mathrm{Deg}((\widetilde{\Psi}_1, D_{\widetilde{\Psi}_1}), U_1, \mathbb{R}^n) + \mathrm{Deg}((\widetilde{\Psi}_2, D_{\widetilde{\Psi}_2}), U_2, \mathbb{R}^n),
\end{aligned}$$

where the equality (2) follows directly from the additivity property of the topological degree for weighted maps and the equality (3) holds by Definition 4.6.4.

(c) Contraction. Let $\varepsilon = \min\{\varepsilon((\Psi, D_\Psi), \overline{U}), \varepsilon((\pi_k \circ \Psi|\overline{U_k}, D'), \overline{U_k})\}$. By Corollary 4.1.8, there exists $\delta'_\varepsilon \leqslant \varepsilon$ such that if $\varphi_1 \in a_w(\Psi_1, \delta'_\varepsilon)$, then $\varphi_1|\overline{U_k} \in a_w(\Psi_1|\overline{U_k}, \varepsilon)$. Moreover, in view of Proposition 4.1.16, there is $\delta''_\varepsilon \leqslant \varepsilon$ such that if $\widetilde{\varphi}_m \in a_w(\pi_k \circ \Psi_m, \delta_\varepsilon)$, then $i \circ \widetilde{\varphi}_m \in a_w(\Psi_m, \varepsilon)$, where $i\colon \mathbb{R}^k \hookrightarrow \mathbb{R}^n$ is the inclusion. Let $\varepsilon_0 := \min\{\delta'_\varepsilon, \delta''_\varepsilon\}$. Let $\varphi_i \in a_w(\Psi_i, \varepsilon_0)$ for $1 \leqslant i \leqslant m-1$ and $\widetilde{\varphi}_m \in a_w(\pi_k \circ \Psi_m, \varepsilon_0)$. Thus, from Definition 4.6.4 we may conclude that

$$\begin{aligned}\operatorname{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) &= \deg((i \circ \widetilde{\varphi}_m) \circ \ldots \circ \varphi_1, U, \mathbb{R}^n) \\ &\overset{(*)}{=} \deg(\pi_k \circ (i \circ \widetilde{\varphi}_m) \circ \ldots \circ \varphi_1 | \overline{U_k}, U_k, \mathbb{R}^k) \\ &= \deg(\widetilde{\varphi}_m \circ \ldots \circ \varphi_1 | \overline{U_k}, U_k, \mathbb{R}^k) \\ &= \operatorname{Deg}((\pi_k \circ \Psi | \overline{U_k}, D'), U_k, \mathbb{R}^k),\end{aligned}$$

where the equality $(*)$ follows from the contraction property of the topological degree for weighted maps.

(d) Linearity. First we observe that $\varphi_1 \in a_w(\Psi_1, \varepsilon)$ if and only if $\alpha\varphi_1 \in a_w(\alpha\Psi_1, \varepsilon)$ and $\{x \in \overline{U} \mid 0 \in \Psi(x)\} = \{x \in \overline{U} \mid 0 \in \alpha\Psi(x)\}$. Let $\varphi_i \in a_w(\Psi_i, \varepsilon)$ for $1 \leqslant i \leqslant m$, where $\varepsilon \leqslant \varepsilon((\Psi, D_\Psi), \overline{U})$. Then, by Definition 4.6.4, we have

$$\begin{aligned}\operatorname{Deg}((\Upsilon, D_\Upsilon), U, \mathbb{R}^n) &= \deg(\varphi_m \circ \ldots \circ (\alpha\varphi_1), U, \mathbb{R}^n) \\ &\overset{(*)}{=} \alpha \cdot \deg(\varphi_m \circ \ldots \circ \varphi_1, U, \mathbb{R}^n) = \alpha \cdot \operatorname{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n),\end{aligned}$$

where the equality $(*)$ follows from the linearity property of the topological degree for weighted maps.

(e) Homotopy invariance. Let D_Ψ and D_Φ be homotopic in $\mathbb{M}(\mathcal{K})_{\partial U}$. Just as in the proof of the existence property we can assume that Ψ and Φ are compositions of two weighted carriers, i.e. $\Psi = \Psi_2 \circ \Psi_1$ and $\Phi = \Phi_2 \circ \Phi_1$. Under these assumptions, there exist two weighted carriers

$$\chi_1 \colon \overline{U_1} \times [0,1] \multimap X_1 \quad \text{and} \quad \chi_2 \colon X_1 \times [0,1] \multimap \mathbb{R}^n \quad (X_1 \in \mathcal{K})$$

such that

- $\chi_1(x,0) = \Psi_1(x)$, for all $x \in \overline{U}$, and $\chi_2(y,0) = \Psi_2(y)$, for all $y \in X_1$,
- $\chi_1(x,1) = \Phi_1(x)$, for all $x \in \overline{U}$, and $\chi_2(y,1) = \Phi_2(y)$, for all $y \in X_1$,
- $\{x \in \overline{U} \mid 0 \in \chi(x,t) \text{ for some } t \in [0,1]\} \cap \partial U = \emptyset$,

where $\chi \colon \overline{U} \times [0,1] \multimap \mathbb{R}^n$ is given by (4.88). From Proposition 4.1.15 it follows that there exists $\delta > 0$ such that if a weighted map $\varphi \colon \overline{U} \times [0,1] \multimap \mathbb{R}^n$ satisfies the following condition $\Gamma_\varphi \subset O_\delta(\Gamma_\chi)$, then

$$\{x \in \overline{U} \mid 0 \in \varphi(x,t) \text{ for some } t \in [0,1]\} \cap \partial U = \emptyset.$$

Consider the following decompositions

$$D'_\Psi \colon \overline{U} \xrightarrow{\overline{\chi}_1 \circ i_0} X_1 \times [0,1] \xrightarrow{\chi_2} \mathbb{R}^n,$$
$$D'_\Phi \colon \overline{U} \xrightarrow{\overline{\chi}_1 \circ i_1} X_1 \times [0,1] \xrightarrow{\chi_2} \mathbb{R}^n,$$

where $i_0\colon \overline{U} \to \overline{U} \times [0,1]$ and $i_1\colon \overline{U} \to \overline{U} \times [0,1]$ are given by $i_0(x) = (x,0)$ and $i_1(x) = (x,1)$, for all $x \in \overline{U}$, respectively. By Proposition 4.6.7, one obtains

$$\operatorname{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) = \operatorname{Deg}((\Psi, D'_\Psi), U, \mathbb{R}^n), \tag{4.92}$$

$$\operatorname{Deg}((\Phi, D_\Phi), U, \mathbb{R}^n) = \operatorname{Deg}((\Phi, D'_\Phi), U, \mathbb{R}^n). \tag{4.93}$$

We can assume that $\delta \leqslant \min\{\varepsilon((\Psi, D'_\Psi), \overline{U}), \varepsilon((\Phi, D'_\Phi), \overline{U})\}$. Now, we shall show that

$$\operatorname{Deg}((\Psi, D'_\Psi), U, \mathbb{R}^n) = \operatorname{Deg}((\Phi, D'_\Phi), U, \mathbb{R}^n).$$

By Lemma 4.1.6, there exists $0 < \gamma \leqslant \delta$ such that

$$O_\gamma(\chi_2) O_\gamma(\overline{\chi}_1)(x,t) \subset O_\delta(\chi_2 \circ \overline{\chi}_1(O_\delta((x,t)))),$$

for each $(x,t) \in \overline{U} \times [0,1]$. Furthermore, in view of Lemma 4.4.4, there exists $\varrho \leqslant \gamma$ such that if $\varphi_{\overline{\chi}_1} \in a_w(\overline{\chi}_1, \varrho)$, then

$$\varphi_{\overline{\chi}_1} \circ i_0 \in a_w(\overline{\chi}_1 \circ i_0, \gamma), \quad \varphi_{\overline{\chi}_1} \circ i_1 \in a_w(\overline{\chi}_1 \circ i_1, \gamma).$$

Let $\varphi_{\overline{\chi}_1} \in a_w(\overline{\chi}_1, \varrho)$ and $\varphi_{\chi_2} \in a_w(\chi_2, \varrho)$. Consequently, from the w-homotopy invariance of the topological degree for weighted maps it follows that

$$\deg(\varphi_{\chi_2} \circ \varphi_{\overline{\chi}_1} \circ i_0, U, \mathbb{R}^n) = \deg(\varphi_{\chi_2} \circ \varphi_{\overline{\chi}_1} \circ i_1, U, \mathbb{R}^n). \tag{4.94}$$

On the other hand, by Definition 4.6.4, we have

$$\operatorname{Deg}((\Psi, D'_\Psi), U, \mathbb{R}^n) = \deg(\varphi_{\chi_2} \circ \varphi_{\overline{\chi}_1} \circ i_0, U, \mathbb{R}^n) \tag{4.95}$$

and

$$\operatorname{Deg}((\Phi, D'_\Phi), U, \mathbb{R}^n) = \deg(\varphi_{\chi_2} \circ \varphi_{\overline{\chi}_1} \circ i_1, U, \mathbb{R}^n). \tag{4.96}$$

Consequently, taking into account (4.94)–(4.96), one obtains

$$\operatorname{Deg}((\Psi, D'_\Psi), U, \mathbb{R}^n) = \operatorname{Deg}((\Phi, D'_\Phi), U, \mathbb{R}^n). \tag{4.97}$$

Finally, in view of (4.92), (4.93) and (4.97), we get

$$\operatorname{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) = \operatorname{Deg}((\Phi, D_\Phi), U, \mathbb{R}^n),$$

as required.

(f) Linear invariance. We start with the following simple observation

$$\{x \in \overline{U} \mid 0 \in \Psi(x)\} \cap \partial U = \emptyset$$
$$\Leftrightarrow \{x \in T(\overline{U}) \mid 0 \in T \circ \Psi \circ T^{-1}(x)\} \cap \partial(T(U)) = \emptyset.$$

Let $\varepsilon = \min\{\varepsilon((\Psi, D_\Psi), \overline{U}), \varepsilon((T \circ \Psi \circ T^{-1}|T(\overline{U}), D'), T(\overline{U}))\}$. In view of Proposition 4.1.7, there exists $\delta'_\varepsilon \leqslant \varepsilon$ such that if $\varphi_1 \in a_w(\Psi_1, \delta'_\varepsilon)$, then $\varphi_1 \circ T^{-1}|T(\overline{U}) \in a_w(\Psi_1 \circ T^{-1}|T(\overline{U}), \varepsilon)$. Additionally, by Proposition 4.1.16, there exists $\delta''_\varepsilon \leqslant \varepsilon$

such that if $\varphi_m \in a_w(\Psi_m, \delta''_\varepsilon)$, then $T\circ\varphi_m \in a_w(T\circ\Psi_m, \varepsilon)$. Let $\delta_\varepsilon := \min\{\delta'_\varepsilon, \delta''_\varepsilon\}$ and let $\varphi_i \in a_w(\Psi_i, \delta_\varepsilon)$ for $1 \leqslant i \leqslant m$. Then, be Definition 4.6.4, we obtain

$$\begin{aligned}\operatorname{Deg}((\Psi, D_\Psi), U, \mathbb{R}^n) &= \deg(\varphi_m \circ \ldots \circ \varphi_1, U, \mathbb{R}^n)\\ &\overset{(*)}{=} \deg((T\circ\varphi_m)\circ \ldots \circ (\varphi_1 \circ T^{-1}|T(\overline{U})), T(U), \mathbb{R}^n)\\ &= \operatorname{Deg}((T\circ\Psi\circ T^{-1}|T(\overline{U})), D'), T(U), \mathbb{R}^n),\end{aligned}$$

where the equality $(*)$ follows immediately from the linear invariance of the topological degree for weighted maps. $\square$

We now extend the topological degree to any k-dimensional normed space E^k. Choose any linear isomorphism $T\colon \mathbb{R}^k \to E^k$.

Definition 4.6.10. Let U be an open subset of E^k such that $\overline{U}$ is a compact ANR. Let $\Psi \in \mathbb{M}(\mathcal{K})_{\partial U}(\overline{U}, E^k)$ and let

$$D_\Psi\colon \overline{U} = X_0 \xrightarrow{\Psi_1}\!\!\circ\, X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\, X_n = E^k \tag{4.98}$$

be a decomposition of Ψ. We define a topological degree of (Ψ, D_Ψ) by the formula

$$\operatorname{Deg}((\Psi, D_\Psi), U, E^k) := \operatorname{Deg}((T^{-1}\circ\Psi\circ T|T^{-1}(\overline{U}), D'), T^{-1}(U), \mathbb{R}^k),$$

where

$$D'\colon T^{-1}(\overline{U}) \xrightarrow{\Psi_1\circ T}\!\!\circ\, X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_{n-1}}\!\!\circ\, X_{n-1} \xrightarrow{T^{-1}\circ\Psi_n}\!\!\circ\, X_n = \mathbb{R}^k.$$

Lemma 4.6.11. *Definition* (4.98) *does not depend on the choice of a linear isomorphism $T\colon \mathbb{R}^k \to E^k$.*

Proof. Let $T_1\colon \mathbb{R}^k \to E^k$, $T_2\colon \mathbb{R}^k \to E^k$ be two linear isomorphisms. We shall show that

$$\begin{aligned}&\operatorname{Deg}((T_1^{-1}\circ\Psi\circ T_1|T_1^{-1}(\overline{U}), D_1), T_1^{-1}(U), \mathbb{R}^k)\\ &\qquad\qquad = \operatorname{Deg}((T_2^{-1}\circ\Psi\circ T_2|T_2^{-1}(\overline{U}), D_2), T_2^{-1}(U), \mathbb{R}^k),\end{aligned}$$

where

$$\begin{aligned}D_1: T_1^{-1}(\overline{U}) \xrightarrow{\Psi_1\circ T_1}\!\!\circ\, X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_{n-1}}\!\!\circ\, X_{n-1} \xrightarrow{T_1^{-1}\circ\Psi_n}\!\!\circ\, X_n = \mathbb{R}^k,\\ D_2: T_2^{-1}(\overline{U}) \xrightarrow{\Psi_1\circ T_2}\!\!\circ\, X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_{n-1}}\!\!\circ\, X_{n-1} \xrightarrow{T_2^{-1}\circ\Psi_n}\!\!\circ\, X_n = \mathbb{R}^k.\end{aligned}$$

Let $T_3\colon \mathbb{R}^k \to \mathbb{R}^k$ be given by $T_3 := T_1^{-1}\circ T_2$. In addition, let

$$D'_1\colon T_3^{-1}\circ T_1^{-1}(\overline{U}) \xrightarrow{\Psi'_1}\!\!\circ\, X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_{n-1}}\!\!\circ\, X_{n-1} \xrightarrow{\Psi'_n}\!\!\circ\, X_n = \mathbb{R}^k,$$

where $\Psi_1' = \Psi_1 \circ T_1 \circ T_3|T_3^{-1} \circ T_1^{-1}(\overline{U})$, $\Psi_n' = T_3^{-1} \circ T_1^{-1} \circ \Psi_n$. Then, by the linear invariance of Deg (see Theorem 4.6.9), one has

$$\begin{aligned}
&\mathrm{Deg}((T_1^{-1} \circ \Psi \circ T_1|T_1^{-1}(\overline{U}), D_1), T_1^{-1}(U), \mathbb{R}^k) \\
&\quad = \mathrm{Deg}((T_3^{-1} \circ T_1^{-1} \circ \Psi \circ T_1 \circ T_3|T_3^{-1} \circ T_1^{-1}(\overline{U}), D_1'), T_3^{-1} \circ T_1^{-1}(U), \mathbb{R}^k) \\
&\quad = \mathrm{Deg}((T_2^{-1} \circ \Psi \circ T_2|T_2^{-1}(\overline{U})), D_2, T_2^{-1}(U), \mathbb{R}^k),
\end{aligned}$$

where the last equality follows from the fact that $D_1' = D_2$ and $T_3^{-1} \circ T_1^{-1} = T_2^{-1}$. □

In the next theorem we collect some properties of Deg.

Theorem 4.6.12. *Let $\Psi, \Phi \in \mathbb{M}(\mathcal{K})_{\partial U}(\overline{U}, E^k)$ have decompositions*

$$\begin{aligned}
D_\Psi : \overline{U} = X_0 \xrightarrow{\Psi_1}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\ X_n = E^k, \\
D_\Phi : \overline{U} = Y_0 \xrightarrow{\Phi_1}\!\!\circ\ Y_1 \xrightarrow{\Phi_2}\!\!\circ \cdots \xrightarrow{\Phi_n}\!\!\circ\ Y_n = E^k,
\end{aligned}$$

respectively.

(a) (Existence) *If* $\mathrm{Deg}((\Psi, D_\Psi), U, E^k) \neq 0$, *then* $0 \in \Psi(x)$ *for some* $x \in U$.

(b) (Additivity) *Let U_1 and U_2 be disjoint open subsets of U such that $\Psi_+^{-1}(0) \subset U_1 \cup U_2$ and let $\widetilde{\Psi}_i$ denote the restriction of Ψ to $\overline{U}_i$, then*

$$\mathrm{Deg}((\Psi, D_\Psi), U, E^k) = \mathrm{Deg}((\widetilde{\Psi}_1, D_{\widetilde{\Psi}_1}), U_1, E^k) + \mathrm{Deg}((\widetilde{\Psi}_2, D_{\widetilde{\Psi}_2}), U_2, E^k),$$

where

$$\begin{aligned}
D_{\widetilde{\Psi}_1} : \overline{U_1} \xrightarrow{\Psi_1|\overline{U}_1}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\ X_n = E^k, \\
D_{\widetilde{\Psi}_2} : \overline{U_2} \xrightarrow{\Psi_1|\overline{U}_2}\!\!\circ\ X_1 \xrightarrow{\Phi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\ X_n = E^k.
\end{aligned}$$

(c) (Contraction) *Let E' be a subspace of E^k, U be an open subset of E^k and $U' := U \cap E'$. If $\Psi \in \mathbb{M}(\mathcal{K})_{\partial U}(\overline{U}, E')$ and $\overline{U'} \in$ ANR, then*

$$\mathrm{Deg}((j_{E'} \circ \Psi, D_{j_{E'} \circ \Psi}), U, E^k) = \mathrm{Deg}((\Psi|\overline{U'}, D'), U', E'),$$

where $j_{E'} : E' \to E^k$ is the inclusion and

$$\begin{aligned}
D_{j_{E'} \circ \Psi} : \overline{U} \xrightarrow{\Psi_1}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_{n-1}}\!\!\circ\ X_{n-1} \xrightarrow{j_{E'} \circ \Psi_n}\!\!\circ\ E^k, \\
D' : \overline{U'} \xrightarrow{\Psi_1|\overline{U'}}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_{n-1}}\!\!\circ\ X_{n-1} \xrightarrow{\Psi_n}\!\!\circ\ E'.
\end{aligned}$$

(d) (Linearity) *Let $\Upsilon = \alpha \Psi$ and*

$$D_\Upsilon : \overline{U} = X_0 \xrightarrow{\alpha\Psi_1}\!\!\circ\ X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\ X_n = E^k.$$

Then

$$\mathrm{Deg}((\Upsilon, D_\Upsilon), U, E^k) = \alpha \cdot \mathrm{Deg}((\Psi, D_\Psi), U, E^k).$$

(e) (Homotopy invariance) *If the decompositions* D_Ψ, D_Φ *are homotopic in* $\mathbb{M}(\mathcal{K})_{\partial U}$, *then*

$$\operatorname{Deg}((\Psi, D_\Psi), U, E^k) = \operatorname{Deg}((\Phi, D_\Phi), U, E^k).$$

Proof. This theorem follows immediately from Definition 4.6.10 and Theorem 4.6.9. $\square$

Now, we are going to show that some requirements of Definition 4.6.10 can be removed. More precisely, it turns out that there is no need to require U to be bounded subset of E^k.

Let E^k be a k-dimensional normed space and let U be an open subset of E^k. Then we put

$$\mathbb{M}(\mathcal{U})_0(U, E^k) := \{\Psi \in \mathbb{M}(\mathcal{U})(U, E^k) \mid \Psi_+^{-1}(0) \text{ is compact}\}.$$

Definition 4.6.13. If $\Phi, \Psi \in \mathbb{M}(\mathcal{U})_0(U, E^k)$ have decompositions

$$D_\Phi: U = X_0 \xrightarrow{\Phi_1}\!\!\circ\, X_1 \xrightarrow{\Phi_2}\!\!\circ \cdots \xrightarrow{\Phi_n}\!\!\circ\, X_n = E^k,$$
$$D_\Psi: U = X_0 \xrightarrow{\Psi_1}\!\!\circ\, X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_m}\!\!\circ\, X_m = E^k,$$

then we say that the compositions D_Φ and D_Ψ are *homotopic* in $\mathbb{M}(\mathcal{U})_0$ if $n = m$, $X_i = X_i'$, and there is a map $\chi_i \in \mathbb{M}(X_{i-1} \times [0,1], X_i)$ with $\chi_i(\,\cdot\,, 0) = \Phi_i$, $\chi_i(\,\cdot\,, 1) = \Psi_i$, $1 \leqslant i \leqslant n$, such that the following set

$$\{x \in U \mid 0 \in \chi(x,t) \text{ for some } t \in [0,1]\}$$

is compact, where $\chi: U \times [0,1] \multimap E^k$ is a homotopy given by

$$\chi(x,t) := \chi_n \circ \overline{\chi}_{n-1} \circ \ldots \circ \overline{\chi}_1(x,t)$$

with $\overline{\chi}_i(x,t) = \chi_i(x,t) \times \{t\}$ for $x \in X_{i-1}$, $t \in [0,1]$, $1 \leqslant i \leqslant n-1$.

Let $\Psi \in \mathbb{M}(\mathcal{U})_0(U, E^k)$. Since $\Psi_+^{-1}(0)$ is compact, it follows that there are open balls $B(x_1, \delta_1), \ldots, B(x_m, \delta_m)$ such that

$$\Psi_+^{-1}(0) \subset \bigcup_{i=1}^m B(x_i, \delta_i) \subset \bigcup_{i=1}^m \overline{B(x_i, \delta_i)} \subset U.$$

Let $\mathbb{U} := \bigcup_{i=1}^m B(x_i, \delta_i)$. Since $\overline{\mathbb{U}}$ is a finite union of compact convex sets, we deduce that $\overline{\mathbb{U}}$ is a compact ANR (see Theorem 1.2.4).

Definition 4.6.14. Let U be an open subset of E^k and $\Psi \in \mathbb{M}(\mathcal{U})_0(U, E^k)$. In addition, let

$$D_\Psi: U = X_0 \xrightarrow{\Psi_1}\!\!\circ\, X_1 \xrightarrow{\Psi_2}\!\!\circ \cdots \xrightarrow{\Psi_n}\!\!\circ\, X_n = E^k \tag{4.99}$$

be a decomposition of Ψ. We define a topological degree of (Ψ, D_Ψ) by the formula

$$\operatorname{Deg}((\Psi, D_\Psi), U, E^k) := \operatorname{Deg}((\Psi|\overline{\mathbb{U}}, D'_\Psi), \mathbb{U}, E^k), \tag{4.100}$$

where $\mathbb{U}$ is as above nad

$$D'_\Psi\colon \overline{\mathbb{U}} \xrightarrow{\Psi_1|\overline{\mathbb{U}}} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{n-1}} X_{n-1} \xrightarrow{\Psi_n} X_n = E^k.$$

Lemma 4.6.15. *Definition* (4.100) *does not depend on the choice of* $\mathbb{U}$.

Proof. Let $\Psi \in \mathbb{M}(\mathcal{U})_0(U, E^k)$ and let

$$D_\Psi\colon U \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{n-1}} X_{n-1} \xrightarrow{\Psi_n} X_n = E^k$$

be a decomposition of Ψ. In addition, let $\mathbb{U}_1$ and $\mathbb{U}_2$ be two open subsets of E^k such that

$$\begin{aligned}
\Psi_+^{-1}(0) \subset \mathbb{U}_1 &= \bigcup_{i=1}^{m_1} B(x_i, \delta_i) \subset \bigcup_{i=1}^{m_1} \overline{B(x_i, \delta_i)} \subset U,\\
\Psi_+^{-1}(0) \subset \mathbb{U}_2 &= \bigcup_{j=1}^{m_2} B(x'_i, \delta'_i) \subset \bigcup_{i=1}^{m_1} \overline{B(x'_j, \delta'_j)} \subset U.
\end{aligned}$$

Let $\mathbb{U}_3 := \mathbb{U}_1 \cup \mathbb{U}_2$. Now, using once again Theorem 1.2.4, we infer that $\overline{\mathbb{U}_3}$ is an ANR. Then, in view of the excision property of Deg (see Theorem 4.6.12), one has

$$\operatorname{Deg}((\Psi|\overline{\mathbb{U}_3}, D_3), \mathbb{U}_3, E^k) = \operatorname{Deg}((\Psi|\overline{\mathbb{U}_1}, D_1), \mathbb{U}_1, E^k), \tag{4.101}$$

$$\operatorname{Deg}((\Psi|\overline{\mathbb{U}_3}, D_3), \mathbb{U}_3, E^k) = \operatorname{Deg}((\Psi|\overline{\mathbb{U}_2}, D_2), \mathbb{U}_1, E^k), \tag{4.102}$$

where

$$\begin{aligned}
D_1 : \overline{\mathbb{U}_1} &\xrightarrow{\Psi_1|\overline{\mathbb{U}_1}} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{n-1}} X_{n-1} \xrightarrow{\Psi_n} X_n = E^k,\\
D_2 : \overline{\mathbb{U}_2} &\xrightarrow{\Psi_1|\overline{\mathbb{U}_1}} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{n-1}} X_{n-1} \xrightarrow{\Psi_n} X_n = E^k,\\
D_3 : \overline{\mathbb{U}_3} &\xrightarrow{\Psi_1|\overline{\mathbb{U}_3}} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{n-1}} X_{n-1} \xrightarrow{\Psi_n} X_n = E^k.
\end{aligned}$$

Consequently, taking into account (4.101)–(4.102), one obtains

$$\operatorname{Deg}((\Psi|\overline{\mathbb{U}_2}, D_2), \mathbb{U}_2, E^k) = \operatorname{Deg}((\Psi|\overline{\mathbb{U}_1}, D_1), \mathbb{U}_1, E^k),$$

which completes the proof. □

We conclude this section with the standard properties of the topological degree.

Theorem 4.6.16. *Let $\Psi, \Phi \in \mathbb{M}(\mathcal{U})_0(U, E^k)$ have decompositions*

$$D_\Psi : U = X_0 \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = E^k,$$
$$D_\Phi : U = Y_0 \xrightarrow{\Phi_1} Y_1 \xrightarrow{\Phi_2} \cdots \xrightarrow{\Phi_n} Y_n = E^k,$$

respectively.

(a) (Existence) *If* $\mathrm{Deg}((\Psi, D_\Psi), U, E^k) \neq 0$, *then* $0 \in \Psi(x)$ *for some* $x \in U$.

(b) (Additivity) *Let* U_1 *and* U_2 *be disjoint open subsets of* U *such that* $\Psi_+^{-1}(0) \subset U_1 \cup U_2$ *and let* $\widetilde{\Psi}_i$ *denote the restriction of* Ψ *to* $\overline{U}_i$, *then*

$$\mathrm{Deg}((\Psi, D_\Psi), U, E^k) = \mathrm{Deg}((\widetilde{\Psi}_1, D_{\widetilde{\Psi}_1}), U_1, E^k) + \mathrm{Deg}((\widetilde{\Psi}_2, D_{\widetilde{\Psi}_2}), U_2, E^k),$$

where

$$D_{\widetilde{\Psi}_1} : U_1 \xrightarrow{\Psi_1|U_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = E^k,$$
$$D_{\widetilde{\Psi}_2} : U_2 \xrightarrow{\Psi_1|U_2} X_1 \xrightarrow{\Phi_2} \cdots \xrightarrow{\Psi_n} X_n = E^k.$$

(c) (Contraction) *Let* E' *be a subspace of* E^k, U *be an open subset of* E^k *and* $U' := U \cap E'$. *If* $\Psi \in \mathbb{M}(\mathcal{U})_0(U, E')$, *then*

$$\mathrm{Deg}((j_{E'} \circ \Psi, D_{j_{E'} \circ \Psi}), U, E^k) = \mathrm{Deg}((\Psi|U', D'), U', E'),$$

where $j_{E'} : E' \to E^k$ *is the inclusion and*

$$D_{j_{E'} \circ \Psi} : U \xrightarrow{\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{n-1}} X_{n-1} \xrightarrow{j_{E'} \circ \Psi_n} E^k,$$
$$D' : U' \xrightarrow{\Psi_1|U'} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_{n-1}} X_{n-1} \xrightarrow{\Psi_n} E'.$$

(d) (Linearity) *Let* $\Upsilon = \alpha\Psi$ *and*

$$D_\Upsilon : U = X_0 \xrightarrow{\alpha\Psi_1} X_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_n} X_n = E^k.$$

Then $\mathrm{Deg}((\Upsilon, D_\Upsilon), U, E^k) = \alpha \cdot \mathrm{Deg}((\Psi, D_\Psi), U, E^k)$.

(e) (Homotopy invariance) *If the decompositions* D_Ψ, D_Φ *are homotopic in* $\mathbb{M}(\mathcal{U})_0$, *then*

$$\mathrm{Deg}((\Psi, D_\Psi), U, E^k) = \mathrm{Deg}((\Phi, D_\Phi), U, E^k).$$

Proof. This theorem follows from Definition 4.6.13 and Theorem 4.6.12. □

Remark 4.6.17. It is possible to define a topological degree for compositions of weighted carriers defined on open subsets of arbitrary normed sapce. The details will be given in a forthcoming paper of the present author.

CHAPTER 5

REMARKS ON THE NIELSEN FIXED POINT THEORY FOR WEIGHTED MAPS

In this chapter we are going to show that the Nielsen fixed point theory cannot be extended to the multivalued weighted case. More precisely, we show that there is no w-homotopy invariant for weighted maps $\varphi\colon X \multimap X$ defined on compact ANR's, denoted by $N_w(\varphi)$, with the following properties:

(a) $\#\mathrm{Fix}(\varphi) \geqslant N_w(\varphi)$,
(b) if $\varphi \sim_w \psi$, then $N_w(\varphi) = N_w(\psi)$,
(c) if f is a continuous single-valued function, then $N_w(f) = N(f)$, where $N(f)$ stands for the Nielsen number for single-valued continuous maps (see [8] where the definition and the properties of N are presented).

Let x_0 be a fixed point of S^0 and let

$$\Sigma_{x_0} S^0 := S^0 \times [0,1]/((S^0 \times \{0,1\}) \cup (\{x_0\} \times [0,1]))$$

be the reduced suspension of S^0. In addition, let $h\colon \Sigma_{x_0} S^0 \to S^1 \subset \mathbb{R}^2$ be a homomorphism given by

$$h([x,t]) = \begin{cases} e^{2\pi i t} & \text{if } x \neq x_0, \\ (1,0) & \text{if } x = x_0. \end{cases} \tag{5.1}$$

Lemma 5.0.1 (see [53] or [40])**.** *Let $\varphi, \psi\colon \Sigma_{x_0} S^0 \multimap \Sigma_{x_0} S^0$ be two weighted maps such that $\varphi([x_0,0]) = \psi([x_0,0]) = [x_0,0]$ and $I_w(\varphi) = I_w(\psi) = 0$. Then $\varphi * \psi \sim_w (\varphi \cup \psi)$, where*

$$\varphi * \psi([x,t]) = \begin{cases} \varphi([x,2t]) & \text{if } 0 \leqslant t \leqslant 1/2, \\ \psi([x,2t-1]) & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

Let $\alpha\colon \Sigma_{x_0} S^0 \to \Sigma_{x_0} S^0$ and $c\colon \Sigma_{x_0} S^0 \to \Sigma_{x_0} S^0$ be given by

$$\alpha([x,t]) = [x,t], \quad c([x,t]) = [x_0,t],$$

for $[x,t] \in S^1$. In addition, let $\beta, \gamma\colon \Sigma_{x_0} S^0 \multimap \Sigma_{x_0} S^0$ be defined by

$$\beta([x,t]) = \begin{cases} (\alpha \cup (-1)c)([x,2t]) & \text{if } 0 \leqslant t \leqslant 1/2, \\ (\alpha \cup (-1)c)([x,2t-1]) & \text{if } 1/2 \leqslant t \leqslant 1, \end{cases}$$

$$\gamma([x,t]) = \begin{cases} \beta([x,2t] & \text{if } 0 \leqslant t \leqslant 1/2, \\ \beta[x,2t-1] & \text{if } 1/2 \leqslant t \leqslant 1, \end{cases}$$

where $(-1)c\colon \Sigma_{x_0} S^0 \to \Sigma_{x_0} S^0$ is a weighted map with $I_w((-1)c) = -1$.

Proposition 5.0.2. *Let $\beta, \gamma\colon \Sigma_{x_0} S^0 \multimap \Sigma_{x_0} S^0$ be as above. Then $\gamma \sim_w \beta \cup \beta$ and hence $\gamma \cup c \sim_w \beta \cup \beta \cup c$*

Proof. This follows from Lemma 5.0.1. □

Lemma 5.0.3. *Let $\gamma\colon \Sigma_{x_0} S^0 \multimap \Sigma_{x_0} S^0, \alpha\colon \Sigma_{x_0} S^0 \to \Sigma_{x_0} S^0$ be as above. Then $\gamma \cup c \sim_w \tau$, where $\tau\colon \Sigma_{x_0} S^0 \to \Sigma_{x_0} S^0$ is given by*

$$\tau([x,t]) = \begin{cases} \alpha([x,4t] & \textit{if } 0 \leqslant t \leqslant 1/4, \\ \alpha[x,4t-1] & \textit{if } 1/4 \leqslant t \leqslant 1/2, \\ \alpha([x,4t-2] & \textit{if } 1/2 \leqslant t \leqslant 3/4, \\ \alpha([x,4t-3] & \textit{if } 3/4 \leqslant t \leqslant 1. \end{cases} \tag{5.2}$$

Proof. It is clear that $\gamma \cup c = \tau \cup 0 \cdot c$. Hence to prove the result it is enough to show that $\tau \cup (0 \cdot c) \sim_w \tau$. To this end, it is sufficient to consider the following w-homotopy

$$\theta(z,t) = \begin{cases} (\tau \cup (0 \cdot c))(z) & \text{if } 0 \leqslant t \leqslant 1/2, \\ \tau(z) & \text{if } 1/2 < t \leqslant 1, \end{cases}$$

which completes the proof. □

Proposition 5.0.4. *Let $\beta\colon \Sigma_{x_0} S^0 \multimap \Sigma_{x_0} S^0$ and $\tau, c\colon \Sigma_{x_0} S^0 \to \Sigma_{x_0} S^0$ be as above. Then $\tau \sim_w \beta \cup \beta \cup c$.*

Proof. This proposition follows immediately from Proposition 5.0.2 and Lemma 5.0.3. □

Lemma 5.0.5 ([8]). *Let X be a compact* ANR *and let $f\colon X \to X$ be a continuous function. In addition, let $h\colon Y \to X$ be a homomorphism. Then*

$$N(f) = N(h^{-1} \circ f \circ h).$$

Lemma 5.0.6 ([8]). *Let $\kappa\colon S^1 \to S^1 \subset \mathbb{C}$ be given by $\kappa(z) = z^4$. Then $N(\kappa) = 3$.*

It is easy to see that the following diagram

$$\begin{tikzcd} \Sigma_{x_0} S^0 \arrow[r, "\tau"] \arrow[d, "h"'] & \Sigma_{x_0} S^0 \arrow[d, "h"] \\ S^1 \arrow[r, "\kappa"'] & S^1 \end{tikzcd}$$

commutes, where τ and h are defined by (5.2) and (5.1), respectively. Since h is a homomorphism, it follows that

$$\tau = h^{-1} \circ \kappa \circ h. \tag{5.3}$$

Consequently, taking into account Lemmas 5.0.5 and 5.0.6 and (5.3), we obtain the following result.

Proposition 5.0.7. *Let* $\tau\colon \Sigma_{x_0} S^0 \to \Sigma_{x_0} S^0$ *be defined by* (5.2). *Then*

$$N(\tau) = 3.$$

Now, we are able to formulate and prove the main result.

Theorem 5.0.8. *The Nielsen fixed point theory for single-valued maps defined on compact* ANR*s cannot be extended to the class of multivalued weighted maps.*

Proof. Let $\beta, \tau, c\colon \Sigma_{x_0} S^0 \to \Sigma_{x_0} S^0$ be as above. Suppose, contrary to our claim, that such a w-homotopy invariant N_w exists. Consequently, from the w-homotopy invariance of N_w and Proposition 5.0.4 we obtain

$$N_w(\beta \cup \beta \cup c) = N_w(\tau).$$

Since τ is a single-valued map, it follows that

$$N_w(\tau) = N(\tau) = 3,$$

where the latter equality holds by Proposition 5.0.7. Hence, a weighted map $\beta \cup \beta \cup c$ should have at least three fixed points. On the other hand, it is easy to see that $\beta \cup \beta \cup c$ has only one fixed point, a contradiction. The proof of the theorem is complete.

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