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GROUP EXTENSION OF DYNAMICAL SYSTEMS IN ERGODIC THEORY AND TOPOLOGICAL DYNAMICS

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INTRODUCTION

Group extensions of dynamical systems form an important and efficiently explored class of extensions. This large class enjoys a great variety of useful properties that make considerations on them quite fruitful. The group extensions are considered in measure-theoretic ergodic theory and in topological dynamics as well. In both cases the results are formally similar however the methods are different. The methods applied in the study of the class of group extensions are usually of algebraic and topological character, therefore some differences in the methods may be observed on appearance (or not) of compactness of the group in the considered group extension. In applications it is rather difficult to exceed behind of the class of locally compact groups – the groups that admit a one-point compactification. This is one of the main reasons to limit the object of research to the locally compact groups, frequently even to the compact ones.

Let us now define more precisely the objects that will appear in this dissertation. By [2], an ergodic extension $\widetilde{T}\colon (Z,\mathcal{A},m)\to(Z,\mathcal{A},m)$ of an automorphism $T\colon (X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is of the form

$$\begin{aligned}\widetilde{T}\colon (X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\otimes\nu)&\to(X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\otimes\nu),\\ \widetilde{T}(x,y)&=(Tx,\psi(x)(y)),\end{aligned}\tag{1}$$

where $\psi\colon X\to \mathrm{Aut}(Y,\nu)$ is a measurable map (and ψ is called a *Rokhlin cocycle*). Some examples of Rokhlin cocycles can be obtained in the following way. First take G a locally compact second countable group and let $\varphi\colon X\to G$ be a cocycle. Then suppose that G acts measurably on $(Y,\mathcal{C},\nu)$ as $G\ni g\mapsto\gamma_g\in\Gamma=\{\gamma_g : g\in G\}$. Then let

$$T_{\varphi,\Gamma}\colon (X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\otimes\nu)\to(X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\otimes\nu)$$

be given by

$$T_{\varphi,\Gamma}(x,y)=(Tx,\gamma_{\varphi(x)}(y)).\tag{2}$$

The extensions of the form (2) seem to be a very particular case of the general situation (1). However, quite surprisingly, as noticed in [14], each Rokhlin extension (1) is isomorphic, as an extension, to (2); moreover, G may be taken countable and amenable.

In this dissertation we consider measure-theoretic dynamical systems and topological systems as well. In both cases the basic tools used in the study are similar: joinings (ergodic in ergodic theory, and either minimal or B-sets in topological dynamics), and the groups of essential values of cocycles. The corresponding results obtained in ergodic theory and in topological dynamics are comparable, not identical. Generally, the universe of group extensions in topological dynamics pictured in this dissertation turns out to be more diverse and containing few regularity – in contrast with the universe of measure-theoretic group extensions.

This dissertation consists of eight chapters. Chapter 2 and Chapter 3 deal with measure-theoretic ergodic theory, in Chapters 4 to 7 topological dynamics is explored, Chapter 8 compares some results and properties in measure-theoretic ergodic theory and in topological dynamics. Chapter 1 contains preliminary notions, definitions, useful facts and theorems applied in the sequel.

The results of Chapter 2 come from a joint with A. del Junco and M. Lemańczyk paper [44]. In [95], Veech proved a theorem describing factors of ergodic 2-fold simple automorphisms in terms of compact subgroups of the centralizer (see also [45]). The property 2-*fold simplicity* is defined by 2-joinings – invariant measures on Cartesian square of the given system, projecting onto the system as the original measures. In particular, each system is a factor any of its joining. In the 2-fold simplicity case, each ergodic 2-self-joining is either a graph measure or the product measure and this property is sufficient to describe all factors. But a graph measure, as a dynamical system, is isomorphic to the original system and the natural projection factor map is one-to-one a.s. with respect to the joining measure. In other words, a graph measure λ is one point extension of the base system X. In particular, the relative product $\lambda \times_X \lambda$ is ergodic. We will use this observation to define a new class of ergodic automorphisms, called semisimple automorphisms. An ergodic automorphism is called semisimple if for each its ergodic self-joining the automorphism corresponding to the self-joining is relatively weakly mixing with respect to the both marginal σ-algebras. It turns out that many classes of automorphisms previously studied are semisimple. Indeed, all discrete spectrum, 2-fold simple, direct products of minimal self-joinings, Gaussian–Kronecker automorphisms are semisimple. We exhibit a structure of factors of semisimple automorphisms; in particular, we prove that one can decompose a given factor map $X \to Y$ of a semisimple X into $X \to \widetilde{Y} \to Y$, where the extension $X \to \widetilde{Y}$ is relatively weakly mixing and $\widetilde{Y} \to Y$ is a compact group extension.

In order to study the structure of factors of a given automorphisms, we introduce the notion of a natural family of factors. A general factorization theorem for an automorphism X possessing a natural family of factors says that if Y is a factor of X then there exists a decomposition $X \to \widetilde{Y} \to Y$ for some natural factor $\widetilde{Y}$ with the remaining properties as above, i.e. $X \to \widetilde{Y}$ is relatively weakly mixing and $\widetilde{Y} \to Y$ is a compact group extension. We also explore ergodic compact group extensions of semisimple automorphisms. In Section 2.5 we describe ergodic joinings of such extensions. In Section 2.6 we apply the concept of a natural family of factors to give a description of factors of group extensions of 2-fold simple automorphisms, generalizing earlier results from [64] and [71]. Finally, we consider the conjecture that if, for an automorphism with a natural family of factors, all natural factors are coalescent then all factors so are. We give the positive answer in case of group extensions of rotations (Theorem 2.6.7).

Chapter 3 contains results from a joint with M. Lemaczyk and H. Nakada paper [66]. It is an important problem in ergodic theory to study classes of automorphisms with a "given" set of self-joinings, see [94]. Historically, such an approach was first presented in [86] by D. Rudolph, where the existence of automorphisms (so called MSJ) with a minimal structure of self-joinings was shown. A generalization of this notion appeared in [95] and then in [45] – the notion of 2-fold simplicity. A further generalization was proposed in Chapter 2, where the notion called semisimplicity was introduced. As proved in Chapter 2, such automorphisms have still strong ergodic properties, and in particular the structure of their factors can be easily described. Based on some earlier results of J.-P. Thouvenot, it was already remarked in Chapter 2 that some Gaussian automorphisms are semisimple (recall that Gaussian automorphisms are never 2-fold simple). In [68] a far reaching study of Gaussian automorphisms with a minimal (in the category of Gaussian automorphisms) set of self-joinings (called GAG) is presented. All GAG systems turn out to be semisimple.

Almost all examples of automorphisms presented above are weakly mixing. In fact, the only exception are ergodic rotations which are 2-fold simple but not weakly mixing. Being more precise, the MSJ property implies weak mixing, while in the class of 2-fold simple automorphisms we have: either such an automorphism is weakly mixing or it is a rotation (see [45]). In the class of semisimple automorphisms it is a question whether the existence of a discrete part in the spectrum forces a decomposition into direct product of the form "discrete spectrum automorphism $\times$ weakly mixing automorphism". The question is natural because it has been noticed in Chapter 2 that an ergodic distal automorphism is semisimple if and only if it is a rotation. It follows that more is true: since each ergodic automorphism is relatively weakly mixing over its maximal distal factor, if an ergodic automorphism is semisimple then it is relatively weakly mixing over its Kronecker factor (see Section 2.4).

In Chapter 3 we will construct semisimple weakly mixing extensions of irrational rotations. The main idea of the construction comes from some papers by D. Rudolph [88] and E. Glasner, B. Weiss [38]. Roughly, we fix a simple (or even semisimple) action of an Abelian locally compact second countable group that will serve as fiber automorphisms of a skew product whose base is an irrational rotation. When some assumptions on the relevant fiber cocycle are put then the skew product turn out to be semisimple (it cannot be 2-fold simple). In order to see that we have constructed a completely new class (in particular, no forementioned direct product decomposition exists) of semisimple automorphisms we use some recent results from [63]: the class we will consider is disjoint in the sense of Furstenberg from all weakly mixing automorphisms, on the other hand the automorphisms from this class are relatively weakly mixing extension of the base irrational rotation.

An essential part of this chapter is to show the existence of some cocycles over irrational rotations, taking values in Abelian locally compact second countable groups and having strong ergodic properties (see Section 3.3). Here, we consider two examples of well known (one being real-valued, described in Subsection 3.3.1, and the second, described in Subsection 3.3.2, integer-valued) cocycles over the rotation by an irrational α, where α has bounded partial quotients.

Chapter 4 is based on [36] (a joint paper with E. Glasner and A. Siemaszko). Given a dynamical system (either measure theoretical or topological), its family of factors can have a rich and complex structure. An interesting step towards a systematic classification of this family (in the measure theoretical case), was taken in Chapter 2. It was shown there that for an ergodic system $(X, \mathcal{B}, \mu, T)$, there always exists a unique minimal natural family of factors, $\mathcal{N}$, that includes all those factors arising from ergodic self-joinings and that has the following property: for every factor sub-algebra $\mathcal{A}$ of $\mathcal{B}$ there exists a natural cover $\widehat{\mathcal{A}} \supset \mathcal{A}$ such that the corresponding factor map from the factor defined by $\widehat{\mathcal{A}}$ to the one defined by $\mathcal{A}$, is a compact group extension. This natural subfamily of factors is strongly related to structure theory and can, in some cases, considerably simplify the study of the family of all factors. Two such cases are studied in Chapter 2: the case of an ergodic group extension of a group rotation and the case of what is called in Chapter 2 semisimple systems. In some cases the minimal family of natural factors coincides with the entire family of factors (see [26], where this is shown for Bernoulli systems; see also [33]). However, even in these cases, the mere fact that $\mathcal{N}$ consists of all factors is of great interest.

The purpose of the investigation is to study several analogies of the notion of natural family of factors in topological dynamics, or more precisely, in the theory of minimal dynamical systems (called here minimal flows). Our first approach (expounded in Section 4.2) is perhaps the most straightforward one. We define a self-joining of the minimal system (X, T) to be any minimal subset of $X \times X$. With this definition of joining we mimic the definition of natural family of factors given in Chapter 2. As in the measure theoretical case we get in this way the existence of a unique smallest natural family of factors $\mathcal{N}$, that includes all the factors arising from self-joinings. This family $\mathcal{N}$ has the property that for every factor Y of X there exists a unique natural cover $\widetilde{Y} \in \mathcal{N}$ such that the map $\widetilde{Y} \to Y$ is a regular (but not necessarily a group) extension. We characterize the least member of the smallest natural family of factors of the system (X, T) as the unique maximal regular factor of (X, T). At the end of Section 4.2 we consider an alternative approach. We call a non-empty subset W of $X \times X$ a B-set if it is closed, $T \times T$ invariant, topologically transitive and such that the union of the minimal subsets of W is dense in W. Now we enlarge the class of admissible self-joinings by allowing all B-sets to be joinings. The corresponding notion of a natural family of factors now has the property that the map $\pi\colon \widetilde{Y} \to Y$ is again regular and in addition admits a decomposition $\pi = \omega \circ \kappa$, where κ is a group extension and ω a proximal one. We show that for this type of natural family the Kronecker factor is the least member of the smallest natural family. By a result of Bronstein, for a PI-flow X, and in particular for a distal flow, a B-set in $X \times X$ is necessarily minimal ([10], see also [7]), and the two notions of natural families coincide. Unlike the situation in ergodic theory, the largest zero entropy factor of a minimal system need not be natural.

Section 4.4, motivated by [44], [64], [71] and [92], deals with natural families of factors for a minimal group extension of a group rotation. We show by direct methods that for such a flow the family $\{X/F : F$ a closed normal subgroup of $G\}$, is a natural family of factors for the G-extension (X, T) of the group rotation $Z = X/G$ (Proposition 4.4.8).

In Chapter 5, based on a joint with M. Lemaczyk paper [65], we will study dynamical properties of extensions by topological cocycles taking values in a locally compact group G. Such a subject is under research mainly in the measure-theoretic setting. K. Schmidt in [89] developed the idea of an essential value of a cocycle as a tool to investigate ergodic properties of extensions by cocycles with values in G. It is also well known that one of the consequences of Dye's theorem ([15]) on orbital equivalence is that the first cohomology group (of cocycles taking values in a fixed locally compact group) is the same for all ergodic systems. In particular, if $G = \mathbb{R}$ then for each ergodic system there exist non-regular (in the sense of [89]) cocycles (these are cocycles ϕ which are not cohomologous to any cocycle taking values in the group of essential values of ϕ). Of course the structure of such cocycles is far from being understood.

In the topological setup Dye's theorem is no more valid and we may hope that for some classes of topological systems the structure of cocycles will be much more clear. In this paper we make first steps following this direction and show that a particularly easy classification appears if we study real cocycles over minimal rotations T. We show that if such a cocycle ϕ is not regular then necessarily $\int \phi \, d\mu \neq 0$ (μ is a unique T-invariant measure). In this case the partition into orbits of the corresponding skew product

$$T_\phi: X \times \mathbb{R} \to X \times \mathbb{R}, \qquad T_\phi(x,t) = (Tx, \phi(x) + t)$$

is the decomposition of T_ϕ into minimal components. Otherwise, when $\int \phi\, d\mu = 0$, the cocycle is regular and moreover, either it is a topological coboundary or T_ϕ is topologically ergodic.

We should emphasize that such a classification is no longer valid for strictly ergodic systems that are not rotations (see Section 5.5).

In [28] and [38], S. Glasner and B. Weiss studied the problem of topological disjointness from the class WM of all minimal weakly mixing topological systems. They considered the following situation. Assume that $T: X \to X$ is a minimal rotation and let $(S_t)_{t\in\mathbb{R}}$ be a weakly mixing flow on a compact Hausdorff space Y. Assume that $\phi: X \to \mathbb{R}$ is a topological cocycle and let

$$\widetilde{T}_\phi: X \times Y \to X \times Y, \qquad \widetilde{T}_\phi(x,y) = (Tx, S_{\phi(x)}(y).$$

Then for ϕ running through a certain generic set of cocycles, the following results have been proved: $\widetilde{T}_\phi$ is not PI but it is disjoint from all weakly mixing transformations, if moreover, $(S_t)_{t\in\mathbb{R}}$ is regular then $\widetilde{T}_\phi$ is a multiplier of the class of topological systems disjoint from WM. We introduce the notion of universally ergodic cocycles and show that the two disjointness results hold under the only assumption of universal ergodicity of ϕ.

The purpose of Chapter 6, containing results of [72], is to describe groups of essential values of continuous cocycles (over minimal rotations) taking values in locally compact Abelian groups whose dual is connected. Recall that in the measure-theoretic context the notion of essential values over ergodic actions has been introduced by K. Schmidt ([89]). In topological dynamics a parallel theory has been developed by G. Atkinson [6], although only for extensions by $\mathbb{R}^m$. An adaptation of Schmidt's concepts was considered in Chapter 5. It was suggested that a full description of all groups of essential values is possible over minimal rotations and indeed, in Chapter 5 it has been shown that the only possible groups of essential values for cocycles taking values in $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$. Here we go further and study the case of cocycles taking values in locally compact Abelian groups without compact subgroups. By a classification of LCA groups ([77, Theorem 25]), such a group is of the form $\mathbb{R}^m \oplus D$, where D is discrete, torsion-free. Our main result shows that a group of essential values is then contained in $\mathbb{R}^m$ and moreover, it must be a linear subspace of $\mathbb{R}^m$. We will also prove that an $\mathbb{R}^m$-extension of a minimal rotation is conservative iff the cocycle has zero mean (with respect to Haar measure), and that topological non-ergodicity of a conservative $\mathbb{R}^m$-extension leads to a functional equation. Both these results are essential improvements of the paper by G. Atkinson [6].

In this chapter we also propose the notion of a regularity of a topological cocycle. Namely, we say that a cocycle φ is regular if it is cohomologous to a cocycle taking all values in the group $E(\varphi)$ of essential values of φ. In this case infinity is not an essential value of the quotient map $\widetilde{\varphi}: X \to G/E(\varphi)$, however the converse does not appear to hold in general. Due to our analysis of possible groups of essential values we show that a cocycle (over a minimal rotation) is regular iff the corresponding extension is conservative.

We should like to emphasize that our analysis of cocycle group extensions essentially exploits the fact that we study cocycles over minimal rotations. It has been already noticed in Chapter 5 that the group of essential values may be $\mathbb{Z}$ for some minimal extensions by $\mathbb{R}$, however in this case the base cannot be a rotation.

Chapter 7, based on [75], a joint paper with A. Siemaszko, is devoted to the problem of minimal subsets of cylinder transformations. Let X be a compact metric space and $T: X \to X$ be a homeomorphism of X. Let $\varphi: X \to \mathbb{R}$ be a continuous function. By a *cylinder transformation* we mean a homeomorphism $T_\varphi: X \times \mathbb{R} \to X \times \mathbb{R}$ (or rather a $\mathbb{Z}$-action generated by it) given by the formula

$$T_\varphi(x, r) = (Tx, \varphi(x) + r).$$

We will also consider the case $\mathbb{R}^m$ instead of $\mathbb{R}$. It is essentially proved by A. S. Besicovitch in [8] that the cylinder transformation cannot itself be minimal. We also mention a deep result of P. Le Calvez and J.-Ch. Yoccoz saying that there is no minimal homeomorphism on the infinite annulus or more generally on the two-dimensional sphere with a finite set of points removed [60]. This of course generalizes Besicovitch's result.

The problem of the minimal subsets of a cylinder transformation turns out to be related to the problem of possible forms of ω-limit sets. H. Poincaré was the first to consider flows (generated by differential equations) on $\mathbb{R}^3$ that had time one homeomorphisms topologically isomorphic to cylinder cocycle extensions over irrational rotations [83]. He made an attempt at classifying possible form of the vertical section of ω-limit sets. His classification turned out to be partial and only A. B. Krygin gave the full classification in [55]. In [56] A. B. Krygin gave a full classification in the differentiable situation proving that actually there are four possibilities: either $\{0\}$ – the case of coboundary, or $\mathbb{R}$ – the case of transitive point, or $\mathbb{R}^+$, or $\mathbb{R}^-$.

In Sections 7.1 and 7.2 of this chapter we show that there are no minimal sets for any transitive cylinder transformation defined by bounded variation cocycles over an irrational rotation on the circle (Theorem 7.1.4) and over adding machines (Theorem 7.2.4). Moreover, the only compact monothetic groups that do not admit transitive cocycles are finite cyclic groups (Theorem 7.3.6).

Chapter 8, based on [74], is devoted to compare some twin notions in measure-theoretic ergodic theory and in topological dynamics. Some notions and theorems in topological dynamics imitate their analogues from measure-theoretic ergodic theory (see [34]). However the structure of objects in topological dynamics is sometimes more complicated than in ergodic theory. In particular the theorem saying that each measure-theoretic dynamical system is built up from ergodic components has no appropriate version in topological dynamics. The two most similar counterparts in topological dynamics of measure-theoretic ergodicity are minimality and topological transitivity (topological ergodicity). Both notions have some properties similar to ergodicity, unfortunately not all of them.

In this chapter we will compare some properties of special cocycle extensions (see (2) above) in measure-theoretic ergodic theory and in topological dynamics. It is known that each measure-theoretic extension is a cocycle extension [2], however the cocycle takes its values in a big Polish group, namely in the group of all automorphisms of a fixed Lebesgue space. In topological dynamics there are extensions that cannot be represented as cocycle extensions (see Example 8.2.1). The special cocycle extensions considered below will strongly depend on cocycles taking values in locally compact groups.

To study them, the main tool we will use is the notion of the group of essential values of a cocycle. This notion was introduced by Klaus Schmidt ([89]) in the measure-theoretic context. A topological version of the notion of group of essential values inherits many properties and consequences of the original Schmidt's definition (see [6], [64]). In this chapter we also work with the problem whether the conjugacy class of the group of essential values is a cohomology invariant in a nonabelian case. In measure-theoretic ergodic theory this is not true – see [5]. We present a counterexample to this guess in topological dynamics (see Example 8.2.3). In [74] a comment on this example was given that this is a topological counterexample to a relevant measure-theoretic theorem [13, Proposition 1.1]. This comment miss the goal as [13, Proposition 1.1] is based on an extra assumption that the cocycle under considerations is regular. It is easy to see that the Danilenko's proof works also in topological dynamics. On the other hand, for some constructions and strong theorems in ergodic theory there is a topological counterpart. In this chapter we compare descriptions of isomorphisms of Rokhlin cocycle extensions in ergodic theory and topological dynamics.

In the topological context we will study only extensions of the form (2) and here Γ is assumed to be a continuous action of a locally compact second countable group G on a compact metric space Y. In the study of extensions of the form (2) an important role is played by associated, so named, cylindrical transformations $T_\varphi\colon X \times G \to X \times G$, $T_\varphi(x,g) = (Tx, \varphi(x)g)$. Similarly to the measure-theoretic situation central object is the set $E_\infty(\varphi)$ of essential values of φ. We will give (Section 8.2) examples that some important properties of $E_\infty(\varphi)$ that hold in ergodic theory are not inherited by topological dynamics. In this paper we also describe (Section 8.3) base preserving equivariant homeomorphisms of Rokhlin cocycle extensions of minimal flows, that means, equivariant homeomorphisms of the form $\widehat{S}\colon (X \times Y_1, \widetilde{T}) \to (X \times Y_2, \overline{T})$, where both $\widetilde{T}$ and $\overline{T}$ are Rokhlin cocycle extensions of a given topologically transitive flows (X, T), and both these extensions are defined by the same cocycle $\varphi\colon X \to G$. The results of this chapter refer to [63, Proposition 5], [14, Theorem 7.3], [67, Proposition 2.1].

CHAPTER 1

PRELIMINARIES

1.1. Measure-theoretic dynamical systems

Let T be an automorphism of a probability Lebesgue space $(X, \mathcal{B}, \mu)$ (some basic information on Lebesgue spaces can be found in Appendix A). Then the quadruple $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ will be called a *measure-theoretic dynamical system*, or shortly a *dynamical system*. In the sequel we will often shortly call T a dynamical system.

One of the most important theorems in ergodic theory is so named the Birkhoff–Khinchin Ergodic Theorem:

Theorem 1.1.1 (Birkhoff–Khinchin Ergodic Theorem). *Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system and $f \in L^1(X, \mathcal{B}, \mu)$. Then for μ-almost every $x \in X$ the following limits exist and are equal to each other*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^{-k} x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} f(T^k x) \stackrel{\text{def}}{=} \overline{f}(x). \tag{1.1}$$

Further $\overline{f}(Tx) = \overline{f}(x)$ whenever the limits above exist. Moreover,

$$\overline{f} \in L^1(X, \mathcal{B}, \mu) \quad \text{and} \quad \int_X \overline{f}(x)\, d\mu = \int_X f(x)\, d\mu. \tag{1.2}$$

The limits that appear in the Birkhoff–Kchinchin Ergodic Theorem are called *time means* or *means along trajectory.*

A measurable set A is called *invariant* with respect to the automorphism T if $\mu(A \triangle TA) = \mu(A \triangle T^{-1}A) = 0$. A measurable function f is said to be *invariant* with respect to the automorphism T if $\mu(\{x \in X : f(x) \neq f(Tx)\}) = 0$.

Now we formulate one of the most important definition in ergodic theory.

Definition 1.1.2. A dynamical system $(X, \mathcal{B}, \mu, T)$ is said to be *ergodic* if for any invariant with respect to T set A, either $\mu(A) = 0$ or $\mu(A^c) = \mu(X \setminus A) = 0$. In such a case T is said to be an *ergodic automorphism.*

Each ergodic dynamical system may be characterized in the following way.

Proposition 1.1.3. *Let T be an automorphism of a probability Lebesgue space $(X, \mathcal{B}, \mu)$. The following statements are equivalent.*

(a) *T is ergodic.*
(b) *For every $A \in \mathcal{B}$ with $\mu(A) > 0$ we have $\mu\left(\bigcup_{n>0} T^{-n}A\right) = 1$.*
(c) *For every $A, B \in \mathcal{B}$ with $\mu(A) > 0$, $\mu(B) > 0$ there exists $n > 0$ with $\mu(T^{-n}A \cap B) > 0$.*

Now we give a characterization of ergodicity in terms of measurable real functions.

Proposition 1.1.4. *Let T be an automorphism of a probability Lebesgue space $(X, \mathcal{B}, \mu)$. The following statements are equivalent.*

(a) *T is ergodic.*
(b) *If f is measurable and $(f \circ T)(x) = f(x)$ a.e. then f is constant a.e.*
(c) *If $f \in L^2(X, \mathcal{B}, \mu)$ and $(f \circ T)(x) = f(x)$ a.e. then f is constant a.e.*

Theorem 1.1.5. *Suppose that T is an automorphism of a probability Lebesgue space $(X, \mathcal{B}, \mu)$. Then T is ergodic if and only if for all $A, B \in \mathcal{B}$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}A \cap B) = \mu(A)\mu(B).$$

Definition 1.1.6. Let T be an automorphism of a probability Lebesgue space $(X, \mathcal{B}, \mu)$.

(a) We say that T is *weakly mixing* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0 \quad \text{for all } A, B \in \mathcal{B}.$$

(b) We say that T is *strongly mixing* or *mixing* if

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{B}.$$

Evidently each strongly mixing transformation is weakly mixing, and each weakly mixing is ergodic. We also have the following characterization of weakly mixing automorphisms.

Theorem 1.1.7. *Suppose that T is an automorphism of a probability Lebesgue space $(X, \mathcal{B}, \mu)$. The following statements are equivalent:*

(a) *T is weakly mixing.*
(b) *There exists a subset $J \subset \mathbb{Z}^+$ of density zero such that*

$$\lim_{J \not\ni n \to \infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{B}.$$

(c) *For each $A, B \in \mathcal{B}$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)|^2 = 0.$$

(d) *$T \times T \colon (X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu) \to (X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu)$ is ergodic.*
(e) *$T \times T \colon (X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu) \to (X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu)$ is weakly mixing.*

(f) *If $0 \neq f \in L^2(X, \mathcal{B}, \mu)$ and $\lambda \in \mathbb{C}$ satisfy $f \circ T = \lambda f$ then $\lambda = 1$ and f is constant a.e.*

If we are given a dynamical system T on a probability space $(X, \mathcal{B}, \mu)$, then other T-invariant probability measures on $(X, \mathcal{B}, \mu)$ may exist. The following theorem gives some information on the structure of the set of such measures.

Theorem 1.1.8. *Suppose T is a dynamical system on a measurable space $(X, \mathcal{B})$ along with two probability T-invariant measures μ and ν on $\mathcal{B}$. Then:*

(a) *If μ is ergodic with respect to T while ν is absolutely continuous with respect to μ, then $\mu = \nu$.*
(b) *If both measures μ and ν are ergodic with respect to T then either $\mu = \nu$, or μ and ν are mutually singular.*

Theorem 1.1.9. *Let T be an automorphism of a probability Lebesgue space $(X, \mathcal{B}, \mu)$. Then there exists a measurable partition $\mathbf{P}$ of X satisfying the following conditions.*

(a) *Each element of the partition $\mathbf{P}$ is a T-invariant set.*
(b) *If $C \in \mathbf{P}$ and μ_C is the conditional measure on C, then T is ergodic on the Lebesgue space $(X, \mathcal{B}, \mu_C)$.*

The partition $\mathbf{P}$ the theorem above is describing is called a *decomposition of T into ergodic components*. By virtue of Theorem A.2.6, such a decomposition is unique. Each system $(X, \mathcal{B}, \mu_C)$ is called an *ergodic component* of the dynamical system $(X, \mathcal{B}, \mu, T)$. The decomposition (see Definition A.2.5)

$$\mu = \int_{X/\mathbf{P}} \mu_C \, d\mu$$

is called the *decomposition of the measure μ into ergodic components* or the *ergodic decomposition*. Denote by $E(X,T)$ the family of all ergodic for T probability measures on X. By Theorem 1.1.9, $E(X,T)$ is non-empty.

Definition 1.1.10. Let $T\colon(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be an automorphism of a probability Lebesgue space. By the *centralizer* $C(T)$ of T we mean the set

$$C(T)=\{S\colon X\to X : S \text{ preserves } \mu \text{ and } ST=TS\}.$$

Note that the centralizer is always a semigroup, not necessarily a group. The notion of coalescence, described below, comes from [80].

Definition 1.1.11. Let $T\colon(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be an ergodic automorphism. We will say that T is *coalescent*, if $C(T)$ is a group.

We equip $C(T)$ with the *weak topology* in the following way. We say that a sequence $(S_n)_{n\geq1}$ of elements of $C(T)$ *converges weakly* to $S\in C(T)$ if

$$\mu(S_n^{-1}(A)\triangle S^{-1}(A))\xrightarrow{n\to\infty}0\quad\text{for each } A\in\mathcal{B}.$$

Definition 1.1.12. Let $(X,\mathcal{B},\mu,T)$ and $(Y,\mathcal{C},\nu,S)$ be two measure-theoretic dynamical systems, and let $\pi\colon X\to Y$ be a measurable map satisfying $\mu(\pi^{-1}(C))=\nu(C)$ for all $C\in\mathcal{C}$. If $S\circ\pi=\pi\circ T$ then we call π a *homomorphism.* In such a case $(Y,\mathcal{C},\nu,S)$ is said to be a *factor* of $(X,\mathcal{B},\mu,T)$, and $(X,\mathcal{B},\mu,T)$ is said to be an *extension* of $(Y,\mathcal{C},\nu,S)$. If π is a conjugacy (i.e. π^{-1} is an isomorphism of the σ-algebras $\mathcal{C}$ and $\mathcal{B}$), then we call π an *isomorphism.*

1.2. Ergodic dynamical systems with discrete spectrum

The content of this section is borrowed from [98, Chapter 3].

Let $(X,\mathcal{B},\mu,T)$ be a dynamical system. Define

$$U_T\colon L^2(X,\mathcal{B},\mu)\to L^2(X,\mathcal{B},\mu)$$

by $U_T(f)=f\circ T$. Then U_T is a unitary operator on $L^2(X,\mathcal{B},\mu)$. It is clear that if $(X,\mathcal{B},\mu,T)$ and $(Y,\mathcal{C},\nu,S)$ are two isomorphic dynamical systems then the corresponding unitary operators U_T and U_S are conjugate, i.e. there exists an invertible linear operator $W\colon L^2(X,\mathcal{B},\mu)\to L^2(Y,\mathcal{C},\nu)$ such that $U_T\circ W=W\circ U_T$ and $\int Wf\cdot\overline{Wg}\,d\nu=\int f\cdot\overline{g}\,d\mu$ for all $f,g\in L^2(X,\mathcal{B},\mu)$ (i.e. W is an isomorphism of Hilbert spaces).

An important role in ergodic theory play eigenvalues of U_T. It is clear that if T and S are isomorphic then U_T and U_S have the same eigenvalues.

Theorem 1.2.1. *Let $(X,\mathcal{B},\mu,T)$ be an ergodic dynamical system and let U_T be the corresponding unitary operator. Then:*

(a) *It $U_Tf=\lambda f$, where $\lambda\in\mathbb{C}$, $f\in L^2(X,\mathcal{B},\mu,T)$, $f\not\equiv0$, then $|\lambda|=1$ and $|f|=\mathrm{const}$ a.e.*

(b) *Eigenvalues corresponding to different eigenvalues of U_T are orthogonal.*

(c) *If f and g are both eigenvalues corresponding to the eigenvalue λ then $f = cg$ a.e. for some $c \in \mathbb{C}$.*

(d) *The eigenvalues of U_T form a subgroup of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.*

Definition 1.2.2. We say that an ergodic dynamical system $(X, \mathcal{B}, \mu, T)$ has *discrete spectrum* (*pure-point spectrum*) if there exists an orthonormal basis for $L^2(X, \mathcal{B}, \mu, T)$ consisting of eigenvalues of T.

Theorem 1.2.3 (Discrete Spectrum Theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$, $(Y, \mathcal{C}, \nu, S)$ be ergodic dynamical systems with discrete spectrum. Then these systems are isomorphic if and only if U_T and U_S have the same eigenvalues.*

Natural examples of ergodic transformations with discrete spectrum are rotations on groups. For a compact Abelian group G, denote by ν the normalized Haar measure on G, and by $\widehat{G}$ the character group of the group G. Let $a \in G$, the automorphism $T: G \to G$ defined by $T(g) = ag$, $g \in G$, is called a *rotation* on the group G. If moreover this automorphism is ergodic with respect to the Haar measure ν, we call T an *ergodic rotation*.

Theorem 1.2.4. *Let T, given by $T(g) = ag$, be an ergodic rotation on a compact Abelian group G. Then T has discrete spectrum. Moreover, every eigenfunction of U_T is a constant multiple of a character, and the eigenvalues of U_T are $\{\gamma(a) : \gamma \in \widehat{G}\}$.*

Theorem 1.2.5 (Representation Theorem)**.** *Every ergodic dynamical system $(X, \mathcal{B}, \mu, T)$ with discrete spectrum is isomorphic to an ergodic rotation on some compact Abelian group. The group is metrizable if and only if $(X, \mathcal{B}, \mu)$ has a countable basis.*

Theorem 1.2.6 (Existence Theorem)**.** *Every subgroup $\Lambda \subset \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the group of eigenvalues of an automorphism with discrete spectrum.*

Consider now an ergodic dynamical system $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ and let $\pi: \mathcal{X} \to \mathcal{Y} = (Y, \mathcal{C}, \nu, S)$ be a homomorphism such that the system $\mathcal{Y}$ has discrete spectrum. Then $\mathcal{Y}$ is a canonical factor of $\mathcal{X}$ in the sense that whenever $\overline{\pi}: \mathcal{X} \to \overline{\mathcal{Y}} = (\overline{Y}, \overline{\mathcal{C}}, \overline{\nu}, \overline{S})$ is another homomorphism such that $\overline{\mathcal{Y}}$ is isomorphic to $\mathcal{Y}$, then $\overline{\pi}^{-1}(\overline{\mathcal{C}}) = \pi^{-1}(\mathcal{C})$, [79]. Using this property one can deduce that each ergodic dynamical system $\mathcal{X}$ is possessed of the largest factor with discrete spectrum, i.e. a factor with discrete spectrum $\mathcal{Y}$ such that whenever $\overline{\mathcal{Y}}$ is another factor with discrete spectrum of $\mathcal{X}$, then $\overline{\mathcal{Y}}$ is a factor of $\mathcal{Y}$.

Definition 1.2.7. Let $\mathcal{X}$ be an ergodic dynamical system. The largest factor with discrete spectrum of $\mathcal{X}$ is called the *Kronecker factor*.

1.3. Measure-theoretical joinings

If $T\colon (X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$, $S\colon(Y,\mathcal{C},\nu)\to(Y,\mathcal{C},\nu)$ are ergodic automorphisms then by a *joining* of T and S we mean any $T\times S$-invariant measure λ on $X\times Y$ such that, for $B\in\mathcal{B}$ and $C\in\mathcal{C}$,

$$\lambda(B\times Y)=\mu(B),\quad \lambda(X\times C)=\nu(C).$$

The set of all joinings of T and S we will denote by $J(T,S)$ or $J(X,Y)$, while the subset of $J(T,S)$ consisting of all $T\times S$-ergodic joinings, by $J^e(T,S)$ or $J^e(X,Y)$. Obviously the product measure $\mu\times\nu$ is a joining of T and S, therefore $J(T,S)\neq\emptyset$.

Proposition 1.3.1. *If $\lambda\in J(T,S)$ and if*

$$\lambda=\int_{E(T,S)}\gamma\,d\tau(\gamma)$$

is its ergodic decomposition, where $E(T,S)$ stands for all $T\times S$-ergodic measures on $X\times Y$, then $\tau(J^e(T,S))=1$.

Proof. As λ is a joining, for any $B\in\mathcal{B}$ we have

$$\mu(B)=\lambda(B\times Y)=\int_{E(T,S)}\gamma(B\times Y)\,d\tau(\gamma).$$

Each measure $\gamma(\,\cdot\times Y)$ is an ergodic measure on $\mathcal{B}$, hence the equality above gives an ergodic decomposition of μ. However μ is already ergodic, so $\gamma(\,\cdot\,\times Y)=\mu$ for τ-a.e. $\gamma\in E(T,S)$. In a similar way we prove that $\gamma(X\times\,\cdot\,)=\nu$ for τ-a.e. $\gamma\in E(T,S)$. Thus $\gamma\in J^e(T,S)$ for τ-a.e. $\gamma\in E(T,S)$. □

Proposition 1.3.1 says that the ergodic decomposition of a joining consists of joinings. In particular $J^e(T,S)\neq\emptyset$.

If $f\colon X\to Y$ is a measurable map then we define a *graph measure* μ_f on $X\times Y$ by

$$\mu_f(A\times B)=\mu(A\cap f^{-1}(B)).$$

It is easy to observe that the μ_f-measure of the graph of the map f in $X\times Y$ is equal to 1 (notice that if $\mu_f\in J(T,S)$ then $S\circ f=f\circ T$).

Lemma 1.3.2. *If $\lambda\in J^e(T,S)$ then*

$$\lambda=\mu_f\iff\underset{C\in\mathcal{C}}{\forall}\;\underset{B\in\mathcal{B}}{\exists}\;\lambda(B\times C^c\cup B^c\times C)=0. \tag{1.3}$$

Proof. If $\lambda=\mu_f$, then for a $C\in\mathcal{C}$ put $B=f^{-1}(C)$. Clearly the equality $\lambda(B\times C^c\cup B^c\times C)=0$ holds.

To prove the converse observe first that for such sets C and B we have $\mu(B)=\nu(C)=\lambda(B\times C)$. If $\lambda(B'\times C^c\cup B'^c\times C)=0$, then $\mu(B'\cap B)=\mu(B)=\mu(B')$, so $B = B'$ almost surely. Thus for a given C the set B is unique up to the measure μ. Define an isomorphism F of the Boolean σ-algebras $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{B}}$ by

$$F(C) = B \;\Leftrightarrow\; \lambda(B \times C^c \cup B^c \times C) = 0.$$

Then F defines an isomorphism of Lebesgue spaces $f\colon (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ such that $f^{-1}(C) = F(C)$ for all $C \in \mathcal{C}$. Clearly $\lambda = \mu_f$ and we are done. □

If $Y = X$ and $f = \mathrm{Id}$, the identity function, then the graph measure μ_{Id} we will call the *diagonal measure*.

Now we present the definition of simple and minimal self-joinings transformation (see [45]). Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. If $S_1, \dots, S_k \in C(T)$ then we call the image of the measure μ under the map

$$X \ni x \mapsto (S_1x, \dots, S_kx) \in \underbrace{X \times \ldots \times X}_{k \text{ times}} = X^k\}$$

an *off-diagonal measure*. Each off-diagonal measure is clearly an ergodic k-joining. By a *product of off-diagonal* (POOD) on X^k we mean that the set $\{1, \dots, k\}$ has been split into t_i-element subsets A_i, $i = 1, \dots, r$, then on each X^{t_i} we put an off-diagonal measure and then take the product of these off-diagonal measures. A POOD is evidently a self-joining of (X, T). Note that both product measure and off-diagonal measures on X^k are POOD. We say that T is *k-simple* if $C(T)$ is a group and each k-self-joining of (X, T) is POOD and T is *simple* if it is k-simple for each positive integer k. If T is simple and additionally $C(T) = \{T^n : n \in \mathbb{Z}\}$ then we say that T has *minimal self-joinings*, (MSJ).

Definition 1.3.3 ([22])**.** Two automorphisms $T_i\colon (X_i, \mathcal{B}_i, \mu_i) \to (X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, are said to be *disjoint* if $J(T_1, T_2) = \{\mu_1 \times \mu_2\}$. We will then write $T_1 \perp T_2$.

The notion of disjointness given in Definition 1.3.3 is also called the *disjointness in Furstenberg sense.*

Definition 1.3.4. If $S\colon (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ is a common factor of

$$T_i\colon (X_i, \mathcal{B}_i, \mu_i) \to (X_i, \mathcal{B}_i, \mu_i), \quad i = 1, 2,$$

and $\lambda \in J(Y, Y)$, by the *relatively independent extension* $\widehat{\lambda} \in J(X_1, X_2)$ of λ we mean the measure

$$\widehat{\lambda}(A_1 \times A_2) = \int_{Y \times Y} E(A_1|Y)(y_1)E(A_2|Y)(y_2)\, d\lambda(y_1, y_2).$$

Denote by $\widehat{\lambda} = \mu_1 \times_{\mathcal{C}} \mu_2$ the relatively independent extension of the diagonal measure on $Y \times Y$. By the *relative product* $T_1 \times_S T_2$ of T_1 and T_2 with respect to S we mean the relatively independent extension of the diagonal measure on Y.

We say that T_1 and T_2 are *relatively disjoint* over S, if the relative product $T_1 \times_S T_2$ is ergodic.

1.4. Group extensions of measure-theoretic dynamical systems

Let $T:(X,\mathcal{B}_0) \to (X,\mathcal{B}_0)$ be an automorphism of a standard Borel space $(X,\mathcal{B}_0)$, that means T is a bijective map such that $T^{-1}\mathcal{B}_0 = \mathcal{B}_0$. Let μ be a probability T-invariant measure on $(X,\mathcal{B}_0)$. Denote by $\mathcal{B}$ the T-invariant σ-algebra of all μ-measurable subsets of X. Then $(X,\mathcal{B},\mu)$ is a probability Lebesgue space with T being an automorphism of it. In what follows all σ-algebras under consideration will be complete with respect to the corresponding measure.

Let G be a locally compact group with the unit element e, equipped with a left-invariant Haar measure $\nu = \nu_G$ defined on the σ-algebra $\mathcal{B}(G)$ of Borel subsets of G. Suppose that $\varphi: X \to G$ is a Borel map. Define a $\mathbb{Z}$-cocycle $\varphi^{(\,\cdot\,)}:\mathbb{Z}\times X \to G$ for the $\mathbb{Z}$-action $n \mapsto T^n$, $n \in \mathbb{Z}$, by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(T^{n-1}x)\varphi(T^{n-2}x)\dots\varphi(Tx)\varphi(x), & n \geqslant 1,\\ e, & n = 0,\\ \varphi(T^n x)^{-1}\varphi(T^{n+1}x)^{-1}\dots\varphi(T^{-1}x)^{-1}, & n \leqslant -1. \end{cases} \tag{1.4}$$

Then the cocycle identity

$$\varphi^{(n+k)}(x) = \varphi^{(n)}(T^k x)\varphi^{(k)}(x) \tag{1.5}$$

is fulfilled. Note that each measurable $\mathbb{Z}$-cocycle $\Phi = \Phi(n,x)$ is of the form (1.4): simply define $\varphi(x) = \Phi(1,x)$. In what follows we will shortly call measurable $\varphi: X \to G$ a *cocycle*. Such a cocycle allows us to define an $T_\varphi: X \times G \to X \times G$ by the formula

$$T_\varphi(x,g) = (Tx, \varphi(x)g). \tag{1.6}$$

Then

$$(T_\varphi)^n(x,g) = (T^n x, \varphi^{(n)}(x)g), \quad n \in \mathbb{Z}. \tag{1.7}$$

The map T_φ preserves the (possible) infinite measure $\mu\times\nu_G$. The dynamical system $(X\times G, \mathcal{B}\otimes\mathcal{B}(G), \mu\times\nu_G, T_\varphi)$ is called a *group extension* of T, or, indicating the group, a *G-extension* of T. If G is compact then T_φ is also called a *compact group extension*. We say that the cocycle φ is *ergodic* if the corresponding group extension T_φ is ergodic, i.e. if for each T_φ-invariant set $A \in \mathcal{B}\otimes\mathcal{B}(G)$, either $(\mu\times\nu_G)(A) = 0$ or $(\mu\times\nu_G)(A^c) = 0$.

For each $g \in G$, let $\sigma_g(x,h) = (x,hg)$. For this right action of G on $X \times G$ we have $T_\varphi\sigma_g = \sigma_g T_\varphi$.

If $H \subset G$ is a closed subgroup then we define $T_{\varphi,H}: X \times G/H \to X \times G/H$ by the formula

$$T_{\varphi,H}(x, gH) = (Tx, \varphi(x)gH). \tag{1.8}$$

If no confusion can arise then we will denote the measure ν restricted to the sets of the form $BH = \bigcup_{b \in B} bH$, $B \subset G$, i.e. to the sets invariant with respect to the right action of H on $X \times G$, again by $\nu = \nu_G$. Let $\widetilde{\mu} = \mu \times \nu$. Denote by $\widetilde{\mathcal{B}}$ the product σ-algebra $\mathcal{B} \otimes \mathcal{B}(G)$. If $p: G \to G/H$ is the natural projection then we denote

$$\widetilde{B}_H = \mathcal{B} \otimes p(\mathcal{B}(D)). \tag{1.9}$$

The factor $T_{\varphi,H}$ of T_φ we will call a *natural factor* of T_φ and $T_{\varphi,H}$ an *isometric extension* of T. If the group H is normal in G, then we call $T_{\varphi,H}$ a *normal natural factor* of T_φ.

Theorem 1.4.1 (Veech's Theorem, [34])**.** *Let us assume that $T: (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic automorphism and that $\mathcal{C}$ is its factor. Let*

$$\mu \otimes_{\mathcal{C}} \mu = \int_{J_2^2(T)} \gamma \, dP(\gamma)$$

be the ergodic decomposition of the relatively independent extension of the diagonal measure on $\mathcal{C} \otimes \mathcal{C}$. If P-a.e. γ is a graph measure, then there exists a compact subgroup $H \subset C(T)$ such that $Y = X/H$, i.e.

$$\mathcal{C} = \{B \in \mathcal{B} : h(B) = B \text{ for all } h \in H\}.$$

In other words, $\mathcal{B}$ is a group extension of $\mathcal{A}$ by the group H.

The proof of the theorem below, that is a relative version of Veech's Theorem, was communicated to the author by M. Lemaczyk.

Theorem 1.4.2. *Suppose that $T: (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic automorphism, G is a compact metric group equipped with the normalized Haar measure ν defined on the σ-algebra $\mathcal{G}$ of Borel subsets of G. Denote $\overline{\mu} = \mu \times \nu$. Let $\varphi: X \to G$ be a measurable cocycle such that T_φ is ergodic. If $\mathcal{A}$ is a factor of T_φ such that $\mathcal{B} \otimes \{\emptyset, G\} \subset \mathcal{A}$, then there exists a compact subgroup $H \subset G$ such that $\mathcal{A} = \mathcal{B} \otimes \mathcal{B}(G)_H$, where $\mathcal{B}(G)_H$ is the Borel structure on the quotient space G/H. In other words, each factor $\mathcal{A}$ satisfying $\mathcal{B} \otimes \{\emptyset, G\} \subset \mathcal{A} \subset \mathcal{B} \otimes \mathcal{B}(G)$ is an isometric extension of $\mathcal{B}$.*

Proof. Let

$$\overline{\mu} \otimes_{\mathcal{A}} \overline{\mu} = \int_{J_2^e(T)} \gamma \, dP(\gamma)$$

be the ergodic decomposition of the relatively independent extension of the diagonal measure $\Delta_{\mathcal{A}}$ on $\mathcal{A} \otimes \mathcal{A}$. Observe that

$$P(\{\gamma \in J_2^e(T) : \gamma|_{\mathcal{A}\otimes\mathcal{A}} = \Delta_{\mathcal{A}}\}) = 1.$$

Now suppose that $\gamma \in J_2^e(T)$ satisfies $\gamma|_{\mathcal{A}\otimes\mathcal{A}} = \Delta_{\mathcal{A}}$, then

$$\gamma|_{(\mathcal{B}\otimes\{\emptyset,G)\otimes(\mathcal{B}\otimes\{\emptyset,G\})} = \Delta_{\mathcal{B}\otimes\{\emptyset,G\}},$$

and consequently

$$\gamma(\{(x,g,x,h) : x \in X,\ g,h \in G\}) = 1.$$

The (measurable) map $(x,g,x,h) \mapsto g^{-1}h$ is $T_\varphi \times T_\varphi$-invariant, hence it is γ-a.e. constant, i.e. there is $g_0 \in G$ such that $g^{-1}h = g_0$ for γ-a.e. (x,g,x,h). This is equivalent to say that γ is a graph joining. By virtue of Veech's Theorem, there exists a compact subgroup $H \subset G$ such that

$$\mathcal{A} = \{B \in \mathcal{B} \otimes \mathcal{B}(G) : Bh = B \text{ for all } h \in H\} = \mathcal{B} \otimes \mathcal{B}(G)_H,$$

which finishes the proof. □

The content of the following can be found e.g. in [50]–[52]. We will list some basic facts concerning the ergodic decomposition of a compact group extension of an ergodic automorphism and, in Section 2.5, apply them in our analysis of ergodic joinings for group extensions of semisimple automorphisms.

Let $(X,\mathcal{B},\mu,T)$ be an ergodic dynamical system. Let G be a compact metric group equipped with the normalized Haar measure ν on the family $\mathcal{B}(G)$ of Borel subsets of G. Assume that $\varphi\colon X \to G$ is a Borel map. Because the G-extension T_φ is not necessarily ergodic with respect to $\widetilde{\mu}$, let

$$\widetilde{\mu} = \int_{E(T_\varphi)} \lambda \, d\gamma(\lambda)$$

be the ergodic decomposition of $\widetilde{\mu}$.

Take any $\lambda \in E(T\varphi)$. Denote by H the *stabilizer* of λ in G, i.e. $H = \{g \in G : \lambda g = \lambda\}$.

Lemma 1.4.3.

(a) *H is a closed subgroup of G.*
(b) *If $(x,g),(x,h) \in Y$, then $hH = gH$.*

Let us decompose λ over the factor (X,μ,T):

$$\lambda = \int_X \lambda_x \, d\mu(x).$$

Let ν_H denote the Haar measure on H.

Lemma 1.4.4. *For almost each $x \in X$ there exists a $g = g_x \in G$ such that*

$$\lambda_x = \delta_x \times g\nu_H.$$

Let us define a function $\tau: X \to G/H$ by

$$\tau(x) = g_x H \tag{1.10}$$

where g_x is defined by Lemma 1.4.4. Then $(X \times G/H, \lambda, T_\varphi)$ is isomorphic to (X, μ, T): the map $p: X \times G/H \to X$, $p(x, gH) = x$ is measurable and λ-a.e. one-to-one. Therefore p is invertible and $p^{-1}(x) = (x, \tau(x))$. It forces τ to be measurable. Also

$$\tau(Tx) = \varphi(x)\tau(x) \tag{1.11}$$

Theorem 1.4.5. *There exists a function $t: X \to G$ such that the system $(X \times G, \lambda, T_\varphi)$ is isomorphic to $(X \times H, \mu \times \nu_H, T_\psi)$, where*

$$\psi(x) = t(Tx)^{-1}\varphi(x)t(x).$$

By [89], ergodicity of T_φ may be described using the notion of essential values of φ. Denote $G_\infty = G \cup \{\infty\}$ to be the one-point compactification of G (if G is compact then $G_\infty = G$).

Definition 1.4.6. A $g \in G_\infty$ is called an *essential value* of φ if for each positive measure set $U \in \mathcal{B}$ and for each open neighbourhood $G_\infty \supset V \ni g$ there exists an integer n such that the set

$$U \cap T^{-n}U \cap \{x \in X : \varphi^{(n)}(x) \in V\}$$

has positive measure. Denote by $E_\infty(\varphi)$ the set of all essential values of φ and set

$$E(\varphi) = E_\infty(\varphi) \cap G.$$

The set $E(\varphi)$ has the following properties.

Proposition 1.4.7.

(a) *$E(\varphi)$ is a closed subgroup of G;*
(b) *φ is a coboundary if and only if $E_\infty(\varphi) = \{0\}$;*
(c) *T_φ is ergodic if and only if $E(\varphi) = G$.*

Given a cocycle $\varphi: X \to G$, let $\varphi^*: X \to G/E(\varphi)$ be the corresponding quotient cocycle.

Lemma 1.4.8. $E(\varphi^*) = \{0\}$.

Definition 1.4.9. Assume that the group G is Abelian. We say that φ is *regular* if it is cohomologous to a an ergodic cocycle ψ taking all values in $E(\varphi)$ i.e. if there exists a measurable function $f: X \to G$ such that all values of the cocycle $\psi(x) = (f(Tx))^{-1}\varphi(x)f(x)$ are in $E(\varphi)$.

By [89], regular cocycles are, in measure-theoretic ergodic theory, characterized by the following property.

Proposition 1.4.10. *A cocycle φ is regular if and only if $E_\infty(\varphi^*) = \{0\}$.*

It follows that a regular φ is cohomologous to a cocycle $\psi: X \to E(\varphi)$ and the latter cocycle is ergodic as a cocycle with values in $E(\varphi)$. In particular, if $E(\varphi)$ is cocompact then φ is regular and as a direct consequence we obtain that all cocycles taking values in compact groups are regular.

Proposition 1.4.11 ([69])**.** *Let T be an ergodic automorphism. Assume that G and H are Abelian locally compact second countable groups and let $\pi: G \to H$ be a continuous group homomorphism. Let $\varphi: X \to G$ be a cocycle. Then*

$$\overline{\pi(E(\varphi))} \subset E(\pi \circ \varphi).$$

Moreover, if φ is regular then

$$\overline{\pi(E(\varphi))} = E(\pi \circ \varphi).$$

1.5. Rokhlin cocycle extensions

Let $(Y, \mathcal{C}, \nu)$ be a probability Lebesgue space, G an Abelian locally compact second countable group; in what follows we will assume that G contains no non-trivial compact subgroup. Let $\{R_g\}_{g \in G}$ be a measurable action of G on $(Y, \mathcal{C}, \nu)$ by automorphisms of the Lebesgue space $(Y, \mathcal{C}, \nu)$, i.e. the following map

$$G \times Y \ni (g, y) \mapsto R_g(y) \in Y$$

is measurable, and satisfies $R_e = \mathrm{Id}_Y$ and $R_{g+h} = R_g \circ R_h$.

Definition 1.5.1. We say that the action $\{R_g\}_{g \in G}$ is *ergodic* if for any $C \in \mathcal{C}$ satisfying $\nu(R_g C \triangle C) = 0$ for all $g \in G$ we have $\nu(C) = 0$ or $\nu(C) = 1$.

Definition 1.5.2 ([67], [90])**.** We say that the action $\{R_g\}_{g \in G}$ is *mildly mixing* if for any sequence $(g_k)_{k \geq 1}$ of elements of G going to infinity in G, and for any $C \in \mathcal{C}$ satisfying $\lim_{k \to \infty} \nu(R_{g_k} C \triangle C) \to 0$ we have $\nu(C) = 0$ or $\nu(C) = 1$.

Definition 1.5.3. We say that the action $\{R_g\}_{g \in G}$ is *weakly mixing* if the action $G \times (Y \times Y) \ni (g, y_1, y_2) \mapsto (R_g(y_1), R_g(y_2)) \in Y \times Y$ is ergodic.

Let $T\colon (X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be an ergodic automorphism, $\mathcal{G}=\{R_g\}_{g\in G}$ an action of G on $(Y,\mathcal{C},\nu)$, where G is an Abelian locally compact second countable group. Assume that $\varphi\colon X\to G$ is a cocycle. We define

$$T_{\varphi,\mathcal{G}}:(X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\times\nu)\to(X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\times\nu),$$
$$T_{\varphi,\mathcal{G}}(x,y)=(Tx,R_{\varphi(x)}(y)).$$

We call $T_{\varphi,\mathcal{G}}$ a *Rokhlin cocycle extension* of T. We will make use of some recent results from [63] (Propositions 1.5.4 and 1.5.5 below).

Proposition 1.5.4. *If $\mathcal{G}$ is ergodic and φ is ergodic then $T_{\varphi,\mathcal{G}}$ is ergodic.*

Proposition 1.5.5.

(a) *If the action $\mathcal{G}$ is mildly mixing and $T_{\varphi,\mathcal{G}}$ is ergodic, then the extension $T_{\varphi,\mathcal{G}}\to T$ is relatively weakly mixing.*
(b) *If $\mathcal{G}$ is weakly mixing, $T_{\varphi,\mathcal{G}}$ is ergodic and the maximal spectral type of $\mathcal{G}$ satisfies the group property then the extension $T_{\varphi,\mathcal{G}}\to T$ is relatively weakly mixing. In particular, the assertion holds whenever the action $\mathcal{G}$ is Gaussian.*

We will also make use of the following relative unique ergodicity result ([63]) for Rokhlin cocycle extensions.

Proposition 1.5.6. *Assume that φ is ergodic and $\mathcal{G}$ is a Borel action on $(Y,\mathcal{C})$. Suppose that ρ is an ergodic $T_{\varphi,\mathcal{G}}$-invariant measure (on $\mathcal{B}\otimes\mathcal{C}$) whose projection on $\mathcal{B}$ equals μ. Then $\rho=\mu\otimes\nu'$, where ν' is $\mathcal{G}$-invariant and ergodic.*

The following disjointness result has been proved in [67].

Proposition 1.5.7. *Suppose that W is an ergodic automorphism. If $T\perp W$, $\varphi\colon X\to G$ is ergodic and the action $\mathcal{G}=\{R_g\}_{g\in G}$ is mildly mixing, then $T_{\varphi,\mathcal{G}}\perp W$.*

1.6. Gauss dynamical systems

The definition of Gauss dynamical system given below comes from [12]. Consider the space M of all bisequences of real numbers, i.e. let $M=\mathbb{R}^{\mathbb{Z}}$. We will use the following notation: if $x\in M$, then let $x[s]$ be the sth position in the sequence x. Suppose that $\mathcal{M}$ is the σ-algebra generated by the cylinder subsets of the space M, i.e. by the sets of the form

$$C_{s,A}=\{x\in M : x[s]\in A\},$$

where $s\in\mathbb{Z}$, $A\subset\mathbb{R}$ is a Borel set. Denote by T the shift transformation in the space M given by $(Tx)[s]=x[s+1]$. A probability measure μ on $\mathcal{M}$ is said to be a *Gauss measure* if the joint distribution of any family of variables

$x[s_1], x[s_2], \dots, x[s_r]$ is an r-dimensional Gauss distribution. It is known that such a probability distribution is well defined by the numbers

$$\begin{aligned} m(s_i) &= \int x[s]\,d\mu(x), && i = 1, \dots, r, \\ b(s_i, s_j) &= \int x[s_i] \cdot x[s_j]\,d\mu(x), && i, j = 1, \dots, r. \end{aligned} \tag{1.12}$$

If μ is a Gauss measure, then $(M, \mathcal{M}, \mu)$ is said to be a Gauss random process. The Gauss measure μ is stationary (that means invariant with respect to T) if

$$m(s) = m = \text{const}, \quad b(s_1, s_2) = b(s_1 + s, s_2 + s), \tag{1.13}$$

for all $s_1, s_2, s \in \mathbb{Z}$, equivalently

$$b(s_1, s_2) = b(0, s_2 - s_1) \stackrel{\text{def}}{=} b(s_2 - s_1), \quad s_1, s_2 \in \mathbb{Z}.$$

One usually assumes that the mean m vanishes, since the transformation $x[s] \mapsto x[s] - m$ maps arbitrary Gauss measure μ into a Gauss measure with zero mean. The function $b(s)$, $s \in \mathbb{Z}$, is said to be the *correlation function* of the Gauss measure. Moreover, the correlation function is positive definite. By the Herglotz theorem (Theorem A.3.3), it may be presented in the form

$$b(s) = \int_{-\pi}^{\pi} e^{i\lambda s}\,d\sigma(\lambda),$$

where σ is a finite positive Borel measure on the circle $\mathbb{T}$. The measure σ is called the *spectral measure* of the Gauss measure μ. If $m = 0$ then the spectral measure σ uniquely determines the original measure μ. Moreover, as $b(s) = b(-s)$, we have $\sigma(A) = \sigma(-A)$ for any Borel set $A \subset \mathbb{T}$.

Definition 1.6.1. The shift transformation on the space $(M, \mathcal{M})$ equipped with a Gauss stationary measure μ is said to be a *Gauss automorphism*.

There is a more abstract equivalent definition of Gauss automorphism. Suppose T is an automorphism of a measure space $(M, \mathcal{M}, \mu)$. The real element $h_0 \in L^2(M, \mathcal{M}, \mu)$ is said to be a *Gauss element with zero mean* if for any collection of integers $n_1, \dots, n_r$ the random variables h_{j_j}, $j = 1, \dots, r$, where $h_n = U_T^n h_0 = h_0 \circ T^n$, have the joint Gauss probability distribution with zero mean. In such a case for Borel sets $C_1, \dots, C_r \subset \mathbb{R}$ we have

$$\begin{aligned} &\mu(\{x : h_{n_1}(x) \in C_1, \dots, h_{n_r}(x) \in C_{n_r}\}) \\ &\qquad = \int_{C_1 \times \dots \times C_r} p(t^{(1)}, \dots, t^{(r)})\,dt^{(1)} \dots dt^{(r)}, \end{aligned}$$

where $p(t^{(1)}, \dots, t^{(r)}) = \text{const} \cdot \exp[-(Dt, t)/2]$, $t = (t^{(1)}, \dots, t^{(r)})$, D is the matrix inverse to the scalar product matrix $B = \big(\|(h_{n_i}, h_{n_j})\|\big)_{i,j}$ and the constant is determined by the normalization condition.

Definition 1.6.2. The automorphism T is called a *Gauss automorphism* if there exists a Gauss element $h_0 \in L^2(M, \mathcal{M}, \mu)$ with zero mean such that the T-invariant minimal σ-algebra $\mathcal{M}_{h_0}$ containing of all sets of the form $B_{n,C} = \{x \in M : h_n(x) \in C\}$, $n \in \mathbb{Z}$, $C \subset \mathbb{R}$, is a Borel set, coincides with $\mathcal{M}$.

In general, if h_0 is a Gauss element, then we refer to the σ-algebra $\mathcal{M}_{h_0}$ as to *Gauss subalgebra*.

1.7. Topological dynamics – definitions and notations

Let X be a locally compact space. By $\mathrm{Hom}(X, X)$ we denote the group of all homeomorphisms of the space X with the uniform convergence topology, making $\mathrm{Hom}(X, X)$ a topological group. For a compact metric (X, d) the topology of uniform convergence is defined by the metric

$$d(p, q) = \sup_{x \in X} d(p(x), q(x)) + \sup_{x \in Y} d(p^{-1}(x), q^{-1}(x))$$

for $p, q \in \mathrm{Hom}(X, X)$.

Let T be a locally compact group acting on X as a group of homeomorphisms $\Gamma = \{\gamma_t : t \in T\} \subset \mathrm{Hom}(X, X)$. More precisely, we consider a map $T \times X \ni (t, x) \mapsto \gamma_t(x) \in X$ that is continuous and satisfies the conditions: $\gamma_{ts}(x) = \gamma_t(\gamma_s(x))$, $\gamma_e(x) = x = \mathrm{Id}_X(x)$, where e denotes the unit of T. In the sequel we will assume that the action Γ is *effective* i.e. $\gamma_t = \mathrm{Id}_X$ if and only if $t = e$. The pair (X, Γ) will be called a *locally compact T-flow*, or shortly a *T-flow*. To emphasize that X is compact we call (X, Γ) a *compact T-flow*. For the case $T = \mathbb{Z}$, any action of $\mathbb{Z}$ is defined by one homeomorphism γ_1; this homeomorphism is traditionally denoted by T and in such a case we will denote a $\mathbb{Z}$-flow by (X, T).

Let (X_1, Γ_1) and (X_2, Γ_2) be two T-flows. By $(X_1 \times X_2, \Gamma_1 \times \Gamma_2)$ we denote the T-flow given by the action $(x_1, x_2, t) \mapsto (\gamma_t^1(x_1), \gamma_t^2(x_2))$.

Let (X, Γ) be a T-flow. For $x \in X$ denote

$$\mathrm{Orb}(x) = \mathrm{Orb}_\Gamma(x) = \{\gamma_t(x) : t \in T\},$$

the *orbit*, and

$$\overline{\mathrm{Orb}}(x) = \overline{\mathrm{Orb}}_\Gamma(x) = \overline{\mathrm{Orb}(x)},$$

the *orbit closure* of the point x. Similarly, for a set $A \subset X$ write $\mathrm{Orb}_\Gamma(A) = \{\gamma_t(x) : x \in A,\ t \in T\}$ for the orbit, and $\overline{\mathrm{Orb}}_\Gamma(A) = \overline{\mathrm{Orb}(A)}$ for the orbit closure of the set A. The flow (X, Γ) is *point transitive*, if there exists $x_0 \in X$ with dense orbit: $\overline{\mathrm{Orb}}(x_0) = X$. A set $A \subset X$ is said to be Γ-*invariant*, if $\gamma_t(A) = A$ for all $t \in T$. We say that a set $M \subset X$ is Γ-*minimal*, if M is closed, Γ-invariant and each nonempty closed invariant subset of M is equal to M. If X is minimal itself, we call the T-flow (X, Γ) a *minimal flow*.

Theorem 1.7.1. *If X is a compact Hausdorff space then there exists a minimal subset of the T-flow (X,Γ).*

Theorem 1.7.1 fails when the phase space X is only locally compact, an example (somewhat artificial) can be found in [7, Chapter 1, pp. 27–28]. In Chapter 7 a large family of quite natural locally compact flows that do not admit minimal subset will be described.

For open sets $U,V\subset X$ the *dwelling set* $D(U,V)\subset T$ is defined by

$$D(U,V)=\{t\in T:\gamma_t(U)\cap V\neq\emptyset\}.$$

For $x\in X$ and an open $U\subset X$ define the dwelling set $D(x,U)$ by

$$D(x,U)=\{t\in T:\gamma_t(x)\in U\}.$$

Clearly X is point transitive if and only if there exists $x_0\in X$ such that $D(x_0,U)\neq\emptyset$ for every nonempty open $U\subset X$. A point $x\in X$ is *almost periodic* if for each nonempty open neighbourhood $U\ni x$ the dwelling set $D(x,U)$ is syndetic (a set $A\subset T$ is *syndetic* whenever there exists a compact subset C of T such that $T=CA$; see e.g. [97, IV(1.2)]). If $T=\mathbb{Z}$, then the notion of a syndetic set coincide with the notion of a relatively dense one: a set $A\subset\mathbb{Z}$ is *relatively dense* if there exists a positive integer N with the property that each $n\in\mathbb{Z}$ has a form $n=k+r$, where $k\in A$ and $0\leqslant r\leqslant N$. Each element of a compact minimal set is almost periodic; for each almost periodic point x_0, the closure orbit $\overline{\mathrm{Orb}_\Gamma}(x_0)$ is a compact minimal set. A flow (X,Γ) is *topologically ergodic* if $D(U,V)\neq\emptyset$ for any non-empty open sets $U,V\subset X$. Equivalently, X is topologically ergodic if and only if each nonempty open invariant subset of X is dense. Each point transitive flow is topologically ergodic, not vice versa. Both these notions coincide however in the case of metric spaces.

Let (X,T) be a $\mathbb{Z}$-flow. An $x\in X$ is called a *recurrent point* if for any open neighbourhood U of x the dwelling set $D(x,U)$ is both upper and lower unbounded. In other words, a point $x\in X$ is recurrent if there exist sequences of integers $n_i\to+\infty$, $m_i\to-\infty$ such that $T^{n_i}x\to x$, $T^{m_i}x\to x$. An $x\in X$ is called a *wandering point* if there exists an open neighbourhood U of x such that $D(U,U)=\{0\}$, i.e. the sets T^nU, $n\in\mathbb{Z}$, are pair-wise disjoint. If X is a complete metric space then the set consisting of all recurrent and wandering points is residual ([40, Theorem 7.24]). By definition, T is *conservative* if for any non-empty open set $U\subset X$, $D(U,U)\setminus\{0\}\neq\emptyset$. Clearly, T is conservative if and only if no point in X is wandering. If (X,T) is point transitive and X is a perfect space then T is conservative. Conservative homeomorphisms are also called *regionally recurrent* ([40]) or *non-wandering* ([97]).

We say that (X,Γ) is *uniformly rigid* (or shortly *rigid*) if there exists a sequence $(t_j)_{j\geqslant1}$ of elements of the group T such that $t_j\xrightarrow{j\to\infty}\infty$ and $\gamma_{t_j}\xrightarrow{j\to\infty}\mathrm{Id}$ uniformly; we will then call $(t_j)_{j\geqslant1}$ a *rigidity time* for T ([37]). The simplest uniformly rigid homeomorphisms are rotations on monothetic groups: if X is a monothetic group with $\overline{\{a^n : n \in \mathbb{Z}\}} = X$, then the homeomorphism $T\colon X \to X$ defined by $T(x) = ax$, $x \in X$, is called a *minimal rotation* (note that such a T is indeed a minimal homeomorphism). If T is such a rotation and $T^{n_t}x \to x$ for some x, then $(n_t)_{t\geqslant 1}$ is a rigidity time for T.

A pair $(x, y) \in X \times X$ is called *distal* if there exists $\delta = \delta(x, y) > 0$ such that $d(\gamma_t(x), \gamma_t(y)) > \delta$ for all $t \in T$. If a pair is not distal, it is said to be *proximal*. A flow (X, Γ) is called *distal* if each pair $(x, y) \in X \times X$, $x \neq y$, is distal. A flow (X, Γ) is called *proximal* if each pair $(x, y) \in X \times X$ is proximal. No Abelian group T admits non-trivial proximal flows (see [27]).

Every distal flow can be decomposed into minimal pieces (see [16]). From this it is easy to deduce Ellis's result that if (X, Γ) is a minimal flow, then the T-flow $(X \times X, \Gamma \times \Gamma)$ is decomposable into minimal pieces if and only if (X, Γ) is distal.

We say that two compact flows T-flows (X_1, Γ_1) and (X_2, Γ_2) are *disjoint*, if for the T-flow $(X_1 \times X_2, \Gamma_1 \times \Gamma_2)$, the only nonempty closed $\Gamma_1 \times \Gamma_2$-invariant subset $D \subset X_1 \times X_2$ satisfying $\pi_i(D) = X_1$, where $\pi_i(x_1, x_2) = x_i$, $i = 1, 2$, is just $X_1 \times X_2$. In such a case we will write $X_1 \perp X_2$.

The *centralizer*, $C(X, \Gamma)$, of (X, Γ) is the set of all continuous $S\colon X \to X$ that commute with the action Γ of T:

$$C(X, \Gamma) = \{S\colon X \to X : S \text{ is continuous and } S \circ \gamma_t = \gamma_t \circ S \text{ for all } t \in T\}.$$

Clearly, $C(X, \Gamma)$ equipped with the topology of uniform convergence is a topological semigroup. The set of all invertible elements of $C(X, \Gamma)$ we denote by $\mathrm{Aut}(X, \Gamma)$. We endow $\mathrm{Aut}(X, \Gamma)$ with the topology induced by the metric given by the formula

$$\overline{d}(f_1, f_2) = \sup_{x \in X} d(f_1(x), f_2(x)) + \sup_{x \in X} (f_1^{-1}(x), f_2^{-1}(x)),$$

that makes $\mathrm{Aut}(X, \Gamma)$ a topological group.

Let (X, Γ) be a T-flow. We say that a T-flow (Y, Δ) is a *factor* of (X, Γ) if there exists a continuous map $\pi\colon X \to Y$ (called a *homomorphism*) such that $\pi(X) = Y$ and $\pi \circ \gamma_t = \delta_t \circ \pi$. In such a case (X, Γ) (or sometimes π) is called an *extension* of (Y, Δ). It is easy to see that any factor of a minimal flow is also minimal. If the map π is also a homeomorphism, we say that (X, Γ) and (Y, Δ) are *isomorphic* and call π an *isomorphism*. If $\pi\colon (X, \Gamma) \to (Y, \Delta)$ is a homomorphism we can define a closed equivalence relation $R_\pi \subset X \times X$ by

$$R_\pi = \{(x, x') \in X \times X : \pi(x) = \pi(x')\}. \tag{1.15}$$

The relation R_π is Γ-invariant, that means $(\gamma_t(x), \gamma_t(x')) \in R_\pi$ for each $t \in T$ whenever $(x, x') \in R_\pi$. Obviously, the quotient space $(X_{R_\pi}, \Gamma_{R_\pi})$ (here Γ_{R_π} denotes the quotient action of T on X_{R_π}) with the quotient topology is isomorphic to (Y, S). This allows us to picture factors of (X, Γ) as $\Gamma \times \Gamma$-invariant, closed equivalence relations (ICER's) on X, also called *factor relations*. Conversely, given such a relation R we can define a homomorphism $\pi\colon (X,\Gamma) \to (X_R, \Gamma_R)$ by $\pi(x) = [x]_R$ (here $[x]_R$ denotes an equivalence class of x). Note that if we have two factor relations R_i, $i = 1, 2$, and a homomorphism $\pi\colon (X_{R_1}, T_{R_1}) \to (X_{R_2}, T_{R_2})$ with $\pi_{R_2} = \pi \circ \pi_{R_1}$, then $R_1 \subset R_2$. If we have a family $\{R_i\}_{i\in I}$ of factor relations, by $\bigvee_{i\in I} R_i$ we denote the smallest factor relation containing all R_i's.

If $Y \subset X$ and R is a factor relation on $X \times X$, then by R_Y we denote the restriction of R to Y (i.e. $R_Y = R \cap (Y \times Y)$).

We say that a T-flow (X,Γ) is *equicontinuous*, if for each $x \in X$ and for each $\alpha \in \mathfrak{A}_X$ – the uniform structure on X (see Section B.1 for the definition of uniform structure), there exists an open neighbourhood U of x such that $\gamma_g(U) \subset \alpha[\gamma_g(x)]$ for all $g \in G$. The standard examples of equicontinuous flows are rotations on topological groups. An extension $\pi\colon (X,\Gamma) \to (Y,\Delta)$ is said to be an *equicontinuous extension* if for each $\alpha \in \mathfrak{A}_X$ there exists $\beta \in \mathfrak{A}_X$ such that $(\gamma_t(x), \gamma_t(x')) \in \alpha$ for all $t \in T$ and for all $(x,x') \in R_\pi$.

Next we describe a special class of extensions – the group extensions. The following definition is a slight modification of [97, V (4.1)].

Definition 1.7.2. An extension $\pi\colon (X,\Gamma) \to (Y,\Delta)$ of T-flows is called a *group extension with group* K whenever the following conditions are fulfilled:

(a) K is a topological group acting continuously on X from the right as a subgroup of $\mathrm{Aut}(X,\Gamma)$ of automorphisms of (X,Γ);
(b) the fibers of π are precisely the K-orbits in X.

An important example of a group extension is a cocycle extension.

Definition 1.7.3. Let (X,Γ) be a T-flow, K a locally compact group, $\Phi\colon T \times X \to K$ a continuous cocycle. Define a T-action $\gamma_\Phi\colon T \times X \times K \to X \times K$ by the formula

$$\gamma_\Phi(g,x,k) = (\gamma_g(x), \Phi(g,x)k). \tag{1.16}$$

Denote $\Gamma_\Phi = \{(\gamma_t(\,\cdot\,), \Phi(t,\,\cdot\,)) : t \in T\}$. The T-flow $(X\times K, \Gamma_\Phi)$ is called a *cocycle extension* of (X,Γ).

If the cocycle extension $(X \times K, \Gamma_\Phi)$ is point transitive, we say that Φ is a *point transitive cocycle*.

In this dissertation we will deal mainly with $T = \mathbb{Z}$. In this case each cocycle is defined by a single continuous map. To be more precise consider a $\mathbb{Z}$-flow (X,T), where $T\colon X \to X$ is a homeomorphism, and a continuous map $\varphi\colon X \to K$, where K is a locally compact group. Define a cocycle $\Phi = \varphi^{(\,\cdot\,)}\colon \mathbb{Z} \times X \to K$ by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(T^{n-1}x)\varphi(T^{n-2}x)\dots\varphi(Tx)\varphi(x), & n \geqslant 1,\\ e, & n = 0,\\ \varphi(T^n x)^{-1}\varphi(T^{n+1}x)^{-1}\dots\varphi(T^{-1}x)^{-1}, & n \leqslant -1. \end{cases} \tag{1.17}$$

Then clearly the cocycle identity $\varphi^{(n+k)}(x) = \varphi^{(n)}(T^k x)\varphi^{(k)}(x)$ is fulfilled. Thus a continuous map φ defines a $\mathbb{Z}$-cocycle $\varphi^{(n)}$. Conversely, each $\mathbb{Z}$-cocycle $\Psi\colon \mathbb{Z} \times X \to G$ is of the form $\Psi(n,x) = \varphi^{(n)}(x)$, where $\varphi(x) = \Psi(1,x)$. Therefore we will call a continuous function $\varphi\colon X \to K$ a $\mathbb{Z}$-cocycle.

Definition 1.7.4. Let (X,T) be a compact $\mathbb{Z}$-flow, (Y,Γ) a compact G-flow, where G is a locally compact Abelian group and $\Gamma = \{\gamma_g : g \in G\}$ an effective continuous left action of G on Y. Assume that $\varphi\colon X \to G$ is a continuous map. We define a homeomorphism $T_{\varphi,\Gamma}\colon X \times Y \to X \times Y$ by

$$T_{\varphi,\Gamma}(x,y) = (Tx, \gamma_{\varphi(x)}(y)), \quad x \in X,\ y \in Y.$$

The $\mathbb{Z}$-flow $(X \times Y, T_{\varphi,\Gamma})$ we will call a *Rokhlin cocycle extension* of T.

A homomorphism $\pi\colon (X,\Gamma) \to (Y,\Delta)$ of T-flows is called *isometric* if there exists a group extension $\rho\colon Z \to Y$ and a homomorphism $\sigma\colon Z \to X$ such that $\pi \circ \sigma = \rho$.

A minimal T-flow (X,Γ) is *regular* if for each almost periodic point $(x,y) \in X \times X$ there exists an $S \in C(X,\Gamma)$ such that $y = S(x)$.

The following theorem is due to W. H. Gottschalk, [39], and J. Auslander, [7].

Theorem 1.7.5. *Every compact regular distal flow is equicontinuous* (*is a group extension of a trivial flow*).

The notions of distality and regularity can be "relativized" (with respect to factor). Let $\pi\colon (X,\Gamma) \to (Y,\Delta)$ be a homomorphism of T-flows. Then π is said to be *distal* (regular) provided the defining conditions from the absolute case hold for every $(x,y) \in R_\pi$. The homomorphism is called *proximal* if every pair $(x,y) \in R_\pi$ is a proximal pair; it is called a *weakly mixing* homomorphism if the flow $(R_\pi, T \times T)$ is point transitive. In any such a case the flow (X,Γ) is called a *distal* (*regular, proximal, weakly mixing*) *extension* of (Y,Δ), respectively.

We will also use the relativized version of Theorem 1.7.5, [29]. This could be considered as a topological version of theorem of Veech (see [95], also [45]; a proof of Veech's theorem is also contained in [61]).

Theorem 1.7.6. *Let (X,T) be a compact minimal $\mathbb{Z}$-flow and let $\pi\colon X \to Y$ be a regular distal homomorphism. Then π is a group extension.*

1.8. Universal flows

Let T be a group with the discrete topology. Suppose that X is a compact Hausdorff space and $\Gamma \subset \mathrm{Hom}(X,X)$ is a continuous left action of T on X, i.e. (X,Γ) is a T-flow. Fix $x_0 \in X$. Such a flow with distinguished point will be called a *pointed flow* and denoted either by (X,Γ,x_0) or shortly by (X,x_0). If we have a family $\{(Z_\sigma, z_\sigma)\}_{\sigma\in\Sigma}$ of pointed minimal flows we may choose $x_0 \in \prod_{\sigma\in\Sigma}$

satisfying $x_0(\sigma) = z_\sigma$ and set

$$\bigvee_{\sigma\in\Sigma} (Z_\sigma, z_\sigma) = (\overline{\mathrm{Orb}}(x_0), x_0). \tag{1.18}$$

For two pointed minimal flows (X, x_0) and (Y, y_0) we will use the notation

$$(X, x_0) \vee (Y, y_0) = (X \vee Y, (x_0, y_0)). \tag{1.19}$$

By Theorem B.2.5, the continuous map $T \ni t \mapsto \gamma_t(x_0) \in X$ can be extended to a continuous map $\beta T \ni p \mapsto px_0 \in X$. In such a way we have defined an action of T on βT by $(t, p) \mapsto tp$. Clearly each map $p \mapsto tp$ is a homeomorphism so we can consider a T-flow $(\beta T, T)$. This flow is evidently point transitive. Now, if (X, Γ, x_0) is a pointed T-flow that is point transitive with $\overline{\mathrm{Orb}}(x_0) = X$, then we are able to define a homeomorphism $\pi\colon (\beta T, T) \to (X, \Gamma)$ in the following way. Extend the map $T \ni t \mapsto \gamma_t(x_0) \in X$ to a continuous map $\beta T \ni p \mapsto px_0 = \pi(x_0) \in X$. Then (X, Γ) is a factor of $(\beta T, T)$. This means that $(\beta T, T)$ is the universal flow in the class of all compact Hausdorff point transitive T-flows. Notice that it is true that for each $x \in X$ the map $\beta T \ni p \mapsto px \in X$ is continuous, however in general the map $X \ni x \mapsto px \in X$ need not be continuous. Observe also that $(pq)x = p(qx)$ for $p, q \in \beta T$, $x \in X$.

Lemma 1.8.1. *Let (X, Γ) be a compact Hausdorff flow and $x \in X$.*

(a) $\overline{\mathrm{Orb}}(x) = (\beta T)x$.
(b) $\overline{\mathrm{Orb}}(x)$ *is minimal if and only if $x \in Mx$ for each minimal ideal $M \subset \beta T$ if and only if in each minimal ideal there is an idempotent v such that $vx = x$.*

Proof. The property (a) is clear.

(b) Assume that $\overline{\mathrm{Orb}}(x) = (\beta T)x$ is a minimal set. Then $(\beta T)x = Mx$ and $x = ex \in (\beta T)x = Mx$, where e is the unit element of the group T, so $x \in Mx$. In particular, $x = m_0 x$ for some $m_0 \in M$. Consider the nonempty set $\{m \in M : mx = x\}$. By Lemma B.2.7, this set contains an idempotent. Suppose now that $vx = x$, where $v \in M$ is an idempotent; then $x = vx \in Mx$. To this end assume that $x \in Mx$ for each minimal ideal $M \subset \beta T$. We will show that Mx is a minimal set. Suppose $A \subset Mx$ is an invariant closed set. Then $(\beta T)A \subset A$. Let $mx \in A$, then $(\beta T)mx \subset Mx$, $(\beta T)m \subset M$, and, by minimality of M, $(\beta T)m = M$. Thus $(\beta T)mx = Mx \subset A$, hence Mx is minimal. As $Mx = (\beta T)x = \overline{\mathrm{Orb}}(x)$, $\overline{\mathrm{Orb}}(x)$ is a minimal set. □

Lemma 1.8.2. *Let (X, Γ) be a compact Hausdorff T-flow, $x, y \in X$. The following conditions are equivalent.*

(a) *x and y are proximal.*
(b) *There exists $p \in \beta T$ such that $px = py$.*
(c) *There is a minimal ideal M such that $px = py$ for every $p \in M$.*

Proof. Suppose x and y are proximal, then there is a net $(t_i)_{i\in I}$ of elements of T such that $\gamma_{t_i}(x) \to z$, $\gamma_{t_i}(y) \to z$. Passing to a subnet if necessary we may assume that $t_i \to p \in \beta T$. Then $px = z = py$.

Suppose that (b) is true, $px = py$. Let M be a minimal ideal, then $N = Mp$ is also a minimal ideal and clearly $qx = qy$ for all $q \in N$.

Clearly (c) implies (a). □

Lemma 1.8.3. *Let (X, Γ) be a minimal compact Hausdorff T-flow, $x \in X$. Then*

$$P(x) = \{y \in X : x \text{ and } y \text{ are proximal}\}$$
$$= \{vx : v \text{ is an idempotent in some minimal ideal of } \beta T\}.$$

Proof. If x and y are proximal, then, by Lemma 1.8.2, there is a minimal ideal M such that $px = py$ for all $p \in M$. As X is minimal, there is an idempotent $v \in M$ such that $y = vy$ (Lemma 1.8.1) and $vx = vy = y$. Conversely, if v is an idempotent in βT then x and vx are proximal since $vx = v(vx)$. □

Lemma 1.8.4. *Let (X, Γ) be a compact Hausdorff T-flow, v an idempotent in some minimal ideal of βT. Then every pair of different points in $vX = \{x \in X : vx = x\}$ is distal.*

Proof. If $x, y \in vX$, then $v(x, y) = (x, y)$ and hence $\overline{\mathrm{Orb}}(x, y) \subset X \times X$ is a minimal set (Lemma 1.8.1(b)). If x and y were proximal, this minimal set would be included in the diagonal. □

Now fix a minimal ideal M in βT. Denote by J the set of all idempotents in M and choose a distinguished idempotent $u \in J$. Denote

$$G = uM.$$

By Proposition B.2.8, G is a group.

Given a compact Hausdorff minimal T-flow (X, Γ), choose a point $x_0 \in uX = \{ux : x \in X\} = \{x : ux = x\}$. Under the map $\beta T \ni p \mapsto px_0 \in X$, the ideal M is mapped onto X and u onto x_0. Thus (M, u) is a universal minimal pointed flow in the sense, that for every minimal flow X there is a point $x_0 \in X$ such that (X, x_0) is a factor of (M, u). Unless we say otherwise the base point x_0 of a minimal pointed flow (X, x_0) will be chosen so that $ux_0 = x_0$.

Definition 1.8.5. Let (X, x_0) be a pointed minimal flow. Define the *Ellis group* of (X, x_0) to be

$$\mathcal{G}(X, x_0) = \{\alpha \in G : \alpha x_0 = x_0\}.$$

It is clear that $\mathcal{G}(X, x_0)$ is a subgroup of G.

Proposition 1.8.6. *Let $\pi: (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal flows.*

(a) $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$.
(b) $\mathcal{G}(X, x_0) = \mathcal{G}(Y, y_0)$ *if and only if π is proximal.*
(c) *If π is proximal then $\pi^{-1}(y) \subset Jx$ for any $x \in \pi^{-1}(y)$.*
(d) *π is distal if and only if for every $y \in Y$ and $p \in M$ with $py_0 = y$ the following holds: $\pi^{-1}(y) = p\mathcal{G}(Y, y_0)x_0$.*

Proof. (a) Let $\alpha \in \mathcal{G}(X, x_0)$, then $\alpha y_0 = \alpha\pi(x_0) = \pi(\alpha x_0) = \pi(x_0) = y_0$.

(b) Suppose $\mathcal{G}(X, x_0) = \mathcal{G}(Y, y_0)$. We show that $x_1, x_2 \in \pi^{-1}(y_0)$ implies x_1 and x_2 are proximal. For such x_1, x_2 we have $x_1 = px_0$, $x_2 = qx_0$ for $p, q \in M$. Denote $\alpha = up^{-1}q$, then $\alpha y_0 = up^{-1}qy_0 = up^{-1}q\pi(x_0) = up^{-1}\pi(qx_0) = up^{-1}\pi(px_0) = upp^{-1}\pi(x_0) = uy_0 = y_0$. Thus $\alpha \in \mathcal{G}(Y, y_0)$ and hence $\alpha \in \mathcal{G}(X, x_0)$ i.e. $up^{-1}qx_0 = x_0$. Thus $ux_1 = upx_0 = up(up^{-1}qx_0) = uqx_0 = ux_2$ and x_1, x_2 are proximal.

Conversely suppose that π is proximal and let $\alpha \in \mathcal{G}(Y, y_0)$. Then $\pi(\alpha x_0) = \alpha y_0 = y_0$ implies αx_0 and x_0 are proximal. On the other hand αx_0 and x_0 are distal ($\alpha = u\alpha$) by Lemma 1.8.2(c), hence $\alpha x_0 = x_0$ and $\alpha \in \mathcal{G}(X, x_0)$.

(c) Suppose π is proximal and let $x, x_1 \in \pi^{-1}(y)$, then x and x_1 are proximal and by Lemma 1.8.2(b) there exists an idempotent v' in some minimal ideal L of βT such that $x_1 = v'x$. Now let $v \in J$ be equivalent to v' (i.e. $vv' = v'$, $v'v = v$, see Lemma B.2.10), then $y = \pi(v'x) = \pi(vv'x) = v\pi(v'x) = vy$, and hence $\pi(vx) = v\pi(x) = vy = y$. It follows that vx and $v'x$ are proximal. But $vv'x = v'x$ ant thus $vx, v'x \in vX$ and by Lemma 1.8.2(c), vx and $v'x$ are also distal. Therefore $x_1 = v'x = vx \in Jx$, and the proof is complete.

(d) Suppose $\pi^{-1}(py_0) = p\mathcal{G}(Y, y_0)x_0$, and let $v \in J$ be such that $vp = p$. Then $p\mathcal{G}(Y, y_0)x_0 \subset vX$ and π is distal by Lemma 1.8.2(c).

Let π be distal. If $y = px_0$ for some $p \in M$ then for $\alpha \in \mathcal{G}(Y, y_0)$ we have $\pi(p\alpha x_0) = \pi(px_0) = py_0 = y$. Thus $p\mathcal{G}(Y, y_0)x_0 \subset \pi^{-1}(y)$. On the other hand if $\pi(x) = y$ then $x = qx_0$ for some $q \in M$ and since $\pi(qx_0) = y = \pi(px_0)$ we conclude as in (b), that $\alpha = up^{-1}q \in \mathcal{G}(Y, y_0)$. If $v \in J$ is such that $vq = q$ then $q = vp\alpha$. Now $y = \pi(x) = \pi(qx_0)$ and $y = py_0 = v\alpha y_0 = \pi(v\alpha x_0)$. Thus $qx_0 = v(p\alpha x_0)$ and $p\alpha x_0$ are both distal and proximal. Hence they are equal and $x = qx_0 = p\alpha x_0 \in p\mathcal{G}(Y, y_0)x_0$. □

Proposition 1.8.7. *Let $\phi: (X, x_0) \to (Y, y_0)$ and $\psi: (Z, z_0) \to (Y, y_0)$ be two distal homomorphisms of compact Hausdorff minimal pointed flows. There exists a homomorphism $\theta: (Z, z_0) \to (X, x_0)$ if and only if $\mathcal{G}(X, x_0) \supset \mathcal{G}(Z, z_0)$.*

Proof. Suppose first that θ exists. Then, by Proposition 1.8.6(a), $\mathcal{G}(X, x_0) \supset \mathcal{G}(Z, z_0)$.

Suppose that $\mathcal{G}(X, x_0) \supset \mathcal{G}(Z, z_0)$. For $p \in M$ define $\theta(pz_0) = px_0$. If $z = pz_0 = qz_0$ for $p, q \in M$ then $up^{-1}q \in \mathcal{G}(Z, z_0)$ and by assumption, $up^{-1}qx_0 = x_0$

and $pp^{-1}qx_0 = vqx_0 = px_0$, where $v = pp^{-1}$ is an idempotent in M. Thus qx_0 and px_0 are proximal. Now $pz_0 = qz_0$ implies $py_0 = qy_0$ and hence $\phi(px_0) = \phi(qx_0)$. This implies that px_0 and qx_0 are also distal i.e. $px_0 = qx_0$ and θ is well defined. Clearly θ is a continuous homomorphism and the proof is complete. □

Let (X, Γ) be a compact Hausdorff T-flow and let 2^X be the family of all closed nonempty subsets of X. Recall that the Vietoris topology on 2^X is defined by a basis consisting of sets of the form

$$\langle U_1, \dots, U_n \rangle = \left\{ A \in 2^X : A \subset \bigcup_{i=1}^{n} U_i,\ A \cap U_i \neq \emptyset,\ i = 1, \dots, n \right\},$$

where $U_1, \dots, U_n$ are open subsets of X. In this topology, if a net $(A_i)_{i\in I}$ converges to A, $\lim A_i = A$, then

$$A = \{\lim x_i : x_i \in A_i,\ i \in I' \text{ and } (A_i)_{i\in I'} \text{ is a subnet of } (A_i)_{i\in I}\}.$$

Because X is assumed to be compact Hausdorff, so is 2^X with Vietoris topology. This topology is metrizable if and only if X is metrizable.

There is a natural T-flow structure on 2^X induced by (X, Γ), namely $(t, A) \mapsto \gamma_t(A)$. The map $T \ni t \mapsto \gamma_t(A) \in 2^X$ can be extended to a map

$$\beta T \ni p \mapsto p \circ A \in 2^X.$$

The following lemma is clear.

Lemma 1.8.8. *Suppose $A \in 2^X$, $p, q \in \beta T$. The following statements hold:*

(a) *$p \circ A$ is the set of all points $x \in X$ such that there exist nets $(x_i)_{i\in I}$ of elements of A and $(t_i)_{i\in I}$ of elements of T for which $\lim t_i = p$ and $\lim \gamma_{t_i}(x_i) = x$,*
(b) *$pA \subset p \circ A$,*
(c) *$p \circ (q \circ A) = (pq) \circ A$.*

For an arbitrary $A \subset X$ (not necessarily closed) define

$$p \circ A = p \circ \overline{A}. \tag{1.20}$$

Recall that $G = uM \subset M$, where M is a fixed minimal ideal in βT and u a fixed idempotent in M; G is a group with the unit element u. One can easily verify the following.

Proposition 1.8.9. *The operation*

$$G \supset A \mapsto (u \circ A) \cap G$$

defines a closure operation on G.

The operation $G \supset A \mapsto (u \circ A) \cap G$ will be denoted by $G \supset A \mapsto \overline{A}^{\tau}$. The topology induced on G by this operation we will call the τ-topology.

Lemma 1.8.10. $\overline{A}^{\tau} = u(u \circ A)$.

Proof. We have $u(u \circ A) \subset G$, $u(u \circ A) \subset u \circ (u \circ A) = u^2 \circ A = u \circ A$. On the other hand if $p \in (u \circ A) \cap G$ then $up = p$ so $p \in u(u \circ A)$. □

The following proposition collects facts proved in [27, IX.1].

Proposition 1.8.11.

(a) *In τ-topology G is a T_1 compact space.*
(b) *τ-topology is weaker than the original topology induced from M.*
(c) *For each $\beta \in G$ the maps*

$$G \ni \alpha \mapsto \alpha\beta \in G \quad \textit{and} \quad G \ni \alpha \mapsto \beta\alpha \in G$$

are homeomorphisms in the τ-topology.
(d) *All the groups of the form $\mathcal{G}(X, x_0)$, where $ux_0 = x_0$, are closed in τ-topology.*

Definition 1.8.12. For every τ-closed subgroup F of G we let

$$F' := \bigcap\{\overline{V}^{\tau} : V \text{ is } \tau\text{-open neighbourhood of } u \text{ in } F\}.$$

Proposition 1.8.13 ([27, Theorem IX.1.9])**.** *Let F be a τ-closed subgroup of the group G.*

(a) *F' is a τ-closed normal subgroup of F. Moreover, F' is invariant under all topological automorphisms of the group F.*
(b) *F/F' with the quotient topology is a compact Hausdorff topological group.*
(c) *If K is a τ-closed subgroup of F then F/K is a Hausdorff space if and only if $F' \subset K$.*

Definition 1.8.14. We say that a compact Hausdorff minimal pointed T-flow (X, Γ, x_0) is *incontractible* if $u \circ Gx_0 = X$. We say that an extension $\phi: (X, x_0) \to (Y, y_0)$ is *relatively incontractible* (RIC), if for every $p \in M$,

$$\phi^{-1}(py_0) = p \circ \mathcal{G}(Y, y_0)x_0.$$

RIC-extensions are open and have a dense set of almost periodic points in the relation R_π. Every distal extension is RIC. Every homomorphism from a minimal flow to the one-point flow is RIC.

Theorem 1.8.15 ([27, Proposition X.3.2])**.** *Let $\phi: X \to Y$ be a homomorphism of compact Hausdorff minimal flows. Then there exists a commutative diagram of minimal flows homomorphisms*

$$\begin{tikzcd} X \arrow[d, "\phi"'] & X^* \arrow[l, "\theta^*"'] \arrow[d, "\phi^*"] \\ Y & Y^* \arrow[l, "\theta"] \end{tikzcd}$$

where θ, θ^* are proximal and ϕ^* is RIC*. The extensions θ and θ^* are isomorphisms if and only if ϕ is already* RIC*. We may also assume that $X^* = X \vee Y^*$, and θ^* and ϕ^* are projections onto the first and the second coordinate respectively.*

Definition 1.8.16. The diagram in Theorem 1.8.15 is called the *shadow diagram* of $\phi\colon X \to Y$.

The main reason that RIC-extensions are so useful is the following theorem (see [18]).

Theorem 1.8.17. *Let $(X,T) \xrightarrow{\pi} (Y,T)$ be a* RIC*-extension of minimal flows. Then there exists a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[d, "\pi"'] & Z \arrow[dl, "\rho"] \\
Y &
\end{tikzcd}$$

where ρ is an isometric extension with $B = F'A$ $(B = \mathcal{G}(Z, z_0), F = \mathcal{G}(Y, y_0)$, and $A = \mathcal{G}(X, x_0))$. The flow Z is the largest isometric extension of Y within X, and ρ is an isomorphism if and only if π is a weakly mixing extension if and only if $B = F$.

Definition 1.8.18. We say that a minimal flow X is *strictly* PI, if there is an ordinal ν and flows $\{W_\alpha : \alpha \le \nu\}$ such that

(a) W_0 is the trivial flow.
(b) For every $\alpha < \nu$ there exists a homomorphism $\phi_\alpha\colon W_{\alpha+1} \to W_\alpha$ which is either proximal or almost periodic.
(c) For a limit ordinal $\alpha \le \nu$, $W_\alpha = \bigvee_{\beta<\alpha} W_\beta$.
(d) $W_\nu = X$.

We say that X is a PI-flow if there exist a strictly PI flow X' and proximal homomorphism $\phi\colon X' \to X$.

Using the shadow construction and Theorem 1.8.17 repeatedly one obtains the following structure theorem.

Theorem 1.8.19 ([18])**.** *Given a* RIC *homomorphism $\pi = \pi_0\colon (X,T) \to (Y,T)$ of metric minimal flows, there exist a countable ordinal η and a canonically defined commutative diagram* (*the canonical* PI*-tower*)*:*

$$\begin{tikzcd}
X = X_0 \arrow[d, "\pi_0"'] \arrow[dr, "\sigma_1"] & & X_1 \arrow[ll, "\widetilde{\theta}_1"'] \arrow[d, "\pi_1"'] & \cdots & X_\nu \arrow[d, "\pi_\nu"'] \arrow[dr, "\sigma_{\nu+1}"] & & X_{\nu+1} \arrow[ll, "\widetilde{\theta}_{\nu+1}"'] \arrow[d, "\pi_{nu+1}"'] & \cdots & X_\eta = X_\infty \arrow[d, "\pi_\infty"] \\
Y = Y_0 & Z_1 \arrow[l, "\rho_1"] & Y_1 \arrow[l, "\theta_1"] & \cdots & Y_\nu & Z_{\nu+1} \arrow[l, "\rho_{\nu+1}"] & Y_{\nu+1} \arrow[l, "\theta_{\nu+1}"] & \cdots & Y_\eta = Y_\infty
\end{tikzcd}$$

where for each $\nu \leq \eta, \pi_\nu$ *is* RIC, ρ_ν *is isometric,* $\theta_\nu, \widetilde{\theta}_\nu$ *are proximal and* π_∞ *is* RIC *and weakly mixing. For a limit ordinal* $\nu, X_\nu, Y_\nu, \pi_\nu$ *are the inverse limits of* $X_\iota, Y_\iota, \pi_\iota$ *for* $\iota < \nu$. *In terms of Ellis groups:* $B = AF^\infty$ *and* $AB' = B$, *where* $A = \mathcal{G}(X), B = \mathcal{G}(Y_\infty)$, *and* $F = \mathcal{G}(Y)$. *The extension* π *is* PI *if and only if* $X_\infty = Y_\infty$.

CHAPTER 2

SEMISIMPLE AUTOMORPHISMS

2.1. Group and isometric extensions, joinings

Let $T\colon (X, \mathcal{B}_0) \to (X, \mathcal{B}_0)$ be an automorphism of a standard Borel space. Let μ be a probability T-invariant measure on $(X, \mathcal{B}_0)$, $\mathcal{B}$ the T-invariant σ-algebra of all μ-measurable subsets of X. Then $(X, \mathcal{B}, \mu)$ is a probability Lebesgue space with T being an automorphism of it. Let $S\colon (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ be a factor of $T\colon (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$. If no confusion can arise, we will often use the following abbreviations: $T \to S$ or $\mathcal{B} \to \mathcal{C}$ or even $X \to Y$. In terms of joinings we can express the fact that the extension $X \to Y$ is a group extension (see Theorem 1.4.1 – the Veech's Theorem).

Suppose now that $\mathcal{B}_1 \subset \mathcal{B}$ is a $T-$invariant sub-σ-algebra (factor), hence giving rise to a factor $\overline{T}\colon (\overline{X}, \mathcal{B}_1, \overline{\mu}) \to (\overline{X}, \mathcal{B}_1, \overline{\mu})$ of T. Note that if we take the family of all factors of T, say $\mathcal{B}_\kappa, \kappa \in \Lambda$, containing $\mathcal{B}_1$ with the property that each $\lambda \in J^e(\mathcal{B}_\kappa, \mathcal{B}_\kappa)$ that projects onto the diagonal measure on $\mathcal{B}_1 \otimes \mathcal{B}_1$ is a graph joining, then the smallest factor of T containing all $\mathcal{B}_\kappa, \kappa \in \Lambda$ enjoys the same property. Hence there exists the maximal factor $\widetilde{\mathcal{B}} \subset \mathcal{B}$ such that $\widetilde{\mathcal{B}} \to \mathcal{B}_1$ is a group extension. Note also that if $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}$ are factors then the smallest factor of $\mathcal{B}$ containing $\mathcal{B}_1$ and $\mathcal{B}_2$ can naturally be identified with an ergodic joining of $\mathcal{B}_1$ and $\mathcal{B}_2$.

Suppose $(X, \mathcal{B}, \mu, T) \to (\overline{X}, \mathcal{B}_1, \overline{\mu}, \overline{T})$ is an extension of ergodic systems. Denote

$$\mu = \int_{\overline{X}} \mu_{\overline{x}}\, d\overline{\mu}(\overline{x})$$

to be the disintegration of μ over $\overline{\mu}$. We have $T = \overline{T}_\theta$, where

$$\overline{T}_\theta(\overline{x}, z) = (\overline{T}\overline{x}, \theta_{\overline{x}}(z))$$

with $X = \overline{X} \times Z$, $\mu = \overline{\mu} \times \nu$ (see [23]). Then $\mu_{\overline{x}}$ can be viewed as a measure on $\mathcal{B}$ just concentrated on the fibers of the natural map $\pi\colon X \to \overline{X}$, i.e. $\mu_{\overline{x}} = \delta_{\overline{x}} \times \nu$.

Definition 2.1.1. Denote $H = L^2(X,\mu)$. We say that a function $f \in H$ is *almost periodic* (AP) if for each $\varepsilon > 0$ there exist $g_1,\dots,g_k \in H$ such that for each $p \in \mathbb{Z}$

$$\min_{1\le j\le k} \|fT^p - g_j\|_{L^2(\mu_{\overline{x}})} < \varepsilon \tag{2.1}$$

for a.a. $\overline{x} \in \overline{X}$.

Definition 2.1.2. If $\mathcal{B}_1$ is a factor of $\mathcal{B}$ then we say that the extension $\mathcal{B} \to \mathcal{B}_1$ is *compact* if the set of AP functions is dense in the Hilbert space $H = L^2(X,\mu)$.

Theorem 2.1.3 ([99]). *An extension $(X,\mathcal{B},\mu,T) \to (\overline{X},\mathcal{B}_1,\overline{\mu},\overline{T})$ is compact if and only if there exists a compact group G and its closed subgroup H such that $Z = G/H$ and $\theta_{\overline{x}} = \varphi(\overline{x})H$ for a measurable map $\varphi\colon \overline{X} \to G$, i.e. the extension $X \to \overline{X}$ is an isometric extension.*

Proposition 2.1.4. *Suppose that $(X,\mathcal{B},T) \to (\overline{X},\overline{\mathcal{B}},\overline{T})$ is an ergodic isometric extension. Then there exists an ergodic extension $(Y,\mathcal{C},S)$ of X such that $Y \to \overline{X}$ is a group extension and moreover for each ergodic extension $(Y',\mathcal{B}',S')$ of X with $Y' \to \overline{X}$ a group extension we have*

$$\begin{tikzcd} Y' \arrow[r] \arrow[dr] & X \arrow[r] & \overline{X} \\ & Y \arrow[u] & \end{tikzcd}$$

Proof. Let $\widetilde{S}\colon(\widetilde{Y},\widetilde{\mathcal{C}},\widetilde{\nu}) \to (\widetilde{Y},\widetilde{\mathcal{C}},\widetilde{\nu})$ be any ergodic extension of X that is a group extension of $\overline{X}$. Take the family of all factors $\widetilde{\mathcal{C}}_\kappa \subset \widetilde{\mathcal{C}}$, $\kappa \in \Lambda$, that are group extensions of $\overline{\mathcal{B}}$ and set

$$\mathcal{C} = \bigcap_{\kappa\in\Lambda} \widetilde{\mathcal{C}}_\kappa.$$

Note that if $\lambda \in J^e(\mathcal{C},\mathcal{C})$ projects onto the diagonal measure on $\overline{\mathcal{B}} \otimes \overline{\mathcal{B}}$ then for any ergodic extension $\widehat{\lambda}$ of λ on $\widetilde{\mathcal{C}} \otimes \widetilde{\mathcal{C}}$ we have that $\widehat{\lambda}$ is a graph measure. Hence if $A \in \mathcal{C}$ then there exists a set $\widetilde{B} \in \widetilde{\mathcal{C}}$ such that

$$\widehat{\lambda}(A \times \widetilde{Y} \triangle \widetilde{Y} \times \widetilde{B}) = 0.$$

Thus, it is clear that $\widetilde{B} \in \widetilde{\mathcal{C}}_\kappa$ for each $\kappa \in \Lambda$ and consequently $\widetilde{B} \in \mathcal{C}$. By Veech's Theorem, $\mathcal{C} \to \overline{\mathcal{B}}$ is a group extension.

Take any ergodic joining of Y' and Y which is diagonal on $\overline{X}$; we get a system Z. Now, Y and Y' are represented in Z by some invariant σ-algebras, say $\mathcal{A}$ and $\mathcal{A}'$. Let $\mathcal{C}_1 = \mathcal{A} \cap \mathcal{A}' \subset \overline{\mathcal{C}} \otimes \overline{\mathcal{C}}$. Take any ergodic self-joinings λ on $\mathcal{C}_1 \otimes \mathcal{C}_1$ that is diagonal on $\overline{X} \times \overline{X}$. Then this joining has an ergodic extension $\widetilde{\lambda}$ to $Z \times Z$.

Take any set $C \in \mathcal{C}$. Because $\mathcal{A}$ and $\mathcal{A}'$ are group extensions of $\overline{X}$, there exist $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$ such that

$$\widetilde{\lambda}(C \times Z \triangle Z \times A) = 0, \quad \widetilde{\lambda}(C \times Z \triangle Z \times A') = 0.$$

Therefore $A = A' \in \mathcal{A} \cap \mathcal{A}' = \mathcal{C}$. Hence λ is a graph joining and consequently $\mathcal{C}_1$ is a group extension of $\overline{X}$. □

Definition 2.1.5. The extension Y of X, defined (up to an isomorphism) by Proposition 2.1.4, will be called the *minimal group cover* of X.

Definition 2.1.6 ([23])**.** An extension $X \to \overline{X}$ is called *distal* if for a certain ordinal η we have a family of factors $\mathcal{B}_\kappa$, $\kappa \leq \eta$, such that $\mathcal{B}_{\kappa+1} \to \mathcal{B}_\kappa$ is compact, and if κ is a limit ordinal then $\mathcal{B}_\kappa = \bigcup_{\kappa' < \kappa} \mathcal{B}_{\kappa'}$.

H. Furstenberg in [23] proved for each factor $\mathcal{B}_1 \subset \mathcal{B}$ the existence of the maximal $\widehat{\mathcal{B}} \subset \mathcal{B}$ such that $\widehat{\mathcal{B}} \to \mathcal{B}_1$ is distal. Actually this follows from the following lemma:

Lemma 2.1.7. *If* $\mathcal{B}_1 \supset \mathcal{B}$ *and* $\mathcal{B}_2 \supset \mathcal{B}$ *are ergodic distal extensions and* $\lambda \in J^e(\mathcal{B}_1, \mathcal{B}_2)$ *satisfies* $\lambda|_{\mathcal{B} \otimes \mathcal{B}} = \Delta$*, then* $(\mathcal{B}_1 \otimes \mathcal{B}_2, \lambda)$ *is a distal extension of* $\mathcal{B}$.

Proof. Let $\lambda \in J^e(\mathcal{B}_1, \mathcal{B}_2)$ and $\lambda|_{\mathcal{B} \otimes \mathcal{B}} = \Delta$. By Theorem 1.4.5, if $\mathcal{B}_1$ and B_2 are group extensions of $\mathcal{B}$ then λ is a group extension of μ because $(\mathcal{B} \otimes \mathcal{B}, \Delta)$ is isomorphic to $(\mathcal{B}, \mu)$. Consequently, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are isometric extensions of $\mathcal{B}$, then, by Theorem 1.4.2, λ is also isometric extension of $\mathcal{B}$.

Now we will use transfinite induction. Assume that $\widetilde{\mathcal{B}}_1$ and $\widetilde{\mathcal{B}}_2$ are ergodic extensions of $\mathcal{B}$ such that each ergodic joining of $\widetilde{\mathcal{B}}_1$ and $\widetilde{\mathcal{B}}_2$ which projects onto $\mathcal{B} \otimes \mathcal{B}$ as the diagonal measure, is a distal extension of $\mathcal{B}$. Let $\mathcal{B}_1 \subset \widetilde{\mathcal{B}}_1$ and $\mathcal{B}_2 \subset \widetilde{\mathcal{B}}_2$ be ergodic isometric extensions. Extend λ to an ergodic joining $\widehat{\lambda}$ of some ergodic group covers of $\mathcal{B}_1$ and $\mathcal{B}_2$. Then $\widehat{\lambda}$ is a group extension of $\mathcal{B}$. Again by the Theorem 1.4.2, λ is an isometric extension of $\mathcal{B}$.

If $\mathcal{B}_1$ and $\mathcal{B}_2$ are inverse limits of consecutive isometric extensions, then by the considerations above λ is a distal extension of $\mathcal{B}$ as an inverse limit of isometric extensions of $\mathcal{B}$. □

Now, let us consider $\lambda \in J(T, S)$, where $T\colon (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, $S\colon (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$. Then there exist the biggest σ-algebras $\mathcal{B}_1(\lambda) \subset \mathcal{B}$, $\mathcal{B}_2(\lambda) \subset \mathcal{C}$ such that λ identifies $\mathcal{B}_1(\lambda) \times Y$ with $X \times \mathcal{B}_2(\lambda)$. Indeed, take the family of all pairs $(\mathcal{B}_1, \mathcal{B}_2)$, $\mathcal{B}_1 \subset \mathcal{B}$, $\mathcal{B}_2 \subset \mathcal{C}$, where λ identifies $\mathcal{B}_1 \times Y$ with $X \times \mathcal{B}_2$. Then the smallest factor containing all of $\mathcal{B}_1$, say $\widetilde{\mathcal{B}}_1$, and the smallest one containing all of $\mathcal{B}_2$, say $\widetilde{\mathcal{B}}_2$, has the property, that $\widetilde{\mathcal{B}}_1 \times Y \stackrel{\lambda}{=} X \times \widetilde{\mathcal{B}}_2$. In fact, consider $\mathcal{B} \times Y$ and $X \times \mathcal{C}$ as two sub-σ-algebras of $\mathcal{B} \times \mathcal{C}$, where the equality between sets is understood $\bmod \lambda$. Then $\mathcal{B} \times Y \cap X \times \mathcal{C}$ is on one hand a sub-σ-algebra of $\mathcal{B} \times Y$, so of the form $\mathcal{B}' \times Y$, and on the other hand, a sub-σ-algebra of $X \times \mathcal{C}$, so of the form $X \times \mathcal{C}'$. We have $\mathcal{B}_1(\lambda) = \mathcal{B}'$ and $\mathcal{B}_2(\lambda) = \mathcal{C}'$.

2.2. Furstenberg decomposition

Definition 2.2.1. Let $T\colon (X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be an ergodic automorphism and $\mathcal{A}\subset\mathcal{B}$ be its T-invariant σ-algebra. We call T *relatively weakly mixing* (rel. w. m.) with respect to $\mathcal{A}$, if the relatively independent extension of the diagonal measure on $\mathcal{A}$, say $\lambda=\mu\times_{\mathcal{A}}\mu$, is ergodic. For short this will be denoted by $\mathcal{B}\to\mathcal{A}$ rel. w. m.

Note that if T_2 is weakly mixing and T_1 is ergodic then clearly $T_1\times T_2\to T_1$ rel. w. m.

Suppose that $\mathcal{B}\to\mathcal{A}_2$ rel. w. m. and we have $\mathcal{B}\supset\mathcal{A}_1\supset\mathcal{A}_2$. Then we can consider the relatively independent extension of the diagonal measure on $\mathcal{A}_2$ in $\mathcal{B}\otimes\mathcal{B}$ as well as in $\mathcal{A}_1\otimes\mathcal{A}_1$. The latter is a factor of the former, so obviously $\mathcal{A}_1\to\mathcal{A}_2$ rel. w. m.

Definition 2.2.2. Let $T\colon (X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be ergodic and $\mathcal{A}$ be a factor of it. Assume that $\mathcal{A}\subset\mathcal{A}_1\subset\mathcal{B}$ is another factor. The decomposition $\mathcal{B}\to\mathcal{A}_1\to\mathcal{A}$ is called a *Furstenberg decomposition* of $\mathcal{B}\to\mathcal{A}$, if $\mathcal{B}\to\mathcal{A}_1$ rel. w. m. and $\mathcal{A}_1\to\mathcal{A}$ is distal.

By the method presented in [23] we know that for each $\mathcal{A}\subset\mathcal{B}$ there exists a Furstenberg decomposition of $\mathcal{B}\to\mathcal{A}$.

Proposition 2.2.3. *For any $\mathcal{A}\subset\mathcal{B}$ there exists only one Furstenberg decomposition of $\mathcal{B}\to\mathcal{A}$.*

Proof. Let $\mathcal{C}$ be the maximal distal extension of $\mathcal{A}$ such that $\mathcal{B}\to\mathcal{C}\to\mathcal{A}$. Take any Furstenberg decomposition $\mathcal{B}\to\widetilde{\mathcal{A}}\to\mathcal{A}$ of $\mathcal{B}\to\mathcal{A}$. Then, by Lemma 2.1.7, each ergodic joining of $\mathcal{C}$ and $\widetilde{\mathcal{A}}$ that projects onto $\mathcal{A}\otimes\mathcal{A}$ as the diagonal measure is a distal extension of $\mathcal{A}$. Therefore $\widetilde{\mathcal{A}}\subset\mathcal{C}$. Conversely, since $\mathcal{B}\to\widetilde{\mathcal{A}}$ is rel. w. m., so is $\mathcal{C}\to\widetilde{\mathcal{A}}$. Hence $\mathcal{C}=\widetilde{\mathcal{A}}$. □

Proposition 2.2.4. *Let $T\colon (X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be ergodic and*

$$T'\colon (X',\mathcal{B}',\mu')\to(X',\mathcal{B}',\mu'),\qquad T_1\colon (X_1,\mathcal{B}_1,\mu_1)\to(X_1,\mathcal{B}_1,\mu_1)$$

be its two ergodic extensions. Suppose that $\lambda\in J^e(T',T_1)$ is such that $\lambda|_{X\times X}=\Delta_X$. Assume moreover, that $(X_1,\mu_1)\to(X,\mu)$ is distal and $(X'\times X_1,\lambda)\to(X',\mu')$ rel. w. m. *Then in $(X'\times X_1,\lambda)$ we have $\mathcal{B}'\times X_1\to X'\times\mathcal{B}_1$.*

Proof. Let us assume that

$$(X'\times X_1,\lambda)\to(\widehat{X}_1,\widehat{\mu}_1)\to(X_1,\mu_1)\quad\text{and}\quad(X',\mu')\to(\widehat{X},\widehat{\mu})\to(X,\mu)$$

are Furstenberg decompositions. It is then clear that the extension $(\widehat{X}_1,\widehat{\mu}_1)\to(X,\mu)$ is distal. By Lemma 2.1.7, the maximality of $\widehat{X}_1$ and the fact that the extension $\widetilde{X}\to X$, where $\widetilde{X}$ is the smallest factor of $(X'\times\widehat{X}_1,\lambda)$ containing $\widehat{X}_1$ and $\widehat{X}$, is distal, we must have $\widehat{X}\subset\widehat{X}_1$. Therefore, (X',μ') and $(\widehat{X}_1,\widehat{\mu}_1)$

are relatively disjoint over $\widehat{X}$. Thus, no harm arises if we assume that λ is the relative product of X' and $\widehat{X}_1$ over $\widehat{X}$. To be more precise, let

$$\begin{aligned} X' &= \widehat{X} \times Z', & T'(\widehat{x}, z') &= (\widehat{T}\widehat{x}, \theta'_{\widehat{x}}(z')), \\ \widehat{X}_1 &= \widehat{X} \times Z_1, & T_1(\widehat{x}, z_1) &= (\widehat{T}\widehat{x}, \theta^1_{\widehat{x}}(z_1)). \end{aligned}$$

Therefore, the relative product $X' \times_{\widehat{X}} \widehat{X}_1$ of X' and $\widehat{X}_1$ over $\widehat{X}$, denote it by $\widetilde{T}\colon (\widehat{X} \times Z' \times Z_1) \to (\widehat{X} \times Z' \times Z_1)$, is defined by the formula

$$\widetilde{T}(\widehat{x}, z', z_1) = (\widehat{T}\widehat{x}, \theta'_{\widehat{x}}(z'), \theta^1_{\widehat{x}}(z_1)).$$

By our assumption, the relative product $\widetilde{\widetilde{T}} = \widetilde{T} \times_{X'} \widetilde{T}\colon \widehat{X} \times Z' \times Z_1 \times Z_1 \to \widehat{X} \times Z' \times Z_1 \times Z_1$ is ergodic. It is clear that

$$\widetilde{\widetilde{T}}(\widehat{x}, z', z_1, z_2) = (\widehat{T}\widehat{x}, \theta'_{\widehat{x}}, \theta^1_{(\widehat{x}, z')}(z_1), \theta^1_{(\widehat{x}, z')}(z_2)),$$

where $\theta^1_{(\widehat{x}, z')}(z_i) = \theta^1_{\widehat{x}}(z_i)$, $i = 1, 2$. Therefore the relative product $\widehat{X}_1 \times_{\widehat{X}} \widehat{X}_1$ which is defined on $\widehat{X} \times Z_1 \times Z_1$ by the formula

$$(\widehat{x}, z_1, z_2) \mapsto (\widehat{T}\widehat{x}, \theta^1_{\widehat{x}}(z_1), \theta^1_{\widehat{x}}(z_2))$$

is a factor of $\widetilde{\widetilde{T}}$, hence is ergodic. This means however that $\widehat{X}_1 = \widehat{X}$ that completes the proof. $\square$

As a consequence we have

Proposition 2.2.5. *Let $T\colon (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be ergodic and $\{\mathcal{A}_i : i \in I\}$ a family of its factors such that $\mathcal{B} \to \mathcal{A}_i$* rel. w. m. *for each $i \in I$. Then*

$$\mathcal{B} \to \mathcal{A} = \bigcap_{i \in I} \mathcal{A}_i \quad \text{rel. w. m.}$$

Proof. Let $\mathcal{A}' \supset \mathcal{A}$ be the maximal distal extension of $\mathcal{A}$ in $\mathcal{B}$. Then $\mathcal{B} \to \mathcal{A}'$ rel. w. m. By virtue of Proposition 2.2.4, $\mathcal{A}' \subset \mathcal{A}_i$ and consequently $\mathcal{A}' \subset \mathcal{A}$, hence $\mathcal{A}' = \mathcal{A}$. $\square$

Proposition 2.2.6. *Suppose that $\mathcal{B} \supset \mathcal{A}_1 \supset \mathcal{A}_2$, and that both extensions $\mathcal{B} \to \mathcal{A}_1$ and $\mathcal{A}_1 \to \mathcal{A}_2$ are relatively weakly mixing. Then the extension $\mathcal{B} \to \mathcal{A}_2$ is relatively weakly mixing as well.*

Proof. Let $\widehat{\mathcal{A}}_2$ be the maximal distal extension of $\mathcal{A}_2$ in $\mathcal{B}$. We have $\widehat{\mathcal{A}}_1 \to \mathcal{A}_2$ is distal while $\mathcal{A}_1 \to \mathcal{A}_2$ rel. w. m. Therefore $\widehat{\mathcal{A}}_2$ and $\mathcal{A}_1$ are disjoint relatively to $\mathcal{A}_2$, so $\widehat{\mathcal{A}}_2 \cap \mathcal{A}_1 = \mathcal{A}_2$. It follows from Proposition 2.2.5 that $\mathcal{B} \to \widehat{\mathcal{A}}_2 \cap \mathcal{A}_1$ rel. w. m. $\square$

2.3. Semisimplicity

Definition 2.3.1. Let $T\colon (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ be ergodic. We say that T is *semisimple* if for every self-joining $\lambda \in J^e(T,T)$ we have

$$(X \times X, \lambda) \xrightarrow{\pi_i} (X,\mu) \text{ rel. w. m.},$$

where $\pi_i\colon X \times X \to X$, $\pi(x_1,x_2) = x_i$, $i = 1,2$.

Below we present some examples.

Example 2.3.2. Suppose that the automorphism T has discrete spectrum. In such a case each joining $\lambda \in J^e(T,T)$ is a graph joining, so T is semisimple.

Example 2.3.3. Assume that T has 2-fold simplicity property, i.e. if $\lambda \in J^e(T,T)$ then either λ is a graph joining or $\lambda = \mu \times \mu$. So, immediately from Definition 2.3.1 we get that T is semisimple.

Example 2.3.4. $T_1,\ldots,T_k$, $1 \le k \le \infty$, with MSJ (for definition see Section 1.3). Then clearly $T_1 \times \ldots \times T_k$ is semisimple.

All the examples above are in some sense pure; they are either weakly mixing or discrete spectrum. Semisimple maps can however have mixed spectrum.

Example 2.3.5. $T = T_1 \times T_2$, where T_1 has discrete spectrum and T_2 has MSJ. Then each $\lambda \in J^e(T,T)$ is either a graph joining (T can be viewed as a group extension with a constant cocycle of T_2) or appears in the ergodic decomposition of $\mu \times \mu$, where $\mu = \mu_1 \times \mu_2$. Any such a λ is isomorphic to $T_1 \times T_2 \times T_2$, so T is semisimple.

Proposition 2.3.6. *Let $T\colon (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ be semisimple and let $\mathcal{A}_1,\mathcal{A}_2 \subset \mathcal{B}$ be factors. Suppose that $\mathcal{B} \to \mathcal{A}_j$* rel. w. m.*, $j = 1,2$. Then, for each $\overline{\lambda} \in J^e(\mathcal{A}_1,\mathcal{A}_2)$, we have*

$$(\mathcal{A}_1 \otimes \mathcal{A}_2, \overline{\lambda}) \to (\mathcal{A}_j,\mu) \quad \text{rel. w. m.,}\ j = 1,2.$$

Proof. Extend $\overline{\lambda}$ to $\lambda \in J^e(T,T)$ whose projection on $\mathcal{A}_1 \otimes \mathcal{A}_2$ is $\overline{\lambda}$. We have that both extensions $(X \times X,\lambda) \to (X,\mu) \to (\mathcal{A}_1,\mu)$ are rel. w. m. By Proposition 2.2.6, $(X \times X,\lambda) \to (\mathcal{A}_1,\mu)$ rel. w. m. But obviously, we have a sequence of factors

$$(X \times X,\lambda) \to (\mathcal{A}_1 \otimes \mathcal{A}_2,\overline{\lambda}) \to (\mathcal{A}_1,\mu),$$

so we must have $(\mathcal{A}_1 \otimes \mathcal{A}_2,\overline{\lambda}) \to (\mathcal{A}_1,\mu)$ rel. w. m. □

Substituting in Proposition 2.3.6, $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$ we obtain the following

Corollary 2.3.7. *Suppose that $T\colon (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ is semisimple and let $\mathcal{A} \subset \mathcal{B}$ be a factor. If $\mathcal{B} \to \mathcal{A}$* rel. w. m.*, then also $\overline{T}\colon (\overline{X},\mathcal{A},\overline{\mu}) \to (\overline{X},\mathcal{A},\overline{\mu})$ is semisimple.*

2.4. Natural factors and the structure of factors for semisimple automorphisms

Definition 2.4.1. Let $T\colon (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ be ergodic. Suppose that $\mathfrak{N}$ is a class of factors satisfying

(2.2) $\mathfrak{N}$ is closed under taking intersections and containing $\mathcal{B}$ and the trivial σ-algebra $\mathcal{N}$.

We will call $\mathfrak{N}$ *natural* if

(N-1) $\mathcal{B}_i(\lambda) \in \mathfrak{N}$ for all $\lambda \in J^e(T,T)$, $i=1,2$.

(N-2) If $\mathcal{A}_1, \mathcal{A}_2 \in \mathfrak{N}$ and $S\colon \mathcal{A}_1 \to \mathcal{A}_2$ establishes an isomorphism then S sends natural factors contained in $\mathcal{A}_1$ into natural factors contained in $\mathcal{A}_2$.

Since $\mathfrak{N}$ is closed under intersections, for each factor $\mathcal{A} \subset \mathcal{B}$ we have a smallest natural factor $\widehat{\mathcal{A}} \in \mathfrak{N}$ with $\widehat{\mathcal{A}} \supset \mathcal{A}$.

Definition 2.4.2. Let $\mathcal{A} \subset \mathcal{B}$ be a factor. The smallest natural factor $\widehat{\mathcal{A}} \in \mathfrak{N}$ such that $\widehat{\mathcal{A}} \supset \mathcal{A}$ will be called the *natural cover* of $\mathcal{A}$.

Remark 2.4.3. Suppose that T is an ergodic automorphism. Then directly from the definition it follows that there exists the smallest family $\mathfrak{N}_0$ of natural factors. Note also that if $\mathcal{B} \to \mathcal{A}$ rel. w. m. then $\mathcal{A} \in \mathfrak{N}_0$. Indeed, in such a case we have $\mu \times_{\mathcal{A}} \mu \in J^e(\mathcal{B},\mathcal{B})$ and obviously $\mathcal{B}_i(\mu \times_{\mathcal{A}} \mu) = \mathcal{A}$, $i=1,2$.

Proposition 2.4.4. *A family $\mathfrak{N}$ satisfying (2.2) is natural if and only if whenever $\lambda \in J^e(T,T)$ and λ restricted to factors $\mathcal{A}_1 \otimes \mathcal{A}_2$ establishes their isomorphism then λ is an isomorphism on the natural covers.*

Proof. Suppose that a family $\mathfrak{N}$ of factors satisfies (2.2).

First assume that $\mathfrak{N}$ is natural. Let $\lambda \in J^e(T,T)$ and $\lambda|_{\mathcal{A}_1\otimes\mathcal{A}_2}$ is an isomorphism. By (N-1) we have that $\widehat{\mathcal{A}_j} \subset \mathcal{B}_j(\lambda)$, $j=1,2$, and (N-2) completes this part of the proof.

To prove the converse take $\lambda \in J^e(T,T)$, then λ establishes an isomorphism between $\mathcal{B}_1(\lambda)$ and $\mathcal{B}_2(\lambda)$. Since these two are the biggest with this property we must have $\widehat{\mathcal{B}_j(\lambda)} = \mathcal{B}_j(\lambda)$, $j=1,2$ and (N-1) follows. Now, let $\mathcal{A}_1, \mathcal{A}_2 \in \mathfrak{N}$ and S be an isomorphism between them. Lift this isomorphism to a $\lambda \in J^e(T,T)$. Take $\mathcal{A} \in \mathfrak{N}$ with $\mathcal{A} \subset \mathcal{A}_1$. Then λ is an isomorphism of $\mathcal{A} = \widehat{\mathcal{A}}$ with $S\mathcal{A}$ but also with $\widehat{S\mathcal{A}}$. Hence $S\mathcal{A} = \widehat{S\mathcal{A}}$ so $S\mathcal{A} \in \mathfrak{N}$ which completes the proof. $\square$

Corollary 2.4.5. *Let $\mathfrak{N}$ be a natural family of factors for T. Then for each factor $\mathcal{A}$ of T the extension $\widehat{\mathcal{A}} \to \mathcal{A}$ is a group extension.*

Proof. Take any ergodic self-joining λ on $\widehat{\mathcal{A}} \otimes \widehat{\mathcal{A}}$ that is diagonal on $\mathcal{A} \otimes \mathcal{A}$. Hence λ establishes an isomorphism of $\mathcal{A}$ with itself (by the identity). From Proposition 2.4.4, λ is an isomorphism of $\widehat{\mathcal{A}}$ with itself, so λ is a graph joining on $\widehat{\mathcal{A}} \otimes \widehat{\mathcal{A}}$. By Veech's Theorem (see Theorem 1.4.1), $\widehat{\mathcal{A}} \to \mathcal{A}$ is a group extension. $\square$

Lemma 2.4.6. *Let $\widehat{T_i}\colon(\widehat{X_i},\widehat{\mathcal{B}_i},\widehat{\mu_i})\to(\widehat{X_i},\widehat{\mathcal{B}_i},\widehat{\mu_i})$, $i=1,2$, be ergodic distal extensions of $T_i\colon(X_i,\mathcal{B}_i,\mu_i)\to(X_i,\mathcal{B}_i,\mu_i)$, $i=1,2$. Assume that $\widehat{\lambda}\in J^e(\widehat{T_1},\widehat{T_2})$ has the property that its restriction λ to $\mathcal{B}_1\otimes\mathcal{B}_2$ is a graph joining and moreover for $i=1,2$ the extension $(\widehat{\mathcal{B}_1}\otimes\widehat{\mathcal{B}_2},\widehat{\lambda})\to(\widehat{\mathcal{B}_i},\widehat{\mu_i})$ is* rel. w. m. *Then $\widehat{\lambda}$ is also a graph joining.*

Proof. Note that in $(\widehat{\mathcal{B}_1}\otimes\widehat{\mathcal{B}_2},\widehat{\lambda})$ the σ-algebras $\mathcal{B}_1=\mathcal{B}_2 \pmod{\widehat{\lambda}}$. Therefore

$$\widehat{\mathcal{B}_1}\otimes\widehat{\mathcal{B}_2}\to\widehat{\mathcal{B}_1}\to\mathcal{B}_1 \quad\text{and}\quad \widehat{\mathcal{B}_2}\otimes\widehat{\mathcal{B}_2}\to\widehat{\mathcal{B}_2}\to\mathcal{B}_2$$

are, by assumption, two Furstenberg decompositions of $\mathcal{B}_1=\mathcal{B}_2$. By Proposition 2.2.3 we have $\widehat{\mathcal{B}_1}=\widehat{\mathcal{B}_2}\pmod{\widehat{\lambda}}$, so $\widehat{\lambda}$ is an isomorphism of $\widehat{\mathcal{B}_1}$ and $\widehat{\mathcal{B}_2}$. $\square$

Below, we will consider a family of natural factors for semisimple maps.

Proposition 2.4.7. *Let $T\colon(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be ergodic and semisimple. Put*

$$\mathfrak{N}=\{\mathcal{A}\subset\mathcal{B}\colon\mathcal{B}\to\mathcal{A}\text{ rel. w. m.}\}\cup\{\mathcal{N}\}.$$

Then the family $\mathfrak{N}$ is natural.

Proof. By Proposition 2.2.5, $\mathfrak{N}$ is closed under intersections. We will prove that if $\lambda\in J^e(T,T)$ establishes an isomorphism of $\mathcal{A}_1$ and $\mathcal{A}_2$ then λ is also an isomorphism of natural covers $\widehat{\mathcal{A}_1}$ and $\widehat{\mathcal{A}_2}$. Now, $\widehat{\mathcal{A}_1}$ and $\widehat{\mathcal{A}_2}$ can be described as the maximal distal extensions of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. By Proposition 2.3.6, if by $\widehat{\lambda}$ we denote the restriction of λ to $\widehat{\mathcal{A}_1}\otimes\widehat{\mathcal{A}_2}$ then $(\widehat{\mathcal{A}_1}\otimes\widehat{\mathcal{A}_2},\widehat{\lambda})\to(\widehat{\mathcal{A}_i},\mu)$ rel. w. m., $j=1,2$. Lemma 2.4.6 finishes the proof. $\square$

By applying Proposition 2.4.7 and Corollary 2.4.5 we obtain the following

Theorem 2.4.8 (Structure Theorem)**.** *Assume that the automorphism T: $(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is ergodic and semisimple. Then for each factor $\mathcal{A}$ there exists an $\widehat{\mathcal{A}}$ with $\mathcal{B}\to\widehat{\mathcal{A}}$* rel. w. m. *such that $\widehat{\mathcal{A}}$ is a group extension of $\mathcal{A}$.*

Remark 2.4.9. If T is 2-fold simple then the only factors with respect to which T is rel. w. m. are the trivial ones, so applying Theorem 2.4.8 we obtain the well known Veech's Theorem on factors of 2-fold simple maps (see [95], [45]).

Reamrk 2.4.10. Applying Theorem 2.4.8 it is very easy to give examples of T that are not semisimple. Indeed, if there are $\mathcal{B}_2\subset\mathcal{B}_1\subset\mathcal{B}$ such that $\mathcal{B}\to\mathcal{B}_1$ isometric, $\mathcal{B}_1\to\mathcal{B}_2$ isometric but $\mathcal{B}\to\mathcal{B}_2$ is not isometric, then $\mathcal{B}$ is not semisimple. Since $\mathcal{B}\to\mathcal{B}_2$ is distal, we must have $\widehat{\mathcal{B}_2}=\mathcal{B}$. If $\mathcal{B}$ were semisimple, then, by Theorem 2.4.8, $\mathcal{B}\to\mathcal{B}_2$ would be a group extension.

Corollary 2.4.11. *If $T\colon(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is ergodic and semisimple then its entropy $h(T)$ is equal to zero.*

Proof. First, note that no Bernoulli $T\colon(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is semisimple. Indeed, take any nontrivial weakly mixing compact group extension $T_\varphi\colon(X\times$

$G, \widetilde{\mathcal{B}}, \widetilde{\mu}) \to (X \times G, \widetilde{\mathcal{B}}, \widetilde{\mu})$ of T. By [87], T_φ is again Bernoulli with the same entropy as T. Now, in $\widetilde{\mathcal{B}}$ we have two factors, namely, $\widetilde{\mathcal{B}}$ and $\mathcal{B}$ isomorphic to T. If T were semisimple, then the smallest factor containing these two factors (equal to $\widetilde{\mathcal{B}}$) would have to be rel. w. m. with respect to $\mathcal{B}$; a contradiction.

Suppose that $h(T) > 0$. Then there exists a Bernoulli factor $\mathcal{A}$ with the same entropy. Take the natural cover $\widehat{\mathcal{A}}$ of $\mathcal{A}$. Then $\widehat{\mathcal{A}} \to \mathcal{A}$ is a compact group extension. If $\widehat{\mathcal{A}}$ is weakly mixing then $\widehat{\mathcal{A}}$ is Bernoulli, so $\widehat{\mathcal{A}}$ is semisimple. In general, $\widehat{\mathcal{A}}$ can be represented as $\widehat{\mathcal{A}} = \widetilde{\mathcal{A}} \otimes \mathcal{K}$, where $\widetilde{\mathcal{A}}$ is Bernoulli and $\mathcal{K}$ is the maximal Kronecker factor of $\widehat{\mathcal{A}}$ (see [87]). Moreover, $\widetilde{\mathcal{A}}$ can be represented as a nontrivial group extension of a Bernoulli factor, say of $\widetilde{\mathcal{A}}_1$. Hence $\widetilde{\mathcal{A}} \otimes \mathcal{K}$ is a nontrivial group extension of $\widetilde{\mathcal{A}}_1 \otimes \mathcal{K}$. But these two automorphisms are isomorphic so the former is not semisimple. $\square$

Remark 2.4.12. Suppose that T is ergodic and distal. Then T is semisimple if and only if T has discrete spectrum. Indeed, if T is semisimple and $\mathcal{K}$ is its maximal Kronecker factor then $\mathcal{B} \to \mathcal{K}$ rel. w. m. ($\widehat{\mathcal{K}}$ is a group extension of $\mathcal{K}$ that is a group extension of one point dynamical system; since $\widehat{\mathcal{K}}$ must be semisimple, we have $\widehat{\mathcal{K}} = \mathcal{K}$).

2.5. Joinings of ergodic group extensions of semisimple automorphisms

Assume that $T: (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ and $S: (Y, \mathcal{C}, m) \to (Y, \mathcal{C}, m)$ are ergodic automorphisms. Let G_1, G_2 be compact metric groups with Haar measures ν_1, ν_2 respectively. Let $\varphi_1: X \to G_1$, $\varphi_2: Y \to G_2$ be such that T_{φ_1} and S_{φ_2} are ergodic.

Suppose that $\lambda \in J^e(T, S)$ has the property that the two extensions

$$(T \times S, \lambda) \to (T, \mu) \quad \text{and} \quad (T \times S, \lambda) \to (S, m)$$

are rel. w. m. The following theorem describes any $\widetilde{\lambda} \in J^e(T_{\varphi_1}, S_{\varphi_2})$ whose projections on $\mathcal{B} \otimes \mathcal{C}$ is λ.

Theorem 2.5.1. *There are normal closed subgroups $H_1 \subset G_1$, $H_2 \subset G_2$, a continuous group isomorphism $v: G_1/H_1 \to G_2/H_2$ and a Borel map $f: X \times Y \to G_2/H_2$ such that for any Borel sets $A \subset X$, $C_1 \subset G_1$, $B \subset Y$, $C_2 \subset G_2$ we have*

$$\begin{aligned}\widetilde{\lambda}(A \times C_1 \times B \times C_2) = \int_{X \times Y \times G_1/H_1} & E(\chi_{A \times Y \times C_1} | H_1)(x, y, g_1 H_1) \\ & \cdot E(\chi_{X \times B \times C_2} | H_2)(x, y, f(x,y) v(g_1 H_1)) \, d(\lambda \times \nu_1)(x, y, g_1 H_1).\end{aligned}$$

The proof of Theorem 2.5.1 is long and is divided into several lemmas.

Let $\pi: X \times G_1 \times Y \times G_2 \to X \times Y$, $\pi(x, g_1, y, g_2) = (x, y)$. Then $\pi^* \widetilde{\lambda} = \lambda$. Let us decompose $\widetilde{\lambda}$ over the factor $(X \times Y, \lambda, T \times S)$:

$$\widetilde{\lambda} = \int_{X \times Y} \widetilde{\lambda}_{(x,y)} \, d\lambda(x, y).$$

Let $H = \{(h_1, h_2) \in G_1 \times G_2 : \widetilde{\lambda}(h_1, h_2) = \widetilde{\lambda}\}$, be the stabilizer of $\widetilde{\lambda}$. By Lemma 1.4.4,

$$\widetilde{\lambda} = \int_{X \times Y} \delta_{(x,y)} \times (g_1, g_2)\nu_H \, d\lambda(x, y),$$

where $(g_1, g_2)H = \tau(x, y)$.

Let $H_1 \subset G_1$, $H_2 \subset G_2$ be given by

$$H_1 = \{g_1 \in G_1 : (g_1, e_2) \in H\}, \quad H_2 = \{g_2 \in G_2 : \ (e_1, g_2) \in H\}$$

where e_i denotes the unit element of the group G_1, $i = 1, 2$. Put $\pi_i: G_1 \times G_2 \to G_i$, $\pi_i(g_1, g_2) = g_i$, $i = 1, 2$.

Lemma 2.5.2. $\pi_i(H) = G_i$, $i = 1, 2$.

Proof. First, we note that $T_{\varphi_1} \times S_{\varphi_2}, X \times G_1 \times Y \times G_2, \widetilde{\lambda})$ is a group extension of $T \times S$, where the group we extend by is H (see Theorem 1.4.5). If we take the projection onto the first three coordinates, then we get a group extension of $(T \times S, X \times Y, \lambda)$ by $\pi_1(H)$. This group extension is ergodic. On the other hand, $(T_{\varphi_1}, X \times G_1, \widetilde{\mu}) \to (T, X, \mu)$ is a group extension and $(T \times S, X \times Y, \lambda) \to (T, X, \mu)$ rel. w. m., so the relative product $T_{\varphi_1} \times_{(X,\mu,T)} (T \times S, \lambda)$ is ergodic. This relative product is equal to $((T \times S)_{\varphi_2}, \widehat{\lambda})_{\varphi_1}$, i.e. it is a group extension of $(T \times S, \lambda)$ via G_1. Since the latter is ergodic, $\pi_1(H) = G_1$.

The proof of the equality $\pi_2(H) = G_2$ is similar. □

The next lemma immediately follows from Lemma 2.5.2.

Lemma 2.5.3. *The subgroups H_1 and H_2 are normal in G_1 and G_2 respectively.*

Lemma 2.5.4.

(a) *If $(g_1, g_2), (g_1, \widetilde{g}_2) \in H$ then $\widetilde{g}_2^{-1} g_2 \in H_2$.*
(b) *If $(g_1, g_2) \in H, (\widetilde{g}_1, g_2) \in H$ then $\widetilde{g}_1^{-1} g_1 \in H_1$.*
(c) *$(g_1, g_2) \in H$ if and only if $g_1 H_1 \times g_2 H_2 \subset H$.*

Proof. (a) Let us assume that $(g_1, g_2) \in H$, $(g_1, \widetilde{g}_2) \in H$. Then $(g_1^{-1}, g_2^{-1}) \in H$ and $H \ni (g_1^{-1}, \widetilde{g}_2^{-1})(g_1, g_2) = (e_1, \widetilde{g}_2^{-1} g_2)$. Therefore $\widetilde{g}_2^{-1} g_2 \in H_2$. The proof of (b) is similar.

(c) Assume that $(g_1, g_2) \in H$. Take $h_1 \in H_1$ and $h_2 \in H_2$. Then $(h_1, e_2) \in H$ and $(e_1, h_2) \in H$. Therefore $(h_1, h_2) \in H$ and, by assumption,

$$H \ni (g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2).$$

Because h_1, h_2 were arbitrary, $g_1H_1 \times g_2H_2 \subset H$. □

We define a map $v: G_1/H_1 \to G_2/H_2$ by the following formula

$$v(g_1H_1) = \pi_2((g_1H_1 \times G_2) \cap H). \tag{2.3}$$

Lemma 2.5.5. *The map v defined by* (2.3) *is a continuous group isomorphism.*

Proof. By Lemma 2.5.4, v is well defined. The continuity of v is evident. Obviously v is bijective. Because $H_1 \times H_2 \subset H$, $v(H_1) = H_2$. We now prove that v is a group homomorphism.

Take $gH_1, \overline{g}H_1 \in G_1/H_1$. Set $v(gH_1\overline{g}H_1) = \widetilde{g}H_2$, $v(gH_1) = g_1H_2$, $v(\overline{g}H_1) = \overline{g}_1H_2$. Then $g\overline{g}H_1 \times \widetilde{g}H_2 \subset H$, $gH_1 \times g_1H_2 \subset H, \overline{g}H_1 \times \overline{g}_1H_2 \subset H$. This implies $g\overline{g}H_1 \times g_1\overline{g}_1H_2 \subset H$. By Lemma 2.5.4, $\widetilde{g}H_2 = g_1\overline{g}_1H_2$, i.e. $v(gH_1\overline{g}H_1) = v(gH_1)v(\overline{g}H_1)$.

Obviously $v(g^{-1}H_1) = v(gH_1)^{-1}$. □

As an immediate consequence of Lemma 2.5.4 and Lemma 2.5.5 we have

Lemma 2.5.6. $H = \bigcup_{g \in G_1} gH_1 \times v(gH_1)$.

Let

$$(T \times S)_{\varphi_i,H_i}: X \times Y \times G_i/H_i \to X \times Y \times G_i/H_i,$$
$$(T \times S)_{\varphi_i,H_i}(x_1, x_2, gH_i) = (Tx_1, Sx_2, \varphi_i(x_i)gH_i), \quad i = 1, 2.$$

Then $(X \times Y \times G_i/H_i, \lambda \times \nu_i, (T \times S)_{\varphi_i,H_i})$, $i = 1, 2$, is an ergodic dynamical system.

Our next aim is to define an isomorphism $\overline{I}$ of $(T \times S)_{\varphi_1,H_1}$ and $(T \times S)_{\varphi_2,H_2}$. It will have the form

$$\overline{I} = I_{f,v}: X \times Y \times G_1/H_1 \to X \times Y \times G_2/H_2,$$
$$I_{f,v}(x, y, gH_1) = (x, y, f(x, y)v(gH_1)),$$

for some measurable map $f: X \times Y \to G_2/H_2$.

Let $\alpha: (G_1 \times G_2)/H \to G_2/H_2$ be the (open) map given by

$$\alpha((g_1, g_2)H) = g_2v(g_1^{-1}H_1). \tag{2.4}$$

We have to prove that α is well defined. Assume that $(g_1, g_2)H = (\widetilde{g}_1, \widetilde{g}_2)H$. Then $(g_1^{-1}\widetilde{g}_1, g_2^{-1}\widetilde{g}_2) \in H$ and therefore

$$v(g_1^{-1}\widetilde{g}_1H_1) = g_2^{-1}\widetilde{g}_2H_2. \tag{2.5}$$

We will show that $(g_2v(g_1^{-1}H_1))^{-1}\widetilde{g}_2v(\widetilde{g}_1H_1) = H_2$.

Indeed, by (2.5),

$$\begin{aligned}(g_2v(g_1^{-1}H_1))^{-1}\widetilde{g}_2v(\widetilde{g}_1^{-1}H_1) &= v(g_1H_1)g_2^{-1}\widetilde{g}_2H_2v(\widetilde{g}_1^{-1}H_1)\\ &= v(g_1H_1)v(g_1^{-1}\widetilde{g}_1H_1)v(\widetilde{g}_1^{-1}H_1) = H_2.\end{aligned}$$

Thus α is well defined.

Having α we can define the desired function $f\colon X \times Y \to G_2/H_2$, by setting

$$f(x,y) = \alpha(\tau(x,y)), \tag{2.6}$$

where τ is defined by (1.10) and it satisfies (1.11) for $\varphi = \varphi_1 \times \varphi_2$.

Now, one easily checks that

$$(T \times S)_{\varphi_2,H_2} \circ \overline{I} = \overline{I} \circ (T \times S)_{\varphi_1,H_1}.$$

We will also use the following

Lemma 2.5.7.

(a) $\tau(x,y) = \bigcup_{g\in G_1} gH_1 \times f(x,y)v(gH_1)$ λ-*a.s.*

(b) $\widetilde{\lambda}\Big(\bigcup_{\substack{(x,y)\in X\times Y\\ g\in G_1}} \{(x,y)\} \times gH_1 \times f(x,y)v(gH_1)\Big) = 1.$

Proof. (a) Fix $(x,y) \in X \times Y$. Set $\tau(x,y) = (a,b)H$. Then by (2.4), (2.6) and Lemma 2.5.6,

$$\begin{aligned}\bigcup_{g\in G_1} gH_1 \times f(x,y)v(gH_1) &= \bigcup_{g\in G_1} gH_1 \times bv(a^{-1}H_1)v(gH_1)\\ &= \bigcup_{g\in G_1} gH_1 \times bv(a^{-1}gH_1) = \bigcup_{g\in G_1} agH_1 \times bv(gH_1)\\ &= (a,b)\bigcup_{g\in G_1} gH_1 \times v(gH_1) = (a,b)H = \tau(x,y).\end{aligned}$$

(b) Using (a) we have

$$\begin{aligned}1 &= \widetilde{\lambda}\Big(\bigcup_{(x,y)\in X\times Y} \{(x,y)\} \times \tau(x,y)\Big)\\ &= \widetilde{\lambda}\Big(\bigcup_{(x,y)\in X\times Y} \Big(\{(x,y)\} \times \bigcup_{g\in G_1} gH_1 \times f(x,y)v(gH_1)\Big)\Big)\\ &= \widetilde{\lambda}\Big(\bigcup_{\substack{(x,y)\in X\times Y\\ g\in G_1}} \{(x,y)\} \times gH_1 \times f(x,y)v(gH_1)\Big). \qquad \square\end{aligned}$$

Proof of Theorem 2.5.1. By Lemma 2.5.7 we can define an isomorphism

$$\begin{aligned}&U\colon (X \times Y \times G_1/H_1 \times G_2/H_2, \widetilde{\lambda}, (T \times S)_{\varphi_1\times\varphi_2,H_1\times H_2}) \to\\ &((X\times Y\times G_1/H_1)\times(X\times Y\times G_2/H_2), (\lambda\times\nu_1)_{\overline{I}}, (T\times S)_{\varphi_1,H_1}\times(T\times S)_{\varphi_2,H_2})\end{aligned}$$

by

$$U(x,y,gH_1,f(x,y)v(gH_1)) = (x,y,gH_1,x,y,f(x,y)v(gH_1)).$$

Then U sends the measure $\widetilde{\lambda}$ to $(\lambda \times \nu_1)_{\overline{I}}$ and we have

$$\widetilde{\lambda}(A \times B \times C) = \int_{X \times Y \times G_1/H_1} \chi_{A \times B}(x, y, gH_1) \cdot \chi_{A \times C}(x, y, f(x,y)v(gH_1))\, d(\lambda \times \nu_1)(x, y, gH_1)$$

for $A \subset X \times Y$, $B \subset G_1/H_1$, $C \subset G_2/H_2$.

Therefore, for $A \subset X \times G_1$, $B \subset Y \times G_2$,

$$\widetilde{\lambda}(A \times B) = \int_{X \times Y \times G_1/H_1} E(\chi_{\overline{A}} \mid H_1)(x, y, gH_1) \cdot E(\chi_{\overline{\overline{B}}} | H_2)(x, y, f(x,y)v(gH_1))\, d\lambda\, d\nu_1$$

which finishes the proof of Theorem 2.5.1. □

Corollary 2.5.8. *Assume $T\colon (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic semisimple automorphism. Let G be a compact metric group equipped with the normalized Haar measure ν, let $\varphi\colon X \to G$ be such that T_φ is ergodic, and suppose $\widetilde{\lambda} \in J^e(T_\varphi, T_\varphi)$ is an extension of some $\lambda \in J^e(T,T)$. Then there are normal closed subgroups $H_1, H_2 \subset G$, a continuous group isomorphism $v\colon G/H_1 \to G/H_2$ and a Borel map $f\colon X \times X \to G/H_2$ such that for any Borel sets $A, B \subset X$ and $C_1, C_2 \subset G$ we have*

$$\widetilde{\lambda}(A \times C_1 \times B \times C_2) = \int_{X \times X \times G/H_1} E(\chi_{A \times X \times C_1} | H_1)(x, y, gH_1) \cdot E(\chi_{X \times B \times C_2} | H_2)(x, y, f(x,y)v(gH_1)) d(\lambda \times \nu)(x, y, gH_1).$$ □

Assume that $T\colon (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic automorphism and $\varphi\colon X \to G$ a cocycle such that T_φ is ergodic. Suppose that $S \in C(T)$ has an extension to $\widetilde{S} \in C(T_\varphi)$. If we assume that additionally S is invertible then it is well known that

$$\widetilde{S}(x, g) = S_{f,v}(x, g) = (Sx, f(x)v(g)), \tag{2.7}$$

where $f\colon X \to G$ is measurable and $v\colon G \to G$ is a continuous group epimorphism (this result can be directly deduced from Theorem 2.5.1). In general we obtain the following:

Proposition 2.5.9. *If $\widetilde{S} \in C(T_\varphi)$ is an extension of some $S \in C(T)$ then $\widetilde{S}$ is of the form* (2.7)*, where $v\colon G \to G$ is a continuous group homomorphism* (*not necessarily onto*).

Proof. Write $\widetilde{S}(x, g) = (Sx, \psi(x, g))$. Since $\widetilde{S} T_\varphi = T_\varphi \widetilde{S}$ we get

$$\psi(T_\varphi(x, g)) = \varphi(Sx)\psi(x, g).$$

Writing $\sigma_g(x, h) = (x, hg)$ we have $\sigma_g \in C(T_\varphi)$. Set

$$F_g(x, h) = \psi(x, h)^{-1} \psi\sigma_g(x, h) = \psi(x, h)^{-1}\psi(x, hg).$$

We have

$$\begin{aligned} F_g T_\varphi(x,h) &= (\psi T_\varphi(x,h)^{-1}\psi\sigma_g T_\varphi(x,h) = (\psi T_\varphi(x,h))^{-1}\psi T_\varphi(x,hg) \\ &= (\varphi(Sx)\psi(x,h))^{-1}\varphi(Sx)\psi(x,hg) = F_g(x,h). \end{aligned}$$

Thus F_g is a constant function. Set

$$v(g) = F_g(\,\cdot\,,\,\cdot\,).$$

Clearly $v: G \to G$ is measurable. We now show that v is a group homomorphism. We have $v(e) = e$ and

$$\begin{aligned} v(g_1 g_2) &= \psi(x,h)^{-1}\psi(x,hg_1g_2) \\ &= \psi(x,h)^{-1}\psi(x,hg_1)\psi(x,hg_1)^{-1}\psi(x,(hg_1)g_2) = v(g_1)v(g_2). \end{aligned}$$

In particular, v is continuous.

Put

$$f(x,h) = \psi(x,h)v(h)^{-1} \quad \text{a.s.}$$

Take any $g \in G$. Then for a.e. (x,h) we have

$$\begin{aligned} f\sigma_g(x,h) &= f(x,hg) = \psi(x,hg)v(hg)^{-1} \\ &= \psi(x,h)\psi(x,h)^{-1}\psi(x,hg)v(g)^{-1}v(h)^{-1} = \psi(x,h)v(h) = f(x,h). \end{aligned}$$

Therefore f depends only on x □

Definition 2.5.10 ([79]). Assume that $\mathcal{A} \subset \mathcal{B}$ is a factor of T. We call it a *canonical* (resp. *weakly canonical*) factor of T if for each isomorphic copy $\mathcal{A}'$ of $\mathcal{A}$ we have $\mathcal{A}' = \mathcal{A}$ (resp. $\mathcal{A}' \subset \mathcal{A}$).

Proposition 2.5.11. *Suppose that $T:(X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ is semisimple. Let $\widehat{T}:(\widehat{X},\widehat{\mathcal{B}},\widehat{\mu}) \to (\widehat{X},\widehat{\mathcal{B}},\widehat{\mu})$ be an arbitrary ergodic distal extension of T. Then T is a weakly canonical factor of $\widehat{T}$.*

Proof. Suppose that $\mathcal{B}'$ is a factor of $\widehat{\mathcal{B}}$ isomorphic to $\mathcal{B}$. Let $\mathcal{A}$ be the smallest factor containing $\mathcal{B}$ and $\mathcal{B}'$. Since T is semisimple, $\mathcal{A} \to \mathcal{B}$ rel. w. m. However, $\widehat{\mathcal{B}} \to \mathcal{A} \to \mathcal{B}$, and $\widehat{\mathcal{B}} \to \mathcal{B}$ is a distal extension. Hence $\mathcal{A}$ and $\widehat{\mathcal{B}}$ are relatively (over $\mathcal{B}$) disjoint, and consequently $\mathcal{B} = \mathcal{A}$. □

Remark 2.5.12. Notice that the centralizer of a semisimple automorphism need not be a group; for instance, take $T = T_1 \times T_1 \times \ldots$, where T_1 has MSJ.

Remark 2.5.13. D. Newton in [79] asked about canonicality of automorphisms, i.e. whether there are automorphisms which are canonical factors in an arbitrary ergodic extension. As shown in [61], the only ones with this property are those with discrete spectrum. Let us ask what is the class of automorphisms which are canonical factors in an arbitrary ergodic distal extension. The above proposition says that semisimple coalescent automorphisms enjoy this property. The question arises whether they are the only ones.

It follows from Proposition 2.5.11 that a semisimple automorphism sits weakly canonically in any of its ergodic group extensions. In particular, if $\widetilde{S} \in C(T_\varphi)$, then $\widetilde{S}^{-1}(\mathcal{B}) \subset \mathcal{B}$ and we can apply Proposition 2.5.9. Hence we obtain the following generalization of the results from [4], [79], [71]:

Corollary 2.5.14. *If $T_\varphi\colon (X \times G, \widetilde{\mu}) \to (X \times G, \widetilde{\mu})$ is an ergodic group extension of a semisimple automorphism and $\widetilde{S} \in C(T_\varphi)$ then there are $S \in C(T)$, a Borel map $f\colon X \to G$ and a continuous group homomorphism $v\colon G \to G$ such that*

$$\widetilde{S}(x, g) = (Sx, f(x)v(g)).$$

If, additionally, T is coalescent, then v is onto.

2.6. Applications of natural families

Lemma 2.6.1. *Assume that $T\colon (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a 2-fold simple weakly mixing automorphism, $\varphi\colon X \to G$ a cocycle such that T_φ is weakly mixing. Let $\widetilde{\lambda} \in J^e(T_\varphi, T_\varphi)$ with $\widetilde{\lambda}|_{\mathcal{B}\otimes\mathcal{B}}$ an isomorphism. Then $\mathcal{B}_1(\widetilde{\lambda}) = \widetilde{\mathcal{B}}_{H_1}$ and $\mathcal{B}_2(\widetilde{\lambda}) = \widetilde{\mathcal{B}}_{H_2}$ for some H_1 and H_2 which are normal.*

Proof. Let $H \subset G \times G$ be the stabilizer of $\widetilde{\lambda}$. By Lemma 2.5.2, $\pi_i(H) = G$, $i = 1, 2$. Since $\mathcal{B}_1(\widetilde{\lambda})$ and $\mathcal{B}_2(\widetilde{\lambda})$ are two factors between $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ ($\widetilde{\lambda}$ is an isomorphism on the base), it follows that $\mathcal{B}_1(\widetilde{\lambda}) = \widetilde{\mathcal{B}}_{H_1}$ and $\mathcal{B}_2(\widetilde{\lambda}) = \widetilde{\mathcal{B}}_{H_2}$, where H_1, H_2 are closed subgroups of G (this easily follows from the relative version of Veech's Theorem). We will prove that for each $g \in G$

$$\widetilde{\mathcal{B}}_{g^{-1}H_1 g} \subset \mathcal{B}_1(\widetilde{\lambda}). \tag{2.8}$$

Fix $g \in G$. Since $\pi_i(H) = G$, $i = 1, 2$, there exists $g_2 \in G$ such that $(g, g_2) \in H$. We have

$$\sigma_g(\widetilde{\mathcal{B}}_{H_1}) = \widetilde{\mathcal{B}}_{g^{-1}H_1 g}, \quad \sigma_{g_2}(\widetilde{\mathcal{B}}_{H_2}) = \widetilde{\mathcal{B}}_{g_2^{-1}H_2 g_2},$$

so (by the definition of $\mathcal{B}_1(\widetilde{\lambda})$ it is enough to show that

$$\sigma_g \widetilde{\mathcal{B}}_{H_1} = \sigma_{g_2} \widetilde{\mathcal{B}}_{H_2} \mod \widetilde{\lambda}.$$

This is however obvious, because if $A \in \widetilde{\mathcal{B}}_{H_1}$, $B \in \widetilde{\mathcal{B}}_{H_2}$ and

$$\widetilde{\lambda}(A \times (X \times G) \triangle (X \times G) \times B) = 0$$

then

$$\widetilde{\lambda}(\sigma_g A \times (X \times G) \triangle (X \times G) \times \sigma_{g_2} B) = \widetilde{\lambda}(A \times (X \times G) \triangle (X \times G) \times B) = 0.$$

Therefore (2.8) follows. The proof is complete by the symmetry of the argument. □

Lemma 2.6.2. *Let T be semisimple and coalescent, $\varphi: X \to G$ ergodic, $H \subset G$ a closed subgroup and $\widehat{S} \in C(T_\varphi)$. Assume that $\widetilde{\mathcal{B}}_H$ is $\widehat{S}$-invariant. If $\widehat{S}$ is invertible on $\widetilde{\mathcal{B}}$ then $\widehat{S}$ so is on $\widetilde{\mathcal{B}}_H$.*

Proof. By Corollary 2.5.14 we have

$$\widehat{S}(x,g) = S_{f,v}(x,g) = (Sx, f(x)v(g)),$$

where $v: G \to G$ is a group automorphism. We have assumed that $\widehat{S}^{-1}\widetilde{\mathcal{B}}_H \subset \widetilde{\mathcal{B}}_H$ which means that $S_{f,v}(x,gH) \in X \times G/H$ for all $(x,g) \in X \times G$. But $S_{f,v}(x,gH) = (Sx, f(x)v(gH))$ so $f(x)v(g)v(H) \in G/H$ for all $(x,g) \in X \times G$, hence $v(H) = (f(x)v(g))^{-1} g_{(x,g)}H$ and $v(H) = g_0H$. But $v(H)$ is a subgroup, so $g_0 = e$ and hence $v(H) = H$. We have achieved that on $X \times G/H$

$$\widehat{S}(x,gH) = S_{f,v,H}(x,gH) = (Sx, f(x)v(g)H)$$

and one directly checks that $S_{f,v,H}$ is invertible. □

Corollary 2.6.3. *If $\overline{T} \to T$ is an isometric ergodic extension, T is semisimple and the group cover of $\overline{T}$ is coalescent then $\overline{T}$ is also coalescent.*

Proposition 2.6.4. *If $T_\varphi: (X \times G, \widetilde{\mu}) \to (X \times G, \widetilde{\mu})$ is a weakly mixing group extension of a weakly mixing 2-fold simple map T, $\mathfrak{N}$ is the natural family of factors for T defined in Proposition 2.4.7 then the family*

$$\mathfrak{N}_G = \{\widetilde{\mathcal{B}}_H : H \textit{ is a normal closed subgroup of } G\} \cup \{\mathfrak{N}\}$$

is a natural family of factors for T_φ.

Proof. Since obviously $\mathfrak{N}$ is closed under taking intersections (the smallest closed subgroup generated by a family of closed normal subgroups is normal) and Lemma 2.6.2 holds true, it remains to show that if $\widetilde{S}: \widetilde{\mathcal{B}}_{H_1} \to \widetilde{\mathcal{B}}_{H_2}$ is an isomorphism of two natural factors then $\widetilde{S}$ sends natural factors contained in $\widetilde{\mathcal{B}}_{H_1}$ to natural factors contained in $\widetilde{\mathcal{B}}_{H_2}$. By Proposition 2.5.9, $\widetilde{S}(x,gH_1) = (Sx, f(x)v(gH_1))$, where $v: G/H_1 \to G/H_2$ is a continuous group isomorphism, $S \in C(T)$ and $f: X \to G/H$ is measurable. If H' is a closed normal subgroup containing H_1 then by the form of $\widetilde{S}$ we have $\widetilde{S}\widetilde{\mathcal{B}}_{H'} = \widetilde{\mathcal{B}}_{v(H'/H_1)}$ and it is clear that $v(H'/H_1)$ is a normal subgroup of G/H_2. □

Remark 2.6.5. From Proposition 2.6.4 and Structure Theorem we get immediately the result on the structure of factors for group extensions of rotations proved in [71].

Remark 2.6.6. If we assume that a 2-fold simple map is not weakly mixing, then in fact it has discrete spectrum (see [45]) and then both Lemma 2.6.1 and Proposition 2.6.4 are valid for each ergodic cocycle $\varphi: X \to G$.

The question whether or not each factor of a coalescent automorphism is again coalescent was stated by D. Newton in 1970, [80], and the negative answer is contained in [61] (see also a recent paper by A. Fieldsteel and D. Rudolph [19]). An ergodic group extension of a rotation need not be coalescent, but we will assume, that this is the case and ask about the coalescence of all factors. Our goal is to prove the following theorem (which is a generalization of a result from [61] for the Abelian case).

Theorem 2.6.7. *If $T_\varphi\colon (X \times G, \widetilde{\mu}) \to (X \times G, \widetilde{\mu})$ is an ergodic group extension of a discrete spectrum T and $\mathfrak{N}$ denotes the natural family of factors defined in Proposition* 2.6.4*, then all factors of T_φ are coalescent whenever all natural factors so are.*

Proof. Let $\mathcal{E}$ be such a factor of T_φ that is isomorphic to its proper factor $\mathcal{E}' \subsetneqq \mathcal{E}$. To simplify the notation we assume that $\widehat{\mathcal{E}} = \widetilde{\mathcal{B}}$. Now, $\mathcal{E}' \subset \mathcal{E}$ and they are isomorphic, so by coalescence property of natural factors we have $\widehat{\mathcal{E}'} = \widehat{\mathcal{E}} = \mathcal{B}$.

Let $\mathcal{H}(\mathcal{E})$ be the compact subgroup contained in $C(T_\varphi)$ that determines $\mathcal{E}$. Let $\overline{S}$ be this (noninvertible) element of the centralizer of $\mathcal{E}$ that gives rise to an isomorphism of $\mathcal{E}$ and $\mathcal{E}'$. Denote by $\widehat{S}$ an extension of $\overline{S}$ to $C(T_\varphi)$. By assumption of this theorem, $\widehat{S}$ is invertible. Moreover the factor $\mathcal{E}' = \overline{S}^{-1}\mathcal{E}$ is determined by $\widehat{S}^{-1}\mathcal{H}(\mathcal{E})\widehat{S}$. Consequently

$$\widehat{S}^{-1}\mathcal{H}(\mathcal{E})\widehat{S} \subset \mathcal{H}(\mathcal{E})$$

and the inclusion is strict. Denote

$$H = \{g \in G : \sigma_g \in \mathcal{H}(\mathcal{E})\},$$

where $\sigma_g(x,h) = (x, hg)$. Note that each $\sigma_g \in C(T_\varphi)$ and it can be written as Id_{g,τ_g}, where $\tau_g(h) = g^{-1}hg$. Now, each element $\widehat{U} \in \mathcal{H}(\mathcal{E})$ is of the form $\widehat{U} = U_{f,v}$ (Proposition 2.5.9) and if two elements $\widehat{U}, \widehat{\widehat{U}} \in \mathcal{H}(\mathcal{E})$ have the same projections on the first coordinate (i.e. they are liftings of the same $U \in C(T)$) then $\widehat{U} = \widehat{\widehat{U}} \circ \sigma_g$ for certain $g \in H$. Suppose that $\widehat{S} = S_{f,w}$, where $w\colon G \to G$ is an automorphism. Then we have

$$\widehat{S}^{-1} = (S^{-1})_{w^{-1}[(fS^{-1})^{-1}],w^{-1}},$$

where w^{-1} denotes the inverse in the sense of composition of maps, and

$$(S_{f,w})^{-1} \circ \sigma_g \circ S_{f,w} = \sigma_{w^{-1}(g)}.$$

Take under consideration the factor $\widetilde{\mathcal{B}}_H$, which is determined by the group $\mathcal{H}(\mathcal{E}) \cap \{\sigma_g\colon g \in G\} \subset C(T_\varphi)$, and consider $\widehat{S}^{-1}\widetilde{\mathcal{B}}_H$. The latter factor is determined by

$$\widehat{S}^{-1}\mathcal{H}(\widetilde{\mathcal{B}}_H)\widehat{S} = \{\sigma_{w^{-1}(g)} : g \in H\}.$$

Denoting $H' = \{g \in G : \sigma_{w^{-1}(g)} \in \widehat{S}^{-1}\mathcal{H}(\mathcal{E})\widehat{S}\}$ we have H is a proper subgroup of H' because $\widehat{S}^{-1}\mathcal{H}(\mathcal{E})\widehat{S}$ determines $\overline{S}^{-1}\mathcal{E} = \mathcal{E}'$ and $\mathcal{E}'$ is a proper factor of $\mathcal{E}$.

Thus $\mathcal{B}_{H'}$ is a proper factor of $\mathcal{B}_H$. Moreover, $\widehat{S}^{-1}(\mathcal{B}_H) = \mathcal{B}_{H'}$, and therefore

$$\{\sigma_{w^{-1}(g)} : g \in H\} = \widehat{S}^{-1}\mathcal{H}(\mathcal{B}_H)\widehat{S} = \mathcal{H}(\widehat{S}^{-1}(\mathcal{B}_H)) = \mathcal{H}(\mathcal{B}_{H'}).$$

It implies that $\widetilde{\mathcal{B}}_H$ has a proper factor $\widetilde{\mathcal{B}}_{w^{-1}(H)}$ isomorphic to it. The result follows from Lemma 2.6.2. □

2.7. Final remarks

In 1997 P. Gabriel, M. Lemańczyk and K. Schmidt showed [26] that for Bernoulli transformations the smallest natural family of factors consists of all factors (later, E. Glasner, [33], gave an alternative proof of this theorem). On the other hand for all 2-fold simple maps the smallest natural family consists only of one element (see Remark 2.4.9). From this point of view Bernoulli shifts are an opposition of 2-simple systems.

For semisimple maps on the set $J^e(T,T)$ there is a natural structure of a monoid (see [35]). Suppose that $\lambda_1, \lambda_2 \in J^e(T,T)$. We have

$$(X \times X, \lambda_1) \xrightarrow{\text{rel. w. m.}} X \xleftarrow{\text{rel. w. m.}} (X \times X, \lambda_2)$$

so the relative product over X is rel. w. m. Since $\lambda_1 \times_X \lambda_2$ is ergodic, hence the projections on the first and on the third coordinate give us an ergodic self-joining obtained by $\lambda_1 \circ \lambda_2 \in J^e(T,T)$. This multiplication is associative and has a unit – the diagonal measure on X. If T is weakly mixing then $\mu \times \mu \in J^e(T,T)$ and $(\mu \times \mu) \circ \lambda = \mu \times \mu$ for each $\lambda \in J^e(T,T)$. More generally, if $\mathcal{A}$ is a factor and $\lambda \in J^e(T,T)$ is diagonal on $\mathcal{A}$ then $(\mu \times_{\mathcal{A}} \mu) \circ \lambda = \mu \times_{\mathcal{A}} \mu$. In particular, the relatively independent extensions of diagonal measures gives us idempotents. The only invertible elements are graph joinings μ_S with $S \in C(T)$ necessarily invertible. In 2003 Y. H. Ahn and M. Lemańczyk proved a much more general theorem ([3, Theorem 1]) saying that an automorphism is semisimple if and only if the set of all ergodic self-joinings forms a semigroup with the circle $\circ$ operation. This theorem suggests that semisimplicity is a quite natural notion.

CHAPTER 3

SEMISIMPLE GROUP EXTENSIONS OF ROTATIONS

In this chapter we show that semisimple actions of locally compact second countable Abelian groups and cocycles with values in such groups can be used to built new examples of semisimple automorphisms ($\mathbb{Z}$-actions) that are relatively weakly mixing extensions of irrational rotations.

3.1. General backgrounds

Recall (see Definition 2.3.1) that T is *semisimple* if for each ergodic $\lambda \in J^e(T)$ the extension $(\mathcal{B} \otimes \mathcal{B}, \lambda) \to (\mathcal{B} \times X, \lambda)$ is relatively weakly mixing (clearly, $(\mathcal{B} \times X, \lambda)$ can be identified with $(\mathcal{B}, \mu)$). It has been noticed in Chapter 2 that an ergodic distal automorphism is semisimple if and only if it is isomorphic to a rotation. Moreover, if T is semisimple and $\mathcal{B} \to \mathcal{A}$ is relatively weakly mixing then $\mathcal{A}$ is also semisimple. It follows that if T is semisimple and $\mathcal{D}$ stands for its maximal distal factor then $\mathcal{D}$ is semisimple because $\mathcal{B} \to \mathcal{D}$ is relatively weakly mixing. We have shown the following.

Proposition 3.1.1. *If T is semisimple then it is a relatively weakly mixing extension of its Kronecker factor.*

Let $(X, \mathcal{B}, \mu)$ be a standard probability space. Let $T\colon (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism. Assume that G is an Abelian locally compact second countable group, $\varphi\colon X \to G$ is a cocycle and consider the group extension

$$T_\varphi\colon (X \times G, \mathcal{B} \otimes \mathcal{G}, \mu \times m_G) \to (X \times G, \mathcal{B} \otimes \mathcal{G}, \mu \times m_G),$$
$$T_\varphi(x, g) = (Tx, \varphi(x) + g).$$

Here $\mathcal{G}$ denotes the σ-algebra of Borel sets of G and m_G stands for a (infinite whenever G is not compact) Haar measure on $\mathcal{B}(G)$. As it follows from [89], ergodicity of φ is "controled" by the group $E(\varphi)$ of *essential values* of φ (see Definition 1.4.6).

Given $T\colon (X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ and $\varphi\colon X\to G$ by $\varphi_*\mu$ we denote the image of μ on G via φ. Recall also that an increasing sequence $(q_n)_{n\geqslant 1}$ of integers is called a *rigidity time* for T if $T^{q_n}\to \mathrm{Id}$ weakly. We will make use of the following essential value criterion.

Proposition 3.1.2 ([69]). *Assume that T is ergodic and let $\varphi\colon X\to G$ be a cocycle with values in an Abelian locally compact second countable group G. Let $(q_n)_{n\geqslant 1}$ be a rigidity time for T. Suppose that $(\varphi^{(q_n)})_*\mu\to\nu$ weakly on $G_\infty=G\cup\{\infty\}$. Then* $\operatorname{supp}(\nu)\subset E_\infty(\varphi)$.

Assume that G is an Abelian locally compact second countable group that contains no non-trivial compact subgroup and let $\mathcal{G}=\{R_g\}_{g\in G}$ be a Borel action of this group on a Borel space $(Y,\mathcal{C})$ (we always suppose that such a space is standard that is, up to isomorphism, Y is a Polish space, while $\mathcal{C}$ stands for the σ-algebra of Borel sets) meaning that the map $G\times Y\ni(g,y)\mapsto R_gy\in Y$ is measurable. If now ν is a probability measure invariant for the action of G then the notions defined in the previous section for $\mathbb{Z}$-actions have their natural extensions to corresponding notions for actions of G on $(Y,\mathcal{C},\nu)$ (see also [45]). In Chapter 1 a definition of mildly mixing action is given (see Definition 1.5.2). An equivalent condition for mild mixing is the following (in the case when G contains no non-trivial compact subgroup): the action $\mathcal{G}$ is *mildly mixing* if for each non-trivial factor $\mathcal{A}\subset\mathcal{C}$ of $\mathcal{G}$ the representation $g\mapsto U_{R_g}\in U(L^2(Y/\mathcal{A},\mathcal{A},\nu))$ ($U_{R_g}f=f\circ R_g$ for $f\in L^2(Y/\mathcal{A},\mathcal{A},\nu)$) is faithful and its image is closed. For other conditions, equivalent to mild mixing, see [76], [24], [90], [63]. In particular, for $\mathbb{Z}$- (or $\mathbb{R}$-) actions, mild mixing is equivalent to the lack of nontrivial rigid factors.

3.2. Self-joinings of Rokhlin cocycles extensions for regular cocycles

By the circle we will mean $\mathbb{T}=[0,1)$ with addition modulo 1. Given $t\in\mathbb{R}$ by $\{t\}$ we will denote its fractional part. Given an irrational α let $[0:a_0,a_1,\dots]$ denote the continued fraction expansion of α (see e.g. [49]). We say that α has *bounded partial quotients* if the sequence $(a_n)_{n\geqslant 1}$ is bounded. We will denote by $(q_n)_{n\geq 1}$ the sequence of denominators of α, that is $q_0=1, q_1=a_1$ and $q_{n+1}=a_{n+1}q_n+q_{n-1}$, $n\geq 2$. We will make use of the following result of C. Kraaikamp and P. Liardet (see also [59]).

Theorem 3.2.1 ([54]). *If α has bounded partial quotients then:*

(a) *for each real $\beta\notin\mathbb{Q}\alpha+\mathbb{Q}$ the set of accumulation points in $\mathbb{T}$ of $(\{q_n\beta\})_{n\geqslant 1}$ is infinite;*

(b) *for each real $\beta\notin\mathbb{Z}\alpha+\mathbb{Z}$ there exists $0<c<1$ in the set of accumulation points in $\mathbb{T}$ of $(\{q_n\beta\})_{n\geqslant 1}$.*

The following well known result follows from the classical Koksma inequality (e.g. [57]) and elementary properties of denominators of α.

Proposition 3.2.2. *If $T\colon \mathbb{T} \to \mathbb{T}$ is given by $Tx = x + \alpha$, $f\colon \mathbb{T} \to \mathbb{R}$ has bounded variation and $\int_{\mathbb{T}} f(t)\,dt = 0$ then $|f^{(q_n)}(t)| \leqslant 2\operatorname{Var}(f)$ for each $t \in \mathbb{T}$ and each $n \geqslant 1$.*

Closed subgroups of $\mathbb{R}^n$ are described in [77, Chapter II]. It follows that if $E \subset \mathbb{R}^2$ is a closed subgroup then it has one of the following forms:

$$
\begin{aligned}
&\text{(a) } E = \{t\vec{v} : t \in \mathbb{R}\}, && \text{where } \vec{v} \in \mathbb{R}^2,\\
&\text{(b) } E = \{n\vec{v}_1 + m\vec{v}_2 : n, m \in \mathbb{Z}\}, && \text{where } \vec{v}_1, \vec{v}_2 \in \mathbb{R}^2,\\
&\text{(c) } E = \{t\vec{v}_1 + k\vec{v}_2 : t \in \mathbb{R},\ k \in \mathbb{Z}\}, && \text{where } \vec{v}_1, \vec{v}_2 \in \mathbb{R}^2,\\
&\text{(d) } E = \mathbb{R}^2.
\end{aligned} \tag{3.1}
$$

Note that subgroups of the form (c) are cocompact whenever $\vec{v}_2 \neq 0$.

Let $\mathcal{G} = \{R_g\}_{g\in G}$ be a weakly mixing action of G on $(Y, \mathcal{C}, \nu)$, where G is Abelian locally compact second countable group. Given $H \subset G \times G$ a closed subgroup satisfying

$$\overline{\pi_1(H)} = G = \overline{\pi_2(H)}, \tag{3.2}$$

where $\pi_i(g_1, g_2) = g_i$, $i = 1, 2$, we denote by $\mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$ the set of all probability measures ρ on $\mathcal{C} \otimes \mathcal{C}$ that are $R_{g_1} \times R_{g_2}$-invariant for all $(g_1, g_2) \in H$ and such that $\rho(C \times Y) = \rho(Y \times C) = \nu(C)$ for all $C \in \mathcal{C}$. If $\rho \in \mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$ and $\rho = \int \rho_\gamma\, dP(\gamma)$ denotes the H-ergodic decomposition of ρ, then $\nu(\,\cdot\,) = \int \rho_\gamma(\,\cdot\, \times Y)\, dP(\gamma)$ and in view of (3.2), for P-a.e. γ, $\rho_\gamma(\,\cdot\, \times Y)$ is $\mathcal{G}$-invariant. Since ν is an extreme point in the simplex of all $\mathcal{G}$-invariant measures, $\rho_\gamma \in \mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$ for P-a.e. γ. It easily follows that $\mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$ is a simplex whose set of extreme points coincides with $\mathcal{M}^e(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$ the set of ergodic members of $\mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$.

In what follows we will keep in mind Remark A.3.17. Moreover, we will use the notion of spectral disjointness.

Definition 3.2.3. Let G be a locally compact group, $\Gamma_1 = \{\gamma_g^1 : g \in G\}$ and $\Gamma_2 = \{\gamma_g^2 : g \in G\}$ be two actions of G on a probability Lebesgue spaces $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ respectively. We say that the actions Γ_1 and Γ_2 are *spectrally disjoint*, if the maximal spectral types of Γ_1 and Γ_2 are mutually singular.

The relations between spectral disjointness and disjointness in the Furstenberg sense (see Definition 1.3.3) is given in the following lemma that is a straightforward generalization of [42, Theorem 2.1].

Lemma 3.2.4. *Let G be a locally compact group, $\Gamma_1 = \{\gamma_g^1 : g \in G\}$ and $\Gamma_2 = \{\gamma_g^2 : g \in G\}$ be two actions of G on a probability Lebesgue spaces $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$, respectively. If Γ_1 and Γ_2 are spectrally disjoint then they are disjoint in the Furstenberg sense.*

A role of our assumption of weak mixing of the G-action $\mathcal{G} = \{R_g\}_{g\in G}$ is easily seen in the following.

Proposition 3.2.5. *An ergodic G-action is weakly mixing if and only if for each sub-action $\mathcal{H}$ of a closed subgroup $H \subset G\times G$ satisfying* (3.2)*, the $\mathcal{H}$-action is still ergodic.*

Proof. The proof we will present was done by F. Parreau. Let $p: \widehat{G}\times\widehat{G} \to \widehat{H}$ be the dual homomorphism corresponding to the natural embedding of H in $G\times G$. Denote

$$\Gamma = H^{\perp} = \{(\gamma_1,\gamma_2) \in \widehat{G}\times\widehat{G} : (\gamma_1,\gamma_2)(H) = 1\}.$$

In view of (3.2),

$$\Gamma \cap \widehat{G}\times\{1\} = \Gamma\cap\{1\}\times\widehat{G} = \{(1,1)\}. \tag{3.3}$$

The maximal spectral type of the product action $\mathcal{G}\otimes\mathcal{G}$ equals $\tau\times\tau$, where τ stands for the maximal spectral type of $\mathcal{G}$. Since $\mathcal{G}$ is weakly mixing, $\tau = \delta_0 + \tau_c$, where τ_c is a continuous measure on $\widehat{G}$ and τ_c is the maximal spectral type of the unitary action of $\mathcal{G}$ on the space $L^2_0(Y,\mathcal{C},\nu)$ of zero mean functions. Now, the maximal spectral type of the product action $\mathcal{G}\otimes\mathcal{G}$ on $L^2_0(Y\times Y,\mathcal{C}\otimes\mathcal{C},\nu\times\nu)$ equals

$$\delta_0\times\tau_c + \tau_c\times\delta_0 + \tau_c\times\tau_c.$$

Since the maximal spectral type of $\mathcal{H}$ on any $\mathcal{G}\otimes\mathcal{G}$-invariant subspace of $L^2_0(Y\times Y, \mathcal{C}\otimes\mathcal{C},\nu\times\nu)$ is the image via p of the maximal spectral type of $\mathcal{G}\times\mathcal{G}$ on that space, all we need to show is that the measures

$$p_*(\delta_0\times\tau_c),\quad p_*(\tau_c\times\delta_0),\quad p_*(\tau_c\times\tau_c)$$

are singular with respect to δ_0. However, directly from (3.3) it follows that for each $\gamma\in\widehat{G}$,

$$\Gamma\cap\widehat{G}\times\{\gamma\} = \Gamma\cap\{\gamma\}\times\widehat{G} \quad \text{has at most one element.}$$

Now, if σ is any finite Borel measure on $\widehat{G}$, then

$$p_*(\tau_c\times\sigma)(\{0\}) = (\tau_c\times\sigma)(\Gamma) = \int_{\widehat{G}} \tau_c(\Gamma\cap\widehat{G}\times\{\gamma\})\,d\sigma(\gamma) = 0,$$

so the result easily follows. □

The following proposition describes the simplex $\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};H)$ in some cases.

Proposition 3.2.6.

(a) $\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};G\times G) = \{\nu\times\nu\}$.
(b) $\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};H) = \{\nu\times\nu\}$ *whenever H is cocompact.*
(c) $\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};\Delta_G) = J(\mathcal{G})$.

Proof. (a) This is a standard argument using spectral disjointness of the trivial identity G-action and any ergodic G-action (see Lemma 3.2.4).

(b) Assume that $\rho_0 \in \mathcal{M}^e(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$. Then for each $(g_1, g_2) \in G \times G$, $(R_{g_1} \times R_{g_2})_* \rho_0 \in \mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$. We put

$$\rho = \int_{(G\times G)/H} (R_{g_1} \times R_{g_2})_* \rho_0 \, d((g_1, g_2)H). \tag{3.4}$$

Then ρ is a well defined $\mathcal{G} \times \mathcal{G}$-invariant measure, hence $\rho = \nu \times \nu$. We look at the decomposition (3.4) as a decomposition of an $\mathcal{H}$-invariant measure $\nu \times \nu$ which is $\mathcal{H}$-ergodic by Proposition 3.2.5. Since in this decomposition all measures are in $\mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$, by extremality, $(R_{g_1} \times R_{g_2})_* \rho_0$ is one and the same measure a.e. and therefore $\rho_0 = \nu \times \nu$.

(c) This part is obvious. □

Assume that $T: (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic automorphism and let $\varphi: X \to G$ be a cocycle. Throughout we suppose that the G-action $\mathcal{G} = \{R_g\}_{g \in G}$ is weakly mixing. Assume that H is a closed subgroup of G. Furthermore, assume that $\psi: X \to H$ is a cocycle cohomologous to φ, i.e. for some measurable $f: X \to G$, $\varphi = f - fT + \psi$. Consider the corresponding Rokhlin cocycle extensions $T_{\varphi,\mathcal{G}}: X \times Y \to X \times Y$ and $T_{\psi,\mathcal{G}}: X \times Y \to X \times Y$, defined by $T_{\varphi,\mathcal{G}}(x, y) = (Tx, R_{\varphi(x)}(y))$, $T_{\psi,\mathcal{G}}(x, y) = (Tx, R_{\psi(x)}(y))$. Then

$$T_{\varphi,\mathcal{G}} \quad \text{and} \quad T_{\psi,\mathcal{G}} \quad \text{are relatively isomorphic,} \tag{3.5}$$

that is, there exists an isomorphism which is the identity on $\mathcal{B} \times Y$. Indeed, the map $X \times Y \ni (x, y) \mapsto (x, R_{f(x)}(y)) \in X \times Y$ establishes a relative isomorphism.

Proposition 3.2.7. *Assume that φ is regular but is not a coboundary. Assume that $\mathcal{G}$ is mildly mixing. Then $T_{\varphi,\mathcal{G}}$ is ergodic.*

Proof. In view of (3.5), one has to note only that for any closed subgroup $H \subset G$, $H \neq \{0\}$, the corresponding action $\mathcal{H}$ is ergodic. □

Remark 3.2.8. The fact above is clearly false whenever G is compact. The reason is that in such a case G acts on itself by translations and for no proper closed subgroup H the action of H by translations on G remains ergodic.

Given $\lambda \in J^e(T)$ denote by $J(T_{\varphi,\mathcal{G}}; \lambda)$ the set of self-joinings of $T_{\varphi,\mathcal{G}}$ whose restriction to $\mathcal{B} \otimes \mathcal{B}$ equals λ. Given a cocycle $\varphi: X \to G$ and $\lambda \in J^e(T)$, consider the cocycle $\varphi \times \varphi = (\varphi \times \varphi)_\lambda$, where

$$(\varphi \times \varphi)(x_1, x_2) = (\varphi(x_1), \varphi(x_2)) \in G \times G.$$

It is considered as a cocycle for the $\mathbb{Z}$-action given by $(T \times T, \lambda)$. Let $H_\lambda \subset G \times G$ be the group of essential values of $(\varphi \times \varphi)_\lambda$, i.e. $H_\lambda = E((\varphi \times \varphi)_\lambda)$. Denote by $\mathcal{H}_\lambda$ the corresponding to H_λ subaction of the product $G \times G$-action $\{R_{g_1} \times R_{g_2}\}_{g_1, g_2 \in G}$.

Theorem 3.2.9. *Let $\varphi\colon X \to G$ be an ergodic cocycle. Assume that $(\varphi\times\varphi)_\lambda$ is regular. Then there exists a measurable $f = (f_1, f_2)\colon X \times X \to G\times G$ (defined λ-a.e.) such that the map Λ_f given by*

$$X \times Y \times X \times Y \ni (x_1, y_1, x_2, y_2) \mapsto (x_1, x_2, R_{f_1(x_1)} \times R_{f_2(x_2)}(y_1, y_2)) \in X \times X \times Y \times Y$$

establishes an affine isomorphism of

$$J(T_{\varphi,\mathcal{G}}; \lambda) \quad \text{and} \quad \lambda \otimes \mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H_\lambda) := \{\lambda \times \rho : \rho \in \mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H_\lambda)\}.$$

More precisely, there exists $\theta\colon X\times X \to H_\lambda$ an ergodic cocycle for $(T\times T, \lambda)$ such that for each $\widetilde{\lambda} \in J(T_{\varphi,\mathcal{G}}; \lambda)$, Λ_f establishes an isomorphism of $(T_{\varphi,\mathcal{G}} \times T_{\varphi,\mathcal{G}}, \widetilde{\lambda})$ and $((T \times T)_{\theta,\mathcal{H}_\lambda}, \lambda \times \rho)$, where $\lambda \times \rho = (\Lambda_f)_*(\widetilde{\lambda})$.

Proof. Since $(\varphi \times \varphi)_\lambda$ is regular, there exists a measurable function $f\colon X \times X \to G \times G$ and an ergodic cocycle $\theta\colon X \times X \to H_\lambda$ (both maps defined λ-a.e.) such that

$$(\varphi \times \varphi)(x_1, x_2) = f(x_1, x_2) - f(Tx_1, Tx_2) + \theta(x_1, x_2) \ \ \lambda - \text{a.e.}$$

It then follows from Proposition 1.4.11 that H_λ has dense projections.

Assume that $\widetilde{\lambda} \in J^e(T_{\varphi,\mathcal{G}}; \lambda)$. If by $\widetilde{\lambda}_1$ we denote the image of $\widetilde{\lambda}$ via the map Λ_f then clearly the commutation relation

$$\Lambda_f(T_{\varphi,\mathcal{G}} \times T_{\varphi,\mathcal{G}}) = (T \times T)_{\theta,\mathcal{H}_\lambda}\Lambda_f$$

gives rise to a measure-theoretic isomorphism of $(T_{\varphi,\mathcal{G}} \times T_{\varphi,\mathcal{G}}, \widetilde{\lambda})$ and $((T \times T)_{\theta,\mathcal{H}_\lambda}, \widetilde{\lambda}_1)$. However θ is ergodic and the projection of $\widetilde{\lambda}_1$ on $X\times X$ equals λ, so by the relative unique ergodicity property (Proposition 1.5.6), $\widetilde{\lambda}_1 = \lambda\times\rho$, where ρ is $\mathcal{H}_\lambda$-invariant and ergodic.

Furthermore, the maps

$$X \times Y \ni (x_i, y_i) \xmapsto{s_i} (x_i, R_{f_i(x_i)}(y_i)) \in X \times Y, \quad i = 1, 2,$$

have the property that $(s_i)_*(\mu \times \nu) = \mu \times \nu$. It follows that the projections of ρ on Y are equal to ν and therefore $\rho \in \mathcal{M}^e(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H_\lambda)$. Since for each $\rho \in \mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H_\lambda)$, $(\Lambda_f^{-1})_*\lambda \times \rho \in J(T_{\varphi,\mathcal{G}}; \lambda)$, the result follows. □

Remark 3.2.10. The above proof tells us that the isomorphism Λ_f of $(T_{\varphi,\mathcal{G}} \times T_{\varphi,\mathcal{G}}, \widetilde{\lambda})$ and $((T \times T)_{\theta,\mathcal{H}_\lambda}, (\Lambda_f)_*(\widetilde{\lambda}))$ is "relative" over $T_{\varphi,\mathcal{G}}$ in the sense that

$$\Lambda_f(\mathcal{B} \otimes \mathcal{C} \times X \times Y) = \mathcal{B} \times X \otimes \mathcal{C} \times Y$$

and the action of $((T\times T)_{\theta,\mathcal{H}_\lambda}, (\Lambda_f)_*(\widetilde{\lambda}))$ restricted to $(\mathcal{B}\times X\otimes\mathcal{C}\times Y)$ is isomorphic to $T_{\varphi,\mathcal{G}}$. It follows that the relative properties of the two automorphisms over $T_{\varphi,\mathcal{G}}$ are the same.

We will now study some particular cases of λ for which $(\varphi\times\varphi)_\lambda$ is indeed regular.

Corollary 3.2.11 (relative self-joinings)**.** *If $\varphi\colon X\to G$ is ergodic then*

$$J^e(T_{\varphi,\mathcal{G}};\Delta_X)=\Delta_X\times J^e(\mathcal{G})$$

up to permutation of coordinates.

Proof. We have $H_{\Delta_X}=\Delta_G$, $f=(0,0)$ and $(\varphi\times\varphi)_{\Delta_X}$ is regular. It follows that, up to permutation of coordinates, the map Λ_f is the identity, and we apply Theorem 3.2.9. □

The corollary above allows us to give a list of relative factors of $T_{\varphi,\mathcal{G}}$, that is all factors that contain $\mathcal{B}\times Y$. The result below generalizes the well known compact group extension case.

Corollary 3.2.12. *Assume that $\varphi\colon X\to G$ is ergodic. Let $\mathcal{B}\times Y\subset\widetilde{\mathcal{A}}\subset\mathcal{B}\otimes\mathcal{C}$ be a factor of $T_{\varphi,\mathcal{G}}$. Then there exists $\mathcal{D}\subset\mathcal{C}$ that is a $\mathcal{G}$-factor and $\widetilde{\mathcal{A}}=\mathcal{B}\otimes\mathcal{D}$.*

Proof. It follows from Corollary 3.2.11 that

$$\mu\times\nu\times_{\widetilde{\mathcal{A}}}\mu\times\nu=\int_{J^e(\mathcal{G})}\Delta_X\times\rho\,dP(\rho).$$

Hence this relative product is invariant under $\mathrm{Id}_X\times R_g\times\mathrm{Id}_X\times R_g$ which means that $\widetilde{\mathcal{A}}$ is invariant for the action $\{\mathrm{Id}_X\times R_g\}_{g\in G}$. Since $\mathcal{B}\times Y\subset\widetilde{\mathcal{A}}$, we obtain a measurable family for $\{Q_x\}_{x\in X}$ of partitions Q_x of Y such that $\{\{x\}\times Q_x\}_{x\in X}$ generates $\widetilde{\mathcal{A}}$. Let $\mathcal{C}_x\subset\mathcal{C}$ denote the σ-algebra generated by Q_x. Since $\widetilde{\mathcal{A}}$ is $\mathrm{Id}_X\times R_g$-invariant, $\mathcal{C}_x$ is a $\mathcal{G}$-factor. But $\widetilde{\mathcal{A}}$ is also $T_{\varphi,\mathcal{G}}$-invariant, so $R_{\varphi(x)}Q_x=Q_{Tx}$ for a.e. $x\in X$ and therefore the map $x\mapsto L^2(\mathcal{C}_x)$ is T-invariant. Since the map $x\mapsto E(\,\cdot\,|\mathcal{C}_x)$ is measurable, $Q_x=\mathrm{const}$ for a.e. $x\in X$ and the result follows. □

The following corollary generalizes Glasner's results of Section 2 from [32].

Corollary 3.2.13 (relative simplicity)**.** *If $\mathcal{G}$ is additionally 2-fold simple then $T_{\varphi,\mathcal{G}}$ is relatively 2-fold simple, that is the only ergodic self-joining of $T_{\varphi,\mathcal{G}}$ that projects onto Δ_X is either a graph or the relatively independent extension of Δ_X.*

Proof. Take $\widetilde{\lambda}\in J^e(T_{\varphi,\mathcal{G}};\Delta_X)$. If $\widetilde{\lambda}$ is not the relative product then $\widetilde{\lambda}=\Delta_X\times\nu_W$ where $W\in C(\mathcal{G})$. Then clearly $\widetilde{\lambda}=(\mu\times\nu)_{\mathrm{Id}\times W}$. □

Now we consider a more general situation of an ergodic self-joining of $T_{\varphi,\mathcal{G}}$ whose projection on $X\times X$ equals μ_S for some $S\in C(T)$.

Denote by $\mathcal{M}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}};\mu)$ the set of all probability $T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}}$-invariant measures on $\mathcal{B}\otimes\mathcal{C}\otimes\mathcal{C}$ whose restrictions to $\mathcal{B}$ equal μ (here $\varphi\times\varphi\circ S: X\to G\times G$, $(\varphi\times\varphi\circ S)(x) = (\varphi(x),\varphi(Sx))$). Then the map Λ given by

$$X\times Y\times X\times Y \ni (x,y_1,Sx,y_2)\mapsto (x,y_1,y_2)\in X\times Y\times Y$$

establishes an affine isomorphism of $J(T_{\varphi,\mathcal{G}};\mu_S)$ and $\mathcal{M}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}};\mu)$. More precisely, for each $\widetilde{\lambda}\in J(T_{\varphi,\mathcal{G}};\mu_S)$, Λ establishes an isomorphism of $(T_{\varphi,\mathcal{G}}\times T_{\varphi,\mathcal{G}},\widetilde{\lambda})$ and $(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}},(\Lambda)_*(\widetilde{\lambda}))$. Moreover, this isomorphism is the identity on the first two coordinates, so it is relative with respect to $T_{\varphi,\mathcal{G}}$. Therefore, in what follows we will identify $J(T_{\varphi,\mathcal{G}};\mu_S)$ with $\mathcal{M}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}},\mu)$.

Assume now that additionally $\varphi\times\varphi\circ S: X\to G\times G$ is regular. From the above it follows that in Theorem 3.2.9 with no loss of generality we can replace $J(T_{\varphi,\mathcal{G}};\mu_S)$ by $\mathcal{M}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}};\mu)$ and if $\widetilde{\lambda}\in\mathcal{M}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}},\mu)$ then by applying Remark 3.2.10 we obtain that the relative properties of $(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}},\widetilde{\lambda})$ and $((T\times T)_{\theta,\mathcal{H}_{\mu_S}},(\Lambda_f)_*(\widetilde{\lambda}))$ over $T_{\varphi,\mathcal{G}}$ are the same. In particular, we obtain the following.

Proposition 3.2.14. *Assume that $\varphi: X\to G$ is an ergodic cocycle, $S\in C(T)$ and $\varphi\times\varphi\circ S: X\to G\times G$ is regular, say*

$$(\varphi,\varphi\circ S) = (f_1\circ T, f_2\circ T) - (f_1,f_2) + \theta,$$

where $\theta: X\to H_{\mu_S}$. Let $\widetilde{\lambda}\in\mathcal{M}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}};\mu)$. Then

(a) *$\widetilde{\lambda}$ is ergodic if and only if $(\Lambda_f)_*(\widetilde{\lambda})$ is ergodic;*
(b) *$\widetilde{\lambda}$ is one point extension of $T_{\varphi,\mathcal{G}}$ (that is, $\widetilde{\lambda}$ is a graph) if and only if $(\Lambda_f)_*(\widetilde{\lambda})$ is one point extension of $T_{\varphi,\mathcal{G}}$;*
(c) *$\widetilde{\lambda} = \mu\times\nu\times\nu$ if and only if $(\Lambda_f)_*(\widetilde{\lambda}) = \mu_S\times\nu\times\nu$;*
(d) *$(T_{\varphi,\mathcal{G}}\times T_{\varphi,\mathcal{G}},\widetilde{\lambda})\to T_{\varphi,\mathcal{G}}$ is relatively weakly mixing if and only if*

$$((T\times T)_{\theta,\mathcal{H}_{\mu_S}},(\Lambda_f)_*(\widetilde{\lambda}))\to T_{\varphi,\mathcal{G}}$$

is relatively weakly mixing.

In general we do not know whether or not given an ergodic cocycle φ, is $\varphi\times\varphi\circ S$ for each $S\in C(T)$ regular (see next section for examples of φ for which the answer is positive), but obviously we have the following.

Remark 3.2.15. The cocycle $(\varphi\times\varphi)_\lambda$ is a regular cocycle when $\lambda = \mu_{T^k}$. Indeed, in such a case $(\varphi\times\varphi)_\lambda$ is cohomologous to $(\varphi\times\varphi)_{\Delta_X}$ since clearly $\varphi\circ T^k$ is T-cohomologous to φ ($\varphi\circ T^k = \varphi + \varphi^{(k)}\circ T - \varphi^{(k)}$).

We will also need the following.

Lemma 3.2.16. *Assume that $\mathcal{G}$ is weakly mixing and φ is ergodic. Let $\mathcal{D} \subset \mathcal{C}$ be a $\mathcal{G}$-factor. Then $T_{\varphi,\mathcal{G}}$ is relatively weakly mixing over $\mathcal{B} \otimes \mathcal{D}$ if and only if $\mathcal{G}$ is relatively weakly mixing action over $\mathcal{D}$.*

Proof. First notice that directly from the definition of conditional expectation if $\mathcal{D} \subset \mathcal{C}$ then (up to permutation of coordinates)

$$\mu \times \nu \times_{\mathcal{B}\otimes\mathcal{D}} \mu \times \nu = \Delta_X \times \nu \times_{\mathcal{D}} \nu.$$

It follows that the relative product $T_{\varphi,\mathcal{G}} \times_{\mathcal{B}\otimes\mathcal{D}} T_{\varphi,\mathcal{G}}$ over the $T_{\varphi,\mathcal{G}}$-factor $\mathcal{B} \otimes \mathcal{D}$ is isomorphic to $T_{\varphi\times\varphi,\{R_g\times R_g\}_{g\in G}|\mathcal{C}\otimes_{\mathcal{D}}\mathcal{C}}$, where $\varphi \times \varphi\colon X \to G \times G$, $(\varphi \times \varphi)(x) = (\varphi(x), \varphi(x))$.

Since $(\varphi, \varphi)\colon X \to \Delta_G$ is ergodic, $T_{\varphi\times\varphi,\{R_g\times R_g\}_{g\in G}|\mathcal{C}\otimes_{\mathcal{D}}\mathcal{C}}$ is ergodic if and only if the diagonal Δ_G-action on $\mathcal{C} \otimes_{\mathcal{D}} \mathcal{C}$ is ergodic itself (see Proposition 1.5.6). □

3.3. Cocycles over irrational rotations

In this section we put $X = \mathbb{T} = [0, 1)$ and we consider $Tx = x+\alpha \pmod 1$ an irrational rotation on X. By μ we denote Lebesgue measure on $\mathbb{T}$. Throughout this section α is assumed to have bounded partial quotients.

3.3.1. A real-valued ergodic cocycle φ for which $\varphi \times \varphi \circ S$ is regular for each $S \in C(T)$. We will consider the real cocycle $\varphi(x) = \{x\} - 1/2$. Let $(q_n)_{n\geq 1}$ be the sequence of denominators of α. Assume that $\beta \in [0, 1)$ and that

$$\{q_{n_k}\beta\} \xrightarrow{k\to\infty} c \in [0, 1).$$

Consider the sequence $(\nu_k)_{k\geq 1}$ of probability measures on $\mathbb{R}^2$ defined by

$$\nu_k := ((\varphi \times \varphi \circ S)^{(q_{n_k})})_*\mu,$$

where $Sx = x + \beta \pmod 1$. Since

$$\underset{x,y\in[0,1)}{\forall} |\varphi^{(q_n)}(x) - \varphi^{(q_n)}(y)| \leq 4 \operatorname{Var}\varphi = 4$$

and $\int_X \varphi\, d\mu = 0$, $\operatorname{Im}(\varphi\times\varphi\circ S)^{(q_n)} \subset [-4, 4]\times[-4, 4]$. It follows that we can select a subsequence from the sequence $(\nu_k)_{k\geq 1}$ that converges weakly to a probability measure ν (which is also concentrated on the above square). No harm arises if we assume that $\nu_k \xrightarrow{k\to\infty} \nu$.

We will now show in what kind of subsets of $\mathbb{R}^2$ the support of ν is contained. To this end let us verify that

$$\varphi^{(q_n)}(x) = q_n x + \frac{q_n(q_n - 1)}{2}\alpha - \frac{q_n}{2} + M(x), \quad \text{where } M(x) \in \mathbb{Z}.$$

It follows that $\varphi^{(q_n)}(x + \beta) = \varphi^{(q_n)}(x) + q_n\beta + M(x + \beta) - M(x)$ if $x + \beta < 1$ or $\varphi^{(q_n)}(\{x + \beta\}) = \varphi^{(q_n)}(x) + (q_n\beta - q_n) + M(\{x + \beta\}) - M(x)$ if $1 \leq x + \beta < 2$.

Consider now the image of measure $(\varphi^{(q_{n_k})} \times \varphi^{(q_{n_k})} \circ S)_*\mu$ via

$$F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{T}, \quad F(x,y) = e^{2\pi i(y-x)}$$

that is, we send ν_k to the circle. However $F \circ (\varphi \times \varphi \circ S)^{(q_{n_k})}(x) = e^{2\pi i q_{n_k}\beta}$, whence $F_*\nu_k$ is the Dirac measure concentrated at $e^{2\pi i q_{n_k}\beta}$. Since $\nu_k \to \nu$ weakly, $F_*\nu_k \to F_*\nu$ (since all these measures are concentrated on a bounded subset of $\mathbb{R}^2$). Since $F_*\nu_k = \delta_{e^{2\pi i q_{n_k}\beta}}$ and $e^{2\pi i q_{n_k}\beta} \to e^{2\pi i c}$, $F_*\nu = \delta_{e^{2\pi i c}}$. It follows that

$$\operatorname{supp} \nu \subset \{(x,y) \in \mathbb{R}^2 : e^{2\pi i(y-x)} = e^{2\pi i c}\} = \{(x,y) \in \mathbb{R}^2 : y - x - c \in \mathbb{Z}\}.$$

Lemma 3.3.1. *If α has bounded partial quotients then the cocycle $\varphi \times \varphi \circ S$ is ergodic for each $\beta \in [0,1)$ satisfying $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$.*

Proof. In view of Proposition 3.1.2, $\operatorname{supp}(\nu) \subset E(\varphi \times \varphi \circ S)$. It follows from [69] that each accumulation point of the sequence $((\varphi^{(q_n)})_*\mu)_{n\geq 1}$ is an absolutely continuous measure (more precisely, it is a measure whose image via exp is Lebesgue measure on the circle). Thus the support of ν which is contained in the union of lines of the form $y = x - c - k$ (with parameter $k \in \mathbb{Z}$) has “absolutely continuous” projections on both coordinates.

Due to Theorem 3.2.1(a), if α has bounded partial quotients then for each $\beta \notin \mathbb{Q}\alpha+\mathbb{Q}$, the set of accumulation points of the sequence $(\{q_n\beta\})_{n\geq 1}$ is infinite.

Since the projections of ν are absolutely continuous and the number of $c \in [0,1)$ under consideration is infinite, $E(\varphi \times \varphi \circ S)$ cannot be of the form (a)–(c) (see (3.1)). It follows that $E(\varphi \times \varphi \circ S) = \mathbb{R}^2$ and thus $\varphi \times \varphi \circ S$ is ergodic. $\square$

Lemma 3.3.2. *If α has bounded partial quotients and $\beta \in (\mathbb{Q}\alpha+\mathbb{Q})\setminus(\mathbb{Z}\alpha+\mathbb{Z})$ then $E(\varphi \times \varphi \circ S)$ is cocompact. In particular, $\varphi \times \varphi \circ S$ is a regular cocycle.*

Proof. It follows from Theorem 3.2.1(b) that there exists $c \in (0,1)$ and a subsequence $(q_{n_k})_{k\geq 1}$ such that $\{q_{n_k}\beta\} \xrightarrow{k\to\infty} c$. As in the proof of Lemma 3.3.1 we get that there are uncountably many essential values of $\varphi \times \varphi \circ S$ in the union of the straight lines $y = x - c - k$ (k as before is an integer-valued parameter). It directly follows that the group $E(\varphi \times \varphi \circ S)$ is either of the form (c) or (d), hence cocompact. $\square$

Lemma 3.3.3. *For each α and $\beta \in \mathbb{Z}\alpha$ the cocycle $\varphi \times \varphi \circ S$ is cohomologous to $\varphi \times \varphi$. In particular, it is regular.*

Proof. For each $n \in \mathbb{Z}$ we simply have $\varphi \times \varphi \circ T^n - \varphi \times \varphi = 0 \times \varphi^{(n)} \circ T - 0 \times \varphi^{(n)}$. The cocycle $\varphi \times \varphi$ is ergodic as a cocycle taking values in the subgroup $\Delta_{\mathbb{R}} = \{(t,t) : t \in \mathbb{R}\}$ since φ was ergodic. $\square$

Collecting the results contained in Lemmas 3.3.1–3.3.3 we have proved the following.

Theorem 3.3.4. *If α has bounded partial quotients and $\beta \in [0,1)$ then $\varphi \times \varphi \circ S$ is a regular cocycle. Moreover, for each $S \in C(T)$, $E(\varphi \times \varphi \circ S)$ is either cocompact or equals $\Delta_{\mathbb{R}}$.*

3.3.2. An integer-valued ergodic cocycle φ such that $\varphi \times \varphi \circ S$ is regular for each $S \in C(T)$. In this subsection will prove that there are cocycles $\varphi\colon X \to \mathbb{Z}$ over irrational rotations such that:

(A) φ is ergodic,
(B) $\underset{S\in C(T)}{\forall}$ $\varphi \times \varphi \circ S \to \mathbb{Z} \times \mathbb{Z}$ is regular,
(C) $\underset{S\in C(T)}{\forall}$ $\varphi \times \varphi \circ S \to \mathbb{Z} \times \mathbb{Z}$ is not ergodic.

Let $G = \{(m,n) \in \mathbb{Z}^2 : m-n \text{ is even}\}$. Then:

- G is a subgroup of $\mathbb{Z}^2$,
- G has index 2 in $\mathbb{Z}^2$; in particular G is cocompact,
- G is generated by $\{(1,1),(1,-1)\}$,
- $\Delta_{\mathbb{Z}} \subset G$.

Assume that $\varphi\colon X \to \mathbb{Z}$,

$$\varphi(x) = \begin{cases} 1 & x \in [0,1/2), \\ -1 & x \in [1/2,1). \end{cases}$$

The fact that φ is ergodic has been shown in [1].

For each $\beta \in [0,1)$, $\mathrm{Im}(\varphi \times \varphi \circ S) \subset G$, where $Sx = x+\beta$; therefore $\varphi \times \varphi \circ S\colon X \to \mathbb{Z} \times \mathbb{Z}$ cannot be ergodic as $E(\varphi \times \varphi \circ S) \subset G$.

Theorem 3.3.5. *There exists an uncountable set $\Sigma \subset [0,1)$ of irrational numbers such that for each $\alpha \in \Sigma$, $\varphi \times \varphi \circ S$ is regular for each $S \in C(T)$ and:*

$$\begin{aligned} E(\varphi \times \varphi \circ S) &\subset G && \textit{for all } \beta \notin \mathbb{Z} \cup \{1/2\}, \\ E(\varphi \times \varphi \circ S) &= \Delta_{\mathbb{Z}} && \textit{for all } \beta \in \mathbb{Z}\alpha, \\ E(\varphi \times \varphi \circ S) &= \widetilde{\Delta}_{\mathbb{Z}} && \textit{if } \beta = 1/2, \ \textit{where } \widetilde{\Delta}_{\mathbb{Z}} = \{(n,-n) : n \in \mathbb{Z}\}. \end{aligned}$$

Proof. The proof of Theorem 3.3.5 will be done in several steps. First of all we define the set Σ.

A number α is in Σ if the following conditions are satisfied:

(i) α is irrational with bounded partial quotients;
(ii) $|\alpha - p_n/q_n| \le 1/3q_n^2$ for each $n \ge 1$;
(iii) q_n is odd for each $n \ge n_0$;
(iv) infinitely many of p_n are odd, and infinitely many are even.

It is clear that Σ is uncountable.

Fix $\alpha \in \Sigma$. Assume that $\beta \in [0,1)$, $Sx = x+\beta$. Suppose that $(q_{n_k})_{k\ge1}$ is a subsequence of the sequence of denominators of α for which

$$\{q_{n_k}\beta\} \xrightarrow{k\to\infty} c \quad \text{with } 0 < c < 1.$$

We have

$$q_{n_k}\left(\beta - \frac{r_k}{q_{n_k}}\right) \xrightarrow{k\to\infty} c, \quad \text{where } r_k = [q_{n_k}\beta].$$

Hence for $\varepsilon > 0$ ($\varepsilon \ll c$) and k large enough,

$$(c-\varepsilon)\frac{1}{q_{n_k}} < \beta - \frac{r_k}{q_{n_k}} < (c+\varepsilon)\frac{1}{q_{n_k}},$$

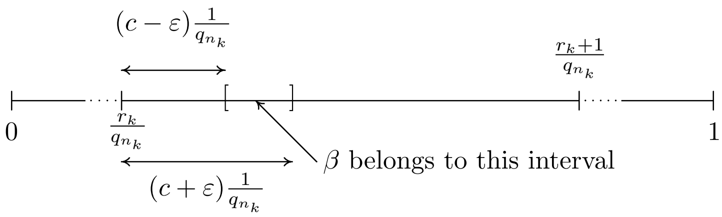

FIGURE 3.1

equivalently (see Figure 3.1)

$$\beta \in \left(\frac{r_k}{q_{n_k}} + (c-\varepsilon)\frac{1}{q_{n_k}}, \frac{r_k}{q_{n_k}} + (c+\varepsilon)\frac{1}{q_{n_k}}\right).$$

Case 1. $0 < c < 1/2$. We assume additionally

$$n_k \text{ are even for } k \geq k_0 \tag{3.6}$$

(for such n_k we have $\alpha > p_{n_k}/q_{n_k}$). We fix $\delta > 0$ such that

$$\delta < 1/2 - c, \tag{3.7}$$

$$\delta < 1/6. \tag{3.8}$$

Choose $0 < \delta' < (1/2 - c) - \delta$ and then $\varepsilon \ll c$ so that

$$0 < \varepsilon < \frac{1}{100}\left(\frac{1}{2} - c - \delta - \delta'\right).$$

For each k large enough ($k \geq k_1$ and k_1 will be specified by the argument below) define

$$A_k^{(i)} = \left[\frac{i}{q_{n_k}}, \frac{i}{q_{n_k}} + \delta\frac{1}{q_{n_k}}\right), \quad i = 0, \ldots, q_{n_k} - 1.$$

For each $j = 0, \ldots, q_{n_k} - 1$, we have $j/(3q_{n_k}^2) \leq 1/(3q_{n_k})$, so in view of (3.8), the interval $T^jA_k^{(i)}$

(3.9) is contained in an interval

$$\left[\frac{s}{q_{n_k}}, \frac{s}{q_{n_k}} + \frac{1}{2}\cdot\frac{1}{q_{n_k}}\right)$$

and the map $\{0, \ldots, q_{n_k} - 1\} \ni j \mapsto s \in \{0, \ldots, q_{n_k}\}$ is 1–1

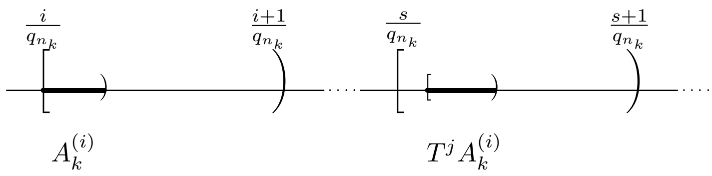

Figure 3.2

(see Figure 3.2).

In view of the inequality (3.7), the interval $SA_k^{(i)}$ is contained in an interval

$$\left[\frac{\widetilde{s}}{q_{n_k}}, \frac{\widetilde{s}}{q_{n_k}} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}\right)$$

(see Figure 3.3).

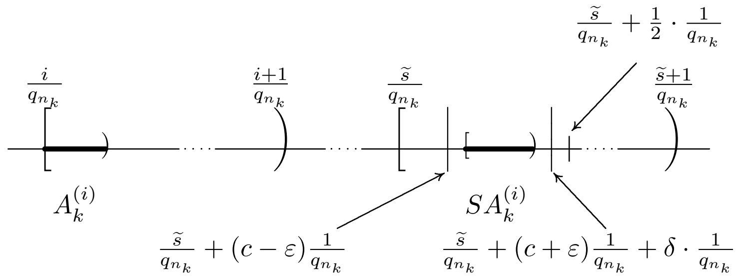

Figure 3.3

For each $j = 0, \ldots, q_{n_k} - 1$, the interval $T^j SA_k^{(i)}$

(3.10) is contained in an interval

$$\left[\frac{t}{q_{n_k}}, \frac{t}{q_{n_k}} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}\right)$$

and the map $\{0, \ldots, q_{n_k} - 1\} \ni j \mapsto s \in \{0, \ldots, q_{n_k}\}$ is 1–1.

For each $i \in \{0, 1, \ldots, q_{n_k} - 1\}$, define

$$b_k(i) = j \Leftrightarrow T^j SA_k^{(i)} \subset \left[\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{q_{n_k}}, \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}\right)$$

(note that since q_{n_k} is odd, the last interval equals $[r'/q_{n_k}, (r'+1)/q_{n_k})$, where $r' = q_{n_k}/2 - 1/2$). By (3.10), the function b_k is well defined on $\{0, \ldots, q_{n_k} - 1\}$ with values in $\{0, \ldots q_{n_k} - 1\}$. In fact

$$b_k \quad \text{is a bijection.} \tag{3.11}$$

Indeed, if $j = b_k(i_1) = b_k(i_2)$, then the intervals $T^jSA_k^{(i_1)}$, $T^jSA_k^{(i_2)}$ are both contained in an interval of length $1/q_{n_k}$, it follows that $A_k^{(i_1)}$ and $A_k^{(i_2)}$ are contained in an interval of length $1/q_{n_k}$ which is an obvious contradiction.

We say that $i \in \{0, \dots, q_{n_k}\}$ is *good* if $0 \le b_k(i) \le 3\delta' q_{n_k}$. In view of (3.11),

$$\operatorname{card}\{i = 0, \dots, q_{n_k} - 1 : i \text{ is good}\} \ge \delta' q_{n_k}. \tag{3.12}$$

We will show that

$$\text{if } i \text{ is good, } x \in A_k^{(i)} \text{ then } (\varphi^{(q_{n_k})}(x), \varphi^{(q_{n_k})}(x+\beta)) = (1,1). \tag{3.13}$$

Indeed, $\varphi^{(q_{n_k})}(x) = 1$ follows directly from (3.9) (and this is true for each $i = 0, \dots, q_{n_k} - 1$). We have

$$T^{b_k(i)}(x+\beta) \in \left[\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{q_{n_k}}, \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}\right)$$

and the distance between $T^{b_k(i)}(x+\beta)$ and $1/2 - 1/(2q_{n_k})$ is estimated by (see Figure 3.3)

$$(c+\varepsilon)\frac{1}{q_{n_k}} + \delta \cdot \frac{1}{q_{n_k}} + \frac{b_k(i)}{3q_{n_k}^2} \le (c+\varepsilon+\delta+\delta') \cdot \frac{1}{q_{n_k}} < \frac{1}{2} \cdot \frac{1}{q_{n_k}}$$

(by our choice of ε, and for k large enough), so $\varphi^{(q_{n_k})}(x+\beta) = 1$.

It follows from (3.12) and (3.13) that

$$\mu\{x \in [0,1) : (\varphi^{(q_{n_k})}(x), \varphi^{(q_{n_k})}(x+\beta)) = (1,1)\} \ge \delta \cdot \delta'$$

for each k sufficiently large, so

$$(1,1) \in E(\varphi \times \varphi \circ S). \tag{3.14}$$

We will now show that $(1,-1) \in E(\varphi \times \varphi \circ S)$. We have assumed that α has bounded partial quotients, so for some $C > 0$,

$$\left|\alpha - \frac{p_n}{q_n}\right| \ge \frac{1}{Cq_n^2}$$

for all $n \ge 1$. Since (ii) holds, for some subsequence $(q_{n_{k_l}})_{l \ge 1}$ we have

$$q_{n_{k_l}}^2 \left|\alpha - \frac{p_{n_{k_l}}}{q_{n_{k_l}}}\right| \xrightarrow{l \to \infty} \frac{1}{D},$$

where $D \ge 3$. But the n_{k_l}'s are still even, so without loss of generality we simply assume that

$$q_{n_k}^2 \left|\alpha - \frac{p_{n_k}}{q_{n_k}}\right| \xrightarrow{k \to \infty} \frac{1}{D}. \tag{3.15}$$

Fix

$$0 < \delta'' < \min\left(\frac{c}{2}, \frac{1}{2} - \frac{1}{D}, \frac{1}{D}\right). \tag{3.16}$$

Let $0 < \varepsilon < \delta''/100$. For k sufficiently large and $i = 0, \ldots, q_{n_k} - 1$ put

$$B_k^{(i)} = \left[\frac{i}{q_{n_k}} + \left(\frac{1}{2} - \frac{1}{D}\right)\frac{1}{q_{n_k}} - \frac{\delta''}{q_{n_k}}, \frac{i}{q_{n_k}} + \left(\frac{1}{2} - \frac{1}{D}\right)\frac{1}{q_{n_k}} - \varepsilon\frac{1}{q_{n_k}}\right)$$

(see Figure 3.4).

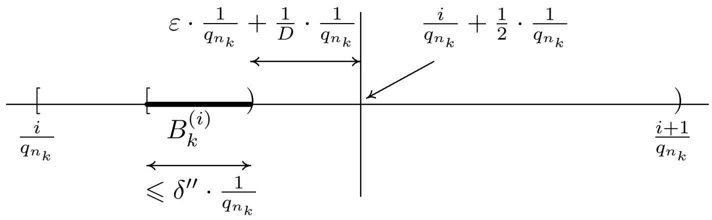

FIGURE 3.4

Then, for every k large enough we have: for each $j = 0, \ldots, q_{n_k} - 1$, for the interval $T^j B_k^{(i)}$

(3.17) (3.9) holds.

The interval $SB_k^{(i)}$ is contained in an interval $[\bar{s}/q_{n_k}, \bar{s}/q_{n_k} + 1/q_{n_k})$ and for each $j = 0, \ldots, q_{n_k} - 1$, for the interval $T^j SB_k^{(i)}$,

(3.18) (3.10) holds.

It follows that the formula

$$c_k(i) = j \Leftrightarrow T^j SB_k^{(i)} \subset \left[\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{q_{n_k}}, \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}\right)$$

defines a bijection $c_k\colon \{0, \ldots, q_{n_k}\} \to \{0, \ldots, q_{n_k}\}$. Notice that in view of (3.16), $ST^{q_{n_k}-1}B_k^{(i)}$ is contained in an interval of the form $[u/q_{n_k}, (u+1)/q_{n_k})$. In fact it is contained in the right half of that interval and more precisely, using (3.15), the distance of $ST^{q_{n_k}-1}B_k^{(i)}$ from $u/q_{n_k} + 1/(2q_{n_k})$ is at least $c/(3q_{n_k})$ (for k large enough) – see Figure 3.5.

It now follows from (3.15) that the interval $ST^{q_{n_k}-j}B_k^{(i)}$ will be contained in the right half of an $[u_j/q_{n_k}, (u_j+1)/q_{n_k})$ whenever

$$j\left(\alpha - \frac{p_{n_k}}{q_{n_k}}\right) < \frac{c}{3} \cdot \frac{1}{q_{n_k}},$$

and thus, using (ii), for all $j \in \{0, \ldots, q_{n_k} - 1\}$ satisfying

$$0 < j < cq_{n_k}.$$

We say that $i \in \{0, \ldots, q_{n_k} - 1\}$ is *good* if $c_k(i) = q_{n_k} - j$ with $0 < j < cq_{n_k}$. The number of good i's is at least cq_{n_k}.

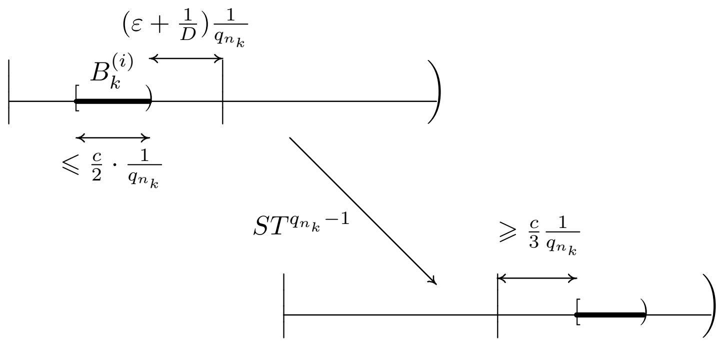

FIGURE 3.5. $T^{q_{n_k}-1}$ shifts $B_k^{(i)}$ close to $1/q_{n_k} + (1/2) \cdot (1/q_{n_k})$, while S shifts $T^{q_{n_k}-1}B_k^{(i)}$ into the right half of $[u/q_{n_k}, (u+1)/q_{n_k})$.

It follows from (3.17), (3.18) and the above discussion that

$$\mu\{x \in [0,1) : (\varphi^{(q_{n_k})}(x), \varphi^{(q_{n_k})}(x+\beta)) = (1,-1)\} \geq c(\delta'' - \varepsilon),$$

so $(1,-1) \in E(\varphi \times \varphi \circ S)$.

In order to conclude Case 1 we have to consider the situation when n_k is odd for all k large enough. First put

$$C_k^{(i)} = \left[\frac{i}{q_{n_k}} + \frac{1}{2} \cdot \frac{1}{q_{n_k}} - \delta_1 \cdot \frac{1}{q_{n_k}}, \frac{i}{q_{n_k}} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}\right)$$

(see Figure 3.6).

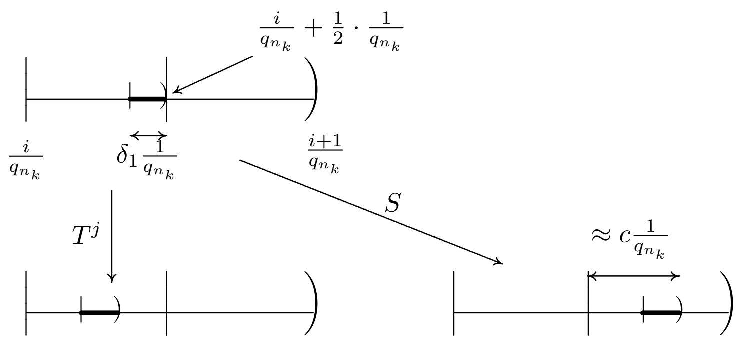

FIGURE 3.6

By consideration similar to those used before, it follows that $(1,-1) \in E(\varphi \times \varphi \circ S)$. If we put

$$D_k^{(i)} = \left[\frac{i}{q_{n_k}} + (1-c-\varepsilon)\frac{1}{q_{n_k}} - \delta_2 \cdot \frac{1}{q_{n_k}}, \frac{i}{q_{n_k}} + (1-c-\varepsilon)\frac{1}{q_{n_k}}\right)$$

then similar arguments show that also $(-1,1) \in E(\varphi \times \varphi \circ S)$ (see Figure 3.7).

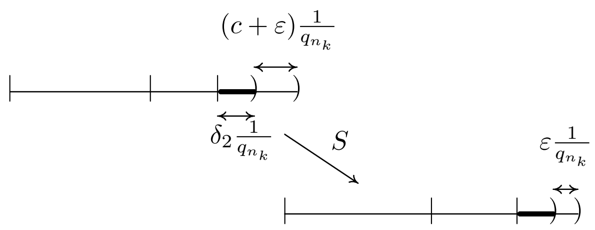

FIGURE 3.7

Case 2. $1/2 < c < 1$. We replace β by $-\beta$ (that is S by S^{-1}) and by Case 1 we obtain

$$(1,1),(1,-1) \in E(\varphi \times \varphi \circ S^{-1}) \tag{3.19}$$

because $\{q_n(-\beta)\} \to -c$ (and $-c = 1-c$, $0 < 1-c < 1/2$). In fact, to obtain (3.19) we have

$$((\varphi \times \varphi \circ S^{-1})^{(q_{n_k})})_*\mu \to \nu$$

and $\nu\{(1,-1)\} > 0$. In other words, there exists a $\kappa > 0$ such that for k large enough,

$$\underset{Y_k \subset [0,1),\, \mu(Y_k) \geq \kappa}{\exists} \ \underset{x \in Y_k}{\forall} (\varphi^{(q_{n_k})}(x), \varphi^{(q_{n_k})}(S^{-1}x)) = (1,-1).$$

Put $Y_k' := S^{-1}Y_k$. For $x \in Y_k'$ we have

$$\varphi^{(q_{n_k})}(x) = -1, \quad \varphi^{(q_{n_k})}(Sx) = 1$$

for all k large enough. It follows that for any limit measure ν' of the distributions $((\varphi \times \varphi \circ S)^{(q_{n_k})})_*\mu$ we have $\nu'\{(-1,1)\} \geq \kappa$ and therefore $(-1,1) \in E(\varphi \times \varphi \circ S)$. We show similarly that $(1,1) \in E(\varphi \times \varphi \circ S)$, so finally $E(\varphi \times \varphi \circ S) = G$.

Case 3. $c = 1/2$. It is clear that the method described in Case 1 (and Case 2) gives $(1,-1) \in E(\varphi \times \varphi \circ S)$.

Case 4. $c = 0$. In this case one shows (by the method of Cases 1 and 2) that $(1,1) \in E(\varphi \times \varphi \circ S)$.

Conclusion of Cases 1–4:

(3.20) If $c \neq 0, 1/2$ belongs to the set of accumulation points of the sequence $A_\alpha(\beta) = \{\{q_n\beta\} : n \geq 1\}$ then $E(\varphi \times \varphi \circ S) = G$, in particular

$$\text{if } \beta \notin \mathbb{Q}\alpha + \mathbb{Q}, \text{ then } E(\varphi \times \varphi \circ S) = G \tag{3.21}$$

(indeed, in this case the set $A_\alpha(\beta)$ has infinitely many limit points). Furthermore,

(3.22) if $\{0, 1/2\}$ is contained in the set of limit points of $A_\alpha(\beta)$, then

$$E(\varphi \times \varphi \circ S) = G.$$

Case 5. $\beta = 1/2$. In this case $\varphi \circ S = -\varphi$ $(Sx = x + 1/2)$ and $\mathrm{Im}(\varphi \times \varphi \circ S) \subset \widetilde{\Delta}_{\mathbb{Z}}$, and since $1/2$ belongs to the set of accumulation points of $A_\alpha(\beta)$ (in view of (iii)), $(1, -1) \in E(\varphi \times \varphi \circ S)$, so $\varphi \times \varphi \circ S$ is ergodic as cocycle taking values in $\widetilde{\Delta}_{\mathbb{Z}}$.

Case 6. $\beta = 1/2^l$, $l \geq 2$ (also $\beta = i/2^l$ with i odd). Because of (iii), $q_n = 2\widetilde{q}_n + 1$ and

$$q_n\beta = \frac{\widetilde{q}_n}{2^{l-1}} + \frac{1}{2^l}.$$

Since the set of accumulation points of $\{\{\widetilde{q}_n/2^{l-1}\} : n \geq n_0\}$ is contained in $\{i/2^{l-1} : i = 0, \dots, 2^{l-1}\}$,

$$\exists_{c \neq 0, 1/2} \; c \text{ belongs to the set of accumulation points of } A_\alpha(\beta), \tag{3.23}$$

so by (3.20), $E(\varphi \times \varphi \circ S) = G$.

Case 7. $\beta = u/v$, $v \neq 2^l$. It follows $v = 2^l w$, where $w \geq 3$ is odd. The set of accumulation points of $A_\alpha(\beta)$ is contained in $\{i/(2^l w) : i = 0, \dots, 2^l w - 1\}$. But w cannot divide all denominators q_n, $n \geq n_0$, because two consecutive denominators are relatively prime. Thus (3.23) also holds.

Case 8. $\beta = (1/2^l)\alpha$, $l \geq 2$ (and $\beta = (1/2^l)\alpha + 1/2$). First note that

$$\left| q_n \frac{\alpha}{2^l} - \frac{p_n}{2^l} \right| = \frac{q_n}{2^l} \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{2^l} \cdot \frac{1}{3q_n},$$

so the set of accumulation points of $A_\alpha(\beta)$ is the same as that of $\{p_n/2^l : n \geq 1\}$ and we are in the situation of Case 6 by (iv).

Case 9. $\beta = (u/v)\alpha$ with u, v relatively prime, $v \neq 2^l$, $l \geq 1$ (and $\beta = (u/v)\alpha + 1/2$). This reduces to the study of the set of accumulation points of $\{p_n \cdot u/v : n \geq 1\}$ and a reasoning as in Case 7 applies.

Case 10. $\beta = \alpha/2$ (and $\beta = \alpha/2 + 1/2$). In this case, we consider $\{p_n \cdot 1/2 : n \geq 1\}$ (and $\{(p_n + q_n) \cdot 1/2 : n \geq 1\}$) and due to (iv), both 0 and $1/2$ are in the set of accumulation points of $A_\alpha(\beta)$; then we apply (3.22).

If none of the above cases holds then it remains to consider the following:

Case 11. $\beta = (u/v)\alpha + s/v$, where

- $(u, v, s) = 1$,

- $0 \le u, s < v$,
- $v \ge 3$.

We study the set of accumulation points of

$$\left\{\frac{up_n + sq_n}{v} : n \ge 1\right\}$$

which is contained in $\{i/v : i = 0, \dots, v-1\}$. If the only accumulation points are 0 or 1/2, then

$$\underset{N}{\exists}\ \underset{n \ge N}{\forall}\ \underset{k_n \in \mathbb{Z}}{\exists}\ 2up_n + 2sq_n = k_n v.$$

Hence for $n \ge N$ we have

$$v\begin{pmatrix} k_n \\ k_{n+1} \end{pmatrix} = \begin{pmatrix} p_n & q_n \\ p_{n+1} & q_{n+1} \end{pmatrix}\begin{pmatrix} 2v \\ 2s \end{pmatrix}$$

where

$$\det\begin{bmatrix} p_n & q_n \\ p_{n+1} & q_{n+1} \end{bmatrix} = 1 \quad \text{or} \quad -1.$$

It follows that

$$v\begin{pmatrix} p_n & q_n \\ p_{n+1} & q_{n+1} \end{pmatrix}^{-1}\begin{pmatrix} k_n \\ k_{n+1} \end{pmatrix} = \begin{pmatrix} 2u \\ 2s \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} p_n & q_n \\ p_{n+1} & q_{n+1} \end{smallmatrix}\right)^{-1}$ is integer-valued, v divides $2u$ and $2s$. However, $v \ge 3$, so we obtain a contradiction. Hence (3.23) must hold in this case.

The proof of Theorem 3.3.5 is complete. □

3.4. Semisimple automorphisms

In this section we return to a general study of automorphisms of the form $T_{\varphi,\mathcal{G}}$, i.e. to Rokhlin cocycle extensions.

We prove a theorem giving rise to new classes of semisimple automorphisms.

Theorem 3.4.1. *Let $\mathcal{G} = \{R_g\}_{g \in G}$ be a mildly mixing action of G. Assume that T is an irrational rotation and $\varphi: X \to G$ is an ergodic cocycle such that for each $S \in C(T)$, the cocycle $\varphi \times \varphi \circ S: X \to G$ is regular. Assume moreover that for each $S \in C(T)$, $E(\varphi \times \varphi \circ S)$ is either cocompact or equals Δ_G.*

(a) *If $\mathcal{G}$ is 2-fold simple then each ergodic self-joining of $T_{\varphi,\mathcal{G}}$ is either a graph or the relatively independent extension of a graph joining of T. Moreover, $T_{\varphi,\mathcal{G}}$ is semisimple.*

(b) *If $\mathcal{G}$ is semisimple then $T_{\varphi,\mathcal{G}}$ is semisimple.*

Proof. (a) Take $\widetilde{\lambda} \in J^e(T_{\varphi,\mathcal{G}})$. Hence, for some $S \in C(T)$, $\widetilde{\lambda}$ is an extension of μ_S and therefore (see the discussion before Proposition 3.2.14) we can assume that $\widetilde{\lambda} \in \mathcal{M}^e(T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}})$.

Assume first that $\varphi \times \varphi \circ S$ is ergodic as a G^2-cocycle. The corresponding $\mathcal{G} \times \mathcal{G}$-action $\{R_{g_1} \times R_{g_2}\}_{g_1, g_2 \in G}$ is uniquely ergodic in the sense of Proposition 3.2.6(a). By Theorem 3.2.9 and Proposition 3.2.14(c), $\widetilde{\lambda} = \mu \times \nu \times \nu$. In view of Proposition 3.2.14(d), it remains to show that the extension

$$(T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}}, \widetilde{\lambda}) \to T_{\varphi, \mathcal{G}}$$

is relatively weakly mixing. But if we put $W = T_{\varphi, \mathcal{G}}$ and we consider the cocycle $\varphi \circ S$ as $\varphi \circ S : X \times Y \to G$ (that is as a cocycle for W) then $T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}}$ and $W_{\varphi \circ S, \mathcal{G}}$ are relatively isomorphic (over the common factor $T_{\varphi, \mathcal{G}}$) and moreover $W_{\varphi \circ S, \mathcal{G}}$ is relatively weakly mixing over the base W because the G-action is mildly mixing and $W_{\varphi \circ S, \mathcal{G}}$ is ergodic (see Proposition 1.5.5(a)).

Consider now a more general case: $E(\varphi \times \varphi \circ S) = H$ is a proper subgroup of $G \times G$. Then the extension

$$(T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}}, \mu \times \nu \times \nu) \to T_{\varphi, \mathcal{G}}$$

is relatively weakly mixing if and only if so is

$$(T_{\theta, \mathcal{G}}, \mu \times \nu \times \nu) \to T_{\varphi, \mathcal{G}},$$

where θ is an ergodic cocycle with values in H cohomologous to $\varphi \times \varphi \circ S$.

Suppose that additionally H is cocompact. Then, in sense of Proposition 3.2.6(b), the corresponding $\mathcal{H}$-action is still uniquely ergodic. Hence $\widetilde{\lambda} = \mu \times \nu \times \nu$ and therefore the same argument as in the previous case shows that $(T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}}, \widetilde{\lambda})$ is relatively weakly mixing over $T_{\varphi, \mathcal{G}}$.

It remains to consider the case $H = \Delta_G$. It follows $\widetilde{\lambda} = \mu \times \rho$ (recall that we still identify $\widetilde{\lambda}$ with an element of $\mathcal{M}^e(T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}})$), where $\rho \in J^e(\mathcal{G})$. If $\rho = \nu \times \nu$ then we are in the situation already considered. Otherwise ρ is a graph and by Proposition 3.2.14(b), so must be $\widetilde{\lambda}$.

(b) The proof is along the same lines as the one of (a) except the case of $H = \Delta_G$. We have to show that the extension

$$(T_{\theta, \Delta_G}, \mu \times \rho) \to T_{\theta, \{(R_g, R_g)_{g \in G}\}}$$

is relatively weakly mixing. As usual we consider $T_{\varphi, \mathcal{G}}$ as a factor of the system $T_{\theta, \{(R_g, R_g)_{g \in G}\}}$ "sitting" on the first two coordinates (note that as a σ-algebra it is equal to $\mathcal{B} \otimes \mathcal{C} \times Y$). By considering θ, $\mathcal{D} = \mathcal{C} \times Y$ and ρ, we are in the situation of Lemma 3.2.16. Because $\mathcal{G}$ is semisimple and $\rho \in J^e(\mathcal{G})$, $(\mathcal{C} \otimes \mathcal{C}, \rho)$ is relatively weakly mixing over $\mathcal{D}$, whence $(T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}}, \widetilde{\lambda})$ is relatively weakly mixing over $T_{\varphi, \mathcal{G}}$ and the result follows. $\square$

Remark 3.4.2. It is now easy to describe the smallest natural family (see Definition 2.4.1) of the semisimple automorphisms arising from Theorem 3.4.1. Indeed, such a family consists of all factors of $T_{\varphi, \mathcal{G}}$ relative to which $T_{\varphi, \mathcal{G}}$ is weakly mixing . First of all, note that T is a maximal distal factor of $T_{\varphi, \mathcal{G}}$.

It follows that if $\mathcal{A}$ is a factor relative to which $T_{\varphi,\mathcal{G}}$ is weakly mixing then $\mathcal{A}$ contains the "first coordinate". We then apply Corollary 3.2.12. We find that the factors $T_{\varphi,\mathcal{G}}$ relative to which $T_{\varphi,\mathcal{G}}$ is weakly mixing are of the form $\mathcal{B}\otimes\mathcal{D}$, where $\mathcal{D}$ is a $\mathcal{G}$-factor. If $\mathcal{D}$ is non-trivial then it is determined by $\{\mathrm{Id}\}\times K$, which is a compact subgroup of $C(T_{\varphi,\mathcal{G}})$, and and it follows that $T_{\varphi,\mathcal{G}}$ is not relatively weakly mixing over $\mathcal{B}\otimes\mathcal{D}$ unless $\mathcal{D}=\mathcal{C}$. We have shown that the smallest natural family equals $\{T,T_{\varphi,\mathcal{G}}\}$.

3.5. Final remarks

The examples of semisimple automorphisms given by Theorem 3.4.1 are weakly mixing extensions of rotations and each such example is disjoint from any weakly mixing automorphism (see Proposition 1.5.5 and Proposition 1.5.7). There are Gaussian actions that are mildly mixing and semisimple (see [68]). By looking at the proof of Theorem 3.4.1, it is clear that the mild mixing assumption can be replaced by being Gaussian semisimple.

If we consider extensions $T_{\varphi,\mathcal{G}}$ of irrational rotations in which we have $G=\mathbb{Z}$ and φ is given by Theorem 3.3.5 then one more assumption on $\{R^n\}_{n\in\mathbb{Z}}$ has to be added. It is is caused by the fact that $\widetilde{\Delta}_{\mathbb{Z}}$ appears as the group of essential values of $\varphi\times\varphi\circ S$ for some $S\in C(T)$. It gives rise to study $J(R,R^{-1})$. In order to obtain Theorem 3.4.1 it is sufficient to assume that the $\mathbb{Z}$-action R satisfies either

(i) $R\perp R^{-1}$ (see [45] for the case of MSJ), or
(ii) R is isomorphic to R^{-1} (which is always the case whenever Gaussian actions are considered).

In Theorem 3.2.9 we deal with self-joinings of order 2. It is clear however that the same results hold for self-joinings of higher degrees. Given $n\geq 2$ denote by $J_n(T_{\varphi,\mathcal{G}})$ the set of n-self-joinings of $T_{\varphi,\mathcal{G}}$. Then Corollary 3.2.11 yields to the following.

Proposition 3.5.1. *If $\varphi\colon X\to G$ is ergodic then the map*

$$(x_1,y_1,\dots,x_n,y_n)\xrightarrow{\Lambda_0^n}(x_1,\dots,x_n,y_1,\dots,y_n)$$

is an affine isomorphism of $J_n(T_{\varphi,\mathcal{G}};\Delta_X)$ and $\{\Delta_X\times\rho:\rho\in J_n(\mathcal{G})\}$.

In [45] the following problem is formulated: Is it true that for each ergodic zero entropy automorphism $W\colon(Z,\mathcal{E},\kappa)\to(Z,\mathcal{E},\kappa)$, if $\rho\in J_3(W)$ is pairwise independent then $\rho=\kappa\times\kappa\times\kappa$? (The problem is open in the weakly mixing case.) An affirmative answer would imply that each 2–fold mixing automorphism is 3-fold mixing (the latter being Rokhlin's well known open problem). We have been unable to answer Junco–Rudolph's question. However, the method of the present paper gives rise to the negative answer to the *relative* version of their problem. Indeed, let T be an ergodic rotation, $\varphi\colon X\to\mathbb{Z}$ an ergodic cocycle and R a Bernoulli automorphism. It is well known that there exists $\rho \in J_3^e(R)$ which is pairwise independent but it is not the product measure $\nu \times \nu \times \nu$. By the above proposition, $(\Lambda_0^3)^{-1}(\mu \times \rho)$ is an ergodic element of $J_3(T_{\varphi,\mathcal{G}})$ (here $T_{\varphi,\mathcal{G}}(x,y) = (Tx, R^{\varphi(x)}(y)))$ which is relatively pairwise independent, but is different from $(\Lambda_0^3)^{-1}(\mu \times (\nu \times \nu \times \nu))$. Moreover, $T_{\varphi,\mathcal{G}}$ being disjoint from the class of weakly mixing transformations, the entropy of $T_{\varphi,\mathcal{G}}$ equals zero and hence also the relative entropy of $T_{\varphi,\mathcal{G}}$ over T equals zero.

CHAPTER 4

NATURAL FAMILIES OF FACTORS IN TOPOLOGICAL DYNAMICS

4.1. General backgrounds

We recall the basic notation and results of the universal theory of topological dynamics (see Section 1.8 for details). As usual $\beta\mathbb{Z}$ denotes the Čech–Stone compactification of $\mathbb{Z}$, I a fixed arbitrary minimal left ideal of the semigroup $\beta\mathbb{Z}$ (replacing the customary M), J the set of idempotents in I, u a fixed arbitrary element of J, and $G = uI$ the maximal subgroup of I corresponding to u. The semigroup $\beta\mathbb{Z}$ acts on every compact flow (Z, T), and $\overline{\mathrm{Orb}}(z) = \{pz : p \in \beta\mathbb{Z}\}$ for $z \in Z$. A necessary and sufficient condition for z to be almost periodic is that $vz = z$ for some $v \in J$. Thus JZ is the collection of all almost periodic points in Z. When z is almost periodic then $\overline{\mathrm{Orb}}(z) = \{pz : p \in I\}$.

Given a minimal flow (Z, T), we shall always choose a distinguished point z_0 in Z, such that $uz_0 = z_0$. Our convention is that under a homomorphism a distinguished point goes to a distinguished point. When (Z, z_0, T) is such a pointed minimal flow, its Ellis group $\mathcal{G}(Z, z_0)$ is defined by $\mathcal{G}(Z) = \mathcal{G}(Z, z_0) = \{\alpha \in G : \alpha z_0 = z_0\}$. The set G is equipped with a compact T_1 topology, called the τ-topology, with respect to which, all groups of the form $\mathcal{G}(Z, z_0)$ are closed. For a given set $A \subset Z$ we denote by $\overline{A}^{\tau}$ the τ-closure of A.

If we have a family $\{(Z_\sigma, z_\sigma)\}_{\sigma\in\Sigma}$ of pointed minimal flows we choose $x_0 \in \prod_{\sigma\in\Sigma} Z_\sigma$, $x_0(\sigma) = z_\sigma$, and set

$$\bigvee_{\sigma\in\Sigma} (Z_\sigma, z_\sigma) = (\overline{\mathrm{Orb}}(x_0), x_0)$$

(see (1.18)). Observe that $\bigvee_{\sigma\in\Sigma}(Z_\sigma, z_\sigma)$ is minimal and

$$\mathcal{G}\Big(\bigvee_{\sigma\in\Sigma} (Z_\sigma, z_\sigma)\Big) = \bigcap_{\sigma\in\Sigma} \mathcal{G}(Z_\sigma, z_\sigma).$$

For every τ-closed subgroup F of G we let

$$F' := \bigcap\{\overline{V}^{\tau} : V \text{ is } \tau\text{-open neighbourhood of } u \text{ in } F\}$$

(see Definition 1.8.12). Then F' is a τ-closed normal subgroup of F characterized as the smallest τ-closed subgroup H of F such that F/H is a compact Hausdorff topological group (see Proposition 1.8.13). One can iterate this operation to obtain the (possibly transfinite) sequence of "derived" groups $F'' = (F')', \dots, F^{\alpha+1} = (F^{\alpha})', \dots$, where for a limit ordinal α, $F^{\alpha} = \bigcap\{F^{\beta} : \beta < \alpha\}$. For some ordinal η this process stabilizes (i.e. $F^{\eta+1} = F^{\eta}$), and we denote $F^{\infty} = F^{\eta}$.

For a pointed minimal distal flow (X, x_0, T), $X = Gx_0$ and the family of pointed factors $(X, x_0, T) \xrightarrow{\pi} (Y, y_0, T)$ is in 1–1 correspondence with the family of τ-closed subgroups $F \supset A$, where $A = \mathcal{G}(X, x_0)$ and $F = \mathcal{G}(Y, y_0)$.

In terms of Ellis groups, a minimal extension $(X, T) \xrightarrow{\pi} (Y, T)$ is proximal if and only if $\mathcal{G}(X, x_0) = \mathcal{G}(Y, y_0)$. It is isometric if nd only if it is distal and $\mathcal{G}(Y, y_0)' \subset \mathcal{G}(X, x_0)$; if moreover $\mathcal{G}(X, x_0) \lhd \mathcal{G}(Y, y_0)$, then π is a group extension (see Proposition 1.8.6).

If $\{A_i\}$ is any collection of τ-closed subgroups of G, we let $\bigvee A_i$ be the smallest τ-closed subgroup of G containing all the A_i's.

In Section 4.2 we will need the following lemma.

Lemma 4.1.1 ([30, Lemma 3])**.** *In the diagram of homomorphisms of compact minimal flows*

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\pi"'] & Z \arrow[dl, "\rho"] \\ Y & \end{tikzcd}$$

suppose π and ρ are regular, σ is proximal, and X is metrizable. Then the topological groups $\Gamma_\pi = \{\psi \in \operatorname{Aut}(X) : \pi \circ \psi = \pi\}$ and $\Gamma_\rho = \{\psi \in \operatorname{Aut}(Z) : \rho \circ \psi = \rho\}$ are isomorphic.

Lemma 4.1.2 ([93, Lemma 2.2.4])**.** *Suppose that $\pi: X \to Y$ and $\theta: Y \to Z$ are homomorphisms of compact minimal flows. If π is regular and θ is proximal then $\theta \circ \pi: X \to Z$ is regular.*

Proof. Suppose $(x, x') \in X \times Y$ is an almost periodic point satisfying $\theta(\pi(x)) = \theta(\pi(x'))$. As θ is proximal, the pair $(\pi(x), \pi(x'))$ is a proximal pair. On the other hand, $(\pi(x), \pi(x'))$ is an almost periodic pair in $Y \times Y$, hence $\pi(x) = \pi(x')$. Since π is regular, there exists $S \in \operatorname{Aut}(X)$ such that $S(x) = S(x')$. □

We also have the following.

Theorem 4.1.3 ([30, Theorem 2])**.** *Let $X \xrightarrow{\pi} Y$ be a regular homomorphism of minimal flows; then each of the homomorphisms onto Y of the flows X_ν, Y_ν, Z_ν, $\nu \le \eta$, constructed in the canonical* PI*-tower for π is regular.*

The following theorem is a part of [27, Theorem 2.1].

Theorem 4.1.4. *Let $X \xrightarrow{\phi} Y$ be a* RIC*-extension. Then there exists the maximal isometric extension $Z \xrightarrow{\psi} Y$ such that the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\phi"'] & Z \arrow[dl, "\psi"] \\ Y & \end{tikzcd}$$

commutes.

Theorem 4.1.5 ([30, Theorem 3])**.** *Let (X, T) be a metrizable minimal flow and $X \xrightarrow{\pi} Y$ a regular* RIC *homomorphism. Let $X \xrightarrow{\omega} Z \xrightarrow{\kappa} Y$, $\pi = \kappa \circ \omega$, where Z is the largest equicontinuous extension of Y under π. Then ω is* RIC *and weakly mixing, and κ is a group homomorphism. In particular, when X is regular, the homomorphism $X \xrightarrow{\omega} Z$ of X onto its largest equicontinuous factor is* RIC *and weakly mixing and Z is a compact group rotation.*

4.2. A natural family of factors defined by minimal joinings

In this section we introduce definition of a natural family of factors (factor relations) for a minimal flow using so named minimal joinings, and prove some basic facts.

Definition 4.2.1. Let T be a homeomorphism of a compact metric space (X, d). We will consider the $\mathbb{Z}$-flow (X, T). The set of all minimal subsets of $(X \times X, T \times T)$ we denote by $\mathcal{M}_2(X, T)$. We call the elements of $\mathcal{M}_2(X, T)$ the *minimal joinings* of (X, T).

The simplest examples of minimal flows are minimal rotations on compact metrizable monothetic groups. If X is such a group and $\overline{\{x_0^n : n \in \mathbb{Z}\}} = X$, then the map $T(x) = x_0 x$ is a minimal homeomorphism. In this case we can easily describe the centralizer of (X, T): $S \in C(X, T)$ if and only if there is an $a \in X$ such that $S(x) = ax$, $x \in X$. Actually, more is true:

$$M \in \mathcal{M}_2(X, T) \Leftrightarrow \underset{S \in C(X,T)}{\exists} M = \mathrm{Graph}(S) = \{(x, S(x)) : x \in X\}.$$

A minimal rotation is an example of a distal flow.

Given two minimal flows (X_i, T_i), and two factor relations R_i on X_i, $i = 1, 2$, the relation $\overline{R}_i$ in $X_1 \times X_2$ is defined by $((x_1, x_2), (y_1, y_2)) \in \overline{R}_i$ if and only if $(x_i, y_i) \in R_i$. The equivalence classes of $\overline{R}_i$ are of the form $[x]_{R_1} \times X_2$ and $X_1 \times [x]_{R_2}$ respectively.

Lemma 4.2.2. *Let $M \subset X_1 \times X_2$ be a minimal set and R_i, $i = 1, 2$, be two factor relations on X_i with natural homomorphisms π_i respectively. Then*

$(\overline{R}_1)_M = (\overline{R}_2)_M$ *if and only if there exists an isomorphism* $S\colon (X_1)_{R_1} \to (X_2)_{R_2}$ *such that*

$$(\pi_1 \times \pi_2)(M) = \mathrm{Graph}(S). \tag{4.1}$$

Proof. When $(\overline{R}_1)_M = (\overline{R}_2)_M$ and the projections of M into both coordinates are onto, for each $x_1 \in X_1$ there exists an $x_2 \in X_2$ with

$$([x_1]_{R_1} \times X_2) \cap M = (X_1 \times [x_2]_{R_2}) \cap M.$$

Define S by setting $S([x_1]_{R_1}) = [x_2]_{R_2}$. Clearly the map S is well defined, Graph(S) is closed hence compact, therefore S is continuous. Since Graph$(S) \subset M$ and M is minimal, (4.1) holds. Clearly, S is a bijection. Now take x_1, x_2, x_2', so that

$$([x_1]_{R_1} \times X_2) \cap M = (X_1 \times [x_2]_{R_2}) \cap M \tag{4.2}$$

and

$$([T_1(x_1)]_{R_1} \times X_2) \cap M = (X_1 \times [x_2']_{R_2}) \cap M. \tag{4.3}$$

We have $(T_2)_{R_2} \circ S([x_1]_{R_1}) = [T_2(x_2)]_{R_2}$ and $S \circ (T_1)_{R_1}([x_1]_{R_1}) = [x_2']_{R_2}$. If we act by $T_1 \times T_2$ on (4.2) and use (4.3), we get

$$[T_2(x_2)]_{R_2} = [x_2^{'}]_{R_2}.$$

Thus S is an isomorphism.

Suppose now that $S\colon (X_1)_{R_1} \to (X_2)_{R_2}$ is an isomorphism satisfying (4.1). Take $(x_1, x_2) \in ([x_1]_{R_1} \times X_2) \cap M$. Then $(\pi_1 \times \pi_2)(x_1, x_2) = ([x_1]_{R_1}, [x_2]_{R_2}) \in$ Graph(S). We get $[x_2]_{R_2} = S([x_1]_{R_1})$, and therefore $(x_1, x_2) \in (X_1 \times S([x_1]_{R_1})) \cap M$. By symmetry of arguments $(\overline{R}_1)_M = (\overline{R}_2)_M$ and the proof is finished. □

Take any minimal set $M \subset X_1 \times X_2$. We will show that there exist two smallest factor relations $P_i(M)$ on X_i respectively, such that

$$(\overline{P}_1(M))_M = (\overline{P}_2(M))_M. \tag{4.4}$$

Indeed, let (R^1_σ, R^2_σ), $\sigma \in \Sigma$, be the family of all pairs of factor relations satisfying

$$(\overline{R^1_\sigma})_M = (\overline{R^2_\sigma})_M.$$

Put

$$P_i(M) = \bigcap_{\sigma \in \Sigma} R^i_\sigma, \quad i = 1, 2.$$

It is clear that $P_i(M)$ satisfy (4.4) since every equivalence class $[x]_{P_i(M)}$ is an intersection of all equivalence classes $[x]_{R^1_\sigma}$ and $[x]_{R^2_\sigma}$, respectively. There are other equivalent ways of defining $P_i(M)$. Note that $P_i(M)$ is the smallest factor relation containing $M \circ M^{-1}$ for $i = 1$ and $M^{-1} \circ M$ for $i = 2$ (here $M^{-1} = \{(x, y) \in X_2 \times X_1 : (y, x) \in M)$. Finally, we can also define $P_i(M)$

using subalgebras of $\mathcal{C}(X_i)$. Consider two subalgebras $\overline{\mathcal{C}}_i \subset \mathcal{C}(M)$, $i = 1, 2$, defined by

$$\overline{\mathcal{C}}_i = \{h \in \mathcal{C}(M) : h \text{ depends only on the } i\text{-th coordinate}\}.$$

The intersection $\overline{\mathcal{C}}_1 \cap \overline{\mathcal{C}}_2$ can be considered as a subalgebra of $\mathcal{C}(X_1)$, in such a case we denote it by $\mathcal{C}_1(M)$, or as a subalgebra of $\mathcal{C}(X_2)$, then it is denoted by $\mathcal{C}_2(M)$. It is not difficult to see that $P_i(M) = R(\mathcal{C}_i(M))$, $i = 1, 2$.

Now, let (X, T) be a minimal flow. Assume that a family $\mathcal{N}$ of factor relations satisfies the following conditions.

(N-1) $\Delta_X \in \mathcal{N}$;
(N-2) $\{R_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{N} \Rightarrow \bigvee_{\lambda \in \Lambda} R_\lambda \in \mathcal{N}$.

Definition 4.2.3. A family $\mathcal{N}$ of factor relations satisfying (N-1) and (N-2) is said to be *natural* if it in addition satisfies the following conditions:

(N-3) $\underset{M \in \mathcal{M}_2(X,T)}{\forall} P_i(M) \in \mathcal{N}$, $i = 1, 2$;
(N-4) $\underset{R,\, R_1,\, R_2 \in \mathcal{N}}{\forall} \; \underset{\substack{S: X_{R_1} \to X_{R_2} \\ \text{an isomorphism}}}{\forall} R_1 \subset R \Rightarrow (S \times S)(R) \in \mathcal{N}$,

where $(x_1', x_2') \in (S \times S)(R)$ if and only if $[x_i']_{R_2} = S[x_i]_{R_1}$, $i = 1, 2$, for some $(x_1, x_2) \in R$.

Definition 4.2.4. For any family of factor relations $\mathcal{N}$ satisfying (N-1) and (N-2) and for each factor relation R there exists the biggest factor relation $\widetilde{R} \in \mathcal{N}$ with $\widetilde{R} \subset R$. When $\mathcal{N}$ is a natural family we will call $\widetilde{R}$ the *natural core* of R. The corresponding factor map $\pi: X_{\widetilde{R}} \to X_R$ is the *natural cover* of X_R.

Remark 4.2.5. Given a natural family of factor relations $\mathcal{N}$ for X, $\bigvee \mathcal{N}$ is the largest element of $\mathcal{N}$ (or the least factor in the corresponding family of factors).

Remark 4.2.6. If we take the intersection of all natural families of factor relations for X, we get the smallest natural family of factor relations.

It is possible to characterize natural family in an alternative way.

Proposition 4.2.7. *Let $\mathcal{N}$ be a family of factor relations satisfying* (N-1) *and* (N-2)*. Then $\mathcal{N}$ is natural if and only if for every factor relations R_1, R_2 and each $M \in \mathcal{M}_2(X, T)$ satisfying $(\pi_{R_1} \times \pi_{R_2})(M) = \mathrm{Graph}(S)$ for some isomorphism $S: X_{R_1} \to X_{R_2}$ we have $(\pi_{\widetilde{R}_1} \times \pi_{\widetilde{R}_2})(M) = \mathrm{Graph}(\widetilde{S})$ for some isomorphism $\widetilde{S}: X_{\widetilde{R}_1} \to X_{\widetilde{R}_2}$.*

Proof. Suppose that $\mathcal{N}$ is natural. Denote $\pi_i = \pi_{R_i}$ and $\widetilde{\pi}_i = \pi_{\widetilde{R}_i}$ for $i = 1, 2$. Assume that $M \in \mathcal{M}_2(X, T)$ and $(\pi_1 \times \pi_2)(M) = \mathrm{Graph}(S)$ for some isomorphism $S: X_{R_1} \to X_{R_2}$. By (N-3), $R_i \supset \widetilde{R}_i \supset P_i(M) \in \mathcal{N}$. Let $\overline{S}: X_{P_1(M)} \to X_{P_2(M)}$ be the isomorphism given by Lemma 4.2.2. It remains to show that $\overline{S}$ acting on $X_{\widetilde{R}_1}$ is an isomorphism between $X_{\widetilde{R}_1}$ and $X_{\widetilde{R}_2}$. By (N-4),

$(\overline{S}\times\overline{S})(\widetilde{R}_1)\in\mathcal{N}$ hence $(\overline{S}\times\overline{S})(\widetilde{R}_1)\subset\widetilde{R}_2$. Thus $\widetilde{R}_1\subset(\overline{S}\times\overline{S})^{-1}(\widetilde{R}_2)$. But, again by (N-4), $(\overline{S}\times\overline{S})^{-1}(\widetilde{R}_2)\in\mathcal{N}$ and therefore $\widetilde{R}_1\supset(\overline{S}\times\overline{S})^{-1}(\widetilde{R}_2)$. We conclude that $(\overline{S}\times\overline{S})(\widetilde{R}_1)=\widetilde{R}_2$. The required isomorphism $\widetilde{S}$ is just $\overline{S}$ considered on $X_{\widetilde{R}_1}$.

In order to prove the converse take any $M\in\mathcal{M}_2(X,T)$. Using Lemma 4.2.2 we get that there exists an isomorphism $S\colon X_{P_1(M)}\to X_{P_2(M)}$ with $(\pi_{P_1(M)}\times\pi_{P_2(M)})(M)=\mathrm{Graph}(S)$. Then for some isomorphism $\widetilde{S}\colon X_{P_1(M)^\sim}\to X_{P_2(M)^\sim}$ satisfying $\widetilde{p}_2\circ\widetilde{S}=S\circ\widetilde{p}_1$, where $\widetilde{p}_i$, $i=1,2$, denote the corresponding homomorphisms from $X_{P_i(M)^\sim}$ onto $X_{P_i(M)}$, we have $(\pi_{P_1(M)^\sim}\times\pi_{P_2(M)^\sim})(M)=\mathrm{Graph}(\widetilde{S})$. But $P_i(M)$ are the smallest with such a property, therefore

$$P_i(M)=P_i(M)^\sim\in\mathcal{N}.$$

Now take $R_i\in\mathcal{N}$, $i=1,2$, and an isomorphism $S\colon X_{R_1}\to X_{R_2}$. Take $M\in\mathcal{M}_2(X,T)$ satisfying (4.1) (any minimal set in $\{(x_1,x_2)\in X\times X\colon\pi_{R_2}(x_2)=S\pi_{R_1}(x_1)\}$). Assume that $R_1\subset R\in\mathcal{N}$. By $S'\colon X_R\to X_{(S\times S)(R)}$ denote the isomorphism S considered on the equivalence classes of R. Then $(\pi_R\times\pi_{(S\times S)(R)})(M)=\mathrm{Graph}(S')$. Therefore $(\pi_{\widetilde{R}}\times\pi_{((S\times S)(R))^\sim})(M)=\mathrm{Graph}(\widetilde{S}')$, where $\widetilde{S}'$ is an isomorphism between $X_{\widetilde{R}}$ and $X_{((S\times S)(R))^\sim}$. Since $\widetilde{R}=R$, we get $\widetilde{S}'=S'$, but this forces $(S\times S)(R)=((S\times S)(R))^\sim\in\mathcal{N}$. □

Corollary 4.2.8. *Let us suppose that (X,T) is a minimal flow with the property that for each two factor relations R_i, $i=1,2$, if (X_{R_1},T_{R_1}) is isomorphic to (X_{R_2},T_{R_2}), then $R_1=R_2$. Then*

$$\mathcal{N}=\Big\{\bigvee_{i\in I}P_1(M_i):\{M_i\}_{i\in I}\subset\mathcal{M}_2(X,T)\Big\}$$

is a natural family of factor relations for (X,T).

Proof. Clearly $\mathcal{N}$ satisfies conditions (N-1) and (N-2). Now, let $M\in\mathcal{M}_2(X,T)$ and R_1,R_2 be two factor relations on X. Suppose that M induces an isomorphism $S\colon X_{R_1}\to X_{R_2}$, hence $R_1=R_2$. By definition of $\mathcal{N}$, $P_i(M)\subset\widetilde{R}_i\subset R_i$, where $\widetilde{R}_i$ is the $\mathcal{N}$-core of R_i. Let $\overline{S}\colon X_{P_1(M)}\to X_{P_2(M)}$ be an isomorphism given by Lemma 4.2.2. Consider $\overline{S}$ on the equivalence classes of $\widetilde{R}_1$ and denote it by $\widetilde{S}$. Since $R_1=R_2$, also $\widetilde{R}_1=\widetilde{R}_2$ and $(\widetilde{S}\times\widetilde{S})\widetilde{R}_1=\widetilde{R}_1=\widetilde{R}_2$ so that $\widetilde{S}$ is the required isomorphism between $X_{\widetilde{R}_1}$ and $X_{\widetilde{R}_2}$. By Proposition 4.2.7, the family $\mathcal{N}$ is natural. □

Definition 4.2.9. A minimal flow (X,T) is called *regular* if for every almost periodic point $(x,y)\in X\times X$ there exists $S\in\mathrm{Aut}(X,T)$ such that $y=S(x)$.

As a corollary from Proposition 4.2.7, we obtain the following result, which is an analog of the one in the measure-theoretic context (see [44]).

Theorem 4.2.10. *Let $\mathcal{N}$ be a natural family of factor relations for a minimal flow (X,T). For each factor relation R on X the homomorphism $\pi\colon X_{\widetilde{R}} \to X_R$ is regular. Furthermore if π is distal, it is a group extension.*

Proof. Take $\widetilde{x}_i \in X_{\widetilde{R}}$, $i = 1,2$ with $\pi(\widetilde{x}_i) = x \in X_R$, where $(\widetilde{x}_1, \widetilde{x}_2)$ is almost periodic. Let $M = \overline{\mathrm{Orb}}(\widetilde{x}_1, \widetilde{x}_2) \in \mathcal{M}_2(X_{\widetilde{R}}, T_{\widetilde{R}})$. Clearly $(\pi \times \pi)(M) = \Delta_{X_R} = \mathrm{Graph}(\mathrm{Id}_{X_R})$. Proposition 4.2.7 yields $\widetilde{S} \in \mathrm{Aut}(X_{\widetilde{R}}, T_{\widetilde{R}})$ with $M = \mathrm{Graph}(\widetilde{S})$. Therefore $\widetilde{S}(\widetilde{x}_1) = \widetilde{x}_2$. Thus π is regular. The second part of the proof follows immediately from Theorem 1.7.6. $\square$

Since any factor of a distal flow is distal, we have the following.

Corollary 4.2.11. *Let $\mathcal{N}$ be a natural family of factor relations for a minimal distal flow (X,T). Then for each factor relation R on X the homomorphism $\pi\colon X_{\widetilde{R}} \to X_R$ is a group extension.*

Lemma 4.2.12. *Let (X,T) be a minimal flow. Let R_1, R_2, K be factor relations on X. Assume that $M \in \mathcal{B}_2(X,T)$. If M induces isomorphisms $S\colon X_{R_1} \to X_{R_2}$ and $\overline{S}\colon X_K \to X_K$, then M induces an isomorphism between $X_{R_1 \cap K}$ and $X_{R_2 \cap K}$.*

Proof. Take $x \in X$ and define $\widetilde{S}([x]_{R_1 \cap K}) = S([x]_{R_1}) \cap \overline{S}([x]_K)$. All we need to show is that $\widetilde{S}$ is indeed a map from $X_{R_1 \cap K}$ to $X_{R_2 \cap K}$. Assume that $S([x]_{R_1}) = [y]_{R_2}$ and $\overline{S}([x]_K) = [\overline{y}]_K$. Since $M \in \mathcal{B}_2(X,T)$, there exists $\widetilde{y} \in X$ such that $(x, \widetilde{y}) \in M$. Since M induces S and $\overline{S}$, we have $[y]_{R_2} = [\widetilde{y}]_{R_2}$ and $[\overline{y}]_K = [\widetilde{y}]_K$. Therefore $\widetilde{S}([x]_{R_1 \cap K}) = [\widetilde{y}]_{R_2} \cap [\widetilde{y}]_K = [\widetilde{y}]_{R_2 \cap K}$. $\square$

Lemma4.2.13. *Let K be a factor relation of a minimal flow (X,T). If X_K is regular then*

$$\mathcal{N} = \{R : R \textit{ is a factor relation on } X \textit{ and } R \subset K\}$$

is a natural family of factor relations.

Proof. Obviously, $\mathcal{N}$ satisfies (N-1) and (N-2) of Definition 4.2.3. Now, if R is a factor relation on X, then the $\mathcal{N}$-core of R is equal to $R \cap K$. To finish the proof take two factor relations R_1, R_2 on X and $M \in \mathcal{M}_2(X,T)$ which induces an isomorphism $S\colon X_{R_1} \to X_{R_2}$. Since M must induce an isomorphism of X_K with itself, by Lemma 4.2.12, M induces an isomorphism $S'\colon X_{R_1 \cap K} \to X_{R_2 \cap K}$. Clearly $S \circ p_1 = p_2 \circ S'$, where p_i denotes the homomorphism $X_{R_i \cap K} \to X_{R_i}$, for $i = 1,2$. In view of Proposition 4.2.7, $\mathcal{N}$ is natural. $\square$

Proposition 4.2.14. *Let (X,T) be a minimal flow.*

(a) *There exists the greatest regular factor Y of X.*
(b) *The factor Y is the least member of the smallest natural family of factors of X.*

Proof. (a) Let X_1 and X_2 be two regular factors of X, corresponding to the factor relations R_1 and R_2. Let $Y = X_1 \vee X_2$ be the factor corresponding to the factor relation $R_1 \cap R_2$. Then Y is isomorphic to the subflow $\overline{\mathrm{Orb}}(x_1, x_2)$ of the product flow $X_1 \times X_2$, where (x_1, x_2) is some almost periodic point in $X_1 \times X_2$. To show that Y is regular we examine an almost periodic point $((z_1, z_2), (w_1, w_2)) \in Y \times Y$. Now (z_i, w_i) is almost periodic point of $X_i \times X_i$, $i = 1, 2$, so by regularity of X_i, there exist $\phi_i \in \mathrm{Aut}(X_i), i = 1, 2$ with $\phi_i(z_i) = w_i$. Let $\phi: Y \to X_1 \times X_2$ be defined by $\phi(u, v) = (\phi_1(u), \phi_2(v))$. Clearly $\phi(Y)$ is a minimal subset of the flow $X_1 \times X_2$ and since $\phi(z_1, z_2) = (w_1, w_2) \in Y$, we conclude that $\phi: Y \to Y$ is an automorphism of Y, and Y is a regular flow. The same argument works for a joining of any family of regular factors. Therefore $Y = \bigvee X_\nu$ is a regular factor of X, where $\{X_\nu\}$ is the collection of all regular factors of X.

(b) Let K be the factor relation corresponding to Y. Since the family $\mathcal{N}$ defined in Lemma 4.2.13 contains only factor relation contained in K, it remains to show that $K \in \mathcal{N}$ for every natural family $\mathcal{N}$. Let $\pi: X_{\widetilde{K}} \to X_K$ be the natural cover with respect to the smallest natural family of factors of X. By regularity of Y, for an almost periodic point $(\widetilde{y}_1, \widetilde{y}_2) \in \widetilde{Y} \times \widetilde{Y}$ ($\widetilde{Y} = X_{\widetilde{K}}$), there exists $\phi \in \mathrm{Aut}(Y)$ with $y_2 = \phi(y_1)$, where $y_i = \pi(\widetilde{y}_i)$. In view of Proposition 4.2.7, there exists an automorphism $\widetilde{\phi} \in \mathrm{Aut}(\widetilde{Y})$ satisfying $\pi \circ \widetilde{\phi} = \phi \circ \pi$. Put $\widetilde{y} = \widetilde{\phi}(\widetilde{y}_1)$, then

$$\pi(\widetilde{y}) = \pi(\widetilde{\phi}(\widetilde{y}_1)) = \phi(\pi(\widetilde{y}_1)) = \phi(y_1) = y_2 = \pi(\widetilde{y}_2).$$

Thus $(\pi \times \pi)(\widetilde{y}_1, \widetilde{y}_2) = (\pi \times \pi)(\widetilde{y}_1, \widetilde{y}) = (y_1, y_2)$ and $(\widetilde{y}_1, \widetilde{y}_2)$, $(\widetilde{y}_1, \widetilde{y})$ are almost periodic. Assume that $v, w \in J$ (J is the set of all idempotents in the fixed minimal ideal I, see page 83) with $v(\widetilde{y}_1, \widetilde{y}) = (\widetilde{y}_1, \widetilde{y})$, $w(\widetilde{y}_1, \widetilde{y}_2) = (\widetilde{y}_1, \widetilde{y}_2)$ and put $\widetilde{y}_0 = v\widetilde{y}_2$. Now, since

$$\pi(\widetilde{y}) = \pi(v\widetilde{y}) = v\pi(\widetilde{y}) = v\pi(\widetilde{y}_2) = \pi(v\widetilde{y}_2) = \pi(\widetilde{y}_0)$$

and $(\widetilde{y}, \widetilde{y}_0)$ is an almost periodic point $(v(\widetilde{y}, \widetilde{y}_0) = (\widetilde{y}, \widetilde{y}_0))$, by Theorem 4.2.10 we obtain that $\widetilde{y}_0 = \widetilde{\psi}(\widetilde{y})$ for some $\widetilde{\psi} \in \mathrm{Aut}(\widetilde{Y})$. Finally, since $wv = w$,

$$\widetilde{y}_2 = w\widetilde{y}_2 = wv\widetilde{y}_2 = w\widetilde{y}_0 = w\widetilde{\psi}(\widetilde{y}) = w\widetilde{\psi} \circ \widetilde{\phi}(\widetilde{y}_1) = \widetilde{\psi} \circ \widetilde{\phi}(w\widetilde{y}_1) = \widetilde{\psi} \circ \widetilde{\phi}(\widetilde{y}_1),$$

so $X_{\widetilde{K}}$ is regular and the proof is complete. □

4.3. A natural family of factors defined by B-joinings

Now we modify our definition of natural family to obtain a stronger result about natural covers. Using this we will be able to identify the least element in this new natural family as the Kronecker factor, i.e. the maximal equicontinuous factor.

Definition 4.3.1. We say that a closed, invariant set $N \subset X$ is a *B-set*, if it is point transitive and it has a dense set of almost periodic points. The family of B-sets of $X \times X$ we denote by $\mathcal{B}_2(X, T)$ and call them *B-joinings*.

Definition 4.3.2. Let (X,T) be a minimal flow. We say that a family $\mathcal{N}$ of factors of (X,T) is *B-natural*, if it satisfies the conditions (N-1), (N-2), (N-4), and moreover

(NB-3) $\displaystyle\underset{N\in\mathcal{B}_2(X,T)}{\forall} \quad P_i(N)\in\mathcal{N},\ i=1,2;$

In other words, in the original definition of natural family one replaces the minimal joinings $M\in\mathcal{M}_2(X,T)$ by joinings $N\in\mathcal{B}_2(X,T)$. Notice that if a family is B-natural, it is natural, therefore all assertions we made for natural family (except of Proposition 4.2.14) remain valid for B-natural family, and that for PI flows, natural families coincide with B-natural ones. We denote the B-natural core of the factor relation R by $\widetilde{R}$.

Lemma 4.3.3. *Let $\pi\colon X\to Y$ be a regular homomorphism of minimal flows. Let Γ_π be the group of automorphisms S of X satisfying $\pi\circ S=\pi$. If Γ_π is compact then the homomorphism $\omega\colon X/\Gamma_\pi\to Y$ is proximal.*

Proof. Put $Z=X/\Gamma_\pi$. Denote the corresponding homomorphism from X to Z by κ, (hence for every $x\in X$, $\kappa(x)=\Gamma_\pi x$ and $\omega(\kappa(x))=\pi(x)$). Take $x\in X$ with $ux=x$ and put $z=\kappa(x)$, $y=\pi(x)$. Denote $B=\mathcal{G}(Y,y)$ and $F=\mathcal{G}(Z,z)$. All we have to show is that $B\subset F$.

Take $\beta\in B$, hence $\pi(\beta x)=\beta\pi(x)=\pi(x)$. Since $(x,\beta x)$ is almost periodic $(u(x,\beta x)=(x,\beta x))$, by regularity of π, there exists $S\in\mathrm{Aut}(X)$ such that $S(x)=\beta x$. We have $\pi(x)=\pi(\beta x)=\pi(S(x))$, hence, by minimality of X, $\pi\circ S=\pi$ and $S\in\Gamma_\pi$. Therefore we have $\beta z=\beta\kappa(x)=\kappa(\beta x)=\kappa(S(x))=\kappa(x)$ and $\beta\in F$. $\square$

Theorem 4.3.4. *Let (X,T) be a compact minimal flow, $\mathcal{N}$ a B-natural family of factors for (X,T). Then for any factor relation R the homomorphism $\pi\colon X_{\widetilde{R}}\to X_R$ decomposes as $\pi=\omega\circ\kappa$, $X_{\widetilde{R}}\xrightarrow{\kappa}Z\xrightarrow{\omega}X_R$; where ω is a proximal extension and κ is a group extension.*

Proof. Let us denote $Y=X_R$, $\widetilde{Y}=X_{\widetilde{R}}$. Consider the shadow diagram

$$\begin{tikzcd} X \arrow[d, "\widetilde{\pi}"'] & \\ \widetilde{Y} \arrow[d, "\pi"'] & (\widetilde{Y})^* \arrow[l, "\theta^*"'] \arrow[d, "\pi^*"] \\ Y & Y^* \arrow[l, "\theta"] \end{tikzcd}$$

FIGURE 4.1

$$\begin{tikzcd} (\widetilde{Y})^* \arrow[r, "\eta^*"] \arrow[d, "\pi^*"'] & Z^* \arrow[dl, "\rho^*"] \\ Y^* & \end{tikzcd}$$

FIGURE 4.2

of the homomorphism π (see Figure 4.1). By construction π^* is RIC. By Theorem 4.2.10 the homomorphism π is regular. First we show that π^* is also regular. Recall that we have denoted by I some minimal ideal in $\beta\mathbb{Z}$, by u an idempotent from I and $UG = uI$ being a group. Choose $\widetilde{y}_0 \in \widetilde{Y}$, $y_0 \in Y$ and $y_0^* \in Y^*$ with $u\widetilde{y}_0 = \widetilde{y}_0$, $uy_0 = y_0$, $uy_0^* = y_0^*$, and such that $\pi(\widetilde{y}_0) = y_0$, $\theta(y_0^*) = y_0$. We may assume that $(\widetilde{Y})^* = \widetilde{Y} \vee Y^* = \overline{\mathrm{Orb}}(\widetilde{y}_0, y_0^*)$, and θ^* and π^* are the projections onto the first and the second coordinate respectively (see Theorem 1.8.15 and (1.19)). Take an almost periodic point $((\widetilde{y}_1, y^*), (\widetilde{y}_2, y^*)) \in R_{\pi^*}$, then clearly $(\widetilde{y}_1, \widetilde{y}_2)$ is an almost periodic point in R_π. As π is regular, there exists an $S \in \mathrm{Aut}(\widetilde{Y})$ such that $S(\widetilde{y}_1) = \widetilde{y}_2$. Now observe that $S \times \mathrm{Id}_{Y^*} \in \mathrm{Aut}((\widetilde{Y})^*)$. Indeed, denoting $S(\widetilde{y}_0) = \alpha\widetilde{y}_0$ for some $\alpha \in G$ (as $S(\widetilde{y}_0) = S(u\widetilde{y}_0)) = uS(\widetilde{y}_0) = u\alpha\widetilde{y}_0$) we get $\alpha \in \mathcal{G}(Y, y_0) = \mathcal{G}(Y^*, y_0^*)$ since $\pi \circ S = \pi$. It follows that for each $q \in I$ we have

$$\begin{aligned}(S \times \mathrm{Id}_{Y^*})(q(\widetilde{y}_0, y_0^*)) &= \big(S(q\widetilde{y}_0), qy_0^*\big) \\ &= (q\alpha\widetilde{y}_0, q\alpha y_0^*) = q\alpha(\widetilde{y}_0, y_0^*) \in \overline{\mathrm{Orb}}(\widetilde{y}_0, y_0^*),\end{aligned}$$

i.e. $S \times \mathrm{Id}_{Y^*} \in \mathrm{Aut}((\widetilde{Y})^*)$ and π^* is regular.

Now use Theorem 4.1.4 to construct the diagram shown at Figure 4.2, where ρ^* is the maximal isometric extension of Y^* under $(\widetilde{Y})^*$. By Theorem 4.1.5, ρ^* is a group extension and η^* is RIC and weakly mixing. In particular then, $R_{\eta^*} \in \mathcal{B}_2((\widetilde{Y})^*)$. Since the image of a B-set under a homomorphism is again a B-set, we conclude that $\widetilde{N} = (\theta^* \times \theta^*)(R_{\eta^*}) \in \mathcal{B}_2(\widetilde{Y})$. Let

$$\mathcal{L} = \big\{L \subset X \times X : L \text{ is closed, invariant and } (\widetilde{\pi} \times \widetilde{\pi})(L) = \widetilde{N}\big\}.$$

Let N be a minimal element in $\mathcal{L}$. We will show that $N \in \mathcal{B}_2(X)$. Indeed, let $\widetilde{n}$ be a transitive point in $\widetilde{N}$ and let $n \in N$ satisfy $(\widetilde{\pi} \times \widetilde{\pi})(n) = \widetilde{n}$. Then $\overline{\mathrm{Orb}}(n) \subset N$ and $\overline{\mathrm{Orb}}(n) \in \mathcal{L}$. Since N is minimal, $\overline{\mathrm{Orb}}(n) = N$. Let M be the set of all almost periodic points in N. Since each almost periodic points of $\widetilde{N}$ is an image via $\widetilde{\pi} \times \widetilde{\pi}$ of some almost periodic point in N, we have

$$\widetilde{N} = \overline{(\widetilde{\pi} \times \widetilde{\pi})(M)} \subset (\widetilde{\pi} \times \widetilde{\pi})(\overline{M}),$$

so $(\widetilde{\pi} \times \widetilde{\pi})(\overline{M}) = \widetilde{N}$, hence $\overline{M} \in \mathcal{L}$.

Again by minimality of N, $\overline{M} = N$ and hence $N \in \mathcal{B}_2(X)$. By the commutative nature of our diagrams it follows that $(\pi \circ \widetilde{\pi} \times \pi \circ \widetilde{\pi})(N) = \Delta_Y$ and we conclude that $\widetilde{N} = (\widetilde{\pi} \times \widetilde{\pi})(N) = \mathrm{Graph}(\widetilde{S})$ for some $\widetilde{S} \in \mathrm{Aut}(\widetilde{Y}, T)$ and in particular $\widetilde{N}$ is minimal. However as $\Delta_{\widetilde{Y}} \subset \widetilde{N}$, we have $\widetilde{N} = \Delta_{\widetilde{Y}}$. Thus $\widetilde{N} = (\theta^* \times \theta^*)(R_{\eta^*}) = \Delta_{\widetilde{Y}}$. Since θ^* is a proximal extension, we conclude that $\Delta_{(\widetilde{Y})^*}$ is the only minimal subset in R_{η^*}. As η^* is RIC – hence R_{η^*} is a B-set – this implies that $R_{\eta^*} = \Delta_{(\widetilde{Y})^*}$, i.e. η^* is an isomorphism and it follows that π^* is a group extension. By Lemma 4.1.2, $\theta \circ \pi^*$ is regular, hence in view of Lemma 4.1.1, the group Γ_π of automorphisms σ of $\widetilde{Y}$ satisfying $\pi \circ \sigma = \pi$ is topologically isomorphic with the group $\Gamma_{\theta\circ\pi^*}$ of automorphisms of $(\widetilde{Y})^*$; the latter is a compact group and so we put $Z = \widetilde{Y}/\Gamma_\pi$ and let

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\kappa"] \arrow[d, "\pi"'] & Z \arrow[dl, "\omega"] \\ Y & \end{tikzcd}$$

be the diagram with the corresponding quotient maps. Now, applying Lemma 4.3.3 we get that ω is proximal. This completes the proof of the theorem. $\square$

We call the maximal equicontinuous factor of a flow the *Kronecker factor.*

Proposition 4.3.5.

(a) *The Kronecker factor, Y, of a minimal flow (X,T) is B-natural, for every B-natural family of factors.*

(b) *The Kronecker factor is the least member of the smallest B-natural family of factors of X.*

Proof. (a) By Theorem 4.3.4 we have $\widetilde{Y} \xrightarrow{\kappa} Z \xrightarrow{\omega} Y$, where κ is a K-extension ($K \subset \mathrm{Aut}(\widetilde{Y})$ is compact) and ω is proximal. Put $\pi = \omega\circ\kappa$. Let $S\colon Y \to Y$ be an automorphism of Y and let $\widetilde{S}\colon \widetilde{Y} \to \widetilde{Y}$ be a lift of S to an automorphism of $\widetilde{Y}$ (use the B-version of Proposition 4.2.7). Then for $\widetilde{y} \in \widetilde{Y}$ and $k \in K$, $(\widetilde{S}\widetilde{y}, \widetilde{S}(\widetilde{y}k))$ is an almost periodic point in $\widetilde{Y} \times \widetilde{Y}$, so that also $(\kappa\times\kappa)(\widetilde{S}\widetilde{y}, \widetilde{S}(\widetilde{y}k))$ is an almost periodic point of $Z \times Z$. On the other hand,

$$\omega(\kappa(\widetilde{S}(\widetilde{y}k))) = \pi(\widetilde{S}(\widetilde{y}k)) = S\pi(\widetilde{y}k) = S\pi(\widetilde{y}) = \pi(\widetilde{S}\widetilde{y}) = \omega(\kappa(\widetilde{S}\widetilde{y})),$$

and it follows that $\kappa(\widetilde{S}(\widetilde{y}k))$ and $\kappa(\widetilde{S}\widetilde{y})$ are also proximal points in Z. Therefore $\kappa(\widetilde{S}(\widetilde{y}k)) = \kappa(\widetilde{S}\widetilde{y})$ and we can define $\widehat{S}\colon Z \to Z$ unambiguously by $\widehat{S}(\kappa\widetilde{y}) = \kappa(\widetilde{S}\widetilde{y})$. Since ω is a proximal extension, it follows that over every minimal set in $Y \times Y$ there exists a unique minimal set in $Z \times Z$, so that our observation above yields an isomorphism of $\mathrm{Aut}(Y)$ onto $\mathrm{Aut}(Z)$. By Lemma 4.1.1this is a topological isomorphism and since $\mathrm{Aut}(Y) = Y$ is a compact group, so is $\mathrm{Aut}(Z)$. We find that $Z/\mathrm{Aut}(Z)$ is a factor of Z which is necessarily a proximal flow. Finally, since a proximal minimal flow is a one point flow, we have $\mathrm{Aut}(Z)$ acting transitively on Z and conclude that ω is 1–1.

Therefore π is a K-extension and we can finish the proof as follows. Take $\widetilde{y}_1, \widetilde{y}_2 \in \widetilde{Y}$ and put $\pi(\widetilde{y}_i) = y_i$. We know that $Sy_1 = y_2$ for some $S \in \mathrm{Aut}(Y)$. Let $\widetilde{S} \in \mathrm{Aut}(\widetilde{Y})$ with $\pi\circ\widetilde{S} = S\circ\pi$. Then $\pi(\widetilde{y}_2) = Sy_1 = \pi(\widetilde{S}\widetilde{y}_1)$ and there exists $k \in K$ such that $(\widetilde{S}\widetilde{y}_1)k = \widetilde{y}_2$. Therefore the group $\mathrm{Aut}(\widetilde{Y})$ acts transitively on $\widetilde{Y}$ and by Theorem 1.7.5, $\widetilde{Y}$ is an equicontinuous factor. Thus $Y = \widetilde{Y}$, which finishes the proof of (a).

(b) Since a B-set in a Kronecker system is necessarily minimal, we see that the proof of Lemma 4.2.13implies that the family of factor relations $\{R : R \subset K\}$ is B-natural. Now conclude as in the proof of Proposition 4.2.14 (b). $\square$

Example 4.3.6. Take (X, T) to be the minimal Chacon transformation. Then (X, T) is a regular flow and also a prime flow (i.e. has no nontrivial factors), and it follows that the smallest natural family of factors for (X, T) is $\mathcal{N} = \{X\}$. In contrast, the smallest B-natural family of factors (as well as the smallest measure theoretical one) is $\mathcal{N}_B = \{X, \text{point}\}$. If we take $Y = X \times Z$, where Z is an irrational rotation, then we still have $\mathcal{N} = \{Y\}$, but $\mathcal{N}_B = \{Y, Z\}$. In general, the Kronecker factor is the least member of the smallest natural family of factors for (X, T) if an onlu if it is also the largest regular factor of X. Comparing Propositions 4.2.14 and 4.3.5 and recalling that for PI-flows the notion of natural coincide with the notion of B-natural, we conclude that for PI-flows, and in particular for distal flows, the Kronecker factor and the largest regular factor coincide.

Example 4.3.7. As established in Chapter 2, in the measure theoretical case the map $\widetilde{Y} \to Y$ is always a group extension. A direct corollary of this is that the Pinsker factor (i.e. the largest zero-entropy factor) is always natural. It turns out that the minimal flow constructed in [9] serves also as an example of a regular, weakly mixing flow (X, T) of positive entropy with natural and B-natural families $\{X\}$ and $\{X, \text{point}\}$ respectively, with the property that its maximal zero entropy factor $X \to Z$ is neither a natural nor a B-natural factor.

Problem 4.3.8. Find a nontrivial example of a minimal flow (X, T) all of whose factors belong to the smallest family of natural factors. Also a B-flow all of whose factors belong to the smallest family of B-natural factors.

4.4. Group extensions of minimal rotations

In this section we describe an example of a natural family for a group extension of minimal rotation. The proof we present here involves, in some sense, only the basic methods.

First we show how to reduce an arbitrary group extension of minimal rotation to a minimal flow (Section 4.4.1). Then we describe minimal subsets in the Cartesian square of such a flow (Section 4.4.2) and using this we construct a natural family of factors (Section 4.4.3).

4.4.1. Minimal subsets of group extensions of minimal rotations. In what follows we let (X, T) be a minimal rotation. We will study properties of a group extension $(\overline{X}, \overline{T})$ (not necessarily minimal) of (X, T).

Suppose that $(\overline{X}, \overline{T})$ is a G-extension of (X, T). Recall that G is a compact (in the topology of uniform convergence) subgroup of $\operatorname{Aut}(\overline{X}, \overline{T})$. Let $M \subset \overline{X}$ be $\overline{T}$-minimal. Denote by $\pi \colon \overline{X} \to X$ the factor homomorphism given by $\pi(\overline{x}) = [\overline{x}]_G$.

Because T is minimal, $\pi(M) = X$. Put

$$H = \{g \in G : g(M) = M\}.$$

Clearly, H is a closed subgroup of G. Observe that if $g \in G$, then $g(M)$ is minimal, hence either $g(M) = M$ or $g(M) \cap M = \emptyset$. Therefore

$$g \in H \text{ if and only if } \underset{\overline{x}\in M}{\exists}\, g(\overline{x}) \in M.$$

For $\overline{x} \in \overline{X}$ put

$$M_{\overline{x}} = \{g \in G \colon g^{-1}(\overline{x}) \in M\}.$$

$M_{\overline{x}}$ is not empty because $\pi(M) = X$ i.e. each $[\overline{x}]_G$ contains at least one element of M ($GM = \overline{X}$ and each minimal subset of $\overline{X}$ is of the form $g(M)$). We also have

Lemma 4.4.1. *For each $\overline{x} \in \overline{X}$ there exists a $g = g_{\overline{x}} \in G$ such that*

$$M_{\overline{x}} = gH.$$

Proof. Fix $\overline{x} \in \overline{X}$. Take $g \in M_{\overline{x}}$. We will show that $M_{\overline{x}} = gH$. Assume that $g' \in M_{\overline{x}}$. Then ${g'}^{-1}(\overline{x}) \in M$ and $g^{-1}(\overline{x}) \in M$. Thus $g^{-1}g'({g'}^{-1}(\overline{x})) = g^{-1}(\overline{x}) \in M$. This and (4.5)imply that $g^{-1}g' \in H$ and therefore $g' \in gH$.

On the other hand if $h \in H$ then it is clear that $(gh)^{-1}(\overline{x}) = h^{-1}g^{-1}(\overline{x}) \in M$. Thus $gh \in M_{\overline{x}}$. □

By Lemma 4.4.1, there exists a map $\beta \colon \overline{X} \to G/H$ given by

$$\beta(\overline{x}) = M_{\overline{x}} = g_{\overline{x}}H. \tag{4.6}$$

Here G/H is understood as the homogeneous space of right cosets since H is not necessarily normal. In what follows we will consider the quotient topology on G/H. The map β has the following obvious properties:

$$\beta(g(\overline{x})) = g\beta(\overline{x}), \quad g \in G, \tag{4.7}$$

$$\beta^{-1}(gH) = gM, \quad g \in G, \tag{4.8}$$

$$\beta \text{ is constant on each minimal subset of } \overline{X}. \tag{4.9}$$

Let $\mathcal{M}(\overline{X})$ denote the set of all minimal subsets of $\overline{X}$. Let $2^{\overline{X}}$ be the space of non–empty, closed subset of $\overline{X}$ with the topology induced by the Hausdorff metric (see Chapter 1for definition). Recall that as X is compact, so is $2^{\overline{X}}$.

Lemma 4.4.2. *If $g_n \to g$ in G, then $g_nM \to gM$ in $2^{\overline{X}}$.*

Proof. Suppose that $g_n \to g$. Take a sequence $\{m_n\}_{n\geq 1}$ of elements from M. Choosing a subsequence if necessary we may assume that $m_n \to m$, for some $m \in M$. Then $g_nm_n \to gm$, which implies that $g_nM \to gM$. □

From this lemma it follows that $\mathcal{M}(\overline{X})$ is closed in $2^{\overline{X}}$ and on $\mathcal{M}(\overline{X})$ we will consider the topology induced from $2^{\overline{X}}$.

Lemma 4.4.3. *The map $\beta\colon \mathcal{M}(\overline{X}) \to G/H$ is a continuous bijection.*

Proof. By (4.9), β is well defined, and by (4.9)and (4.7), it is 1–1. It follows from (4.8)that β is onto.

Now suppose that $g_n M \to gM$ and $\beta(g_n M) = g_n H \to g'H$. Choosing a subsequence we can assume that $g_n \to g'h$ for some $h \in H$. By Lemma 4.4.2 we have that $g_n M \to g'hM = g'M$, then $gM = g'M$. Therefore $\beta(g_n M) \to g'H = \beta(g'M) = \beta(gM)$. $\square$

Define a metric $\widetilde{d}$ on $\mathcal{M}(\overline{X})$ setting

$$\widetilde{d}(gM, g'M) = d(\beta(gM), \beta(g'M)) = d(gH, g'H),$$

where d is the quotient metric on G/H induced by an invariant metric on G. As a corollary of the above consideration we have the following.

Proposition 4.4.4. *The metric $\widetilde{d}$ is equivalent to the Hausdorff metric on $\mathcal{M}(\overline{X})$ and $\beta\colon (\mathcal{M}(\overline{X}), \widetilde{d}) \to (G/H, d)$ is an isometry.*

Put $\widehat{T} = \overline{T}|_M$. Note that $H \subset \mathrm{Aut}(\widehat{T})$.

Theorem 4.4.5. *$\widehat{T}$ is an H-extension of T.*

Proof. It is enough to show that $\overline{T}_G$ is isomorphic to $\widehat{T}_H$. Let $\psi\colon M_H \to \overline{X}_G$ be given by the formula $\psi([\overline{x}]_H) = [\overline{x}]_G$. First we will show that ψ is one-to-one. If $\overline{x}, \overline{y} \in M$ and $[\overline{x}]_G = [\overline{y}]_G$, then $\overline{y} = g(\overline{x})$ for some $g \in G$. This implies $g \in H$ (use (4.5)) and consequently $[\overline{x}]_H = [\overline{y}]_H$.

Now take $[\overline{y}]_G \in \overline{X}_G$. Because $\pi(M) = X$, there is an $\overline{x} \in M$ such that $[\overline{x}]_G = [\overline{y}]_G$. This implies that ψ is a bijection.

The continuity of ψ is obvious because $[\overline{x}]_H \subset [\overline{x}]_G$, $\overline{x} \in M$. Since M_H is compact, ψ is a homeomorphism. Next,

$$\psi \circ \widehat{T}_H([\overline{x}]_H) = \psi([T\overline{x}]_H) = [T\overline{x}]_G,$$

$$\overline{T}_G \circ \psi([\overline{x}]_H) = \overline{T}_G([\overline{x}]_G) = [T\overline{x}]_G.$$

Thus $\widehat{T}_H$ is isomorphic to $\overline{T}_G$ and the proof is complete. $\square$

4.4.2. Minimal subsets of Cartesian squares of minimal group extensions of rotations. Assume that $(\overline{X}, \overline{T})$ is a minimal G-extension of a rotation (X, T) by a homomorphism π. Consider $(\overline{X} \times \overline{X}, \overline{T} \times \overline{T})$. Take $M \in \mathcal{M}_2(\overline{X}, \overline{T})$. Clearly, the projection of M onto the first and onto the second coordinate is just $\overline{X}$. Besides $(\pi \times \pi)(M) = \mathrm{Graph}(S)$ for some $S \in \mathrm{Aut}(X)$. Obviously $(\mathrm{Graph}(S), T \times T)$ is isomorphic to (X, T). Observe that if $\overline{W} = (\pi \times \pi)^{-1}(\mathrm{Graph}(S))$ and if $\tau\colon \overline{W} \to X$ is given by $\tau(\overline{x}, \overline{y}) = \pi(\overline{x})$, then τ is a homomorphism and $\tau^{-1}(x)$ is a $G \times G$-orbit of an $(\overline{x}, \overline{y}) \in \tau^{-1}(x)$. Thus $(\overline{W}, \overline{T} \times \overline{T})$ is a free $G \times G$-extension of (X, T) and we can apply results of Section 4.4.1.

Put $H = \{(g, g') \in G \times G : (g, g')(M) = M\}$. By results of Section 4.4.1, $\mathcal{M}(\overline{W})$ with the topology induced by the Hausdorff metric is homeomorphic to $(G \times G)/H$. Now, we will examine the structure of the group H.

Denote by $\pi_i: G \times G \to G$ the projection $\pi_i(g_1, g_2) = g_i$, $i = 1, 2$. Then

$$\pi_i(H) = G, \quad i = 1, 2. \tag{4.10}$$

Indeed, fix $g \in G$ and $(\overline{x}, \overline{x}') \in M$. Put $x = \pi(\overline{x})$, $x' = \pi(\overline{x}')$. There exists $\overline{x}'' \in \overline{X}$ such that $(g\overline{x}, \overline{x}'') \in M$. Since $\pi(g\overline{x}) = \pi(\overline{x}) = x$, $\pi(\overline{x}'') = x'$. Thus $\overline{x}'' = g'\overline{x}'$ for some $g' \in G$ and we have $(g\overline{x}, g'\overline{x}') = (g, g')(\overline{x}, \overline{x}') \in M$. This implies $(g, g') \in H$, hence $\pi_1(H) = G$. Similarly, $\pi_2(H) = G$.

Put

$$H_1 = \{g \in G : (g, e) \in H\}, \qquad H_2 = \{g \in G : (e, g) \in H\}.$$

Fix $h_1 \in H_1$. Given $g \in G$, by (4.10), we can find $j \in G$ so that $(g, j) \in H$, hence $(gh_1, j) \in H$. Since $(g^{-1}, j^{-1}) \in H$, $(gh_1g^{-1}, e) \in H$. Consequently,

$$H_1 \text{ and } H_2 \text{ are normal closed subgroups of } G. \tag{4.11}$$

Besides we have

$$\text{if } (g_1, g_2), (g_1', g_2) \in H \text{ then } g_1'g_1^{-1} \in H_1, \tag{4.12}$$

and

$$\text{if } (g_1, g_2), (g_1, g_2') \in H \text{ then } g_2'g_2^{-1} \in H_2, \tag{4.13}$$

besides

$$(g_1, g_2) \in H \text{ iff } g_1H_1 \times g_2H_2 \subset H. \tag{4.14}$$

Define ξ by

$$\xi(gH_1) = \pi_2((gH_1 \times G) \cap H).$$

Now, (4.11)–(4.14) imply directly that

Lemma 4.4.6. *The map ξ establishes an isomorphism of topological groups between G/H_1 and G/H_2 and moreover*

$$H = \bigcup_{g \in G} gH_1 \times \xi(gH_1).$$

Let us define a map $\overline{S}: \overline{X}_{H_1} \to 2^{\overline{X}}$ by

$$\overline{S}([\overline{x}]_{H_1}) = \Pi_2(([\overline{x}]_{H_1} \times \overline{X}) \cap M),$$

where $\Pi_2: \overline{X} \times \overline{X} \to \overline{X}$ denotes the projection onto the second coordinate.

Proposition 4.4.7.

$$M = \bigcup_{\overline{x} \in \overline{X}} [\overline{x}]_{H_1} \times \overline{S}([\overline{x}]_{H_1}),$$

moreover $\overline{S}$ *establishes an isomorphism between* $\overline{X}_{H_1}$ *and* $\overline{X}_{H_2}$.

Proof. First, we show that $\overline{S}$ is a map between $\overline{X}_{H_1}$ and $\overline{X}_{H_2}$. Take $(\overline{x}, \overline{y}) \in ([\overline{x}]_{H_1} \times \overline{X}) \cap M$. We show that $\overline{S}([\overline{x}]_{H_1}) = [\overline{y}]_{H_2}$. To this end take another $\overline{y}'$ with $(\overline{x}, \overline{y}') \in ([\overline{x}]_{H_1} \times \overline{X}) \cap M$. Obviously

$$(\pi \times \pi)(\overline{x}, \overline{y}) = (\pi \times \pi)(\overline{x}, \overline{y}') = [\overline{x}]_G \times S([\overline{x}]_G),$$

where $\mathrm{Graph}(S) = (\pi \times \pi)(M)$. From the proof of Theorem 4.4.5 we deduce that there exists $(g_1, g_2) \in H$ such that $(g_1, g_2)(\overline{x}, \overline{y}) = (\overline{x}, \overline{y}')$. Thus $g_1 = e$ and $g_2 \in H_2$. This forces $\overline{S}([\overline{x}]_{H_1}) \subset [\overline{y}]_{H_2}$. The opposite inclusion is obvious. Then $\overline{S}$ is obviously a bijection that commutes with $\overline{T}$ (considered on $\overline{X}_{H_1}$ and $\overline{X}_{H_2}$ respectively). Since M is compact and $\mathrm{Graph}(\overline{S})$ is equal to the natural projection of M to $\overline{X}_{H_1} \times \overline{X}_{H_2}$, $\overline{S}$ is continuous, hence it is an isomorphism of flows. □

4.4.3. Natural family of factors for minimal group extensions of rotations. Now we are in a position to construct a natural family of factors for $(\overline{X}, \overline{T})$. Put

$$\mathcal{N} = \{R_F \subset \overline{X} \times \overline{X} : F \text{ is closed normal subgroup of } G\}.$$

Proposition 4.4.8. *The family* $\mathcal{N}$ *is a natural family of factor relations for* $(\overline{X}, \overline{T})$.

Proof. First, $\Delta_{\overline{X}} = R_{\{e\}} \in \mathcal{N}$. Then notice that if $R_{F_i} \in \mathcal{N}$, $i \in I$, then

$$\bigvee_{i \in I} R_{F_i} = R_F,$$

where F is the closed subgroup generated by the union of the groups F_i, $i \in I$. Since F is normal, the condition (N-2) of the definition of natural family is satisfied.

Now take $M \in \mathcal{M}_2(\overline{X}, \overline{T})$. From Proposition 4.4.7 we get that $P_i(M) = R_{H_i}$, where H_i are defined in Section 4.4.2for M. Thus $P_i(M) \in \mathcal{N}$ and condition (N-3) is fulfilled.

To prove (N-4) take $R_{F_i} \in \mathcal{N}$, $i = 1, 2$, an isomorphism $\overline{S}\colon \overline{X}_{R_{F_1}} \to \overline{X}_{R_{F_2}}$ and $R_{F_1} \subset R_F \in \mathcal{N}$. Assume that $M \in \mathcal{M}_2(\overline{X}, \overline{T})$ is such that the natural projection of it onto $\overline{X}_{R_{F_1}} \times \overline{X}_{R_{F_2}}$ is equal to $\mathrm{Graph}(\overline{S})$. Lemma 4.4.6 yields a topological group isomorphism $\xi\colon G/F_1 \to G/F_2$. Now consider $\overline{S}$ acting on the equivalence classes of R_F. By the form of S, $(S \times S)(R_F) = R_{\xi(F/F_1)}$ and since $\xi(F/F_1)$ is a normal subgroup of G/F_2, the condition (N-4) is also satisfied. □

Corollary 4.4.9. *For any factor $(\overline{X}_R, \overline{T}_R)$ of $(\overline{X}, \overline{T})$ there exists a largest closed normal subgroup F of G such that*

$$\pi\colon (\overline{X}_F, \overline{T}_F) \to (\overline{X}_R, \overline{T}_R),$$

(*where π denotes the corresponding homomorphism*) *is a group extension.*

Proof. The group F is defined by taking the natural core R_F of R. By Corollary 4.2.11, π is a group extension. $\square$

It is not difficult to observe (see Figure 4.3), that π is a Γ_π-extension, where $\Gamma_\pi = \{S \in \mathrm{Aut}(X) : \pi \circ S = \pi\}$.

$$\begin{tikzcd}
\overline{X} \arrow[rr, "G\text{-ext.}"] \arrow[dr, "F\text{-ext.}"] \arrow[dd] & & X \\
& \overline{X}_F \arrow[ur, "G/F\text{-ext.}"'] \arrow[dl, "\Gamma_\pi\text{-ext.}"] & \\
\overline{X}_R & &
\end{tikzcd}$$

Figure 4.3.

CHAPTER 5

REAL COCYCLE EXTENSIONS OF MINIMAL ROTATIONS

In this chapter we prove that for each minimal rotation $T: X \to X$ on a compact Hausdorff space and each topological cocycle $\phi: X \to \mathbb{R}$ either ϕ is a topological coboundary or T_φ is topologically ergodic or the partition into orbits is the decomposition of T_ϕ into minimal components. As an application, we generalize a result by E. Glasner and show that if $(S_t)_{t\in\mathbb{R}}$ is a minimal topologically weakly mixing flow then whenever ϕ is universally ergodic the map

$$X \times Y \ni (x, y) \mapsto (Tx, S_{\phi(x)}(y)) \in X \times Y$$

is not PI but is disjoint from all minimal topologically weakly mixing systems.

5.1. Existence of almost periodic points

Following [7, p. 11], if x is an almost periodic point in a locally compact flow, then $\overline{\mathrm{Orb}}(x)$ is a compact minimal set. To see this take a compact neighbourhood U of x and let $A = \{n \in \mathbb{Z} : T^n x \in U\}$. Since x is almost periodic, the dwelling set $D(x, U)$ is relatively dense, that is for some $N \in \mathbb{N}$, $A \cup (A+1) \cup \ldots \cup (A+N) = \mathbb{Z}$. Thus $\mathrm{Orb}(x) = \bigcup_{i=0}^{N} \{T^{n+i}x : n \in A\} \subset U \cup TU \cup \ldots \cup T^N U$, that is compact, so $\overline{\mathrm{Orb}}(x)$ is compact.

Now we will study locally compact group extensions of compact minimal flows.

Proposition 5.1.1. *Let (X, T) be a minimal compact flow, G a locally compact group, $\varphi: X \to G$ a continuous map. Assume that there is an almost periodic point in $(X \times G, T_\varphi)$. Then there exists a compact subgroup H of G and a continuous map $f: X \to G/H$ such that*

$$f(Tx) = \varphi(x) f(x) \quad \textit{for all } x \in X.$$

Proof. Let $(\widetilde{x}, \widetilde{g}) \in X \times G$ be an almost periodic point, and denote $M = \overline{\mathrm{Orb}}(\widetilde{x}, \widetilde{g})$. Then M is minimal, compact and it projects onto X.

Let $H = \{g \in G : Mg = M\}$. Clearly H is a group. For $x \in X$ denote $M_x = \{g \in G : (x, g) \in M\}$, $H_x = \{h \in G : M_x h = M_x\}$. Obviously $H \subset H_x$ for each $x \in X$. Conversely, if $h \in H_x$ then $M \cap Mh$ is nonempty because $M_x \subset M \cap Mh$. Minimality of M gives $M = Mh$, so $h \in H$. Thus $H = H_x$ for each $x \in X$. Now we will show that

$$\forall_{x \in X} \forall_{g_0 \in M_x} M_x = g_0 H. \tag{5.1}$$

Indeed, fix $x \in X$, $g_0 \in M_x$. Now, if $h \in H$, then $(x, b_0) \in M$ and $(x, g_0 h) \in M$, so $g_0 h \in M_x$. On the other hand, if $g \in M_x$, then $(x, g) \in M$ and $(x, g) = (x, g_0) g_0^{-1} g \in M g_0^{-1} g$. Thus $M \cap M g_0^{-1} g \neq \emptyset$, hence $g_0^{-1} g \in H$. Therefore $g = g_0 (g_0^{-1} g) \in g_0 H$ and (5.1) is proved.

Since each M_x is compact, so is H. Equip $G/H = \{gH : g \in G\}$ with the quotient topology. Define $f: X \to G/H$ by

$$f(x) = M_x.$$

To prove that f is continuous take an open set $\widetilde{U} \subset G/H$. Then $\widetilde{U} = UH$ for some open set $U \subset G$. Clearly $f^{-1}(UH) = \pi_X((X \times UH) \cap M)$, where $\pi_X: X \times G \to X$ is the projection. Assume that $x_i \notin f^{-1}(\widetilde{U})$, $x_i \to x_0$, where $(x_i)_{i \in I}$ is a net. Then $(x_i, g_i) \in M$ for some $g_i \in f(x_i)$. Choose a convergent subnet $(x_j, g_j) \to (x_0, g) \in M$ (M is compact). Since $x_i \notin \pi_X((X \times UH) \cap M)$, $g_i H \notin \widetilde{U}$. Thus $gH \notin \widetilde{U}$ and $x_0 \notin f^{-1}(\widetilde{U})$. This proves that $f^{-1}(\widetilde{U})$ is open.

The equality $f(Tx) = \varphi(x) f(x)$ is clear. □

Remark 5.1.2. If G has no nontrivial compact subgroups (e.g. $G = \mathbb{R}^n$, $G = \mathbb{Z}^n$), then the above proposition says that φ is a coboundary. In particular, if $G = \mathbb{R}$ we obtain the classical Gottschalk–Hedlund result ([40, Theorem 14.11]).

5.2. Essential values of a cocycle

In this section we will quickly adapt Schmidt's methods [89] of essential values for measurable cocycles to the continuous case. Assume that (X, T) is a compact flow. Let G be a locally compact group with the unit element e. By G_∞ we denote the Aleksandroff compactification of G: $G_\infty = G \cup \{\infty\}$. In further we extend the group operation from the group G onto the set G_∞ by $g \cdot \infty = \infty$ for all $g \in G_\infty$. In this way the operation $G_\infty \times G_\infty \ni (g_1, g_2) \to g_1 g_2 \in G_\infty$ is continuous.

Definition 5.2.1. Let $\varphi: X \to G$ be a cocycle. We say that $r \in G_\infty$ is an *essential value* of φ if for each nonempty open $U \subset X$ and each neighbourhood V of r there exists $N \in \mathbb{Z}$ such that

$$U \cap T^{-N} U \cap \{x \in X : \varphi^{(N)}(x) \in V\} \neq \emptyset. \tag{5.2}$$

The set of all essential values of φ will be denoted by $E_\infty(\varphi)$. Put also $E(\varphi) = E_\infty(\varphi) \cap G$.

Proposition 5.2.2. *The sets $E_\infty(\varphi)$ and $E(\varphi)$ have the following properties*:

(a) *$E(\varphi)$ is a closed subgroup of G.*
(b) *If G is Abelian and $\psi(x) = (\xi(Tx))^{-1}\varphi(x)\xi(x)$, where $\psi, \xi\colon X \to G$ are continuous maps, then $E_\infty(\varphi) = E_\infty(\psi)$.*

Proof. (a) Since $\varphi^{(0)} \equiv e$, $e \in E(\varphi)$. Assume $g \in E(\varphi)$. To show that $g^{-1} \in E(\varphi)$ take nonempty open sets $U \subset X$ and $g^{-1} \in V \subset G$. Then $g \in V^{-1}$. Since $g \in E(\varphi)$, there exists an integer N such that $U \cap T^{-N}U \cap (\varphi^{(N)})^{-1}(V^{-1}) \neq \emptyset$. For $x \in U \cap T^{-N}U \cap (\varphi^{(N)})^{-1}(V^{-1})$ we will show that $y = T^N x \in U \cap T^N U \cap (\varphi^{(-N)})^{-1}(V)$. Clearly $y \in U \cap T^N U$. Next $x \in (\varphi^{(N)})^{-1}(V^{-1})$ if and only if $(\varphi^{(N)}(x))^{-1} \in V$. The cocycle condition yields $e = \varphi^{(-N)}(T^N x)\varphi^{(N)}(x)$ so $\varphi^{(-N)}(y) = (\varphi^{(N)}(x))^{-1} \in V$. Thus $y \in U \cap T^N U \cap (\varphi^{(-N)})^{-1}(V)$ and $g^{-1} \in E(\varphi)$ follows.

Assume now that $g, h \in E(\varphi)$. Let $U \subset X$ and $V \subset G$ be open, nonempty and such that $gh \in V$. There are open sets $V_1, V_2 \subset G$ such that $g \in V_1$, $h \in V_2$, $V_1V_2 \subset V$. Since $g, h \in E(\varphi)$, there exist integers N_1, N_2 such that

$$U_1 = U \cap T^{-N_1}U \cap (\varphi^{(N_1)})^{-1}(V_1) \quad \text{and} \quad U_2 = U_1 \cap T^{-N_2}U_1 \cap (\varphi^{(N_2)})^{-1}(V_2)$$

are nonempty. We will show that $U_2 \subset U \cap T^{-N_1-N_2}U \cap (\varphi^{(N_1+N_2)})^{-1}(V)$. Assume that $x \in U_2$. Then $x \in U_1 \subset U$. Since $x \in T^{-N_2}U_1 \subset T^{-N_2}(T^{-N_1}U)$, $x \in T^{-N_1-N_2}U$, so $x \in U \cap T^{-N_1-N_2}U$. Since $x \in U_2$, $\varphi^{(N_2)}(x) \in V_2$ and $T^{N_2}(x) \in U_1$. Thus $\varphi^{(N_1)}(T^{N_2}x) \in V_1$, and we have

$$\varphi^{(N_1+N_2)}(x) = \varphi^{(N_1)}(T^{N_2}x)\varphi^{(N_2)}(x) \in V_1V_2 \subset V.$$

Therefore $E(\varphi)$ is a group. The fact that $E(\varphi)$ is closed follows directly from definition.

(b) We will show that $E_\infty(\varphi) \subset E_\infty(\psi)$. Because G is Abelian,

$$\psi^{(N)}(x) = (\xi(T^N x))^{-1}\varphi^{(N)}(x)\xi(x) = \varphi^{(N)}(x)\xi(x)(\xi(T^N x))^{-1}.$$

Assume that $r \in E_\infty(\varphi)$. Let $U \subset X$, $V \subset G$ be nonempty open sets with $r \in V$. Find open sets $V_0, V_1 \subset G$ such that $r \in V_1$, $e \in V_0$, $V_1V_0 \subset V$. Let $W \subset U$ be an open nonempty set satisfying $\xi(x)\xi(y)^{-1} \in V_0$ for all $x, y \in W$. Now it follows from $r \in E_\infty(\varphi)$ that there exist an integer N and $x_0 \in X$ such that $x_0 \in W \cap T^{-N}W \cap \{x \in X : \varphi^{(N)}(x) \in V_1\}$. Then $x_0 \in U \cap T^{-N}U$ and $\psi^{(N)}(x_0) = \varphi^{(N)}(x_0)\xi(x_0)(\xi(T^N x_0))^{-1} \in V_1V_0 \subset V$.

By symmetry we have $E_\infty(\varphi) = E_\infty(\psi)$. □

Proposition 5.2.3. *Assume that (X, T) is topologically ergodic. Then $(X \times G, T_\varphi)$ is topologically ergodic if and only if $E(\varphi) = G$.*

Proof. Assume that $(X \times G, T_\varphi)$ is topologically ergodic. Suppose that $g \in G$, $U \subset X$ and $V \subset G$ are open nonempty sets such that $g \in V$. Fix open nonempty sets $V_0, V_1 \subset G$ satisfying $V_1 V_0^{-1} \subset V$. Since $(X \times G, T_\varphi)$ is topologically ergodic, $D(U \times V_0, U \times V_1) \neq \emptyset$ and we can find an integer N, $x_0 \in X$, $g_0 \in G$ satisfying $(x_0, g_0) \in U \times V_0 \cap T_\varphi^{-N}(U \times V_1)$. Then $x_0 = T^N x_1$ for some $x_1 \in U$, $g_0 \in \varphi^{(N)}(x_1)V_1$, in particular $x_0 \in T^N U$. Thus $x_0 \in U \cap T^N U$. Since $g_0 \in \varphi^{(N)}(x_1)V_1$, there exists $g_1 \in V_1$ such that $g_0 = \varphi^{(N)}(T^{-N}x_0)g_1 = (\varphi^{(-N)}(x_0))^{-1}g_1$. This implies $\varphi^{(-N)}(x_0) = g_1 g_0^{-1} \in V_1 V_0^{-1} \subset V$ and $x_0 \in (\varphi^{(-N)})^{-1}(V)$. Thus $g \in E(\varphi)$.

Assume now that $E(\varphi) = G$. Let $U_1, U_2 \subset X$, $V_1, V_2 \subset G$ be nonempty open sets. We will show that $D(U_1 \times V_1, U_2 \times V_2) \neq \emptyset$. First find nonempty open sets $\widetilde{V}_1 \subset V_1$ and $W \subset G$ such that $W\widetilde{V}_1 \subset V_2$. Because T is topologically ergodic, there exists an integer n such that $U_1 \cap T^{-n}U_2 \neq \emptyset$. There are nonempty open sets $U \subset U_1 \cap T^{-n}U_2$ and $V \subset G$ such that $T_\varphi^n(U \times V) \subset U_2 \times W$. Since $E(\varphi) = G$, we can find an integer N such that $Y = U \cap T^{-N}U \cap (\varphi^{(N)})^{-1}(V)$ is nonempty. We will show that $Y \times \widetilde{V}_1 \subset (U_1 \times V_1) \cap T_\varphi^{-n-N}(U_2 \times V_2)$. Take $(y, g) \in Y \times \widetilde{V}_1$. Then $y \in U \subset U_1 \cap T^{-n}U_2$, $g \in \widetilde{V}_1 \subset V_1$, hence $(y, g) \in U_1 \times V_1$. We have $T_\varphi^{n+N}(y, g) = (T^{n+N}y, \varphi^{(n+N)}(y)g)$. Since $y \in Y$, $T^N y \in U$ and $T^{n+N}y \in U_2$. Moreover, $\varphi^{(N)}(y) \in V$ and $\varphi^{(n)}(T^N y)v \in W$ for each $v \in V$, so $\varphi^{(n)}(T^N y)\varphi^{(N)}(y) \in W$. Thus

$$\varphi^{(n+N)}(y)g = (\varphi^{(n)}(T^N y)\varphi^{(N)}(y))g \in W\widetilde{V}_1 \subset V_2.$$

We have shown that $T_\varphi^{n+N}(y, g) = (T^{n+N}y, \varphi^{(n+N)}(y)g) \in U_2 \times V_2$. □

Remark 5.2.4. It follows from Proposition 5.2.3 and [89, Section 3] that whenever a continuous cocycle is ergodic with respect to a measure which is positive on open sets, then it is topologically ergodic. In particular continuous cocycles from [82], [69] and [21] are topologically ergodic.

From now on we will assume that G is Abelian with the group operations written additively.

Proposition 5.2.5. *Let (X, T) be topologically ergodic, $\varphi: X \to G$ a cocycle. Assume that $K \subset G$ is compact and $K \cap E(\varphi) = \emptyset$. Then for each nonempty open $U \subset X$ there exists a nonempty open set $V \subset U$ satisfying*

$$\bigcup_{n \in \mathbb{Z}} (V \cap T^{-n}V \cap \{x : \varphi^{(n)}(x) \in K\}) = \emptyset.$$

Proof. For each $r \in K$ we can find an open neighbourhood M_r of zero in G and an open set $U_r \subset X$ such that

$$\bigcup_{n \in \mathbb{Z}} (U_r \cap T^{-n}U_r \cap \{x : \varphi^{(n)}(x) \in r + M_r\}) = \emptyset. \tag{5.3}$$

Let N_r be a symmetric open neighbourhood of zero in G satisfying $N_r + N_r \subset M_r$. By compactness of K there exist $r_1, \ldots, r_m \in K$ such that

$$(r_1 + N_{r_1}) \cup (r_2 + N_{r_2}) \cup \ldots \cup (r_m + N_{r_m}) \supset K.$$

Now, let $U \subset X$ be a nonempty open set. Put $U_0 = U$. We will construct an open set $U_1 \subset U_0$ such that if $x, T^n x \in U_1$, then $\varphi^{(n)}(x) \notin r_1 + N_{r_1}$. Since T is topologically ergodic, $U_{r_1} \cap T^{n_0} U_0 \neq \emptyset$ for some n_0. Denote $\widetilde{U} = U_0 \cap T^{-n_0} U_{r_1}$. Then $\widetilde{U} \subset U_0$, $T^{n_0}\widetilde{U} \subset U_{r_1}$. Because the map $f(x,y) = \varphi^{(n_0)}(x) - \varphi^{(n_0)}(y)$ is continuous and it satisfies $f(x,x) = 0$ for all $x \in X$, the set $f^{-1}(N_{r_1})$ is open, nonempty and $(\widetilde{U} \times \widetilde{U}) \cap f^{-1}(N_{r_1}) \neq \emptyset$. Thus there exists a nonempty open $U_1 \subset \widetilde{U}$ such that $U_1 \times U_1 \subset f^{-1}(N_{r_1})$, i.e.

$$\varphi^{(n_0)}(x) - \varphi^{(n_0)}(y) \in N_{r_1} \quad \text{for } x, y \in U_1. \tag{5.4}$$

By (5.4), if $x, T^n x \in U_1$, then $\varphi^{(n_0)}(x) - \varphi^{(n_0)}(T^n x) \in N_{r_1}$ which implies $T^{n_0}x, T^{n+n_0}x \in U_{r_1}$. In view of (5.3), $\varphi^{(n)}(T^{n_0}x) \notin r_1 + M_{r_1}$. Because $\varphi^{(n)}(T^{n_0}x) = \varphi^{(n)}(x) + (\varphi^{(n_0)}(T^n x) - \varphi^{(n_0)}(x))$, $\varphi^{(n)}(x) \notin r_1 + N_{r_1}$.

We iterate this procedure for (U_1, N_{r_2}), $\ldots$, (U_{m-1}, N_{r_m}) to obtain open sets $U \supset U_1 \supset \ldots \supset U_m$ such that $\varphi^{(n)}(x) \notin \bigcup_{i=1}^{k}(r_i + N_{r_i})$ for x, $T^n x \in U_k$, $k = 1, \ldots, m$. Put $V = U_m$. Then

$$\bigcup_{n \in \mathbb{Z}} (V \cap T^{-n}V \cap \{x : \varphi^{(n)}(x) \in K\}) = \emptyset$$

and the proposition is proved. □

In the following lemma we assume that the flow (X,T) is minimal. For $A \subset G$ and $k \geqslant 1$ denote $kA = \underbrace{A + \ldots + A}_{k \text{ times}}$, and $0A = \{0\}$.

Lemma 5.2.6. *Let $\varphi: X \to G$ be a cocycle, where (X,T) is a compact minimal flow and G is a locally compact Abelian group.*

(a) *If $\infty \notin E_\infty(\varphi)$, then there exists a compact set $C \subset G$ such that $\varphi^{(n)}(x) \in C$ for all $x \in X$, $n \in \mathbb{Z}$.*
(b) *If $E(\varphi) = \{0\}$ and there exists a compact $C \subset G$ such that $\varphi^{(n)}(x) \in C$ for all $x \in X$, $n \in \mathbb{Z}$, then φ is a coboundary.*

Proof. (a) If $\infty \notin E_\infty(\varphi)$, then there exist an open set $U \subset X$ and a compact $K \subset G$ such that

$$\underset{n \in \mathbb{Z}}{\forall}\; U \cap T^{-n}U \cap \{x \in X : \varphi^{(n)}(x) \notin K\} = \emptyset. \tag{5.5}$$

Since (X,T) is compact and minimal, there exists a positive integer N such that $\bigcup_{i=0}^{N-1} T^{-i}U = X$. For $0 \leqslant i, j < N$ denote

$$K_{i,j} = i\varphi(X) + K + j\varphi(X).$$

Clearly each $K_{i,j}$ is a compact subset of G. Define $C \subset G$ by

$$C = \bigcup_{i=0}^{N-1}\bigcup_{j=0}^{N-1} K_{i,j} \cup \bigcup_{i=0}^{N-1}\bigcup_{j=0}^{N-1} (-K_{i,j}) \cup \bigcup_{i=0}^{N-1} i\varphi(X) \cup \bigcup_{i=0}^{N-1} i\big(-\varphi(X)\big).$$

Then C is compact.

Assume that $x \in X$, $n \in \mathbb{Z}$, $n \geqslant N$. Consider the sequence

$$x, Tx, \dots, T^{i-1}x, T^i x, \dots, T^{n-j-1}x, T^{n-j}x, \dots, T^{n-1}x,$$

where i is the smallest nonnegative integer such that $T^i x \in U$, and j is the smallest nonnegative integer such that $T^{n-j}x \notin U$. Then $T^{n-j-i-1}(T^i x) = T^{n-j-1}x \in U$, and by (8.17), $\varphi^{(n-j-i-1)}(T^i x) \in K$. Thus

$$\varphi^{(n)}(x) = \varphi^{(i)}(x) + \varphi^{(n-j-i-1)}(T^i x) + \varphi^{(j)}(T^{n-j-1}x) \in K_{i,j} \subset C.$$

If $n \leqslant -N$ then $\varphi^{(n)}(x) = -\varphi^{(-n)}(T^n x) \in -K_{i,j} \subset C$ for some i, j.

(b) Since $\varphi^{(n)}(x) \in C$, $x \in X$, $n \in \mathbb{Z}$, there exists a compact T_φ-invariant subset of $X \times G$. This implies that there exists a compact T_φ-minimal subset $M \subset X \times G$. It follows from $\pi_X(M) = X$ that $\bigcup_{g \in G}(M + g) = X \times G$. Hence all points $(x, g) \in X \times G$ are almost periodic.

Given a compact minimal set $M \subset X \times G$ let

$$H(M) = \{g \in G : M + g = M\}.$$

Clearly $H(M)$ is a closed subgroup of G. Now, fix a compact minimal M_0 and let $H = H(M_0)$. If M is another minimal subset of $X \times G$ then $M = M_0 + g$ for some $g \in G$ and it is easy to see that $H(M) = H$ for all minimal $M \subset X \times G$. We intend to prove that $H = \{0\}$. To this end suppose $H \neq \{0\}$ and choose $g_0 \in H$, $g_0 \neq 0$. There exists a compact neighbourhood K of g_0 such that $0 \notin K$. If follows from Proposition 5.2.5 that we can find an open nonempty $U \subset X$ satisfying

$$\forall_{n \in \mathbb{Z}}\ x \in U \cap T^{-n}U \Rightarrow \varphi^{(n)}(x) \notin K. \tag{5.6}$$

If $x_0 \in U$ then $(x_0, g_0) = (x_0, 0) + g_0 \in \overline{\mathrm{Orb}}(x_0, 0)$, so there exists a sequence $(n_i)_{i \geqslant 1}$ such that $T_\varphi^{n_i}(x_0, 0) \to (x_0, g_0)$. Thus $T^{n_i}x_0 \to x_0$, $\varphi^{(n_i)}(x_0) \to g_0$ and there exists i_0 such that $T^{n_i}x_0 \in U$ for all $i \geqslant i_0$. Hence for $i \geqslant i_0$ we have $x_0 \in U \cap T^{-n_i}U$ and, by (8.18), $\varphi^{(n_i)}(x_0) \notin K$. This gives rise to a contradiction because $\varphi^{(n_i)}(x_0) \to g_0$.

Therefore $H = \{0\}$ and M is a graph of some continuous $\xi: X \to G$. Since M is T_φ-invariant, $\varphi(x) = \xi(Tx) - \xi(x)$, $x \in X$, so φ is a coboundary. $\square$

Proposition 5.2.7. *Let $\varphi: X \to G$ be a cocycle, where (X, T) is a compact minimal flow and G is a locally compact Abelian group. Then $E_\infty(\varphi) = \{0\}$ iff φ is a coboundary.*

Proof. If $E_\infty(\varphi) = \{0\}$, then by Lemma 5.2.6 φ is a coboundary. Conversely, if $\varphi = f \circ T - f$ for some continuous f, then taking an open neighbourhood $V \subset G$ of zero and an open $U \subset X$ such that $x', x'' \in U$ implies $f(x') - f(x'') \in V$ we get that whenever $U \cap T^{-n}U \neq \emptyset$, then $f(T^n x) - f(x) \in V$ for each $x \in U \cap T^{-n}U$. Therefore $E_\infty(\varphi) = \{0\}$. $\square$

Due to this proposition we can define (similarly to [89]) the notion of quasi regular cocycle.

Definition 5.2.8. Let (X,T) be a minimal flow, G a locally compact Abelian group and $\varphi\colon X \to G$ a cocycle. We say that φ is *quasi regular* if the cocycle $\varphi^*\colon X \to G/E(\varphi)$ given by $\varphi^*(x) = \varphi(x) + E(\varphi)$ satisfies $E_\infty(\varphi^*) = \{0\}$, that is φ^* is a $G/E(\varphi)$ coboundary.

5.3. Characterization of essential values for minimal rotations

Let X be a compact metric monothetic group, $Tx = ax$, where $\{a^n : n \in \mathbb{Z}\}$ is dense in X. It follows that T is minimal. Assume that G is a locally compact Abelian group and $\varphi\colon X \to G$ is a cocycle.

Proposition 5.3.1. *Assume $g \in G_\infty$. Then $g \in E_\infty(\varphi)$ if and only if there exists a rigidity time $(n_t)_{t\geqslant 1}$ and a sequence $(x_t)_{t\geqslant 1}$ of elements of X such that $\varphi^{(n_t)}(x_t) \to g$.*

Proof. Assume that $g \in E_\infty(\varphi)$. Choose a sequence of open sets $X \supset W_1 \supset W_2 \supset \dots$ with $\bigcap_t W_t = \{e\}$. There are open symmetric sets $U_t \subset X$, $t \geqslant 1$, such that $U_1 \supset U_2 \supset \dots$ and $U_t U_t^{-1} \subset W_t$, $t \geqslant 1$. Choose open sets $G_\infty \supset V_1 \supset V_2 \supset \dots$ with $\bigcap_t V_t = \{g\}$. Then there exist integers n_t and $x_t \in X$, $t \geqslant 1$, such that

$$x_t \in U_t \cap T^{-n_t} U_t \cap \{x : \varphi^{(n_t)}(x) \in V_t\}.$$

Therefore $x_t \in U_t$ and $T^{n_t} x_t = a^{n_t} x_t \in U_t$ for $t \geqslant 1$, which implies $a^{n_t} = a^{n_t} x_t x_t^{-1} \in U_t U_t^{-1} \subset W_t$. Since $\bigcap W_t = \{e\}$, $a^{n_t} \to e$, i.e. $T^{n_t} \to \mathrm{id}$ uniformly. Moreover, $\varphi^{(n_t)}(x_t) \in V_t$, $t \geqslant 1$, which forces $\varphi^{(n_t)}(x_t) \xrightarrow{t\to\infty} g$.

Assume now that $\varphi^{(n_t)}(x_t) \xrightarrow{t\to\infty} g \in G_\infty$, where $x_t \in X$, $t \geqslant 1$, and $(n_t)_{t\geqslant 1}$ is a rigidity time for T. Let U be a nonempty open subset of X and $V \subset G_\infty$ a neighbourhood of g. We will show that for some t the following holds:

$$U \cap T^{-n_t} U \cap \{x : \varphi^{(n_t)}(x) \in V\} \neq \emptyset. \tag{5.7}$$

Let W be a nonempty open set satisfying $W \subset \overline{W} \subset U$. As $T^{n_t} \to \mathrm{Id}$ uniformly,

$$\underset{t_1}{\exists}\ \underset{t\geqslant t_1}{\forall}\ W \subset U \cap T^{-n_t} U. \tag{5.8}$$

Since T is minimal,

$$\underset{k>0}{\exists}\ \underset{x\in X}{\forall}\ \{x, Tx, \dots, T^{k-1}x\} \cap W \neq \emptyset. \tag{5.9}$$

Find open sets $0 \in V_0 \subset G$ and $g \in V_1 \subset G_\infty$ such that $kV_0 + V_1 \subset V$. Then

$$\begin{gathered} \underset{t_2}{\exists}\ \underset{t\geqslant t_2}{\forall}\ \varphi^{(n_t)}(x_t) \in V_1, \\ \underset{t_3}{\exists}\ \underset{t\geqslant t_3}{\forall}\ \underset{x\in X}{\forall}\ \varphi(T^{n_t}x) - \varphi(x) \in V_0. \end{gathered} \tag{5.10}$$

Take $t \geqslant \max\{t_1, t_2, t_3\}$. By (5.9), $T^i x_t \in W$ for some i, $0 \leqslant i < k$. Then $T^i x_t \in U$ and, by (5.8), $T^{n_t}(T^i x_t) \in U$. By virtue of (5.10) we have

$$\begin{aligned}\varphi^{(n_t)}(T^i x_t) &= \varphi^{(n_t)}(x_t) + (\varphi^{(n_t)}(T^i x_t) - \varphi^{(n_t)}(x_t)) \\ &= \varphi^{(n_t)}(x_t) + \sum_{m=0}^{i-1} (\varphi(T^{n_t+m} x_t) - \varphi(T^m x_t)) \\ &\in \varphi^{(n_t)}(x_t) + iV_0 \subset V_1 + kV_0 \subset V.\end{aligned}$$

Thus $T^i x_t \in U$, $T^{n_t}(T^i x_t) \in U$, $\varphi^{(n_t)}(T^i x_t) \in V$ and this completes the proof.$\square$

Remark 5.3.2. It follows from Proposition 5.3.1, that if $(n_t)_{t\geqslant 1}$ is a rigidity time for T, then all cluster points in the Vietoris topology on 2^{G_∞} of the net $\varphi^{(n_t)}(X)$, $t \geqslant 1$ are subsets of $E_\infty(\varphi)$.

5.4. Classification of continuous real cocycles over minimal rotations

During this section we will assume that $G = \mathbb{R}$, X is a compact metric monothetic group, $T: X \to X$, $Tx = ax$, where $\{a^n : n \in \mathbb{Z}\}$ is dense in X. Thus (X, T) is strictly ergodic, i.e. it is minimal and Haar measure μ is its unique probability invariant measure. In particular, for each continuous function $f: X \to \mathbb{R}$,

$$\frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n \to \int_X f \, d\mu \quad \text{uniformly on } X. \tag{5.11}$$

For a continuous function $f: X \to \mathbb{R}$ set

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

Lemma 5.4.1. *If $\varphi: X \to \mathbb{R}$ is a cocycle and $\int_X \varphi \, d\mu \neq 0$, then $E_\infty(\varphi) = \{0, \infty\}$.*

Proof. Assume that $\int_X \varphi \, d\mu = c \neq 0$. By (5.11),

$$\frac{1}{N} \sum_{n=0}^{N-1} \varphi \circ T^n = \frac{1}{n} \varphi^{(n)} \to \int_X \varphi \, d\mu = c \quad \text{uniformly on } X.$$

Thus $|\varphi^{(n)}| \to +\infty$ uniformly. In particular, for each sequence $(x_t)_{t\geqslant 1}$ of elements of X, $|\varphi^{(n)}(x_t)| \to +\infty$, so, in view of Proposition 5.3.1, $E(\varphi) = \{0\}$. Choosing any rigidity time $(n_t)_{t\geqslant 1}$ and any $x \in X$ we get that $|\varphi^{(n_t)}(x)| \to +\infty$, so $\infty \in E_\infty(\varphi)$. $\square$

Remark 5.4.2. It follows that if $\int_X \varphi \, d\mu \neq 0$, then each orbit in $X \times \mathbb{R}$ is closed, hence minimal. Thus $X \times \mathbb{R}$ is a disjoint union of closed orbits.

Lemma 5.4.3. *If $\varphi: X \to \mathbb{R}$ is a cocycle and $\varphi^{(n_t)} \to 0$ uniformly for each rigidity time $(n_t)_{t\geqslant 1}$, then $E_\infty(\varphi) = \{0\}$ and there exists a continuous function $\xi: X \to \mathbb{R}$ such that $\varphi(x) = \xi(Tx) - \xi(x)$, $x \in X$.*

Proof. Choose any rigidity time $(n_t)_{t\geq 1}$ and any sequence $(x_t)_{t\geqslant 1}$ from X. Then $\varphi^{(n_t)}(x_t) \to 0$. By Proposition 5.3.1, $E_\infty(\varphi) = \{0\}$. The rest follows from Proposition 5.2.7. □

Proposition 5.4.4. *If $\varphi: X \to \mathbb{R}$ is a coboundary, then the sets*

$$\Gamma_r = \{(x, \xi(x) + r) : x \in X\}, \quad r \in \mathbb{R},$$

are compact and minimal; thus each point in $X \times \mathbb{R}$ is almost periodic and $X \times \mathbb{R}$ is a union of minimal sets. □

Lemma 5.4.5. *Let $\varphi: X \to \mathbb{R}$ be a cocycle satisfying $\int_X \varphi \, d\mu = 0$. If there exist a $c > 0$ and a rigidity time $(n_t)_{t\geqslant 1}$ such that $\|\varphi^{(n_t)}\| > c$ for all $t \geqslant 1$, then $E(\varphi) = \mathbb{R}$.*

Proof. Passing to a subsequence of $(n_t)_{t\geqslant 1}$ and replacing φ by $-\varphi$ if necessary we may assume that

$$\underset{t\geqslant 1}{\forall} \sup\{\varphi^{(n_t)}(x) : x \in X\} > c.$$

Take $r \in \mathbb{R}$, $r \in (0, c)$. We will show, that $r \in E(\varphi)$. Choose $\varepsilon > 0$ such that $(r-2\varepsilon, r+2\varepsilon) \subset (0, c)$ and find $\delta > 0$ for which if $d(x, x') < \delta$ then $|\varphi(x) - \varphi(x')| < \varepsilon$. There exists t such that $d(T^{n_t}x, x) < \delta$ for each $x \in X$. Find $y, z \in X$ satisfying $\varphi^{(n_t)}(y) < 0$, $\varphi^{(n_t)}(z) > c$. Then there exists a positive integer k such that $\varphi^{(n_t)}(T^k y) > c - \varepsilon > r + \varepsilon$. Because $|\varphi^{(n_t)}(Tx) - \varphi^{(n_t)}(x)| = |\varphi(T^{n_t}x) - \varphi(x)| < \varepsilon$ for each $x \in X$, so for each $0 < i < k$, $|\varphi^{(n_t)}(T^i y) - \varphi^{(n_t)}(T^{i-1}y)| < \varepsilon$. Therefore we can find $0 < i < k$ such that $\varphi^{(n_t)}(T^i y) \in (r - \varepsilon, r + \varepsilon)$. Thus $r \in E(\varphi)$. By Proposition 5.2.2, $E(\varphi) = \mathbb{R}$. □

It follows from Proposition 5.2.3, Lemmas 5.4.1, 5.4.3, 5.4.5, Remarks 5.4.2 and 5.4.4 that the following theorem holds.

Theorem 5.4.6. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi: X \to \mathbb{R}$ a continuous cocycle. Then either T_φ is topologically ergodic or $X \times \mathbb{R}$ is a union of closed orbits or φ is a coboundary.*

Remark 5.4.7. Theorem 5.4.6 is strictly related to a G. Atkinson's result [6, Theorem 1]. Atkinson's theorem is formulated for a rotation on a multidimensional torus, and for a cocycle with values in $\mathbb{R}^m$. Taking $m = 1$, as we have assumed in this chapter, Atkinson's theorem gives a part of Theorem 5.4.6 for the first and for the third case.

Directly from Theorem 5.4.6 we obtain the following.

Corollary 5.4.8. *If T is a minimal rotation and $\varphi: X \to \mathbb{Z}$ is a cocycle with zero mean then φ is a coboundary.*

Collecting results of Remark 5.4.2, Proposition 5.4.4 and Theorem 5.4.6 we get the following.

Corollary 5.4.9. *Let T be a minimal rotation on a compact metric monothetic group X equipped with a probability Haar measure μ, $\varphi: X \to \mathbb{R}$ a continuous map.*

If $\int_X \varphi\, d\mu \neq 0$, then $E_\infty(\varphi) = \{0, \infty\}$ and each point in $X \times \mathbb{R}$ is wandering.

If $\int_X \varphi\, d\mu = 0$, then φ is either topologically ergodic (with $E(\varphi) = \mathbb{R}$) or a coboundary (with $E_\infty(\varphi) = \{0\}$).

In Section 5.5 we will show that such a simple classification of real cocycles is no longer true if a rotation T is replaced by a general strictly ergodic flow.

Remark 5.4.10. Let (X, T) be a minimal flow and G be either $\mathbb{Z}$ or $\mathbb{R}$. Consider the Banach space $C(X, G)$ of all continuous functions $\phi: X \to G$ with the supremum norm. Let $C_0(X, G)$ be a (closed) subspace of $C(X, G)$ consisting of these ϕ for which $\int_X \phi\, d\lambda = 0$ for all T-invariant Borel probability measures λ on X. Denote by $E(X, G)$ the set of all topologically ergodic cocycles $\phi: X \to G$. It follows from [46, Proposition 9.12] that coboundaries form a dense subset of $C_0(X, G)$. Therefore if $C_0(X, G)$ contains at least one topologically ergodic cocycle then the set $E(X, G) \cap C_0(X, G)$ is dense in $C_0(X, G)$. In general, $E(X, G)$ is not contained in $C_0(X, G)$. However if (X, T) is a minimal rotation, the inclusion $E(X, G) \subset C_0(X, G)$ holds. Assume now that T is a minimal rotation and $E(X, G) \neq \emptyset$. For each $m > 0$ set

$$C_m = \{\phi \in C_0(X, G) : \|\phi^{(n)}\| \leqslant m \text{ for all } n \in \mathbb{Z}\}.$$

Then C_m is closed and, since $E(X, G) \subset C_0(X, G) \setminus C_m$, C_m has empty interior, so it is nowhere dense in $C_0(X, G)$. By Gottschalk–Hedlund Theorem [40, Theorem 14.11], C_m consists solely of coboundaries. It follows from Theorem 5.4.6 that

$$E(X, G) = \bigcap_{m=1}^{\infty} C_m^c$$

is a $\mathcal{G}_\delta$ dense subset of $C_0(X, G)$.

5.5. Zero-time cocycle for Morse shifts

In contrast with Corollary 5.4.8, below, we will give examples of integer-valued cocycles with $E_\infty(\varphi) = \{0, \infty\}$ and $\int_X \varphi = 0$. They will be considered over so called Morse shifts introduced in [47] (we will recall the definition below). Morse shifts are subshifts of the full shift $(\{0, 1\}^{\mathbb{Z}}, T)$, they are minimal and monoergodic. For details we refer to [58].

By *blocks* we will mean finite sequences of zeros and ones, in notation $B = b_0 \dots b_{n-1}$, where $b_i \in \{0,1\}$. The number $|B| = n$ we will call the *length* of B. If $B = b_0 \dots b_{k-1}$ is a block and $i \in \{0,1\}$, then set $B + i = (b_0 + i) \dots (b_{k-1} + i)$, where the summation $b_j + i$, $i = 0, \dots, k-1$, is taken modulo 2. In particular denote $\widetilde{B} = B + 1$, the "mirror" of B. For blocks $B = b_0 \dots b_{k-1}$ and $C = c_0 \dots c_{l-1}$ put

$$B \times C = (B + c_0)(B + c_1) \dots (B + c_{l-1}).$$

If $x \in \{0,1\}^{\mathbb{Z}}$, $x = \dots x_{-3}x_{-2}x_{-1}x_0x_1x_2x_3 \dots$, then for integers $k \leqslant n$ denote $x[k,n] = x_k x_{k+1} \dots x_n$. For $B = b_0 \dots b_{k-1}$ and for $0 \leqslant i \leqslant j \leqslant k-1$ denote $B[i,j] = b_i b_{i+1} \dots b_j$. Given block B define

$$[B] = \{x \in \{0,1\}^{\mathbb{Z}} : x[0, |B| - 1] = B\}.$$

Now let $A = a_0 \dots a_{n-1}$ be a block satisfying $A[0] = a_0 = 0$. Additionally we will assume that A is neither of the form $010101 \dots 010$ nor $00 \dots 0$. Define a sequence C_t, $t \geqslant 1$, of blocks by

$$C_1 = A, \qquad C_{t+1} = C_t \times A, \quad t = 1, 2, \dots$$

Then $|C_t| = n^t$, $t \geqslant 1$ and $C_{t+s}[0, n^t - 1] = C_t$ for $t, s \geqslant 1$. Thus the intersection $\bigcap_{t \geqslant 1}[C_t] \subset \{0,1\}^{\mathbb{Z}}$ is a nonempty closed set. Choosing any $y \in \bigcap_{t \geqslant 1}[C_t]$ we can find $x_0 \in \overline{\mathrm{Orb}}(y)$ satisfying

$$\begin{aligned} \text{either} \quad x_0[-n^t, n^t - 1] &= C_t C_t, \quad t \geqslant 1 \\ \text{or} \quad x_0[-n^t, n^t - 1] &= \widetilde{C}_t C_t, \quad t \geqslant 1. \end{aligned}$$

Put

$$X = \overline{\mathrm{Orb}}(x_0).$$

Then (X, T) is a *Morse shift*. Take $x \in X$. Then for all $t \geqslant 1$, x is an infinite concatenation of C_t and $\tilde{C}_t$ and this structure is unique in the following meaning: there exists a unique sequence of integers $(p_t(x))_{t \geqslant 1}$ such that

$$\begin{gathered} -n^t \leqslant p_t(x) < 0, \quad t \geqslant 1, \\ x[kn^t + p_t(x), kn^t + p_t(x) + n^t - 1] = C_t \text{ or } \widetilde{C}_t \quad \text{for all } k \in \mathbb{Z} \end{gathered}$$

(see [58]). Moreover, a Morse shift is *recognizable* in the sense that there exists a positive integer L, called a *recognizability constant*, such that for each $x \in X$ and for each $t \geqslant 1$, if $x[p, p + Ln^t - 1] = (C_t + i_1)(C_t + i_2) \dots (C_t + i_L)$, then $p = p_t(x) + kn^t$ for some integer k.

Let $\{0,1\}^*$ be the set of all blocks. Define $\psi \colon \{0,1\}^* \to \mathbb{Z}$ by

$$\psi(B) = \sum_{i=0}^{|B|-1} (-1)^{B[i]}.$$

Then

$$\psi(BC) = \psi(B) + \psi(C), \quad \psi(\widetilde{B}) = -\psi(B), \quad \psi(B \times C) = \psi(B)\psi(C).$$

Let $\varphi: X \to \mathbb{Z}$ be defined by the formula

$$\varphi(x) = (-1)^{x[0]}, \quad x \in X.$$

The cocycle corresponding to φ is called the *zero-time cocycle*. Note that $\varphi^{(k)}(x) = \psi(x[0, k-1])$ for $k \geqslant 1$ and $\int_X \varphi \, d\mu = 0$.

We will show the following.

Proposition 5.5.1.

(a) *If $\psi(A) = 0$, then φ is a coboundary.*
(b) *If $|\psi(A)| = 1$, then T_φ is topologically ergodic.*
(c) *If $|\psi(A)| \geqslant 2$, then $E_\infty(\varphi) = \{0, \infty\}$. In particular, φ is not quasi regular.*

Proof. (a) Clearly $\psi(\widetilde{A}) = 0$ and if $x \in X$, $k > 0$, then

$$x[0, k-1] = (A[r, n-1] + i_0)(A + i_1) \dots (A + i_{l-1})(A[0, s] + i_l),$$

where $0 < r \leqslant n-1$, $0 \leqslant s < n-1$. Thus

$$|\varphi^{(k)}(x)| = |\psi(x[0, k-1])| = |\psi(A[r, n-1] + i_0) + \psi(A[0, r] + i_l)| < 2n.$$

Since $\varphi^{(-k)}(x) = -\varphi^{(k)}(T^{-k}x)$ for $k > 0$, the cocycle $\varphi^{(\cdot)}$ is bounded on $\mathbb{Z} \times X$, so φ is a coboundary.

(b) First assume that $\psi(A) = 1$. Then $\psi(C_t) = 1$, $\psi(\widetilde{C}_t) = -1$, $t \geqslant 1$. We will show that either $1 \in E(\varphi)$ or $-1 \in E(\varphi)$. It follows from $\psi(A) = 1$ and $A \neq 0101 \dots 010$, that either

$$\exists_{0 \leqslant k < n-1} \psi(A[0, k]) = 1, \quad A[k+1] = 0, \tag{5.12}$$

or

$$\exists_{0 < k < n-1} \psi(A[0, k]) = -1, \qquad A[k+1] = 0. \tag{5.13}$$

Indeed, by looking at $\psi(A[0, i])$, $i = 0, \dots, n-1$, we get that either there exists $j > 0$ with $\psi(A[0, j]) = 2$ or there exists $j > 0$ with $\psi(A[0, j]) = -1$. In the first case, choosing the smallest j with this property we get $\psi(A[0, j-1]) = 1$, $A[j] = 0$. In the second case, choosing the last j with this property we get $\psi(A[0, j]) = -1$, $A[j+1] = 0$. Now, take any nonempty open set $U \subset X$. There exists $t \geqslant 1$ such that $T^p[C_t + i] \subset U$ for some $0 \leqslant p < n^t$, $i \in \{0, 1\}$. Let $B = (A \times \ldots \times A) \times (C_t + i)$, where in the definition of B the number of A's is such that $|A \times \ldots \times A| > L$, L is the recognizability constant. Then B starts with the block $C_t + i$ (recall that A starts with 0) and hence $T^p[B] \subset T^p[C_t + i] \subset U$. Actually we know more, namely, for each $x \in T^p[B]$ we have $p_t(x) = p_{t+1}(x) = p$

(because of the recognizability property). Suppose (5.12) is valid and take $x \in T^p[B]$. Then$T^{kn^t}x \in T^p[C_t + i] \subset U$ and

$$\varphi^{(kn^t)}(x) = \psi(x[0, kn^t - 1]) = 1.$$

Thus we have proved that $1 \in E(\varphi)$, so $E(\varphi) = \mathbb{Z}$.

The case (5.13) is similar and gives $\varphi^{(kn^t)}(x) = -1$, thus $E(\varphi) = \mathbb{Z}$.

If $\psi(A) = -1$, then we can either repeat our previous considerations or observe that the block $A \times A$ defines the same system X as A and $\psi(A \times A) = 1$. Thus $E(\varphi) = \mathbb{Z}$.

It follows from Proposition 5.2.3 that T_φ is topologically ergodic.

(c) Fix a positive integer M and given an open nonempty set $U \subset X$ find $x \in U$ and $t > M$ with $T^{p_t}x \in [C_t + i] \subset T^{p_t}U$, where $(p_t)_{t \geqslant 1}$ is the sequence defining the unique structure of C_t's and $\widetilde{C}_t$'s on x. Find $k > 0$ such that $x[p_t + kn^t, p_t + (k+1)n^t - 1] = C_t + i$. Since $|\psi(A)| \geqslant 2$ and $\psi(C_t) = \pm\psi(A)^t$, the number $\psi(A)^t$ divides $\varphi^{(k)}(x)$. Clearly $|\psi(A)^t| > t > M$. Thus $\infty \in E_\infty(\varphi)$. To show that $E(\varphi) = \{0\}$, suppose that there is a positive $m \in E(\varphi)$. Fix t satisfying $|\psi(A)|^t > m$. Let $U = [C_{t+s}]$, where $sn > L$ (L is the recognizability constant for X). There is k such that $U \cap T^{-k}U \cap \{x : \varphi^{(k)}(x) = m\} \neq \emptyset$. By the recognizability, n^t divides k. Thus $\varphi^{(k)}(x)$ is of the form $\varphi^{(k)}(x) = l|\psi(A)|^t$ for some $l \in \mathbb{Z}$. Since $|\psi(A)|^t > m > 0$, $\varphi^{(k)}(x) \neq m$, a contradiction. Thus $E(\varphi) = \{0\}$. $\square$

According to a private letter from A. Forrest ([20]), he proved the following theorem.

Suppose that (X, T) is minimal Cantor and that $\varphi\colon X \to \mathbb{Z}$ is a cocycle such that its mean with respect to every T-invariant measure equals zero. Then either

(i) φ is a coboundary or
(ii) φ is topologically ergodic or
(iii) $E_\infty(\varphi) = \{0, \infty\}$ or
(iv) φ is cohomologous to a cocycle $n\psi$ for some $n \geqslant 1$ with $\psi\colon X \to \mathbb{Z}$ topologically ergodic.

5.6. An application – a disjointness theorem

Assume that G is a locally compact group acting continuously on a compact Hausdorff space Y, $G \times Y \ni (g, y) \to g(y) \in Y$; thus all g's are homeomorphisms. Then the pair (Y, G) is said to be a *G-flow*. A subset $D \subset Y$ is called *G-minimal*, if D is closed, G-invariant ($gD = D$ for all $g \in G$) and the only proper subset of D with these properties is the empty set. A G-flow (Y, G) is *G-minimal* if Y is G-minimal.

Now, let (X,T) be a compact minimal $\mathbb{Z}$-flow, G a locally compact group acting on a compact Hausdorff space Y, $\varphi: X \to G$ a cocycle. Define a homeomorphism $\widetilde{T}_\varphi: X \times Y \to X \times Y$ by the formula

$$\widetilde{T}_\varphi(x,y) = (Tx, \varphi(x)(y)).$$

We will explore $\mathbb{Z}$-flows of the form $(X \times Y, \widetilde{T}_\varphi)$.

Proposition 5.6.1. *Let (X,T) be a compact minimal flow, G a locally compact group acting on a compact Hausdorff space Y, $\varphi: X \to G$ a cocycle such that T_φ is point transitive.*

(a) *If $M \subset X \times Y$ is $\widetilde{T}_\varphi$-minimal, then there exists a compact G-invariant $Y_0 \subset Y$ such that $M = X \times Y_0$. Moreover (Y_0, G) is point transitive.*
(b) *If the G-flow (Y,G) is minimal, then $\widetilde{T}_\varphi$ is also minimal.*
(c) *If the G-flow (Y,G) is point transitive then $\widetilde{T}_\varphi$ is also point transitive.*

Proof. (a) If (x_0, g_0) has dense orbit in $X \times G$, then also (x_0, e) has dense orbit in $X \times G$, where e is the unit element of G. Since X is minimal and $X \times Y$ is compact, the projection $\pi_X: X \times Y \to X$ maps M onto X, $\pi_X(M) = X$. Find $y_0 \in Y$ such that $(x_0, y_0) \in M$. Define

$$D = \{(x,g) \in X \times G : (x, g(y_0)) \in M\}.$$

Then $(x_0, e) \in D$, D is closed and T_φ-invariant, so $D = X \times G$. Put

$$Y_0 = \overline{G(y_0)},$$

where $G(y_0) = \{g(y_0) : g \in G\}$. Since $D = X \times G$, $X \times G(y_0) \subset M$. Clearly $X \times G(y_0)$ is $\widetilde{T}_\varphi$-invariant, hence the minimality of M yields $X \times Y_0 = M$.

(b) In this case $Y_0 = \overline{G(y_0)} = Y$.

(c) Assume that (x_0, e) has dense orbit in $X \times G$ and that $y_0 \in Y$ has dense orbit in Y i.e. $\overline{G(y_0)} = Y$. We will show that (x_0, y_0) has dense orbit in $X \times Y$ (for $\widetilde{T}_\varphi$). Take any nonempty open sets $U \subset X$, $V \subset Y$. There exists a nonempty open $W \subset G$ such that $g \in W$ implies $g(y_0) \in V$. We can find $n \in \mathbb{Z}$ with $T_\varphi^n(x_0, e) \in U \times W$. Then $T^n x_0 \in U$, $\varphi^{(n)}(x_0) \in W$ and $\widetilde{T}_\varphi^n(x_0, y_0) = (T^n x_0, \varphi^{(n)}(x_0)(y_0)) \in U \times V$. $\square$

Recall that two compact flows (X_1, T_1) and (X_2, T_2) are *disjoint*, if the only nonempty closed $T_1 \times T_2$-invariant subset $D \subset X_1 \times X_2$ satisfying $\pi_i(D) = X_i$, where $\pi_i(x_1, x_2) = x_i$, $i = 1, 2$, is just $X_1 \times X_2$. In such a case we will write $T_1 \perp T_2$.

Proposition 5.6.2. *Assume that (X,T) and (Z,S) are compact minimal $\mathbb{Z}$-flows such that $T \perp S$. Let G be an Abelian locally compact group acting on a compact Hausdorff space Y in such a way that the G-flow (Y,G) is minimal.*

Let $\varphi: X \to G$ *be a* $\mathbb{Z}$*-cocycle such that* $T_\varphi \times S$ *is point transitive. Then* $\widetilde{T}_\varphi \times S$ *is point transitive and* $\widetilde{T}_\varphi \perp S$.

Proof. Let (x_0, g_0, z_0) be a point with dense orbit in $(X \times G) \times Z$. Then also $((x_0, e), z_0)$ has dense orbit in $(X \times G) \times Z$. Assume that $D \subset (X \times Y) \times Z$ is a closed, $\widetilde{T}_\varphi \times S$-invariant set with $\pi_{X \times Y}(D) = X \times Y$, $\pi_Z(D) = Z$. Let $\widetilde{D} = \pi_{X \times Z}(D)$. Then $\pi_X(\widetilde{D}) = X$, $\pi_Z(\widetilde{D}) = Z$. Clearly $\widetilde{D}$ is closed and $T \times S$-invariant. From $T \perp S$ we get $\widetilde{D} = X \times Z$, which implies $(x_0, z_0) \in \widetilde{D}$, and therefore there exists $y \in Y$ with $(x_0, y, z_0) \in D$. Define $F = F_y: (X \times G) \times Z \to (X \times Y) \times Z$ by $F(x, g, z) = (x, g(y), z)$. Since the G-flow (Y, G) is minimal, $F((X \times G) \times Z)$ is dense in $(X \times Y) \times Z$. Moreover, $F \circ (T_\varphi \times S) = (\widetilde{T}_\varphi \times S) \circ F$ and therefore, for each $(x, g, z) \in (X \times G) \times Z$, $F(\mathrm{Orb}(x, g, z)) = \mathrm{Orb}(F(x, g, z))$. Thus $F(x_0, e, z_0)$ has dense orbit in $(X \times Y) \times Z$, $(X \times Y) \times Z = \overline{\mathrm{Orb}}(F(x_0, e, z_0)) = \overline{\mathrm{Orb}}(x_0, e(y), z_0) = \overline{\mathrm{Orb}}(x_0, y, z_0)$, which implies that $\widetilde{T}_\varphi \times S$ is point transitive. Since $(x_0, y, z_0) \in D$, $D = (X \times Y) \times Z$ and $\widetilde{T}_\varphi \perp S$. □

Definition 5.6.3. Let (X, T) be a compact minimal flow. A cocycle $\varphi: X \to \mathbb{R}$ is called *universally ergodic*, if for each compact flow (Y, S) such that $T \times S$ is topologically ergodic, the flow $T_\varphi \times S$ is also topologically ergodic.

We will now show the existence of universally ergodic cocycles over irrational rotations.

Let $\mathbb{T} = \{x \in \mathbb{C} : |x| = 1\}$ be the unit circle on the complex plane, $T: \mathbb{T} \to \mathbb{T}$, $Tx = \exp(2\pi i \alpha)x$, where α is irrational. Denote by dx the probability Haar measure on $\mathbb{T}$.

Lemma 5.6.4. *Let* $\varphi: \mathbb{T} \to \mathbb{R}$ *be a cocycle such that* $\int_{\mathbb{T}} \varphi \, dx = 0$ *and let* $(n_t)_{t \geqslant 1}$ *be a sequence of integers.*

(a) *If* $\underset{c>0}{\exists} \underset{t \geqslant 1}{\forall} \sup\{\varphi^{(n_t)}(x) : x \in \mathbb{T}\} > c$ *then*

$$\underset{0<r<c}{\forall} \ \underset{(x_t)_{t \geqslant 1}}{\exists} \ \underset{t \geqslant 1}{\forall} \varphi^{(n_t)}(x_t) = r.$$

(b) *If* $\underset{c<0}{\exists} \underset{t \geqslant 1}{\forall} \inf\{\varphi^{(n_t)}(x) : x \in \mathbb{T}\} < c$ *then*

$$\underset{c<r<0}{\forall} \ \underset{(x_t)_{t \geqslant 1}}{\exists} \ \underset{t \geqslant 1}{\forall} \varphi^{(n_t)}(x_t) = r.$$

Proof. We will prove only the first statement. Take $0 < r < c$ and denote $\varepsilon = (c - r)/2$. Since $\int_{\mathbb{T}} \varphi \, dx = 0$, there exists a sequence $(y_t)_{t \geqslant 1}$ such that $\varphi^{(n_t)}(y_t) = \int_{\mathbb{T}} \varphi^{(n_t)} \, dx = 0$, $t \geqslant 1$. By assumption, we can find a sequence $(z_t)_{t \geqslant 1}$ such that $\varphi^{(n_t)}(z_t) \geqslant r + \varepsilon$. By continuity of $\varphi^{(n_t)}$, $t \geqslant 1$, there exists a sequence $(x_t)_{t \geqslant 1}$ satisfying $\varphi^{(n_t)}(x_t) = r$, $t \geqslant 1$. □

Proposition 5.6.5. *Let $T:\mathbb{T}\to\mathbb{T}$ be an irrational rotation, $\varphi:\mathbb{T}\to\mathbb{R}$ a cocycle. If*

$$\int_{\mathbb{T}}\varphi(x)\,dx=0 \quad \textit{and} \quad \mathop{\exists}_{c>0}\mathop{\forall}_{n\geqslant 1}\|\varphi^{(n)}\|>c,$$

then φ is universally ergodic.

Proof. Let (Y,S) be a compact flow such that $T\times S$ is topologically ergodic. Define $\widetilde{\varphi}:\mathbb{T}\times Y\to\mathbb{R}$ by the formula $\widetilde{\varphi}(x,y)=\varphi(x)$. Then $T_\varphi\times S$ is clearly isomorphic to $(T\times S)_{\widetilde{\varphi}}$. We will show that $E(\widetilde{\varphi})=\mathbb{R}$.

Let $U\subset\mathbb{T}$ and $V\subset Y$ be nonempty open sets. Consider a sequence of open sets $U\supset U_1\supset U_2\supset\ldots$ such that $\bigcap_t U_t$ is a one point set. Since $T\times S$ is topologically ergodic, for each t there is n_t satisfying

$$(U_t\times V)\cap(T\times S)^{-n_t}(U_t\times V)\neq\emptyset.$$

Then $(n_t)_{t\geqslant 1}$ is a rigidity time for T. Moreover

$$\mathop{\forall}_{t}(U\times V)\cap(T\times S)^{-n_t}(U\times V)\neq\emptyset.$$

Replacing φ by $-\varphi$ and passing to a subsequence if necessary we may assume that

$$\mathop{\forall}_{t\geqslant 1}\sup\{\varphi^{(n_t)}(x):x\in\mathbb{T}\}>c.$$

Now choose any $r\in(0,c)$. By Lemma 5.6.4, there exists a sequence $(x_t)_{t\geqslant 1}$ such that

$$\mathop{\forall}_{t\geqslant 1}\varphi^{(n_t)}(x_t)=r.$$

Fix an arbitrary $\varepsilon>0$. We will show that

$$(U\times V)\cap(T\times S)^{-n_t}(U\times V)\cap\{(x,y):|\widetilde{\varphi}^{(n_t)}(x,y)-r|<\varepsilon\}\neq\emptyset.$$

Let $W\subset\overline{W}\subset U$ be a nonempty open set. Then

$$\mathop{\exists}_{t_1}\mathop{\forall}_{t\geqslant t_1}W\subset U\cap T^{-n_t}U,$$

and since T is minimal,

$$\mathop{\exists}_{k>0}\mathop{\forall}_{x\in\mathbb{T}}\{x,Tx,,\ldots,T^{k-1}x\}\cap W\neq\emptyset.$$

Furthermore

$$\mathop{\exists}_{t_2}\mathop{\forall}_{t\geqslant t_2}\mathop{\forall}_{x\in\mathbb{T}}|\varphi(T^{n_t}x)-\varphi(x)|<\frac{\varepsilon}{k}.$$

Take $t \geqslant t_0 = \max\{t_1, t_2\}$. Denote $y_t = T^i x_t \in W$, where $0 \leqslant i < k$. Then $y_t \in U \cap T^{-n_t}U$. Moreover

$$\begin{aligned}
|\varphi^{(n_t)}(y_t) - r| &= |\varphi^{(n_t)}(y_t) - \varphi^{(n_t)}(x_t)| \\
&= \Big| \sum_{m=0}^{i-1} \varphi(T^{m+n_t}x_t) - \varphi(T^m x_t) \Big| \\
&\leqslant \sum_{m=0}^{i-1} |\varphi(T^{m+n_t}x_t) - \varphi(T^m x_t)| < \varepsilon.
\end{aligned}$$

Therefore $\widetilde{\varphi}^{(n_t)}(y_t, z) \in (r-\varepsilon, r+\varepsilon)$ for each $z \in Y$. Now choose $z_t \in V \cap S^{-n_t}V$. Then

$$(y_t, z_t) \in (U \times V) \cap (T \times S)^{-n_t}(U \times V) \cap \{(x, y) : |\widetilde{\varphi}^{(n_t)}(x, y) - r| < \varepsilon\},$$

which completes the proof. □

Now we will argue that cocycles satisfying assumptions of Proposition 5.6.5 do exist. Suppose that $\varphi\colon \mathbb{T} \to \mathbb{R}$ is a cocycle for $Tx = \exp(2\pi i\alpha)x$ with zero mean. Moreover suppose that the condition

$$\|\varphi^{(n)}\| \geqslant c > 0 \quad \text{for all } n \geqslant 1 \tag{5.14}$$

is not valid. It follows that there exists an increasing sequence $(k_n)_{n\geqslant 1}$ of positive integers with $\varphi^{(k_n)} \to 0$ uniformly, so $\exp(2\pi i\varphi^{(k_n)}) \to 1$ uniformly. Consider an operator $V_\varphi\colon L^2(\mathbb{T}, dx) \to L^2(\mathbb{T}, dx)$ given by the formula $V_\varphi F(x) = \exp(2\pi i\varphi(x))F(Tx)$. Then clearly $V_\varphi^k F(x) = \exp(2\pi i\varphi^{(k)}(x))F(T^k x)$. Taking $F \equiv 1$ we get

$$\widehat{\sigma}[k] = \langle V_\varphi^k 1, 1\rangle = \int_{\mathbb{T}} \exp(2\pi i\varphi^{(k)}(x))\, dx.$$

It follows that if the condition (5.14) is not satisfied, then there exists a spectral measure σ of the operator V_φ which is a Dirichlet measure: $\widehat{\sigma}[k_n] \to \sigma(\mathbb{T})$ for some sequence (k_n). If for instance the maximal spectral type of V_φ is a Rajchman measure then (5.14) holds. In particular (5.14) holds if the maximal spectral type of V_φ is Lebesgue measure. Examples of continuous functions φ for which the maximal spectral type of V_φ is Lebesgue can be found in [11], [43], [69]. Because each $f \in L^1(\mathbb{T})$ is cohomologous to a continuous function [53], also many Anzai cocycles determines $\varphi \in C(\mathbb{T})$ satisfying (5.14). M. Lemaczyk has recently proved [62] that if a special flow with a function f over rotation is mixing then the function $f_0 = f - \int f$ satisfies $|f_0^{(n)}| \to \infty$ in measure. It follows that there is a continuous function φ in the cohomology class of f_0 such that $\|\varphi^{(n)}\|_{C(\mathbb{T})} \to \infty$.

In what follows we will briefly describe notions of RIC and PI flows. For more details we refer to [28] or [97].

Recall that a flow (Z,S) is *topologically weakly mixing*, if the product flow $(Z\times Z, S\times S)$ is topologically ergodic. It is well known that all minimal rotations are disjoint from all weakly mixing minimal flows [22]. For a factor map $\pi\colon(Z,S)\to(X,T)$ denote $R_\pi=\{(z_1,z_2):\pi(z_1)=\pi(z_2)\}$. The map π is called *weakly mixing*, if the flow $(R_\pi, S\times S)$ is topologically ergodic. If Z is minimal then π is said to be *relatively incontractible*, RIC for short, whenever all proximal minimal extensions of X are disjoint from π. For π which is RIC there exists a commutative diagram

$$\begin{tikzcd}
Z \arrow[dd, "\pi"'] \arrow[dr, "\widetilde{\pi}"] & \\
 & \widetilde{X} \arrow[dl, "\rho"] \\
X &
\end{tikzcd}$$

where ρ is the largest equicontinuous extension of X that is a factor of Z. The above decomposition $\pi=\rho\widetilde{\pi}$ we will call a RIC-*decomposition* of π. For any minimal flow (Z,S) one can construct a diagram called the *canonical* PI-*tower*. This diagram is presented at Figure 5.1.

$$\begin{tikzcd}
Z \arrow[d] & & Z_1 \arrow[ll, "\widetilde{\theta}_1"'] \arrow[dl, "\pi_1"'] \arrow[d, "\widetilde{\pi}_1"] & & Z_2 \arrow[ll, "\widetilde{\theta}_2"'] \arrow[dl, "\pi_2"'] \arrow[d, "\widetilde{\pi}_2"] & \cdots \arrow[l] & Z_\infty \arrow[l] \arrow[d, "\pi_\infty"] \\
* & X_1 \arrow[l, "\theta_1"] & Y_1 \arrow[l, "\rho_1"] & X_2 \arrow[l, "\theta_2"] & Y_2 \arrow[l, "\rho_2"] & \cdots \arrow[l] & Y_\infty \arrow[l]
\end{tikzcd}$$

FIGURE 5.1

In the picture all θ_i, $\widetilde{\theta}_i$ are proximal, all π_i are RIC and all decompositions $\pi_i=\rho_i\widetilde{\pi}_i$ are the RIC-decompositions. If the map π_∞ is an isomorphism, then (Z,S) is called a PI-*flow*. If $\pi\colon Z\to X$, where (X,T) is a rotation, is RIC and weakly mixing then X is the maximal equicontinuous factor of Z and the canonical PI-tower has the form shown at Figure 5.2.

$$\begin{tikzcd}
Z \arrow[d] & & Z \arrow[ll, "\mathrm{Id}"'] \arrow[dl, "\pi"'] \arrow[d, "\pi"] \\
* & X \arrow[l] & X \arrow[l, "\mathrm{Id}"]
\end{tikzcd}$$

FIGURE 5.2

In such a case Z is not PI provided π is not an isomorphism.

Theorem 5.6.6. *Let (X, T) be a minimal rotation, $\varphi: X \to \mathbb{R}$ a universally ergodic cocycle, $(Y, \mathbb{R})$ a compact minimal $\mathbb{R}$-flow. Then $\widetilde{T}_\varphi \perp S$ for each weakly mixing compact minimal flow (Z, S). If additionally $(Y, \mathbb{R})$ is weakly mixing and Y is metric then $\widetilde{T}_\varphi$ is not* PI.

Proof. Let (Z, S) be a weakly mixing compact minimal flow. Then $T \perp S$ and, by assumption, $T_\varphi \times S$ is topologically ergodic hence point transitive. By Proposition 5.6.2, $\widetilde{T}_\varphi \perp S$.

Now assume that the $\mathbb{R}$-flow $(Y, \mathbb{R})$ is weakly mixing. First we will show that the extension $\pi: (X \times Y, \widetilde{T}_\varphi) \to (X, T)$, where $\pi(x, y) = x$, is weakly mixing. Denote

$$R_\pi = \{(x, y, x, y') : x \in X, y, y' \in Y\} \cong X \times Y \times Y.$$

It suffices to show that the flow $(R_\pi, \widetilde{T}_{\varphi\times\varphi})$ is point transitive, where

$$\widetilde{T}_{\varphi\times\varphi}(x, y, y') = (Tx, \varphi(x)(y), \varphi(x)(y')).$$

Since (Y, G) is weakly mixing, $(Y \times Y, G)$ is point transitive (Y is metric). The cocycle $\varphi \times \varphi: X \to \Delta_\mathbb{R} = \{(r, r) : r \in \mathbb{R}\}$ is clearly topologically ergodic. By Proposition 5.6.1, $\widetilde{T}_{\varphi\times\varphi}: R_\pi \to R_\pi$ is point transitive, thus the extension $\pi: (X \times Y, \widetilde{T}_\varphi) \to (X, T)$ is weakly mixing. It follows from [28, Proposition 2.1] that π is RIC. In particular (X, T) is the maximal equicontinuous factor of $\widetilde{T}_\varphi$. Thus the canonical PI tower for π is reduced to $X \times Y \xrightarrow{\pi} Y$, so $\widetilde{T}_\varphi$ is not PI.□

Denote by $\mathcal{W}$ the class of all compact minimal weakly mixing flows and by $\mathcal{W}^\perp$ the class of all compact minimal flows, which are disjoint from all flows belonging to $\mathcal{W}$. Following [38] we say that a compact minimal flow (X, T) is a *multiplier of the class of topological systems disjoint from weakly mixing*, if for each compact minimal flow $(Z, S) \in \mathcal{W}^\perp$ and for each minimal subset $M \subset X \times Z$, the flow $(M, T \times S)$ is in $\mathcal{W}^\perp$. Recall that an extension $\pi: (Y, G) \to (Y_1, G)$ is *regular* if for each almost periodic pair $(y_1, y_2) \in Y \times Y$ satisfying $\pi(y_1) = \pi(y_2)$ there exists a homeomorphism $S: Y \to Y$ such that $S \circ g = g \circ S$ for all $g \in G$ (we call such an S an *automorphism* of (Y, G)) and satisfying $S(y_1) = y_2$. A flow (Y, G) is *regular* if treated as an extension of the one point flow $(*, G)$ it is regular.

Assume now that $(Y, \mathbb{R})$ is regular and let $\widetilde{T}_\varphi: X \times Y \to X \times Y$ satisfy the assumptions of Theorem 5.6.6, i.e. $\varphi: X \to \mathbb{R}$ is a universally ergodic cocycle over a minimal rotation (X, T) and $(Y, \mathbb{R})$ is a compact minimal $\mathbb{R}$-flow. Thus $(X \times Y, \widetilde{T}_\varphi) \in \mathcal{W}^\perp$. Let $R: Z \to Z$ be a homeomorphism of a compact Hausdorff space such that the flow (Z, R) is minimal. Assume that (Z, R) is in $\mathcal{W}^\perp$. Let $M \subset (X \times Y) \times Z$ be a minimal subset. We will show that $(M, \widetilde{T}_\varphi \times R)$ is in $\mathcal{W}^\perp$ giving explicite a family of examples of multipliers of $\mathcal{W}^\perp$, precising a result of E. Glasner and B. Weiss, [38].

First observe that the extension $(X \times Y, \widetilde{T}_\varphi) \to (X, T)$ is regular. Indeed, if $\big((x, y_1), (x, y_2)\big)$ is an almost periodic element in $\big((X \times Y) \times (X \times Y), \widetilde{T}_\varphi \times \widetilde{T}_\varphi\big)$, then for some nontrivial sequence of integers $(\widetilde{T}_\varphi^{n_i} \times \widetilde{T}_\varphi^{n_i})\big((x, y_1), (x, y_2)\big) \to \big((x, y_1), (x, y_2)\big)$, hence $\varphi^{(n_i)}(x)(y_1) \to y_1$, $\varphi^{(n_i)}(x)(y_2) \to y_2$ and (y_1, y_2) is almost periodic in $(Y \times Y, \mathbb{R})$. Since $(Y, \mathbb{R})$ is regular, there is an automorphism of $(Y, \mathbb{R})$ such that $S(y_1) = y_2$. Define $\widetilde{S}: X \times Y \to X \times Y$ by $\widetilde{S}(x, y) = (x, S(y))$. Clearly $\widetilde{S}(x, y_1) = (x, y_2)$. Thus the extension $(X \times Y, \widetilde{T}_\varphi) \to (X, T)$ is regular and we can apply Proposition 4.1 in [38]. Since rotations are multipliers of the class of topological systems disjoint from weakly mixing, so is $\widetilde{T}_\varphi$.

CHAPTER 6

ESSENTIAL VALUES OF TOPOLOGICAL COCYCLES OVER MINIMAL ROTATIONS

In this chapter a theory of essential values of cocycles over minimal rotations with values in locally compact Abelian groups, especially $\mathbb{R}^m$ will be developed. We will give some criteria for such cocycles to be conservative and describe the group of essential values of such ones.

6.1. Essential values of a cocycle

The following proposition is a topological version of a similar one from [69].

Proposition 6.1.1. *Let (X,T) be a flow. Assume that G, H are locally compact Abelian groups and let $\pi\colon G \to H$ be a continuous group homomorphism. If $\varphi\colon X \to G$ is a continuous map, then*

$$\overline{\pi(E(\varphi))} \subset E(\pi \circ \varphi).$$

Proof. Let $h \in \pi(E(\varphi))$, then $h = \pi(g)$ for some $g \in E(\varphi)$. Fix any open $U \subset X$ and an open neighbourhood V of $h = \pi(g)$ in H. Then $\pi^{-1}(V)$ is an open neighbourhood of g in G and there exists an integer N such that

$$U \cap T^{-N}U \cap \{x \in X : \varphi^{(N)}(x) \in \pi^{-1}(V)\} \neq \emptyset.$$

Using the identity $(\pi \circ \varphi)^{(N)} = \pi \circ \varphi^{(N)}$ we have

$$\begin{aligned} U \cap T^{-N}U \cap \{x \in X : (\pi \circ \varphi)^{(N)}(x) \in V\} \\ = U \cap T^{-N}U \cap \{x \in X : \varphi^{(N)}(x) \in \pi^{-1}(V)\} \neq \emptyset, \end{aligned}$$

and therefore $h \in E(\pi \circ \varphi)$. Since $E(\pi \circ \varphi)$ is closed, the result follows. $\square$

Proposition 6.1.2. *Assume that (X,T) is a topological flow, G a locally compact Abelian group, $\varphi\colon X\to G$ a continuous map, $H\subset E(\varphi)$ a closed subgroup. Let $\varphi_H\colon X\to G/H$, $\varphi_H(x)=\varphi(x)+H$. Then $E(\varphi_H)=E(\varphi)/H$.*

Proof. By Proposition 6.1.1, $E(\varphi)/H\subset E(\varphi_H)$. To prove the converse choose $g+H\in E(\varphi_H)$. We will show that $g\in E(\varphi)$. Take an open set $U\subset X$ and an open neighbourhood V of zero in G. Let $V_0\subset G$ be such an open neighbourhood of zero that $V_0+V_0\subset V$. Since $g+H\in E(\varphi_H)$, there exist an integer N and $x_0\in X$ such that

$$x_0\in U\cap T^{-N}U\cap\{x\in X:\varphi_H^{(N)}(x)\in g+H+V_0\}.$$

Then $\varphi^{(N)}(x_0)=g+h+v_0$ for some $h\in H$, $v_0\in V_0$. Denote

$$W=U\cap T^{-N}U\cap\{x\in X:\varphi^{(N)}(x)\in g+h+V_0\}.$$

Clearly W is an open subset of X, $x_0\in W$, so $W\neq\emptyset$. Since $-h\in H\subset E(\varphi)$, there exists an integer M such that

$$W\cap T^{-M}W\cap\{x\in X:\varphi^{(M)}(x)\in -h+V_0\}\neq\emptyset.$$

We will show that each x from the above intersection is in

$$U\cap T^{-N-M}U\cap\{x\in X:\varphi^{(N+M)}(x)\in g+V\}.$$

Since $x\in W\cap T^{-M}W$, $x\in U\cap T^{-N-M}U$. Moreover, $\varphi^{(N+M)}(x)=\varphi^{(N)}(T^Mx)+\varphi^{(M)}(x)$ and $\varphi^{(M)}(x)\in -h+V_0$. On the other hand $T^Mx\in W$, so $\varphi^{(N)}(T^Mx)\in g+h+V_0$. Thus

$$\begin{aligned}\varphi^{(N+M)}(x)=\varphi^{(N)}(T^Mx)+\varphi^{(M)}(x)&\in g+h+V_0-h+V_0\\ &=g+V_0+V_0\subset g+V\end{aligned}$$

and the result follows. □

Definition 6.1.3. Let (X,T) be a flow, G a locally compact Abelian group, $\varphi\colon X\to G$ a continuous map. We say that the cocycle φ is *regular* if there exists a continuous map $f\colon X\to G$ such that all values of the cocycle $\psi=\varphi+f\circ T-f$ are in $E(\varphi)$.

From Proposition 6.1.2 we have the following corollary.

Corollary 6.1.4. *Assume that (X,T) is a flow, G a locally compact Abelian group and $\varphi\colon X\to G$ a continuous cocycle. Let $\widetilde{\varphi}\colon X\to G/E(\varphi)$ be given by $\widetilde{\varphi}(x)=\varphi(x)+E(\varphi)$. Then $E(\widetilde{\varphi})=\{0\}$. If additionally φ is regular, then also $E_\infty(\widetilde{\varphi})=\{0\}$.*

Proof. The first assertion follows from Proposition 6.1.2. If φ is regular then $\varphi=\psi+f-f\circ T$, where $\psi\colon X\to E(\varphi)$. Therefore $\widetilde{\varphi}=\widetilde{\psi}+\widetilde{f}-\widetilde{f}\circ T=\widetilde{f}-\widetilde{f}\circ T$ is a coboundary as $\widetilde{\psi}=0$. By Proposition 5.2.7, $E_\infty(\widetilde{\varphi})=\{0\}$. □

Theorem 6.1.5. *Let (X,T) be a topologically ergodic flow, G a locally compact Abelian group, $\varphi\colon X \to G$ a regular cocycle. Then $(X \times G, T_\varphi)$ is a disjoint union of topologically ergodic subflows, each of them isomorphic to $(X \times E(\varphi), T_\psi)$, where $\psi\colon X \to E(\varphi)$, $\psi = \varphi + f \circ T - f$.*

Proof. For each $g \in G$ define a map $J_g\colon X \times E(\varphi) \to X \times G$ by setting

$$J_g(x,v) = (x, v + g - f(x)).$$

We will show that the following conditions hold:

(a) T_ψ is topologically ergodic.
(b) Each J_g is a homeomorphic embedding and $J_g T_\psi = T_\varphi J_g$.
(c) $J_g(X \times E(\varphi))$ is a closed and T_φ-invariant subset of $X \times G$ for each $g \in G$.
(d) $J_g(X \times E(\varphi)) = J_h(X \times E(\varphi))$ if and only if $g - h \in E(\varphi)$.
(e) $J_g(X \times E(\varphi)) \cap J_h(X \times E(\varphi)) = \emptyset$ if and only if $g - h \notin E(\varphi)$.

ad (a). Assume that $v \in E(\varphi)$ and let $U \subset X$ be an non-empty open set, $V, V_1 \subset E(\varphi)$ open neighbourhoods of zero with $V_1 + V_1 \subset V$. Fix an open $U_1 \subset U$ such that $x', x'' \in U_1$ implies $f(x') - f(x'') \in V_1$. Then there exists an integer n such that

$$W = U_1 \cap T^{-n} U_1 \cap \{x \in X : \varphi^{(n)}(x) \in v + V_1\} \neq \emptyset.$$

We will show that

$$W \subset U \cap T^{-n} U \cap \{x \in X : \psi^{(n)}(x) \in v + V\}.$$

Take $x \in W$, then $x, T^n x \in U_1 \subset U$, so $f(T^n x) - f(x) \in V_1$. Thus

$$\psi^{(n)}(x) = \varphi^{(n)}(x) + f(T^n x) - f(x) \in v + V_1 + V_1 \subset v + V.$$

We have proved that $v \in E(\psi)$ which implies $E(\psi) = E(\varphi)$ and therefore T_ψ is topologically ergodic.

ad (b). The equality $J_g T_\psi = T_\varphi J_g$ and continuity of J_g is clear. Since $(J_g)^{-1}\colon J_g(X \times E(\varphi)) \to X \times E(\varphi)$ is given by the formula $(J_g)^{-1}(x,h) = (x, h - g + f(x))$, $(J_g)^{-1}$ is continuous.

ad (c). Since $J_g T_\psi = T_\varphi J_g$, $J_g(X \times E(\varphi))$ is T_φ-invariant. To show that $J_g(X \times E(\varphi))$ is closed assume that $(J_g(x_i, v_i))_{i \in I}$ is a convergent net of elements of $J_g(X \times E(\varphi))$, $J_g(x_i, v_i) = (x_i, v_i + g - f(x_i)) \to (x,h)$. Then $x_i \to x$, $v_i + g - f(x_i) \to h$. As f is continuous, $f(x_i) \to f(x)$ and therefore $v_i \to h - g + f(x)$. Since $E(\varphi)$ is a closed subgroup of G, $h - g + f(x) \in E(\varphi)$ and $(x,h) = J_g(x, h - g + f(x))$, so $J_g(X \times E(\varphi))$ is closed.

ad (d). If $g - h \in E(\varphi)$ and $(x,v) \in X \times E(\varphi)$, then

$$J_g(x,v) = (x, v + g - f(x)) = (x, v + (g-h) + h - f(x)) = J_h(x, v + g - h),$$

thus $J_g(X\times E(\varphi)) \subset J_h(X\times E(\varphi))$. By symmetry of arguments, $J_g(X\times E(\varphi)) = J_h(X \times E(\varphi))$. Conversely, if $J_g(X \times E(\varphi)) = J_h(X \times E(\varphi))$ then $J_g(x, 0) \in J_h(X \times E(\varphi))$ i.e. $(x, g - f(x)) = (x, v + h - f(x))$ for some $v \in E(\varphi)$, hence $g - h = v \in E(\varphi)$.

ad (e). Suppose $(x, v) \in J_g(X \times E(\varphi)) \cap J_g(X \times E(\varphi))$. Then $(x, v) = (x, u + g - f(x)) = (x, w + h - f(x))$ for some $u, w \in E(\varphi)$, hence $g - h = w - u \in E(\varphi)$ and, by (c), $g - h \in E(\varphi)$. The opposite implication is clear in view of (c) and (d) is proved.

It follows from (a)–(e) that $\{J_g(X \times E(\varphi) : g \in G\}$ is a family of pair-wise disjoint T_φ-invariant subsets of $X \times G$. Clearly these subsets cover whole $X \times G$ and all J_g's are homeomorphisms satisfying $J_gT_\psi = T_\varphi J_g$, so each $J_g(X \times E(\varphi))$ is topologically ergodic. □

Remark 6.1.6. If G a locally compact Abelian group, then there exists a closed-open subgroup H of G, that is a direct sum of a compact group and $\mathbb{R}^m$ (see for instance [77, Theorem 25]). If additionally G has no compact subgroups, then $H = \mathbb{R}^m$. Because $\mathbb{R}^m$ is a divisible group, G is a direct sum of $\mathbb{R}^m$ and $G/\mathbb{R}^m$. Note that the latter group is always discrete.

Lemma 6.1.7. *Let (X,T) be a compact flow and $\varphi: X \to \mathbb{R}^m$ a cocycle. If for some unbounded sequence $(n_t)_{t\geqslant 1}$ of integers, the sequence $\left(\varphi^{(n_t)}\right)_{n\geqslant 1}$ converges uniformly to a constant $a \in \mathbb{R}^m$, then $a = 0$.*

Proof. Let μ be a probability T-invariant measure on X. Then clearly $\int_X \varphi^{(n_t)}\, d\mu \to a$ as $t \to \infty$. Because $\int_X \varphi^{(n_t)}\, dx = n_t \int_X \varphi\, dx$, $n_t \int_X \varphi\, dx \to a$. This implies $a = 0$. □

The following lemma will be essentially helpful in proof of the very important Lemma 6.2.4.

Lemma 6.1.8. *Let (X,T) be a compact flow and G a locally compact Abelian group with no non-trivial compact subgroup. Assume that $\varphi: X \to G$ is continuous. If $\varphi^{(n_t)} \to g \in G$ uniformly for some unbounded sequence of integers $(n_t)_{t\geqslant 1}$, then $g = 0$.*

Proof. Let $G = \mathbb{R}^m \oplus D$, where D is a discrete group without compact subgroups. Denote $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1: X \to \mathbb{R}^m$, $\varphi_2: X \to D$. Then $\varphi^{(k)} = \varphi_1^{(k)} + \varphi_2^{(k)}$ for any integer k and therefore $\varphi_1^{(n_t)} \to g_1$, $\varphi_2^{(n_t)} \to g_2$, where $g = g_1 + g_2$, $g_1 \in \mathbb{R}^m$, $g_2 \in D$. By Lemma 6.1.7, $g_1 = 0$. Consider now $\varphi_2: X \to D$. For the converse suppose $g_2 \neq 0$. We have $\varphi_2^{(n_t)} \to g_2$ uniformly. Since D is discrete, $\varphi_2^{(n_t)} \equiv g_2$ for t large enough. Fix such a t. For $k \geqslant 1$ we can find integers $s = s_k$, $r = r_k$ such that $n_{t+k} = sn_t + r$. Then applying s times the cocycle equality $\psi^{(m+n)} = \psi^{(m)} \circ T^n + \psi^{(n)}$ we get

$$
\begin{aligned}
g_2 &= \varphi_2^{(n_{t+k})} = \varphi_2^{(sn_t+r)} \\
&= \varphi_2^{(n_t)} \circ T^{(s-1)n_t+r} + \varphi_2^{(n_t)} \circ T^{(s-2)n_t+r} + \ldots + \varphi_2^{(n_t)} \circ T^r + \varphi_2^{(r)} \\
&= sg_2 + \varphi_2^{(r)}.
\end{aligned}
$$

Because r is just the remainder of the division by n_t, there exist only finitely many values of r and therefore the set of values of all $\varphi_2^{(r)}$, $r = 0, 1, \ldots, n_t$, is bounded. On the other hand, the group D has no compact subgroups and therefore $sg_2 \to \infty$, which gives a contradiction and the lemma is proved. □

Lemma 6.1.9. *Let $v \in \mathbb{R}^m$, $v \neq 0$. There exists an open bounded set V, $\mathbb{R}^m \supset V \ni v$, such that for each bounded set $A \subset \mathbb{R}^n$ there exists a positive integer $p > 0$ satisfying the following condition:*

If $s_1, s_2 \in \mathbb{N}$, $s_2 \geqslant 3s_1 + p$, then $A \cap (s_2V - s_1V) = \emptyset$.

Proof. Let V be an open ball with center in v and of radius $r = \|v\|/2$. Then we have

$$
\sup\{\|x\| : x \in V\} = \frac{3}{2}\|v\|, \quad \inf\{\|x\| : x \in V\} = \frac{1}{2}\|v\|.
$$

Let $A \subset \mathbb{R}^m$ be a bounded set. Denote $M = \sup\{\|x\| : x \in A\}$. Choose p satisfying $(1/2)\|v\| \cdot p > M$. Now, let s_1, s_2 be such integers that $s_2 \geqslant 3s_1 + p$. We will show that $A \cap (s_2V - s_1V) = \emptyset$. To do this take $v_1, \ldots, v_{s_2}, u_1, \ldots, u_{s_1} \in V$. Then $v_1/s_2 + \ldots + v_{s_2}/s_2 \in V$ and we have

$$
\begin{aligned}
\|v_1 + \ldots + v_{s_2} - u_1 - \ldots - u_{s_1}\| &\geqslant \|v_1 + \ldots + v_{s_2}\| - \|u_1 + \ldots + u_{s_1}\| \\
&= s_2 \|\frac{1}{s_2} v_1 + \ldots + \frac{1}{s_2} v_{s_2}\| - \|u_1 + \ldots + u_{s_1}\| \\
&\geqslant s_2 \cdot \frac{1}{2}\|v\| - \|u_1\| - \ldots - \|v_{s_1}\| \\
&\geqslant s_2 \cdot \frac{1}{2}\|v\| - s_1 \cdot \frac{3}{2}\|v\| = (s_2 - 3s_1) \cdot \frac{1}{2}\|v\| \geqslant p \cdot \frac{1}{2}\|v\| > M.
\end{aligned}
$$

Thus $v_1 + \ldots + v_{s_2} - u_1 - \ldots - u_{s_1}$ cannot belong to A and the proof is complete. □

Lemma 6.1.10. *Let (X, T) be a compact flow, G a locally compact Abelian group, $G_0 = K \oplus \mathbb{R}^m$ an open subgroup of G with K compact. Assume that $\varphi: X \to G$ is a continuous map. If $\varphi^{(n_t)} \to g \in G_0$ uniformly for some increasing (decreasing) sequence $(n_t)_{t \geqslant 1}$ of integers, then $g \in K$.*

Proof. Let $g = k + v$, where $k \in K$, $v \in \mathbb{R}^m$. Assume that $v \neq 0$. Then we can find an open bounded set V, $v \in V \subset \mathbb{R}^m$, satisfying Lemma 6.1.9. Let $U \subset K$ be an open neighbourhood of k. There exists a $t_0 > 0$ such that

$$
\varphi^{(n_t)}(X) \subset U \oplus V. \tag{6.1}
$$

for any $t \geqslant t_0$. Fix $t \geqslant t_0$ and take $l > 0$. Then $n_{t+l} = s_l n_t + r_l$, where $0 \leqslant r_l < n_t$. Moreover, by the cocycle equation, for each $x \in X$

$$\varphi^{(n_{t+l})}(x) = \varphi^{(n_t)}(T^{(s_l-1)n_t+r_l}x) + \varphi^{(n_t)}(T^{(s_l-2)n_t+r_l}x) + \ldots + \varphi^{(n_t)}(T^{r_l}x) + \varphi^{(r_l)}(x),$$

and by (6.1),

$$\varphi^{(n_{t+l})}(x) \in U \oplus V, \qquad \varphi^{(n_{t+l})}(x) \in s_l(U \oplus V) + \varphi^{(r_l)}(x).$$

Because r_l takes only finitely many values, so does $\varphi^{(r_l)}(x)$, $l \geqslant 1$. Therefore we can find $0 < l_1 < l_2$ such that $l_2 - 3l_1 > p_0$, where p_0 is given by Lemma 6.1.9 for $A = V - V$, and $r_{l_1} = r_{l_2}$. Then

$$\varphi^{(n_t+l_1)}(x) \in s_{l_1}(U \oplus V) + \varphi^{(r_{l_1})}(x),$$
$$\varphi^{(n_t+l_2)}(x) \in s_{l_2}(U \oplus V) + \varphi^{(r_{l_2})}(x).$$

This implies

$$\varphi^{(n_t+l_2)}(x) - \varphi^{(n_t+l_1)}(x) \in s_{l_2}(U \oplus V) - s_{l_1}(U \oplus V)$$

and therefore

$$(V - V) \cap (s_{l_2}V - s_{l_1}V) \neq \emptyset,$$

which contradicts Lemma 6.1.9. Thus $v = 0$ and $g \in K$. □

Lemma 6.1.11. *Let (X, T) be a compact flow, G a locally compact Abelian group, $G_0 = K \oplus \mathbb{R}^m$ an open subgroup of G with K compact, $\varphi: X \to G$ a continuous map. Assume that $(n_t)_{t\geqslant 1}$ is an increasing (decreasing) sequence of integers such that $\varphi^{(n_t)} \to g \in G$ uniformly. Then $rg \in K$ for some non-zero integer r.*

Proof. Let $\overline{\varphi}: X \to G/G_0$, $\overline{\varphi}(x) = \varphi(x) + G_0$. Then $\overline{\varphi}^{(n_t)} \to g + G_0$ uniformly. Since G/G_0 is discrete, $\overline{\varphi}^{(n_t)}(X) = \{g + G_0\}$ for t large enough. Fix such a t, then for each $l = 1, 2, \ldots$, $n_{t+l} = s_l n_t + r_l$, $0 \leqslant r_l < n_t$. By the cocycle equation,

$$\varphi^{(n_{t+l})}(x) = \varphi^{(n_t)}(T^{(s_l-1)n_t+r_l}x) + \varphi^{(n_t)}(T^{(s_l-2)n_t+r_l}x) + \ldots + \varphi^{(n_t)}(T^{r_l}x) + \varphi^{(r_l)}(x),$$

hence

$$g + G = s_l(g + G_0) + \overline{\varphi}^{(r_l)}(x), \quad l = 1, 2, 3, \ldots$$

Choose $l_1 < l_2$ such that $r_{l_1} = r_{l_2}$. Then $0 = (s_{l_2} - s_{l_1})(g + G_0)$, so $(s_{l_2} - s_{l_1})g \in G_0$. Denote $r = s_{l_2} - s_{l_1} > 0$. Let $\psi: X \to G$, $\psi(x) = r\varphi(x)$, $x \in X$. Then $\psi^{(n_t)} = r\varphi^{(n_t)} \to rg \in G_0$ uniformly. By Lemma 6.1.10, $rg \in K$. □

Definition 6.1.12. If A, B are topological spaces, $\pi: A \to B$ a continuous map with $\pi(A) = B$, then a continuous map $s: B \to A$ is called a *continuous selector* for π if s satisfies $\pi(s(y)) = y$ for all $y \in B$.

Lemma 6.1.13. *Let (X,T) be a flow, G a locally compact Abelian group, $\varphi: X \to G$ a continuous map. Let $\widetilde{\varphi}: X \to G/E(\varphi)$, $\widetilde{\varphi}(x) = \varphi(x) + E(\varphi)$. If $E_\infty(\widetilde{\varphi}) = \{0\}$ and there exists a continuous selector for the natural quotient map $G \to G/E(\varphi)$, then φ is regular.*

Proof. Let s be a selector for the quotient map $G \to G/E(\varphi)$. Since $E_\infty(\varphi) = \{0\}$, $\widetilde{\varphi}$ is a coboundary (Proposition 5.2.7), $\widetilde{\varphi} = \widetilde{f} \circ T - \widetilde{f}$, where $\widetilde{f}: X \to G/E(\varphi)$. Define $f: X \to G$ by $f(x) = s(\widetilde{f}(x))$. Then $\varphi(x) - f(Tx) + f(x) = \varphi(x) - s(\widetilde{f}(Tx)) + s(\widetilde{f}(x)) \in E(\varphi)$ and φ is regular. □

Remark 6.1.14. In the following cases the natural quotient maps $G \to G/H$, where G, $H \subset G$ are topological Abelian groups, admit continuous selectors:

(a) G/H is a discrete group;
(b) $H = \mathbb{R}^m$ for some integer $m \geqslant 0$.

6.2. The groups of essential values for extensions of minimal rotations

In this section we will concentrate on the following situation: $T: X \to X$ is a minimal rotation, X a compact metric monothetic group, G a locally compact Abelian group.

Lemma 6.2.1. *Let T be a minimal rotation on a compact metric monothetic group X, D a discrete Abelian group, $\varphi: X \to D$ a continuous map. If $(n_t)_{t\geqslant 1}$ is a rigidity time for T then*

$$\underset{t_0}{\exists}\ \underset{t\geqslant t_0}{\forall}\ \underset{d_t}{\exists}\ \underset{x\in X}{\forall}\ \varphi^{(n_t)}(x) = d_t$$

i.e. each $\varphi^{(n_t)}$ is a constant function for t large enough.

Proof. Let $(n_t)_{t\geqslant 1}$ be a rigidity time for T. Then $T^{n_t} \to \mathrm{Id}$ uniformly. Since φ is continuous and D is discrete, there exists a t_0 such that $\varphi(T^{n_t}x) = \varphi(x)$ for all $t \geqslant t_0$ and each $x \in X$. Fix $x_0 \in X$ and $t \geqslant t_0$. Then $\varphi^{(n_t)}(T^{i+1}x_0) - \varphi^{(n_t)}(T^i x_0) = \varphi(T^{n_t+i}x_0) - \varphi(T^i x_0) = 0$ for all $i \in \mathbb{Z}$, so $\varphi^{(n_t)}(T^i x_0) = \varphi^{(n_t)}(T^{i+1}x_0)$, $i \in \mathbb{Z}$. It follows from the minimality of T that $\varphi^{(n_t)} = \mathrm{const} = d_t \in D$. □

The characterization below of essential values can be found in [65].

Proposition 6.2.2. *Let T be a minimal rotation on a compact metric monothetic group X, G a locally compact Abelian group, $\varphi: X \to G$ a continuous map. Assume that $0 \neq g \in G_\infty$. Then $g \in E_\infty(\varphi)$ if and only if there exists a rigidity time $(n_t)_{t\geqslant 1}$ and a sequence $(x_t)_{t\geqslant 1}$ of elements of X such that $\varphi^{(n_t)}(x_t) \to g$.*

Proposition 6.2.3. *Let T be a minimal rotation on a compact metric monothetic group X, G a locally compact Abelian group, $\varphi\colon X \to G$ a continuous map. Let $H \subset G$ be a closed subgroup such that $H/E(\varphi) \cap H$ is compact. Put $\varphi_H\colon X \to G/H$, $\varphi_H(x) = \varphi(x) + H$. Then $E(\varphi_H)$ is naturally isomorphic to $E(\varphi)/E(\varphi) \cap H$. Moreover, $\infty \in E_\infty(\varphi_H)$ if and only if $\infty \in E_\infty(\varphi_{E(\varphi)\cap H})$.*

Proof. Let $H_0 = E(\varphi) \cap H$ and let $\pi\colon G/H_0 \to G/H$ be the natural quotient map. We will show that π restricted to $E(\varphi)/H_0$ is a group isomorphism of $E(\varphi)/H_0$ and $E(\varphi_H)$. Suppose first that $g+H_0 \in E(\varphi)+H_0$, then $g \in E(\varphi)$ and, by Proposition 6.2.2, there exists a rigidity time $(n_t)_{t\geqslant 1}$ and a sequence $(x_t)_{t\geqslant 1}$ of elements of X such that $\varphi^{(n_t)}(x_t) \to g$. Then $\varphi^{(n_t)}(x_t)+H \to g+H = \pi(g+H_0)$ and $\pi(g+H_0) \in E(\varphi_H)$. Now, let $g \in E(\varphi)$ and $\pi(g + H_0) = g + H = H$. Then $g \in H$ so $g \in E(\varphi) \cap H = H_0$. Thus π is one-to-one on $E(\varphi)/H_0$. To show that $\pi(E(\varphi)/H_0) = E(\varphi_H)$ take $g + H \in E(\varphi_H)$. Then $\varphi^{(n_t)}(x_t) + H \to g + H$ for some rigidity time $(n_t)_{t\geqslant 1}$ and a sequence $(x_t)_{t\geqslant 1}$ of elements of X. Let V be a neighbourhood of $g + H$ in G/H with $\overline{V}$ being compact. Since the kernel H/H_0 of π is compact, $\pi^{-1}(\overline{V}) \subset G/H_0$ is compact as well. Therefore there exists a converging subsequence $\big(\varphi^{(n_{s_t})}(x_{s_t})+H_0\big)_{t\geqslant 1}$ of the sequence $\big(\varphi^{(n_t)}(x_t)+H_0\big)_{t\geqslant 1}$, $\varphi^{(n_{s_t})}(x_{s_t}) + H_0 \to g_0 + H_0 \in G/H_0$. By Proposition 6.2.2, $g_0 + H_0 \in E(\varphi_{H_0}) = E(\varphi)/H_0$ (Proposition 6.1.2). Since $g_0 + H = g + H$, $g + H = \pi(g_0 + H_0)$ with $g_0+H_0 \in E(\varphi)/H_0$, π restricted to $E(\varphi)/H_0$ is onto $E(\varphi_H)$. Collecting results we get that $E(\varphi_H)$ is naturally isomorphic to $E(\varphi)/E(\varphi)\cap H$ and the first assertion is proved.

We will show the second assertion. Assume that $\infty \in E_\infty(\varphi_{H_0})$. Then $\varphi^{(n_t)}(x_t) + H_0 \to \infty$ for some rigidity time $(n_t)_{t\geqslant 1}$ and $x_t \in X$, $t \geqslant 1$. Now, if $(h_t)_{t\geqslant 1}$ is an arbitrary sequence of elements of H, then, as H/H_0 is compact, we may assume that $(h_t + H_0)_{t\geqslant 1}$ converges, $h_t + H_0 \to h + H_0$. Then $\varphi^{(n_t)}(x_t)+h_t+H_0 \to h+\infty = \infty$. Since $(h_t)_{t\geqslant 1}$ was arbitrary, $\varphi^{(n_t)}(x_t)+H \to \infty$, hence $\infty \in E_\infty(\varphi_H)$. Conversely, if $\infty \in E_\infty(\varphi_H)$ then $\varphi^{(n_t)}(x_t) + H \to \infty$ for some rigidity time $(n_t)_{t\geqslant 1}$ and $x_t \in X$, $t \geqslant 1$, hence $\varphi^{(n_t)}(x_t) + H_0 \to \infty$ and $\infty \in E_\infty(\varphi_{H_0})$. □

Lemma 6.2.4. *Let T be a minimal rotation on a compact metric monothetic group X, G a locally compact Abelian group with no non-trivial compact subgroup, $\varphi\colon X \to G$ a continuous map. If $E(\varphi) \neq \{0\}$, then no point in $E(\varphi)$ is isolated.*

Proof. Assume that $0 \neq g \in G$ is an isolated element of $E(\varphi)$. By Proposition 6.2.2, $g = \lim_{t\to\infty} \varphi^{(n_t)}(x_t)$, where $(n_t)_{t\geqslant 1}$ is a rigidity time for T, $x_t \in X$, $t \geqslant 1$. It follows from Lemma 6.1.8 that $\varphi^{(n_t)} \not\to g$ uniformly. As g is isolated, there exists an open set $0 \in V \subset G$ such that $\overline{V}$ is compact and $E(\varphi) \cap [g + (V + V)] = \{g\}$. Moreover, we may assume that

$$\forall_{t\geqslant 1}\ \exists_{z\in X}\ \varphi^{(n_t)}(z) \notin g + (V + V). \tag{6.2}$$

Find an open symmetric set V_1, $0 \in V_1 \subset G$, such that $V_1 + V_1 \subset V$ and put

$$K = g + (\overline{V} \setminus V_1).$$

Clearly K is a compact set, $K \cap E(\varphi) = \emptyset$. By Proposition 5.2.5, there exists an open non-empty set $U \subset X$ such that

$$\forall_{n\in\mathbb{Z}} \; U \cap T^{-n}U \cap \{x \in X : \varphi^{(n)} \in K\} = \emptyset. \tag{6.3}$$

Let d be an invariant metric on X. Fix $x_0 \in U$ and $\delta > 0$ such that the ball with center in x_0 and of radius 2δ is included in U, and the following condition

$$d(x, x') < \delta \Rightarrow \varphi(x) - \varphi(x') \in V_1$$

is valid. Let B be the ball with center in x_0 and of radius $\delta/2$. Then B with its $(3/2)\delta$-neighbourhood is included in U. Let M be such a positive integer that

$$\forall_{x\in X} \; \exists_{0\leqslant i\leqslant M-1} \; T^i x \in B. \tag{6.4}$$

Such an M exists as T is minimal.

Let $W \subset G$ be such an open symmetric neighbourhood of zero that $M \cdot W \subset V_1$. Fix a t satisfying

$$\forall_{x\in X} \; \varphi(T^{n_t}x) - \varphi(x) \in W, \tag{6.5}$$

$$\forall_{x\in X} \; d(T^{n_t}x, x) < \delta, \tag{6.6}$$

$$\varphi^{(n_t)}(x_t) \in g + W. \tag{6.7}$$

Let z be given by (6.2) for the fixed t. Since the positive part of the orbit of x_t is dense in X ($\overline{\{T^n x_t : n \geqslant 1\}} = X$), there exists a positive l such that $\varphi^{(n_t)}(T^l x_t) - \varphi^{(n_t)}(z) \in V_1$. Then $\varphi^{(n_t)}(T^l x_t) \notin g + V$. Let k be the smallest positive integer such that $\varphi^{(n_t)}(T^k x_t) \notin g + V$. Then $\varphi^{(n_t)}(T^{k-1}x_t) \in g + V$. For each i we have

$$\varphi^{(n_t)}(T^{k-i+1}x_t) - \varphi^{(n_t)}(T^{k-i}x_t) = \varphi(T^{n_t+k-i}x_t) - \varphi(T^{k-i}) \in W \tag{6.8}$$

by (6.5).

Now observe that $k > M$. Indeed, if this is not the case then

$$\begin{aligned}\varphi^{(n_t)}(T^k x_t) - \varphi^{(n_t)}(x_t) &= \sum_{j=0}^{k-1}[\varphi^{(n_t)}(T^{j+1}x_t) - \varphi^{(n_t)}(T^j x_t)]\\ &= \sum_{j=0}^{k-1}\left[\varphi(T^j x_t) - \varphi(T^{n_t+j}x_t)\right] \in k \cdot W \subset M \cdot W \subset V_1\end{aligned}$$

by (6.8). Then, by (6.7),

$$\varphi^{(n_t)}(T^k x_t) \in \varphi^{(n_t)}(x_t) + V_1 \subset g + W + V_1 \subset g + V_1 + V_1 \subset g + V,$$

which is impossible because of the choice of k. Therefore $k > M$.

Consider now the points $T^{k-i}x_t$, $i = 1, \dots, M$. Since $k > M$, all differences $k - i$, $i = 1, \dots, M$, are positive and $\varphi^{(n_t)}(T^{k-i}x_t) \in g + V$, $i = 1, \dots, M$. By (6.4), at least one of these $T^{k-i}x_t$ is in B, say $T^{k-j}x_t \in B$. Put $y = T^{k-j}x_t$. We will show that

$$y \in U \cap T^{-n_t}U \cap \{x \in X : \varphi^{(n_t)} \in K\},$$

which will give a contradiction to (6.3). By our choice of y, $y \in B \subset U$. By (6.6), $d(T^{n_t}y, y) < \delta$ so $T^{n_t}y$ belongs to the δ-neighbourhood of B. By definition of δ, $T^{n_t}y \in U$ i.e. $y \in T^{-n_t}y$. To finish the proof observe that $\varphi^{(n_t)}(y) \notin g + V_1$. Indeed, $y = T^{k-j}x_t$, where $j \leqslant M$. By (6.8),

$$\varphi^{(n_t)}(y) - \varphi^{(n_t)}(T^k x_t) = \varphi^{(n_t)}(T^{k-j}x_t) - \varphi^{(n_t)}(T^k x_t) \in j \cdot W \subset M \cdot W \subset V_1,$$

so $\varphi^{(n_t)}(T^k x_t) \in \varphi^{(n_t)}(y) + V_1$.

If $\varphi^{(n_t)}(y) \in g + V_1$ then $\varphi^{(n_t)}(T^k x_t) \in g + V_1 + V_1 \subset g + V$, that is not true. Thus $\varphi^{(n_t)}(y) \notin g + V_1$, $\varphi^{(n_t)}(y) \in g + V$, so

$$\varphi^{(n_t)}(y) \in g + (V \setminus V_1) \subset g + (\overline{V} \setminus V_1) = K,$$

which finishes the proof. □

Now we are in a position to formulate a theorem describing all possible groups of essential values for cocycles $\varphi: X \to G$ over minimal rotations in the case when G has no compact subgroups. By Remark 6.1.6, such a group is a direct sum of $\mathbb{R}^m$ and a discrete group.

Remark 6.2.5. If G is a closed subgroup of $\mathbb{R}^m$, then, by [77, Theorem 6], G is of the form

$$G = \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_l \oplus \mathbb{R}v_1 \oplus \ldots \oplus \mathbb{R}v_k,$$

where $w_1, \dots, w_l, v_1, \dots, v_k \in \mathbb{R}^m$ are linearly independent vectors.

Theorem 6.2.6. *Assume that T is a minimal rotation on a compact metric monothetic group X, G a locally compact Abelian group without compact subgroups. If $\varphi: X \to G$ is a continuous map then $E(\varphi)$ is a linear subspace of $\mathbb{R}^m \subset G$.*

Proof. First we will show that $E(\varphi) \subset \mathbb{R}^m$, where $\mathbb{R}^m \subset G$ is an open subgroup of G. To do this we will use Proposition 6.1.2 for $H = E(\varphi) \cap \mathbb{R}^m$. Since $G/\mathbb{R}^m$ is discrete, so is $E(\varphi)/H$. By Lemma 6.2.4, $E(\varphi) = H \subset \mathbb{R}^m$ and, by Remark 6.2.5,

$$E(\varphi) = \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_l \oplus \mathbb{R}v_1 \oplus \ldots \oplus \mathbb{R}v_k$$

for some linearly independent vectors $w_1, \dots, w_l, v_1, \dots, v_k$, $l + k \leqslant m$.

Now apply Proposition 6.1.2 for $H = \mathbb{R}v_1 \oplus \ldots \oplus \mathbb{R}v_k$ to get that $E(\widetilde{\varphi}) = \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_l$, i.e. that $E(\widetilde{\varphi})$ turns out to be a discrete group. By Lemma 6.2.4, $l = 0$ and $E(\varphi) = \mathbb{R}v_1 \oplus \ldots \oplus \mathbb{R}v_k$ is a linear subspace of $\mathbb{R}^m$. □

6.3. Atkinson's theorem and regularity of cylinder flows

In the main theorem in [6, Theorem 1] a condition for a conservative cylinder flow over a minimal rotation on a torus to be point transitive is given. We will generalize this theorem for cylinder flows being extension of any minimal rotation on a compact monothetic metric group. In Atkinson's proof the fact that torus is a connected space was used. In our proof we will omit this property using a method of "short steps", introduced in Chapter 5 in the proof of Proposition 5.3.1. Nevertheless, both our proof of Proposition 6.3.7 and of the lemmas proceeding this theorem are modifications of the ones in [6]. After showing Proposition 6.3.7 we will use it to prove Theorem 6.3.8 giving several conditions equivalent to conservativity.

We start with a version of [6, Lemma 4]. The differences are that a torus is replaced with any minimal rotation and a sphere $\{v \in \mathbb{R}^m : \|v\| = r\}$ by a ring $K(a,b) = \{v \in \mathbb{R}^m : a \leqslant \|v\| \leqslant b\}$, where $\|\cdot\|$ denotes a norm in $\mathbb{R}^m$. Our assumption that $E(\varphi) = \{0\}$ is not essential in view of Theorem 6.2.6.

Lemma 6.3.1. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi\colon X \to \mathbb{R}^m$ a continuous map such that $E(\varphi) = \{0\}$. Then for any positive real numbers $a < b$ there exists a positive $\delta = \delta(a,b)$ such that if $d(T^n, \mathrm{Id}) < \delta$ then either $\varphi^{(n)}(X) \subset B(0,a)$ or $\varphi^{(n)}(X) \subset \overline{B(0,b)}^c = X \times \mathbb{R}^m \setminus \overline{B(0,b)}$.*

Proof. For any $x_0 \in X$ and $i = 1,2,3,\ldots$ let

$$A_i(x_0) = \overline{\{\varphi^{(n)}(x) : x \in B(x_0, i^{-1}),\ d(T^n, \mathrm{Id}) < i^{-1}\}},$$

$$A(x_0) = \bigcap_{i=1}^{\infty} A_i(x_0).$$

By virtue of Proposition 5.3.1, $A(x_0) \subset E(\varphi)$, so either $A(x_0) = \emptyset$ or $A(x_0) = \{0\}$. In particular, if we put

$$K(a,b) = \{v \in \mathbb{R}^m : a \leqslant \|v\| \leqslant b\} = \overline{B(0,b)} \setminus B(0,a),$$

then $A(x_0) \cap K(a,b) = \emptyset$ i.e.

$$\bigcap_{i=1}^{\infty} [A_i(x_0) \cap K(a,b)] = \emptyset.$$

Since $A_1(x_0) \supset A_2(x_0) \supset A_3(x_0) \supset \ldots$ and $K(a,b)$ is a compact set, some of the sets $A_i(x_0) \cap K(a,b)$ must be empty. Let $A_j \cap K(a,b) = \emptyset$. In particular

$$\{\varphi^{(n)}(x) : x \in B(x_0, j^{-1}),\ d(T^n, \mathrm{Id}) < j^{-1}\} \cap K(a,b) = \emptyset.$$

This is true for each $x_0 \in X$, because

$$X = \bigcup_{x \in X} B(x, i_x^{-1}),$$

where for each $x \in X$, $\{\varphi^{(n)}(y) : y \in B(x, i_x^{-1}),\ d(T^n, \mathrm{Id}) < i_x^{-1}\} \cap K(a,b) = \emptyset$. Since X is compact, there are points $x_1, \dots, x_k$ and integers $i_1, \dots, i_k$ such that

$$X = \bigcup_{j=1}^{k} B(x_j, i_j^{-1})$$

and for each $j = 1, \dots, k$,

$$\{\varphi^{(n)}(y) : y \in B(x_j, i_j^{-1}),\ d(T^n, \mathrm{Id}) < i_j^{-1}\} \cap K(a,b) = \emptyset.$$

The map φ is uniformly continuous, thus there exists a $\delta_1 > 0$ such that for any $x', x'' \in X$, if $d(x', x'') < \delta_1$ then $\|\varphi(x') - \varphi(x'')\| < b - a$. Let

$$\delta = \delta(a,b) = \min\{\delta_1, i_1^{-1}, \dots, i_k^{-1}\}.$$

Then $\{\varphi^{(n)}(y) : y \in B(x_j, i_j^{-1}),\ d(T^n, \mathrm{Id}) < \delta\} \cap K(a,b) = \emptyset,\ j = 1, \dots, k$. Since the balls $B(x_j, i_j^{-1})$, $j = 1, \dots, k$, cover whole X, for each $x \in X$ and for each n satisfying $d(T^n, \mathrm{Id}) < \delta$ we have $\varphi^{(n)}(x) \notin K(a,b)$.

Now fix n satisfying $d(T^n, \mathrm{Id}) < \delta$ and assume that $\|\varphi^{(n)}(x)\| < a$ for some $x \in X$. Then for any integer j,

$$\varphi^{(n)}(T^{j+1}x) - \varphi^{(n)}(T^j x) = \varphi(T^{n+j}x) - \varphi(T^j x) = \varphi(T^n(T^{j+1}x)) - \varphi(T^j x).$$

Since $d(T^n, \mathrm{Id}) < \delta$, $d(T^n(T^j x), T^j x) < \delta$, hence

$$\|\varphi(T^n(T^{j+1}x)) - \varphi(T^j x)\| < b - a,$$

and therefore $\|\varphi^{(n)}(T^{j+1}x) - \varphi^{(n)}(T^j x)\| < b - a$. This means that the distances between consecutive points of the sequence $(\varphi^{(n)}(T^j x))_{j \in \mathbb{Z}}$ are less than $b - a$. As none of these points is in $K(a,b)$ and $\|\varphi^{(n)}(x)\| < a$, all $\varphi^{(n)}(T^j x)$, $j \in \mathbb{Z}$, are in $B(0,a)$. As $\varphi^{(n)}$ is a continuous map, $\varphi^{(n)}(X) \subset B(0,a)$. □

The next lemma is a version of [6, Theorem 2] for $m = 1$ only, however for an arbitrary minimal rotation on a compact monothetic group. Moreover, we formulate our lemma as a necessary and sufficient condition and do not assume that the map φ has bounded variation. In Theorem 6.3.8 we will prove such a result for any m.

Lemma 6.3.2. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi\colon X \to \mathbb{R}$ be a continuous map. Then T_φ is conservative if nd only if $\int_X \varphi\, d\mu = 0$, where μ is the normalized Haar measure on X.*

Proof. Assume that $\int_X \varphi\, d\mu \neq 0$, for instance that $\int_X \varphi\, d\mu > 0$. We will show that some point in $X \times \mathbb{R}$ is wandering. Fix an arbitrary $x \in X$; we will show that the point $(x,0) \in X \times \mathbb{R}$ is wandering for T_φ. Since $(1/n)\varphi^{(n)} \to \int_X \varphi\, d\mu$ uniformly,

$$\underset{n_0>0}{\exists} \; \underset{n>n_0}{\forall} \; \underset{y\in X}{\forall} \; \varphi^{(n)}(y) > 1. \tag{6.9}$$

Find an open set $U \subset X$ such that $x \in U$ and the sets T^iU, $0 \leqslant i \leqslant n_0$, are pair-wise disjoint. Let $V = (-1/3, 1/3) \subset R$. We will show that the sets $T_\varphi^n(U \times V)$, $n \geqslant 0$, are pair-wise disjoint. Suppose this is not the case and denote by k a positive integer satisfying

$$(U \times V) \cap T_\varphi^k(U \times V) \neq \emptyset.$$

Then $U \cap T^kU \neq \emptyset$, so $k > n_0$. Moreover, there exist $v, v' \in V$ and $x' \in U$ such that $v = \varphi^{(k)}(x') + v'$, equivalently, $\varphi^{(k)}(x') = v - v'$. It follows from (6.9), that $\varphi^{(k)}(x') > 1$. On the other hand $v - v' < 1/3 - (-1/3) = 2/3 < 1$, which gives a contradiction. Thus the sets $T_\varphi^n(U \times V)$, $n \geqslant 0$, are pair-wise disjoint, so $(x,0)$ is a wandering point for T_φ. We have proved that T_φ is not conservative.

Assume now that $\int_X \varphi\, d\mu = 0$. Then φ is either a coboundary or ergodic (Corollary 5.4.9). If φ is a coboundary, $X \times \mathbb{R}$ is a union of compact minimal sets, in particular each point of $X \times \mathbb{R}$ is almost periodic. Such points can not be wandering, so T_φ is conservative. In the later case (φ is ergodic), T_φ is point transitive – there exists a point (x_0, r_0) with dense orbit. Let $U \subset X \times \mathbb{R}$ be any non-empty open set. Without loosing generality we may assume that $(x_0, r_0) \in U$. Then, because $X \times \mathbb{R}$ is a perfect set, $T_\varphi^n(x_0, r_0) \in U$ for some $n \neq 0$, thus $U \cap T_\varphi^n U \neq \emptyset$. Since U was arbitrary, T_φ is conservative. □

To make this chapter self-contained we will give proofs of the next three lemmas, that are generalizations of [6, Lemma 5, Lemma 6, Lemma 7] to the case of any minimal rotation on a compact metric monothetic group, despite, by virtue of Lemma 6.3.1, the original proofs work also in this case.

Lemma 6.3.3. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi: X \to \mathbb{R}^m$ a continuous map such that $E(\varphi) = \{0\}$, $(n_t)_{t\geqslant 1}$ a sequence of integers. The following conditions are equivalent:*

(a) *For all $(x,v) \in X \times \mathbb{R}^m$ the sequence $(T_\varphi^{n_t}(x,v))_{t\geqslant 1}$ converges.*
(b) *For some $(x_0,v_0) \in X \times \mathbb{R}^m$ the sequence $(T_\varphi^{n_t}(x_0,v_0))_{t\geqslant 1}$ converges.*
(c) *The sequence of functions $(\varphi^{(n_t)})_{t\geqslant 1}$ converges uniformly and $(T^{n_t})_{t\geqslant 1}$ converges.*

Proof. The implications from (a) to (b) and from (c) to (a) are clear. Suppose that (b) is true, then the sequences $(\varphi^{(n_t)}(x_0))_{t\geqslant 1}$ and $(T^{n_t}x_0)_{t\geqslant 1}$ converge. In particular, they are Cauchy sequences. Now fix $\varepsilon > 0$ and, using Lemma 6.3.1, find $\delta = \delta(\varepsilon, 2\varepsilon)$. Let N be such that for $i, j > N$ the following equalities hold:

$$\|\varphi^{n_i}(x_0) - \varphi^{(n_j)}(x_0)\| < \varepsilon, \quad d(T^{n_i}x_0, T^{n_j}x_0) < \delta.$$

Since $d(T^{n_i}x_0, T^{n_j}x_0) = d(T^{n_i-n_j}, \mathrm{Id})$ and, by the cocycle identity, $\varphi^{(n_i)}(x_0) - \varphi^{(n_i)}(x_0) = \varphi^{(n_i-n_j)}(T^{n_j}x_0)$,

$$\|\varphi^{(n_i-n_j)}(T^{n_j}x_0)\| < \varepsilon, \quad d(T^{n_i-n_j}, \mathrm{Id}) < \varepsilon, \quad i, j > N.$$

By virtue of Lemma 6.3.1, $\varphi^{(n_i-n_j)}(X) \subset B(0, \varepsilon)$ for $i, j > N$. This implies that the sequence $(\varphi^{(n_t)})_{t\geqslant 1}$ is uniformly Cauchy, therefore uniformly convergent. □

Lemma 6.3.4. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi: X \to \mathbb{R}^m$ a continuous map such that $E(\varphi) = \{0\}$. Then every orbit closure under T_φ is minimal.*

Proof. Let $(x, r) \in \overline{\mathrm{Orb}}(x_0, v_0)$, then $T_\varphi^{n_i}(x_0, v_0) \to (x, v)$, that means $T^{n_i}x_0 \to x$ and $\varphi^{(n_i)}(x_0) + v_0 \to v$. Since T is a rotation, there exists an $a \in X$ such that $Tz = az$ for all $z \in X$. Now, $T^{n_i}x_0 = a^{n_i}x_0 \to x$, so $a^{n_i} \to xx_0^{-1}$, hence $a^{-n_i} \to x_0x^{-1}$, which is equivalent to $T^{-n_i}x = a^{-n_i}x \to x_0$. We will prove that $\varphi^{(-n_i)}(x) + v \to v_0$ which implies that $T_\varphi^{-n_i}(x, v) \to (x_0, v_0)$. As $T_\varphi^{n_i}(x_0, v_0) \to (x, v)$, $T_\varphi^{n_i}(x_0, 0) \to (x, v - v_0)$. By virtue of Lemma 6.3.3, $\varphi^{n_i} \to h$ uniformly for some continuous function $h: X \to \mathbb{R}^m$. By the cocycle identity, $0 = \varphi^{(n_i-n_i)}(x) = \varphi^{(n_i)}(T^{-n_i}x) + \varphi^{(-n_i)}(x)$ and the fact that $\varphi^{(n_i)} \to h$ uniformly, $T^{-n_i}x \to x_0$ and $h(x_0) = \lim \varphi^{(n_i)}(x_0) = v - v_0$ we have

$$\begin{aligned}\|\varphi^{(-n_i)}(x) - (v_0 - v)\| &= \|-\varphi^{(n_i)}(T^{-n_i x}) - (v_0 - v)\| = \|\varphi^{(n_i)}(T^{-n_i x}) - (v - v_0)\| \\ &\leqslant \|\varphi^{(n_i)}(T^{-n_i}) - h(T^{-n_i}x)\| + \|h(T^{-n_i}x) - (v - v_0)\| \xrightarrow{i\to\infty} 0.\end{aligned}$$

Thus

$$T_\varphi^{-n_i}(x, v) = (T^{-n_i}x, \varphi^{(-n_i)}(x) + v) \xrightarrow{i\to\infty} (x_0, v_0 - v + v) = (x_0, v_0).$$

We have proved that if $(x, v) \in \overline{\mathrm{Orb}}(x_0, v_0)$ then $(x_0, v_0) \in \overline{\mathrm{Orb}}(x, v)$, hence $\overline{\mathrm{Orb}}(x, v) = \overline{\mathrm{Orb}}(x_0, v_0)$, so each orbit closure is minimal. □

Lemma 6.3.5. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi: X \to \mathbb{R}^m$ a continuous map such that $E(\varphi) = \{0\}$ and T_φ is conservative. Then φ is a coboundary.*

Proof. By [40, Theorem 7.24], the set of all recurrent points in $X \times \mathbb{R}^m$ is residual, in particular non-empty. Let $(x_0, v_0) \in X \times \mathbb{R}^m$ be a recurrent point. Then there exist sequences of integers $k_t \to -\infty$ and $n_t \to +\infty$ such that $T_\varphi^{k_t}(x_0, v_0) \to (x_0, v_0)$ and $T_\varphi^{n_t}(x_0, v_0) \to (x_0, v_0)$. Thus $T^{k_t} \to \mathrm{Id}$, $T^{n_t} \to \mathrm{Id}$ uniformly and $\varphi^{(k_t)}(x_0) \to 0$, $\varphi^{(n_t)}(x_0) \to 0$. By Lemma 6.3.3, the sequences of functions $(\varphi^{(k_t)})_{t\geqslant 1}$ and $(\varphi^{(n_t)})_{t\geqslant 1}$ are uniformly convergent. By virtue of Lemma 6.3.1, both $\varphi^{(k_t)} \to 0$ and $\varphi^{(n_t)} \to 0$ uniformly and therefore $T_\varphi^{k_t} \to \mathrm{Id}$, $T_\varphi^{n_t} \to \mathrm{Id}$ uniformly. In particular each point $(x, v) \in X \times \mathbb{R}^m$ is recurrent under T_φ. It follows from Lemma 6.3.4 that $X \times \mathbb{R}^m$ is a (disjoint) union of minimal sets. By virtue of [40, Theorem 7.05], each point in $X \times \mathbb{R}^m$ is almost periodic. Denoting $\varphi = (\varphi_1, \dots \varphi_m)$ we get the following property.

(6.10) For each $i = 1, \dots, m$, all points $(x, v) \in X \times \mathbb{R}$ are almost periodic under T_{φ_i}.

Since T_φ is conservative, all T_{φ_i}'s are conservative. By Lemma 6.3.2, $\int_X \varphi_i \, d\mu = 0$, $i = 1, \dots, m$. Then for each $i = 1, \dots, m$, either φ_i is a coboundary or T_{φ_i} is point transitive. Suppose that some T_{φ_i} is point transitive and $(x_0, v_0) \in X \times \mathbb{R}$ has dense orbit under T_{φ_i}. Then (x_0, v_0) cannot be almost periodic, which is a contradiction with (6.10). Thus all φ_i's are coboundaries, hence also φ is a coboundary. □

Now we are able to exhibit a key property of conservative cocycles.

Proposition 6.3.6. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi: X \to \mathbb{R}^m$ a continuous map such that T_φ is conservative. Then there exits a basis of $\mathbb{R}^m$ such that $\varphi = (\varphi_1, \dots, \varphi_m)$, $E(\varphi) = E(\varphi_1, \dots, \varphi_k) = \mathbb{R}^k$ and $(\varphi_{k+1}, \dots, \varphi_m)$ is a coboundary.*

Proof. If T_φ is point transitive then $E(\varphi) = \mathbb{R}^m$ and the assertions of this proposition easily follow. Assume that T_φ is not point transitive. Then, by virtue of Theorem 6.2.6, $E(\varphi)$ is a k-dimensional linear subspace of $\mathbb{R}^m$, and, by Proposition 5.2.3, $k < m$. Changing a basis of $\mathbb{R}^m$ if necessary we may assume that $\varphi = (\varphi_1, \dots, \varphi_m)$ and $E(\varphi) = E(\varphi_1, \dots, \varphi_k) = \mathbb{R}^k$. By virtue of Proposition 6.1.2, $E(\varphi_{k+1}, \dots, \varphi_m) = \{0\}$. By Lemma 6.3.5, $(\varphi_{k+1}, \dots, \varphi_m)$ is a coboundary. □

Now we are in a position to generalize [6, Theorem 1] to any minimal rotation on a compact metric monothetic group.

Proposition 6.3.7. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi: X \to \mathbb{R}^m$ a continuous map such that T_φ is conservative. Then T_φ is not point transitive if and only if there exist non-zero linear functional $L: \mathbb{R}^m \to \mathbb{R}$ and continuous function $f: X \to \mathbb{R}$ satisfying the functional equation*

$$L \circ \varphi + f - f \circ T = 0. \tag{6.11}$$

Proof. Assume that T_φ is not point transitive. By Propositions 6.3.6 and 5.2.3 we may assume that $\varphi = (\varphi_1, \dots, \varphi_m)$, $E(\varphi) = E(\varphi_1, \dots, \varphi_k) = \mathbb{R}^k$ and $(\varphi_{k+1}, \dots, \varphi_k)$ is a coboundary, where $k < m$.

Let $g = (g_{k+1}, \dots, g_m): X \to \mathbb{R}^{m-k}$ be such a continuous function that $(\varphi_{k+1}, \dots, \varphi_m) = (g_{k+1} \circ T, \dots, g_m \circ T) - (g_{k+1}, \dots, g_m)$. Define $L: \mathbb{R}^m \to \mathbb{R}$ by $L(v_1, \dots, v_m) = v_m$. Let $f: X \to \mathbb{R}$, $f(x) = g_m(x)$. Then clearly $L \circ \varphi + f - f \circ T = 0$ and the necessity is proved.

To prove the sufficiency suppose that T_φ is point transitive and $L\circ\varphi+f-f\circ T=0$ for a non-zero linear functional $L:\mathbb{R}^m\to\mathbb{R}$ and a continuous function $f:X\to\mathbb{R}$. By virtue of Proposition 6.1.1, $L(E(\varphi))\subset E(L\circ\varphi)$. Since $L\neq 0$, $L(E(\varphi))=L(\mathbb{R}^m)=\mathbb{R}$, so $E(L\circ\varphi)=\mathbb{R}$ and $T_{L\circ\varphi}$ is point transitive, hence not a coboundary, which finishes the proof. □

In [6] a sufficient condition for T_φ to be conservative, where φ is a cocycle defined on the one-dimensional torus, with values in $\mathbb{R}^m$, is given ([6, Theorem 2]). The next theorem contains a generalization of the one from [6]. Note that our theorem is formulated as a sufficient and necessary condition.

Theorem 6.3.8. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi:X\to\mathbb{R}^m$ be a continuous map. Then the following conditions are equivalent:*

(a) *T_φ is conservative.*
(b) *φ is regular.*
(c) *If $\widetilde{\varphi}:X\to\mathbb{R}^m/E(\varphi)$ is given by $\widetilde{\varphi}(x)=\varphi(x)+E(\varphi)$, then $E_\infty(\widetilde{\varphi})=\{0\}$.*
(d) *$\int_X\varphi\,d\mu=0$, where μ is the normalized Haar measure on X.*

Proof. Assume that (a) is true, i.e. T_φ is conservative. We will show that φ is regular. Clearly, if T_φ is point transitive then φ is regular. Suppose T_φ is not point transitive. By virtue of Propositions 5.2.3 and 6.3.6 we may assume that $\varphi=(\varphi_1,\dots,\varphi_m)$, $E(\varphi)=E(\varphi_1,\dots,\varphi_k)=\mathbb{R}^k$, $k<m$, $(\varphi_{k+1},\dots,\varphi_m)$ is a coboundary i.e. $\varphi_j=f_j\circ T-f_j$ for some continuous functions $f_j:X\to\mathbb{R}$, $j=k+1,\dots,m$. Define $f:X\to\mathbb{R}^m$ by

$$f(x)=(0,\dots,0,f_{k+1}(x),\dots,f_m(x)).$$

Then $\psi=\varphi+f-f\circ T:X\to E(\varphi)$ and φ is regular.

The implication from (b) to (c) is a part of Corollary 6.1.4.

Assume now that the condition (c) is true. Suppose for the contrary that $\int_X\varphi\,d\mu\neq 0$. Then, denoting $\varphi=(\varphi_1,\dots,\varphi_m)$ we may assume that for instance $\int_X\varphi_1\,d\mu>0$. Since $\varphi_1^{(n)}/n\to\int_X\varphi_1\,d\mu$ uniformly, $\varphi_1^{(n_t)}\to+\infty$ uniformly for each rigidity time $(n_t)_{t\geqslant 1}$, hence $E_\infty(\varphi)=\{0,\infty\}$ and $E(\varphi)=\{0\}$. In particular $E_\infty(\widetilde{\varphi})=\{0,\infty\}$, which is a contradiction.

Assume now that (d) is true; we will show (a). If $\int_X\varphi\,d\mu=0$, then $\int_X\varphi_i\,d\mu=0$, $i=1,\dots,m$. Changing a basis of $\mathbb{R}^m$ if necessary we may assume that $\varphi_{k+1},\dots,\varphi_m$ are coboundaries and $T_{(\varphi_1,\dots,\varphi_k)}$ is point transitive. Then $T_{(\varphi_1,\dots,\varphi_k,\varphi_{k+1},\dots,\varphi_m)}$ is isomorphic to $T_{(\varphi_1,\dots,\varphi_k,0,\dots,0)}$ (as homeomorphims of $X\times\mathbb{R}^m$). By virtue of Theorem 6.1.5, the flow $(X\times\mathbb{R}^m,T_{(\varphi_1,\dots,\varphi_k,0,\dots,0)})$ is a disjoint union of point transitive subflows, where each of them is isomorphic to $(X\times\mathbb{R}^k,T_{(\varphi_1,\dots,\varphi_k)})$. The space $X\times\mathbb{R}^k$ is perfect and the flow $T_{(\varphi_1,\dots,\varphi_k)}$ is point transitive, hence conservative and so is T_φ. □

Corollary 6.3.9. *Let T be a minimal rotation on a compact metric monothetic group X, $\varphi\colon X \to \mathbb{R}^m$ be a continuous map. Then the following conditions are equivalent:*

(a) *T_φ is not conservative.*
(b) *φ is not regular.*
(c) *$E_\infty(\widetilde{\varphi}) = \{0, \infty\}$.*
(d) *$\int_X \varphi\, d\mu \neq 0$, where μ is the normalized Haar measure on X.*
(e) *$E_\infty(\varphi) = \{0, \infty\}$*
(f) *All points in $X \times \mathbb{R}^n$ are T_φ-wandering.*

Proof. Equivalence of (a), (b), (c) and (d) is clear because of Theorem 6.3.8. Since $\varphi^{(n)}/n \to \int_X \varphi\, d\mu$ uniformly, (f) is a consequence of (d). If (f) is true then obviously also (e) is true (Proposition 6.2.2) and (e) implies (c). □

6.4. A more general case

Lemma 6.4.1. *Let T be a minimal rotation on a compact metric monothetic group X, D is a discrete group, $\varphi\colon X \to D$ a continuous map. Moreover, let $\widetilde{\varphi}\colon X \to D/E(\varphi)$, $\widetilde{\varphi}(x) = \varphi(x) + E(\varphi)$. Then:*

(a) *If $d \in D_\infty$, $d \neq 0$ then $d \in E_\infty(\varphi)$ if and only if $\varphi^{(n_t)} \to d$ uniformly for some rigidity time $(n_t)_{t\geqslant 1}$.*
(b) *If $d \in E(\varphi)$ then d has finite order, i.e. $rd = 0$ for some non-zero integer r. In particular, if D possesses no compact subgroups then $E(\varphi) = \{0\}$.*
(c) *$E(\varphi)$ is a finite group.*
(d) *$\infty \in E_\infty(\widetilde{\varphi})$ if and only if $\infty \in E_\infty(\varphi)$.*
(e) *$E_\infty(\varphi) \subset D$ if and only if there exist a rigidity time $(n_t)_{t\geqslant 1}$ and a $d \in D$ such that $\varphi^{(n_t)} \equiv d$ for all $t \geqslant 1$.*
(f) *φ is regular if and only if $E_\infty(\varphi) \neq \{0, \infty\}$.*
(g) *Either $E_\infty(\varphi) = \{0, \infty\}$ or $E_\infty(\varphi) \subset D$.*

Proof. (a) This is a straightforward consequence of Proposition 5.3.1 and of Lemma 6.2.1 as D is discrete.

(b) Let $d \in E(\varphi)$, $d \neq 0$. Then by (a), $\varphi^{(n_t)} \to d$ uniformly for some rigidity time $(n_t)_{t\geqslant 1}$, so, by virtue of Lemma 6.1.11, d has finite order.

(c) Assume $E(\varphi) \neq \{0\}$ and fix $0 \neq d_0 \in E(\varphi)$. It follows from (a) that $\varphi^{(n_t)} \equiv d_0$ for some rigidity time $(n_t)_{t\geqslant 1}$. Let $n = n_1$. Consider now any $0 \neq d \in E(\varphi)$. Then, by (a), $d \equiv \varphi^{(m_t)}$ for some rigidity time $(m_t)_{t\geqslant 1}$. Let $m_t = s_t n + r_t$, $0 \leqslant r_t < n$, $t \geqslant 1$. Then, by the cocycle identity, for any $x \in X$,

$$d = \varphi^{(m_t)}(x) = \varphi^{(s_t n + r_t)}(x) = \sum_{i=0}^{s_t - 1} \varphi^{(n)}(T^{in + r_t} x) + \varphi^{(r_t)}(x) = s_t d_0 + \varphi^{(r_t)}(x).$$

By (b), the set $D_0 = \{sd_0 + \varphi^{(r)}(x) : s \in \mathbb{Z}, 0 \leqslant r < n, x \in X\}$ is finite. Clearly $d \in D_0$. As d was arbitrary, $E(\varphi) \subset D_0 \cup \{0\}$ and therefore $E(\varphi)$ is finite.

(d) Observe that as $E(\varphi)$ is finite, $\widetilde{\varphi}^{(n_t)} \to \infty$ uniformly for some rigidity time $(n_t)_{t\geqslant 1}$ if and only if $\varphi^{(n_t)} \to \infty$ uniformly, so $\infty \in E_\infty(\widetilde{\varphi})$ if and only if $\infty \in E_\infty(\varphi)$.

(e) Assume first that $E_\infty(\varphi) \subset D$ i.e. $\infty \notin E_\infty(\varphi)$. If $E_\infty(\varphi) = \{0\}$, then φ is a coboundary (Proposition 5.2.7) and therefore $\varphi^{(n_t)} \to 0$ uniformly for each rigidity time $(n_t)_{t\geqslant 1}$ and we are done. If $E_\infty(\varphi) \neq \{0\}$ then also $E(\varphi) \neq \{0\}$ and, by (a), $\varphi^{(n_t)} \to d$ uniformly for some rigidity time $(n_t)_{t\geqslant 1}$, where d is an element of $E(\varphi)$. Then $\varphi^{(n_t)} \equiv d$ for t large enough and we may assume that $\varphi^{(n_t)} \equiv d$ for all t.

Assume now that $\varphi^{(n_t)} \equiv d$, $t \geqslant 1$, for some rigidity time $(n_t)_{t\geqslant 1}$. In such a case $\varphi^{(-n_t)} \equiv -d \in E(\varphi)$, $t \geqslant 1$ and therefore we may assume that $n_1 > 0$. Let $n = n_1$ and suppose that $\infty \in E_\infty(\varphi)$. Then, by (a), $\varphi^{(m_t)} \to \infty$ uniformly for some rigidity time $(m_t)_{t\geqslant 1}$. Let $m_t = s_t n + r_t$, $0 \leqslant r_t < n$, $t \geqslant 1$. Then $\varphi^{(m_t)} \in D_0$ for all $t \geqslant 1$, where D_0 is the finite set defined in the proof of (c). In particular $\varphi^{(m_t)} \not\to \infty$, a contradiction. Thus $\infty \notin E_\infty(\varphi)$.

(f) Assume that φ is regular and suppose for the contrary that $E_\infty(\varphi) = \{0,\infty\}$. Then $E(\varphi) = \{0\}$, hence $E_\infty(\widetilde{\varphi}) = \{0,\infty\}$ and we have got a contradiction with Corollary 6.1.4. Thus $E_\infty(\varphi) \neq \{0,\infty\}$.

Assume now that $E_\infty(\varphi) \neq \{0,\infty\}$. Then either $\infty \notin E_\infty(\varphi)$ or $E(\varphi) \neq \{0\}$. Consider the first case. Then $E_\infty(\widetilde{\varphi}) = \{0\}$ and there exists a continuous selector for the natural quotient map $D \to D/E(\varphi)$ (Remark 6.1.14). By virtue of Lemma 6.1.13, φ is regular. Consider now the second case. Let $0 \neq d \in E(\varphi)$. It follows from (a) that $\varphi^{(n_t)} \equiv d$ for some rigidity time $(n_t)_{t\geqslant 1}$. By (e), $\infty \notin E_\infty(\varphi)$ and by (d), $\infty \notin E_\infty(\widetilde{\varphi})$. As $D/E(\varphi)$ is discrete, there exists a continuous selector for the quotient map $D \to D/E(\varphi)$ and, by Lemma 6.1.13, φ is regular.

(g) If $\infty \in E_\infty(\varphi)$ then, by (e), $E(\varphi) = \{0\}$. $\square$

Lemma 6.4.2. *Let T be a minimal rotation on a compact metric monothetic group X, D a discrete group, $\varphi_d: X \to D$, $\varphi_r: X \to \mathbb{R}^m$ be continuous maps.*

(a) *If $\varphi_d + \varphi_r: X \to D \oplus \mathbb{R}^m$ is regular then both φ_d and φ_r are also regular.*
(b) *If φ_d and φ_r are regular then there exists a subgroup $D_0 \subset D$ such that $E(\varphi_d + \varphi_r) = D_0 \oplus E(\varphi_r)$.*

Proof. (a) Suppose that $\varphi_d + \varphi_r$ is regular, then $\psi = \varphi_d + \varphi_r + f \circ T - f: X \to E(\varphi_d + \varphi_r)$ for some continuous $f: X \to D \oplus \mathbb{R}^m$. Write $f = f_d + f_r$, $\psi = \psi_d + \psi_r$, where $\psi_d: X \to E(\varphi_d)$, $\psi_r: X \to E(\varphi_r)$, because $E(\varphi_d + \varphi_r) \subset E(\varphi_d) \oplus E(\varphi_r)$. Therefore both φ_d and φ_r are regular.

(b) Assume now that φ_d and φ_r are regular, then we may assume that $E(\varphi_d) = D$ with D finite and $E(\varphi_r) = \mathbb{R}^m$. Suppose that $v \in E(\varphi_d)$. Then, by Proposition 5.3.1, $\varphi_r^{(n_t)}(x_t) \to v$ for some rigidity time $(n_t)_{t\geqslant 1}$. As $E(\varphi_d)$

is discrete, we may assume that $\varphi_d^{(n_t)} \equiv \text{const}$ for each individual t large enough, and, since $E(\varphi_d)$ is finite (Lemma 6.4.1(c)), passing to a subsequence of the sequence $(n_t)_{t\geqslant 1}$ if necessary, $\varphi_d^{(n_t)} \equiv d$, $t \geqslant 1$, for some $d \in E(\varphi_d)$. Thus $d + v \in E(\varphi_d + \varphi_r)$. We have shoved that for each $v \in E(\varphi_r)$ there exists a $d \in E(\varphi_d)$ such that $d + v \in E(\varphi_d + \varphi_r)$. Let M be such a positive integer that $Md = 0$ for all $d \in E(\varphi_d)$. Take $v \in E(\varphi_r) = \mathbb{R}^m$, then also $v/m \in E(\varphi_r)$, and find $d \in E(\varphi_d)$ such that $d + v/M \in E(\varphi_d + \varphi_r)$. Then $M(d + v/m) = v \in E(\varphi_d + \varphi_r)$ which implies $E(\varphi_r) \subset E(\varphi_d + \varphi_r)$. Let $D_0 = D \cap E(\varphi_d + \varphi_r)$, then $E(\varphi_d + \varphi_r) = D_0 \oplus E(\varphi_r)$ and lemma is proved. □

Theorem 6.4.3. *Let T be a minimal rotation on a compact metric monothetic group X, G a locally compact Abelian group such that $\mathbb{R}^m \subset G$ is an open subgroup, $\varphi: X \to G$, a continuous map. Then there exist a finite group D_0 and a linear subspace $V \subset \mathbb{R}^m$ such that $E(\varphi) = D_0 \oplus V$.*

Proof. Since $\mathbb{R}^m \subset G$ is an open subgroup, $G = D \oplus \mathbb{R}^m$, where D is a discrete Abelian group. Put $\varphi = \varphi_d + \varphi_r$, where $\varphi_d: X \to D$, $\varphi_r: X \to \mathbb{R}^m$. Observe that if either $E_\infty(\varphi_d) = \{0, \infty\}$ or $E_\infty(\varphi_r) = \{0, \infty\}$ then $E_\infty(\varphi) = \{0, \infty\}$. Indeed, in the first case, by Lemma 6.4.1(e), $\varphi_d^{(n_t)}(x_t) \to \infty$ for each rigidity time $(n_t)_{t\geqslant 1}$ and $x_t \in X$, $t \geqslant 1$ and therefore $E_\infty(\varphi) = \{0, \infty\}$. In the second case, by Corollary 6.3.9, all points in $X \times \mathbb{R}^m$ are T_{φ_r}-wandering, hence $E_\infty(\varphi) = \{0, \infty\}$.

If $E_\infty(\varphi) = \{0, \infty\}$ then theorem is true with $D_0 = \{0\}$, $V = \{0\}$. Assume that $E_\infty(\varphi) \neq \{0, \infty\}$. Then $E_\infty(\varphi_d) \neq \{0, \infty\}$ and $E_\infty(\varphi_r) \neq \{0, \infty\}$. By virtue of Theorem 6.3.8, φ_r is regular and, by Lemma 6.4.1, φ_d is regular. It follows from Lemma 6.4.2 that $E(\varphi) = D_0 \oplus E(\varphi_r)$, where D_0 is a finite subgroup of D. By virtue of Theorem 6.2.6, $V = E(\varphi_r)$ is a linear subspace of $\mathbb{R}^m$, which finishes the proof. □

Theorem 6.4.4. *Let T be a minimal rotation on a compact metric monothetic group X, G a locally compact Abelian group such that $\mathbb{R}^m \subset G$ is an open subgroup, and $\varphi: X \to G$ a continuous map. Let $\widetilde{\varphi}: X \to G/E(\varphi)$, $\widetilde{\varphi}(x) = \varphi(x) + E(\varphi)$. Then the following conditions are equivalent:*

(a) $E_\infty(\widetilde{\varphi}) = \{0\}$.
(b) φ *is regular.*
(c) T_φ *is conservative.*

Proof. Since $\mathbb{R}^m$ is a divisible group, $G = D \oplus \mathbb{R}^m$, where D is a discrete group, $D = G/\mathbb{R}^m$. Thus $\varphi = \varphi_d + \varphi_r$, where $\varphi_d: X \to D$, $\varphi_r: X \to \mathbb{R}^m$. Moreover, $E(\varphi) = E(\varphi_d + \varphi_r) \subset E(\varphi_d) \oplus E(\varphi_r)$.

(a)⇒(b). First we will show that both φ_d and φ_r are regular. φ_r is regular by Corollary 6.3.9. If $E(\varphi) = \{0\}$, then $\varphi = \varphi_d + \varphi_r$ is a coboundary and so are both φ_d and φ_r. Assume that $E(\varphi) \neq \{0\}$. Then $\varphi^{(n_t)}(x_t) \to d + v$ for some rigidity time $(n_t)_{t\geqslant 1}$ and $x_t \in X$, $t \geqslant 1$, where $0 \neq d + v \in E(\varphi)$. Consequently $\varphi_d^{(n_t)}(x_t) \to d$ and we may assume that $\varphi^{(n_t)}(x_t) \equiv d$, $t \geqslant 1$.

By virtue of Lemma 6.4.1(e)–(f), φ_d is regular. We have shoved that when (a) is true then both φ_d and φ_r are regular. It follows from Theorem 6.4.3 that there exist a finite subgroup $D_0 \subset D$ and a linear subspace $V \subset \mathbb{R}^m$ such that $E(\varphi) = D_0 \oplus V$. Changing a basis of $\mathbb{R}^m$ if necessary we may assume that $V = \mathbb{R}^k$ for some $0 \leqslant k \leqslant m$. Then $G/E(\varphi) = (D/D_0) \oplus \mathbb{R}^{m-k}$. Observe that there exist a continuous selector s for the quotient map $G \to G/E(\varphi)$. By virtue of Lemma 6.1.13, $\varphi = \varphi_d + \varphi_r$ is regular.

(b)$\Rightarrow$(c). Suppose $\varphi = \varphi_d + \varphi_r$ is regular. By virtue of Lemma 6.4.2 we may assume that both T_{φ_d} and T_{φ_r} are topologically ergodic (then $E(\varphi_d) = D$, $E(\varphi_r) = \mathbb{R}^m$), $E(\varphi) = D_0 \oplus \mathbb{R}^m$ for some subgroup $D_0 \subset D$, $\varphi\colon X \to D_0 \oplus \mathbb{R}^m$, and that $T_\varphi\colon X \times E(\varphi) \to X \times E(\varphi)$ is point transitive. Since either $D_0 \oplus \mathbb{R}^m$ is a perfect space ($m > 0$) or $D_0 \oplus \mathbb{R}^m$ is finite ($m = 0$), T_φ is conservative.

(c)$\Rightarrow$(a). Suppose T_φ is conservative. If $U \subset X$, $V \subset \mathbb{R}^m$ are open non-empty sets, $d \in D$, then the dwelling sets satisfy

$$D(U \times (\{d\} \oplus V), U \times (\{d\} \oplus V)) \subset D(U \times \{d\}, U \times \{d\}) \cap D(U \times V, U \times V)$$

so both T_{φ_d} and T_{φ_r} are conservative. By virtue of Theorem 6.3.8, $E_\infty(\widetilde{\varphi}_r) = \{0\}$. Suppose for the contrary that $E_\infty(\widetilde{\varphi}) = \{0, \infty\}$. Then $\widetilde{\varphi}^{(n_t)}(x_t) \to \infty \in G_\infty$ i.e. $\varphi_d^{(n_t)}(x_t) + d_t + \varphi_r^{(n_t)}(x_t) + v_t \to \infty$ for each $d_t + v_t \in E(\varphi)$. Since D is finite, $\varphi_r^{(n_t)}(x_t) + v_t \to \infty$ for any sequence $(v_t)_{t \geqslant 1}$ of elements from $\mathbb{R}^m$, equivalently, $\widetilde{\varphi}_r^{(n_t)}(x_t) \to \infty$ and $\infty \in E_\infty(\widetilde{\varphi}_r)$, a contradiction. Thus $E_\infty(\varphi) = \{0\}$ and the proof is complete. $\square$

CHAPTER 7

CYLINDER COCYCLE EXTENSIONS OF ROTATIONS

In this chapter we will concentrate on the following situation. The flows (X, T) will be minimal rotations on compact metric monothetic groups, $G = \mathbb{R}$ or $G = \mathbb{R}^m$ for some positive integer m. In such cases the group $E(\varphi)$ of essential values of φ is a linear subspace of $\mathbb{R}^m$ (Theorem 6.2.6), and, whenever φ has zero mean, φ is regular (Theorem 6.3.8). In particular, for $\varphi\colon X \to \mathbb{R}$ there is a trichotomy:

(a) either $\int_X \varphi\, d\mu \neq 0$ – then T_φ is transient: all orbits are discrete;
(b) or T_φ is point transitive;
(c) or φ is a coboundary.

Let us recall now the Denjoy–Koksma inequality. For the definition of discrepancy and for the proof of Theorem 7.0.1 below we refer to [57].

Theorem 7.0.1 ([57, Theorem 5.1, Chapter 2])**.** *Let φ be a function of bounded variation on $[0,1]$ and $x_1, \dots, x_N \in [0,1)$. Then*

$$\left|\frac{1}{N}\sum_{n=1}^{N} \varphi(x_n) - \int_0^1 \varphi(t)\, dt\right| \leq \operatorname{Var}(\varphi) D_N^*,$$

where D_N^* denotes a discrepancy of the sequence $\{x_1, \dots, x_N\}$.

We use this theorem in the particular case when $\int_0^1 \varphi(t)\, dt = 0$, $x_n = x + n\alpha \bmod 1$ (α is irrational), $N = q_k$, where $(q_k)_{k\geq 1}$ is the sequence of the denominators of the continued fraction expansion of α. Since, in such a case, the inequality $D_{q_k}^* \leq 1/q_k + 1/q_{k+1}$ holds (see e.g. [57, (3.17), Chapter 2]), we obtain the following estimation

$$\|\varphi^{(q_k)}\| \leq 2\operatorname{Var}(\varphi), \tag{7.1}$$

(here, as well as in the sequel, $\|\cdot\|$ denotes the supremum norm in the space of bounded variation functions) that we will use in proof of Theorem 7.1.4 below.

7.1. The problem of minimality for cylinder extensions of minimal rotations on a circle

In this section we will use the following standard notations. Let $\mathbb{T}$ be the unit circle on the complex plane with its natural topological group structure; we will often identify $\mathbb{T}$ with the interval $[0,1) \bmod 1$. Then denote by $CBV(\mathbb{T})$ the Banach space of continuous real functions on $\mathbb{T}$ with bounded variation: $\psi \in CBV(\mathbb{T})$ if and only if ψ is continuous and $\mathrm{Var}(\psi) < \infty$. Let $CBV_0(\mathbb{T})$ be the subspace of $CBV(\mathbb{T})$ consisting of all functions of zero mean with respect to the Lebesgue measure on $\mathbb{T}$. Note, that if $\|\cdot\|$ denotes the supremum norm on the space $C(\mathbb{T})$, then the norm Var on $CBV_0(\mathbb{T})$ satisfies $\|\psi\| \leq \mathrm{Var}(\psi)$, in particular the usual norm $\|\cdot\| + \mathrm{Var}(\,\cdot\,)$ on $CBV(\mathbb{T})$ is equivalent on $CBV_0(\mathbb{T})$ to $\mathrm{Var}(\,\cdot\,)$. Let $AC_0(\mathbb{T})$ be the subspace of $CBV_0(\mathbb{T})$ of all absolutely continuous functions of zero mean.

The following theorem was essentially proved by A. S. Besicovitch in [8]. Although in [8] it is considered the case of $X = \mathbb{T}$, it is an immediate observation that Besicovitch used only compactness of $\mathbb{T}$. We repeat the proof of Besicovitch in our general situation for the chpter to be more self-contained.

Theorem 7.1.1 ([8])**.** *Let (X,T) be a compact metric flow, $\varphi\colon X \to \mathbb{R}$ a continuous map. Then $T_\varphi\colon X \times \mathbb{R} \to X \times \mathbb{R}$ is not minimal.*

Proof. We may assume that (X,T) is minimal and T_φ is point transitive. Let $(x_0,0) \in X \times \mathbb{R}$ be a transitive point for T_φ, i.e. $\overline{\mathrm{Orb}}(x_0,0) = X \times \mathbb{R}$. By [40, Theorem 9.23] we may assume that $(x_0,0)$ is extensively transitive that means both positive and negative semi-orbits of $(x_0,0)$ are dense in $X \times \mathbb{R}$:

$$\overline{\{T_\varphi^n(x_0,0) : n \geq 0\}} = X \times \mathbb{R}, \qquad \overline{\{T_\varphi^n(x_0,0) : n \leq 0\}} = X \times \mathbb{R}.$$

Thus we are able to find three sequences of integers $(m_j)_{j\geq 1}$, $(n_j)_{n\geq 1}$, $(s_j)_{j\geq 1}$ such that $m_j < s_j < n_j$, $j \geq 1$, and

$$\varphi^{(m_j)}(x_0) < -j, \quad \varphi^{(n_j)}(x_0) < -j, \quad \varphi^{(s_j)}(x_0) > j, \quad j \geq 1.$$

As φ is continuous and X is compact,

$$m_j - s_j \to -\infty, \qquad n_j - s_j \to \infty. \tag{7.2}$$

We may assume that

$$\varphi^{(s_j)}(x_0) = \max\{\varphi^{(n)}(x_0) : m_j \leq n \leq n_j\}, \quad j \geq 1. \tag{7.3}$$

Consider the points

$$\begin{aligned}(x_j^n, r_j^n) = T_\varphi^{s_j+n}(x_0, -\varphi^{(s_j)}(x_0)) &= (T^{s_j+n}x_0, \varphi^{(s_j+n)}(x_0) - \varphi^{(s_j)}(x_0)) \\ &= (T^{s_j+n}x_0, \varphi^{(n)}(T^{s_j}x_0)), \quad n \in \mathbb{Z},\ j \geq 1.\end{aligned}$$

Then, for $m_j - s_j \leq n \leq n_j - s_j$, $j \geq 1$, we have

$$r_j^0 = 0, \quad T_\varphi(x_j^n, r_j^n) = (x_j^{n+1}, r_j^{n+1}). \tag{7.4}$$

Take a convergent subsequence $x_{j_k}^0 \to \widetilde{x}$. Then, by (7.4), for each $n \in \mathbb{Z}$ the subsequence

$$(x_{j_k}^n, r_{j_k}^n) = (x_{j_k}^n, \varphi^{(n)}(T^{s_{j_k}} x_0)) = T_\varphi^n(x_{j_k}^0, 0), \; k \geq 1$$

is also convergent, $(x_{j_k}^n, r_{j_k}^n) \to (T^n\widetilde{x}, \varphi^{(n)}(\widetilde{x}))$. By (7.2), for each given integer n the inequalities $m_{j_k} - s_{j_k} < n < n_{j_k} - s_{j_k}$ hold for k large enough, hence, by (7.4) and by (7.3), $\varphi^{(n)}(\widetilde{x}) \leq 0$. In particular for each $(x, r) \in \overline{\mathrm{Orb}}(\widetilde{x}, 0)$ we have $r \leq 0$, therefore $\mathrm{Orb}(\widetilde{x}, 0)$ is not dense in $X \times \mathbb{R}$. □

Corollary 7.1.2. *Let (X, T) be a compact metric flow, $\varphi: X \to \mathbb{R}^m$ a continuous map. Then $T_\varphi: X \times \mathbb{R}^m \to X \times \mathbb{R}^m$ is not minimal.*

Proof. Let $\varphi = (\varphi_1, \dots, \varphi_m): X \to \mathbb{R}^m$ be a continuous map. By the above there exists a point $\widetilde{x} \in X$ such that the orbit of $(\widetilde{x}, 0)$ via T_{φ_1} is not dense in $X \times \mathbb{R}$, hence the orbit of $(\widetilde{x}, 0, \dots, 0)$ via T_φ is not dense in $X \times \mathbb{R}^m$. In particular T_φ is not minimal. □

Our next aim is to show that no continuous bounded variation cocycle on $\mathbb{T}$ admits minimal subsets (Theorem 7.1.4). The method of the proof of this theorem is similar to the proof of [56, Proposition 2] of Krygin's paper on the Poincaré sets for smooth cocycles – the vertical sections of limits sets in $\mathbb{T} \times \mathbb{R}$. Together with Lemma 7.1.3 below some ideas of Krygin's proof of [56, Proposition 2], after modifications, will give the proof of our result.

Lemma 7.1.3. *Let (X, T) be a minimal rotation on a compact metric monothetic group, $\varphi: X \to \mathbb{R}^m$ a continuous map. Suppose $M \subset X \times \mathbb{R}^m$ to be T_φ-minimal set. Denote $M_x = (\{x\} \times \mathbb{R}^m) \cap M$. Then $\operatorname{card} M_x \leq 1$ for every $x \in X$.*

Proof. First consider the case $m = 1$. If T_φ is not point transitive, then either it is a coboundary or T_φ is transient. In the first case M is a graph of some continuous function $f: X \to \mathbb{R}$ (see [64, Proposition 5.1]), in the second one M is equal to orbit via T_φ of some point (see [64, Remark 4]). In both cases $\operatorname{card} M_x \leq 1$. Thus we may assume that T_φ is point transitive.

Observe that as T is minimal, the set $D = \{x \in X : M_x \neq \emptyset\}$ is dense in X. Put $H = \{r \in \mathbb{R} : M + r = M\}$ (here $M + r = \{(x, s + r) : (x, s) \in M\}$). It is easy to see that H is a closed subgroup. Similarly as in [36, Lemma 3.1] or [93, Lemma 2.6.1] we see that if $M_x \neq \emptyset$ then $M_x = r + H$ for every $r \in \mathbb{R}$ such that $(x, r) \in M$.

First assume that $H = \mathbb{R}$. Then for $x \in D$ we have $M_x = \mathbb{R}$ that implies $M = X \times \mathbb{R}$, a contradiction with Theorem 7.1.1.

Now let $H = a\mathbb{Z}$. Take a T_φ-transitive point $(x, 0) \in X \times \mathbb{R}$. Find a sequence $x_i \in D$, $i \geq 1$, that converges to x, and numbers $r_i \in \mathbb{R}$ such that $(x_i, r_i) \in M$. Since $H = a\mathbb{Z}$, the numbers r_i may be chosen from $[0, a)$, thus by passing to a subsequence, if necessary, we may assume that points (x_i, r_i) converge to $(x, r) \in M$. But (x, r) is a transitive point, a contradiction.

It remains the case $H = \{0\}$ that gives the result.

Suppose now m is arbitrary. For any linear functional $L: \mathbb{R}^m \to \mathbb{R}$ define a factor map $\widetilde{L}: X \times \mathbb{R}^m \to X \times \mathbb{R}$ setting $\widetilde{L}(x, r) = (x, L(r))$. Then the set $N^L = \widetilde{L}(M)$ is minimal and $\operatorname{card} N^L_x \leq 1$ for each $x \in X$. Suppose $r, s \in M_x$, $r = (r_1, \dots, r_m)$, $s = (s_1, \dots, s_m)$. Fix i, $1 \leq i \leq m$, and take $L = p_i$, the projection onto ith coordinate. Then $\widetilde{p_i}(x, r) = (x, r_i) \in N^{p_i}_x$ and $\widetilde{p_i}(x, s) = (x, s_i) \in N^{p_i}_x$, hence $r_i = s_i$. We have shown $r = s$ and the result follows. □

Theorem 7.1.4. *Let T be a minimal rotation on $\mathbb{T}$. If $\varphi \in CBV_0(\mathbb{T})$ and φ is not coboundary then T_φ has no minimal subsets.*

Proof. Identify $\mathbb{T}$ with $[0, 1) \bmod 1$ and let $Tx = x + \alpha \mod 1$, where α is an irrational number. Let $(q_n)_{n\geq 1}$ be the sequence of denominators in the continued fraction expansion of α.

Let us assume that $M \subset \mathbb{T} \times \mathbb{R}$ is a T_φ-minimal set and $(x, 0) \in M$. By Lemma 7.1.3 we find $\delta > 0$ and $\varepsilon > 0$ such that the positive semi-orbit $\{(T_\varphi)^n(x, 0) : n > 0\}$ of $(x, 0)$ intersects neither $B^- = (x - \delta, x + \delta) \times (-\varepsilon - 2\operatorname{Var}(\varphi), -\varepsilon)$ nor $B^+ = (x - \delta, x + \delta) \times (\varepsilon, \varepsilon + 2\operatorname{Var}(\varphi))$.

There exists an interval $I \subset (x - \delta, x + \delta)$ with $x \in I$, and a positive integer n such that every point of the orbit of x under T has the first return time to I equal either to q_n or to q_{n+1}. Now, by (7.1), we have $|\varphi^{(l)}(x)| \leq \varepsilon$ whenever $T^l x \in I$, $l > 0$ since the positive semi-orbit of $(x, 0)$ does not intersect $B^- \cup B^+$. Moreover, the set $\{l > 0 : T^l_\varphi(x, 0) \in I \times [-\varepsilon, \varepsilon]\} \subset \mathbb{N}$ has bounded gaps, thus the positive semi-orbit of $(x, 0)$ is bounded. Therefore, by [40, Theorem 14.11], φ is a coboundary. We have got a contradiction. □

Remark 7.1.5. In [91] E. A. Sidorov constructs for each irrational rotation on $\mathbb{T}$ point transitive cocycle without discrete orbits (recall that a discrete orbit is always a minimal set). Below, using rather standard methods, we generalize this showing that over every irrational rotation there exists a cocycle without minimal sets.

On the other hand A. S. Besicovitch [8] constructs a particular irrational rotation and a point transitive cocycle that admits a discrete orbit. There remains an open problem whether there exist point transitive cylinder cocycles with minimal sets other than discrete orbits.

Now we will show the following lemma.

Lemma 7.1.6. *For every minimal rotation T on $\mathbb{T}$ there exists $\varphi \in AC_0(\mathbb{T})$ that is point transitive.*

Proof. Assume that T is a minimal rotation on $\mathbb{T}$, $Tx = x + \alpha$, such that all $\varphi \in AC_0(\mathbb{T})$ are coboundaries, i.e. for every $\varphi \in AC_0(\mathbb{T})$ there exists $g_\varphi \in C(\mathbb{T})$ such that $\varphi = g_\varphi - g_\varphi \circ T$. By minimality of T we may assume that g_φ is a zero mean function. We have obtained a well defined linear map $CBV_0(\mathbb{T}) \supset AC_0(\mathbb{T}) \ni \varphi \longmapsto g_\varphi \in C_0(\mathbb{T})$. For the purpose of this proof we consider the space $AC_0(\mathbb{T})$ with the variation norm Var. With this norm the map $\varphi \longmapsto g_\varphi$ is continuous by the Closed Graph Theorem. Thus there is a constant $M \geq 0$ such that

$$\|g_\varphi\| \leq M \cdot \mathrm{Var}(\varphi) \tag{7.5}$$

for every $\varphi \in AC_0(\mathbb{T})$. However, we will see that there exists a sequence $(P_N)_{N\geq 0}$ of real polynomials on $\mathbb{T}$ such that

$$\lim_{N\to\infty} \frac{\|P_N\|}{\mathrm{Var}(P_N - P_N \circ T)} = \infty \tag{7.6}$$

(what will give a contradiction to (7.5)). To see this consider

$$P_N(x) = \sum_{n=-N}^{N} a_n e^{2\pi i n x}, \quad N \geq 0,$$

such that $a_n = a_{-n}$ for $n \geq 0$, $a_n = 0$ for $n \neq q_k$, $a_{q_k} > 0$ (here $(q_k)_{k\geq 1}$ is the sequence of denominators of the continued fraction expansion of α), and

$$\lim_{N\to\infty} \frac{\sum_{q_k \leq N} a_{q_k}}{(\sum_{q_k\leq N} a_{q_k}^2)^{1/2}} = \infty. \tag{7.7}$$

We have

$$P_N' - P_N' \circ T = 4\pi i \sum_{q_k \leq N} q_k a_{q_k} (1 - e^{2\pi i q_k \alpha}) e^{2\pi i q_k x}.$$

Since $|1 - e^{2\pi i q_k \alpha}| \leq \|q_k \alpha\|$ and $q_k \|q_k\alpha\| \leq 1$ (here $\|\beta\|$ denotes the distance of the number β from the set of integers), we have

$$\begin{aligned} \mathrm{Var}(P_N - P_N \circ T) &= \|P_N' - P_N' \circ T\|_{L^1} \\ &\leq \|P_N' - P_N' \circ T\|_{L^2} \leq \mathrm{const} \cdot \left(\sum_{q_k \leq N} a_{q_k}^2 \right)^{1/2}. \end{aligned} \tag{7.8}$$

Now, the equality $\|P_N\| = 2\sum a_{q_k}$ (recall that $\|\cdot\|$ denotes the supremum norm), (7.8), and the assumption (7.7) imply (7.6). We obtained a contradiction with (7.5) and we are done. □

For more general results on the existence of point transitive cocycles over irrational rotations see [70, Theorem 3 and 6].

Remark 7.1.7. Consider a linear map $C_0(\mathbb{T}) \ni f \overset{\Phi}{\longmapsto} f - f \circ T \in C_0(\mathbb{T})$. Assume for a moment that $\Phi(C_0(\mathbb{T})) \subset CBV_0(\mathbb{T})$. Then, by the Closed Graph Theorem, the map $\Phi\colon C_0(\mathbb{T}) \to CBV_0(\mathbb{T})$ is continuous. Thus there is a constant

$M > 0$ such that $\mathrm{Var}(f - f \circ T) \le M\|f\|$ for all $f \in C_0(\mathbb{T})$. On the other hand it is not difficult to find $f \in C_0(\mathbb{T})$ with $\|f\| = 1$ and arbitrary large variation of $f - f \circ T$, which is a contradiction. This rather standard consideration shows the following.

Lemma 7.1.8. *For every minimal rotation T on $\mathbb{T}$ there exists $f \in C_0(\mathbb{T})$ such that $f - f \circ T$ is not of bounded variation.*

Theorem 7.1.4 together with Remarks 7.1.5 and 7.1.7 show that there are also unbounded variation cocycles without minimal sets.

7.2. The problem of minimality for cylinder extensions of adding machines

Let $\overline{r} = (r_n)_{n\ge1}$ be a sequence of integers such that $r_n \ge 2$, $n \ge 1$. Set $\lambda_0 = 1$, $\lambda_n = r_1 \cdot \ldots \cdot r_n$, $n \ge 1$. Let

$$\mathbb{Z}(\overline{r}) = \left\{ \sum_{n=0}^{\infty} a_n\lambda_n : a_n \in \{0, \ldots , r_{n+1} - 1\} \right\} \tag{7.9}$$

be the compact group of $\overline{r}$-adic numbers with the product topology induced from $\prod_{n=0}^{\infty}\{0, \ldots , r_n - 1\}$. This topology may be defined by the metric $d(\sum a_n\lambda_n, \sum b_n\lambda_n) = 1/\lambda_m$, where $m = \min\{n : a_n \ne b_n\}$.

For $m \ge 1$ and $0 \le k < \lambda_m$ define the sets $W_k^m = [a_0\, a_1\, \ldots\, a_{m-1}]$ by

$$W_k^m = \left\{ x \in \mathbb{Z}(\overline{r}) : x = \sum_{n=0}^{\infty} x_n\lambda_n,\ x_i = a_i,\ i = 0, \ldots , m - 1 \right\}, \tag{7.10}$$

where $a_i \in \{0, \ldots , r_{i+1} - 1\}$ are such that $\sum_{i=1}^{m} a_i\lambda_i = k$. Let $\mathcal{W}^m = \{W_0^m, \ldots , W_{\lambda_m-1}^m\}$. Clearly the sets W_k^m are closed-open and $\bigcup \mathcal{W}^m = \mathbb{Z}(\overline{r})$. Let μ denote the normalized Haar measure on $\mathbb{Z}(\overline{r})$. Observe that $\mathrm{diam}(W_k^m) = \mu(W_k^m) = 1/\lambda_m$. We define a homeomorphism $T\colon \mathbb{Z}(\overline{r}) \to \mathbb{Z}(\overline{r})$ setting $Tx = x+1$ obtaining a minimal rotation on a compact metric monothetic group $\mathbb{Z}(\overline{r})$. Then the metric d defined above as well as the measure μ are T-invariant. Moreover $TW_k^m = W_{k+1}^m$, where $k + 1$ is taken mod λ_m. The flow $(\mathbb{Z}(\overline{r}), T)$ is called an *adding machine.*

Denote by $C(\mathbb{Z}(\overline{r}))$ the space (algebra) of all continuous real functions on $\mathbb{Z}(\overline{r})$. Equip $C(\mathbb{Z}(\overline{r}))$ with the topology of uniform convergence. Observe that each real function that is constant on elements of some $\mathcal{W}^m$ is continuous.

Definition 7.2.1. We say that $\varphi \in C(\mathbb{Z}(\overline{r}))$ has *bounded variation* if

$$\mathrm{Var}(\varphi) := \sup_{m\ge0} \mathrm{V}_m(\varphi) < \infty, \quad \text{where } \mathrm{V}_m(\varphi) = \sum_{k=0}^{\lambda_m-1} (\max_{W_k^m} \varphi - \min_{W_k^m} \varphi).$$

The family of such functions we denote by $CBV(\mathbb{Z}(\overline{r}))$, and as usual, $CBV_0(\mathbb{Z}(\overline{r}))$ stands for the subfamily of $CBV(\mathbb{Z}(\overline{r}))$ consisting of all functions with zero mean with respect μ.

Remark 7.2.2. Let (X, d) be a compact metric space and $\varphi: X \to \mathbb{R}$ be a continuous function. Recall that a function $M_\varphi = M: \mathbb{R}_+ \to \mathbb{R}$ defined by

$$M(h) = \sup_{d(x,y) \le h} |\varphi(x) - \varphi(y)|$$

is called a *continuity modulus* of φ. Now let us take $X = \mathbb{Z}(\overline{r})$ and consider a family of functions $\varphi \in C(\mathbb{Z}(\overline{r}))$ such that $M_\varphi(1/\lambda_k) = O(1/\lambda_k)$ (i.e. the sequence $(\lambda_k M_\varphi(1/\lambda_k))_{k \ge 1}$ is bounded). Since obviously

$$\mathrm{V}_m(\varphi) \le \lambda_m M(1/\lambda_m),$$

this family is contained in $CBV(X)$; actually this inclusion may be strict in general.

We intend to prove that a point transitive bounded variation cocycle over an adding machine does not admit minimal subset. Such a cocycle is constructed in Example 7.3.2. We start with a lemma, that contains a kind of the Denjoy–Koksma inequality for cocycles over adding machines.

Lemma 7.2.3. *Let $\varphi \in CBV(\mathbb{Z}(\overline{r}))$. Then, for every $x \in \mathbb{Z}(\overline{r})$,*

$$\left| \frac{1}{\lambda_m} \varphi^{(\lambda_m)}(x) - \int_{\mathbb{Z}(\overline{r})} \varphi(t)\, dt \right| \le \frac{1}{\lambda_m} \operatorname{Var}(\varphi).$$

Proof. Fixing $m \ge 1$ and $x \in \mathbb{Z}(\overline{r})$ we may assume that $x \in W_0^m$. Then we have

$$\begin{aligned} \left| \varphi^{(\lambda_m)}(x) - \lambda_m \int_{\mathbb{Z}(\overline{r})} \varphi(t)\, dt \right| &\le \sum_{k=0}^{\lambda_m - 1} \left| \varphi(T^k x) - \lambda_m \int_{W_k^m} \varphi(t)\, dt \right| \\ &\le \sum_{k=0}^{\lambda_m - 1} \lambda_m \int_{W_k^m} |\varphi(T^k x) - \varphi(t)|\, dt \\ &\le \sum_{k=0}^{\lambda_m - 1} \lambda_m \int_{W_k^m} (\max_{W_k^m} \varphi - \min_{W_k^m} \varphi)\, dt = \mathrm{V}_m(\varphi) \le \operatorname{Var}(\varphi). \qquad \square \end{aligned}$$

In case of $\varphi \in CBV_0(X)$ the inequality from Lemma 7.2.3 takes the form

$$\|\varphi^{(\lambda_m)}\| \le \operatorname{Var}(\varphi). \tag{7.11}$$

Theorem 7.2.4. *Let (X, T) be an adding machine. If $\varphi \in CBV_0(X)$ and φ is not a coboundary then T_φ admits no minimal subsets.*

Proof. The proof is similar to the proof of Theorem 7.1.4. We consider the rigidity time $(\lambda_m)_{m \ge 1}$ instead of $(q_k)_{k \ge 1}$ and use (7.11) instead of (7.1). The interval I is replaced by one of the levels of $\mathcal{W}^n$ for appropriate n. $\square$

7.3. Existence of point transitive cocycles over compact monothetic groups

Our next aim is to show that all minimal rotations on compact infinite metrizable monothetic groups admit point transitive real cocycles. W. H. Gottschalk and G. A. Hedlund [40] have developed a theory of real cocycles over minimal rotations on connected and locally connected monothetic groups. However also rotations on disconnected monothetic groups may admit point transitive cocycles.

Example 7.3.1. Let $X = \mathbb{Z}_2 \times [0,1)$ and $T(i,x) = (i+1, x+\alpha)$, where the addition on the first coordinate is taken mod 2 while the addition on the second coordinate is taken mod 1. Take a continuous function $\varphi\colon [0,1) \to \mathbb{R}$ and define $\widetilde{\varphi}\colon X \to \mathbb{R}$ by $\widetilde{\varphi}(i,x) = \varphi(x)$. Assume $\widetilde{\varphi}$ to be coboundary, i.e. there exists a continuous $\widetilde{g}\colon X \to \mathbb{R}$ with $\widetilde{\varphi}(i,x) = \widetilde{g}(i+1, x+\alpha) - \widetilde{g}(i,x)$. Defining $g\colon [0,1) \to \mathbb{R}$ by $g(x) = (\widetilde{g}(0,x) + \widetilde{g}(1,x))/2$ we have

$$g(x+\alpha) - g(x) = \frac{1}{2}(\widetilde{\varphi}(1,x) + \widetilde{\varphi}(0,x)) = \varphi(x).$$

Now, taking a point transitive cocycle φ on $([0,1), \alpha)$ we get, by the construction above, a point transitive cocycle $\widetilde{\varphi}$ over a disconnected, locally connected monothetic group.

Notice that in the example above, g is an integral of $\widetilde{g}$ with respect to the normalized Haar measure on the kernel of the projection onto the second coordinate. This simple observation gives rise to Lemma 7.3.3

Now we show that also adding machines, that are not locally connected, admits point transitive cocycles.

Example 7.3.2. Let $X = \mathbb{Z}(\overline{r})$ and T be a minimal rotation on X. Put

$$\chi_s\bigg(\sum_{n=0}^{\infty} a_n \lambda_n\bigg) = \exp\bigg(2\pi i \frac{\sum_{n=0}^{s-1} a_n \lambda_n}{\lambda_s}\bigg).$$

Observe that (the character group) $\widehat{X} = \{\chi_s^l : s \geq 1,\ 0 \leq l \leq \lambda_s - 1\}$. Define $\varphi\colon X \to \mathbb{R}$ by

$$\varphi(x) = \sum_{s=1}^{\infty} \varphi_s = \sum_{s=1}^{\infty} \frac{\chi_s + \chi_s^{-1}}{\lambda_s}.$$

Clearly $\varphi \in C_0(X)$. Assume that φ is a coboundary, $\varphi = g \circ T - g$ for some $g \in C_0(X)$. Represent $g = \sum_{s \geq 1} \sum_{0 < t < \lambda_s} a_{s,t} \chi_s^t$. For $s \geq 1$ we have

$$a_{s,t} = \begin{cases} \dfrac{1}{\lambda_s(\exp(2\pi i/\lambda_s) - 1)} & \text{if } t = 1, \\ 0 & \text{if } 1 < t < \lambda_s - 1, \\ \dfrac{1}{\lambda_s(\exp(-2\pi i/\lambda_s) - 1)} & \text{if } t = \lambda_s - 1. \end{cases}$$

Simple calculations show that

$$|a_{s,1}| = |a_{s,\lambda_s-1}| = \frac{1}{2\lambda_s \sin(\pi/\lambda_s)} \geq \frac{1}{2\pi},$$

which is impossible. Therefore φ is a point transitive cocycle over the adding machine. It turns out that $\varphi \in CBV_0(X)$. To see this observe first that χ_s, hence also φ_s, is constant on the levels of $\mathcal{W}^t$ for $t \geq s$. Moreover, φ_s takes on levels of $\mathcal{W}^s$ the values $2\cos(2\pi l/\lambda_s)/\lambda_s$, $0 \leq l < \lambda_s$, as χ_s takes the values $\exp(2\pi i l/\lambda_s)$, $0 \leq l < \lambda_s$. Therefore

$$\max_{W_k^m} \varphi - \min_{W_k^m} \varphi \leq \sum_{s>m} \bigl(\max_{W_k^m} \varphi_s - \min_{W_k^m} \varphi_s\bigr) \leq \sum_{s>m} \frac{4}{\lambda_s}$$

for $m \geq 1$. Thus

$$\mathrm{V}_m(\varphi) = \sum_{j=0}^{\lambda_m-1} \bigl(\max_{W_k^m} \varphi - \min_{W_k^m} \varphi\bigr) \leq 4 \sum_{s>m} \frac{\lambda_m}{\lambda_s} < \frac{8}{r_{m+1}} \leq 4.$$

Consequently

$$\mathrm{Var}(\varphi) = \sup_{m\geq 1} \mathrm{V}_m(\varphi) \leq 4$$

and φ has bounded variation.

We intend to prove that each minimal rotation on a compact metric monothetic group admits a point transitive real cocycle (Theorem 7.3.6). Moreover, for each $m \geq 1$ and for each linear subspace $V \subset \mathbb{R}^m$ there exists a continuous cocycle with range in $\mathbb{R}^m$ such that $E(\varphi) = V$ (Corollary 7.3.7). To get these results we proceed as follows.

Lemma 7.3.3. *Let $\pi: X \to Y$ be a continuous group epimorphism of compact metric monothetic groups. Let $T: X \to X$, $Tx = x + \alpha$, $S: Y \to Y$, $Sy = y + \beta$, where $\beta = \pi(\alpha)$, be minimal rotations. Then for any point transitive cocycle $\varphi: Y \to \mathbb{R}$ the cocycle $\widetilde{\varphi}: X \to \mathbb{R}$ defined by $\widetilde{\varphi}(x) = \varphi(\pi(x))$ is also point transitive.*

Proof. Assume that $\varphi: Y \to \mathbb{R}$ is a point transitive cocycle. Then $\int \varphi\, d\mu_Y = 0$, hence $\int \widetilde{\varphi}\, d\mu_X = 0$, where μ_X and μ_Y denote the normalized Haar measures on X and Y, respectively. By [64, Theorem 1], either $T_{\widetilde{\varphi}}$ is point transitive or $\widetilde{\varphi}$ is a coboundary. Assume $\widetilde{\varphi}: X \to \mathbb{R}$, $\widetilde{\varphi} = \varphi \circ \pi$ to be a coboundary over the rotation by $\alpha \in X$, i.e. there exists a continuous $\widetilde{g}: X \to \mathbb{R}$ with $\widetilde{\varphi}(x) = \widetilde{g}(x + \alpha) - \widetilde{g}(x)$. Denote $K = \ker \pi$ and identify Y with X/K; then $\beta = \alpha + K$. Define a continuous function $g: Y \to \mathbb{R}$ by $g(x + K) = \int_K \widetilde{g}(x + k)\, dk$. We have

$$\begin{aligned} g(x + \alpha + K) - g(x + K) &= \int_K (\widetilde{g}(x + \alpha + k) - \widetilde{g}(x + k))\, dk \\ &= \int_K \widetilde{\varphi}(x + k)\, dk = \int_K \varphi(x + K)\, dk = \varphi(x + K). \end{aligned}$$

Thus we have shown φ to be a coboundary over the rotation by β, which is a contradiction. □

In the following theorem we generalize Lemma 7.3.3.

Theorem 7.3.4. *Let $\pi\colon X \to Y$ be a continuous group epimorphism of compact metric monothetic groups, and let $T\colon X \to X$, $S\colon Y \to Y$, where $\pi \circ T = S \circ \pi$, be minimal rotations. Then $E(\varphi) = E(\varphi \circ \pi)$ for each continuous cocycle $\varphi\colon Y \to \mathbb{R}^m$.*

Proof. If φ is transient, then clearly $\varphi \circ \pi$ is also transient. Suppose S_φ is conservative, hence regular (see [72, Theorem 4.9]). Let $L\colon \mathbb{R}^m \to \mathbb{R}$ be linear. By [72, Theorem 3.5], both $E(\varphi)$ and $E(\varphi\circ\pi)$ are linear subspaces of $\mathbb{R}^m$. Then, by [74, Proposition 3.1],

$$E(L \circ \varphi) = L(E(\varphi)), \quad E(L \circ \varphi \circ \pi) = L(E(\varphi \circ \pi)).$$

By Lemma 7.3.3, $E(L \circ \varphi) = E(L \circ \varphi \circ \pi)$ so $L(E(\varphi)) = L(E(\varphi \circ \pi))$. As L is arbitrary, $E(\varphi) = E(\varphi \circ \pi)$. □

Remark 7.3.5. Define $\varphi\colon [0,1) \to \mathbb{R}$ setting

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{1}{q_k} \cos 2\pi q_k x = \frac{1}{2} \sum_{k\geq 1} \frac{1}{q_k^2} (e^{2\pi i q_k x} + e^{-2\pi i q_k x})$$

(by Euler's formula). Since $\sum 1/q_k$ converges, φ is a well defined continuous (and zero mean) function. We will show that φ is point transitive (compare [40, 14.14]). Suppose for the contrary that φ is a coboundary, i.e. $\varphi = g - g \circ T$ for some continuous function g. Let $a_n = \int_{\mathbb{T}} g(x) e^{-2\pi n i x}\, dx$, $n \in \mathbb{Z}$. By Lebesgue–Riemann Lemma $\lim a_n = 0$. Simple calculations show that

$$\begin{aligned} \frac{1}{2} \cdot \frac{1}{q_k} &= a_{\pm q_k}(1 - e^{\pm 2\pi i q_k \alpha}), && k \geq 1, \\ a_n &= 0, && n \neq \pm q_k,\ k \geq 1. \end{aligned}$$

However $|e^{2\pi i q_k \alpha} - 1| < 8/q_{k+1}$ and it follows that

$$16|a_{q_k}| = \frac{8}{q_k} \cdot \frac{1}{|e^{2\pi i q_k \alpha} - 1|} > \frac{q_{k+1}}{q_k} \geq 1,$$

which gives a contradiction. Thus the cocycle φ is not a coboundary, hence φ is point transitive.

Using the point transitive cocycle φ we have defined above one may construct for any m a point transitive cocycle $\widetilde{\varphi}\colon \mathbb{T} \to \mathbb{R}^m$. To see this consider m pair-wise disjoint subsequences $(c_{k,j})_{k\geq 1}$, $j = 1, \dots, m$ of the sequence $(q_k)_{k\geq 1}$ such that no of the sequences $((1/c_{k,j})|e^{2\pi i c_{k,j}\alpha} - 1|)_{k\geq 1}$, $j = 1, \dots, m$, is convergent. Setting

$$\varphi_j(x) = \sum_{k\geq 1} \frac{1}{c_{k,j}} \cos 2\pi c_{k,j} x, \quad j = 1, \dots, m$$

we get that no non-zero combination $b_1\varphi_1 + \ldots + b_m\varphi_m$ is a coboundary. By Atkinson's theorem ([6, Theorem 1] or [72, Proposition 4.8]), the cocycle $\widetilde{\varphi} = (\varphi_1, \ldots, \varphi_m)$ is point transitive.

Using Example 7.3.2, Lemma 7.3.3 and Remark 7.3.5 we get the following.

Theorem 7.3.6. *Assume that X is an infinite compact metric monothetic group. Let $T\colon X \to X$ be a minimal rotation on X. Then (X, T) admits a point transitive real cocycle.*

Remark 7.3.5 allows us to give a slight generalization of Theorem 7.3.6.

Corollary 7.3.7. *Assume that X is an infinite compact metric monothetic group. Let $T\colon X \to X$ be a minimal rotation. Then for each integer $m \geq 1$ and for each linear subspace $V \subset \mathbb{R}^m$ there exists a continuous cocycle $\varphi\colon X \to \mathbb{R}^m$ such that $E(\varphi) = V$.*

Proof. Take a linear subspace $V \subset \mathbb{R}^m$. Let $\psi = (\varphi_1, \ldots, \varphi_m)\colon X \to \mathbb{R}^m$ be a point transitive cocycle. If $\dim V = 0$, then each coboundary is good for us. Suppose $\dim V = k > 0$. Denote $\overline{e}_1, \ldots, \overline{e}_m$ to be the standard base of $\mathbb{R}^m$. Without loosing of generality we may assume that V is generated by $\overline{e}_1, \ldots, \overline{e}_k$. Indeed, by [72, Theorem 4.9], all zero mean cocycles with values in $\mathbb{R}^m$ are regular, and application of [74, Proposition 3.1] finishes the argumentation. Let $\phi = (\varphi_1, \ldots, \varphi_k, 0, \ldots, 0)$. Again by [74, Proposition 3.1], we have that $E(\varphi) = V$. $\square$

CHAPTER 8

SOME APPLICATIONS OF GROUPS OF ESSENTIAL VALUES OF COCYCLES IN TOPOLOGICAL DYNAMICS

8.1. Preliminaries

8.1.1. Measure-theoretic context. It is easy to observe that if G is Abelian and φ is cohomologous to ψ, then $E(\varphi) = E(\psi)$. This fails when G is not Abelian, nevertheless A. Danilenko has shown that in measure-theoretic ergodic theory groups of essential values of cocycles have the following property.

Theorem 8.1.1 ([13, Proposition 1.1])**.** *If the cocycles φ and ψ are regular and cohomologous, then the groups $E(\varphi)$ and $E(\psi)$ are conjugate in G, i.e. $E(\psi) = g^{-1}E(\varphi)g$ for some $g \in G$.*

In Section 8.2 we will give an example that in topological dynamics Theorem 8.1.1 is not true if we omit the assumption of regularity (Example 8.2.3). This a topological version of the result of [5].

In measure-theoretic ergodic theory regular cocycles are characterized by Proposition 1.4.10 saying that a cocycle $\varphi : X \to G$, where G is a locally compact Abelian group, is regular if and only if $E_\infty(\widetilde{\varphi}) = \{0\}$, where $\widetilde{\varphi}: X \to G/E(\varphi)$ is defined by by $\widetilde{\varphi}(x) = \varphi(x)E(\varphi)$. Clearly, always $E_\infty(\widetilde{\varphi}) \subset \{0, \infty\}$. The equivalence in Proposition 1.4.10 is shown making use of Proposition 1.4.7(b) and of the existence of a measurable selector for the quotient map $G \to G/E(\varphi)$. In the topological case continuous selectors may not exist. We will show that Proposition 1.4.10 is not true in topological dynamics – see Proposition 8.2.2.

Assume now that $(Y, \mathcal{C}, \nu)$ is a standard probability space. Consider the set $\mathrm{Aut}(Y, \mathcal{C}, \nu)$ of all automorphisms of $(Y, \mathcal{C}, \nu)$. Then considering the map

$$\mathrm{Aut}(Y, \mathcal{C}, \nu) \ni S \mapsto U_S: L^2(Y, \mathcal{C}, \nu) \to L^2(Y, \mathcal{C}, \nu), \quad U_S(f) = f \circ S$$

we may see $\mathrm{Aut}(Y, \mathcal{C}, \nu)$ as a closed subset of the group $U(L^2(Y, \mathcal{C}, \nu))$ of unitary operators on $L^2(Y, \mathcal{C}, \nu)$ in the strong operator topology. With this topology the set $\mathrm{Aut}(Y,\mathcal{C},\nu)$ is a Polish space. Given a locally compact group G, its representation $\Gamma=\{\gamma_g : g\in G\}$ and a measurable cocycle $\varphi: X\to G$ we may consider the Rokhlin cocycle extension $T_{\varphi,\Gamma}$ defined by $T_{\varphi,\Gamma}(x,y)=(Tx,\gamma_{\varphi(x)}(y))$ – see (2). For cohomologous cocycles φ and ψ, $\psi(x)=(f(Tx))^{-1}\varphi(x)f(x)$, the corresponding skew products $T_{\varphi,\Gamma}$ and $T_{\psi,\Gamma}$ are isomorphic via the map $(x,y)\mapsto(x,f(x)^{-1}(y))$. A complete description of all invertible elements from the centralizer of the automorphism $T_{\varphi,\Gamma}$ for a locally compact second countable group G is given in the following.

Proposition 8.1.2 ([63, Proposition 5]). *Let $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be an ergodic automorphism of a probability standard space, $(Y,\mathcal{C},\nu)$ a probability standard space, Γ a closed locally compact second countable subgroup of $\mathrm{Aut}(Y,\mathcal{C},\nu)$. Let $\varphi: X\to G$ be an ergodic cocycle. Then each invertible element $\widetilde{R}$ of $C(T_{\varphi,\Gamma})$ is of the form*

$$\widetilde{R}(x,y)=(Rx,f(x)\circ W(y)),$$

where $R\in C(T)$, $f: X\to G$ is measurable and $W\in\mathrm{Aut}(Y,\mathcal{C},\nu)$ normalizes the group Γ in $\mathrm{Aut}(Y,\mathcal{C},\nu)$.

We will give analogous characterization of invertible elements of $C(T_{\varphi,\Gamma})$ in topological dynamics context (see Theorem 8.3.10).

8.1.2. Topological dynamics context. Assume that G is a locally compact group with the unit element e, X a compact Hausdorff space and let $\Gamma=\{\gamma_g : g\in G\}$ be a left continuous action of G on X, i.e. there is a continuous map $\gamma: G\times X\to X$ satisfying the following conditions:

$$\gamma(e,x)=x,\quad\text{for all } x\in X,\tag{8.1}$$

$$\gamma(g_1g_2,x)=\gamma(g_1,\gamma(g_2,x))\quad\text{for all } g_1,g_2\in G,\ x\in X.\tag{8.2}$$

As usual we denote $\gamma(g,\cdot)=\gamma_g$. In what follows we will assume that all actions of topological group we consider are effective, i.e. $\gamma_g=\mathrm{Id}_X$ implies $g=e$.

For an Abelian group G, cohomologous cocycles have the same essential values (see Proposition 5.2.2(b)):

Proposition 8.1.3. *Let (X,T) be a compact flow, G a locally compact Abelian group. If $\varphi,\xi: X\to G$ are continuous maps, then*

$$E_\infty(\varphi)=E_\infty((\xi\circ T)^{-1}\varphi\xi).$$

This is not true when G is not Abelian, even for the groups of essential values – see Example 8.2.3.

For regular cocycle φ the following equality $E(\widetilde{\varphi})=\{0\}$ holds (see Corollary 6.1.4), where $\widetilde{\varphi}(x)=\varphi(x)E(\varphi)\in G/E(\varphi)$. In measure-theoretic case the equality $E(\widetilde{\varphi})=\{0\}$ is equivalent to regularity of φ. Proposition 8.2.2 shows that this is not true in topological dynamics.

8.2. Counterexamples in topological dynamics

First we present a simple example of an extension $\widetilde{T} \to T$ of topological flows such that $\widetilde{T}$ is not of the form (1) (see page 7).

Example 8.2.1. Let $\mathbb{T}$ be the unit circle represented as the interval $[0,1)$. Consider $\widetilde{T}: \mathbb{T} \to \mathbb{T}$, $\widetilde{T}x = x + \alpha \pmod 1$, where α is irrational. Then $T = \widetilde{T}^2: \mathbb{T} \to \mathbb{T}$, $Tx = x + 2\alpha \pmod 1$, is a factor of $\widetilde{T}$ with two-point fibers. It is easy to check that $\widetilde{T}$ and T are not isomorphic. Clearly $\widetilde{T}$ is not isomorphic to any skew product $(\mathbb{T} \times Y, T_\psi)$ with continuous $\psi: \mathbb{T} \to \mathrm{Hom}(Y,Y)$.

The following proposition defines a family of topological counterexamples for valid in ergodic theory Proposition 1.4.10.

Proposition 8.2.2. *Assume that (X,T) is a compact metric minimal flow, $\varphi: X \to \mathbb{R}^m$ an ergodic cocycle. Define*

$$\overline{T}: X \times \mathbb{R}^m/\mathbb{Z}^m \to X \times \mathbb{R}^m/\mathbb{Z}^m,$$
$$\overline{T}(x, g+\mathbb{Z}^m) = (Tx, \varphi(x) + g + \mathbb{Z}^m).$$

Let $\psi: X \times \mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}^m$, $\psi(x, g+\mathbb{Z}^m) = \varphi(x)$. Then $E(\psi) = \mathbb{Z}^m$, $E_\infty(\widetilde{\psi}) = \{0\}$, where $\widetilde{\psi}: X \times \mathbb{R}^m/\mathbb{Z}^m \to \mathbb{R}^m/E(\psi) = \mathbb{R}^m/\mathbb{Z}^m$, $\widetilde{\psi}(x, g+\mathbb{Z}^m) = \psi(x, g+\mathbb{Z}^m) + \mathbb{Z}^m$, and ψ is not regular. If moreover (X,T) is distal, then $(X \times \mathbb{R}^m/\mathbb{Z}^m, \overline{T})$ is minimal.

Proof. Clearly $\overline{T}$ is topologically ergodic. If moreover T is distal, $\overline{T}$ is also distal. Thus $\overline{T}$ is minimal provided T is distal. Let us compute $E(\psi)$. Let $g_0 \in E(\psi)$, $g_0 \neq 0$. Then for any nonempty open sets $U \subset X$, $W \subset \mathbb{R}^m/\mathbb{Z}^m$, $g_0 \in V \subset \mathbb{R}^m$, we can find an $n \in \mathbb{Z}$ such that

$$(U \times W) \cap \overline{T}^{-n}(U \times W) \cap \{(x, g+\mathbb{Z}^m) \colon \psi^{(n)}(x, g\mathbb{Z}^m) \in V\} \neq \emptyset.$$

Now, if $(x, g+\mathbb{Z}^m)$ belongs to the set above, then $\psi^{(n)}(x, g+\mathbb{Z}^m) \in V$, $x, T^n x \in U$, $g+\mathbb{Z}^m$, $\varphi^{(n)}(x) + g + \mathbb{Z}^m \in W$, $\varphi^{(n)}(x) \in V$. Consider the sequences

$$U = U_1 \supset U_2 \supset \dots, \quad W = W_1 \supset W_2 \supset \dots, \quad V = V_1 \supset V_2 \supset \dots$$

of open sets with

$$\bigcap_{n \geqslant 1} U_i = \{x\}, \quad \bigcap_{i \geqslant 1} W_i = \{g + \mathbb{Z}^m\}, \quad \bigcap_{i \geqslant 1} V_i = \{g_0\}.$$

Thus we can choose $x_i \in U_i$ with x_i, $T^{n_i} x_i \to x$, $g_i + \mathbb{Z}^m, \varphi^{(n_i)}(x_i) + g_i + \mathbb{Z}^m \to g + \mathbb{Z}^m$, $\varphi^{(n_i)}(x_i) \to g_0$. This implies $\varphi^{(n_i)}(x_i) + \mathbb{Z}^m \to \mathbb{Z}^m$, so $g_0 \in \mathbb{Z}^m$.

Suppose now that $h \in \mathbb{Z}^m$. Take nonempty open sets $U \subset X$, $W \subset \mathbb{R}^m/\mathbb{Z}^m$, and fix $h \in V \subset \mathbb{R}^m$. Find open sets $h \in V_0 \subset V$ and $W_0 \subset W$ such that $V_0 + W_0 \subset W$. As φ is ergodic, there exists $n \in \mathbb{Z}$ such that the set $U \cap T^{-1}U \cap \{x : \varphi^{(n)}(x) \in V_0\}$ is non-empty, say z belongs to it.

Let $w_0+\mathbb{Z}^m \in W_0$. Then $z, T^n z \in U$, $w_0+\mathbb{Z}^m \in W$, $\varphi^{(n)}(z)+w_0+\mathbb{Z}^m \in V_0+W_0 \subset W$ and $\varphi^{(n)}(z) \in V_0 \subset V$. Thus

$$(z, w_0+\mathbb{Z}^m) \in (U\times W)\cap \widetilde{T}^{-n}(U\times W)\cap\{\psi^{(n)}\in V\} \quad \text{and} \quad h\in E(\psi).$$

We have shown that $E(\psi)=\mathbb{Z}^m$.

Now, if ψ were regular, ψ would have the form

$$\varphi(x)=\psi(x,g+\mathbb{Z}^m)=F(x,g+\mathbb{Z}^m)-F\circ\overline{T}(x,g+\mathbb{Z}^m)+\chi(x,g+\mathbb{Z}^m),$$

where $\chi\colon X\times\mathbb{R}^m/\mathbb{Z}^m\to\mathbb{Z}^m$. Integrating both sides of the above equation over $\mathbb{R}^m/\mathbb{Z}^m$ with respect to the normalized Haar measure we get

$$\varphi(x)=f(x)-f(Tx)+\chi_0(x),$$

where $\chi_0(x)$ takes its values in $\mathbb{Z}^m$ (since $\mathbb{R}^m/\mathbb{Z}^m$ is connected), which is impossible as φ is ergodic. Thus ψ is not regular.

Now observe that since $\mathbb{R}^m/\mathbb{Z}^m$ is compact, $\infty\notin E_\infty(\widetilde{\psi})$. □

It follows from Theorem 6.3.8 that each zero mean cocycle defined on a minimal rotation on a compact monothetic metric group and with values in $\mathbb{R}^m$, is regular. Taking in Proposition 8.2.2 a minimal rotation on a circle as (X,T) with topologically ergodic continuous $\varphi\colon\mathbb{T}\to\mathbb{R}$ we get that the compact flow $(\mathbb{T}\times\mathbb{R}/\mathbb{Z},\widetilde{T})$ is minimal and distal. Moreover, this flow admits a non-regular real cocycle ψ with $E_\infty(\widetilde{\psi})=\{0\}$. This shows that the theory of topological cocycles is more complex than this theory in measure-theoretic context.

Below we will present an example of two cohomologous cocycles with values in (non-abelian) group $SL(2,\mathbb{R})$ such that their groups of essential values are not conjugate.

Example 8.2.3. Let $X=\{0,1\}^{\mathbb{Z}}$ be the set of all 0–1 bisequences with product topology. For $x\in X$ denote by $x[n]$ the nth coordinate of x and let $x[n,m]=x[n]x[n+1]\dots x[m]$ for $m\geqslant n$. The product topology on X is defined by the metric

$$d(x,y)=\big(1+\min\{|n| : x[n]\neq y[n]\}\big)^{-1}.$$

Let $T\colon X\to X$ be left side shift, $Tx[n]=x[n+1]$. Then the flow (X,T) is topologically ergodic. Define $f\colon X\to\mathbb{Z}$, $f(x)=(-1)^{x[0]}$. Clearly f is continuous and has zero mean with respect to the Bernoulli probability measure $(1/2,1/2)$ on X. Now we will show that f is ergodic i.e. $T_f\colon X\times\mathbb{Z}\to X\times\mathbb{Z}$ is point transitive. To do this take an arbitrary positive integer m and fix $B=a_{-m}a_{-m+1}\dots a_0a_1\dots a_m$, where all a_i are either zero or one. Let $U=\{x\in X : x[-m,m]=B\}$. Set $n=4m+3$. Denote $\widetilde{B}=\widetilde{a}_{-m}\dots\widetilde{a}_m$, where $\widetilde{a}=1-a$ for $a\in\{0,1\}$. Choose an $x_0\in X$ satisfying

$$\begin{aligned} x_0[-m,m]&=B, & x_0[m+1,3m+1]&=\widetilde{B},\\ x_0[3m+2]&=0, & x_0[3m+3,5m+3]&=B. \end{aligned}$$

Then $x_0 \in U$, $T^n x_0 \in U$. We also have

$$\begin{aligned} f^{(n)}(x_0) &= \sum_{i=0}^{4m+2} (-1)^{x_0[i]} \\ &= \sum_{i=0}^{m} (-1)^{x_0[i]} + \sum_{i=m+1}^{3m+1} (-1)^{x_0[i]} + (-1)^{x_0[3m+2]} + \sum_{i=3m+3}^{4m+3} (-1)^{x_0[i]} \\ &= \sum_{i=0}^{m} (-1)^{a_i} + \sum_{i=-m}^{m} (-1)^{1-a_i} + 1 + \sum_{i=-m}^{-1} (-1)^{a_i} = 1. \end{aligned}$$

As U was arbitrary, we conclude that $1 \in E(f)$. Since $E(f)$ is a group and f is integer-valued, $E(f) = \mathbb{Z}$. By Proposition 5.2.3, f is ergodic.

Now we define a continuous map $\varphi\colon X \to SL(2,\mathbb{R})$ setting

$$\varphi(x) = \begin{bmatrix} 1 & f(x) \\ 0 & 1 \end{bmatrix}.$$

Then clearly

$$E(\varphi) = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

Define a continuous map $\xi\colon X \to SL(2,\mathbb{R})$ by

$$\xi(x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } x[0] = 0, \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} & \text{if } x[0] = 1. \end{cases}$$

Let $\psi\colon X \to SL(2,\mathbb{R})$ be defined by

$$\psi(x) = (\xi(Tx))^{-1} \varphi(x) \xi(x).$$

We will show that $E(\psi)$ is trivial, hence not conjugate to $E(\varphi)$. To prove this take $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in E(\psi)$. Then for each $x \in X$ there exists a sequence $(n_i)_{i \geqslant 1}$ of integers and a sequence $(x_i)_{i \geqslant 1}$ such that

$$x_i \to x, \quad T^{n_i} x_i \to x, \quad \psi^{(n_i)}(x_i) \to \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Suppose first that $x[0] = 0$, we may assume that $x_i[0] = T^{n_i}[0] = 0$ for all $i \geqslant 1$. Then

$$\begin{aligned} \psi^{(n_i)}(x_i) &= (\xi(T^{n_i} x_i)^{-1} \varphi(x_i) \xi(x_i) \\ &= \varphi^{(n_i)}(x_i) = \begin{bmatrix} 1 & f^{(n_i)}(x_i) \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned}$$

that means $c = 0$, $a = 1$, $d = 1$, i.e. $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$.

Assume now that $x[0]=1$. Then we may assume that $x_i[0]=T^{n_i}[0]=1$ for all $i\geqslant 1$. Then

$$\begin{aligned}\psi^{(n_i)}(x_i) &= \xi(T^{n_i}x_i)^{-1}\varphi(x_i)\xi(x_i)\\ &= \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & \varphi^{(n_i)}(x_i)\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}1-\varphi^{(n_i)}(x_i) & \varphi^{(n_i)}(x_i)\\ -1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1-\varphi^{(n_i)}(x_i) & \varphi^{(n_i)}(x_i)\\ -\varphi^{(n_i)}(x_i) & 1+\varphi^{(n_i)}(x_i)\end{bmatrix}\to\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}.\end{aligned}$$

Therefore $\varphi^{(n_i)}(x_i)\to 0$ and $b=0$.

8.3. Isomorphisms of Rokhlin cocycle extensions of point transitive flows

The following proposition is a topological version of Proposition 1.4.11.

Proposition 8.3.1. *Let (X,T) be a $\mathbb{Z}$-flow. Assume that G, H are locally compact Abelian groups and let $\pi\colon G\to H$ be a continuous group homomorphism. If $\varphi\colon X\to G$ is a continuous map, then*

$$\overline{\pi(E(\varphi))}\subset E(\pi\circ\varphi).$$

If additionally φ is regular, then

$$\overline{\pi(E(\varphi))}=E(\pi\circ\varphi).$$

Proof. The inclusion is clear. Assume now that φ is regular, that means $\varphi=(f\circ T)^{-1}\psi f$, where $f\colon X\to G$, $\psi\colon X\to E(\varphi)$ are continuous maps. Let $g\in E(\pi\circ\varphi)$. To prove that $g\in\overline{\pi(E(\varphi))}$, fix an open neighbourhood V of the unit element in H. We will show that $(gV)\cap\pi(E(\varphi))\neq\emptyset$. Let V_0 be an open symmetric neighbourhood of the unit element in H such that $V_0V_0\subset V$. Take an open $U\subset X$ such that $x,y\in U$ implies $\pi(f(y)^{-1})\pi(f(x))\in V_0$. Now, as $g\in E(\pi\circ\varphi)$, there exists n such that the set $U\cap T^{-n}U\cap\{\pi\circ\varphi^{(n)}\in gV_0\}$ is nonempty, say x belongs to it. Then $x\in U$, $T^nx\in U$, and, by our assumption, $\pi(f(T^nx)^{-1})\pi(f(x))\in V_0$. Moreover

$$gV_0\ni\pi\circ\varphi^{(n)}(x)=\pi(f(T^nx)^{-1})\pi(f(x))\pi\circ\psi^{(n)}(x)$$

and we get

$$\pi\circ\psi^{(n)}(x)\in gV_0V_0\subset gV\quad\text{and}\quad\pi\circ\psi^{(n)}(x)\in\pi(E(\varphi)),$$

which finishes the proof. $\square$

Theorem 8.3.2. *Let (X,T) be a compact metric point transitive flow, G a locally compact second countable Abelian group, Y a compact metric space, $\Gamma = \{\gamma_g : g \in G\}$ an effective left continuous actions of G on Y, $\varphi: X \to G$ a continuous map such that T_φ is point transitive. Assume moreover that $\Gamma \subset \mathrm{Hom}(Y,Y)$ is a closed subgroup. Let $\widehat{S} \in C(T_{\varphi,\Gamma})$ be an invertible extension of some $S \in C(T)$. Then there exist: $p \in \mathrm{Hom}(Y,Y)$, a topological group automorphism $v: G \to G$ and a continuous map $\psi: X \to G$ such that*

$$\widehat{S}(x,y) = (Sx, \gamma_{\psi(x)} \circ p(y)), \tag{8.3}$$

and p satisfies

$$\gamma_{v(g)} = p \circ \gamma_g \circ p^{-1}, \quad g \in G. \tag{8.4}$$

Proof. Let $\widehat{S}(x,y) = (Sx, \kappa(x,y))$, where $\kappa: X \times Y \to Y$ is a continuous map. Because $\widehat{S}$ commutes with $T_{\varphi,\Gamma}$, we have

$$\gamma_{\varphi(Sx)}\kappa(x,y) = \kappa(Tx, \gamma_{\varphi(x)}(y)). \tag{8.5}$$

For $x \in X$ let $\kappa_x: Y \to Y$, $\kappa_x(y) = \kappa(x,y)$. Then (8.5) may be written as

$$\gamma_{\varphi(Sx)} \circ \kappa_x = \kappa_{Tx} \circ \gamma_{\varphi(x)}. \tag{8.6}$$

Consider now the map

$$X \ni x \mapsto \kappa_x \in \mathrm{Hom}(Y,Y).$$

We will show that the map above is continuous. Take $\varepsilon > 0$. Find $\delta_1 > 0$ such that $d((x,y),(x',y')) < \delta_1$ implies both $d(\widehat{S}(x,y), \widehat{S}(x',y')) < \varepsilon/2$ and $d(\widehat{S}^{-1}(x,y), \widehat{S}^{-1}(x',y')) < \varepsilon/2$. Now, find $\delta > 0$ such that $\delta < \delta_1$ and if $d(x,x') < \delta$ then $d(Sx, Sx') < \delta_1$. Now assume that $d(x',x) < \delta$. Then

$$\begin{aligned} d(\kappa_x, \kappa_{x'}) &= \sup_{y \in Y} d(\kappa_x(y), \kappa_{x'}(y)) + \sup_{y \in Y} d(\kappa_x^{-1}(y), \kappa_{x'}^{-1}(y)) \\ &\leq \sup_{y \in Y} d(\widehat{S}(x,y), \widehat{S}(x',y)) + \sup_{y \in Y} d(\widehat{S}^{-1}(x,y), \widehat{S}^{-1}(x',y)) < \varepsilon. \end{aligned}$$

Define a (continuous) map $F: X \times G \to \mathrm{Hom}(Y,Y)$ by $F(x,g) = \kappa_x \circ \gamma_g$. Then, by (8.6), $F \circ T_\varphi(x,g) = \kappa_{Tx} \circ \gamma_{\varphi(x)} \circ \gamma_g = \gamma_{\varphi(Sx)} \circ F(x,g)$. Considering the identity

$$F \circ T_\varphi(X,g) = \gamma_{\varphi(Sx)} \circ F(x,g) \tag{8.7}$$

in the quotient space $\mathrm{Hom}(Y,Y) \setminus \Gamma$ of left cosets of Γ in $\mathrm{Hom}(Y,Y)$ we get $\Gamma F \circ T_\varphi(x,g) = \Gamma F(x,g)$. As T_φ is topologically ergodic, the map $\Gamma F: X \to \mathrm{Hom}(Y,Y) \setminus \Gamma$ is constant, $\Gamma F(x,y) = \Gamma p$ for some $p \in \mathrm{Hom}(Y,Y)$. This means that

$$F(x,g) = \gamma_{\overline{\psi}(x,g)} \circ p, \tag{8.8}$$

where $\overline{\psi}: X \times G \to G$ is a continuous map. By (8.7) we have

$$\gamma_{\varphi(Sx)} = \gamma_{\overline{\psi}(Tx,\varphi(x)g)\cdot\overline{\psi}(x,g)^{-1}}$$

which gives

$$\varphi(Sx) = \overline{\psi} \circ T_\varphi(x,g) \cdot \overline{\psi}(x,g)^{-1}. \tag{8.9}$$

Now, for arbitrary $h \in G$ consider a map $A_h: X \times G \to G$, $A_h(x,g) = \overline{\psi}(x,gh)\overline{\psi}(x,g)^{-1}$. Then, by (8.9),

$$\begin{aligned} A_h \circ T_\varphi(x,g) &= \overline{\psi}(Tx,\varphi(x)gh)\overline{\psi}(Tx,\varphi(x)g)^{-1} \\ &= (\overline{\psi}(Tx,\varphi(x)gh)\overline{\psi}(x,gh)^{-1}) \\ &\quad \cdot (\overline{\psi}(x,gh)\overline{\psi}(x,g)^{-1})(\overline{\psi}(x,g)\overline{\psi}(Tx,\varphi(x)g)) \\ &= \varphi(Sx)A_h(x,g)\varphi(Sx)^{-1} = A_h(x,g). \end{aligned}$$

Since T_φ is topologically ergodic, A_h is constant:

$$\overline{\psi}(x,gh)\overline{\psi}(x,g)^{-1} = v(h), \quad \text{where } v: G \to G. \tag{8.10}$$

Clearly v is a continuous group homomorphism. In particular

$$v(h) = \overline{\psi}(x,h)\overline{\psi}(x,e)^{-1}$$

i.e.

$$\overline{\psi}(x,h) = v(h)\overline{\psi}(x,e) = v(h)\psi(x). \tag{8.11}$$

By (8.8), $\kappa_x = \gamma_{\psi(x)} \circ \gamma_{v(g)} \circ p \circ \gamma_g^{-1}$ and κ_x does not depend on g, so $\gamma_{v(g)}p\gamma_g^{-1}$ also does not depend on g. In particular, taking $g = e$ we get

$$\gamma_{v(g)}p\gamma_g^{-1} = p \quad \text{and} \quad \kappa_x = \gamma_{\psi(x)} \circ p. \tag{8.12}$$

Therefore $\widetilde{S}(x,y) = (Sx, \gamma_{\psi(x)} \circ p(y))$. By (8.12), $\gamma_{v(g)} = p \circ \gamma_g \circ p^{-1}$.

To finish the proof observe that as the action Γ is effective, v is a monomorphism. It remains to show that v is onto. By virtue of (8.9) and (8.11) we have

$$\varphi(Sx) = \overline{\psi}(Tx,\varphi(x))\overline{\psi}(x,e)^{-1} = v(\varphi(x)\psi(Tx)\psi(x)^{-1}.$$

Thus we have obtained

$$(\varphi, \varphi \circ S) = (\varphi, v \circ \varphi) \cdot (e, \psi \circ T \cdot \psi^{-1}), \tag{8.13}$$

an equation giving that the cocycle $\varphi \times \varphi \circ S: X \to G \times G$ is cohomologous to $\varphi \times v \circ \varphi$. In particular we have equality of the groups of essential values: $E(\varphi \times \varphi \circ S) = E(\varphi \times v \circ \varphi)$. On the other hand, it is easy to see that

$$E(v \circ \varphi) = \overline{v(G)}, \quad E(\varphi \times v \circ \varphi) = \overline{\{(g, v(g)) : g \in G\}} = \overline{\Delta_v}$$

and the cocycle $\varphi \times v \circ \varphi$ takes all values in the group Δ_v. Thus the cocycle $\varphi \times \varphi \circ S$ is regular and, by Proposition 8.3.1,

$$G = E(\pi_2 \circ (\varphi \times \varphi \circ S)) = E(\pi_2 \circ (\varphi \times v \circ \varphi)) = \overline{E(v \circ \varphi)} = \overline{v(G)}$$

i.e. $v(G)$ is dense in G.

If G is compact, then $v(G)$ is a closed subset of g, hence $v(G) = G$. If G is connected, then $G = \mathbb{R}^m \oplus K$, where K is compact group. In such a case, as v is a monomorphism, $v(\mathbb{R}^m) = \mathbb{R}^m$ and $v(K) = K$, so $v(G) = v(\mathbb{R}^m \oplus K) = \mathbb{R}^m \oplus K = G$. If G is an arbitrary locally compact Abelian group, then G possesses an open subgroup of the form $\mathbb{R}^m \oplus K$ for some compact group K. Then clearly $v(\mathbb{R}^m \oplus K) = \mathbb{R}^m \oplus K$. As $G/\mathbb{R}^m \oplus K$ is discrete, $v(G/\mathbb{R}^m \oplus K) = G/\mathbb{R}^m \oplus K$ (since $\overline{v(G)} = G$) and therefore v is onto and the result follows. □

If $\Gamma \subset \mathrm{Hom}(Y, Y)$ is closed and acts effectively, then the following corollary from the proof of Theorem 8.3.2 holds.

Corollary 8.3.3. *If $S \in C(T)$ can be lifted to an invertible $\widehat{S} \in C(T_{\varphi,\Gamma})$, then the cocycle $\varphi \times \varphi \circ S$ is regular and $E(\varphi \times \varphi \circ S) = \Delta_v$ for some topological group automorphism v of G. In particular, both projections of $E(\varphi \times \varphi \circ S)$ are equal to G.*

In view of Theorem 8.3.2 observe, that if the actions of Γ on Y is not uniformly rigid, i.e. $\gamma_{g_n} \not\to \mathrm{Id}$ uniformly for any sequence $G \ni g_n \to \infty$ (see e.g. [37] for this and other related notions of rigidity in topological dynamics), then $\Gamma \subset \mathrm{Hom}(Y, Y)$ is closed. Indeed, if Γ is not closed in $\mathrm{Hom}(Y, Y)$, then $\gamma_{g_n} \to \gamma \notin \Gamma$. As $\gamma \notin \Gamma$, we have $g_n \to \infty$. Therefore $\gamma_{g_n}^{-1} = \gamma_{-g_n} \to \gamma^{-1} \notin \Gamma$. Taking a subsequence g_{k_n} such that $h_n = g_n - g_{k_n} \to \infty$ we get $\gamma_{h_n} = \gamma_{g_n}\gamma_{g_{k_n}}^{-1} \to \mathrm{Id}_Y$, so the action of Γ on Y is uniformly rigid.

In general, for an element S of $C(T)$ that can be lifted to an $\widehat{S} \in C(T_{\varphi,\Gamma})$, the following lemma is true.

Lemma 8.3.4. *Let (X, T) be a $\mathbb{Z}$-flow, G a locally compact Abelian group, and $\varphi\colon X \to G$ a cocycle. Let $\Gamma \subset \mathrm{Hom}(Y, Y)$ be a continuous representation of G, where Y is a compact Hausdorff space. If $S \in C(T)$ can be lifted to a $\widehat{S} \in C(T_{\varphi,\Gamma})$ and the cocycle $\varphi \times \varphi \circ S$ is regular, then both projections of $E(\varphi \times \varphi \circ S)$ are dense in G.*

Proof. If $\pi_i\colon G \times G \to G$ denotes the projection onto the ith coordinate, then, by Proposition 8.3.1, $\overline{\pi_i(E(\varphi \times \varphi \circ S))} = E(\pi(\varphi \times \varphi \circ S)) = E(\varphi) = G$, which finishes the proof. □

The requirement of full projections of the group of essential values of the cocycle $\varphi \times \varphi \circ S$ has the following algebraic interpretation.

Lemma 8.3.5. *Assume that (G, e) is a group and $H \subset G \times G$ a subgroup. Consider the natural action of $G \times G$ on $(G \times G)/H$ given by*

$$((\widetilde{g}_1, \widetilde{g}_2), (g_1, g_2)H) \mapsto (\widetilde{g}_1 g_1, \widetilde{g}_2 g_2)H.$$

Then the natural action of $\{e\} \times G$ on $(G \times G)/H$ is transitive if and only if the projection of H on the first coordinate is equal to G. Similarly, the natural action of $G \times \{e\}$ on $(G \times G)/H$ is transitive if and only if the projection of H on the second coordinate is equal to G.

Proof. Assume that the action of $\{e\} \times G$ on $(G \times G)/H$ is transitive. Then

$$\{(e, g)H : g \in G\} = (G \times G)/H. \tag{8.14}$$

Given a $g_1 \in G$, we will find a $g_2 \in G$ such that $(g_1, g_2) \in H$. In view of (8.14), there exists a $g \in G$ such that $(e, g)H = (g_1, e)H$. In particular $(g_1, e) = (h_1, gh_2)$ for some $(h_1, h_2) \in H$, so $(g_1, gh_2) = (g_1, g_2) \in H$.

Conversely, assume that the projection of H on the first coordinate is equal to G. Fix $(g_1, g_2) \in G \times G$. By assumption, there exists an $h \in G$ such that $(g_1, h) \in H$. Let $g = g_2 h^{-1}$. Then $(e, g)H = (e, g)(g_1, h)H = (g_1, g_2)H$ and we are done. □

Motivated by Corollary 8.3.3, Lemma 8.3.4 and Lemma 8.3.5, we will weaken the assumption of Theorem 8.3.2 by skipping the requirement that Γ is closed in $\mathrm{Hom}(Y, Y)$, and replacing it by regularity of $\varphi \times \varphi \circ S$ and full projections of $E(\varphi \times \varphi \circ S)$ in G (Theorem 8.3.8). These two conditions are indeed weaker than the requirement that Γ be closed. For instance, if $G = \mathbb{Z}$ and $\Gamma = \{\gamma_n : n \in \mathbb{Z}\} \subset \mathrm{Hom}(\mathbb{T}, \mathbb{T})$, where $\mathbb{T}$ denotes the unit circle, and $\gamma_n(y) = y + n\alpha \bmod 1$ for some irrational α, then clearly Γ is not closed in $\mathrm{Hom}(\mathbb{T}, \mathbb{T})$. On the other hand, for any extension $\widehat{\mathrm{Id}}_X$ of Id_X, the group $E(\varphi \times \varphi \circ \mathrm{Id}) = E(\varphi \times \varphi) = \Delta_{\mathbb{Z}}$ has full projections and the cocycle $\varphi \times \varphi \circ \mathrm{Id}$ is regular.

In our considerations we need a generalization of Proposition 5.6.1. First, following [72] we define the notion of relatively minimal extensions of topological flows. We say, that if $\pi: X \to Y$ is a factor map of topological flows, then Y is a *relatively minimal extension* of X if for each closed and invariant $Y_0 \subset Y$ satisfying $\pi(Y_0) = X$, we have $Y_0 = Y$.

Proposition 8.3.6. *Let (X, T) be a point transitive flow, G a locally compact Abelian group, $\varphi: X \to G$ a continuous map such that T_φ is point transitive. Let Y be a compact Hausdorff space and $\Gamma = \{\gamma_g : g \in G\}$ a left continuous action of G on Y.*

If $M \subset X \times Y$ is a $T_{\varphi,\Gamma}$-invariant closed set that is a relatively minimal extension of X via the natural projection, then there exists a closed set $Y_0 \subset Y$ such that $M = X \times Y_0$. Moreover, the G-flow (Y_0, Γ) is point transitive.

Proof. By assumptions of this proposition, we can find an $x_0 \in X$ such that $\overline{\mathrm{Orb}}(x_0, e) = X \times G$. Since M is an extension of X via the natural projection, there exists a $y_0 \in Y$ such that $(x_0, y_0) \in M$. Put $D = \{(x, g) : (x, \gamma_g(y_0)) \in M\}$. Clearly $(x_0, e) \in D$, D is closed and $T_{\varphi,\Gamma}$-invariant, hence $D = X \times G$. Let

$$Y_0 = \overline{\mathrm{Orb}}_\Gamma(y_0) = \overline{\{\gamma_g(y_0) : g \in G\}}.$$

Since $D = X \times G$, $X \times Y_0 \subset M$. By assumption of this proposition, the extension $\Pi_X : M \to X$ is relatively minimal, therefore $M = X \times Y_0$. □

The proposition below is a topological counterpart of Theorem 3.2.9.

Proposition 8.3.7. *Let (X, T) be a compact point transitive flow, G a locally compact Abelian group, Y, Z compact Hausdorff spaces, $\Gamma = \{\gamma_g : g \in G\}$, $\Lambda = \{\lambda_g : g \in G\}$ left effective continuous actions of G on Y and Z respectively, $\varphi: X \to G$ a continuous map such that T_φ is point transitive. Assume that $M \subset (X \times Y) \times (X \times Z)$ is a $T_{\varphi,\Gamma} \times T_{\varphi,\Lambda}$-invariant closed set that is point transitive and the extension $\pi_{X\times X}: M \to \pi_{X\times X}(M) = M_0$ is relatively minimal. Assume moreover that the restriction $(\varphi \times \varphi)_{M_0}$ of $\varphi \times \varphi$ to M_0 is regular i.e. there exist functions $f_1, f_2: M_0 \to G \times G$, $\eta_1, \eta_2: M_0 \to E((\varphi \times \varphi)_{M_0})$ such that*

$$\begin{aligned}(\varphi(x_1), \varphi(x_2)) = {} & (f_1(x_1, x_2), f_2(x_1, x_2)) \\ & - (f_1(Tx_1, Tx_2), f_2(Tx_1, Tx_2)) + (\eta_1(x_1, x_2), \eta_2(x_1, x_2))\end{aligned}$$

for all $(x_1, x_2) \in M_0$.

Then there exists a compact $E((\varphi\times\varphi)_{M_0})$-invariant set $A \subset Y \times Z$ such that the map $J: M \to M_0 \times (Y \times Z)$ given by

$$J(x_1, y, x_2, z) = (x_1, x_2, \gamma_{f_1(x_1,x_2)}(y), \lambda_{f_2(x_1,x_2)}(z))$$

is an isomorphism of $(M, T_{\varphi,\Gamma} \times T_{\varphi,\Lambda})$ and $(M_0 \times A, (T \times T)_{(\theta_1,\theta_2),H})$, where $H = \{(\gamma_{g_1}, \lambda_{g_2}) : (g_1, g_2) \in E\big((\varphi, \varphi)_{M_0}\big)\}$.

Proof. Clearly $J \circ (T_{\varphi,\Gamma} \times T_{\varphi,\Lambda}) = (T \times T)_{(\theta_1,\theta_2),H} \circ J$ on M. Thus $J: M \to J(M)$ is an isomorphism and, by [73, Proposition 2.3], $J(M)$ is a relatively minimal extension of M_0. By virtue of Proposition 8.3.6, there exists a closed set $Y_0 \subset Y$ such that $M = X \times A$ and the $E((\varphi, \varphi)_{M_0})$-flow (A, H) is point transitive. □

Theorem 8.3.8. *Let (X, T) be a compact point transitive flow, G a locally compact Abelian group, Y, Z compact Hausdorff spaces, $\Gamma = \{\gamma_g : g \in G\}$, $\Lambda = \{\lambda_g : g \in G\}$ left effective continuous actions of G on Y and Z respectively, $\varphi: X \to G$ a continuous map such that T_φ is point transitive. Assume that $\widehat{S}: X \times Y \to X \times Z$ is an isomorphism of $(X \times Y, T_{\varphi,\Gamma})$ and $(X \times Z, T_{\varphi,\Lambda})$, that is an extension of some $S \in C(T)$. Assume moreover, that the cocycle $\varphi \times \varphi \circ S$ is regular and that both projections of $E(\varphi \times \varphi \circ S)$ are equal to G.*

Then there exist: a homeomorphism $p: Y \to Z$, a topological group automorphism $v: G \to G$ and a continuous map $\psi: X \to G$ such that

$$\widehat{S}(x, y) = (Sx, \lambda_{\psi(x)} \circ p(y)), \tag{8.15}$$

and p satisfies

$$p\circ\gamma_g(y)=\lambda_{v(g)}\circ p(y),\quad g\in G,\ y\in Y. \tag{8.16}$$

Proof. By Proposition 8.3.7, $J(\Delta_{\widehat{S}})=\Delta_S\times A$ and $A\subset Y\times Z$ is a compact, H-invariant set, where $H=\{(\gamma_{g_1},\lambda_{g_2}):(g_1,g_2)\in E(\varphi\times\varphi\circ S)\}$. Therefore

$$\widehat{S}(x,y)=(Sx,\kappa(x,y)) \tag{8.17}$$

for some continuous map $\gamma\colon X\times Y\to Z$.

First we define the topological group automorphism $v\colon G\to G$. To do this take $(g,g_1),(g,g_2)\in E(\varphi\times\varphi\circ S)$. Since A is H-invariant, for each $y\in Y$ we have $\lambda_{g_1}\circ\kappa(x,y)=\kappa(x,\gamma_g(y))=\lambda_{g_2}\circ\kappa(x,y)$, and therefore $\lambda_{g_1g_2^{-1}}=\mathrm{Id}_Z$, i.e. $g_1=g_2$. This implies that there exists a map $v\colon G\to G$ such that

$$E(\varphi\times\varphi\circ S)=\{(g,v(g)):g\in G\}=\Delta_v. \tag{8.18}$$

As $E(\varphi\times\varphi\circ S)$ is a group, v is a group homomorphism. By the assumption that both projections of $E(\varphi\times\varphi\circ S)$ are equal to G, v is onto. In particular v is continuous. Since $\widehat{S}$ is an isomorphism, in a similar way we show that if $(g_1,g),(g_2,g)\in E(\varphi\times\varphi\circ S)$, then $g_1=g_2$, i.e. v is a topological group automorphism.

Because the cocycle $\varphi\times\varphi\circ S$ is regular, there exist functions $f_1,f_2,\theta\colon X\to G$ such that

$$\varphi=f_1\cdot(f_1\circ T)^{-1}\cdot\theta, \tag{8.19}$$

$$\varphi\circ S=f_2\cdot(f_2\circ T)^{-1}\cdot(v\circ\theta). \tag{8.20}$$

Now we are able to prove the existence of the map $p\colon Y\to Z$. More precisely, we will show that

$$A=\{(y,p(y)):y\in Y\}\quad\text{and}\quad\lambda_{v(g)}\circ p=p\circ\gamma_g,\ g\in G. \tag{8.21}$$

Indeed, as $J(\Delta_{\widehat{S}})=\Delta_S\times A$, the set A is a graph of some continuous map $p\colon Y\to Z$. As $\widehat{S}$ is an isomorphism, p is a homeomorphism. To prove that $\lambda_{v(g)}\circ p=p\circ\gamma_g$ fix $(x,y)\in X\times Y$ and denote $\overline{y}=\lambda^{-1}_{f_1(x)}y$. Then $(\gamma_{f_1(x)}\overline{y},l_{f_2(x)}\kappa(x,\overline{y}))\in A$, hence $p(y)=\lambda_{f_2(x)}\kappa(x,\gamma^{-1}_{f_1(x)}y)$, equivalently

$$\kappa(x,\gamma^{-1}_{f_1(x)}y)=\lambda^{-1}_{f_2(x)}p(y). \tag{8.22}$$

Since A is Δ_v-invariant, for each $g\in G$ we have

$$(\gamma_g\circ\gamma_{f_1(x)}\overline{y},\lambda_{v(g)}\circ\lambda_{f_2(x)}\circ\kappa(x,\overline{y}))\in A,$$

i.e.

$$\lambda_{v(g)}\circ\lambda_{f_2(x)}\circ\kappa(x,\gamma^{-1}_{f_1(x)}y)=p(\gamma_g\circ\gamma_{f_1(x)}\overline{y})=p\circ\gamma_g(y).$$

By (8.22), $\lambda_{v(g)}\circ p(y)=p\circ\gamma_g(y)$ and (8.21) is proved.

To finish the proof let $(\gamma_{f_1(x)}y, \lambda_{f_2(x)}\kappa(x,y)) \in A$. By (8.21),

$$\lambda_{f_2(x)} \circ \kappa(x,y) = p \circ \gamma_{f_1(x)}(y) = \lambda_{v(f_1(x))} \circ p(y),$$

hence

$$\kappa(x,y) = \lambda_{f_2(x)}^{-1} \circ \lambda_{v(f_1(x))} \circ p(y).$$

Denote $\psi(x) = v(f_1(x))f_2(x)^{-1}$. Then $\lambda_{\psi(x)} = \lambda_{f_2(x)}^{-1} \circ \lambda_{v(f_1(x))}$ and

$$\widehat{S}(x,y) = (Sx, \kappa(x,y)) = (Sx, \lambda_{\psi(x)} \circ p(y))$$

and the proof is complete. □

Since the isomorphism $\widehat{S}$ from Theorem 8.3.8 satisfies $\widehat{S} \circ T_{\varphi,\Gamma} = T_{\varphi,\Lambda} \circ \widehat{S}$, $\kappa(Tx, \gamma_{\varphi(x)}(y)) = \lambda_{\varphi(Sx)} \circ \kappa(x,y)$. By Theorem 8.3.8,

$$\lambda_{\psi(Tx)} \circ \lambda_{v(\varphi(x))} \circ p(y) = \lambda_{\psi(Tx)} \circ p \circ \gamma_{\varphi(y)}(y) = \lambda_{\psi(Sx)} \circ \lambda_{\psi(x)} \circ p(y),$$

hence

$$\psi \circ T \cdot v \circ \varphi = \varphi \circ S \cdot \psi$$

and we have the following

Corollary 8.3.9. *Under the assumptions of Theorem* 8.3.8*, the cocycles* $\varphi \circ S$ *and* $v \circ \varphi$ *are cohomologous.*

Directly from Theorem 8.3.8 we get the following description of the elements of the centralizer $C(T_{\varphi,\Gamma})$.

Theorem 8.3.10. *Let* (X,T) *be a compact minimal flow,* G *a locally compact Abelian group,* Y *a compact Hausdorff space,* $\Gamma = \{\gamma_g : g \in G\}$ *a left action of* G *on* Y*,* $\varphi: X \to G$ *a continuous map such that* T_φ *is point transitive.*

If $\widehat{S} \in C(T_{\varphi,\Gamma})$ *is such an invertible extension of* $S \in C(T)$ *that the cocycle* $\varphi \times \varphi \circ S$ *is regular and that both projections of* $E(\varphi \times \varphi \circ S)$ *are equal to* G*, then there exist: a homeomorphism* $p: Y \to Y$*, topological group automorphism* $v: G \to G$ *and a continuous map* $\psi: X \to G$ *such that*

$$\widehat{S}(x,y) = (Sx, \gamma_{\psi(x)} \circ p(y)) \quad \textit{and} \quad p \circ \gamma_g = \gamma_{v(g)} \circ p,\ g \in G. \tag{8.23}$$

Corollary 8.3.11. *If, under assumptions of Theorem* 8.3.10*,* $\widehat{S} \in C(T_{\varphi,\Gamma})$ *is invertible, then* p *normalizes* Γ *in* $\mathrm{Hom}(Y,Y)$*.*

Corollary 8.3.12. *If* $\widehat{\mathrm{Id}} \in C(T_{\varphi,\Gamma})$ *is an extension of the identity map on* X*, then* $\widehat{\mathrm{Id}}(x,y) = (x, p(y))$*, where* $p \in \mathrm{Hom}(Y,Y) \cap C(K)$*.*

Proof. In this case $E(\varphi \times \varphi) = \Delta_G$, hence the cocycle $\varphi \times \varphi$ is regular, $f_1 = f_2 \equiv 0$, $v = \mathrm{Id}_G$ and $\gamma_g \circ p = p \circ \gamma_g$, $g \in G$. Thus $\psi \equiv 0$ and $\widehat{\mathrm{Id}}(x,y) = (x, p(y))$.□

If $\widehat{S}: X \times Y \to X \times Z$ is a factor map of flows, where (X,T) is a minimal rotation, then the set $\Pi_{X\times X}(\Delta_{\widehat{S}}) \subset X \times X$ is minimal, so $\Pi_{X\times X}(\Delta_{\widehat{S}}) = \Delta_S$ for some $S \in C(T)$.

Theorem 8.3.13. *Let T be a minimal rotation on a compact metric monothetic group X, $\Gamma = \{\gamma_g : g \in \mathbb{R}^m\}$ a left continuous action of $\mathbb{R}^m$ on a compact metric space Y, $\varphi: X \to \mathbb{R}^m$ a continuous map such that T_φ is point transitive. If $\widehat{S} \in C(T_{\varphi,K})$ is invertible, then there exist: $S \in C(T)$, $p \in \mathrm{Hom}(Y,Y)$, topological group automorphism $v: \mathbb{R}^m \to \mathbb{R}^m$ and a continuous map $\psi: X \to \mathbb{R}^m$ such that*

$$\widehat{S}(x,y) = (Sx, \gamma_{\psi(x)} \circ p(y))$$

and $p \circ \gamma_g = \gamma_{v(g)} \circ p$, $g \in \mathbb{R}^m$.

Proof. Clearly $\widehat{S}$ is an extension of some $S \in C(T)$. φ is ergodic, so by [mentzen, Theorem 4.9], $\varphi \times \varphi \circ S$ is regular and $E(\varphi \times \varphi \circ S)$ is a linear subspace of $\mathbb{R}^m \times \mathbb{R}^m$. As in the proof of Theorem 8.3.8 we deduce that $E(\varphi \times \varphi \circ S)$ has dense both projections on $\mathbb{R}^m$, hence the projections are equal to $\mathbb{R}^m$. An application of Theorem 8.3.8 finishes the proof. □

8.4. A remark on some recent results

In 2005 G. Greschonig and U. Haböck in [41] completely solved the problem of cohomological invariancy of the sets of essential values in the case of non-abelian groups. They changed a little bit the topological version of the Schmidt's definition of the set of essential values (called in [41] an *essential range*) passing from a global notion to a notion depending on a point in the base. In Abelian case both definitions coincide – the group of essential values does not depend on the point. In non-abelian case this is not true. G. Greschonig and U. Haböck found an exact topological version of the measure-theoretical notion of regularity of a cocycle. The definition of regular cocycles from this dissertation is essentially stronger than the G. Greschonig and U. Haböck's one: all regular cocycles in sense of this chapter are regular in the sense of G. Greschonig and U. Haböck, not *vice versa*.

APPENDIX A

LEBESGUE SPACES AND THEIR PROPERTIES

The following chapter is based on [12], [85] and on [98].

A.1. Point and set maps of measure spaces

Let X be a non-empty set, $\mathcal{A}$ a σ-algebra of subsets of X and μ a measure on $\mathcal{A}$. Then we will call $(X, \mathcal{A}, \mu)$ a *measure space*. If additionally $\mu(A) = 1$, then $(X, \mathcal{A}, \mu)$ is called a *probability measure space*.

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. A map $f\colon X \to Y$ is called *measurable* if $f^{-1}(B) \in \mathcal{A}$ for each $B \in \mathcal{B}$. Such an f is said to be a *measure preserving map* if $\mu(f^{-1}(B)) = \nu(B)$ for all $B \in \mathcal{B}$.

Definition A.1.1. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. We say that $(Y, \mathcal{B}, \nu)$ is a *factor* of $(X, \mathcal{A}, \mu)$ if there exist $X' \in \mathcal{A}$, $Y' \in \mathcal{B}$ with $\mu(X \setminus X') = 0$, $\nu(Y \setminus Y') = 0$ and a measure-preserving map $f\colon X' \to Y'$.

Definition A.1.2. Two measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are said to be *isomorphic* if there exist $X' \in \mathcal{A}$, $Y' \in \mathcal{B}$ with $\mu(X \setminus X') = 0$, $\nu(Y \setminus Y') = 0$ and a measure-preserving bijective map $f\colon X' \to Y'$. The map f will be called an *isomorphism* of the measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$. In the case $(Y, \mathcal{B}, \nu) = (X, \mathcal{A}, \mu)$ such an isomorphism will be called an *automorphism* of $(X, \mathcal{A}, \mu)$, and the measure μ is said to be an *invariant measure* for the automorphism f.

For a given measure space $(X, \mathcal{A}, \mu)$ we define a Boolean σ-algebra $\widetilde{\mathcal{A}}$ by the following way. Let $\sim$ be an equivalent relation on $\mathcal{A}$ defined by $A_1 \sim A_2$ if and only if $\mu(A_1 \triangle A_2) = 0$, where $\triangle$ denotes the symmetric difference of sets: $A_1 \triangle A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$. Let $\widetilde{\mathcal{A}} = \mathcal{A}/\sim$. Then $\widetilde{\mathcal{A}}$ is a Boolean σ-algebra. If additionally $(X, \mathcal{A}, \mu)$ is a probability measure space then the Boolean σ-algebra $\widetilde{\mathcal{A}}$ enjoys a structure of a metric space with the distance ρ defined by $\rho(\widetilde{A}_1, \widetilde{A}_2) = \mu(A_1 \triangle A_2)$.

If $(Y,\mathcal{B},\nu)$ is a factor of $(X,\mathcal{A},\mu)$ with the factor map f, then $f^{-1}(\mathcal{B})$ is a sub-σ-algebra of $\mathcal{A}$ and the map $f^{-1}\colon\widetilde{\mathcal{B}}\to f^{-1}(\mathcal{B})^{\sim}\subset\widetilde{\mathcal{A}}$ is a bijection. If f is an isomorphism then $f^{-1}(\mathcal{B})^{\sim}=\widetilde{\mathcal{A}}$.

Definition A.1.3. Given measure spaces $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ we say that the Boolean σ-algebra $\widetilde{\mathcal{B}}$ is a *factor* of the Boolean σ-algebra $\widetilde{\mathcal{A}}$ if there exists an isometric map of metric spaces $F\colon\widetilde{\mathcal{B}}\to\widetilde{\mathcal{A}}$ satisfying $F(\widetilde{\emptyset})=\widetilde{\emptyset}$. If the map F is additionally onto, then F is called an *isomorphism*, the σ-algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic* and the measure spaces $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ are said to be *conjugate*.

Clearly, if the measure spaces $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ are isomorphic via a map $f\colon X'\to Y'$, $X'\subset X$, $Y'\subset Y$, where $\mu(X\setminus X')=0$, $\nu(Y\setminus Y')=0$, then they are also conjugate via $F=f^{-1}\colon\widetilde{\mathcal{B}}\to\widetilde{\mathcal{A}}$; not *vice versa*. However, the converse is true under some natural assumptions.

Theorem A.1.4 ([85], [98]). *Let X_1, X_2 be complete separable metric spaces, let $\mathcal{B}(X_1),\mathcal{B}(X_2)$ be their σ-algebras of Borel subsets and let m_1,m_2 be probability measures on $\mathcal{B}(X_1),\mathcal{B}(X_2)$. Let $\Phi\colon\widetilde{\mathcal{B}}(X_2)\to\widetilde{\mathcal{B}}(X_1)$ be an isomorphism of Boolean σ-algebras. Then there exist $M_1\in\mathcal{B}(X_1)$, $M_2\in\mathcal{B}(X_2)$ with $m_1(M_1)=m_2(M_2)=1$ and an invertible measure-preserving transformation $\phi\colon M_1\to M_2$ such that $\Phi(\widetilde{B})=(\phi^{-1}(B\cap M_2))^{\sim}$ for each $B\in\mathcal{B}(X_2)$. If ψ is any other isomorphism from $(X_1,\mathcal{B}(X_1),m_1)$ to $(X_2,\mathcal{B}(X_2),m_2)$ which induces Φ, then $m_1(\{x\in X_1:\phi(x)\neq\psi(x)\})=0$.*

A.2. Probability Lebesgue spaces

Definition A.2.1. A probability measure space $(X,\mathcal{A},\mu)$ is a *Lebesgue space* if it is isomorphic to a probability space that is a disjoint union of a countable (or finite) number of points $\{x_1,x_2,x_3,\dots\}$ each of positive measure, and the space $([0,s],\mathcal{L}([0,s]),\lambda)$, where $\mathcal{L}([0,s])$ is the σ-algebra of Lebesgue measurable subsets of the interval $[0,s]$ and λ is Lebesgue measure. Here $s=1-\sum_n p_n$, where p_n is the measure of the point y_n.

An wide class of Lebesgue spaces is provided by the following theorem.

Theorem A.2.2 ([85], [98]). *Let X be a complete separable metric space, let $\mathcal{B}(X)$ be its σ-algebra of Borel sets and let m be a probability measure on $\mathcal{B}(X)$ with $m(\{x\})=0$ for each set $\{x\}$ consisting of a single point $x\in X$. Let $([0,1],\mathcal{B}([0,1]),\lambda)$ denote the closed unit interval with its σ-algebra of Borel sets and Lebesgue measure λ. Then the measure spaces $(X,\mathcal{B}(X),m)$, $([0,1],\mathcal{B}([0,1]),\lambda)$ are isomorphic. If $(X,\mathcal{B}_m(X),m)$ denotes the completion of $(X,\mathcal{B}(X),m)$ then $(X,\mathcal{B}_m(X),m)$ is isomorphic to $([0,1],\mathcal{L}([0,1]),\lambda)$, where $\mathcal{L}([0,1])$ is the σ-algebra of Lebesgue measurable sets on $[0,1]$.*

Clearly Theorem A.1.4 is true for Lebesgue spaces (i.e. set maps are always induced by point maps).

Following Rokhlin ([84]) we introduce the notion of a measurable partition of a Lebesgue space.

Definition A.2.3. Let $(X, \mathcal{A}, \mu)$ be a Lebesgue space. A family of measurable sets $\mathbf{P} = \{P_\gamma : \gamma \in \Gamma\}$ is called a *partition* of $(X, \mathcal{A}, \mu)$ if all $P_\gamma \in \mathbf{P}$ are measurable, pair-wise disjoint and $\mu(X \setminus \bigcup \mathbf{P}) = 0$.

Definition A.2.4. Let $(X, \mathcal{A}, \mu)$ be a Lebesgue space. A partition $\mathbf{P} = \{P_\gamma : \gamma \in \Gamma\}$ of $(X, \mathcal{A}, \mu)$ is said to be *measurable* if there exists a countable family $\mathcal{D}$ of measurable sets, each of them being a union of elements of $\mathbf{P}$, such that for any two distinct $A, B \in \mathbf{P}$ there exists a set $D \in \mathcal{D}$ such that exactly one among A, B is included in D.

We will often call a (measurable) partition of a Lebesgue space $(X, \mathcal{A}, \mu)$ shortly a (measurable) partition of X. Clearly each finite partition of a Lebesgue space is measurable. For a measurable partition $\mathbf{P}$ of a Lebesgue space $(X, \mathcal{A}, \mu)$ the space $X/\mathbf{P} = \{A \in \mathcal{A} : A \in \mathbf{P}\}$ is a Lebesgue space as well.

Definition A.2.5. Let $\mathbf{P}$ be a measurable partition of a Lebesgue space $(X, \mathcal{A}, \mu)$. By a *canonical system of conditional measures with respect to the partition* $\mathcal{P}$ we understand a system $\{\mu_C : C \in \mathbf{P}\}$ of measures satisfying the following conditions:

(a) Each measure μ_C is defined on a σ-algebra $\mathcal{A}_C$ of subsets of the set C, where $\mathcal{B}_C = \{A \cap C : A \in \mathcal{A}\}$.
(b) The measure space $(C, \mathcal{A}_C, \mu_C)$ is a Lebesgue space.
(c) For any $A \in \mathcal{A}$ the function $C \mapsto \mu_C(A \cap C)$ is measurable on $X/\mathbf{P}$, and
$$\mu(A) = \int_{X/\mathbf{P}} \mu_C(A \cap C)\, d\mu.$$

The following theorem characterizes measurable partitions.

Theorem A.2.6 ([84]). *A partition of a Lebesgue space is measurable if and only if there exists a canonical system of conditional measures with respect to this partition. Such a system is unique up to set of measure zero.*

A.3. Spectral theory of unitary operators

The contents of this section is borrowed from [81].

Definition A.3.1. A sequence $(r_n)_{n\in\mathbb{Z}}$ of complex numbers is said to be *positive definite* if
$$\sum_{n,m=0}^{N} r_{n-m} a_n \overline{a}_m \geq 0$$

for all sequences $(a_n)_{n\in\mathbb{N}}$ of complex numbers and all non-negative integers N.

The most important example of a positive definite sequence contains the following lemma.

Lemma A.3.2. *If U is a unitary operator on a Hilbert space $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ and $x \in H$, then the sequence $r_n = \langle U^n x, x\rangle$ for $n \in \mathbb{Z}$ is positive definite.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Then for nonnegative integer N we have

$$0 \le \left\langle \sum_{n=0}^{N} a_n U^n x, \sum_{m=0}^{N} a_m U^m x \right\rangle = \sum_{n,m=0}^{N} \langle U^{n-m}x, x\rangle a_n \overline{a}_m, \tag{A.1}$$

so the sequence $(r_n)_{n\in\mathbb{Z}}$ is positive definite. □

Theorem A.3.3 ([Herglotz])**.** *If $(r_n)_{n\in\mathbb{Z}}$ is a positive definite sequence then there is a unique finite non-negative measure μ on $\mathbb{T} = \{z : |z| = 1\}$ (or on $[0,1)$) such that*

$$r_n = \int_{\mathbb{T}} z^n \, d\mu = \int_0^1 \exp(2\pi i n x)\, d\mu, \quad n \in \mathbb{Z}.$$

Conversely, if μ is a non-negative measure on $\mathbb{T}$ then the sequence $r_n = \int_{\mathbb{T}} z^n \, d\mu$, $n \in \mathbb{Z}$, is a positive definite sequence.

Proof. Directly from the definition of a positive definite sequence we get that $r_0 \ge 0$. Now fix a complex λ, and a positive integer n. Taking the sequence $(a_m)_{m\ge 0}$ given by $a_0 = 1$, $a_n = \lambda$ and $a_m = 0$ for other m for $N = n$ we get that

$$0 \le \sum_{k,m\le n}^{n} r_{k-m} a_k \overline{a}_m = (1 + |\lambda|^2) r_0 + r_n \lambda + r_{-n}\overline{\lambda}.$$

Hence $r_n\lambda + r_{-n}\overline{\lambda}$ is real for all complex λ, which implies that $r_{-n} = \overline{r}_n$, $n \ge 0$. Now, for fixed n, let $\lambda = \theta \overline{r}_n$. Then we get that

$$(1 + |\theta|^2 |r_n|^2) r_0 + \theta |r_n|^2 + \overline{\theta} |r_n|^2 \ge 0$$

for all complex θ. For real θ we have a quadratic in θ which is never negative. The (non-positive) discriminant of this quadratic shows that $|r_n| \le r_0$ for all n. In particular, the sequence $(r_n)_{n\in\mathbb{Z}}$ is bounded. Unless $r_0 = 0$ (then all r_n's are zero), without loss of generality we may assume $r_0 = 1$. Let $0 < s < 1$, then positive definiteness yields

$$f_s(z) = \sum_{n,m=0}^{\infty} r_{n-m} s^{n+m} z^{m-n} \ge 0$$

for all complex z with $|z| = 1$. Because

$$f_s(z) = \sum_{n=-\infty}^{\infty} r_n z^{-n} \sum_{m=0}^{\infty} s^{|n|+2m} = \sum_{n=-\infty}^{\infty} r_n z^{-n} s^{|n|} \frac{1}{1-s^2},$$

we get
$$\int_{\mathbb{T}} f_s(z)z^{-n}\,dz = \frac{r_{-n}s^{|n|}}{1-s^2}.$$
Define μ_s by
$$\frac{d\mu_s}{dz} = (1-s^2)f_s(z) \geq 0$$
so that
$$\int_{\mathbb{T}} z^{-n}\,d\mu_s = r_{-n}s^{|n|}, \quad \mu_s(\mathbb{T}) = r_0 = 1.$$
Choose a sequence $(0,1) \ni s_m \to 1$, then $\int_{\mathbb{T}} z^{-k}\,d\mu_{s_m} \to r_{-k}$ for all $k \in \mathbb{Z}$. Hence $\int_{\mathbb{T}} p(z)\,d\mu_{s_m}$ converges as $m \to \infty$ for all polynomials $p(z)$. Since the polynomials are dense in the space $C(\mathbb{T})$ of all continuous complex functions on the circle, we see that $\int_{\mathbb{T}} f(z)\,d\mu_{s_m}$ converges for all $f \in C(\mathbb{T})$ to, say, $J(f)$. But $d\mu_s/dz \geq 0$ implies $J(f) \geq 0$ when $f \geq 0$ and therefore $J(f) = \int_{\mathbb{T}} d\,d\mu$ for some probability μ on $\mathbb{T}$. We conclude that
$$\int_{\mathbb{T}} z^{-k}\,d\mu = \lim_{m\to\infty} \int_{\mathbb{T}} z^{-k}\,d\mu_{s_m} = r_{-k}$$
and the existence part of the theorem is complete.

The measure μ such that $\int_{\mathbb{T}} z^k\,d\mu = r_k$ is unique since $\int_{\mathbb{T}} z^k\,d\mu = \int_{\mathbb{T}} z^k\,d\nu$ for all $k \in \mathbb{Z}$ implies $\mu \equiv \nu$.

If μ is a probability measure on $\mathbb{T}$ and $r_n = \int_{\mathbb{T}} z^k\,d\mu$, $k \in \mathbb{Z}$, then
$$\sum_{n,m=0}^{N} r_{n-m}a_n\overline{a}_m = \sum_{n,m=0}^{N} a_n\overline{a}_m \int_{\mathbb{T}} z^{n-m}\,d\mu = \int_{\mathbb{T}} \left|\sum_{n=0}^{N} a_n z^n\right|^2 d\mu \geq 0.$$
Thus the sequence $(r_n)_{n\in\mathbb{Z}}$ is positive definite. $\square$

Theorem A.3.4 (Wiener)**.** *Let m be a finite Borel measure defined on the circle $\mathbb{K}$. If H is a closed subspace of $L^2(\mathbb{K},m)$ which is invariant with respect to the unitary operator $V(f)(z) = zf(z)$ (i.e. $VH = H$) then*
$$H = \chi_B L^2(\mathbb{K},m) = \{f \in l^2(\mathbb{K},m) : f = 0 \text{ on } B^c\}$$
for some Borel subset B.

Proof. Let $1 = k + h$, where $k \in H^{\perp}$, $h \in H$. Then $k \perp V^n h$ for all n i.e. $\int_{\mathbb{K}} k(z)\cdot\overline{h(z)}z^n\,dm = 0$, $n \in \mathbb{Z}$. Therefore $k(z)h(z) = 0$ and $1 = |k|+|h|$ m-almost everywhere. Since k, h have disjoint "supports" ($k = \chi_A k$, $h = \chi_{A^c}h$), $|k| = 1$ on A and $|h| = 1$ on A^c m-almost everywhere. But $1 = k + h$ implies $k = 1$ on A and $h = 1$ on A^c. In other words $1 = \chi_A + \chi_{A^c}$ is the decomposition of 1 with respect to $H^{\perp}$, H. hence $z^n\chi_{A^c}(z) \in H$ for $n \in \mathbb{Z}$ and we conclude that $\chi_{A^c}L^2(\mathbb{T},m) \subset H$, $\chi_A L^2(\mathbb{T},m) \subset H^{\perp}$ i.e. $\chi_{A^c}L^2(\mathbb{T},m) = H$. $\square$

Definition A.3.5. For $i = 1, 2$ let U_i be unitary operator of Hilbert space H_i. The operators U_1, U_2 are said to be *unitarily* (or *spectrally*) *equivalent* if there exists an isometry W of H_1 onto H_2 such that $WU_1 = U_2W$. In this case we write $U_1 \simeq U_2$.

Let U be a unitary operator on the Hilbert space H. For $x \in H$ let $Z(x)$ denote the cyclic subspace generated by x which is the closure of the linear span of $\{U^n x : n \in \mathbb{Z}\}$.

Lemma A.3.6. *The restriction $U|_{Z(x)}$ is spectrally equivalent to the operator $V_x \colon L^2(\mathbb{T}, \widetilde{x}) \to L^2(\mathbb{T}, \widetilde{x})$ defined by $(V_x f)(z) = zf(z)$, where $\widetilde{x}$ denotes the spectral measure of x, i.e. such that $\langle U^n x, x\rangle = \int_{\mathbb{T}} z^n\, d\widetilde{x}$ for all $n \in \mathbb{Z}$.*

Proof. Define $W(U^n x) = z^n \in L^2(\mathbb{T}, \widetilde{x})$, then W is an isometry on $\{U^n x : n \in \mathbb{Z}\}$ since $\langle U^m x, U^n x\rangle = \int_{\mathbb{T}} z^m z^{-n}\, d\widetilde{x}$. Hence W extends to an isometry of $Z(x)$ onto $L^2(\mathbb{T}, \widetilde{x})$. Clearly $WU = V_x W$; thus U is spectrally equivalent to V_x.□

Lemma A.3.7. *$U|_{Z(x)}$ is spectrally equivalent to $U|_{Z(y)}$ if and only if the spectral measures $\widetilde{x}$ and $\widetilde{y}$ are equivalent.*

Proof. Suppose $WV_x = V_y W$ for some isometry W and write $f(z) = W(1)$, then $WV_x^n 1 = V_y f$, i.e. $W(z^n) = f(z)z^n$. Hence W is the multiplication operator $g \mapsto fg$ and if $B \subset \mathbb{T}$ is a Borel set then χ_B in $L^2(\mathbb{T}, \widetilde{x})$ has the same norm as $f\chi_B$ in $L^2(\mathbb{T}, \widetilde{y})$, i.e. $\widetilde{x}(B) = \int_B |f|^2\, d\widetilde{y}$. Therefore $\widetilde{x}$ is absolutely continuous with respect to $\widetilde{y}$, $\widetilde{x} \ll \widetilde{y}$. A similar argument shows that $\widetilde{y} \ll \widetilde{x}$; and hence $\widetilde{x}$ is equivalent to $\widetilde{y}$.

Conversely, if $\widetilde{x}$ is equivalent to $\widetilde{y}$, then define $W \colon L^2(\mathbb{T}, \widetilde{x}) \to L^2(\mathbb{T}, \widetilde{y})$ by $Wg = g \cdot (d\widetilde{x}/d\widetilde{y})^{1/2}$. W is an isometry and $WV_x = V_y W$. □

Lemma A.3.8. *If $x \in H$ and $\mu \ll \widetilde{x}$ is a finite non-negative Borel measure on $\mathbb{T}$ then there exists $y \in Z(x)$ with $\widetilde{y} = \mu$.*

Proof. It suffices to show the existence of $f \in L^2(\mathbb{T}, \widetilde{x})$ with $\widetilde{f} = \mu$. Let $f = (d\mu/d\widetilde{x})^{1/2}$, then f satisfies

$$\langle V_x^n f, f\rangle = \int_{\mathbb{T}} z^n \frac{d\mu}{d\widetilde{x}} = \int_{\mathbb{T}} z^n\, d\mu = \int_{\mathbb{T}} z^n\, d\widetilde{f}$$

for all $n \in \mathbb{Z}$, i.e. $\mu = \widetilde{f}$. □

Lemma A.3.9. *If $x, y \in Z(z)$ and $Z(x) \perp Z(y)$ then $\widetilde{x} \perp \widetilde{y}$. If in addition $z = x + y$ then $Z(z) = Z(x) \oplus Z(y)$.*

Proof. It suffices to show that if $f, g \in L^2(\mathbb{T}, \widetilde{z})$ and $Z(f) \perp Z(g)$ then $\widetilde{f} \perp \widetilde{g}$. By Wiener's theorem we get $Z(f) = \chi_A \cdot L^2(\mathbb{T}, \widetilde{z})$ and $Z(g) = \chi_B \cdot L^2(\mathbb{T}, \widetilde{z})$, and orthogonality ensures that $\widetilde{z}(A \cap B) = 0$. Since $\int_{\mathbb{T}} z^n\, d\widetilde{f} = \langle V_z^n f, f\rangle = \int_{\mathbb{T}} z^n |f|^2\, d\widetilde{z}$, $d\widetilde{f} = |f|^2\, d\widetilde{z}$, $d\widetilde{g} = |g|^2\, d\widetilde{z}$ and we get that $\widetilde{f} \perp \widetilde{g}$. If we assume now that $z = x + y$ then $1 = f + g$ and $Z(1) = L^2(\mathbb{T}, \widetilde{z}) = Z(f) + Z(g)$. □

Lemma A.3.10. *If $y \in Z(x)$ then $\widetilde{y} \ll \widetilde{x}$ with equivalence holding when and only when $Z(y) = Z(x)$.*

Proof. Map $Z(x)$ to $L^2(\mathbb{T}, \widetilde{x})$ by sending $U^n x$ to z^n and let f denote the image of y. We will show that $\widetilde{f} \ll \widetilde{x}$ with equivalence holding if and only if $Z(f) = Z(1)$ with respect to V_x. But $\int_{\mathbb{T}} z^n \, d\widetilde{f} = \langle V_z^n f, f\rangle = \int_{\mathbb{T}} z^n |f|^2 \, d\widetilde{z}$, hence $d\widetilde{f} = |f|^2 \, d\widetilde{z} \ll d\widetilde{x}$. If $Z(y) = Z(x)$ then $U|_{Z(y)} \simeq U|_{Z(x)}$ and we have seen that $\widetilde{x}$ is equivalent to $\widetilde{y}$. If $Z(y)$ is a proper subspace of $Z(x)$ then $Z(f)$ is a proper subspace of $L^2(\mathbb{T}, \widetilde{x})$ invariant under V_x. By Wiener's theorem $Z(f) = \chi_B L^2(\mathbb{T}, \widetilde{x})$, where $\widetilde{x}(B) < \widetilde{x}(\mathbb{T})$ and hence $\widetilde{x}(B^c) > 0$, $\widetilde{f}(B^c) = 0$ i.e. $\widetilde{y}$ and $\widetilde{x}$ are not equivalent. ☐

Lemma A.3.11. *If $\widetilde{x} \perp \widetilde{y}$ then $Z(x) \perp Z(y)$.*

Proof. Write $y = y_0 + y_1$ with $y_1 \in Z(x)$, $y_0 \perp Z(x)$ so that $Z(y_0) \perp Z(x)$. As $\langle U^n y, y\rangle = \langle U^n y_0, y_0\rangle + \langle U^n y_1, y_1\rangle$ we get $\int_{\mathbb{T}} z^n \, d\widetilde{y} = \int_{\mathbb{T}} z^n \, d\widetilde{y}_0 + \int_{\mathbb{T}} z^n \, d\widetilde{y}_1$. Hence $\widetilde{y} = \widetilde{y_0} + \widetilde{y_1} \perp \widetilde{x}$. But $y_1 \in Z(x)$ implies $\widetilde{y}_1 \ll \widetilde{x}$. Therefore $\widetilde{y}_1 = 0$ and hence $Z(x) \perp Z(y)$. ☐

Lemma A.3.12. *If $\widetilde{x} \perp \widetilde{y}$ then $\widetilde{x+y} = \widetilde{x} + \widetilde{y}$ and $Z(x+y) = Z(x) \oplus Z(y)$.*

Proof. By Lemma A.3.11, $Z(x) \perp Z(y)$, hence

$$\langle U^n(x+y), x+y\rangle = \langle U^n x, x\rangle + \langle U^n y, y\rangle,$$

i.e.

$$\int_{\mathbb{T}} z^n \, d(\widetilde{x+y}) = \int_{\mathbb{T}} z^n \, d\widetilde{x} + \int_{\mathbb{T}} z^n \, d\widetilde{y}$$

so that $\widetilde{x+y} = \widetilde{x} + \widetilde{y}$. Now $d\widetilde{x}/d(\widetilde{x+y}) \in L^2(\mathbb{T}, (\widetilde{x+y}))$ so that for $\varepsilon > 0$ there exists a polynomial $p(z, z^{-1})$ with

$$\int_{\mathbb{T}} \left| \frac{d\widetilde{x}}{d(\widetilde{x+y})} - p(z, z^{-1}) \right|^2 d(\widetilde{x+y}) < \varepsilon.$$

Hence

$$\begin{aligned}
&\|x - p(U,U^{-1})(x+y)\|^2 \\
&\qquad = \langle x, x\rangle - 2\,\mathrm{Re}\langle x, p(U, U^{-1})(x+y)\rangle + \|p(U, U^{-1})(x+y)\|^2 \\
&\qquad = \int_{\mathbb{T}} 1 \, d\widetilde{x} - 2\,\mathrm{Re}\langle x, p(U, U^{-1})(x+y)\rangle + \int_{\mathbb{T}} |p(z, z^{-1})|^2 \, d(\widetilde{x+y}) \\
&\qquad = \int_{\mathbb{T}} 1 \, d\widetilde{x} - 2\,\mathrm{Re} \int_{\mathbb{T}} p(z, z^{-1}) \, d\widetilde{x} + \int_{\mathbb{T}} |p(z, z^{-1})|^2 \, d(\widetilde{x+y}) \\
&\qquad = \int_{\mathbb{T}} 1 \, d\widetilde{x} - \int_{\mathbb{T}} \frac{d\widetilde{x}}{d(\widetilde{x+y})} \, d\widetilde{x} + \int_{\mathbb{T}} \left| \frac{d\widetilde{x}}{d(\widetilde{x+y})} - p(z, z^{-1}) \right|^2 d(\widetilde{x+y}) \\
&\qquad = \int_{\mathbb{T}} \left| \frac{d\widetilde{x}}{d(\widetilde{x+y})} - p(z, z^{-1}) \right|^2 d(\widetilde{x+y}) < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary $x \in Z(x+y)$. In the same way $y \in Z(x+y)$. Therefore $Z(x) \oplus Z(y) \subset Z(x+y)$. The opposite inclusion is clear. $\square$

Definition A.3.13. A cyclic space $Z(x)$ is said to be *maximal* if it is contained in no larger cyclic subspace. In such a case $\widetilde{x}$ is called a *maximal spectral type.*

Clearly $Z(x)$ is a maximal cyclic space if and only if $\widetilde{x} \gg \widetilde{y}$ for all $y \in H$.

Lemma A.3.14. *If U is a unitary operator of a separable Hilbert space H, then there exists a maximal cyclic subspace. Moreover, each $x \in H$ is contained in some maximal cyclic space.*

Proof. Apply Kuratowski–Zorn's lemma to all cyclic subspaces containing x. $\square$

Lemma A.3.15. *If U_i are unitary operators on H_i, $i = 1, 2$, such that $U_1 \simeq U_2$ and $U|_{Z(x)} \simeq U_2|_{Z(y)}$ then $U_1|_{Z(x)^\perp} \simeq U_2|_{Z(y)^\perp}$.*

Proof. The lemma can be transferred to one space. Then we get a new formulation: if $U|_{Z(x)} \simeq U|_{Z(y)}$ then $U|_{Z(x)^\perp} \simeq U|_{Z(y)^\perp}$. It suffices to show that

$$U|_{\overline{Z(x)+Z(y)}} \ominus Z(x) \simeq U|_{\overline{Z(x)+Z(y)}} \ominus Z(y)$$

since $U|_{\overline{Z(x)+Z(y)}} \simeq U|_{\overline{Z(x)+Z(y)}}$. In other words we may assume that

$$\overline{Z(x)+Z(y)} = H.$$

Let $y = y_0 + y_1$, $y_0 \perp Z(x)$, $y_1 \in Z(x)$; then $H = Z(x) \oplus Z(y_0)$. Indeed, suppose $z \perp Z(x) \oplus Z(y_0)$, then $z \perp Z(y_1) \oplus Z(y_0)$, thus $z \perp Z(y)$, and consequently $z = 0$ because $\overline{Z(x)+Z(y)} = H$. By the symmetry of arguments we get that there x_0, y_0 such that

$$H = Z(x) \oplus Z(y_0) = Z(y) \oplus Z(x_0)$$

and all we have to prove is that $\widetilde{x_0}$ is equivalent to $\widetilde{y_0}$. By assumptions of this lemma we have that $\widetilde{x} \equiv \widetilde{y}$ and $\widetilde{x} + \widetilde{y_0} \equiv \widetilde{y} + \widetilde{x_0}$. Suppose that $\widetilde{y_0}$ is not equivalent to $\widetilde{x_0}$. Then there is a nonzero positive measure ν satisfying $\nu \ll \widetilde{x} \equiv \widetilde{y}$, $\nu \ll \widetilde{y_0}$, $\nu \perp \widetilde{x}_0$ (decompose simply $\widetilde{y}_0 = \nu + \mu$, where $\nu \perp \widetilde{x}_0$, $\mu \ll \widetilde{x}_0$). We find $z_1 \in Z(x)$, $z_2 \in Z(y_0)$ with $\widetilde{z}_1 = \nu = \widetilde{z}_2$. Then necessarily $Z(z_1) \perp Z(x_0)$, $Z(z_2) \perp Z(x_0)$ and $z_1, z_2 \in Z(y)$. Thus we get $Z(z_1) \oplus Z(z_2) \subset Z(y)$ that forces $\nu = \widetilde{z}_1 \perp \widetilde{z}_2 = \nu$. Hence $\nu = 0$ and we are done. $\square$

As a simple consequence of the lemmas above we have the following.

Theorem A.3.16. *If U is a unitary operator on a separable Hilbert space H then H can be decomposed into an orthogonal sum of cyclic spaces $H = \bigoplus_{n \geq 1} Z(x_n)$ with $\widetilde{x}_1 \gg \widetilde{x}_2 \gg \cdots$.*

Remark A.3.17. A similar theory can be built for a more general case. Let G be a locally compact second countable group. Denote by $\widehat{G}$ the dual group, i.e. $\widehat{G}$ is the group of all continuous group homomorphisms defined on G and with range in the circle $\mathbb{T}$. We consider $\widehat{G}$ as a topological group with the compact-open topology. Suppose that G acts as a group of automorphisms on a probability Lebesgue space $(X, \mathcal{B}, \mu)$, $G \times X \ni (g, x) \mapsto \gamma_g(x) \in X$. Then we may consider the group $\{U_g : g \in BG\}$ of unitary operators on $L^2(X, \mathcal{B}, \mu)$, where $U_g(f) = f \circ \gamma_g$ for $f \in L^2(X, \mathcal{B}, \mu)$. In this situation Theorem A.3.16 is still valid.

APPENDIX B

TOPOLOGICAL TOPICS

B.1. Uniform structures

In this section we recall the notion of uniform space. We will use terminology and notations from [48, Chapter 6].

Let X be a nonempty set. We will deal with relations on X i.e. with nonempty subsets of the Cartesian product $X \times X$. If U is such a relation then by U^{-1} we denote the relation

$$U^{-1} = \{(y, x) \in X \times X : (x, y) \in U\}.$$

Obviously this operation is involutory in the sense that $(U^{-1})^{-1} = U$. If $U^{-1} = U$ then U is called *symmetric*. If U and V are relations, then the composition $U \circ V$ is defined by

$$U \circ V = \{(x, z) \in X \times X : \underset{y \in X}{\exists} (x, y) \in U, (y, z) \in V\}.$$

Composition is associative, that is, $U \circ (V \circ W) = (U \circ V) \circ W$, and it is always true that $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$. Not that in general $U \circ V$ need not be equal to $V \circ U$. The set

$$\Delta_X = \Delta = \{(x, x) : x \in X\}$$

is called the *identity relation*, or the *diagonal*. For each subset $A \subset X$ and each relation $U \subset X \times X$ define

$$U_A = \{y \in X : \underset{x \in A}{\exists} (x, y) \in U\}.$$

For $x \in X$ denote $U_x = U_{\{x\}}$. For each relations U, V on X and each set $A \subset X$ the following holds: $(U \circ V)_A = U_{V_A}$.

Definition B.1.1. A non–void family $\mathfrak{U}$ of relations on X that satisfies the following conditions

(a) if $U \in \mathfrak{U}$ and $U \subset V \subset X \times X$, then $V \in \mathfrak{U}$;
(b) if $U, V \in \mathfrak{U}$, then $U \cap V \in \mathfrak{U}$;

(c) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
(d) if $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$;
(e) $\bigcap \mathcal{U} = \Delta$.

is called a *uniformity* on X. If $\mathcal{U}$ is a uniformity on X, then the pair $(X, \mathcal{U})$ is said to be a *uniform space.*

Definition B.1.2. Let $(X, \mathcal{U})$ be a uniform space. The family

$$\mathcal{T} = \{T \subset X : \underset{x \in X}{\forall} \ \underset{U \in \mathcal{U}}{\exists} \ U_x \subset T\}$$

is called the *uniform topology.*

Simple calculations show that the uniform topology is a topology. Actually we have more.

Theorem B.1.3 ([48, Corollary 6.17])**.** *The uniform topology is a Tychonoff topology. Conversely, for each Tychonoff topology $\mathcal{T}$ on X there exists a uniformity $\mathcal{U}$ on X such that $\mathcal{T}$ is the uniform topology defined by $\mathcal{U}$.*

B.2. The Čech–Stone compactification of a discrete topological group

In this section we briefly describe the Čech–Stone compactification of a discrete topological group, paying special attention on a semitopological semigroup structure of this compactification we are able to endow it with. We start with the general notions and facts. The reader can find more information on compactifications of a topological spaces in [48, Chapter 5], the following is based on.

Definition B.2.1. Let X be a topological space. By a *compactification* of X we mean a pair (f, Y), where Y is a compact Hausdorff space and $f: X \to Y$ is a homeomorphism of X onto a dense subspace of Y.

The class of all compactifications of X may be endowed with a relation that turns out to be a partial order (see Theorem B.2.2 below). Namely, we say that $(f, Y) \geqq (g, Z)$ if and only if there is a continuous map $h: X \to Z$ that is onto and satisfies $h \circ f = g$. Equivalently $(f, Y) \geqq (g, Z)$ if and only if the function $g \circ f^{-1}: f(X) \to Z$ has a continuous extension h which carries Y onto Z. If the function h can be taken to be a homeomorphism, then (f, Y) and (g, Z) are said to be *topologically equivalent.* In this case both the relations $(f, Y) \geqq (g, Z)$ and $(g, Z) \geqq (f, Y)$ hold, for h^{-1} is a continuous map of Z onto Y such that $f = h^{-1} \circ g$.

Theorem B.2.2 ([48, Theorem 22])**.** *The collection of all compactifications of a topological space is partially ordered by $\geqq$.*

We intend to find a maximal with respect to $\geqq$ compactification of a given space. In order to do this, for a given topological space X denote

$$F(X) = \{\gamma : X \to [0,1] : \gamma \text{ is continuous}\}.$$

By Tychonoff theorem, the Tychonoff cube $[0,1]^{F(X)}$ is a compact Hausdorff space. Consider the evaluation map $e\colon X \to [0,1]^{F(X)}$, $e(x) = (f(x))_{f\in F(X)}$. Directly from the definition of the product topology we get that the evaluation map e is continuous. If moreover X is a Tychonoff space then the evaluation map e is also open.

Definition B.2.3. Let X be a Tychonoff space. Then the pair $(e, \overline{e(X)})$ is called the *Čech–Stone compactification* of X.

Denote $\beta(X) = \overline{e(X)}$, $\beta X = (e, \beta(X))$.

We intend to show that each continuous map on a Tychonoff space X with range in a compact Hausdorff space can be extended to a continuous map on the Čech–Stone compactification of X. We start with a lemma that directly follows from the definition of the product topology.

Lemma B.2.4. *Let A and B be two nonempty sets. If $f\colon A \to B$ and $f^*\colon [0,1]^B \to [0,1]^A$ is defined by $f^*(y) = y \circ f$ for all $y \in [0,1]^B$, then f^* is continuous.*

Theorem B.2.5 (Čech–Stone, see e.g. [48, Theorem 5.24])**.** *If X is a Tychonoff space, Y a compact Hausdorff space, and $f\colon X \to Y$ is a continuous function, then there exists a continuous extension $\widetilde{f}\colon \beta X \to Y$ of f i.e. $\widetilde{f}$ satisfies $\widetilde{f}|_{e(X)} = f \circ e^{-1}$.*

Proof. Given f define $f_*\colon F(Y) \to F(X)$ by letting $f_*(a) = a\circ f$. Then, define $f^*\colon [0,1]^{F(X)} \to [0,1]^{F(Y)}$ by letting $f^*(q) = q \circ f_*$. Let e be the evaluation map of X into $[0,1]^{F(X)}$ and let ε be the evaluation map of Y into $[0,1]^{F(Y)}$ (see the diagram below).

$$\begin{array}{ccc}
\beta(X) \subset [0,1]^{F(X)} & \xrightarrow{f^*} & [0,1]^{F(Y)} \supset \beta(Y) \\
{\scriptstyle e}\big\uparrow & & \big\uparrow{\scriptstyle \varepsilon} \\
X & \xrightarrow[f]{\qquad\qquad} & Y
\end{array}$$

The map e is a homeomorphism of X and $e(X)$, and the map ε is a homeomorphism of Y onto $\beta(Y)$ because Y is a compact Hausdorff space. By Lemma B.2.4, the map f^* is continuous. We will prove that $\varepsilon^{-1} \circ f^*$ is the required continuous extension of $f \circ e^{-1}$. To do this we will show that $f^* \circ e = \varepsilon \circ f$. Let $x \in X$, $h \in F(Y)$, then $(f^* \circ e)(x)(h) = (e(x) \circ f_*)(h) = e(x)(h \circ f) = h \circ f(x) = \varepsilon(f(x))(h) = (\varepsilon \circ f)(x)(h)$ because of the definitions of f^*, f_*, e and ε respectively. The theorem follows. $\square$

It follows from Theorem B.2.5 that the Čech–Stone compactification βX is the largest Hausdorff compactification in the class of Tychonoff spaces. In particular βX is unique up to topological equivalence. It also follows that the compactification βX is uniquely defined by the extension property of Theorem B.2.5.

Suppose now that T is an abstract group. Endow T with the discrete topology and consider the Čech–Stone compactification βT of T. We shall now describe, following [27], the algebraic structure of a semitopological semigroup on βT. For a fixed $t \in T$ consider the map $T \ni s \mapsto ts \in T \subset \beta T$ (as T is homeomorphically embedded in βT). By Theorem B.2.5, this map can be extended to a continuous map $\beta T \ni p \mapsto tp \in \beta T$. Next, consider, for a fixed $p \in \beta T$, the map $T \ni t \mapsto tp \in \beta T$. This again can be extended to a continuous map $\beta T \ni q \mapsto qp \in \beta T$. This gives a semigroup structure on βT. This semigroup is semi semitopological in the sense that the multiplication defined above is continuous from the left.

A subset E of βT is called a *left ideal*, if $(\beta T)E \subset E$. By Kuratowski–Zorn's lemma, βT always contains minimal (with respect to inclusion) ideals.

Lemma B.2.6. *Each minimal left ideal of the semigroup βT is a closed set.*

Proof. Let $M \subset \beta T$ be a minimal left ideal. Then $(\beta T)M = M$. We will show, that $\overline{TM} = \beta TM$. Let $q \in \beta T$, $m \in M$. Let $(g_i)_{i\in I}$ be a net of elements of T that converges to q, $t_i \xrightarrow{i} q$ in βT. Then $t_i m \xrightarrow{i} qm$, so $qm \in \overline{TM}$. Thus $\overline{TM} = M$ and lemma is proved. □

From now on we will deal only with left minimal ideals of βT and we will refer to them as minimal ideals.

An element $p \in \beta T$ that satisfies $p^2 = p$ is called an *idempotent*.

Lemma B.2.7. *Let E be a compact Hausdorff topological space provided with a semigroup structure such that the maps $y \mapsto yx$ are continuous, for all $x \in E$. Then there exists an idempotent in E.*

Proof. Let $\mathcal{S}$ be the collection of all closed nonempty subsets S of E with the property $S^2 = \{s_1 s_2 : s_1, s_s \in S\} \subset S$. As $E \in \mathcal{S}$, $\mathcal{S}$ is nonempty. By Kuratowski–Zorn's lemma there exists a minimal (under inclusion) element in $\mathcal{S}$, say S_0. If $x \in S_0$ then $S_0 x$ is closed, nonempty and $(S_0 x)(S_0 x) \subset S_0^3 x \subset S_0 x$. Hence $S_0 x \in \mathcal{S}$ and since $S_0 x \subset S_0$ it follows that $S_0 x = S_0$.

Let $S = \{y \in S_0 : yx = x\}$, then S is closed non-empty and clearly $S^2 \subset S \subset S_0$. Hence $S = S_0$, and $x^2 = x$ is an idempotent. □

Proposition B.2.8. *Let M be a minimal ideal of βT, and let J be the set of idempotents in M. Then:*

(a) $J \neq \emptyset$.
(b) *If $v \in J$ and $p \in M$ then $pv = p$.*

(c) *For each* $v \in J$*, the set* $vM = \{p \in M : vp = p\}$ *is a subgroup of* M *with identity element* v*. The map* $p \mapsto wp$ *is a group isomorphism of* vM *onto* wm *for every idempotent* $w \in J$*.*

(d) *The collection* $\{vM : v \in J\}$ *is a partition of* M*.*

Proof. (a) follows from Lemma B.2.7.

(b) Since Mv is a minimal ideal that is contained in M, $Mv = M$. If $p \in M$ then there exists $q \in M$ such that $qv = p$. Now $pv = (qv)v = qv^2 = qv = p$.

(c) The only group property we have to show is that each element of vM has an inverse in vM. Let $p \in vM$ then, as in (b), $Mp = M$. Hence there exists $q \in M$ such that $qp = v$. Also $Mq = M$ and there exists $r \in M$ such that $rq = v$. Now $p = vp = rqp = rv = r$ and $vp = qpq = qrq = qv = q$. Thus $q \in vM$ and $qp = pq = v$ i.e. p has an inverse in vM. Denote $q = p^{-1}$.

Now we will prove that the map $vM \ni p \mapsto wp \in wM$ is an isomorphism. Observe that $(wp^{-1})(wp) = wp^{-1}p = wv = w$. Thus $wp^{-1} = (wp)^{-1}$, also $w(pq) = (wp)wq$ and $v(wp) = vp = p$.

(d) If $p \in M$ then $Mp = M$. Hence the set $A = \{q \in M : qp = p\}$ is closed and nonempty, and $A^2 \subset A$. By Lemma B.2.7 there is an idempotent $w \in A$, and thus $p \in wM$, i.e. $M = \bigcup\{vM : v \in J\}$. If $p \in vM \cap wM$ then $w = pp^{-1} = v$, hence the union is disjoint. □

Corollary B.2.9. *Let* M *be a minimal ideal of* βT*, and let* J *be the set of idempotents in* M*. Choose* $u \in J$ *and denote* $G = uM$*. Then every element* p *of* M *has unique representation* $p = v\alpha$ *for* $v \in J$ *and* $\alpha \in G$*. Moreover* $p^{-1} = v\alpha^{-1}$*.*

Lemma B.2.10. *Let* L *and* M *be minimal ideals in* βT*. Let* $v \in M$ *be an idempotent. Then there exists a unique idempotent* $v' \in L$ *such that* $vv' = v'$ *and* $v'v = v$*.*

Proof. Let $v \in M$ be an idempotent, then $Lv = M$ and by Lemma B.2.7 we conclude that the set $\{q \in L : qv = v\}$ contains an idempotent v'. Similarly $Mv' = L$ and we conclude that there exists an idempotent $v_1 \in M$ such that $v_1v' = v'$. Now $v = v'v = v_1v'v = v_1v = v_1$, so we have $v'v = v$, $vv' = v'$. This also shows that v' is unique. □

If $v \in M$ and $v' \in L$ are idempotents satisfying $vv' = v'$, $v'v = v$ then we write $v \sim v'$ and say that v' is *equivalent* to v.

Lemma B.2.11. *If* K, L, M *are minimal ideals in* βT *and* $v \in M$*,* $v' \in L$*,* $v'' \in K$ *are idempotents such that* $v \sim v'$ *and* $v' \sim v''$ *then* $v \sim v''$*.*

Proof. We have $vv'' = v(v'v'') = (vv')v'' = v'v'' = v''$ and similarly $v''v = v$. Thus $v \sim v''$. □

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