Juliusz Schauder Center for Nonlinear Studies
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Juliusz Schauder Center Winter School on

# METHODS
# IN MULTIVALUED ANALYSIS

Toruń, February 15–18, 2006

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## FOREWORD

This volume contains lectures delivered during the Winter School on Methods in Multivalued Analysis organized in February 15–18, 2006 in Toruń by the Juliusz Schauder Center for Nonlinear Studies at the Faculty of Mathematics and Computer Sciences of the Nicolaus Copernicus University. Apart from the plenary lectures by L. Górniewicz, W. Kryszewski, S. Plaskacz and L. Rybiński, the series of short talks have been communicated during the afternoon meetings. Full texts of these lectures and communications are included into the present volume.

The school has gathered 70 participants from different universities of Poland; among them were students, graduate students and scientists working in the area of set-valued analysis.

The organizers and the lecturers hope that the publication of their lectures will be welcomed by the community of Polish and foreign mathematicians. The organizers and the lecturers wish to thank all the people who contributed the success of the school.

*Organizers*

Toruń, June 2006

## CONTENTS

# INTRODUCTION

The interest in the theory of set-valued mappings is caused by many reasons. This intensely developing branch of mathematics has lot in common with topology, nonlinear, convex and non-smooth analysis, the theory of functions and ordinary or partial differential equations. Many results of this theory have found interesting applications in game theory, mathematical economics and control theory. A substantial part of this theory is the study of fixed points and the solvability of generalized equations involving set-valued maps. Such equations, or so-called 'inclusions', are solved by the use of abstract methods of algebraic topology or various approximation techniques.

The algebraic methods, especially those based on the homology or cohomology theory are carefully studied in the lecture by L. Górniewicz, while approximation techniques are presented in the lecture by W. Kryszewski. Such methods lead to diverse results showing the existence of fixed points and/or solutions to many problems arising in nonlinear analysis such as differential equations (without the uniqueness of solutions), differential or integro-differential inclusions and many others. The lecture by L. Rybiński is dealing with various types of selection problems that arise while studying set-valued maps. In particular the problem of the existence of measurable, continuous and Carathéodory selections is thoroughly studied. The powerful tools for studying differential inclusions and related problems in the theory of optimal control or the viability theory are delivered by the techniques of convex and nonsmooth analysis. These methods, in particular the generalized differentiability, are presented in the last lecture by S. Plaskacz.

The contents of the volume is far from being a complete presentation of all methods of the widely understood multlivalued analysis. It rather reflects the personal viewpoint of the authors and, to some extent, shows sometimes their recent contribution to the theory of set-valued maps. However it is a hope of the authors that the presented material may be a convienient starting point for all people wanting to go deeper into the theory and to learn about the beauty of the outlined subject.

# PART I

# LECTURES

Juliusz Schauder Center Winter School
on Methods in Multivalued Analysis
Lecture Notes in Nonlinear Analysis
Volume 8, 2006, 11–66

# HOMOLOGICAL METHODS IN FIXED POINT THEORY OF MULTIVALUED MAPPINGS

Lech Górniewicz

Abstract. In this lecture we would like to present a systematic study of the fixed point point theory for multivalued maps by using homological methods. Homological methods were initiated in 1946 by S. Eilenberg and D. Montgomery in their celebrated paper [11]. Note that by using homological methods it is possible to obtain stronger results than those obtained by means of another methods, for example, approximation methods (comp. [1], [14], [5]).

In this work we shall use some notions and results contained in [18].

## 1. Homology

In this section we consider the Čech homology functor $H$ with compact carriers and those of its properties which are of importance in the fixed-point theory of multi-valued maps. Therefore all facts concerning $H$ are formulated only in the form necessary in the material which follows. The Čech homology and cohomology are of auxiliary importance.

By a *pair* of spaces $(X, X_0)$ we understand a pair consisting of a Hausdorff topological space $X$ and of one its subsets $X_0$. A pair of the form $(X, \emptyset)$ will be identified with the space $X$. Let $(X, X_0), (Y, Y_0)$ be two pairs; if $X \subset Y$ and $X_0 \subset Y_0$, then we say that $(X, X_0)$ is a *subpair* of $(Y, Y_0)$ and indicate this by writing $(X, X_0) \subset (Y, Y_0)$.

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*Key words and phrases.* Lefschetz number, fixed points, CAC-maps, condensing maps, ANR-spaces, fixed point index.

A pair $(X, X_0)$ is called *compact* if $X$ is a compact space and closed subset of $X$.

By a *map* $f\colon (X, X_0) \to (Y, Y_0)$ we understand a continuous (single-valued) map $f\colon X \to Y$ satisfying the condition $f(X_0) = Y_0$. The category of all pairs and maps will be denoted by $\mathcal{C}$. By $\widetilde{\mathcal{C}}$ will be denoted the subcategory of $\mathcal{C}$ consisting of all compact pairs and maps of such pairs. Two maps $f, g\colon (X, X_0) \to (Y, Y_)$ are said to be *homotopic* (written $f \sim g$) provided that there is a map $h\colon (X \times \langle 0,1\rangle, X_0 \times \langle 0,1\rangle) \to (Y, Y_0)$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for each $x \in X$.

We observe that if $(X, X_0)$ is a pair in $\widetilde{\mathcal{C}}$, then $(X \times \langle 0,1\rangle, X_0 \times \langle 0,1\rangle)$ is also in $\widetilde{\mathcal{C}}$.

By $H_*$ $(H^*)$ we denote the *Čech homology* (*cohomology*) *functor* with the coefficients in the field of rational numbers $\mathbb{Q}$ from the category $\widetilde{\mathcal{C}}$ to the category $\mathcal{A}$ of graded vector spaces over $\mathbb{Q}$ and linear maps of degree zero. Thus, for a pair $(X, X_0)$,

$$H_*(X, X_0) = \{H_q(X, X_0)\}, \quad (H^*(X, X_0) = \{H^q(X, X_0)\}),$$

is a graded vector space and, for a map $f\colon (X, X_0) \to (Y, Y_0)$, $H_*(f)$ $(H^*(f))$ is the induced linear map

$$\begin{gathered} f_* = \{f_{*q}\}\colon H_*(X, X_0) \to H_*(Y, Y_0) \\ (f^* = \{f^{*q}\}\colon H^*(X, X_0) \to H^*(Y, Y_0)), \end{gathered}$$

where $f_{*q}\colon H_q(X, X_0) \to (H_q(Y, Y_0)$, $(f^{*q}\colon H^q(X, X_0) \to (H^q(Y, Y_0))$.

We assume as known that the functor $H_*$, $(H^*)$ satisfies all of the Eilenberg–Steenrod axioms for homology (cohomology). Recall that a Čech homology (cohomology) theory can be defined on the category $\mathcal{A}$. Then the Čech cohomology satisfies all of the Eilenberg–Steenrod axioms; however, the Čech homology satisfies all of the Eilenberg–Steenrod axioms except that of exactness.

By $\mathrm{Hom}_{\mathbb{Q}}\colon \mathcal{A} \to \mathcal{A}$ we denote the contravariant functor which to a graded vector space $E = \{E_q\}$ assigns the conjugate graded space $\mathrm{Hom}_{\mathbb{Q}}(E) = \{\mathrm{Hom}(E_q, \mathbb{Q})$ and to a linear map $l\colon E_1 \to E_2$ between graded spaces assigns the conjugate map

$$\mathrm{Hom}_{\mathbb{Q}}(l)\colon \mathrm{Hom}_{\mathbb{Q}}(E_2) \to \mathrm{Hom}_{\mathbb{Q}}(E_1)$$

given by the formula

$$\mathrm{Hom}_{\mathbb{Q}}(l)(u) = u \circ l, \quad \text{for every } u \in \mathrm{Hom}_{\mathbb{Q}}(E_2).$$

We now formulate the Duality Theorem between the Čech homology and cohomology.

**(1.1) Theorem.** *On the category $\widetilde{\mathcal{C}}$ the functors $H_*$ and $\mathrm{Hom}_{\mathbb{Q}} \circ H^*$ are naturally isomorphic; in other words, for every $f\colon (X,X_0) \to (Y,Y_0)$ in $\widetilde{\mathcal{C}}$ we have the commutative diagram*

$$\begin{tikzcd}
H_*(X,X_0) \arrow[r, "\sim"] \arrow[d, "f_*"'] & \mathrm{Hom}_{\mathbb{Q}}(H^*(X,X_0)) \arrow[d, "\mathrm{Hom}_{\mathbb{Q}}(f^*)"] \\
H_*(Y,Y_0) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathbb{Q}}(H^*(Y,Y_0)).
\end{tikzcd}$$

A graded vector space $E = \{E_q\}$ in $\mathcal{A}$ is said to be of *finite type* provided:

(i) $\dim E_q < \infty$ for all $q$ and

(ii) $E_q = 0$ for almost all $q$.

The following fact is well known:

(1.2) *If $E$ is a graded vector space of a finite type, then the graded vector space $\mathrm{Hom}_{\mathbb{Q}}(E)$ is isomorphic to $E$; in particular, $\mathrm{Hom}_{\mathbb{Q}}(E)$ is also of a finite type.*

A pair $(X,X_0)$ in $\widetilde{\mathcal{C}}$ is said to be of *finite type with respect to $H_*$ ($H^*$)* provided the graded vector space $H_*(X,X_0)$ ($H^*(X,X_0)$) is of finite type.

From (1.1) and (1.2) we instantly obtain:

(1.3) *A pair $(X,X_0)$ in $\widetilde{\mathcal{C}}$ is of finite type with respect to $H_*$ if and only if $(X,X_0)$ is of a finite type with respect to $H^*$.*

For pairs $(X,X_0)$, $(Y,Y_0)$ in $\mathcal{C}$ we define the Cartesian product as the pair given by $(X,X_0)\times(Y,Y_0) = (X\times Y, X\times Y_0 \cup X_0 \times Y)$, where in $X \times Y$ the Cartesian product topology is given.

Given maps $f\colon (X,X_0) \to (Y,Y_0)$ and $g\colon (X',X_0') \to (Y',Y_0')$, we can define the product map $f\times g\colon (X,X_0)\times(X',X_0') \to (Y,Y_0)\times(Y',Y_0')$ by letting

$$(f\times g)(x,x') = (f(x), g(x')), \quad \text{for every } x\in X \text{ and } x'\in X'.$$

**(1.4) Theorem** (Küneth Theorem)**.** *For every two pairs $(X,X_0)$, $(X',X_0')$ in $\widetilde{\mathcal{C}}$, there is a linear isomorphism*

$$\mathcal{L}\colon H^*((X,X_0)\times(X',X_0')) \to H^*(X,X_0)\otimes H^*(X',X_0')$$

*such that if $f\colon (X,X_0)\to(Y,Y_0)$ and $g\colon (X',X_0')\to(Y,Y_0')$ in $\widetilde{\mathcal{C}}$, then the following diagram commutes:*

$$\begin{tikzcd}
H^*((X,X_0)\times(X',X_0')) \arrow[d, "\mathcal{L}"'] & H^*((Y,Y_0)\times(Y',Y_0')) \arrow[l, "(f\times g)^*"'] \arrow[d, "\mathcal{L}"] \\
H^*(X,X_0)\otimes H^*(X',X_0') & H^*(Y,Y_0)\otimes H^*(Y',Y_0'). \arrow[l, "f^*\otimes g^*"]
\end{tikzcd}$$

From (1.1), (1.4) and the conimutativity of functors $\otimes$ and $\mathrm{Hom}_{\mathbb{Q}}$ for graded, vector spaces of finite type, we have:

**(1.5) Theorem.** *For every two pairs of finite type $(X, X_0)$, $(X', X'_0)$ in $\widetilde{\mathcal{C}}$, there is a linear isomorphism*

$$\overline{\mathcal{L}}\colon H_*((X, X_0) \times (X', X'_0)) \to H_*(X, X_0) \otimes H_*(X', X'_0)$$

*such that if $f\colon (X, X_0) \to (Y, Y_0)$ and $g\colon (X', X'_0) \to (Y', Y'_0)$ are two maps of pairs of finite type, then the following diagram commutes:*

$$\begin{tikzcd}
H_*((X, X_0) \times (X', X'_0)) \arrow[d, "\overline{\mathcal{L}}"'] & H_*((Y, Y_0) \times (Y', Y'_0)) \arrow[l, "(f\times g)^*"'] \arrow[d, "\overline{\mathcal{L}}"] \\
H_*(X, X_0) \otimes H^*(X', X'_0) \arrow[r, "j_* \otimes g_*"'] & H_*(Y, Y_0) \otimes H_*(Y', Y'_0)
\end{tikzcd}$$

Now, we prove the following theorem:

**(1.6) Theorem.** *Let $(X, d)$ be a compact metric space of finite type with respect to $H^*$. Then there exists an $\varepsilon > 0$ such that for every two maps $f, g\colon Y \to X$, where $Y$ is a compact space, the condition*

$$d(f(y), g(y)) < \varepsilon, \quad \text{for each } y \in Y,$$

*implies $f^* = g^*$.*

First we prove the following lemma:

**(1.7) Lemma.** *Let $X$ be a normal topological space and $\alpha = \{U_1, \ldots, U_n\}$ a finite covering of $X$ by open sets. Then there exists a covering $\beta = \{V_1, \ldots, V_n\}$ of $X$ by open sets, such that for each $i = 1, \ldots, n$, $\overline{V}_i \subset U_i$ ($\overline{V}_i$ denotes the closure of $V_i$ in $X$).*

*Proof.* Consider the following two closed subsets of $X$: $F = X \setminus U_i$, $F' = X \setminus \bigcup_{j=1, j\neq i}^n U_j$, where $i = 1, \ldots, n$ is an arbitrary but fixed number. Since $F \cap F' = \emptyset$, by the normality of $X$ we find open subsets $U$ and $V_i$ of $X$ such that:

(i) $F \subset U$,
(ii) $F' \subset V_i$ and
(iii) $U \cap V_i = \emptyset$.

Since $X \setminus \bigcup_{j=1, j\neq i}^n U_j \subset V_i$, we infer that the family $\{U_1, \ldots, U_{i-1}, V_i, U_{i+1}, \ldots, U_n\}$ is a covering of $X$ by open subsets and $\overline{V}_i \subset U_i$.

Applying the above construction successively for each $i = 1, \ldots, n$, we obtain a covering $\beta = \{V_1, \ldots, V_n\}$ of $X$ by open sets such that $\overline{V}_i \subset U_i$ for each $i = 1, \ldots, n$, and the proof of (1.7) is completed. □

In the proof of (1.6) we will establish the following conventions. By a *covering* of $X$ we understand a finite covering of $X$ by open sets. If $\alpha, \beta$ are two coverings of $X$, then the symbol $\alpha \geq \beta$ means that $\alpha$ *refines* $\beta$. If $\alpha$ is a covering of $X$, then $N(\alpha)$ will stand for the finite simplicial complex which is the *nerve* of $\alpha$ and $H^*(N(\alpha))$ is the simplicial cohomology of $N(\alpha)$ with coefficientes in $\mathbb{Q}$. If $\alpha, \beta$ are two coverings of $X$ and $\alpha \geq \beta$, then by $i_{\alpha\beta}\colon N(\alpha) \to N(\beta)$ we denote a *simplicial map* given by a vertex transformation from $N(\alpha)$ to $N(\beta)$ taking a set $V$ in $\alpha$ to a set $U$ in $\beta$ such that $V \subset U$. It is well known that $i^*_{\alpha\beta}\colon H^*(N(\beta)) \to H^*(N(\alpha))$ is independent of the choice of vertex transformations used to define $i_{\alpha\beta}$. Finally, for a map $f\colon Y \to X$ and a covering $\alpha = \{U_1, \dots, U_n\}$ of $X$, we denote by $f^{-1}(\alpha)$ the covering of $Y$ of the form

$$f^{-1}(\alpha) = \{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$$

and by $f_\alpha\colon N(f^{-1}(\alpha)) \to N(\alpha)$ a simplicial map given by the following vertex transformation:

$$f_\alpha(f^{-1}(U_i)) = U_i, \quad \text{for each } i = 1, \dots, n.$$

*Proof of Theorem* (1.6). Let $[u_{\alpha_1}], \dots, [u_{\alpha_k}]$ be a basis of $H^*(X)$, where $u_{\alpha_i} \in H^*(N(\alpha_i))$ for each $i = 1, \dots, k$. We choose a covering $\alpha = \{U_1, \dots, U_n\}$ of $X$ such that $\alpha \geq \alpha_i$, for all $i = 1, \dots, k$. Consider simplicial maps $i_{\alpha\alpha_i}\colon N(\alpha) \to N(\alpha_i)$ for each $i = 1, \dots, k$. Then

$$v^i_\alpha = i^*_{\alpha\alpha_i}(u_{\alpha_i}) \in [u_{\alpha_i}], \quad \text{for each } i.$$

Applying Lemma (1.7) to the covering $\alpha$, we obtain a covering $\beta = \{V_1, \dots, V_n\}$ such that $\overline{V}_i \subset U_i$ for each $i = 1, \dots, n$. Let $i_{\beta\alpha}\colon N(\beta) \to N(\alpha)$ be a simplicial map given by the vertex transformation $i_{\beta\alpha}(V_i) = U_i$ for each $i$. Then

$$w^i_\beta = i^*_{\beta\alpha}(v^i_\alpha) \in [u_{\alpha_i}], \quad \text{for each } i = 1, \dots k.$$

Let $\varepsilon = \min_i \operatorname{dist}(\overline{V}_i, X \setminus U_i)$. We may assume without loss of generality that $U_i \neq X$ for each $i$. Since $\overline{V}_i \cap X \setminus U_i = \emptyset$ and $\overline{V}_i$, $X \setminus U_i$ are compact, non-empty sets, we deduce that $e$ is a positive real number.

Let $Y$ be a compact space and let $f, g\colon Y \to X$ be two maps such that $d(f(y), g(y)) < \varepsilon$ for each $y \in Y$. We assert that $f^* = g^*$. Consider the coverings $\gamma = f^{-1}(\alpha)$ and $\delta = g^{-1}(\beta)$. It is easy to see that

$$g^{-1}(V_i) \subset f^{-1}(U_i), \quad \text{for each } i = 1, \dots, n \text{ and } \delta \geq \gamma.$$

Let $i_{\delta\gamma}\colon N(\delta) \to N(\gamma)$ be a simplicial map given by the vertex transformation $i_{\delta\gamma}(g^{-1}(V_i)) = f^{-1}(U_i)$ for each $i = 1, \dots, n$. We have the following commutative diagram:

$$\begin{tikzcd}
N(\gamma) \arrow[r, "f_\alpha"] & N(\alpha) \\
N(\delta) \arrow[u, "i_{\delta\gamma}"] \arrow[r, "g_\beta"'] & N(\beta) \arrow[u, "i_{\beta\alpha}"']
\end{tikzcd}$$

This implies that $i^*_{\beta\gamma} f^*_\alpha(v^i_\alpha) = g^*_\beta(w^i_\beta)$ for each $i = 1, \dots, k$ and hence we obtain $[f f^*_\alpha(v^i_\alpha] = [g^*_\beta(w^i_\beta)]$. Since $g^*([u_{\alpha_i}]) = [g^*_\beta(w^i_\beta)]$ and $f^*([u_{\alpha_i}]) = [f^*_\alpha(v^i_\alpha)]$, we find that the maps $f^*, g^*$ are equal on a basis of $H^*(X)$. Finally, from this we deduce that $f^* = g^*$ and the proof of (1.6) is completed. □

Using (1.1) we deduce that (1.6) is equivalent to the following:

**(1.8) Theorem.** *Let $(X, d)$ be a compact metric space of finite type with respect to $H_*$. Then there exists an $\varepsilon > 0$ such that for every two maps $f, g: Y \to X$, where $Y$ is a compact space, the condition:*

$$d(f(y), g(y)) < \varepsilon, \quad \textit{for each } y \in Y,$$

*implies $f_* = g_*$.*

**Remark.** Note that Theorem (1.6) remains true in the case where $Y$ is an arbitrary Hausdorff space.

Let $(X, X_0)$ be an arbitrary pair in $\mathcal{C}$. We shall denote by $\mathcal{M} = \{(A_\alpha, A_{0\alpha})\}$ the directed set of all compact pairs such that $(A_\alpha, A_{0\alpha}) \subset (X, X_0)$ for each $\alpha$, with the natural quasi-order relation $\leq$ defined by the condition

$$(A_\alpha, A_{0\alpha}) \leq (A_\beta, A_{0\beta}) \quad \text{if and only if } (A_\alpha, A_{0\alpha}) \subset (A_\beta, A_{0\beta}).$$

If $(A_\alpha, A_{0\alpha}) \leq (A_\beta, A_{0\beta})$, then we shall denote by $i_{\alpha\beta}: (A_\alpha, A_{0\alpha}) \to (A_\beta, A_{0\beta})$ the inclusion map. For each pair $(A_\alpha, A_{0\alpha})$ consider the graded vector space $H_*(A_\alpha, A_{0\alpha})$, together with the linear map $i_{\alpha\beta *}$ given for $(A_\alpha, A_{0\alpha}) \leq (A_\beta, A_{0\beta})$. Then the family $\{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}$ is a direct system in the category $\mathcal{A}$ over $\mathcal{M}$. We define a graded vector space

$$H(X, X_0) = \varinjlim_{\alpha} \{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}.$$

It is easy to see that

$$H_q(X, X_0) = \{H_q(X, X_0)\},$$

where

$$H_q(X, X_0) = \varinjlim_{\alpha} \{H_q(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}, \quad \text{for each } q.$$

Let $f: (X, X_0) \to (Y, Y_0)$ be a map. Consider the directed sets $\mathcal{M} = \{(A_\alpha, A_{0\alpha})\}$ and $\mathcal{N} = \{B_\gamma, B_{0\gamma})\}$ for $(X, X_0)$ and $(Y, Y_0)$, respectively. We define $F: \mathcal{M} \to \mathcal{N}$ by the formula

$$F((A_\alpha, A_{0\alpha})) = (f(A_\alpha), f(A_{0\alpha})), \quad \text{for each } (A_\alpha, A_{0\alpha}) \in \mathcal{M}.$$

We observe that if $(A_\alpha, A_{0\alpha}) \leq (A_\beta, A_{0\beta})$ then

$$F((A_\alpha, A_{0\alpha})) \leq F((A_\beta, A_{0\beta})).$$

For each $\alpha$, by $f_\alpha: (A_\alpha, A_{0\alpha}) \to (f(A_\alpha), f(A_{0\alpha}))$ we denote a map given by $f_\alpha(x) = f(x)$ for each $x \in A$. Then the map $F$ and the family $\{f_{\alpha *}\}$ is a map of directed systems $\{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}$ and $\{H_*(B_\gamma, B_{0\gamma}), i_{\delta\gamma *}\}$. We define the induced linear map for $f$, $H(f)$, by putting

$$H(f) = f_* = \varinjlim_{\alpha}\{f_{*\alpha}\}.$$

Then we have $f_{*q} = \varinjlim_{\alpha}\{f_{\alpha * q}\}$ for every $q$.

From the functoriality of $\varinjlim$ we deduce that $H\colon \mathcal{C} \to \mathcal{A}$ is a covariant functor. The functor $H$ is said to be the Čech homology functor with compact carriers.

We note that if $(X, X_0)$ is a compact pair, then the family consisting of the single pair $(X, X_0)$ is a cofinal subset of $\mathcal{M} = \{(A_\alpha, A_{0\alpha})\}$ for $(X, X_0)$, and hence we obtain $H_*(X, X_0) = H(X, X_0)$. Similarly, if $f\colon (X, X_0) \to (Y, Y_0)$ is a map of compact pairs, then $H_*(f) = H(f)$.

The following properties of $H$ clearly follow from the Eilenberg–Steenrod axioms for $H_*$ and the simple properties of $\varinjlim_{\alpha}$.

(1.9) *If $f, g\colon (X, X_0) \to f(Y, Y_0)$ are homotopie maps, then the induced linear maps are equal, that is, $f_* = g_*$.*

(1.10) *Let $(X, X_0)$ be a pair in $\mathcal{C}$ and let $i\colon X_0 \to X$, $j\colon X \to (X, X_0)$ be inclusions. Then there exists a linear map*

$$\delta_q\colon H_q(X, X_0) \to H_{q-1}(X_0), \quad \textit{for each } q,$$

*so that*

$$\cdots \longrightarrow H_q(X_0) \xrightarrow{i_{*q}} H_q(X) \xrightarrow{j_{*q}} H_q(X, X_0) \xrightarrow{\delta_q} H_{q-1}(X_0) \longrightarrow \cdots$$

*is exact.*

The linear map $\delta_q$ has the additional property of being natural in the following sense:

(1.11) *Given a map $f\colon (X, X_0) \to (Y, Y_0)$ in $\mathcal{C}$, the diagram*

$$\begin{array}{ccc} H_q(X, X_0) & \xrightarrow{\delta_q} & H_{q-1}(X_0) \\ {\scriptstyle f_{*q}}\big\downarrow & & \big\downarrow{\scriptstyle (f_{X_0})_{*q-1}} \\ H_q(Y, Y_0) & \xrightarrow[\delta_q]{} & H_{q-1}(Y_0) \end{array}$$

*commutes for all $q$, where $f_{X_0}\colon X_0 \to Y_0$ is given by the formula $f_{X_0}(x) = f(x)$, for each $x \in X_0$.*

A pair $(X, X_0)$ of finite type with respect to $H$ is called *simply* of finite type.

We prove the following

**(1.12) Theorem** ([1])**.** *Let $(X,d)$ be a compact metric space of finite type. Then there exists an $\varepsilon > 0$ such that, for every two maps $f,g\colon Y \to X$, where $Y$ is a Hausdorff space, the condition*

$$d(f(y),g(y)) < \varepsilon, \quad \textit{for each } y \in Y$$

*implies $f_* = g_*$.*

*Proof.* Let $\varepsilon$ be as in (1.8). Consider two maps $f,g$ from a Hausdorff space $Y$ to $X$. Let $A$ be a compact subset of $Y$ and let $f_A, g_A\colon A \to X$ be given by $f_A(y) = f(y)$, $g_A(y) = g(y)$ for each $y \in A$. We observe that $f_A, g_A$ satisfies the assumptions of (1.8). So, we have $(f_A)_* = (g_A)_*$. Since $f_* = \varinjlim_A\{(f_A)_*\}$ and $g_* = \varinjlim_A\{(g_A)_*\}$, we infer that $f_* = g_*$ and the proof of (1.12) is completed. □

A space $X$ is *acyclic* provided:

(i) $X$ is non-empty,
(ii) $H_q(X) = 0$ for all $q \geq 1$, and
(iii) $H_0(X) \approx \mathbb{Q}$.

A map $f\colon (X,X_0) \to (Y,Y_0)$ is *proper* provided for any compact $B$ the counter image $f^{-1}(B)$ is also compact. A map $f\colon (X,X_0) \to (Y,Y_0)$ is said to be a *Vietoris map* provided the following conditions are satisfied:

(i) $f$ is proper,
(ii) $f^{-1}(Y_0) = X_0$,
(iii) the set $f^{-1}(y)$ is acyclic for every $y \in Y$.

The following evident remark is of importance:

(1.13) *If $f\colon (X,X_0) \to (Y,Y_0)$ is a Vietoris map and $(B,B_0) \subset (Y,Y_0)$, then the map $\widetilde{f}\colon (f^{-1}(B), f^{-1}(B_0)) \to (B,B_0)$ is also a Vietoris map, where $\widetilde{f}(x) = f(x)$ for each $x \in f^{-1}(B)$.*

We shall require the following classical result:

**(1.14) Theorem** (Vietoris–Begle Mapping Theorem)**.** *Let $X,Y$ be compact spaces. If $f\colon X \to Y$ is a Vietoris map, then the induced map $f_*\colon H_*(X) \to H_*(Y)$ is a linear isomorphism.*

The Vietoris–Begle Mapping Theorem and the five lemma gives:

**(1.15) Theorem.** *Let $(X,X_0)$, $(Y,Y_0)$ be compact pairs. If $f\colon (X,X_0) \to (Y,Y_0)$ is a Vietoris map, then $f_*\colon H_*(X,X_0) \to H_*(Y,Y_0)$ is a linear isomorphism.*

Now, from (1.15) we deduce the following theorem for non-compact pairs.

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[1] Theorem (1.12) is a generalized version (for arbitrary topological spaces) of (1.18).

**(1.16) Theorem.** *If $f: (X, X_0) \to (Y, Y_0)$ is a Vietoris map, then the induced map $f_*: H(X, X_0) \to H(Y, Y_0)$ is a linear isomorphism.*

*Proof.* Consider $\mathcal{M} = \{(A_\alpha, A_{0\alpha})\}$ and $\mathcal{N} = \{(B_\gamma, B_{0\gamma})\}$ for $(X, X_0)$ and $(Y, Y_0)$, respectively. Let $\mathcal{M}_0 = \{(f^{-1}(B_\gamma), f^{-1}(B_{0\gamma})); (B_\gamma, B_{0\gamma}) \in \mathcal{N}\}$. Since $f$ is a proper map, we have $\mathcal{M}_0 \subset \mathcal{M}$. It is easy to see that $\mathcal{M}_0$ is a cofinal subset of $\mathcal{M}$. Therefore we may assume without loss of generality that

$$H(X, X_0) = \varinjlim_{\alpha \in \mathcal{M}_0} \{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}.$$

Then for each $y \in Y$ the map

$$f_\gamma: (f^{-1}(B_\gamma), f^{-1}(B_{0\gamma})) \to (B_\gamma, B_{0\gamma})$$

is a Vietoris map of compact pairs. Using (1.15) we infer that

$$f_{\gamma *}: H_*(f^{-1}(B_\gamma), f^{-1}(B_{0\gamma})) \to H_*(B_\gamma, B_{0\gamma})$$

is a linear isomorphism. Consequently, the linear map $f_* = \varinjlim_{\gamma \in \mathcal{N}} \{f_{\gamma *}\}$ is an isomorphism. The proof of (1.16) is completed. $\square$

In what follows the symbol $p: X \Longrightarrow Y$ will be used for Vietoris mappings. Below, we shall list some properties of Vietoris mappings.

**(1.17) Lemma** (comp. (1.13)).

(a) *If $X \overset{p_1}{\Longrightarrow} Y \overset{p_2}{\Longrightarrow} Z$ are Vietoris maps, then the composition $p_2 \circ p_1: X \Longrightarrow Z$ of $p_1$ and $p_2$ is a Vietoris map too;*

(b) *if $p: X \Longrightarrow Y$ is a Vietoris map and $B \subset Y$, then the map $\widetilde{p}: p^{-1}(B) \Longrightarrow B$, $\widetilde{p}(x) = p(x)$ for every $x \in p^{-1}(B)$ is a Vietoris map;*

(c) *consider a diagram of continuous maps*

$$\begin{tikzcd}
X \arrow[r, "f"] & Y & Z \arrow[l, Rightarrow, "p"'] \\
 & X \oplus Z \arrow[ul, Rightarrow, "p_1"] \arrow[u, "g"] \arrow[ur, "g_1"'] &
\end{tikzcd}$$

*in which $X \oplus Z = \{(u, v) \in X \times Z \mid f(x) = p(v)\}$, $g(u, v) = f(u) = p(u)$, $g_1(u, v) = v$, $p_1(u, v) = v$,*

*then $p$ to be a Vietoris maps implies that $p_1$ is a Vietoris map, too.*

**Remark.** Let us remark that the map $p_1$ in the above diagram is called the fiber product of $f$ and $p$.

Consider the subcategory $\mathcal{C}_0 \subset \mathcal{C}$ consisting of all pairs $(U, V)$ such that $U$ and $V$ are open subsets in the Euclidean space $\mathbb{R}^n$ for some $n$, or $V$ is a finite polyhedron and $V$ is an open subset of $U$, and all maps of such pairs.

Since the family of all pairs of finite polyhedra $\{(K, K_0)\}$ is cofinal in the family of all compact pairs $\{(A, A_0)\}$ contained in $(U, V)$, we obtain the following:

(1.18) *On the category* $\mathcal{C}_1$ *the functors* $H$ *and* $\overline{H}$ *are naturally isomorphic* ($\overline{H}$ denotes the singular homology functor with coefficients in $\mathbb{Q}$).

Let $A \subset U \subset \mathbb{R}^n$, where $A$ is compact and $U$ is open in $\mathbb{R}^n$. We identify the $n$th sphere $S^n = \{x \in \mathbb{R}^n \mid \|X\| = 1\}$ and $\mathbb{R}^n \cup \{\infty\}$. Then from the excision axiom for singular homology and (1.18) we deduce:

(1.19) *The inclusion* $j\colon (U, U \setminus A) \to (S^n, S^n \setminus A)$ *induces an isomorphism*

$$j_*\colon H(U, U \setminus A) \to H(S^n, S^n \setminus A).$$

Let $K$ be a finite polyhedron and $U$ an open subset of $\mathbb{R}^n$, where $K \subset U$. Consider a Vietoris map $p\colon Y \Longrightarrow U$ and a map $q\colon Y \to K$ from a Hausdorff space $Y$ to $K$. We prove the following:

(1.20) *There are isomorphisms* $a_1$, $a_2$, $a_3$ *such that the following diagram commutes*:

$$\begin{tikzcd}
H(U,U\setminus K)\otimes H(U) & H(U,U\setminus K)\otimes H(Y) \arrow[l, "\mathrm{id}\otimes p_*"'] \arrow[r, "\mathrm{id}\otimes q_*"] & H(U,U\setminus K)\otimes H(K) \\
H((U,U\setminus K)\times U) \arrow[u, "a_1"'] & H((U,U\setminus K)\times Y) \arrow[l, "(\mathrm{id}\times p)_*"] \arrow[r, "(\mathrm{id}\times q)_*"'] \arrow[u, "a_2"'] & H((U,U\setminus K)\times K) \arrow[u, "a_3"']
\end{tikzcd}$$

*Proof.* It is easy to see that the families

$$\{(M, M_0) \times L\}, \quad \{(M, M_0) \times p^{-1}(L)\}, \quad \{(M, M_0) \times K\},$$

where $M$, $M_0$, $L$ are finite polyhedra, are cofinal in families of all compact pairs contained in $(U, U \setminus K) \times U$, $(U, U \setminus K) \times Y$ and $(U, U \setminus K) \times K$, respectively. We observe that for every $L$ the space $p^{-1}(L)$ is of finite type ($p$ is a Vietoris map), so we may apply (1.5) and have the commutative diagram

$$\begin{tikzcd}
H_*(M,M_0)\otimes H_*(L) & H_*(M,M_0)\otimes H_*(p^{-1}(L)) \arrow[l, "\mathrm{id}\otimes(p_L)_*"'] \arrow[r, "\mathrm{id}\otimes(q_{p^{-1}(L)})_*"] & H_*(M,M_0)\otimes H_*(K) \\
H_*((M,M_0)\times L) \arrow[u, "f"] & H_*((M,M_0)\times p^{-1}(L)) \arrow[l, "(\mathrm{id}\times p_L)_*"] \arrow[r, "(\mathrm{id}\times q_{p^{-1}(L)})_*"'] \arrow[u, "f"] & H_*((M,M_0)\times K)
\end{tikzcd}$$

From the commutativity of the above diagram and the commutativity of $\varinjlim$ and $\otimes$ we simply deduce (1.20). $\square$

Consider the diagram

$$U \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} K,$$

where $p$ and $q$ are as in (1.20). With the above diagram we associate the following:

$$(U, U \setminus K) \stackrel{\overline{p}}{\Longleftarrow} (Y, Y \setminus p^{-1}(K)) \xrightarrow{\overline{q}} (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}),$$

where $\overline{p}(y) = p(y)$ and $\overline{q}(y) = p(y) - q(y)$ for each $y \in Y$. We observe that $\overline{p}$ is a Vietoris map. Let $\Delta\colon (U, U \setminus K) \to (U, U \setminus K) \times U$ be a map given by $\Delta(x) = (x, x)$ and let $d\colon (U, U \setminus K) \times K \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ be given by $d(x, x') = x - x'$, for each $x \in U$ and $x' \in K$.

**(1.21) Lemma.** *The following diagram commutes*

$$\begin{tikzcd}
H(U, U \setminus K) \arrow[r, "\Delta_*"] \arrow[dr, "\overline{q}_* \overline{p}_*^{-1}"'] & H(U, U \setminus K) \otimes H(U) \arrow[r, "\mathrm{id} \otimes q_* p_*^{-1}"] & H(U, U \setminus K) \otimes H(K) \arrow[dl, "d_*"] \\
& H(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) &
\end{tikzcd}$$

*Proof.* Consider the diagram

$$\begin{tikzcd}
(U, U \setminus K) \times U & (U, U \setminus K) \times Y \arrow[l, "\mathrm{id} \times p"'] \arrow[r, "\mathrm{id} \times q"] & (U, U \setminus K) \times K \arrow[d, "d"] \\
(U, U \setminus K) \arrow[u, "\Delta"] & (Y, Y \setminus p^{-1}(K)) \arrow[l, "\overline{p}"] \arrow[u, "f"] \arrow[r, "\overline{q}"] & (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})
\end{tikzcd}$$

where the map $f$ is given by $f(y) = (p(y), y)$ for each $y \in Y$. From the comtnutativity of the above diagram and (1.20) we obtain (1.21). $\square$

Let us fix for each $n$ an orientation $1 \in H_n(S^n) \approx \mathbb{Q}$ of the $n$th sphere $S^n = \mathbb{R}^n \cup \{\infty\}$. Consider the diagram

$$S^n \xrightarrow{i} (S^n, S^n \setminus A) \xleftarrow{j} (U, U \setminus A)$$

in which $A$ is a compact subset of $U$ and $U$ is open in $\mathbb{R}^n$; $i, j$ are inclusions. From (1.19) we infer that $j_*$ is an isomorphism. We define the fundamental class $O_A$ of the pair $(U, A)$ by the equality $O_A = j_{*n}^{-1} i_{*n}(1)$.

**(1.22) Lemma.** *Let $A \subset A_1 \subset V \subset U \subset \mathbb{R}^n$, where $A, A_1$ are compact, $V$, $U$ are open subsets of $\mathbb{R}^n$ and let $k\colon (V, V \setminus A_1) \to (U, U \setminus A)$ be the inclusion map. Then we have $k_{*n}(O_{A_1}) = O_A$.*

*Proof.* Consider the commutative diagram

$$\begin{tikzcd}
S^n \arrow[r, "i"] \arrow[dr, "i_1"'] & (S^n, S^n \setminus A) & (U, U \setminus A) \arrow[l, "j"'] \\
& (S^n, S^n \setminus A_1) \arrow[u, "k_1"] & (V, V \setminus A_1) \arrow[l, "j_1"] \arrow[u, "k"]
\end{tikzcd}$$

in which $j_1$, $i_1$, $k_1$ are inclusion maps. Applying $H_n$ to the above diagram, we obtain (1.22). $\square$

Finally, we formulate Dold's Lemma in terms of Čech homology with compact carriers. Let $K \subset U \subset \mathbb{R}^n$, where $K$ is a finite polyhedron and $U$ an open subset of $\mathbb{R}^n$. We define the following maps:

$$\begin{gathered} t\colon U \times K \to K \times U, \quad t(x,x') = (x',x), \quad \text{for each } x \in U \text{ and } x' \in K, \\ O_K^{\times}\colon H(K) \to H(U, U \setminus K) \otimes H(K), \quad O_K^{\times}(u) = O_K \otimes u, \quad \text{for each } u \in H(K), \\ \times\colon \mathbb{Q} \otimes H(U) \to H(U), \quad \times(q \otimes u) = q \cdot u, \quad \text{for each } u \in H(U),\ q \in \mathbb{Q}. \end{gathered}$$

**(1.23) Lemma.** *The composite*

$$\begin{aligned} l = l(K,U)\colon H(K) &\xrightarrow{O_K^{\times}} H(U, U \setminus K) \otimes H(K) \\ &\xrightarrow{\Delta_* \oplus \mathrm{id}} H(U, U \setminus K) \otimes H(U) \otimes H(K) \\ &\xrightarrow{\mathrm{id} \otimes t_*} H(U, U \setminus K) \otimes H(K) \otimes H(U) \\ &\xrightarrow{d_* \otimes \mathrm{id}} \mathbb{Q} \otimes H(U) \xrightarrow{\ \times\ } H(U) \end{aligned}$$

*coincides with the linear map $i_*\colon H(K) \to H(U)$.*

**Remark.** Lemma (1.23), in view of (1.18), clearly follows from the original statement of Dold's Lemma. For the proof of this lemma se [8] or [15].

## 2. The Lefschetz number

In what follows the vector spaces are taken over $\mathbb{Q}$.

Let $f\colon E \to E$ be an endomorphism of a finite-dimensional vector space $E$. If $v_1, \dots, v_n$, is a basis for $E$, then we can write

$$f(v_i) = \sum_{j=1}^{n} a_{ij} v_j, \quad \text{for all } i = 1, \dots, n.$$

The matrix $[a_{ij}]$ is called the matrix of $f$ (with respect to the basis $v_1, \dots, v_n$). Let $A = [a_{ij}]$ be an $(n \times n)$-matrix; then the trace of $A$ is defined as $\sum_{i=1}^{n} a_{ii}$. If $f\colon E \to E$ is an endomorphism of a finite-dimensional vector space $E$, then the trace of $f$, written $\mathrm{tr}(f)$, is the trace of the matrix of $f$ with respect to some basis for $E$. If $E$ is a trivial vector space then, by definition, $\mathrm{tr}(f) = 0$. It is a standard result that the definition of the trace of an endomorphism is independent of the choice of the basis for $E$.

We recall the following two basic properties of the trace:

**(2.1) Property.** *Assume that in the category of finite-dimensional vector spaces the following diagram commutes*

$$\begin{tikzcd} E' \arrow[r, "f"] & E'' \arrow[dl, phantom] \\ E' \arrow[u, "f'"] \arrow[r, "f"'] & E'' \arrow[u, "f''"'] \arrow[ul, "g"'] \end{tikzcd}$$

*Then* $\mathrm{tr}(f') = \mathrm{tr}(f'')$*; in other words* $\mathrm{tr}(gf) = \mathrm{tr}(fg)$.

**(2.2) Property.** *Given a commutative diagram of finite-dimensional vector spaces with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r] \arrow[d, "f'"'] & E \arrow[r] \arrow[d, "f"'] & E'' \arrow[r] \arrow[d, "f''"'] & 0 \\ 0 \arrow[r] & E' \arrow[r] & E \arrow[r] & E'' \arrow[r] & 0 \end{tikzcd}$$

*we have* $\mathrm{tr}(f) = \mathrm{tr}(f') + \mathrm{tr}(f'')$.

Let $E = \{E_q\}$ be a graded vector space in $\mathcal{A}$ of finite type. If $f = \{f_q\}$ is an endomorphism of degree zero of such a graded vector space, then the (*ordinary*) *Lefschetz number* $\lambda(f)$ of $f$ is defined by

$$\lambda(f) = \sum_q (-1)^q \mathrm{tr}(f_q).$$

Let $E$ be a finite-dimensional vector space and $v_1, \dots, v_n$ a basis for $E$. We define a basis $v^1, \dots, v^n$ for $\mathrm{Hom}_{\mathbb{Q}}(E)$ by putting

$$v^i(v_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

The basis $v^1, \dots, v^n$ is called the *conjugate basis* to $v_1, \dots, v_n$. For a vector space $E$ and any integer $q$, define a linear map $\Theta_q\colon \mathrm{Hom}_{\mathbb{Q}}(E) \otimes E \to \mathrm{Hom}(E, E)$ by letting

$$\Theta_q(u \otimes v)(v') = (-1)^q u(v') \cdot v, \quad \text{for } u \in \mathrm{Hom}_{\mathbb{Q}}(E),\ v, v' \in E$$

and extend $\Theta_q$ to all $\mathrm{Hom}_{\mathbb{Q}}(E) \otimes E$.

**(2.3) Lemma.** *If the vector space $E$ is finite-dimensional, then $\Theta_q$ is an isomorphism.*

*Proof.* Let $v_1, \dots, v_n$ be a basis for $E$ and $v^1, \dots, v^n$ the conjugate basis to $v_1, \dots, v_n$. Then every element $a$ in $\mathrm{Hom}_{\mathbb{Q}}(E) \otimes E$ has the following form:

$$a = \sum_{i,j=1}^{n} a_{ij} v^i \otimes v_j.$$

If $\Theta_q(a) = 0$, then

$$\Theta_q(a)(v_k) = (-1)^q \sum_{j=1}^{n} a_{kj} v^k(v_k) \cdot v_j = (-1)^q \sum_{j=1}^{n} a_{kj} \cdot v_j = 0,$$

so $a_{kj} = 0$ for all $k, j$, which implies that $a = 0$. To prove $\Theta_q$ is onto, let $\in \mathrm{Hom}(E, E)$. Then we can write

$$f(v_j) = a_{j1} v_1 + \ldots + a_{jn} v_n, \quad \text{for } j = 1, \ldots, n.$$

Let $a = (-1)^q \sum_{m,k=1}^{n} a_{mk} v^m \otimes v_k$. For each $j = 1, \ldots, n$ we see that

$$\Theta_q(a)(v_j) = (-1)^{2q} \sum_{k=1}^{n} a_{jk} \cdot v_k = f(v_j).$$

So $f$ and $\Theta_q(a)$ agree on a basis for $E$, which implies that $\Theta_q$, is onto. The proof of (2.3) is completed. □

Define $e\colon \mathrm{Hom}_{\mathbb{Q}}(E) \otimes E \to Q$ as the evaluation map

$$e(u \otimes v) = u(v), \quad \text{for } u \in \mathrm{Hom}_{\mathbb{Q}}(E),\ v \in E.$$

**(2.4) Lemma.** *If $E$ is a finite-dimensional vector and $f\colon E \to E$ is a linear map, then*

$$e(\Theta_q^{-1}(f)) = (-1)^q \mathrm{tr}(f).$$

*Proof.* Take a basis $v_1, \ldots, v_n$ for $E$ and write

$$f(v_j) = \sum_{k=1}^{n} a_{jk} v_k, \quad \text{for } j = 1, \ldots, n.$$

From the proof of (2.3) we know that

$$\Theta_q^{-1}(f) = (-1)^q \sum_{m,k=1}^{n} a_{mk} (v^m \otimes v_k),$$

so

$$e(\Theta_q^{-1}(f)) = (-1)^q \sum_{m,k=1}^{n} a_{mk} (v^m \otimes v_k) = (-1)^q \sum_{k} a_{kk} = (-1)^q \mathrm{tr}(f)$$

and the proof of (2.4) is completed. □

Let $E = \{E_q\}$ be a graded vector space of finite type. Define the following graded vector spaces:

(1) $E^* = \{E_q^*\}$, where $E_q^* = \mathrm{Hom}_{\mathbb{Q}}(E_{-q})$,
(2) $\mathrm{Hom}(E, E) = \{(\mathrm{Hom}(E, E))_k\}$, where

$$(\mathrm{Hom}(E, E))_k = \bigoplus_{-q+i=k} \mathrm{Hom}(E_q, E_i),$$

(3) $E^* \otimes E = \{(E^* \otimes E)\}$, where $(E^* \otimes E)_k = \bigoplus_{q+i=k} E_q^* \otimes E_i$.

Define $\Theta\colon (E^* \otimes E)_0 \to (\mathrm{Hom}(E,E))_0$ by letting

$$\Theta(u_q \otimes v_i) = \Theta_q(u_q \otimes v_i), \quad \text{for } u_q \in \mathrm{Hom}_{\mathbb{Q}}(E_q), v_i \in E_i, q = i$$

and extend $\Theta_q$ to all $(E^* \otimes E)_0$; and $e\colon (E^* \otimes E)_k \to \mathbb{Q}$ by letting

$$e(u_q \otimes v_i) = u_q(v_i), \quad \text{for } u_q \in \mathrm{Hom}_{\mathbb{Q}}(E_q), v_i \in E_i, q = i$$

and extend $e$ to all $(E^* \otimes E)_0$.

It is immediate from Lemma (2.4) that

**(2.5) Theorem.** *If $f\colon E \to E$ is a linear map of degree zero on a graded vector space of finite type $E$, then $e(\Theta^{-1}(f)) = \lambda(f)$.*

Let $f\colon E \to E$ be an endomorphism of an arbitrary vector space $E$. Denote by $f^{(n)}\colon E \to E$ the $n$th iterate of $f$ and observe that the kernels

$$\mathrm{Ker} f \subset \mathrm{Ker} f^{(2)} \subset \ldots \subset \mathrm{Ker} f^{(n)} \subset \ldots$$

form an increasing sequence of subspaces of $E$. Let us now put

$$N(f) = \bigcup_n \mathrm{Ker} f^{(n)} \quad \text{and} \quad \widetilde{E} = E/N(f).$$

Clearly, $f$ maps $N(f)$ into itself and therefore induces the endomorphism $\widetilde{f}\colon \widetilde{E} \to \widetilde{E}$ on the factor space $\widetilde{E} = E/N(f)$.

**(2.6) Lemma.** *We have $f^{-1}(N(f)) = N(f)$; consequently, the kernel of the induced map $\widetilde{f} : \widetilde{E} \to \widetilde{E}$ is trivial, i.e. $f$ is a monomorpism.*

*Proof.* If $v \in f^{-1}(N(f))$, then $f(v) \in N(f)$. This implies that for some $n$ we have $f^{(n)}(f(v)) = 0 = f^{(n+1)}(v)$ and $v \in N(f)$. Conversely, if $v \in N(f)$, then $f^{(n)}(v) = 0$ for some $n$; then $f^{(n)}(f(v)) = 0$ and hence $f(v) \in N(f)$, i.e., $v \in f^{-1}(N(f))$. □

Let $f\colon E \to E$ be an endomorphism of a vector space $E$. Assume that $\dim \widetilde{E} < \infty$; in this case we define the generalized trace $\mathrm{Tr}(f)$ of $f$ by putting $\mathrm{Tr}(f) = \mathrm{tr}(\widetilde{f})$.

**(2.7) Lemma.** *Let $f\colon E \to E$ be an endomorpism. If $\dim E < \infty$, then* $\mathrm{Tr}(f) = \mathrm{tr}(f)$.

*Proof.* We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N(f) & \longrightarrow & E & \longrightarrow & E/N(f) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \overline{f}} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \widetilde{f}} & & \\
0 & \longrightarrow & N(f) & \longrightarrow & E & \longrightarrow & E/N(f) & \longrightarrow & 0
\end{array}$$

in which $\overline{f}$ is induced by $f$. Applying (2.2), to the above diagram, we obtain

$$(*) \qquad \operatorname{tr}(f) = \operatorname{tr}(\overline{f}) + \operatorname{tr}(\widetilde{f}), \quad \text{where } \operatorname{tr}(\widetilde{f}) = \operatorname{Tr}(f).$$

We prove that $\operatorname{tr}(\overline{f}) = 0$. Since $\dim E < \infty$, we may assume that $N(f) = \operatorname{Ker} f^{(n)}$ for some $n \geq 1$. Now consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Ker}(f) \arrow[r] \arrow[d, "0=\overline{f}_1"'] & \operatorname{Ker}(f^{(2)}) f_1 \arrow[r] \arrow[d, "\overline{f}_2"'] \arrow[dl] & \cdots \arrow[r] & \operatorname{Ker}(f^{(n-1)}) \arrow[r] \arrow[d, "\overline{f}_{n-1}"'] & \operatorname{Ker}(f^{(n)}) \arrow[d, "\overline{f}_n=\overline{f}"] \arrow[dl, "f_n"'] \\
\operatorname{Ker}(f) \arrow[r] & \operatorname{Ker}(f^{(2)}) \arrow[r] & \cdots \arrow[r] & \operatorname{Ker}(f^{n-1)}) \arrow[r] & \operatorname{Ker}(f^{(n)})
\end{tikzcd}$$

where the maps $\overline{f}_i$, $f_i$, $i = 1, \dots, n$, are given, by $f$ (observe that if $v \in \operatorname{Ker}(f^{(i)})$, then $f(v) \in \operatorname{Ker}(f^{(i-1)})$, for every $i > 1$). Then, from (2.1) we infer

$$\operatorname{tr}(\overline{f}) = \operatorname{tr}(\overline{f}_{n-1}) = \ldots = \operatorname{tr}(\overline{f}_2) = \operatorname{tr}(\overline{f}_1) = 0.$$

Finally, from $(*)$ we obtain $\operatorname{Tr}(f) = \operatorname{tr}(\widetilde{f}) = \operatorname{tr}(f)$ and the proof is completed. $\square$

Let $f = \{f_q\}$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We say that $f$ is a *Leray endomorphism* provided the graded vector space $\widetilde{E} = \{\widetilde{E}_q\}$ is of finite type. For such an $f$ we define the (*generalised*) *Lefschets number* $\Lambda(f)$ of $f$ by putting

$$\Lambda(f) = \sum_q (-1)^q \operatorname{Tr}(f_q).$$

It is immediate from (2.7) that

**(2.8) Lemma.** *Let $f: E \to E$ be an endomorphism of degree zero. If $E$ is a graded vector space of finite type, then $\Lambda(f) = \lambda(f)$.*

The following property of the Leray endomorphism is of importance:

**(2.9) Property.** *Assume that in the category $\mathcal{A}$ the following diagram commutes:*

$$\begin{tikzcd}
E' \arrow[r, "f"] & E'' \\
E' \arrow[r, "f"'] \arrow[u, "f'"] & E'' \arrow[u, "f''"'] \arrow[ul, "g"']
\end{tikzcd}$$

*Then if either $f'$ or $f''$ is a Leray endomorphism, then so is the other, and in that case $\Lambda(f') = \Lambda(f'')$.*

*Proof.* By assumption we have, for each $q$, the following commutative diagram in the category of vector spaces:

$$\begin{tikzcd} E'_q \arrow[r, "f_q"] & E''_q \arrow[dl, leftarrow, "g_q"] \\ E'_q \arrow[u, "f'_q"] \arrow[r, "f_q"'] & E''_q . \arrow[u, "f''_q"'] \end{tikzcd}$$

For the proof it is sufficient to show that if either $\mathrm{Tr}(f'_q)$ or $\mathrm{Tr}(f''_q)$ is defined, then so is the other trace, and in that case $\mathrm{Tr}(f'_q) = \mathrm{Tr}(f''_q)$. We observe that the commutativity of the above diagram implies that the following diagram commutes:

$$\begin{tikzcd} E'_q/N(f'_q) \arrow[r, "\widetilde{f}_q"] & E''_q/N(f''_q) \arrow[dl, leftarrow, "\widetilde{g}_q"] \\ E'_q/N(f'_q) \arrow[u, "\widetilde{f}'_q"] \arrow[r, "\widetilde{f}_q"'] & E''_q/N(f''_q) \arrow[u, "\widetilde{f}''_q"'] \end{tikzcd}$$

Since $\widetilde{f}_q$ and $\widetilde{g}_q$ are monomorphisms, the commutativity of the above diagram implies that $\dim(E'_q/N(f'_q)) < \infty$ if and only if $\dim(E''_q/N(f''_q)) < \infty$, and hence we conclude that $\mathrm{Tr}(f'_q)$ is defined, if and only if $\mathrm{Tr}(f''_q)$ is defined. Moreover, from (2.1) we deduce that $\mathrm{Tr}(f'_q) = \mathrm{Tr}(f''_q)$, if $\mathrm{Tr}(f'_q)$ or $\mathrm{Tr}(f''_q)$ is defined. The proof of (2.9) is completed. $\square$

A linear endomorphism $f: E \to E$ is called *weakly nilpotent* provided for every $x \in E$ there exists $n = n_x$ such that $f^n(x) = 0$.

Observe that if $f: E \to E$ is weakly nilpotent then $N(f) = E$ and consequently $\mathrm{Tr}(f) = 0$. Assume that $E = \{E_q\}$ is a graded vector space and $f = \{f_q\}: E \to E$ is an endomorphism. We say that $f$ is weakly nilpotent if and only if $f_q$ is weakly nilpotent for every $q$.

**(2.10) Remark.** Any weakly nilpotent endomorphism $f: E \to E$ is a Leray endomorphism and $\Lambda(f) = 0$.

## 3. The coincidence problem

A natural generalization of the well known fixed point problem is the coincidence problem. Assume we have two metric spaces $(X, d)$, $(Y, d_1)$ and two continuous mappings $p, q: Y \to X$.

We shall say that $p$ and $q$ have a *coincidence* provided there exists a point $x \in X$ such that $p(x) = q(x)$. In the case when $X = Y$ and $p = \mathrm{id}_X$ is the identity map the coincidence problem for $p$ and $q$ reduces to the fixed point problem of $q$.

Observe that for arbitrary $p$ and $q$ usually we do not have a coincidence. Therefore in what follows we can assume that $p$ is a Vietoris map and $q\colon Y \to X$ is a compact map, i.e. $\overline{q(Y)}$ is a compact subset of $X$.

We assume first that $X = U$ is an open subset of $\mathbb{R}^n$.

**(3.1) Lemma.** *Consider the diagram*

$$U \overset{p}{\Longleftarrow} Y \xrightarrow{q} U$$

*in which $p$ is Vietoris and $q$ is compact. Then the set $\chi_{p,q} = \{x \in U \mid x \in q(p^{-1}(x))\}$ is compact.*

*Proof.* Consider a sequence $\{x_n\} \subset U$ such that $x_n \in q(p^{-1}(x_n))$ for every $n$.

For every $n$ we choose $y_n \in p^{-1}(x_n)$ such that $q(y_n) = x_n$. It means that $\{x_n\} \subset \overline{q(Y)}$, and hence $\{x_n\}$ contains a convergent subsequence and the proof is completed. $\square$

We shall now apply the Čech homology with compact carriers to the theory of Lefschetz number and establish a general coincidence theorem, which contains the classical Lefschetz Fixed Point theorem (cf. [15]) as a special case.

Let $U$ by an open subset of the $n$-dimensional euclidean space $\mathbb{R}^n$. Consider the diagram:

$$U \overset{p}{\Longleftarrow} Y \xrightarrow{q} U \tag{3.2}$$

in which $p$ is a Vietoris map and $q$ is a compact map. With the above diagram we associate the diagram:

$$(U, U \setminus \chi_{p,q}) \overset{\overline{p}}{\Longleftarrow} (Y, Y \setminus p^{-1}(\chi_{p,q})) \xrightarrow{\overline{q}} (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}), \tag{3.3}$$

where $\overline{p}(y) = p(y)$ and $\overline{q}(y) = p(y) - q(y)$ for every $y \in Y$.

Now we define the *index of coincidence* $\mathrm{I}(p,q)$ of the pair $(p,q)$ by putting (cf. Section 1):

$$\mathrm{I}(p,q) = \overline{q}_*(\overline{p}_*)^{-1}(O_{\chi_{p,q}}) \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \approx Q. \tag{3.4}$$

**(3.5) Proposition.** *If $\mathrm{I}(p,q) \neq 0$, then there is a $y \in Y$ such that $p(y) = q(y)$.*

*Proof.* Indeed, if $p(y) \neq q(y)$ for each $y \in Y$, then $\chi_{p,q} = \emptyset$ an hence we have:

$$\mathrm{I}(p,q) = \overline{q}_*(\overline{p}_*)^{-1}(O_{\chi_{p,q}}) = \overline{q}_*(\overline{p}_*)^{-1}(O) = 0,$$

observe that then we have $H_n(U,U) = 0$. $\square$

From (1.5) clearly follows:

**(3.6) Proposition.** *If $A$ is a compact set such that $\chi_{p,q} \subset A \subset U$, then $\mathrm{I}(p,q) = \widetilde{q}_*(\widetilde{p}_*)^{-1}(O_A)$, where $\widetilde{p}$, $\widetilde{q}$ are defined by the same formulae as $\overline{p}$ and $\overline{q}$.*

Now we prove the following:

**(3.7) Proposition.** *Let $K$ be a finite polyhedron such that $q(Y)\subset K\subset U$. Then there exists an element $a\in(H(K))^*\otimes H(K)$ such that $\mathrm{I}(p,q)=e(a)$.*

*Proof.* Consider the diagram

$$
\begin{tikzcd}
H_u(U,U/K) \arrow[r, "\Delta_*"] \arrow[dr, "\tilde{q}_*\tilde{p}_*^{-1}"'] & (H(U,U/K)\otimes H(U))_u \arrow[r, "\mathrm{id}\otimes q_{1*}p_*^{-1}"] \arrow[d, phantom, "(\mathrm{I})"] & (H(U,U/K)\otimes H(K))_0 \arrow[dl, "d_*"] \arrow[d, "\widehat{d}\otimes\mathrm{id}"] \arrow[d, phantom, "(\mathrm{II})"'] \\
 & Q\approx H_u(\mathbb{R}^n,\mathbb{R}^n\{0\} & ((H(K)^*\otimes H(K))_0 \arrow[l, "e"]
\end{tikzcd}
$$

in which $q_1\colon Y\to K$ is the contraction of $q$ to the pair $(Y,K)$ and

$$\widehat{d}\colon H(U,U\setminus K)\to(H(K))^*$$

is a linear map of degree $(-n)$ given by:

$$\widehat{d}(u)(v)=d_*(u\otimes v)\quad\text{for } u\in H(U,U\setminus K)\text{ and } v\in H(U\setminus K)$$

and the notations are the same as in Section 1. The subdiagram (I) commutes.

The commutativity of (II) follows by an easy computation. We let:

$$a=(\widehat{d}\otimes\mathrm{id})\circ(\mathrm{id}\otimes q_{1*}p_*^{-1})(\Delta_*(O_K)).$$

Then from the commutativity of the above diagram we get $\mathrm{I}(p,q)=e(a)$ and the proof is completed. $\square$

Now, we are able to prove the following

**(3.8) Theorem** (First Coincidence Theorem)**.** *If we have diagram* (3.2), *then $q_*p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_*p_*^{-1})\neq 0$ implies that $p$ and $q$ have a coincidence.*

*Proof.* Since $q$ is a compact map, there exists a finite polyhedron $K$ such that $q(Y)\subset K\subset U$. We have the commutative diagram

$$
\begin{tikzcd}
H(K) \arrow[rr, "i_*"] & & H(U) \\
H(p^{-1}(K)) \arrow[u, "q'_*"] \arrow[r, "j_*"'] & H(Y) \arrow[ul, "q_{1*}"'] \arrow[r, "\mathrm{id}"] & H(Y) \arrow[u, "q_*"'] \\
H(K) \arrow[u, "(p'_*)^{-1}"] \arrow[rr, "i_*"'] & & H(U) \arrow[u, "p_*^{-1}"'] \arrow[ul, "p_*^{-1}"]
\end{tikzcd}
$$

in which $i_*$, $j_*$ are linear maps induced by inclusions $i\colon K \to U$ and $j\colon p^{-1}(K) \to Y$, respectively, and $q'_*$, $q_{1*}$, $p'_*$ are linear maps induced by the contractions of $q$ and $p$, respectively. The commutativity of the above diagram and (2.9) imply

$$\Lambda(q_* p_*^{-1}) = \lambda(q'_*(p'_*)^{-1}),$$

and hence $q_* p_*^{-1}$ is a Leray endomorphism.

Assume that $\Lambda(q_* p_*^{-1}) \neq 0$. For the proof it is sufficient to show that

$$\lambda(q'_*(p'_*)^{-1}) = \mathrm{I}(p,q) \tag{3.8.1}$$

(cf. also Section 1).

Consider the following diagram:

$$\begin{tikzcd}
H(U,U\setminus K)\otimes H(U)\otimes H(K) \arrow[r, "\widehat{d}\otimes q_{1*}p_*^{-1}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes t_*"'] & (H(K))^*\otimes H(K)\otimes H(K) \arrow[d, "\mathrm{id}\otimes t_*"] \\
H(U,U\setminus K)\otimes H(K)\otimes H(U) \arrow[r, "\widehat{d}\otimes\mathrm{id}\otimes q_{1*}p_*^{-1}"] \arrow[d, "d_*\otimes\mathrm{id}"'] & (H(K))^*\otimes H(K)\otimes H(K) \arrow[d, "e\otimes\mathrm{id}"] \\
H(U)\approx Q\otimes H(U) \arrow[r, "q_{1*}p_*^{-1}"'] & Q\otimes H(K)\approx H(K)
\end{tikzcd}$$

The commutativity of the above diagram is obtained by simple calculation. Let

$$a = (\widehat{d}\otimes \mathrm{id})(\mathrm{id}\otimes q_{1*}p_*^{-1})\Lambda_*(O_K) \in \mathrm{Hom}_Q(H(K))\otimes H(K).$$

Since $e(a) = \mathrm{I}(p,q)$ (see (9.4)), for the proof of (3.8.1) it is sufficient to show that

$$\Theta(a) = q'_*(p'_*)^{-1} \tag{3.8.2}$$

(cf. Section 1).

If we follow $\Delta_*(O_K)\otimes u \in H(U,U\setminus K)\otimes H(U)\otimes H(K)$ along $\to\downarrow\downarrow$, we obtain $(\Theta(a))(u)$. If we follow it along $\downarrow\downarrow$, by Dold's Lemma (1.6) we obtain $i_*(u)$. Therefore, for the proof of (3.8.2) it is sufficient to show that

$$q_{1*}p_*^{-1} i_* = q'_*(p'_*)^{-1}. \tag{3.8.3}$$

Consider the following commutative diagram:

$$\begin{tikzcd}
U & Y \arrow[l, "p"'] \arrow[r, "q_1"] & K \\
K \arrow[u, "i"] & p^{-1}(K) \arrow[l, "p'"] \arrow[u, "j"'] \arrow[ur, "q'"'] &
\end{tikzcd}$$

Applying to the above diagram the functor $H$, we obtain (3.8.3) and the proof of the First Coincidence Theorem is completed. $\square$

To generalize (3.8) we need the Schauder Approximation Theorem.

**(3.9) Theorem** (Schauder Approximation Theorem). *Let $U$ be an open subset of a normed space $E$ and let $f\colon X \to U$ be a compact map. Then for every $\varepsilon > 0$ there exists a finite dimensional subspace $E^{n(\varepsilon)}$ of $E$ and a compact map $f_\varepsilon\colon X \to U$ such that:*

(a) $\|f(x) - f_\varepsilon(x)\| < \varepsilon$, *for every* $x \in X$,
(b) $f_\varepsilon(X) \subset E^{n(\varepsilon)}$,
(c) *the maps* $f_\varepsilon, f\colon X \to U$ *are homotopic.*

*Proof.* Given $\varepsilon > 0$ (we can assume to be sufficiently small) $f(X)$ is contained in the union of open balls $B(y_i, \varepsilon)$ with $B(y_i, 2\varepsilon) \subset U$, $i = 1, \dots, k$.

For every $i = 1, \dots, k$ we define $\lambda_i\colon X \to \mathbb{R}_+$, $\lambda_i(x) = \max\{0, \varepsilon - \|f(x) - y_i\|\}$ and

$$\mu_i\colon X \to [0,1], \quad \mu_i(x) = \frac{\lambda_i(x)}{\sum_{j=1}^k \lambda_j(x)}.$$

Now, we define $f_\varepsilon\colon X \to U$ by putting

$$f_\varepsilon(x) = \sum_{i=1}^k \mu_i(x) \cdot y_i.$$

Let $E^{n(\varepsilon)}$ be a subspace of $E$ spanned by vectors $y_1, \dots, y_n$, i.e.

$$E^{n(\varepsilon)} = \operatorname{span}\{y_1, \dots, y_k\}.$$

Then $f_\varepsilon(X) \subset \operatorname{conv}\{y_1, \dots, y_n\}$ so $f_\varepsilon$ is a compact map. We have:

$$\|f(x) - f_\varepsilon(x)\| \le \sum_{i=1}^k \mu_i(x)\|f(x) - y_i\| < \varepsilon.$$

Moreover, the map $h\colon X \times [0,1] \to U$,

$$h(x,t) = tf(x) + (1-t)f_\varepsilon(x)$$

is a good homotopy joining $f$ and $f_\varepsilon$ and the proof is completed. □

Now, we prove the following:

**(3.10) Theorem** (Second Coincidence Theorem). *Assume that we have a diagram:*

$$U \overset{p}{\Longleftarrow} Y \xrightarrow{q} U,$$

*in which $U$ is an open subset of a normed space $E$, $p$ is Vietoris and $q$ compact. Then $q_* p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* p_*^{-1}) \neq 0$ implies that $p$ and $q$ have a coincidence.*

*Proof.* Since $q\colon Y \to U$ is compact, in view of the Schauder Approximation Theorem for every $n$ we get a finite dimensional subspace $E^n \subset E$ and a compact map $q_n\colon Y \to U$ such that:

(3.10.1) $\|q(y) - q_n(y)\| < 1/n$,

(3.10.2) $q_n(Y) \subset E^n$, and
(3.10.3) $q \sim q_n$.

We let $U_n = U \cap E^n$.

Now, for every $n$, we consider the following commutative diagram:

$$\begin{tikzcd}
U_n \arrow[rr, "i_n"] & & U \\
Y_n \arrow[u, "q'_n"] \arrow[r, "j_n"'] \arrow[d, Rightarrow, "p_n"'] & Y \arrow[ul, "\overline{q}_n"'] \arrow[r, "\mathrm{id}_Y"] \arrow[dr, Rightarrow, "p"] & Y \arrow[u, "q_n"'] \arrow[d, Rightarrow, "p"] \\
U_n \arrow[rr, "i_n"'] & & U
\end{tikzcd}$$

where $q'_n(y) = q_n(y)$, $\overline{q}_n(y) = q(y)$, $p_n(y) = p(y)$, $i_n(x) = x$, $j_n(y) = y$ for respective $y$ and $x$.

Consequently, its image under $H$ is also a commutative diagram:

$$\begin{tikzcd}
H(U_n) \arrow[r, "i_{n*}"] & H(U) \\
H(U_n) \arrow[u, "q'_{n*} \circ p_{n*}^{-1}"] \arrow[r, "i_{n*}"'] & H(U) \arrow[ul, "\overline{q}_{n*} \circ p_*^{-1}"'] \arrow[u, "q_{n*} \circ p_*^{-1}"']
\end{tikzcd}$$

Now, it follows from the First Coincidence Theorem that $q'_{n*} \circ p_{n*}^{-1}$ is a Leray endomorphism. So, by the commutativity property $q_{1*}p_*^{-1}$ is a Leray endomorphism and because $q_{n*} = q_*$ (cf. (3.10.3)) we obtain:

$$\Lambda(q'_{n*}p_{n*}^{-1}) = \Lambda(q_{n*}p_*^{-1}) = \Lambda(q_*p_*^{-1}). \tag{3.10.4}$$

Now, let us assume that $\Lambda(q_*p_*^{-1}) \neq 0$. Then, in view of (3.10.4), by the First Coincidence Theorem we deduce that $p(y_n) = q_n(y_n)$ for every $n$.

Let $x_n = p(y_n) = q_n(y_n)$ for every $n$. We put $q(y_n) = \overline{x}_n$, $n = 1, 2, \ldots$ Since $q$ is compact, we may assume without loss of generality that $\lim_n \overline{x}_n = x \in U$.

We have $\|x_n - \overline{x}_n\| = \|q_n(y_n) - q(y_n)\| < 1/n$ for every $n$ (cf. (3.10.1)) and hence $\lim_n x_n = x$. Then $x \in q(p^{-1}(x))$ and consequently there exists $y \in p^{-1}(x)$ such that $p(y) = q(y) = x$; the proof is completed. $\square$

**(3.11) Theorem** (Coincidence Theorem for arbitrary ANRs)**.** *Consider a diagram:*

$$X \overset{p}{\Longleftarrow} Y \xrightarrow{q} X,$$

*in which $X$ is a retract of some open set in a normed space* (2)*, $p$ is Vietoris and $q$ is compact. Then $q_* \circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* \circ p_*^{-1}) \neq 0$ implies that $p$ and $q$ have a coincidence.*

(2) We shall see in next section that such a space $X$ is an ANR-space.

*Proof.* By assumptions there exists an open subset $U$ of a normed space $E$ such that $X \subset U$ is a retract of $U$. Let $r\colon U \to X$ be the retraction map and $i\colon X \to U$ the inclusion. Of course the following diagram is commutative:

$$\begin{tikzcd}
H(U) \arrow[r, "r_*"] & H(X) \arrow[dl, "i_* q_* p_*^{-1}"'] \\
H(U) \arrow[u, "i_* q_* p_*^{-1} r_*"] \arrow[r, "r_*"'] & H(X) \arrow[u, "q_* p_*^{-1}"']
\end{tikzcd}$$

By applying (3.10) the Second Coincidence Theorem we would like to deduce that $i_* q_* p_*^{-1} r_*$ is a Leray endomorphism.

Now, by considering the fibre product and pull-back construction we obtain the following commutative diagram.

$$\begin{tikzcd}
U \arrow[d, "r"'] & \\
X & U \boxtimes_X Y \arrow[l, "f"'] \arrow[ul, Rightarrow, "\overline{p}"'] \arrow[dl, "\overline{r}"] \arrow[dddl, "\overline{q} \equiv i \circ q \circ \overline{r}"] \\
Y \arrow[u, Rightarrow, "p"] \arrow[d, "q"'] & \\
X \arrow[d, "i"'] & \\
U &
\end{tikzcd}$$

where $\overline{p}(u, y) = u$, $\overline{r}(u, y) = y$, $f(u, y) = r(u) = p(y)$. Then $i_* q_* p_*^{-1} r_* = \overline{q}_* \circ \overline{p}_*^{-1}$ and moreover there is a coincidence point for $p$ and $q$ if and only if it is for $\overline{p}$ and $\overline{q}$. Consequently our result follows from the commutativity property of the Leray endomorphisms and the Second Coincidence Theorem, the proof is completed. $\square$

There are many consequences of Theorem (3.11). Before we state them we need a simple observation.

**(3.12) Property.** *Assume we have a diagram*

$$X \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} X,$$

*in which $X$ is acyclic, $p$ Vietoris and $q$ compact. Then $q_* \circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* \circ p_*^{-1}) = 1$.*

*Proof.* In fact, from the acyclicity of $X$ we deduce that $q_* \circ p_*^{-1} = \mathrm{id}_{H(X)}$ but

$$H_n(X) = \begin{cases} 0 & \text{for } n > 0, \\ Q & \text{for } n = 0, \end{cases}$$

so, our claim follows. □

From (3.11) and (3.12) we obtain:

**(3.13) Corollary.** *If we have the diagram:*

$$X \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} X,$$

*in which* $X \in \mathrm{AR}$, *$p$ is Vietoris and $q$ compact, then there exists a point $y \in Y$ such that $p(y) = q(y)$.*

Now, if we let $Y = X$ and $p = \mathrm{id}_X$ then from (3.11) we deduce the generalized Lefschetz fixed point theorem, proved by A. Granas in 1967 (see [15]):

**(3.14) Corollary.** *If $X \in \mathrm{AR}$ and $f: X \to X$ is a compact map then $f_*: H(X) \to H(X)$ is a Leray endomorphism and $\Lambda(f_*) \neq 0$ implies that $f$ has a fixed point.*

Finally, from (3.14) we deduce the following generalized version of the Schauder fixed point theorem:

**(3.15) Corollary.** *If $X \in \mathrm{AR}$ and $f: X \to X$ is a compact map then $f$ has a fixed point.*

We recommend [18] for details concerning multivalued mappings.

A u.s.c. map $\varphi: X \multimap Y$ is said to be acyclic provided the set $\varphi(x)$ is acyclic for every point $x \in X$.

**(3.16) Lemma.** *If $\varphi: X \multimap Y$ is an acyclic map, then the natural projection $p_\varphi: \Gamma_\varphi \Longrightarrow X$ is a Vietoris map, where $\Gamma_\varphi\{(x, y) \in X \times Y \mid y \in \varphi(x)\}$ and $q_\varphi(x, y) = y$.*

Using Theorem (1.16) for an acyclic map $\varphi: X \multimap Y$, we define the linear map $\varphi_*: H(X) \to H(Y)$ by putting

$$\varphi_* = (q_\varphi)_* \circ [(p_\varphi)_*]^{-1}.$$

$\varphi_*$ is said to be induced by the multi-valued map $\varphi$. It is easy to see that if $\varphi = f$ (i.e. $\varphi$ is a single-valued continuous map), then $\varphi_* = f_*$.

Let $\varphi: X \multimap Y$ be a multi-valued map. A pair $(p, q)$ of single-valued, continuous maps of the form $X \overset{p}{\Longleftarrow} Z \overset{q}{\longrightarrow} Y$ is called a *selected pair* of $\varphi$ (written $(p, q) \subset \varphi$) if the following two conditions are satisfied:

(i) $p$ is a Vietoris map,
(ii) $q(p^{-1}(x)) \subset \varphi(x)$ for each $x \in X$.

**(3.17) Remark.** We observe that if $\varphi$ is a compact map and $(p,q) \subset \varphi$, then $q$ is a compact map.

**(3.18) Proposition.** *If $\varphi\colon X \multimap Y$ is an acyclic map and $(p,q) \subset \varphi$, then $q_* p_*^{-1} = \varphi_*$.*

*Proof.* Let $(p,q)$ be a selected pair of $\varphi$ of the form $X \overset{p}{\Longleftarrow} Z \xrightarrow{q} Y$. Consider the commutative diagram

$$\begin{tikzcd}
 & \Gamma_\varphi \arrow[dl, Rightarrow, "p_\varphi"'] \arrow[dr, "q_\varphi"] & \\
X & & Y \\
 & Z \arrow[ul, Rightarrow, "p"] \arrow[ur, "q"'] \arrow[uu, "f"] &
\end{tikzcd}$$

in which $f(z) = (p(z), q(z))$ for every $z \in Z$.

The condition $q(p^{-1}(x)) \subset \varphi(x)$ implies that $(p(z), q(z)) \in \Gamma_\varphi$. Applying to the above diagram the functor $H$, we obtain $q_* p_*^{-1} = (q_\varphi)_* \circ [(p_\varphi)_*]^{-1}$, and the proof is completed. $\square$

From (3.8) and (3.18) we simply deduce

**(3.19) Proposition.** *If $p\colon Z \to X$ is a Vietoris map from $Z$ onto a metric space $X$, then the map $\varphi_p\colon X \multimap Z$ is acyclic and $(\varphi_p)_* = p_*^{-1}$, where $\varphi_p(x) = p^{-1}(x)$.*

**(3.20) Definition.** A multi-valued map $\varphi\colon X \multimap Y$ is called *admissible* provided there exists a selected pair $(p,q)$ of $\varphi$.

We observe that if $\varphi$ has an acyclic selector or, in particular, a continuous single-valued selector, then $\varphi$ is an admissible map.

**(3.21) Definition.** An admissible map $\varphi\colon X \multimap Y$ is called *strongly admissible* (*s-admissible*) provided there exists a selected pair $(p,q)$ of $\varphi$ such that $q(p^{-1}(x)) = \varphi(x)$ for each $x \in X$.

**(3.22) Examples.**

(a) Every acyclic map is not only admissible but also $s$-admissible. For example, the pair $(p_\varphi, q_\varphi)$ is a selected pair of acyclic map $\varphi$ such that $(p_\varphi, q_\varphi) = \varphi$.

(b) We observe that if $\varphi\colon X \multimap Y$ is an $s$-admissible map, then $\varphi(x)$ is a compact and connected set for each $x \in X$.

The map $\varphi\colon [0,1] \to [0,1]$ given by

$$\varphi(t) = \begin{cases} t & \text{for } t \neq 0, \\ \{0,1\} & \text{for } t = 0, \end{cases}$$

is an admissible map but $\varphi$ is not an $s$-admissible map.

**(3.23) Theorem.** *Let $\varphi\colon X \to X_1$ and $\psi\colon X_1 \to X_2$ be two admissible maps. Then the composition $\psi\circ\varphi\colon X \to X_2$ is an admissible map, and for every selected pair $(p_1, q_{)} \subset \varphi$ and $(p_2, q_2) \subset \psi$ there exists a selected pair $(p, q)$ of $\psi\circ\varphi$ such that*

$$q_{2*} \circ (p_{2*})^{-1} \circ q_{1*} \circ (p_{1*})^{-1} = q_* \circ p_*^{-1}.$$

*Proof.* Let $(p_1, q_1) \subset \varphi$ and $(p_2, q_2) \subset \psi$. Consider the commutative diagram

$$\begin{tikzcd}
X & Z_1 \arrow[l, Rightarrow, "p_1"'] \arrow[r, "q_1"] & X_1 & Z_2 \arrow[l, Rightarrow, "p_2"'] \arrow[r, "q_2"] & X_2 \\
& & Z = Z_1 \otimes Z_2 \arrow[ull, Rightarrow, "p"] \arrow[ul, Rightarrow, "f_1"'] \arrow[u, "g"'] \arrow[ur, "f_2"] \arrow[urr, "q"'] & &
\end{tikzcd}$$

in which

$$Z = \{(z_1, z_2) \in Z_1 \times Z_2;\ q_1(z_1) = p_2(z_2);\ p(z_1, z_2) = p_1(z_1)\},$$
$$q(z_1, z_2) = q_2(z_2), \quad f_1(z_1, z_2) = z_1, \quad f_2(z_1, z_2) = z_2, \quad g(z_1, z_2) = q_1(z_1),$$

for each $(z_1, z_2) \in Z$ (comp. (1.17)). Moreover, we have $q(p^{-1}(x)) \subset \psi(\varphi(x))$ for each $x \in X$. Applying to the above diagram the functor $H$, we obtain

$$q_{2*}(p_{2*})^{-1} \circ q_{1*}(p_{1*})^{-1} = q_* p_*^{-1}$$

and the proof of (3.23) is completed. □

**(3.24) Theorem.** *If $\varphi\colon X \multimap X_1$ and $\psi\colon X_1 \multimap X_2$ are two $s$-admissible maps, then the composition $\psi\circ\varphi\colon X \to X_2$ is an $s$-admissible map and for every $(p_1, q_1) = \varphi$ and $(p_2, q_2) = \psi$ there exists a $(p, q) = \psi\circ\varphi$ such that*

$$q_{2*}(p_{2*})^{-1} \circ q_{1*}(p_{1*})^{-1} = q_* p_*^{-1}.$$

The proof of (3.24) is analogous to the proof of (3.23). Theorem (3.24) implies that the composition of two acyclic maps is an $s$-admissible map.

Let $\varphi\colon X \multimap Y$ be an admissible map. Define the set $\{\varphi\}_*$ of linear maps from $H(X)$ to $H(Y)$ by putting

$$\{\varphi\}_* = \{q_* \circ p_*^{1} \mid (p, q) \subset \varphi\};$$

$\{\varphi\}_*$ is said to be an induced set of linear maps by the map $\varphi$. From (3.18) we infer that if $\varphi$ is an acyclic map then $\{\varphi\}_* = \{\varphi_*\}$.

**(3.25) Theorem.** *Let $\varphi, \psi\colon X \multimap Y$ be two admissible maps. If $\varphi \subset \psi$, then $\{\varphi\}_* \subset \{\psi\}_*$.*

For the proof of (3.25) we observe that if $(p, q) \subset \varphi$, then $(p, q) \subset \psi$.

From (3.25) and (3.18) we obtain

**(3.26) Corollary.** *Let $\psi\colon X \multimap Y$ be an acyclic map and $\varphi\colon X \multimap Y$ an admissible map. If $\varphi \subset \psi$, then $\{\varphi\}_* = \{\psi\}_*$.*

**(3.27) Example.** Let $S^n$ denote the unit $n$-sphere in the Euclidean space $\mathbb{R}^{n+1}$. Define the map $\varphi\colon S^n \multimap S^n$ by $\varphi(x) = S^n$ for each $x \in S^n$. It is easy to see that $\varphi$ is an admissible map and hence every continuous (single-valued) map $f\colon S^n \to S^n$ is a selector of $\varphi$. Therefore Theorem (3.25) implies that $\{\varphi\}_*$ is an infinite set. Moreover, we assert that $\varphi$ is an $s$-admissible map and in this case, if the dimension of $S^n$ is even, there exist two selected pairs, $(p,q) = \varphi$ and $(p',q') = \varphi$, such that $q_*p_*^{-1} \neq q'_*(p'_*)^{-1}$. In order to show this, we define the maps $\psi_1, \psi_2\colon S^n \multimap S^n$ by

$$\psi_1(x) = \left\{ y \in S^n \;\middle|\; \|x-y\| \leq \frac{3}{2} \right\} \quad \text{and} \quad \psi_2(x) = \psi_1(-x), \quad \text{for each } x \in S^n.$$

We have

$$\varphi(x) = \psi_1(\psi_1(x)) = \psi_2(\psi_1(x)), \quad \text{for each } x \in S^n$$

and (3.24) implies that $\varphi$ is an $s$-admissible map. Since $\mathrm{id}_{S^n} \subset \psi_1$ and $(-\mathrm{id}_{S^n}) \subset \psi_2$, from (3.26) we infer that $\psi_{1*} = \mathrm{id}_{H(S^n)}$ and $(-\mathrm{id}_{S^n})_* = \psi_{2*}$. Applying Theorem (3.24) again, we deduce that there exist two selected pairs, $(p,q) = \varphi$ and $(p',q') = \psi$, such that $q_*p_*^{-1} = \psi_{1*} \circ \psi_{1*}$ and $q'_*(p'_*)^{-1} = \psi_{2*} \circ \psi_{1*}$. Finally, this implies that $q_*p_*^1 \neq q'_*(p'_*)^{-1}$ for $\varphi\colon S^{2k} \to S^{2k}$.

**(3.28) Definition.** Two admissible maps $\varphi, \psi\colon X \multimap Y$ are called *homotopic* (written $\varphi \sim \psi$) provided there exists an admissible map $\chi\colon X \times I \to Y$, where $I = [0,1]$, such that

$$\chi(x,0) \subset \varphi \quad \text{and} \quad \chi(x,1) = \psi(x), \quad \text{for each } x \in X.$$

**(3.29) Theorem.** *Let $\varphi, \psi\colon X \multimap Y$ be two admissible maps. Then $\varphi \sim \psi$ implies that there exist selected pairs $(p,q) \subset \varphi$ and $(\overline{p},\overline{q}) \subset \psi$ such that $q_* \circ p_*^{-1} = \overline{q}_* \circ \overline{p}_*^{-1}$.*

*Proof.* Let $(\widetilde{p},\widetilde{q}) \subset \chi$. Consider the commutative diagram

$$\begin{tikzcd}
X & \widetilde{p}^{-1}(i_0(X)) \arrow[l, "p"'] \arrow[d, "j_0"] \arrow[dr, "\widetilde{q}\cdot j_0 = q"] & \\
X \times I \arrow[from=u, "i_0"'] & Z \arrow[l, "\widetilde{p}"'] \arrow[r, "\widetilde{q}"] & Y \\
X \arrow[u, "i_1"] & \widetilde{p}^{-1}(i_1(X)) \arrow[l, "\overline{p}"'] \arrow[u, "j_1"'] \arrow[ur, "\widetilde{q}\cdot j_1 = \overline{q}"'] &
\end{tikzcd}$$

in which $i_0(x) = (x,0)$, $i_1(x) = (x,1)$ for each $x \in X$, $j_0, j_1$ are inclusions and $p, \overline{p}$ are given as the first coordinates of $p(z)$ for every $z \in \widetilde{p}^{-1}(i_0(X))$ and $z \in \widetilde{p}^{-1}(i_1(X))$, respectively. Then $p, \overline{p}$ are Vietoris maps and we have

$(p,q) \subset \varphi$, $(\overline{p},\overline{q}) \subset \psi$. We observe that $i_{0*} = i_{1*}$ is a linear isomorphism. This and the commutativity of the above diagram imply $q_* \circ p_*^{-1} = \overline{q}_* \circ \overline{p}_*^1$. This proves Theorem (3.29). □

**(3.30) Corollary.** *Let $\varphi, \psi: X \multimap Y$ be two admissible maps. Then $\varphi \sim \psi$ implies $\{\varphi\}_* \cap \{\psi\}_* \neq \emptyset$.*

**(3.31) Corollary.** *Let $\varphi, \psi: X \multimap Y$ be two acyclic maps. Then $\varphi \sim \psi$ implies $\varphi_* = \psi_*$.*

**(3.32) Example.** Let $\varphi, \psi_1: S^n \multimap S^n$ be as in (3.27). Define the map $\chi: S^n \times I \to S^n$ by $\chi(x,t) = \psi_1(x)$. Then $\chi$ is a homotopy joining $\varphi$ with $\psi_1$ but $\{\psi_1\}_* = \{\psi_{1*}\}$ is a set consisting of one element; however, $\{\varphi\}_*$ is an infinite set.

An admissible map $\varphi: X \multimap X$ is called a *Lefschetz map* provided for each selected pair $(p,q) \subset \varphi$ the linear map $q_* p_*^{-1}: H(X) \to H(X)$ is a Leray endomorphism.

For every Lefschetz map $\varphi: X \multimap X$ we may define the Lefschetz set

$$\Lambda(\varphi) = \{\Lambda(q_* p_*^{-1}) \mid (p,q) \subset \varphi\}.$$

The following facts are simple consequences of (3.25), (3.30) and (3.31), respectively:

**(3.33) Proposition.** *Let $\varphi, \psi: X \multimap X$ be two Lefschetz maps. Then $\varphi \subset \psi$ implies $\Lambda(\varphi) \subset \Lambda(\psi)$.*

**(3.34) Proposition.** *Let $\varphi, \psi: X \multimap X$ be two Lefschetz maps. Then $\varphi \sim \psi$ implies $\Lambda(\varphi) \cap \Lambda(\psi) \neq \emptyset$.*

**(3.35) Proposition.** *Let $\varphi, \psi: X \multimap X$ be two acyclic maps. If $\varphi \subset \psi$ or $\varphi \sim \psi$, then $\varphi$ is a Lefschetz map if and only if $\psi$ is a Lefschetz map and in this case $\Lambda(\varphi) = \Lambda(\psi)$.*

**(3.36) Example.** Let $X$ be a space which is not of finite type. Define the maps $f, \varphi: X \to X$ by $\varphi(x) = X$, $f(x) = x_0$ for each $x \in X$. Then $\varphi$ is an admissible map. We have $f \subset \varphi$ and $\mathrm{id}_X \subset \varphi$ but $f_*$ is a Leray endomorphism and $\mathrm{id}_{H(X)}$ is not a Leray endomorphism.

## 4. ANR-s, AANR-s and w-AANR-s

In this chapter we recall the notions and basic properties which are essential in the fixed-point theory of multi-valued maps, of ANR-s, AANR-s and w-AANR-s.

A single-valued continuous map $f: X \to Y$ is said te be an *$r$-map* if there is a continuous single-valued map $g: Y \to X$ which is a right inverse of $f$, that is such that the composition $f \circ g\colon Y \to Y$ is the identity map $\mathrm{id}_Y$. If there exists an $r$-map $f\colon X \to Y$, then the space $Y$ is called an *$r$-image* of the space $X$.

The maps called retractions are a special kind of $r$-maps. Suppose that $Y$ is a subset of $X$. Then map $f\colon X \to Y$ is said to be a *retraction* if the inclusion $i\colon Y \to X$ is a right inverse of $f$, i.e. $f(x) = x$ for all points $x \in X$. A subset $X_0$ of a space $X$ is said to be a *retract* of $X$ if there is a retraction of $X$ onto $X_0$. A closed subset $X_0$ of a space $X$ is said to be a *neighbourhood retract* in the space $X$ provided $X_0$ is the retract of an open subset of $X$ which contains $X_0$.

We denote by ANR the class of metrizable absolute neighbourhood retracts. A metrizable space $X$ belongs to ANR provided, for each homeomorphism $h$ mapping $X$ onto a closed subset $h(X)$ of a metrizable space $Y$, the set $h(X)$ is a neighbourhood retract in $Y$.

In what follows we shall make use of the following facts from general topology:

**(4.1) Theorem** (Kuratowski Theorem). *Every metrizable space is embeddable isometrically into a Banach space; in particular, any topologically complete metrizable space can be embedded as a closed subset of a Banach space.*

**(4.2) Theorem** (Arens–Eells Theorem). *Every metrizable space can be embedded as a closed subset of a normed space.*

We prove the following

**(4.3) Theorem.** *In order that $X \in$* ANR *it is necessary and sufficient that $X$ be an $r$-image of an open subset of a normed space.*

*Proof.* Let $X \in$ ANR. By Theorem (4.2) there exists an embedding $h\colon X \to E$ of $X$ into a normed space $E$ such that $h(X)$ is closed in $E$. Then there is a retraction $r\colon U \to h(X)$ of an open subset $U$ of $E$ which contains $h(X)$. Then $h^{-1} \circ r\colon U \to X$ is clearly an $r$-map. Now suppose that $X$ is an $r$-image of a get $U$ which is open in a normed space $E$. Let $f\colon U \to X$ be an $r$-map and $g\colon X \to U$ a right inverse for $f$. Consider a homeomorphism $h$ mapping $X$ onto a closed subset of a metric space $Y$. Then $g_1 = g \circ h^{-1}$ maps $h(X)$ into $U \subset E$ and so, by the generalized theorem of Tietze, there is a continuous extension $\widetilde{g}$ of $g_1$ mapping $Y$ into $E$. Let $U'$ be the counter-image of $U$ under $\widetilde{g}_1$. Then $U'$ is a neighbourhood of $h(X)$ in $Y$. Setting $r(y) = h \circ f \circ \widetilde{g}_1(y)$ for $y \in Y$, we obtain a retraction map $r$ and the proof is completed. $\square$

By applying (4.1), instead of (4.2), we obtain analogously

**(4.4) Theorem.** *A metrizable space is a topologically complete* ANR *if and only if it is an $r$-image of an open set in a Banach space.*

From (4.3) clearly follows

**(4.5) Corollary.** *Every open subset of an* ANR *is* ANR.

Similarly (4.4) implies

**(4.6) Corollary.** *Every open subset of a topologically complete* ANR *is a topologically complete* ANR.

The following facts are well known:

**(4.7) Properties.**

(a) *Every (finite) polyhedron is a compact* ANR.
(b) *Every compact* ANR *is a space of finite type.*
(c) *Every convex subset of a normed space is an* ANR.
(d) *Suppose that the metrizable space $X$ is the union of two closed subsets $X_1$ and $X_2$ and that $X_0 = X_1 \cap X_2$. If $X_0, X_1, X_2 \in$* ANR*, then $X \in$* ANR.

Now, we prove the following geometrical fact:

**(4.8) Lemma.** *If $U$ is open in a Banach space $E$ and $X \subset U$ is compact, then there exists a compact $C \in$* ANR *such that $X \subset C \subset U$.*

*Proof.* Cover $X$ by a finite number of closed balls $W_1, \ldots, W_n \subset U$ and denote by $C_i$ the convex closure of the compact set $X \cap W_i$ for each $i = 1, \ldots, n$. By the Mazur Lemma, every $C_i$ is compact. From the inclusions $C_i \subset W_i \subset U$ we conclude that $X$ is contained in the compact set $C = \bigcup_{i=1}^{n} C_i \subset U$.

Now, we show by induction on $n$ that the union of $n$ compact, convex sets is an ANR. The statement is true if $n = 1$ (comp. (4.7)(c)). Assume that the result is true for any integer less than $n$. By hypothesis $Y = \bigcup_{i=1}^{n-1} C_i$ and $C_n$ are ANR-s. Further,

$$Y \cap C_n = \left( \bigcup_{i=1}^{n-1} C_i \right) \cap C_n = \bigcup_{i=1}^{n-1} (C_i \cap C_n),$$

which by the induction hypothesis is an ANR. Thus $C = Y \cup C_n$ is the union of two ANR-s whose intersection is an ANR and (4.7)(d) implies that $C$ is an ANR. This completes the induction and shows (4.8). $\square$

The class of AANR-s was first studied by H. Noguchi.

**(4.9) Definition.** Let $(X, A)$ be a pair of metric spaces and let $\varepsilon$ be a positive real number. A continuous (single-valued) map $r_\varepsilon \colon X \to A$ is called an *$\varepsilon$-retraction* provided $d(r_\varepsilon(a), a) < \varepsilon$ for all $a \in A$.

A subspace $A$ of a metric space $X$ is said to be an *approximative retract* of $X$ provided for each $\varepsilon > 0$ there exists an $\varepsilon$-retraction $r_\varepsilon \colon X \to A$.

**(4.10) Definition.** A metrizable space $X$ is said to be an *approximative* ANR (AANR) provided for each homeomorphism $h$ mapping $X$ onto a closed subset $h(X)$ of a metric space $Y$, the set $h(X)$ is an approximative retract of some open set $U$ in $Y$.

Although not necessarily locally connected, the AANR-s enjoy many familiar properties of ANR spaces. In particular:

**(4.11) Property.** *Every compact* AANR $X$ *is of finite type.*

**(4.12) Definition.** An AANR $X$ is said to be *admissible* provided there exist a homeomorphism $h$ mapping $X$ onto a closed subset $h(X)$ of a normed space $E$ and an open neighbourhood $U$ of $h(X)$ in $E$ such that the following two conditions are satisfied:

(a) $h(X)$ is an approximative retract of $U$,
(b) the inclusion $i: h(X) \to U$ induces a monomorphism $i_*: H(h(X)) \to H(U)$.

**(4.13) Proposition.** *Every* ANR *in an admissible* AANR.

*Proof.* Let $X \in$ ANR. Using the Arens–Eells embedding theorem, we obtain a homeomorphism $h$ mapping $X$ into a normed, space $E$ such that

(a) $h(X)$ is closed of $E$,
(b) there exists a retraction $r: U \to h(X)$, where $U$ is an open neighbourhood of $h(X)$ in $E$.

Then the inclusion $i: h(X) \to U$ is the right inverse of $r$ and we have $ri = \mathrm{id}_{h(X)}$. Hence we infer that $r_* i_* = \mathrm{id}_{H(h(X))}$ and this implies that $i_*$ is a monomorphism. $\square$

**(4.14) Proposition.** *Every compact* AANR *is an admissible* AANR.

*Proof.* Using the Arens–Eells embedding theorem (or the Kuratowski embedding theorem), we may assume without loss of generality that $X$ is an approximative retract of some open neighbourhood $U$ of $X$ in a normed space $E$. Since $X$ is of finite type, from Theorem (1.12) we deduce that there exists an $\varepsilon_0 > 0$ such that for every two maps $f, g: X \to X$, the condition $\|f(x) - g(x)\| < \varepsilon_0$ implies $f_* = g_*$.

Choose an $\varepsilon > 0$ such that $\varepsilon < \varepsilon_0$ and consider the two maps $\mathrm{id}, r_\varepsilon \circ i: X \to X$, where $r: U \to X$ is an $\varepsilon$-retraction and $i: X \to U$ is an inclusion map. By Theorem (1.12) we infer that $\mathrm{id}_{H(X)} = (r_\varepsilon)_* \cdot i_*$, and this implies that $i_*: H(X) \to H(U)$ is a monomorphism. $\square$

**(4.15) Proposition.** *Every acyclic* AANR *is an admissible* AANR.

For the proof of (4.15) observe that if $X$ is an acyclic space and $X \subset Y$, then the inclusion $i: X \to Y$ induces a monomorphism $i_*: H(X) \to H(Y)$.

The following lemma is of importance:

**(4.16) Lemma.** *Let $X$ be an* AANR. *Assume that $X$ is an approximative retract of an open subset $U$ in a normed space $E$ and $i: X \to U$ induces a monomorphism $i_*: H(X) \to H(U)$. Then for every compact subset $K \subset X$ there* *exists a positive real number $\varepsilon(K)$ such that for every $\varepsilon < \varepsilon(K)$ and for every $\varepsilon$-retraction $r_\varepsilon: U \to X$ we have*

$$(r_\varepsilon)_* i_* j_* = j_* \quad \textit{where } j: K \to X \textit{ is the inclusion map.}$$

*Proof.* Let $\varepsilon(K) > 0$ be a number smaller than the distance $\operatorname{dist}(K, \partial U)$ from the compact set $K$ to the boundary $\partial U$ of $U$ in $E$. From the definition of $\varepsilon(K)$ we infer that for each $x \in X$ and $\varepsilon < \varepsilon(K)$ the interval $t \cdot i r_\varepsilon i j(x) + (1-t) \cdot ij(x)$, where $0 \le t \le 1$, is entirely contained in $U$. This implies that $i r_\varepsilon i j$ and $ij$ are homotopic for every $\varepsilon < \varepsilon(K)$. Since $i_*$ is a monomorphism, we get $(r_\varepsilon)_* i_* j_* = j_*$ for each $\varepsilon < \varepsilon(K)$ and the proof is completed. □

A closed subspace $X$ of a metric space $Y$ is called a *weak approximative neighbourhood retract* in $Y$ provided for every $\varepsilon > 0$ there exist an open neighbourhood $U_\varepsilon$ of $X$ in $Y$ and an $\varepsilon$-retraction $r_\varepsilon: U_\varepsilon \to X$.

**(4.17) Definition.** A metrizable space $X$ is said to be *weakly* AANR (w-AANR) provided for each emedding $h: X \to Y$, $Y$ being a metric space and $h(X)$ being closed in $Y$, the space $h(X)$ is a weak approximative neighbourhood retract in $Y$.

It is easy to see that there exists a compact w-AANR which is not of finite type.

We prove the following simple geometrical fact:

**(4.18) Lemma.** *Let $X$ be a weak approximative neighbourhood retract in a normed space $E$ and let $K$ be a compact subset of $X$. Then for each open neighbourhood $W$ of $K$ in $X$ there exists a positive real number $\partial(W)$ such that $K \subset r_\varepsilon^{-1}(W)$ for each $0 < \varepsilon < \partial(W)$, where $r_\varepsilon$ denotes any $\varepsilon$-retraction related to $X$.*

*Proof.* Let $W$ be an open neighbourhood of $K$ in $X$ and let $\partial(W)$ denote the boundary of $W$ in $X$. Then $K \cap \partial(W) = \emptyset$. We define $f(x) = \inf_{y \in \partial(W)} \|x - y\|$. Since $\partial(W)$ as a closed subset of $X$ is closed in $E$, then $K \cap \partial(W) = \emptyset$ implies $f(x) > 0$ for every $x \in K$ and thus $f: K \to (0, \infty)$. Since $K$ is compact, we deduce that $\partial(W) = \inf_{x \in K}\{f(x)\}$ is a positive real number. Then for every $0 < \varepsilon < \partial(W)$ we have $K \subset r_\varepsilon^{-1}(W)$ and the proof is completed. □

## 5. Fixed-Point Theorem for compact admissible maps

We shall now propose the application of the Čech homology with compact carriers and the theory of Lefschetz number by establishing a general fixed-point theorem for admissible maps, which contains the classical Lefschetz Fixed-Point Theorem (for single-valued maps) and the well-known Eilenberg–Montgomery Fixed-Point Theorem for acyclic maps. The principal results of this chapter are Theorems (5.1) and (5.9).

Now, we shall state the principal result of this paper.

**(5.1) Theorem.** *Let $X$ be an admissible* AANR *and let $\varphi: X \multimap X$ be an admissible compact map. Then:*

(a) *$\varphi$ is a Lefschetz map, and*
(b) $\Lambda(\varphi) \neq \{0\}$

*implies that $\varphi$ has a fixed point.*

*Proof.* Since $X$ is an admissible AANR, we may assume that there exists an open subset of a normed space $E$ such that the following two conditions are satisfied:

(i) $X$ is an approximative retract of $U$,
(ii) the inclusion $i: X \to U$ induces a monomorphism $i_*: H(X) \to H(U)$.

Let $r_n: U \to X$ be a $(1/n)$-retraction. We have

(iii) $\|r_n(x) - x\| < 1/n$ for each $x \in X$ and for every $n$.

Let $p, q: Y \to X$ be a pair of maps such that $(p,q) \subset \varphi$. Consider for each $n$ an admissible compact map $\psi_n: U \to U$ given by $\psi_n = i_* q_* \varphi_{*p} r_n$. Using (3.23) and (3.18), we choose a selected pair $(p_n, q_n) \subset \psi_n$ such that

(iv) $q_{n*} p_{n*}^{-1} = i_* q_* p_*^{-1} r_{n*}$, for each $n$.

Since $q$ is a compact map, we infer that the set $A = \overline{q(Y)}$ is compact. Consider for each $n$ the diagram

$$\begin{tikzcd} H(U) \arrow[r, "r_{n*}"] & H(X) \arrow[dl, phantom] \\ H(U) \arrow[u, "i_* q_* p_*^{-1} r_{n*}"] \arrow[r, "r_{n*}"'] & H(X) \arrow[u, "q_* p_*^{-1}"'] \arrow[ul, "i_* j_* q'_* p_*^{-1}"'] \end{tikzcd}$$

where $q': Y \to A$ is given by $q'(y) = q(y)$ for each $y \in Y$ and $j: A \to X$ is an inclusion. From Lemma (4.16) we obtain $r_{n*} i_* j_* = j_*$ for all $n > n_0$. Since $j_* q'_* = (j \circ q')_* = q_*$, we deduce that the above diagram commutes for each $n > n_0$. Consequently, from (2.9), (iv) and coincidence theorem we conclude that $q_* p_*^{-1}$ is a Leray endomorphism. Thus the assertion (a) is proved.

To prove (b) assume that $\Lambda(\varphi) \neq \{0\}$. Then there exists a selected pair $(p,q) \subset \varphi$ such that $\Lambda(q_* p_*^{-1}) \neq 0$. Let $(p_n, q_n) \subset \psi_n$ where $p_n, q_n$ and $\psi_n$ are obtained as in first part of the proof. Then, from (2.9) and (iv) we have

$$\Lambda(q_{n*} p_{n*}^{-1}) = \Lambda(i_* q_* p_*^{-1} r_{n*}) = \Lambda(q_* p_*^{-1}) \neq 0, \quad \text{for each } n > n_0.$$

This, in view of coincidence theorem, implies that $\psi_n$ has a fixed point for each $n > n_0$. We find a sequence $\{x_n\}$ in the compact set $A$ such that:

(v) $x_n \in \psi_n(x_n)$ for each $n > n_0$.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

(vi) $\lim_k x_{n_k} = x$.

Then from (iii) we obtain

(vii) $\lim_k r_{n_k}(x_{n_k}) = x$.

Conditions (v)–(vii) give

(viii) $\{r_{n_k}(x_{n_k})\} \to x$, $x_{n_k} \in q\varphi_p r_{n_k}(x_{n_k})$ and $\{x_{n_k}\} \to x$.

Finally, the u.s.c. of $\psi = q \circ \varphi_p$ (3.8), in view of (viii) and (3.1), implies $x \in \psi(x) = q \circ \varphi_p(x) = qp^{-1} \subset \varphi(x)$ and the proof of Theorem (5.1) is completed.□

We now draw a few immediate consequences of Theorem (5.1).

**(5.2) Corollary.** *Let $X$ be an* ANR *or a compact* AANR *and let $\varphi: X \multimap X$ be an admissible compact map. Then*

(a) *$\varphi$ is a Lefschetz map, and*
(b) $\Lambda(\varphi) \neq \{0\}$

*implies that $\varphi$ has a fixed point.*

Observe that for $X \in$ ANR the above Corollary immediately follows from Coincidence Theorem.

For acyclic maps we obtain the following

**(5.3) Corollary.** *Let $X$ be an admissible* AANR *or, in particular, either of the following:*

(a) *an* ANR*,*
(b) *a compact* AANR*.*

*If $\varphi: X \multimap X$ is a compact acyclic map, then*

(i) *$\varphi$ is a Lefschetz map, and*
(ii) $\Lambda(\varphi) \neq 0$

*implies that $\varphi$ has a fixed point.*

From (5.3) and (3.35) we deduce

**(5.4) Corollary.** *Let $X$ be an admissible* AANR *and let $\varphi, \psi: X \multimap X$ be two compact acyclic maps which satisfy one of the following conditions:*

(a) *$\varphi$ is a selector of $\psi$,*
(b) *$\varphi$ is homotopic to $\psi$.*

*Then both $\varphi$ and $\psi$ are Lefschetz maps, $\Lambda(\varphi) = \Lambda(\psi)$, and $\Lambda(\psi) \neq 0$ implies that $\varphi$ has a fixed point.*

**(5.5) Corollary.** *Let $X$ he an admissible* AANR *and $\varphi: X \multimap X$ an admissible compact map. Assume further that $\varphi(X)$ is contained in an acyclic subset $X_0$ of $X$. Then $\Lambda(\varphi) = \{1\}$ and $\varphi$ has a fixed point.*

*Proof.* Let $p, q\colon Y \to X$ be a pair of maps such that $(p, q) \subset \varphi$. Write the diagram

$$\begin{tikzcd}
X_0 \arrow[rr, "i"] & & X \\
p^{-1}(X_0) \arrow[u, "\overline{q}"] \arrow[d, "\overline{p}"'] \arrow[r, "j"] & Y \arrow[ul, "q_1"'] \arrow[r, "\mathrm{id}"] \arrow[dr, "p"] & Y \arrow[u, "q"'] \arrow[d, "p"] \\
X_0 \arrow[rr, "i"'] & & X
\end{tikzcd}$$

in which $p, q, q_1$ are contractions of $p$ and $q$, respectively and $i, j$ are inclusions. Then its image under $H$ also commutes. Since $\Lambda(\overline{q}_* p_*^{-1}) = 1$ from (2.9), we have $\Lambda(q_* p_*^{-1}) = 1$ for every $(p, q) \subset \varphi$, and from Theorem (5.1) we obtain (5.5). □

A space $X$ has the fixed-point property within the class of admissible compact maps provided any admissible compact map $\varphi\colon X \to X$ has a fixed, point.

**(5.6) Corollary.** *Let $X$ be an acyclic* AANR *or, in particular, either of the following:*

(a) *an acyclic* ANR*,*
(b) *a contractible open set in a normed space.*

*Then $X$ has the fixed-point property within the class of admissible compact maps.*

This simply follows from (5.5) and (4.15). Similarly, from (5.5) and (4.7)(c), we have

**(5.7) Corollary** (The Schauder Fixed-Point Theorem)**.** *Let $X$ be a convex subset of a normed space. Then $X$ has the fixed-point property within the class of admissible compact maps.*

Finally, we prove the following proposition, well-known for single-valued maps:

**(5.8) Proposition.** *Assume that a space $X$ has the fixed-point property within the class of admissible, compact maps. Then every retract of $X$ has the fixed-point property within the class of admissible compact maps.*

*Proof.* Assume that $X$ has the fixed-point property within the class of admissible compact maps. Let $A \subset X$ be a retract of $X$ and let $r\colon X \to A$ be the corresponding retraction. Let $\varphi\colon A \to A$ be an admissible compact map. Define the map $\psi\colon X \to X$ by putting $\psi = i\varphi r$, where $i\colon A \to X$ is the inclusion map. From (3.23) we deduce that $\varphi$ is an admissible compact map. By assumption, there exists a point $x$ such that $x \in \varphi(x)$, but $\psi(X) \subset A$, and therefore $x \in A$. Since $r$ is a retraction map, we have $r(x) = x$ and hence $x \in \varphi(x)$. This completes the proof. □

Now, we prove the following

**(5.9) Theorem.** *Let $X$ be a compact* w-AANR *of finite type and let $\varphi\colon X \to X$ be an admissible map. Then $\Lambda(\varphi) \neq \{0\}$ implies that $\varphi$ has a fixed point.*

*Proof.* We may assume without loss of generality that $X$ is a weak approximative neighbourhood retract in a Banach space $E$. For each $n = 1, 2, \ldots$ let $r_n\colon U_n \to X$ be a $(1/n)$-retraction from an open neighbourhood of $X$ in $E$ to $X$. We have

$$\|x - r_n(x)\| < 1/n, \quad \text{for all } x \in X. \tag{5.9.1}$$

For each $n$ let $i_n\colon X \to U_n$ be the inclusion map. By assumption we infer that there exists a selected pair $(p, q) \subset \varphi$ such that $\Lambda(q_* p_*^{-1}) \neq 0$. Let $\psi\colon X \to X$ be a map given by $\psi = q \cdot \varphi$. Then $\psi$ is an admissible map and hence $(p, q) = \psi$. Define for each $n$ a map $\psi_n\colon U_n \to U_n$ by putting

$$\psi_n = i_n \psi r_n.$$

From (3.23) and (3.18) we deduce that for each $n$ there exists a selected pair $(p_n, q_N) \subset \psi_n$ such that

$$q_{n*} p_{n*}^{-1} = i_{n*} q_* p_*^{-1} r_{n*}. \tag{5.9.2}$$

Consider for each $n$ the diagram

$$\begin{tikzcd}
H(X) \arrow[r, "i_{n*}"] & H(U_n) \\
H(X) \arrow[u, "q_* p_*^{-1}"] \arrow[r, "i_{n*}"'] & H(U_n) \arrow[u, "i_{n*} q_* p_*^{-1} r_{n*}"'] \arrow[ul, "q_* p_*^{-1} r_{n*}"]
\end{tikzcd}$$

Since $X$ is a compact space of finite type, we deduce from Theorem (1.12) that

$$r_{n*} i_{n*} = \mathrm{id}_{H(X)}, \quad \text{for all } n > n_0.$$

This implies that for each $n > n_0$ the above diagram commutes and hence (5.9.2) and (2.9) gives

$$\lambda(q_* p_*^{-1} = \Lambda(q_{n*} p_{n*}^{-1}) \neq 0, \quad \text{for all } n > n_0.$$

Thus coincidence theorem implies that $\psi_n$ has a fixed point for each $n > n_0$. Using the procedure followed in the proof of (5.1), we obtain a fixed point of $\varphi$, and the proof is completed. □

**(5.10) Corollary.** *If $X$ is an acyclic compact* w-AANR*, then $X$ has the fixed-point property within the class of admissible maps.*

In particular, for acyclic maps Theorem (5.9) and (3.35) give

**(5.11) Corollary.** *Let $X$ be a compact* w-AANR *and let $\varphi, \psi\colon X \multimap X$ be two acyclic maps which satisfy one of the following conditions:*

(a) *$\varphi$ is a selector of $\psi$,*
(b) *$\varphi$ is homotopic to $\psi$.*

*Then $\lambda(\varphi) = \lambda(\psi)$ and $\lambda(\psi) \neq 0$ implies that $\varphi$ has a fixed point.*

Let $A$ be a non-empty subset of a space $X$ and let $i\colon A \to X$ be the inclusion map; call $A$ a *homologically trivial subset* of $X$ provided

(i) $\dim \operatorname{Im} i_{*0} = 1$, and
(ii) $i_{*k} = 0$ for all $k \geq 1$.

We note the following evident facts:

**(5.12) Lemma.**

(a) *If $A \subset X \subset Y$ and $A$ is a homologically trivial subset of $X$, then $A$ is a homologically trivial subset of $Y$.*
(b) *If $A_0 \subset A \subset X$ and $A$ is a homologically trivial subset of $X$, then $A_0$ is a homologically trivial subset of $X$.*
(c) *If $A \subset X$ and $A$ or $X$ is an acyclic space, then $A$ is a homologically trivial subset of $X$.*

**(5.13) Theorem.** *Let $X$ be a metric space and assume that the Lefschetz Fixed-Point Theorem for $X$, within the class of admissible compact maps, holds. If $\varphi\colon X \multimap X$ is an admissible compact map and for some $m \geq 1$ the set $\varphi^m(X)$ is a homologically trivial subset of $X$, then*

(a) *$\Lambda(\varphi) = \{1\}$, and*
(b) *$\varphi$ has a fixed point.*

*Proof.* Assume that $\varphi^m(X)$, $m \geq 1$, is a homologically trivial subset of $X$. Let $\varphi^m(X) = X_0$ and let $i\colon X_0 \to X$ denote the inclusion map. First we observe that $\varphi^m\colon X \to X$, as the composition of admissible maps, is also admissible. Let $\widetilde{\varphi}^m\colon X \to X_0$ be the contraction map of $\varphi^m$ to the pair $(X, X_0)$. It is easy to see that $\widetilde{\varphi}^m$ is an admissible map. We have $\varphi^m = i \circ \widetilde{\varphi}^m$. Let $(p, q)$ be a selected pair of $\varphi$. Then, in view of (3.23), there exists a selected pair $p', q'\colon Y \to X$ of $\varphi^m$ such that

$$q'_* {p'}_*^{-1} = \underbrace{q_* p_*^{-1} \ldots q_* p_*^{-1}}_{m\text{th}}. \tag{5.13.1}$$

Observe that the pair $(p', \overline{q}) \subset \widetilde{\varphi}^m$, where $\overline{q}\colon Y \to X_0$ is the contraction map of $q'$ to the pair $(Y, X_0)$, is a selected pair of $\widetilde{\varphi}^m$. We assert that

$$q'_* {p'}_*^{-1} = i_* \overline{q}'_* {p'}_*^{-1}. \tag{5.13.2}$$

In this order, consider the following commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[r, "q'"] \arrow[dl, "p'"'] & X \\
X & & \\
 & Y \arrow[uu, "\mathrm{id}"] \arrow[ul, "p'"] \arrow[r, "\bar{q}"'] & X_0 \arrow[uu, "i"']
\end{tikzcd}$$

Applying to the above diagram the functor $H$, we obtain (5.13.2). For $n \geq 1$, $i_{*n} = 0$ and hence we have

$$i_{*n} q_{*n} (p'_{*n})^{-1} = q'_{*n} (p'_{*n})^{-1} = q_{*n} p_{*n}^{-1} \dots q_{*n} p_{*n}^{-1}.$$

Since $q_{*n} p_{*n}^{-1}$ is nilpotent for $n \geq 1$, it follows that

$$\mathrm{Tr}(q_{*n} p_{*n}^{-1}) = 0, \quad \text{for } n \geq 1.$$

For $n = 1$, it follows that since the rank of $i_{*n}$ is 1, the rank of $q_{*0} p_{*0}^{-1}$ must be 1. Hence

$$\Lambda(q_* p_*^{-1}) = \mathrm{tr}(q_{*0} p_{*0}^{-1}) = 1$$

and the proof of (a) is completed; (b) simply follows from (a). □

**(5.14) Theorem.** *Let $X$ be a topologically complete* ANR *and $\varphi \colon X \multimap X$ an admissible compact map. Let $K$ be a compact subset of $X$ which is invariant under $\varphi$. Suppose also that $C_\infty = \bigcap_{m \geq 1} \varphi^m(X)$ is contained in $K$ and that each compact subset of $C_\infty$ is a homologically trivial subset of $K$. Then $\varphi$ has a fixed point.*

Let $X$ be a compact space, $A$ a closed subset of $X$ and $i \colon A \to X$ the inclusion map. Then from Theorem (1.1) we obtain

**(5.15) Lemma.** *$A$ is a homologically trivial subset of $X$ if and only if $A$ is a cohomologically trivial subset of $X$, i.e.,* $\dim \mathrm{Im}\, i^{*0} = 1$ *and $i^{*n} = 0$ for each $n \geq 1$.*

**(5.16) Lemma.** *Let $X$ be a compact space of finite type and let $\{A_n\}_{n \geq 1}$ be a sequence of closed, non-empty subsets of $X$ such that $A_{n+1} \subset A_n$ for each $n \geq 1$. Assume further that the set $A = \bigcap_{n \geq 1} A_n$, is a cohomologically trivial subset of $X$. Then there exists a number $n \geq 1$ such that $A_n$ is a cohomologically trivial subset of $X$.*

*Proof.* By the continuity of the Čech cohomology we have

$$H^m(A) = \varinjlim \{H^m(A_n)\}.$$

By assumption, each reduced, cohomology class of $H^m(X)$ is annihilated by the maps induced by the respective inclusions. Hence, there exists for each such class $v$ an integer $n(v)$ such that $v$ is annihilated by the map $i^{*m}_{n(v)}$, where $i_{n(v)}\colon A_{n(v)} \to X$ denote the inclusion map. Since $X$ is of finite type, we infer that $H^*(X)$ has a finite basis. Thus there must exist an integer $n = n(v)$ for all reduced cohomology classes $v$ in such a finite basis. For this $n$, however, $A_n$ is a cohomologically trivial subset of $X$ and the proof of (5.16) is completed. $\square$

*Proof of Theorem (5.14).* By (4.1) we may assume without loss of generality that $X$ is a retract of an open subset $U$ in a Banach space $E$. Let $r\colon U \to X$ be a retraction and $i\colon X \to U$ the inclusion map. Define an admissible map (see (3.23)) $\psi_1\colon U \to U$ by putting $\psi_1 = i\circ\varphi\circ r$. Then $K$ is an invariant subset under $\psi_1$ and moreover,

$$\bigcap_{m\geq 1} \psi_1^m(U) = \bigcap_{m\geq 1} \varphi^m(X).$$

Let $(p,q)$ be a selected pair of $\varphi_1$. Define a map $\psi\colon U \to U$ by putting $\psi = q\circ\varphi$. Then $\psi$ is a u.s.c., compact, admissible map (comp. (3.8)). We have

$$\bigcap_{m\geq 1} \psi^m(U) \subset \bigcap_{m\geq 1} \psi_1^m(U).$$

It is easy to see that $K$ is an invariant subset under $\psi$. Since $\psi$ is a compact map, we infer that the set $A = K\cup(\overline{\psi(U)})$ is a compact subset of $U$. Applying Lemma (4.8) to the pair $(U,A)$, we obtain a compact ANR C such that $A \subset C \subset U$. Then the contraction $\widetilde{\psi}$ of $\psi$ to the pair $(C,C)$ is an admissible map. From (3.4) we infer that $C'_\infty = \bigcap_{m\geq 1}\widetilde{\psi}^m(C)$ is a compact and non-empty subset of $C$. Since $C'_\infty \subset K$, from the assumption and (5.12) we conclude that $C'_\infty$ is a homologically trivial subset of $C$. Hence we infer from (5.15) that $C'_\infty$ is a cohomologically trivial subset of $C$. Applying Lemma (5.16) to the pair $(C, C'_\infty)$, we infer that there exists an integer $m \geq 1$ such that $\widetilde{\psi}^m(C)$ is a cohomologically trivial subset of $C$. Then, in view of (5.15), we deduce that $\widetilde{\psi}^m(C)$ is a homologically trivial subset of $C$ for some $m \geq 1$. Hence Theorem (5.13) implies that $\widetilde{\psi}$ has a fixed point and therefore $\varphi$ has a fixed point. The proof of Theorem (5.14) is completed. $\square$

**(5.17) Corollary.** *Let $X$ be a compact* ANR *and let $\varphi\colon X \to X$ be an admissible map. If the set $C_\infty = \bigcap_{m\geq 1}\varphi^m(X)$ is a homologically trivial subset of $X$, then $\varphi$ has a fixed point.*

In what follows, all spaces will be assumed, to be compact Hausdorff.

For a space $X$ we denote by $\mathrm{Cov}(X)$ the directed set of all finite open coverings of $X$. Let $\varphi\colon X \to X$ be a multivalued map and $\alpha \in \mathrm{Cov}(X)$. A point $x \in X$ is said to be an *$\alpha$-fixed point* for $\varphi$ provided there exists a member $U \in \alpha$ such that

(i) $x \in U$ and

(ii) $\varphi(x) \cap U \neq \emptyset$.

Clearly, if $\alpha, \beta \in \mathrm{Cov}(X)$ and $\alpha$ refines $\beta$, then every $\alpha$-fixed point for $\varphi$ is also a $\beta$-fixed point for $\varphi$.

**(5.18) Lemma.** *Let $\varphi\colon X \to X$ be a u.s.c. map. Assume that there exists a cofinal family of coverings $\mathcal{D} = \{\alpha\} \subset \mathrm{Cov}(X)$ such thai $\varphi$ has an $\alpha$-fixed point for every $\alpha \in \mathcal{D}$. Then $\varphi$ has a fixed point.*

*Proof.* Suppose that $\varphi$ has no fixed points. Then for each $x \in X$ there are open neighbourhoods $V_x$ and $U_{\varphi(x)}$ of $x$ and $\varphi(x)$, respectively, such that $V_x \cap U_{\varphi(x)} = \emptyset$. From the u.s.c. of $\varphi$, we deduce that the set $V = \varphi^{-1}(U_{\varphi(x)})$ is an open neighbourhood of $x$ in $X$. Let $W_x = V_x \cap V$; then we have

(i) $\varphi(W_x) \subset U_{\varphi(x)}$ and
(ii) $W_x \cap U_{\varphi(x)} = \emptyset$.

Since $X$ is a compact space, we infer that there exists a finite number of sets $W_{x_1}, \dots, W_{x_n}$ i "c such that $X = \bigcup_{i=1}^n W_{x_i}$. Putting $\beta = \{W_{x_1}, \dots, W_{x_n}\}$, we get a covering of $X$ such that $\varphi$ has no $\beta$-fixed point. If $\alpha$ is a member of $\mathcal{D}$ that refines $\beta$, then $\varphi$ has no $\alpha$-fixed point, and thus we obtain a contradiction. □

Let $\{X_i\}_{i \in I}$ be a family of compact spaces indexed by an infinite set $I$ and let $X = \prod_{i \in I} X_i$ be their topological product. Denote by $\mathcal{P} = \{J\}$ the family of all finite subsets of $I$; given $J \in \mathcal{P}$, we put $X_J = \prod_{i \in J} X_i$.

**(5.19) Theorem.** *The infinite product $X = \prod_{i \in I} X_i$ of compact spaces has the fixed-point property within the class of admissible maps if and only if every finite product $X_J = \prod_{i \in J} X_i$ ($J \in \mathcal{P}$) has the fixed-point property within the class of admissible maps.*

*Proof.* Choose in each $X_i$ a point $x_i^0$ and define $\widetilde{X}_J \subset X$ as follows:

$$\{x_i\} \in \widetilde{X}_J \Leftrightarrow \begin{cases} x_i \in X_i & \text{for } i \in J, \\ x_i = x_i^0 & \text{for } i \notin J. \end{cases}$$

Clearly we may identify $\widetilde{X}_J$ with $X_J$. Next we define a subset $\mathcal{D} = \{\alpha\} \subset \mathrm{Cov}(X)$ as follows: $\alpha \in \mathcal{D}$ provided $\alpha$ is a finite covering consisting of open sets of the form $U_J = \prod_{i \in J} U_i$ with $U_i$ open in $X_i$ and $U_i = X_i$ for all $i \notin J$. By the Theorem of Tychonoff and taking into account the definition of the product topology, we conclude that $\mathcal{D}$ is cofinal in $\mathrm{Cov}(X)$. Let $\alpha \in \mathcal{D}$; it follows from the definition of the set $\mathcal{D}$ that $\alpha$ determines a finite set of essential indices $J(\alpha)$. Take $r_a\colon X \to \widetilde{X}_{J(\alpha)}$ to be the projection and $s_\alpha\colon \widetilde{X}_{J(\alpha)} \to X$ the inclusion.

Assume that every finite product $X_J = \prod_{i \in J} X_i$ has the fixed-point property within the class of admissible maps. Let $\varphi\colon X \to X$ be an admissible map. We prove that $\varphi$ has a fixed point. Let $p, q\colon Y \to X$ be a selected pair of $\varphi$. Consider the map $\psi\colon X \to X$ given by $\psi = q \circ \varphi_p$. Then from (3.6), (3.3) and (3.23) we deduce that $\psi$ is a u.s.c., admissible map. For each $\alpha \in \mathcal{D}$, consider the map

$\psi_\alpha: \widetilde{X}_{J(\alpha)} \to \widetilde{X}_{J(\alpha)}$ given by $\psi_\alpha = r_\alpha \psi s_\alpha$. Then (3.3) and (3.23) imply that $\psi$ is a u.s.c., admissible map far each $\alpha \in \mathcal{D}$. By assumption, there exists a point $x^\alpha \in \widetilde{X}_{J(\alpha)}$ such that

$$x^\alpha \in \psi_\alpha(x^\alpha) = r_a \psi s_\alpha(x^\alpha) = r_\alpha \psi(x^\alpha), \quad \text{for each } \alpha \in \mathcal{D}. \tag{5.19.1}$$

Let $U$ be a member of $\alpha$ such that $x^\alpha \in U$. Then from (5.19.1) we deduce that $\psi(x^\alpha) \cap U \neq \emptyset$. This implies that $x^\alpha$ is an $\alpha$-fixed point of $\psi$, and hence from (5.18) we infer that $\psi$ has a fixed point. Finally, since $\psi(x) \subset \varphi(x)$ for each $x \in X$, we conclude that $\varphi$ has a fixed point.

Conversely, assume that $X$ has the fixed-point property within the class of admissible maps and that there exists a finite set $J \in \mathcal{P}$ such $X_J$ has no fixed-point property within the class of admissible maps. We may assume without loss of generality that there is an admissible $\psi: \widetilde{X}_J \to \widetilde{X}_J$ such that $x \notin \psi(x)$, for each $x \in \widetilde{X}_J$. Let $r_J: X \to \widetilde{X}_J$ be projection and $s_J: \widetilde{X}_J \to X$ the inclusion. Then we have the admissible map $\varphi: X \to X$ given by $\varphi = s_J \psi r_J$. By assumption there exists a point $x \in X$ such that

$$x \in \varphi(x) = s_J \psi r_J(x).$$

This implies that $r_J(x) \in r_J s_J \psi(r_J(x))$ and thus we obtain a contradiction. The proof of (5.19) is completed. □

From (5.6) and (5.19) we obtain

**(5.20) Corollary.** *An arbitrary Tychonoff cube has the fixed-point property within fhe class of admissible maps.*

Corollary (5.20) and Proposition (5.8) give

**(5.21) Corollary.** *Every retract of a Tychonoff cube has the fixed-point property within tne class of admissible maps.*

## 6. The Lefschetz fixed point theorem for non-compact admissible mappings

The aim of this section is to extend the Lefschetz fixed point theorem onto a class of non-compact mappings: the class of compact absorbing contractions. We define:

**(6.1) Definition.** A multivalued map $\varphi: X \multimap X$ is called a *compact absorbing contraction*, if there exists an open set $U \in X$ such that $\operatorname{cl} \varphi(U)$ is a compact subset of $U$ and $X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(U)$.

Evidently, any compact map $\varphi: X \multimap X$ is a compact absorbing contraction; then we can take $U = X$.

In what follows we will use the following notion: $\varphi \in \mathrm{CAC}(X)$ if and only if $\varphi: X \multimap X$ is admissible and a compact absorbing contraction.

**(6.2) Proposition.** *If $\varphi \in \mathrm{CAC}(X)$ then for every selected pair $(p,q) \subset \varphi$ the homomorphism:*

$$\widetilde{q}_* \circ \widetilde{p}_*^{-1}\colon H(X,U) \to H(X,U)$$

*is weakly nilpotent, where for $p,q\colon \Gamma \to X$ we define $\widetilde{p},\widetilde{q}\colon (\Gamma, p^{-1}(U)) \to (X,U)$, $\widetilde{p}(u) = p(u)$ and $\widetilde{q}(u) = q(u)$ for every $u \in \Gamma$.*

*Proof.* For any compact $K \subset X$ one can find $n$ such that $(qp^{-1})^n(K) \subset U$. Since we consider the Čech homology functor with compact carriers then our claim holds true. □

Now, we shall prove the following:

**(6.3) Theorem.** *Let $X \in$ ANR and $\varphi \in \mathrm{CAC}(X)$. Then $\varphi$ is a Lefschetz map and $\Lambda(\varphi) \neq \{0\}$ implies that $\mathrm{Fix}(\varphi) \neq \emptyset$.*

*Proof.* Let $\varphi\colon X \multimap X$ be an admissible compact absorbing contraction map. Since $\varphi(U) \subset \mathrm{cl}\,\varphi(U) \subset U$, consider $\varphi'\colon U \multimap U$, $\varphi'(x) = \varphi(x)$. Let $(p,q) \subset \varphi$ be a selected pair of $\varphi$. Then $q(p^{-1}(U)) \subset \varphi(U)$. Let $p,q\colon Y \to X$. Then we define $q',p'\colon p^{-1}(U) \to U$, $p'(u) = p(u)$, $q'(u) = q(u)$. Observe that $(p',q') \subset \varphi'$. Since $\varphi'$ is compact, in view of (5.1), $q'_*(p'_*)^{-1}$ is a Leray endomorphism. Consider the maps $p'',q''\colon (Y, p^{-1}(U)) \to (X,U)$; $p''$ is a Vietoris map and, in view of (6.2) $q''_* \circ (p''_*)^{-1}$ is weakly nilpotent. Consequently, from (2.9), (2.10) and (5.8) we deduce that $\Lambda(q_*p_*^{-1}) = \Lambda(q'_*(p'_*)^{-1})$. So, $\varphi$ is a Lefschetz map.

Now, if we assume that $\Lambda(q_*p_*^{-1}) \neq 0$ for some $(p,q) \subset \varphi$, then $\Lambda(q'_*(p'_*)^{-1}) \neq 0$ and by using once again (5.8) we get $\mathrm{Fix}(\varphi') \neq \emptyset$ but it implies that $\mathrm{Fix}(\varphi) \neq \emptyset$ and the proof is completed. □

Now, we would like to show how large the class $\mathrm{CAC}(X)$ is.

**(6.4) Definition.** An u.s.c. multivalued map $\varphi\colon X \multimap Y$ is called *locally compact* provided that, for each $x \in X$, there exists a subset $V$ of $X$ such that $x \in V$, and the restriction $\varphi|_V$ is compact.

**(6.5) Definition.** A multivalued locally compact map $\varphi\colon X \multimap X$ is called *eventually compact* if there exists an iterate $\varphi^n\colon X \multimap X$ of $\varphi$ such that $\varphi^n$ is compact.

**(6.6) Definition.** A multivalued locally compact map $\varphi\colon X \multimap X$ is called a *compact attraction* if there exists a compact $K$ of $X$ such that for each open neighbourhood $V$ of $K$ we have $X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(V)$ and $\varphi^n(x) \subset V$ implies that $\varphi^m(x) \subset V$ for every $m \geq n$ and every $x \in X$, the compact $K$ is then called an *attractor* for $\varphi$.

**(6.7) Definition.** A multivalued locally compact map $\varphi\colon X \multimap X$ is called *asymptotically compact* if the set $C_\varphi = \bigcap_{n=0}^{\infty} \varphi^n(X)$ is a nonempty, relatively compact subset of $X$. The set $C_\varphi$ is called the *center* of $\varphi$.

Note that any multivalued eventually compact map is a compact attraction and asymptotically compact map.

**(6.8) Lemma.** *Any eventually compact map is a compact absorbing contraction map.*

*Proof.* Let $\varphi\colon X \multimap X$ be an eventually compact map such that $K' = \overline{\varphi^n(X)}$ is compact. Define $K = \bigcup_{i=0}^{n-1} \varphi^i(K')$, we have

$$\varphi(K) \subset \bigcup_{i=1}^{n} \varphi^i(K') \subset K \cup \varphi^n(X) \subset K \cup K' \subset K.$$

Since $\varphi$ is locally compact, there exists an open neighbourhood $V_0$ of $K$ such that $L = \overline{\varphi(V_0)}$ is compact, where $\overline{\varphi(V_0)} = \operatorname{cl} \varphi(V_0)$.

There exists a sequence $\{V_1, \ldots, V_n\}$ of open subsets of $X$ such that $L \cap \overline{\varphi(V_i)} \subset V_{i-1}$ and $K \cup \varphi^{n-i}(L) \subset V_i$ for all $i = 1, \ldots, n$. In fact, if $K \cup \varphi^{n-i}(L) \subset V$, and $0 \leq i < n$, since $K \cup \varphi^{n-i}(L)$ and $CV_i \cap L$ are disjoint compact sets of $X$, there exists an open subset $W$ of $X$ such that

$$K \cup \varphi^{n-i}(L) \subset W \subset \overline{W} \subset V_i \cup CL.$$

Define $V_{i+1} = \varphi^{-1}(W)$; since $\varphi(K) \cup \varphi(\varphi^{n-(i+1)}(L)) \subset K \cup \varphi^{n-i}(L) \subset W$, we have $K \cup \varphi^{n-(i+1)}(L) \subset V_{i+1}$, and $\varphi(V_{i+1}) \subset \overline{W} \subset V_i \cup CL$ implies $L \cap \overline{\varphi(V_{i+1})} \subset V_i$. Beginning with $K \cup \varphi^n(L) \subset K \subset V_0$, we define, by induction $V_1, \ldots, V_n$ with the desired properties.

Putting $U = V_0 \cap V_1 \cap \ldots \cap V_n$, we have $K' \subset K \subset U$ and

$$\varphi(U) \subset \varphi(V_0) \cap \varphi(V_1) \cap \ldots \cap \varphi(V_n) \subset L \cap \overline{\varphi(V_1)} \cap \ldots \cap \overline{\varphi(V_n)},$$

hence

$$\overline{\varphi(U)} \subset (L \cap \overline{\varphi(V_1)}) \cap \ldots \cap (L \cap \overline{\varphi(V_n)}) \cap L \subset V_0 \cap \ldots \cap V_{n-1} \cap V_n = U,$$

but $\overline{\varphi(U)}$ is compact since $\overline{\varphi(U)} \subset L$. Moreover,

$$X = \bigcup_{i=1}^{n} \varphi^{-i}(K') \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(U). \qquad \square$$

**(6.9) Proposition.** *Any compact attraction map is a compact absorbing contraction map.*

*Proof.* Let $\varphi\colon X \multimap X$ be a compact attraction map, $K$, a compact attractor for $\varphi$ and $W$, an open set of $X$ such that $K \subset W$ and $L = \overline{\varphi(W)}$ is compact. We have $L \subset X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(W)$ hence, since $L$ is compact, there exists $n \in N$ such that $L \subset \bigcup_{i=0}^{n} \varphi^{-i}(W)$. Define $V = \bigcup_{i=0}^{n} \varphi^{-i}(W)$. Then

$$X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(W) \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(V),$$

$$\varphi(V) \subset \bigcup_{i=0}^{n} \varphi^{-i+1}(W) \subset \varphi(W) \cup V \subset L \cup V \subset V$$

and

$$\varphi^{n+1}(V) \subset \bigcup_{i=0}^{n} \varphi^{n-i+1}(W) = \bigcup_{j=0}^{n} \varphi^{j+1}(W) \subset \bigcup_{j=0}^{n} \varphi^{j}(L),$$

which is compact and included in $V$, since $L \subset V$ and $\varphi(V) \subset V$ implies that $\varphi^j(L) \subset V$ for all $j \in N$. Consider the restriction $\varphi'\colon V \to V$ of $\varphi$. $\varphi'\colon V \to V$ is an eventually compact map, since $V$ is an open set. By Lemma (6.8), there exists an open subset $U$ of $V$, hence of $X$, such that $\operatorname{cl}\varphi'(U) = \operatorname{cl}\varphi(U)$ is a compact subset of $U$ and $V \subset \bigcup_{n=0}^{\infty} {\varphi'}^{-n}(U) \subset \bigcup_{n=0}^{\infty} \varphi^{-n}(U)$. Hence

$$X \subset \bigcup_{n=0}^{\infty} \varphi^{-n}(W) \subset \bigcup_{n=0}^{\infty} \varphi^{-n}(V) \subset \bigcup_{n=0}^{\infty} \varphi^{-n}(U). \qquad \square$$

For more information about the above class of mappings and open problems see: [AG] and [13]. Note also that a fixed point index can be defined for CAC-mappings. We shall end this section by introducing the class of condensing mappings.

Let $E$ ne a Banach space and $\mathcal{B}(E)$ be the set of all bounded nonempty subset of $E$.

We shall define the *measure of noncompactness* on $\mathcal{B}(E)$. We shall say that a subset $A \subset E$ is relatively compact provided the set $\operatorname{cl} A$ is compact.

**(6.10) Definition.** Let $E$ be a Banach space and $\mathcal{B}(E)$ the family of all bounded subsets of $E$. Then the function: $\alpha\colon \mathcal{B}(E) \to \mathbb{R}_+$ defined by:

$$\alpha(A) = \inf\{\varepsilon > 0 \mid A \text{ admits a finite cover by sets of diameter } \leq \varepsilon\}$$

is called the (Kuratowski) *measure of noncompactness*, the $\alpha$-MNC for short.

Another function $\beta\colon \mathcal{B}(E) \to \mathbb{R}_+$ defined by:

$$\beta(A) = \inf\{r > 0 \mid A \text{ can be covered by finitely many balls of radius } r\}$$

is called the (Hausdorff) *measure of noncompactness*.

Definition (6.10) is very useful since $\alpha$ and $\beta$ have interesting properties, some of which are listed in the following

**(6.11) Proposition.** *Let $E$ be a Banach space with* $\dim E = +\infty$ *and* $\gamma\colon \mathcal{B}(E) \to \mathbb{R}_+$ *be either $\alpha$ or $\beta$. Then:*

(a) $\gamma(A) = 0$ *if and only if $A$ is relatively compact,*
(b) $\gamma(\lambda A) = |\lambda|\gamma(A)$ *and* $\gamma(A_1 + A_2) \leq \gamma(A_1) + \gamma(A_2)$, *for every* $\lambda \in \mathbb{R}$ *and* $A, A_1, A_2 \in \mathcal{B}(E)$,
(c) $A_1 \subset A_2$ *implies* $\gamma(A_1) \leq \gamma(A_2)$,
(d) $\gamma(A_1 \cup A_2) = \max\{\gamma(A_1), \gamma(A_2)\}$,
(e) $\gamma(A) = \gamma(\operatorname{conv}(A))$,
(f) *the function* $\gamma\colon \mathcal{B}(E) \to \mathbb{R}_+$ *is continuous* (*with respect to the metric* $d_H$ *on* $\mathcal{B}(E)$).

*Proof.* You will have no difficulty in checking (a)–(d) and (f) ny means of Definition (6.10).

Concerning (e), we only have to show that $\gamma(\operatorname{conv}(A)) \le \gamma(A)$, since $A \subset \operatorname{conv}(A)$ and therefore $\gamma(A) \le \gamma(\operatorname{conv}(A))$. Let $\mu > \gamma(A)$ and $A \subset \bigcup_{i=1}^{m} M_i$ with $\delta(M_i) \le \mu$ if $\gamma = \alpha$ and $M_i = B(x_i, \mu)$ if $\gamma = \beta$. Since $\delta(\operatorname{conv}(\mu_i)) \le \mu$ and $B(x_i, \mu)$ are convex, we may assume that the $M_i$ are convex. Since

$$\begin{aligned}\operatorname{conv}(A) &\subset \operatorname{conv}\left[M_1 \cup \operatorname{conv}\left(\bigcup_{i=2}^{m} M_i\right)\right] \\ &\subset \operatorname{conv}\left[M_1 \cup \operatorname{conv}\left[M_2 \cup \operatorname{conv}\left(\bigcup_{i=3}^{m} M_i\right)\right]\right] \subset \dots,\end{aligned}$$

it suffices to show that

$$\gamma(\operatorname{conv}(C_1 \cup C_2)) \le \max\{\gamma(C_1), \gamma(C_2)\} \quad \text{for convex } C_1 \text{ and } C_2.$$

Now, we have

$$\operatorname{conv}(C_1 \cup C_2) \subset \bigcup_{0 \le \lambda \le 1} [\lambda C_1 + (1-\lambda) C_2],$$

and since $C_1 - C_2$ is bounded there exists an $r > 0$ such that $\|x\| \le r$ for all $x \in (C_1 - C_2)$.

Finally, given $\varepsilon > 0$, we find $\lambda_1, \dots, \lambda_p$ such that

$$[0,1] \subset \bigcup_{i=1}^{p} \left(\lambda_i - \frac{\varepsilon}{r}, \lambda_i + \frac{\varepsilon}{r}\right)$$

and therefore

$$\operatorname{conv}(C_1 \cup C_2) \subset \bigcup_{i=1}^{p} [\lambda_i C_1 + (1-\lambda_i) C_2 + \operatorname{cl} B(0, \varepsilon)].$$

Hence, (b)–(d) and the obvious estimate $\gamma(\operatorname{cl} B(0,\varepsilon)) \le 2\varepsilon$ imply

$$\gamma(\operatorname{conv}(C_1 \cup C_2)) \le \max\{\gamma(C_1), \gamma(C_2)\} + 2\varepsilon,$$

for every $\varepsilon > 0$. Consequently the proof is completed. □

Now, let us state the following obvious observation.

**(6.12) Remark.** For every $A \in \mathcal{B}(E)$ we have $\beta(A) \le \alpha(A) \le 2\beta(A)$.

We shall end this section by considering two examples and by formulating a generalization of the Cantor theorem.

**(6.13) Example.** Assume that $\dim E = +\infty$. Now, let us complete the measures of a ball $B(x_0, r) = \{x_0\} + r \cdot B(0,1)$. Evidently,

$$\gamma(B(x_0, r)) = r\gamma(\operatorname{cl} B(0,1)) = r\gamma(S),$$

where $S = \delta B(0,1) = \{x \in E \mid \|x\| = 1\}$.

Furthermore, $\alpha(S) \leq 2$ and $\beta(S) \leq 1$. Suppose $\alpha(S) < 2$. Then $S = \bigcup_{i=1}^{n} M_i$ with the closed sets $M_i$ and $\delta(M_i) < 2$. Let $E^n$ be an $n$-dimensional subspace of $E$. Then

$$S \cap E^n = \bigcup_{i=1}^{n} M_i \cap E^n$$

and in view of the Lusternik–Schnirelman–Borsuk theorem (see [De3-M, p. 22] or [9, p. 43]) there exists $i$ such that the set $M_i \cap E^n$ contains a pair of antipodal points, $x$ and $-x$. Hence $\delta(M_i) \geq 2$ for this $i$, a contradiction. Thus $\alpha(S) = 2$ and

$$1 = \frac{\alpha(S)}{2} \leq \beta(S) \leq 1,$$

i.e. we have $\alpha(B(x_0, r)) = 2r$ and $\beta(B(x_0, r)) = r$ provided $\dim E = +\infty$.

**(6.14) Example.** Let $r: E \to \operatorname{cl} B(0,1)$ be the retraction map defined as follows:

$$r(x) = \begin{cases} x & \text{if } \|x\| \leq 1, \\ \dfrac{x}{\|x\|} & \text{if } \|x\| > 1. \end{cases}$$

Let $A \in \mathcal{B}(E)$. Since $r(A) \subset \operatorname{conv}(A \cup \{0\})$, we obtain $\gamma(r(A)) \leq \gamma(A)$. In other words we can say that $r$ is a nonexpansive map with respect to the Kuratowski or Hausdorff measure of noncompactness.

Finally, note that the following version of the Cantor theorem holds true.

**(6.15) Theorem.** *If $\gamma = \alpha$ or $\gamma = \beta$ and $\{A_n\}$ is a decreasing sequence of closed nonempty subsets in $B(E)$ such that $\lim_n \gamma(A_n) = 0$. Then $A = \bigcap_{n=1}^{\infty} A_n$ is a nonempty and compact subset of $E$.*

To learn about condensing maps it is useful to start with the notion of $k$-set contraction and condensing pairs of maps. As in Section 4, by a pair $(p, q)$ we mean the following diagram:

$$X \overset{p}{\Longleftarrow} \Gamma \xrightarrow{q} Y$$

in which $p$ is Vietoris and $q$ continuous. Such a pair $(p, q)$ is called compact provided $q$ is compact.

Let $E$ be a Banach space. By $\gamma: \mathcal{B}(E) \to \mathbb{R}_+$ we will denote the measure of non-compactness function, i.e. $\gamma$ is a function satisfying all properties of (4.10). In particular, we can let $\gamma = \alpha$ to be the Kuratowski measure of compactness or $\gamma = \beta$ to be the Hausdorff measure of non-compactness (see Section 4).

**(6.16) Definition.** Let $A$ and $C$ be two subsets of $E$. A pair $A \stackrel{p}{\Longleftarrow} \Gamma \stackrel{q}{\longrightarrow} C$ is called a *$k$-set contraction pair*, if there exists a real number $k$, $0 \le k < 1$, such that for every bounded $B \subset A$ the following condition is satisfied:

$$\gamma(q(p^{-1}(B))) \le k \cdot \gamma(B); \tag{6.16.1}$$

$(p,q)$ is called a *condensing pair*, if for every bounded and no relatively compact $B \subset A$ we have

$$\gamma(q(p^{-1}(B))) < \gamma(B). \tag{6.16.2}$$

It is evident that any compact pair is $k$-set contraction with $k = 0$ and any $k$-set contraction pair is condensing. Moreover, let us observe that if $(p,q)$ is a condensing pair then for any bounded $B \subset A$ the set $q(p^{-1}(B))$ is bounded.

**(6.17) Proposition.** *Let $A \stackrel{p}{\Longleftarrow} \Gamma \stackrel{q}{\longrightarrow} C$ be a condensing pair, where $A$ is a bounded and closed subset of $E$. Then* $\mathrm{Fix}(p,q)$ *is a compact set, where as before* $\mathrm{Fix}(p,q) = \{x \in A \mid x \in q(p^{-1}(x)\}$.

*Proof.* Indeed, we have $\mathrm{Fix}(p,q) \subset q(p^{-1}(\mathrm{Fix}(p,q)))$, hence

$$\gamma(\mathrm{Fix}(p,q)) \le \gamma(q(p^{-1}(\mathrm{Fix}(p,q)))) < \gamma(\mathrm{Fix}(p,q)).$$

So, by (4.10) we deduce that $\overline{\mathrm{Fix}(p,q)}$ is compact. Because $\mathrm{Fix}(p,q) = \overline{\mathrm{Fix}(p,q)}$ the proof is completed. □

We will say that the pair $(p,q)$ satisfies the Palais–Smale condition provided for every sequence $\{u_n\} \subset \Gamma$, the property

$$\lim_n (p(u_n) - q(u_n)) = 0$$

implies that there exists a convergent subsequence of $\{u_n\}$.

**(6.18) Proposition.** *Let* $(p,q)$ *be the same as in* (6.17). *Then the pair* $(p,q)$ *satisfies the Palais–Smale condition.*

*Proof.* Let $\lim_n (p-q)(y_n) = 0$. We put $x_n = p(y_n) - q(y_n)$, $u_n = p(y_n)$. Then $\{x_n\} \subset E$ and $\{u_n\} \subset A$. By assumption $\gamma(\{x_n\}) = 0$. We will show that $\gamma(\{u_n\}) = 0$. Because $q(y_n) \in q(p^{-1}(u_n))$ we have

$$\gamma(q(\{y_n\})) \le \gamma(q(p^{-1}(\{u_n\}))) \le k \cdot \gamma(\{u_n\}).$$

On the other hand, $u_n = x_n + q(y_n)$ so, in view of (4.10.2), we obtain

$$\gamma(\{u_n\}) \le \gamma(\{x_n\}) + \gamma(\{q(y_n)\}) = \gamma(q(\{y_n\})).$$

The above two inequalities imply that $\gamma(\{u_n\}) = 0$. Therefore the set $p^{-1}\overline{(\{u_n\})}$ is compact ($p$ is proper!), so from the sequence $\{y_n\}$ in $p^{-1}(\{u_n\})$ we can choose a convergent subsequence and the proof is completed. □

Let $A$ be a bounded closed subset of $E$ and let $C$ be a convex closed subset of $E$. Consider a $k$-set contraction pair $(p,q)$ from $A$ to $C$. We will associate with such a pair $(p,q)$ a compact pair $(\widetilde{p},\widetilde{q})$ such that $\mathrm{Fix}(p,q)=\mathrm{Fix}(\widetilde{p},\widetilde{q})$. In order to do it we define a decreasing sequence $\{K_n\}$ of closed bounded and convex subsets of $C$ by putting

$$K_1=\overline{\mathrm{conv}}(q(p^{-1}(A))),\dots,K_n=\overline{\mathrm{conv}}(q(p^{-1}(A\cap K_{n-1}))),\dots$$

It is evident that $q(p^{-1}(K_n\cap A))\subset K_{n+1}$ and $\mathrm{Fix}(p,q)\subset K_n$ for every $n$. There are two possibilities, namely,

(6.19) $\qquad K_n\neq\emptyset,\quad$ for each $n$,

(6.20) $\qquad K_i\neq\emptyset,\quad$ for $i=1,\dots,m$ and $K_{m+j}=\emptyset$, for each $j$.

If (6.20) holds then we choose a point $x_0\in K_m$ and we define

(6.21) $\qquad \widetilde{q}\colon\Gamma\to C\quad$ by putting $q(y)=x_0$ and $\widetilde{p}=p$.

Then $(\widetilde{p},\widetilde{q})$ is a compact pair such that $\mathrm{Fix}(p,q)=\mathrm{Fix}(\widetilde{p},\widetilde{q})=\emptyset$.

**(6.22) Lemma.** *Assume that* (6.20) *holds and let $x_1\in K_m$. Then there exists a compact homotopy $h\colon\Gamma\times[0,1]\to C$ joining $\widetilde{q}$ with $\widetilde{q}_1$ such that*

$$\mathrm{Fix}(p,h)=\{x\in A\mid x\in h(p^{-1}((x)\times\{t\})),\ \textit{for every } t\}=\emptyset,$$

*where $\widetilde{q}_1\colon\Gamma\to C$ is given by the formula $\widetilde{q}_1(y)=x_1$.*

For the proof of Lemma (6.22) it is sufficient to consider a homotopy $h\colon\Gamma\times[0,1]\to C$ given as follows:

$$h(y,t)=(1-t)x_0+tx_1.$$

**(6.23) Remark.** By comparing (6.21) and (6.22) we can say that, if (6.20) holds, then the pair $(\widetilde{p},\widetilde{q})$ is defined uniquely up to homotopy.

**(6.24) Lemma.** *If* (6.19) *holds, then $K_\infty=\bigcap_{n=1}^{\infty}K_n$ is a compact convex and nonempty set which contains* $\mathrm{Fix}(p,q)$.

*Proof.* First, we claim that

(6.24.1) $\gamma(K_n)\leq k^n\cdot\gamma(A)$, for each $n$, where $k$ is given for considered $k$-set contraction pair $(p,q)$.

We prove (6.24.1) by induction. Since

$$\gamma(K_1)=\gamma(\overline{\mathrm{conv}}(q(p^{-1}(A))))=\gamma(q(p^{-1}(A)))\leq k\cdot\gamma(A),$$

our assertion holds for $n=1$. Now assume that (6.24.1) is true for every $m<n$. Then we obtain:

$$\begin{aligned}\gamma(K_n)&=\gamma(\overline{\mathrm{conv}}(q(p^{-1}(A\cap K_{n-1}))))=\gamma(q(p^{-1}(A\cap K_{n-1})))\\&\leq k\cdot\gamma(A\cap K_{n-1})\leq k\cdot\gamma(K_{n-1})\leq k\cdot k^{n-1}\gamma(A)=k^n\gamma(A)\end{aligned}$$

and thus finish the proof of (6.24.1).

Now, from (6.24.1) it follows that $\lim_n \gamma(K_n) = 0$. Therefore, our claim follows from (4.14). □

We associate with given $k$-set contraction pair $(p,q)\colon A \overset{p}{\Longleftarrow} \Gamma \xrightarrow{q} C$ the pair $(\widetilde{p},\widetilde{q})$:

$$A \cap K_\infty \overset{\widetilde{p}}{\Longleftarrow} p^{-1}(A \cap K_\infty) \xrightarrow{\widetilde{q}} K_\infty \tag{6.25}$$

by putting $\widetilde{p}(u) = p(u)$ and $\widetilde{q}(u) = q(u)$. Since $\widetilde{q}\,\widetilde{p}^{-1}(A \cap K_\infty) \subset K_\infty$, in view of (6.24) we get $\mathrm{Fix}(p,q) = \mathrm{Fix}(\widetilde{p},\widetilde{q})$. Observe, that if $A = C$, then the condition (6.20) cannot occur.

Since $(\widetilde{p},\widetilde{q})$ is a compact pair, then from the Lefschetz fixed point theorem for admissible (or determined by morphisms) maps we obtain:

**(6.26) Proposition.** *If $C$ is a bounded closed and convex subset of $E$ and $(p,q)$ is a $k$-set contraction pair from $C$ to $C$, then* $\mathrm{Fix}(p,q) \neq \emptyset$.

We prove:

**(6.27) Theorem.** *If $C$ is a bounded closed and convex subset of $E$ and $(p,q)$ is a condensing pair from $C$ to $C$, then* $\mathrm{Fix}(p,q) \neq \emptyset$.

For the proof of (6.27) we need some additional facts. Let $\varepsilon > 0$. A point $u \in \Gamma$ is called an *$\varepsilon$-coincidence* for $(p,q)$, if $\|p(u) - q(u)\| < \varepsilon$.

**(6.28) Lemma.** *If $(p,q)$ has an $\varepsilon$-coincidence for every $\varepsilon > 0$ and satisfies the Palais–Smale condition, then* $\mathrm{Fix}(p,q) \neq \emptyset$.

*Proof.* Let $\varepsilon_n = 1/n$ and $\{u_n\} \subset \Gamma$ be a sequence of $\varepsilon_n$-coincidence points of $(p,q)$, $n = 1,2,\ldots$. Then $\lim_n(p(u_n) - q(u_n)) = 0$. So, from the Palais–Smale condition we obtain that there exists a convergent subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

Let $u = \lim_k u_{n_k}$. Then $p(u) = q(u)$, so the set $\varkappa(p,q)$ of coincidence points is nonempty and consequently $\mathrm{Fix}(p,q) \neq \emptyset$. □

*Proof of* (6.27). We can assume without loss of generality, that $0 \in C$. For each $n = 1,2,\ldots$ we define a map $q_n\colon \Gamma \to C$ by putting:

$$q_n(u) = \left(1 - \frac{1}{n}\right) \cdot q(u).$$

Then $(p,q)$ is an $(1-(1/n))$-set contraction, $n \geq 2$. So, from (6.26) for every $n \geq 2$ we obtain a point $u_n \in \Gamma$ such that $p(u_n) = q(u_n)$. On the other hand we have:

$$\begin{aligned}\|p(u_n) - q(u_n)\| &\leq \|p(u_n) - q_n(u_n)\| + \|q(u_n) - q_n(u_n)\| \\ &= \frac{1}{n}\|q(u_n)\| \leq \frac{1}{n} \cdot \mathrm{diam}(C),\end{aligned}$$

where $\operatorname{diam}(C)$ denotes the diameter of $C$. It implies that $(p, q)$ has $\varepsilon$-coincidence for every $\varepsilon > 0$ and hence our theorem follows from (6.28) and (6.19); the proof is completed. $\square$

Let $C$ be a convex closed subset of $E$ and $(p, q)\colon \overline{U} \overset{p}{\Longleftarrow} \Gamma \xrightarrow{q} C$ be a $k$-set contraction pair such that $\operatorname{Fix}(p, q) \cap \partial U = \emptyset$, i.e. $(p, q)$ has no fixed points on the boundary $\partial U$ of $U$ in $C$, where $U$ is an open subset of $C$.

Following (6.25) we obtain a compact pair

$$(\widetilde{p}, \widetilde{q})\colon \overline{U} \cap K_\infty \overset{\widetilde{p}}{\Longleftarrow} p^{-1}(\overline{U} \cap K'_\infty) \xrightarrow{\widetilde{q}} C \cap K_\infty.$$

For simplicity let us denote $U_1 = U \cap K_\infty$ and $C_1 = C \cap K_\infty$, $\Gamma_1 = p^{-1}(\overline{U} \cap K_\infty)$. Then we have a compact pair

$$\overline{U}_1 \overset{\widetilde{p}}{\Longleftarrow} \Gamma_1 \xrightarrow{\widetilde{q}} C_1,$$

where $U_1$ is open in $C_1$ and $C_1$ is a convex nonempty compact subset of $E$.

Now, by using the Schauder Approximation Theorem, for given $\varepsilon > 0$ we can find a $n(\varepsilon)$-dimensional subspace $E^{n(\varepsilon)}$ of $E$ and an $\varepsilon$-approximation $q_\varepsilon\colon \Gamma_1 \to E^{n(\varepsilon)}$ of $\widetilde{q}$. We let $V = U_1 \cap E^{n(\varepsilon)}$ and $C_\varepsilon = C_1 \cap E^{n(\varepsilon)}$. Then we obtain a diagram:

$$\overline{V} \overset{p_1}{\Longleftarrow} \widetilde{p}^{-1}(\overline{V}) \xrightarrow{q_\varepsilon} C_\varepsilon.$$

It is easy to see that for sufficiently small $\varepsilon > 0$ such that $\operatorname{Fix}(p_1, q_\varepsilon) \cap \partial V = \emptyset$ and $q_\varepsilon$ is homotopic to $q_{\varepsilon'}$ for $\varepsilon, \varepsilon' \leq \varepsilon_0$, for some $\varepsilon > 0$.

Let $r\colon \mathbb{R}^n \to C_\varepsilon$ be a retraction ($C_\varepsilon$ is convex and closed, so $C_\varepsilon \in \mathrm{AR}$). Then $r^{-1}(V)$ is an open subset of $\mathbb{R}^n$ and we have the following commutative diagram:

$$\begin{tikzcd}
\overline{r^{-1}(V)} \arrow[r, "r"] & \overline{V} & \widetilde{p}^{-1}(V) \arrow[l, Rightarrow, "p_1"'] \arrow[r, "q_\varepsilon"] & C_\varepsilon \arrow[d, "i"] \\
& \Gamma_\varepsilon \arrow[ul, Rightarrow, "p_\varepsilon"] \arrow[u, "f"'] \arrow[ur, "g"'] \arrow[rr, "\overline{q}_\varepsilon = i\circ q_\varepsilon \circ g"'] & & E^{n(\varepsilon)}
\end{tikzcd}$$

in which $\Gamma_\varepsilon = \{(x, y) \in \overline{r^{-1}(V)} \times \widetilde{p}^{-1}(V) \mid r(x) = p_1(y)\}$, $p_\varepsilon(x, y) = x$, $f(x, y) = r(x)$, $g(x, y) = y$. Moreover, we obtain:

$$\operatorname{Fix}(p_\varepsilon, \overline{q}_\varepsilon) = \operatorname{Fix}(p_1, q_\varepsilon) \subset V.$$

But for the pair $(p_\varepsilon, \overline{q}_\varepsilon)$ the coincidence index $I(p_\varepsilon, \overline{q}_\varepsilon)$ is well defined (see (3.4)).

We let:

$$I(p, q) = I(p_\varepsilon, \overline{q}_\varepsilon). \tag{6.29}$$

Then $I(p, q)$ is called the coincidence index for the $k$-set contraction pair $(p, q)$. Note that by a standard argument, used already several times, we can see that definition (6.29) is correct for a given retraction $r$.

The following problem remains open (see [14]).

(6.30) Does Definition (6.29) depend on the choice of a retraction map $r$?

Note that (6.30) is a slight reformulation of the definition of a topological degree for $n$-admissible mappings.

We shall make use of the following two properties of the coincidence index defined in (6.29).

**(6.31) Property** (Existence). *If* $I(p,q) \neq 0$, *then* $\mathrm{Fix}(p,q) \neq \emptyset$.

**(6.32) Property** (Homotopy). *Let $U$ be an open subset of $C$, where $C$ is a convex closed subset of a normed space $E$. Let $p\colon \Gamma \Rightarrow \overline{U}$ be a Vietoris map and let $h\colon \Gamma \times [0,1] \to C$ be a continuous map. Assume further that the following two conditions are satisfied:*

- (a) $\mathrm{Fix}(p,h) \cap \partial U = \emptyset$, *where* $\mathrm{Fix}(p,h) = \{x \in U \mid x \in h(p^{-1}(x,t))$ *for some* $t \in [0,1]\}$,
- (b) $\gamma(h(p^{-1}(B \times [0,1]))) \leq k \cdot \gamma(B)$ *for every* $B \subset \overline{U}$ *and some* $0 \leq k < 1$.

*Then* $I(p,h_0) = I(p,h_1)$, *where* $h_i(x) = h(x,i)$, $i = 0,1$.

The standard proofs of (6.31) and (6.32) are left to the reader.

Now we will generalize the non-linear alternative and the Leray–Schauder alternative from the case of $k$-set contraction singlevalued maps to the case of $k$-set contraction pairs. Till the end of this section we will assume that $C$ is a convex and closed subset of $E$ which contains the zero point $0$ of $E$.

**(6.33) Theorem** (The Non-Linear Alternative). *Let $U$ be an open bounded subset of $C$ such that $0 \in U$ and let $(p,q)$ be a $k$-set contraction pair from $\overline{U}$ to $C$. Then at least one of the following properties holds:*

- (a) $\varkappa(p,q) \neq \emptyset$,
- (b) *there is an* $x \in \partial U$ *such that* $x \in (\lambda \cdot q(p^{-1}(x)))$ *for some* $\lambda > 1$.

*Proof.* We can assume without loss of generality, that $\mathrm{Fix}(p,q) \cap \partial U = \emptyset$. For the proof consider a homotopy $h\colon \Gamma \times [0,1] \to C$ defined by the formula $h(y,t) = t \cdot q(y)$. Then $h$ satisfies (6.32)(b) and it is a homotopy joining $q$ with the constant map $q_1, q_1(y) = 0$. If $\mathrm{Fix}(p,h) \cap \partial U = \emptyset$, then from (6.32) and (6.31) we deduce that $\mathrm{Fix}(p,q) \neq \emptyset$, so (a) holds. If $\mathrm{Fix}(p,h) \cap \partial U \neq \emptyset$, then we can take a point $x_0 \in \partial U$ such that $x_0 \in (t_0 \cdot q(p^{-1}(x_0)))$ for some $0 < t_0 < 1$. Consequently, for $\lambda = 1/t_0 > 1$ we have $x_0 \in \lambda \cdot q(p^{-1}(x_0)))$ and the proof is completed. □

**(6.34) Corollary.** *Assume $(p,q)$ is as in* (6.33). *Assume further that for every $x \in \partial U$ and for every $u \in q(p^{-1}(x))$ one of the following conditions holds:*

- (a) $\|u\| \leq \|x\|$,
- (b) $\|u\| \leq \|x - u\|$,
- (c) $\|u\|^2 \leq \|x\|^2 + \|x - u\|^2$.

*Then* $\varkappa(p,q) \neq \emptyset$.

For the proof of (6.34) it is sufficient to note that each of conditions (a)–(c) implies that the second property of the non-linear alternative cannot occur.

For a pair $(p,q)$ from $C$ to $C$ and for a subset $A \subset C$, by $(p_A, q_A)$ we will denote a pair defined as follows:

$$\begin{aligned} p_A &: p^{-1}(A) \Rightarrow A, \quad p_A(y) = p(y), \\ q_A &: p^{-1}(A) \to C, \quad q_A(y) = q(y). \end{aligned}$$

**(6.35) Theorem** (The Leray–Schauder Alternative)**.** *Let* $(p,q)$ *be a pair from* $C$ *to* $C$ *such that for any open and bounded* $U \subset C$ *the pair* $(p_U, q_U)$ *is a* $k$*-set contraction. Let* $G(p,q) = \{x \in C \mid x \in (\lambda \cdot q(p^{-1}(x)))$*, for some* $0 < \lambda < 1\}$*. Then either* $G(p,q)$ *is unbounded or* $\varkappa(p,q) \neq \emptyset$*.*

*Proof.* Assume $G(p,q)$ is bounded. We choose an open ball $B(0,r)$ in $E$ containing $G(p,q)$ in its interior. Let $U = B(0,r) \cap C$. Then $(p_U, q_U) \in \mathcal{C}(\overline{U}, C)$ and no $x \in \partial U$ can satisfy the second property of the non-linear alternative. By using once again (6.33) to the pair $(p_U, q_U)$ we have $\emptyset \neq \varkappa(p_U, q_U) \subset \varkappa(p,q)$ and the proof is completed. □

**(6.36) Remark.** Finally, let us remark that all results of this section can be formulated for $k$-set contraction and condensing admissible maps or morphisms; $\varphi$ is a $k$-set contraction (condensing) admissible map if there exists a $k$-set contraction (condensing) pair $(p,q)$ such that $(p,q) \subset \varphi$. In the definition of the $k$-set contraction (condensing) morphism we consider the equivalence relation in family of all $k$-set contraction (condensing) pairs $(p,q)$.

**(6.37) Remark.** Note that next we will continue the study of $k$-set contraction and condensing maps in the framework of so called compacting mappings.

**(6.38) Definition.** A closed subset of $X$ of a Banach space $E$ is called a *special* ANR provided there exists a family $\{C_j\}_{j \in J}$ of closed convex subsets of $E$ such that $X = \bigcup_{j \in J} C_j$ and this union is locally finite, i.e. for every $x \in X$ there exists a finite set $J_x \subset J$, such that $x \notin C_j$ for every $j \in J \setminus J_x$ (written $X \in$ s-ANR).

Note that a special ANR is an ANR-space.

If $X \in$ s-ANR and $X$ is a finite union $X = \bigcup_{j=1}^{n} C_j$ of closed convex subsets in $E$, then we will write $X \in$ sf-ANR.

We shall use the following lemma:

**(6.39) Lemma.** *Let* $C \in$ sf-ANR *be the union* $C = \bigcup_{j=1}^{n} C_j$ *of closed convex subsets of* $E$*. Then there exists a polyhedron* $P$ *such that* $P = \bigcup_{i=1}^{m} P_i$*, where* $P_i = \operatorname{conv}\{x_1, \ldots, x_{m_i}\}$ *for some* $x_1, \ldots, x_{m_i} \in C$ *and a continuous map* $\pi: C \to C$ *such that* $\pi(C_i) \subset P_i \subset C_i$ *for all* $i \leq n$*.*

Lemma (6.39) is strictly technical so the proof is omitted here. For details see [14].

**(6.40) Definition.** Let $X \in$ s-ANR and $U$ be an open subset of $X$. An admissible map $\varphi: U \multimap X$ is called *compacting* provided there exists an open set $W \subset U$ and a sequence $\{K_n\}$ such that $K_n \in$ sf-ANR for every $n$ and the following conditions are satisfied:

(a) $\operatorname{Fix}(\varphi) \subset W \subset \overline{W} \subset U$,
(b) $W \subset K_1 \subset X$,
(c) $\varphi(W \cap K_n) \subset K_{n+1} \subset K_n$ for any $n \geq 1$,
(d) $\lim_{n\to\infty} \gamma(K_n) = 0$, where $\gamma$ denotes the measure of non-compactness.

We shall prove the following:

**(6.41) Proposition.** *Suppose that $X \in$ s-ANR, $U$ is an open subset of $X$ and $f: U \to X$ is a continuous map such that $S = \{x \in U \mid f(x) = x\}$ is compact. Assume that there is an open neighbourhood $W$ of $S$ such that $f|_W$ is a $k$-set-contraction with $k < 1$. Then $f$ is compacting.*

*Proof.* $X$ has a locally finite covering $\{C_\alpha \mid \alpha \in A\}$ by closed, convex hull of $B$ in the overlying Banach space. By the local finiteness of the covering and the *compactness of* $\overline{\mathrm{co}}\, f(W)$, there exists a neighbourhood $W_1$ of $S$, $\overline{W}_1 \subset W$, such that $(W_1 \cup \overline{\mathrm{co}}\, f(W_1)) \cap C_\alpha$ is empty except for $\alpha$ in a finite index set $A_1$. Define $K_1 \in \mathcal{F}_0$ by $K_1 = \bigcup_{\alpha\in A_1} C_\alpha$ and for $n \geq 1$ define $\{K_n\}$ inductively by $K_{n+1} = (\overline{\mathrm{co}}\, f(W_1 \cap K_n)) \cap X$. Since $f$ is a $k$-set-contraction, $k < 1$, $\gamma(K_{n+1}) \leq k^n \gamma(W_1) \to 0$. It is also not hard to see that $K_n \supset K_{n+1}$, $f(W_1 \cap K_n) \subset K_{n+1}$ and $W_1 \subset K_1$. Thus $f$ is compacting; the proof is completed. $\square$

Now, assume that $\varphi: W \multimap X$ is compacting with $W$ and $\{K_n\}$ satisfying the conditions of (6.40). Since $K_i \in$ sf-ANR there exists $m(i)$ and $C_{i1}, \dots, C_{im(i)}$ closed convex such that $K_i = \bigcup_{j=1}^{m(i)} C_{ij}$ and $\partial(C_{ij}) < \gamma(K_i) + i^{-1}$.

We may now choose $n$ and by Lemma (6.39), $\pi_n: K_1 \to K_1$ such that $\pi_n(K_1) \subset P_n \subset X$, where $P_n$ is a polyhedron such that

$$P_n = \bigcup_{i=1}^{m} \bigcup_{j=1}^{m(i)} P_{ij} \quad \text{and} \quad \pi_n(C_{ij}) \subset P_{ij} \subset C_{ij}.$$

Thus $\pi_n(K_i) \subset K_i$ for any $i \leq n$ and if $x \in K_i$ we have $\|\pi_n(x) - x\| \leq \gamma(K_i) + i^{-1}$. Since the fixed point index for maps determined by morphisms on polyhedra is defined, as we have already observed, we can let:

$$i(X, \varphi, U) = \lim_{n\to\infty} i(P_n, \pi_n \circ \varphi, W \cap P_n) = i(P_n, \pi_n \circ \varphi, W \cap P_n) \tag{6.42}$$

if $n$ is big enough. The proof of a correctness of the above definition is quite long and technically complicated.

We shall restrict our considerations to the case of the fixed point index defined in (6.29) having the following properties:

- existence,
- excision,
- additivity,
- homotopy,
- commutativity,
- mod $p$.

To obtain the normalization property of the above fixed point index one more assumption about $\varphi$ is needed. We have to assume that $\varphi$ is a compact absorbing contraction and compacting mapping. We have proved that for compact absorbing contractions the Lefschetz fixed point theorem is true, so it is sufficient to see that the respective Lefschetz number and the fixed point index are equal.

There are still some open problems concerning compacting and compact absorbing contractions:

One of those is to find relations between:

- compacting, condensing, $k$-set contraction mappings on one hand;
- eventually compact mappings with compact attractors, compact absorbing contractions, asymptotically compact — on the other hand.

For details we recommend [1] and [14].

## 7. Remarks and comments

In this material we concentrated our considerations to the Lefschetz Foxed Point Theorem. Note that more generalized results can be obtained. In Section 3 we defined the index of coincidence $I(p,q)$ for a pair

$$U \overset{p}{\Longleftarrow} Y \xrightarrow{q} \mathbb{R}^n,$$

where $U$ is an open subset of the Euclidean space $\mathbb{R}^n$ (see (3.4)). This definition can be taken up for the pair:

$$U \overset{p}{\Longleftarrow} Y \xrightarrow{q} X,$$

where $U$ is an open subset of an ANR-space $X$ and $q$ is a compact map. It allows us to define the index set $I(\varphi)$ of $\varphi$ for an admissible and compact map $\varphi\colon U \multimap X$. Then using obtained results it is possible to generalize this index to the case when $\varphi\colon U \multimap X$ is a CAC-map. For details we recommend [14]. Moreover, we would like to remark that some special results concerning fixed point index can be done also for condensing and compacting mappings (see again [14]).

Using homological methods it is possible to study the fixed point theory for the fixed point theory for the other classes of multivalued mappings, namely:

- spheric mappings (see [21], [22]);

- mappings with values consisting from one or $n$ acyclic components (see [14] or [10]);
- whitehead mappings (see [Ski]).

The next possibility is connected with non-metric case. Let $E$ be a topological vector space; $E$ is called *Klee admissible* provided for every compact set $K \subset E$ and for every open neighbourjood $V$ of $o$ in $E$ there exists a map $\pi_V\colon K \to E$ such that $\pi_V(K) \subset E^n \subset E$, where $E^n$ is $n$-dimensional subspace of $E$ and $\pi_V(x) \in (X + V)$ for every $x \in K$.

Note that, in particular, any locally convex space $E$ is Klee admissible.

Now, all results presented in this work can be generalized for the respective classes of multivalued mappings of retracts of open sets in Klee admissible spaces. For details concerning non-metric case see [1].

Finally, let us remark that using homological methods it is possible to study also (for details see: [12]–[14], [15]):

- relative versions of the Lefschetz Fixed Point Theorem;
- periodic problems;
- Nielsen theory for multivalued mappings.

Instead of [1] and [14] for further studies of homological methods in fixed point theory we recommend [5], where a review of modern methods in fixed point theory is presented.

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# APPROXIMATION METHODS IN THE THEORY OF SET-VALUED MAPS

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Abstract. In the lecture we shall present some methods allowing to investigate fixed points and the solvability of generalized equations involving set-valued maps.

Generally speaking there are two concurrent attitudes to the problem: topological (or more precisely homological and/or homotopical) approach and the approximation one ([1]). However essentially different, these two attitudes are most often combined and intertwined: sometimes there are sufficiently close single-valued approximations (understood in an appropriate sense) whose topological behavior reflects the properties of the studied map. Thus approximation techniques may interact with topological methods and bring a deeper insight into the theory. Whereas purely topological and algebraic methods seem to be more universal, approximation approach is frequently substantially simpler and, at most occasions, sufficient for a vast area of applications.

## 1. Preliminaries

In this section we shall provide the exposition of some rudimentary material concerning set-valued maps (see e.g. [9], [45], [4], [52] and [46]).

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([1]) Speaking of approximations we have also selections in mind.

**1.1. Set-valued maps.** From the formal point of view, a set-valued map is a simple generalization of an ordinary (single-valued) mapping. To support this statement let us introduce some notation and elementary concepts. Assume that sets $X$, $Y$ are nonempty and let $\varphi \subset X \times Y$ be a relation. Then

$$\varphi^{-1} := \{(y,x) \in Y \times X \mid (x,y) \in \varphi\}$$

is the *inverse* relation. If $A \subset X$, then

$$\varphi(A) := \{y \in Y \mid \exists\, x \in A\ (x,y) \in \varphi\}$$

is the *image* of $A$ *through* $\varphi$; in particular, if $B \subset Y$, then

$$\varphi^{-1}(B) = \{x \in X \mid \exists\, y \in B\ (x,y) \in \varphi\}$$

(sometimes we say that $\varphi^{-1}(B)$ is the *preimage* of $B$ *through* $\varphi$). If $Z$ is another set and a relation $\psi \subset Y \times Z$, then

$$\psi \circ \varphi := \{(x,z) \in X \times Z \mid \exists\, y \in Y\ (x,y) \in \varphi,\ (y,z) \in \psi\}$$

is the *composition* of $\varphi$ and $\psi$.

Let $\pi_X: X \times Y$ and $\pi_Y: X \times Y$ be projections, i.e. $\pi_X(x,y) = x$ and $\pi_Y(x,y) = y$ for any $(x,y) \in X \times Y$. According to the Peano definition, a relation $\varphi \subset X \times Y$ is a *function* or a *map* if

$$\pi_X(\varphi) = X \tag{1.1}$$

and

$$\varphi \circ \varphi^{-1} \subset \Delta_Y := \{(y,y') \in Y \times Y \mid y = y'\}. \tag{1.2}$$

**Definition 1.1.1.** If a relation $\varphi \subset X \times Y$ satisfies condition (1.1), then we say that $\varphi$ is a *set-valued map*.

As in the case of functions, given a set-valued map $\varphi \subset X \times Y$, we write $\varphi: X \multimap Y$ and say that $\varphi(x) := \{y \in Y \mid (x,y) \in \varphi\} = \varphi(\{x\})$ is the *value* of $\varphi$ at $x \in X$; in view of (1.1), $\varphi(x) \neq \emptyset$ for all $x \in X$ ([2]). It is clear that any map $f: X \to Y$ (we then speak of a *single-valued map* in order to make it clear that a map whose values are singletons id considered) is a particular case of a set-valued map; in this case the value $f(x)$ is identified with the singleton $\{f(x)\}$.

In order to incorporate this, perhaps more intuitive approach, one often identifies a set valued map $\varphi: X \multimap Y$ with an ordinary map $\varphi: X \to \mathcal{P}(Y) \setminus \{\emptyset\}$,

(2) In what follows we shall usually make some additional assumptions concerning the values of $\varphi: X \multimap Y$ intimately connected with an additional structure (linear, topological, etc.) imposed onto $Y$.

where $\mathcal{P}(Y)$ stand for the family of all subsets of $Y$. From this point of view it is convenient to define the *graph*

$$\mathrm{Gr}(\varphi) := \{(x,y) \in X \times Y \mid y \in \varphi(x)\}$$

of $\varphi$, although, as we see, the distinction between $\varphi$ and $\mathrm{Gr}(\varphi)$ is only formal.

Given a set-valued map $\varphi: X \multimap Y$ and $B \subset Y$, apart from the preimage $\varphi^{-1}(B)$, it is convenient to define the *strict* or the *small preimage* of $B$

$$\varphi^{+1}(B) := \{x \in X \mid \varphi(x) \subset B\}.$$

Clearly $\varphi^{+1}(B) \subset \varphi^{-1}(B)$.

If $\varphi': X' \multimap Y'$, then one defines the *Cartesian product* $\varphi \times \varphi': X \times X' \multimap Y \times Y'$ by

$$\varphi \times \varphi'(x,x') := \varphi(x) \times \varphi'(x'), \quad (x,x') \in X \times X'.$$

Apart from some obvious properties of images, preimages and strict preimages of set-valued maps, it is not difficult to establish the following issues.

**Proposition 1.1.2.** *Given a set-valued map $\varphi: X \multimap Y$, let $A \subset X$, $B \subset Y$. Then:*

(a) $\varphi^{+1}(Y \setminus B) = X \setminus \varphi^{-1}(B)$ *and* $\varphi^{-1}(Y \setminus B) = X \setminus \varphi^{+1}(B)$*;*
(b) $A \subset \varphi^{+1}(\varphi(A)) \subset \varphi^{-1}(\varphi(A))$*;*
(c) $\varphi(\varphi^{+1}(B)) \subset B \subset \varphi(\varphi^{-1}(B))$*;*
(d) $(\varphi^{-1})^{-1}(A) = \varphi(A)$*;*
(e) $\varphi(A) = \pi_Y(\mathrm{Gr}(\varphi) \cap A \times Y)$*;* $\varphi^{-1}(B) = \pi_X(\mathrm{Gr}(\varphi) \cap X \times B)$.

*If $\psi: Y \multimap Z$ is another set-valued map and $C \subset Z$, then:*

(f) $(\psi \circ \varphi)^{\pm 1}(C) = \varphi^{\pm 1}(\psi^{\pm 1}(C))$*;*
(g) $\mathrm{Gr}(\psi \circ \varphi) = (\varphi \times \mathrm{id}_Z)^{-1}(\mathrm{Gr}(\psi)) = (\mathrm{id}_X \times \psi)(\mathrm{Gr}(\varphi))$ [3].

To facilitate the notation, in what follows we put

$$p_\varphi := \pi_X|_{\mathrm{Gr}(\varphi)}, \quad q_\varphi := \pi_Y|_{\mathrm{Gr}(\varphi)}.$$

Then, for $A \subset X$ and $B \subset Y$,

$$\mathrm{Gr}(\varphi) \cap A \times Y = p_\varphi^{-1}(A), \quad \mathrm{Gr}(\varphi) \cap X \times Y = q_\varphi^{-1}(B).$$

Thus, for example, equalities (e) read

$$\varphi(A) = q_\varphi(p_\varphi^{-1}(A)), \quad \varphi^{-1}(B) = p_\varphi(q_\varphi^{-1}(B)).$$

**1.2. Continuity of set-valued maps.** Assume that $X$ and $Y$ are Hausdorff topological spaces (only such spaces will be considered below) and let $\varphi: X \multimap Y$ be a set-valued map.

[3] $\mathrm{id}_Z$ (resp. $\mathrm{id}_X$) stands for the identity on $Z$ (resp. $X$).

**Definition 1.2.1.** Let $x \in X$. We say that $\varphi$ is *upper* (resp. *lower*) *semicontinuous at* $x$ if, for any open set $V$, if $\varphi(x) \subset V$ (resp. $\varphi(x) \cap V \neq \emptyset$), then $\varphi^{+1}(V)$ (resp. $\varphi^{-1}(V)$) is a neighbourhood of $x$. As usual we say that $\varphi$ is *upper* (resp. *lower*) *semicontinuous* if so it is at any point $x \in X$. A set-valued map being simultaneously upper and lower semicontinuous is called *continuous*.

It is easy to see that in case of a single-valued map, lower and upper semicontinuity coincide with the ordinary continuity.

By the very definition and in view of Proposition 1.1.2(a) and (f), we have:

**Proposition 1.2.2.** *Let $\varphi: X \multimap Y$. The following conditions are equivalent:*

(a) *$\varphi$ is upper semicontinuous;*
(b) *for any open $V \subset Y$, $\varphi^{+1}(V)$ is open;*
(c) *for any closed $C \subset Y$, $\varphi^{-1}(C)$ is closed.*

*In a similar manner, the following conditions are equivalent:*

(i) *$\varphi$ is lower semicontinuous;*
(ii) *for any open $V \subset Y$, $\varphi^{-1}(V)$ is open;*
(iii) *for any closed $C \subset Y$, $\varphi^{+1}(C)$ is closed.*

*The composition of upper* (*resp. lower*) *semicontinuous is upper* (*resp. lower*) *semicontinuous.*

**Example 1.2.3.** (a) Let $J \subset \mathbb{R}$ and $\varphi: \mathbb{R} \multimap \mathbb{R}$ be given by

$$\varphi(x) = \begin{cases} \{0\} & \text{if } x \in \mathbb{R} \setminus J, \\ [-1, 1] & \text{if } x \in J. \end{cases}$$

If $J = [a, b]$, then $\varphi$ upper semicontinuous; if $J = (a, b)$, then $\varphi$ is lower semicontinuous; if e.g. $J = [a, b)$, then $\varphi$ is neither upper nor lower semicontinuous.

(b) Let $f, g: X \to \mathbb{R}$ be lower and upper semicontinuous real functions, respectively, such that $f(x) \leq g(x)$ for all $x \in X$. Then a map $\varphi: X \multimap \mathbb{R}$, given by $\varphi(x) = [f(x), g(x)]$ for $x \in X$, is upper semicontinuous. If $g(x) \leq f(x)$ on $X$, then $\psi: X \multimap \mathbb{R}$, given by $\varphi(x) := [g(x), f(x)]$ for $x \in X$, is lower semicontinuous.

(c) If $p: Y \to X$ is a surjection, then $\varphi: X \multimap Y$ given by $\varphi(x) = p^{-1}(x)$ for $x \in X$, is upper (resp. lower) semicontinuous if and only if $p$ is closed (resp. open). To see this recall that, for any $B \subset Y$, $p(B) = (p^{-1})^{-1}(B) = \varphi^{-1}(B)$.

(d) Let $Z$ be a set and let $p: Z \to X$, $q: Z \to Y$. If $p$ is a surjection, then a set-valued map $\varphi: X \multimap Y$ given by $\varphi(x) = q(p^{-1}(x))$, $x \in X$, is upper (resp. lower) semicontinuous if and only if $p$ is closed (resp. open) with respect to the weakest topology on $Z$ under which $q$ is continuous. In view of the sufficiency part of (c) it is enough to show the necessity: we are to prove that $p$ is closed (resp. open) provided $\varphi$ is upper (resp. lower) semicontinuous. In view of the necessity part of (c) we shall show that $p^{-1}: X \multimap Z$ is upper (resp. lower) semicontinuous. Let

$U$ be open in $Z$; without loss of generality we may assume that $U = q^{-1}(V)$ where $V$ is open in $Y$. Thus $(p^{-1})^{\pm 1}(U) = (q \circ p^{-1})^{\pm 1}(V)$ is open.

(e) If in (d) $Z$ is a topological space, $q$ is continuous and $p$ is closed (resp. open), then $\varphi$ is upper (resp. lower) semicontinuous. This follows from (c) and the last part of Proposition 1.2.2.

**Remark 1.2.4.** The above example (d) is in a sense universal. Given a set-valued map $\varphi\colon X \multimap Y$, we see that, for each $x \in X$, $\varphi(x) = q_\varphi(p_\varphi^{-1}(x))$. Hence if $p_\varphi$ is closed (resp. open), then $\varphi$ is upper (resp. lower) semicontinuous. If $\varphi$ is upper (resp. lower) semicontinuous, then $p_\varphi$ is closed (resp. open) with respect to the weakest topology on $\mathrm{Gr}(\varphi)$ under which $q_\varphi$ is continuous (see also [45], [46]).

Let us now collect some other properties of upper (or lower) semicontinuous maps.

**Proposition 1.2.5.**

(a) *If $Y$ is regular, $\varphi\colon X \multimap Y$ is upper semicontinuous and has closed values, then its graph $\mathrm{Gr}(\varphi)$ is closed (in $X \times Y$)* (4).
(b) *A set-valued map $\varphi\colon X \to Y$ has compact values and is upper semicontinuous if and only if $p_\varphi$ is perfect* (5).
(c) *If $\varphi\colon X \multimap Y$ is upper semicontinuous and has compact values, $A \subset X$ is compact, then so is $\varphi(A)$.*
(d) *If $\varphi$ is as above, $\psi\colon X \multimap Y$ has closed graph and, for each $x \in X$, $\varphi(x) \cap \psi(x) \neq \emptyset$, then $\varphi \cap \psi$ is upper semicontinuous.*

*Proof.* Condition (a) is easy. We shall prove (b). If $p_\varphi$ is perfect, then for each $x \in X$, $\varphi(x) = q_\varphi(p_\varphi^{-1}(x))$ is compact; moreover $p_\varphi$ is closed, so $\varphi$ is upper semicontinuous. If $\varphi$ is upper semicontinuous and has compact values, then for each $x \in X$, $p_\varphi^{-1} = \{x\} \times \varphi(x)$ is compact. Let $B \subset \mathrm{Gr}(\varphi)$ be closed and let $x \notin p(B)$, then $\{x\} \times \varphi(x) \cap B = \emptyset$. The compactness of $\varphi(x)$ implies that there are open sets $U' \subset X$ and $V \subset Y$ such that $x \in U'$, $\varphi(x) \subset V$ and $U' \times V \cap B = \emptyset$. The upper semicontinuity implies that $U'' := \varphi^{+1}(V)$ is open. Let $U := U' \cap U''$. Then $x \in U$ and $U \cap p(B) = \emptyset$. This shows that $p(B)$ is closed.

To see (c) observe that $\varphi(A) = q_\varphi(p_\varphi^{-1}(A))$. Since $p_\varphi$ is perfect, it is proper; hence $p_\varphi^{-1}(A)$ is compact.

(d) Assume that $B$ is closed in $\mathrm{Gr}(\varphi \cap \psi) = \mathrm{Gr}(\varphi) \cap \mathrm{Gr}(\psi)$. Since $\mathrm{Gr}(\psi)$ is closed, $B$ is closed in $\mathrm{Gr}(\varphi)$. Hence $p_{\varphi\cap\psi}(B) = p_\varphi(B)$ is closed and, thus, $\varphi \cap \psi$ is upper semicontinuous. $\square$

(4) Clearly if the $\mathrm{Gr}(\varphi)$ is closed, then so are the values of $\varphi$.

(5) Recall that a map $f\colon X \to Y$ between topological spaces is *perfect* if it is closed and, for each $y \in Y$, the fiber $f^{-1}(y)$ is compact. Each perfect map is *proper*, i.e. such that, for any compact $K \subset Y$, $f^{-1}(K)$ is compact. If $Y$ is a $k$-space (i.e. $Y$ is compactly generated), then $f$ is perfect if and only if it is proper; in particular this holds if $Y$ is a metric space.

**Example 1.2.6.** Generally speaking set-valued maps with closed graphs are not upper semicontinuous. For instance let $\varphi\colon \mathbb{R} \multimap \mathbb{R}^2$ be given by $\varphi(a) = \{(x,y) \in \mathbb{R}^2 \mid y = ax\}$ for $a \in \mathbb{R}$. Then $\mathrm{Gr}(\varphi)$ is closed, but for $a_0 = 0$ the inclusion $\varphi(a) \subset B(\varphi(0),\varepsilon)$ holds for neither $a \neq 0$ nor $\varepsilon > 0$.

**Definition 1.2.7.** We say that $\varphi\colon X \multimap Y$ is *compact* (resp. *locally compact*) if the closure $\mathrm{cl}\,\varphi(X)$ of $\varphi(X)$ is compact (resp. each $x \in X$ has a neighbourhood $U$ such that $\mathrm{cl}\,\varphi(U)$ is compact).

By Proposition 1.2.5(d), if $\varphi$ is locally compact and has closed graph, then $\varphi$ is upper semicontinuous.

**1.3. Continuity in metric spaces.** Suppose now that $X$, $Y$ are metric spaces and let $\varphi\colon X \multimap Y$. As a simple consequence of Definition 1.2.1 and Proposition 1.2.5 we get

**Corollary 1.3.1.**

(a) *The map $\varphi$ is lower semicontinuous at $x_0 \in X$ if and only if, for any $y_0 \in \varphi(x_0)$ and a sequence $x_n \to x_0$, there is a sequence $y_n \to y_0$ such that $y_n \in \varphi(x_n)$ for all $n \in \mathbb{N}$.*

(b) *The map $\varphi$ is upper semicontinuous and has compact values if and only if, given a sequence $(x_n, y_n) \in \mathrm{Gr}(\varphi)$, if $x_n \to x_0$, then there is a subsequence $(y_{n_k})$ such that $y_{n_k} \to y_0 \in \varphi(x_0)$.*

**Example 1.3.2.** Let $I = [0,T]$, $T > 0$. Consider a Carathéodory function $f\colon I \times \mathbb{R}^N \to \mathbb{R}^N$ (i.e. for almost all $t \in I$, $f(t,\cdot\,)$ is continuous and, for all $x \in \mathbb{R}^N$, $f(\,\cdot\,,x)$ is measurable). Assume that there is $c \in L^1(I,\mathbb{R})$ such that $\|f(t,x)\| \leq c(t)$ for almost all $t \in I$ and all $x \in \mathbb{R}^N$. It is well-known that, for any $a \in E$, the set $S(a)$ of all solutions to the problem $x' = f(t,x)$, $x(0) = a$ (i.e. $x \in S(x_0)$ if and only if $x\colon I \to \mathbb{R}^N$ is continuous and $x(t) = a + \int_0^t f(s,x(s))\,ds$ for all $t \in I$) is nonempty. We shall see that the map $\mathbb{R}^N \ni x_0 \multimap S(x_0) \subset C(I,\mathbb{R}^N)$ has compact values and is upper semicontinuous. To this end let $(a_n, x_n) \in \mathrm{Gr}(S)$ and suppose that $a_n \to a \in \mathbb{R}^N$. Then, for all $t \in I$, $x_n(t) = a_n + \int_0^t y_n(s)\,ds$ where $y_n(s) = f(s,x_n(s))$ for $s \in I$. Clearly the sequence $(x_n)$ is uniformly bounded and equicontinuous. By the Ascoli–Arzela theorem, there is a subsequence $x_{n_k} \to x \in C(I,\mathbb{R}^N)$ uniformly. For almost all $s \in I$,

$$\lim_{k\to\infty} y_{n_k}(s) = \lim_{k\to\infty} f(s,x_{n_k}(s)) = f(s,x(s)).$$

Thus, by the Lebesgue theorem,

$$x(t) = \lim_{k\to\infty} x_{n_k}(t) = a + \lim_{k\to\infty} \int_0^t y_{n_k}(s)\,ds = a + \int_0^t f(s,x(s)\,ds$$

and $x \in S(a)$.

Continuity of set valued maps in metric spaces may be studied in terms of the so-called lower and upper set-limits. Assume that $A \subset X$, $\varphi\colon A \multimap Y$ and let $x_0 \in \operatorname{cl} A$. Then, the (topological) *lower set-limit* $\operatorname{Lim\,inf}_{x\to x_0} \varphi(x)$ and the *upper set-limit* $\operatorname{Lim\,sup}_{x\to x_0} \varphi(x)$ are subsets of $Y$ defined as follows:

$$y \in \operatorname*{Lim\,inf}_{x\to x_0} \varphi(x) \ \Leftrightarrow\ \forall\, (x_n)_{n=1}^{\infty},\ x_n \ \xrightarrow{A}\ x_0\ \exists\, y_n \in \varphi(x_n),\ y_n \to y$$

and

$$y \in \operatorname*{Lim\,sup}_{x\to x_0} \varphi(x) \ \Leftrightarrow\ \exists\, (x_n)_{n=1}^{\infty},\ x_n \ \xrightarrow{A}\ x_0\ \exists\, y_n \in \varphi(x_n),\ y_n \to y,$$

where the notation $x_n \xrightarrow{A} x_0$ means that $x_n$ tends to $x_0$ staying in $A$. These limits admit also the following description

$$\operatorname*{Lim\,inf}_{x\to x_0} \varphi(x) = \bigcap_{\varepsilon>0} \bigcup_{\eta>0} \bigcap_{x\in B_A(x_0,\eta)} B(\varphi(x),\varepsilon),$$

$$\operatorname*{Lim\,sup}_{x\to x_0} \varphi(x) = \bigcap_{\varepsilon>0} \bigcap_{\eta>0} \bigcup_{x\in B_A(x_0,\eta)} B(\varphi(x),\varepsilon) = \bigcap_{\eta>0} \operatorname{cl}\Bigg( \bigcup_{x\in B_A(x_0,\eta)} \varphi(x)\Bigg),$$

where

$$B_A(x_0,\eta) := \{x \in A \mid d(x,x_0) < \eta\} \text{ and } B(\varphi(x),\varepsilon) := \{y \in Y \mid d(y,\varphi(x)) < \varepsilon\}.$$

In a particular discrete case: given a sequence $(A_n)_{n=1}^{\infty}$ of subsets in $X$, then

$$\begin{aligned}\operatorname*{Lim\,inf}_{n\to\infty} A_n &:= \Big\{x \in X \ \Big|\ \lim_{n\to\infty} d(x,A_n) = 0\Big\},\\ \operatorname*{Lim\,sup}_{n\to\infty} A_n &:= \Big\{x \in X \ \Big|\ \liminf_{n\to\infty} d(x,A_n) = 0\Big\}.\end{aligned}$$

If $\operatorname{Lim\,inf}_{n\to\infty} A_n = \operatorname{Lim\,sup}_{n\to\infty} A_n$, then this set is denoted by $\operatorname{Lim}_{n\to\infty} A_n$ and called the *limit* of the sequence $(A_n)$.

**Remark 1.3.3.** One has to be cautious when dealing with set-limits. Our definitions may differ from those provided elsewhere. Namely, sometimes different authors define them as:

$$\begin{aligned}\operatorname*{Lim\,inf}_{x\to x_0} \varphi(x) &= \bigcap_{\varepsilon>0} \bigcup_{\eta>0} \bigcap_{x\in B_A(x_0,\eta)\setminus\{x_0\}} B(\varphi(x),\varepsilon),\\ \operatorname*{Lim\,sup}_{x\to x_0} \varphi(x) &= \bigcap_{\varepsilon>0} \bigcap_{\eta>0} \bigcup_{x\in B_A(x_0,\eta)\setminus\{x_0\}} B(\varphi(x),\varepsilon)\\ &= \bigcap_{\eta>0} \operatorname{cl}\Bigg( \bigcup_{x\in B_A(x_0,\eta)\setminus\{x_0\}} \varphi(x)\Bigg)\end{aligned}$$

or, equivalently,

$$\operatorname*{Lim\,inf}_{x\to x_0} := \Big\{ y \in Y \;\Big|\; \lim_{x\to x_0} d(y,\varphi(x)) = 0 \Big\},$$
$$\operatorname*{Lim\,sup}_{x\to x_0} := \Big\{ y \in Y \;\Big|\; \liminf_{x\to x_0} d(y,\varphi(x) \Big\}.$$

Obviously, if $x_0 \in \operatorname{cl} A$ but $x_0 \notin A$ or in the discrete case, then our definitions and those above coincide (compare [4]).

It is easy to see that both upper and lower limits are closed,

$$\operatorname*{Lim\,inf}_{x\to x_0} \varphi(x) \subset \operatorname*{Lim\,sup}_{x\to x_0} \varphi(x)$$

and, if $x_0 \in A$, then

$$\operatorname*{Lim\,inf}_{x\to x_0} \varphi(x) \subset \operatorname{cl}\varphi(x_0) \subset \operatorname*{Lim\,sup}_{x\to x_0} \varphi(x).$$

By the very definition we have

**Proposition 1.3.4.** *Let $\varphi: X \multimap Y$.*

(a) *The map $\varphi$ is lower semicontinuous at $x_0 \in X$ if and only if*

$$\varphi(x_0) \subset \operatorname*{Lim\,inf}_{x\to x_0} \varphi(x).$$

(b) *The map $\varphi$ has closed graph if and only if, for each $x_0 \in X$,*

$$\operatorname*{Lim\,sup}_{x\to x_0} \varphi(x) = \varphi(x_0).$$

We thus see that lower limits help to study lower semicontinuity, while upper limits may be used to establish the closeness of the graph.

Finally let us mention the following result (see [4]). Recall that a set $A \subset X$ is *residual* if $A = \bigcap_{n=1}^{\infty} A_n$ and, for all $n \in \mathbb{N}$, $A_n$ is open and dense in $X$ (i.e. $X \setminus A_n$ is a nowhere dense). In other words $A$ is residual if its complement is contained a set of the first Baire category. Countable intersections of residual sets are residual. In view of the Baire theorem residual subsets of a complete space are dense. A property that holds along a residual set is called a *generic* property.

**Proposition 1.3.5** (Generic continuity)**.** *Let $X$, $Y$ be complete metric spaces and let $\varphi: X \multimap Y$. Then:*

(a) *If $\varphi$ is upper (or lower) semicontinuous, then it is continuous on some residual subset of $X$.*

(b) *If $\varphi$ has closed values and is lower semicontinuous, then there is a residual set $R \subset X$ such that, for all $x_0 \in X$,*

$$\operatorname*{Lim\,sup}_{x\to x_0} \varphi(x) = \varphi(x_0).$$

Given nonempty sets $A, B \subset X$, let

$$h(A,B) := \sup_{a\in A} d(a,B) \in [0,\infty].$$

It is clear that if $A$ is bounded, then $h(A,B) < \infty$. Observe that

$$h(A,B) = \sup_{x\in X}(d(x,B) - d(x,A)).$$

Let, the *Hausdorff distance*

$$d_H(A,B) := \max\{h(A,B), h(B,A)\}.$$

We easily see that

$$d_H(A,B) = \sup_{x\in X}|d(x,A) - d(x,B)| = d_H(B,A).$$

If the sets $A, B$ are bounded, then $d_H(A,B) < \infty$ (not necessarily conversely); if $A = B$, then $d_H(A,B) = 0$; if $d_H(A,B) = 0$ and $A, B$ are closed, then $A = B$. It is also easy to see that, for any $C \subset X$, $d_H(A,B) \le d_H(A,C) + d_H(C,B)$. Thus $d_H$ is a metric in a (hyper)space $\mathcal{B}C(X)$ of all bounded closed subsets of $X$. It is also easy to show that $\mathcal{B}C(X)$ together with $d_H$ is complete (resp. compact) provided so is $X$.

Suppose now that a sequence $(A_n)$ of sets in $X$ is given and let $A \subset X$. Suppose that $h(A, A_n) < \infty$ (resp. $h(A_n, A) < \infty$) for all $n \in \mathbb{N}$. The the following implications hold:

(1) *if* $\lim_{n\to\infty} h(A, A_n) = 0$, *then* $\operatorname{cl} A \subset \operatorname{Lim\,inf}_{n\to\infty} A_n$*;*
(2) *if* $\lim_{n\to\infty} h(A_n, A) = 0$, *then* $\operatorname{Lim\,sup}_{n\to\infty} A_n \subset \operatorname{cl} A$.

Hence

(3) *if* $\lim_{n\to\infty} d_H(A_n, A) = 0$, *then* $\operatorname{cl} A = \operatorname{Lim}_{n\to\infty} A_n$.

Converse implications does not hold in general (it is easy to provide counterexamples). The reason is quite simple: Hausdorff limits are of the 'uniform' character, while set-limit have rather the 'pointwise' character. As concerns (1), the converse statement is true provided $A$ is compact (or relatively compact). In order to have statements converse to (2) (or (3)) one has to assume e.g. that $X$ is compact. Facts analogous to the above ones are true also if a discrete sequence is replaced with a 'continuous' family, i.e. a set-valued map $\varphi\colon B \multimap X$ defined on a subset $B \subset Y$ and respective limits considered when $y \xrightarrow{Y} y_0$ where $y_0 \in \operatorname{cl} B$.

The above discussion corresponds well to the notion of Hausdorff continuity of set-valued maps. Suppose that $\varphi\colon X \multimap Y$ is a set-valued map and let $x_0 \in X$.

**Definition 1.3.6.** We say that $\varphi$ is *$H$-upper* (resp. *lower*) *semicontinuous at* $x_0$ provided, for each $\varepsilon > 0$, there is $\delta > 0$ such that if $x \in X$ and $d(x, x_0) < \delta$, then $h(\varphi(x), \varphi(x_0)) < \varepsilon$ (resp. $h(\varphi(x_0), \varphi(x)) < \varepsilon$). Obviously we say that $\varphi$ is

*$H$-continuous* at $x_0$ if it is $H$-upper and lower semicontinuous at $x_0$, simultaneously. The map is *$H$-upper* (resp. *lower*) *semicontinuous* or *$H$-continuous* if so it is at every $x_0 \in X$.

It is easy to establish the following fact.

**Proposition 1.3.7.** *If $\varphi$ is upper semicontinuous at $x_0$, then it is $H$-upper semicontinuous at $x_0$. If $\varphi$ is $H$-lower semicontinuous at $x_0$, then it is lower semicontinuous at $x_0$. The converse implications hold if $\varphi(x_0)$ is compact. In particular $H$-continuity in equivalent to the continuity in case of set-valued maps with compact values.*

Finally we say that $\varphi: X \multimap Y$ is Lipschitz (precisely *$L$-Lipschitz*, where $L \geq 0$) if, for all $x, y \in X$,

$$d_H(\varphi(x), \varphi(y)) \leq Ld(x, y).$$

In a similar manner one may define the *local Lipschitz continuity* of a set-valued map. We sat that $\varphi$ is a *contraction* (precisely *$k$-contraction*) if it is $k$-Lipschitz with $0 \leq k < 1$.

The notions of upper and lower set-limits may be defined for set-valued maps defined and/or having values in topological spaces. For that reason suppose that $A$ is a subset of a Hausdorff topological space $X$, $x_0 \in \operatorname{cl} A$, and let $\varphi: A \multimap Y$ where $Y$ is another Hausdorff topological space. By definition $y \in \operatorname{Lim\,inf}_{x \to x_0} \varphi(x)$ (resp. $y \in \operatorname{Lim\,sup}_{x \to x_0} \varphi(x)$) if for every (resp. there is a) generalized sequence $(x_\lambda)_{\lambda \in \Lambda}$ such that $x_\lambda \xrightarrow{A} x_0$, for each $\lambda \in \Lambda$, there is $y_\lambda \in \varphi(x_\lambda)$ such that $y_\lambda \to y$ (6). Results similar to described above are the still true.

In particular, given a metric space $X$, $A \subset X$, $x_0 \in \operatorname{cl} A$, a normed space $E$ and a set-valued map $N: X \multimap E^*$ (where $E^*$ denotes the (topological) dual of $E$) one may speak of the weak$^*$-set-limits $w^*\text{-}\operatorname{Lim\,inf}_{x \to x_0} \varphi(x)$ and $w^*\text{-}\operatorname{Lim\,sup}_{x \to x_0} \varphi(x)$, i.e. set-limits in $E^*$ when $E^*$ is endowed with the weak$^*$-topology.

In order to illustrate these issues let us discuss the following

**Lemma 1.3.8** (Walkup–Wets formula [71])**.** *Let $T: X \multimap E$ be a set-valued map such that, for each $x \in X$, $T(x)$ is a closed convex cone. Let $x_0 \in X$ and suppose that, for each $x \in X$,*

$$N(x) := T(x)^\perp := \{p \in E^* \mid \langle p, y \rangle \leq 0 \text{ for all } y \in T(x)\}.$$

*Then $N: X \multimap E^*$ and*

$$\operatorname*{Lim\,inf}_{x \to x_0} T(x) = \Big[w^*\text{-}\operatorname*{Lim\,sup}_{x \to x_0} N(x)\Big]^\perp.$$

(6) Clearly, if $X$, $Y$ are metric spaces, then both notions provided here and above coincide.

*Proof.* Let $y_0 \in \operatorname{Lim\,inf}_{x\to x_0} T(x)$ and let $p_0 \in w^*\text{-}\operatorname{Lim\,sup}_{x\to x_0} N(x)$: we are to show that $\langle p_0, y_0\rangle \le 0$. There is a (generalized) sequence $x_\lambda \to x_0$, $\lambda \in \Lambda$, and $p_\lambda \in N(x_\lambda)$ such that $p_\lambda \to p_0$ weakly*. There also exists a sequence $y_\lambda \in T(x_\lambda)$ such that $y_\lambda \to y_0$. Hence $\langle p_\lambda, y_\lambda\rangle \le 0$ and, therefore, $\langle p_0, y_0\rangle \le 0$.

Conversely suppose that $y_0 \in [w^*\text{-}\operatorname{Lim\,sup}_{x\to x_0} N(x)]^\perp$ but $y_0$ is not contained in $\operatorname{Lim\,inf}_{x\to x_0} T(x)$. Hence there is $\varepsilon > 0$ and a (generalized) sequence $x_\lambda \to x_0$, $\lambda \in \lambda$, such that $B(y_0, \varepsilon) \cap T(x_\lambda) = \emptyset$ for all $\lambda \in \Lambda$. The separation theorem implies that, for each $\lambda \in \Lambda$, there is a form $p_\lambda \in E^*$ such that

$$\sup_{y\in T(x_\lambda)} \langle p, y\rangle \le \langle p_\lambda, y_0\rangle - \varepsilon \|p_\lambda\|.$$

We may assume that $\|p_\lambda\| = 1$; then

$$\sup_{y\in T(x_\lambda)} \langle p_\lambda, y\rangle \le \langle p_\lambda, y_0\rangle - \varepsilon.$$

Since $T(x_\lambda)$ is a cone, we infer that $\sup_{y\in T(x_\lambda)}\langle p_\lambda, y\rangle = 0$, i.e. $p_\lambda \in N(x_\lambda)$ and $\varepsilon \le \langle p_\lambda, y_0\rangle$. By the Alaoglu theorem, we may assume without loss of generality that $p_\lambda \to p_0$ weakly* and $\|p_0\| \le 1$. Clearly $p_0 \in w^*\text{-}\operatorname{Lim\,sup}_{x\to x_0} N(x)$; thus $\langle p_0, y_0\rangle \le 0$. But $\varepsilon \le \langle p_0, y_0\rangle$: contradiction. □

**Corollary 1.3.9.** *Under the above assumption $T$ is lower semicontinuous if and only if* $\operatorname{Gr}(N)$ *is closed in* $X\times E^*$ (*where* $E^*$ *is endowed with weak*-topology*).

*Proof.* We show necessity. Let a (generalized) sequence $(x_\lambda, p_\lambda) \in \operatorname{Gr}(N)$, $\lambda \in \Lambda$, and let $x_\lambda \to x$ and $p_\lambda \to p$ weakly*. Suppose that $p \notin N(x)$. Hence there is $y \in T(x)$ such that $\langle p, y\rangle > 0$. Since $T$ is lower semicontinuous, there is a sequence $(y_\lambda)$ in $E$ such that $y_\lambda \in T(x_\lambda)$ and $y_\lambda \to y$. It is clear that then $\langle p_\lambda, y_\lambda\rangle \le 0$ and $\langle p_\lambda, y_\lambda\rangle \to \langle p, y\rangle$ in $\mathbb{R}$: contradiction.

To see sufficiency, observe that the closeness of $\operatorname{Gr}(N)$ implies that, for each $x_0 \in X$, $w^*\text{-}\operatorname{Lim\,sup}_{x\to x_0} N(x) \subset N(x_0)$. Therefore, by the Walkup-Wets formula

$$T(x_0) = N(x_0)^\perp \subset \Big[w^*\text{-}\operatorname*{Lim\,sup}_{x\to x_0} N(x)\Big]^\perp = \operatorname*{Lim\,inf}_{x\to x_0} T(x). \qquad \square$$

**Remark 1.3.10.** An immediate application of Corollary 1.3.9 concerns regularity of the so-called tangent cones (see [3], [4] and [64] where the finite-dimensional situation has been discussed). Suppose that $K \subset E$, where $E$ is a normed space, is closed and let $x \in K$. By the *Bouligand tangent cone to $K$ at $x$* we mean the set

$$T_K(x) := \operatorname*{Lim\,sup}_{h\to 0^+} \frac{K - x}{h}.$$

In other words $v \in T_K(x)$ if and only if there are sequences $h_n \to 0^+$ and $v_n \to v$ such that $v_n \in (K - x)/h_n$, i.e. $x + h_n v_n \in K$, for all $n \ge 1$. By the *Clarke* *tangent cone to* $K$ *at* $x$ we mean the set

$$C_K(x) := \operatorname*{Lim\,inf}_{y \xrightarrow{K} x,\, h\to 0^+} \frac{K-y}{h}.$$

Hence $v \in C_K(x)$ if and only if, for any sequences $y_n \to x$ and $h_n \to 0^+$, there is a sequence $v_n \to v$ such that $v_n \in (K-y_n)/h_n$, i.e. $y_n + h_n v_n \in K$, for all $n \ge 1$.

It is easy to see that, for each $x \in K$, $T_K(x)$ and $C_K(x)$ are closed cones (i.e. given $v \in T_K(x)$ and $\lambda \ge 0$, $\lambda v \in T_K(x)$ and the same for $C_K(x)$). Additionally, $C_K(x)$ is convex. Moreover, $C_K(x) \subset T_K(x)$. An important result states that

$$\operatorname*{Lim\,inf}_{y \xrightarrow{K} x} T_K(y) \subset C_K(x)$$

with equality in case $\dim E < \infty$.

By the *normal cone to* $K$ *at* $x$ we mean the set

$$N_K(x) := C_K(x)^{\perp} := \{p \in E^* \mid \langle p, v\rangle \le 0 \text{ for all } v \in C_K(x)\}.$$

Suppose that $K \subset E$ is closed convex. It is an easy exercise to show that in this case

$$T_K(x) = C_K(x) = S_K(x) := \operatorname{cl}\biggl(\bigcup_{h>0} \frac{K-x}{h}\biggr). \tag{$*$}$$

By $(*)$, $N_K(x) = \{p \in E^* \mid \sup_{y\in K}\langle p, y-x\rangle \le 0\}$. It is easy to see that the graph $\operatorname{Gr}(N_K)$ of the set-valued map $K \ni x \multimap N_K(x)$ is closed in $K \times E^*$ where $E^*$ has the weak$^*$-topology. Indeed, assume that a (generalized) sequence $(x_\lambda, p_\lambda) \in \operatorname{Gr}(N_K)$, $\lambda \in \Lambda$, is given and $(x_\lambda, p_\lambda) \to (x,p)$, i.e. $x_\lambda \to x$ and $p_\lambda \to p$ weakly$^*$. Hence, for each $y \in K$, $\langle y - x_\lambda, p_\lambda\rangle \le 0$ and, therefore, $\langle y-x, p\rangle = \lim_{\lambda\in\lambda}\langle y-x_\lambda, p_\lambda\rangle \le 0$, i.e. $p \in N_K(x)$.

In view of Corollary 1.3.9, the cone map $K \ni x \multimap T_K(x)$ is lower semicontinuous (see also [64]).

Apart from the above notion of the normal cone $N_K(x)$, the notion of the *proximal normal cone* is sometimes studied. Let, as above, $K \subset E$ be closed. The map $\pi_K$ assigning to each $y \in E$ the (possibly empty) set $\pi_K(y) := \{z \in K \mid \|y-z\| = d_K(y)\}$, called the *metric projection*, has closed graph. If $E$ is a Hilbert space and $K$ is closed convex, then $\pi_K(y)$ is a singleton for any $y \in E$, but, in general, even if $\dim E < \infty$, $\pi_K(y)$ is a set. If $x \in K$, then the set $\pi_K^{-1}(x)$ is always nonempty (since $\pi_K(x) = x$). Let

$$PN_K(x) := \bigcup_{\lambda>0} \lambda(\pi_K^{-1}(x) - x)$$

be the *proximal normal cone*. Observe that if $E$ is a Hilbert space with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ ($K$ is arbitrary), then $\langle v, u\rangle \le 0$ for any $v \in PN_K(x)$

and $u \in \operatorname{cl}\operatorname{conv} T_K(x)$, i.e. $\operatorname{cl}\operatorname{conv} T_K(x) \subset PN_K(x)^\perp$ (see also [64]); moreover, $C_K(x) = N(x,K)^\perp$ where

$$N(x,K) = \operatorname*{Lim\,sup}_{y \xrightarrow{K} x} PN_K(y).$$

For a general discussion of similar problems — see [58].

Below we shall establish the lower semicontinuity of $T_K(\,\cdot\,)$ as a byproduct of some approximation issues.

## 2. Approximations and selections

As mentioned above, from a formal point of view, single-valued maps are particular cases of set-valued ones. But the significance of single-valued maps is evidently more essential in the theory of set-valued maps. A lot of set-valued constructions and issues may be reduced to analogous facts for suitably chosen single valued single-valued maps or sequences of such maps.

**Remark 2.0.1.** Let us make here a general remark. In what follows we shall often make use of partitions of unity. Recall that if $X$ is a paracompact space, then any open cover $\mathfrak{A}$ of $X$ admits a partition of unity subordinated to $\mathfrak{A}$. This means that there is a family $\{\lambda_s\}_{s\in S}$ of continuous functions $\lambda_s\colon X \to [0,1]$ such that $\{\operatorname{supp}\lambda_s\}_{s\in S}$, where $\operatorname{supp}\lambda_s := \operatorname{cl}\{x \in X \mid \lambda_s \neq 0\}$, is a locally-finite refinement of $\mathfrak{A}$, i.e. it is a (closed) covering of $X$, for each $s \in S$, there is $U_s \in \mathfrak{A}$ such that $\operatorname{supp}\lambda_s \subset U_s$ and each point $x \in X$ has a neighbourhood $V$ such that $\#\{s \in S \mid \operatorname{supp}\lambda_s \cap V \neq \emptyset\} < \infty$.

It is well-known that any metric space is paracompact. In particular open coverings of a metric space admit partitions of unity. However in this case one can do better. Namely it is easy to show that if $X$ is a metric space, then there are partitions of unity $\{\lambda_s\}_{s\in S}$ such that, for each $s \in S$, $\lambda_s$ is a Lipschitz function. Given a Lipschitz partition of unity $\{\lambda_s\}_{s\in S}$, a point $y_s \in E$, $s \in S$, where $E$ is a metric vector space, the function

$$f(x) := \sum_{s\in S} \lambda_s(x) y_s, \quad x \in X,$$

is well-defined continuous (this is evident) and *locally Lipschitz*. The reader is kindly asked to take this observation into account always when partition of unity arguments for maps between metric spaces are used.

It is also worthwhile to recall that given a paracompact space and an open cover $\mathfrak{A}$ of $X$, there exists an open *point-star-refinement* $\mathfrak{B}$ of $\mathfrak{A}$, i.e. an open cover $\mathfrak{B}$ of $X$ such that, for each $x \in X$, there is $U_x \in \mathfrak{A}$, such that the *star* $\operatorname{st}(x,\mathfrak{B}) := \bigcup\{V \in \mathfrak{B} \mid x \in V\} \subset U_x$. Similarly there is an open *star-refinement* $\mathfrak{B}$ of $\mathfrak{A}$, i.e. an open cover $\mathfrak{B}$ of $X$ such that the *stars* $\operatorname{st}(V,\mathfrak{B}) := \bigcup\{W \in \mathfrak{B} \mid W \cap V \neq \emptyset\}$, $V \in \mathfrak{B}$, refine $\mathfrak{A}$ (see [37]).

**2.1. Selections.** Suppose that $\varphi: X \multimap Y$ is set-valued map between two sets $X$ and $Y$. By a *section* or a *selection of* $\varphi$ we understand a single-valued map $f: X \to Y$ such that $f(x) \in \varphi(x)$ for all $x \in X$. Obviously we are interested in the existence of selections satisfying additional properties (continuity, measurability etc.) for the very existence is guaranteed by the axiom of choice. The best know result in this direction is the celebrated Michael theorem (compare [68]).

**Theorem 2.1.1** (Michael, [62]). *Suppose that $X$ is a paracompact space, $E$ is a Fréchet space (a completely metrizable locally convex linear topological space) and let $\varphi: X \multimap E$ be lower semicontinuous with closed convex values. Then there is a continuous selection $f: X \to E$ of $\varphi$.*

*Proof.* Choose a translation invariant metric $d$ on $E$ such that balls are convex.

*Step 1.* Assume that $\psi: X \multimap E$ is lower semicontinuous with convex values. Let $\varepsilon > 0$. Then there is a continuous $f: X \to E$ such that $f(x) \in B(\psi(x), \varepsilon) := \psi(x) + B(0, \varepsilon)$ for all $x \in X$. To this end, for any $y \in E$, let $U_y := \psi^{-1}(B(y, \varepsilon))$. Then $\mathfrak{A} := \{U_y\}_{y \in E}$ is an open covering of $X$. Let $\{\lambda_s\}_{s \in S}$ be a partition of unity subordinated to $\mathfrak{A}$, i.e. for each $s \in S$, there is $y_s \in E$ such that $\operatorname{supp} \lambda_s \subset \psi^{-1}(B(y_s, \varepsilon))$. We define

$$f(x) := \sum_{s \in S} \lambda_s(x) y_s, \quad x \in X.$$

Then $f$ is continuous and, for $x \in X$, let $S(x) := \{s \in S \mid \lambda_s(x) \neq 0\}$. For any $s \in S(x)$, there is $y'_s \in \psi(x) \cap B(y_s, \varepsilon)$; hence $d(y'_s - y_s, 0) = d(y'_s, y_s) < \varepsilon$, i.e. $y'_s - y_s \in B(0, \varepsilon)$. Let $y := \sum_{s \in S(x)} \lambda_s(x) y'_s$. The convexity of $B(0, \varepsilon)$ implies that $y - f(x) = \sum_{s \in S(x)} \lambda_s(x)(y'_s - y_s) \in B(0, \varepsilon)$, i.e. $d(f(x), y) < \varepsilon$. The convexity of $\psi(x)$ implies that $y \in \psi(x)$. Therefore

$$d(f(x), \psi(x)) \leq d(f(x), y) < \varepsilon.$$

*Step 2.* We shall construct a sequence $(f_n: X \to E)_{n=1}^{\infty}$ of continuous maps such that

(1) $d(f_{n+1}(x), f_n(x)) < 2^{-(n-1)}$;
(2) $f_n(x) \in B(\varphi(x), 2^{-n})$ for all $x \in X$ and $n \in \mathbb{N}$.

The existence of $f_1$ follows from Step 1 with $\varepsilon = 2^{-1}$. Suppose that $f_1, \ldots, f_n$ are already constructed. Let $\psi(x) := \varphi(x) \cap B(f_n(x), 2^{-n})$ for $x \in X$. Then $\psi$ has convex values and, by (2), $\psi(x) \neq \emptyset$. It is easy to prove that $\psi$ is lower semicontinuous. Again by Step 1, there is $f_{n+1}: X \to E$ such that $f_{n+1}(x) \in B(\psi(x), 2^{-(n+1)})$. Then $f_{n+1}(x) \in B(\varphi(x), 2^{-(n+1)})$ and

$$d(f_{n+1}(x), f_n(x)) < 2^{-n} + 2^{-(n+1)} < 2^{-(n-1)}.$$

This inductively establishes the construction. Condition (1), together with the completeness of $E$ shows that $f_n \to f$ uniformly on $X$. Clearly $f$ is continuous. By (2), $f(x) \in \varphi(x)$ for all $x \in X$. $\square$

The Michael theorem is very strict: if we abandon any of its assumptions, the existence of continuous sections may not occur (see [62] and [12]).

**Remark 2.1.2.** In the context of the Michael theorem it is also worthwhile to observe that it may be shown that if $X$ is a Hausdorff space $X$, such that any lower semicontinuous map $\varphi\colon X \multimap E$ (where $E$ is a Banach space) with closed convex values possesses a continuous section, then $X$ is paracompact.

The next result also seems to be interesting.

**Theorem 2.1.3** (Browder, [16])**.** *Assume that $X$ is a paracompact space, $Y$ is a topological vector space and let $\varphi\colon X \multimap Y$. If, for each $x \in X$, $\varphi(x)$ is convex and, for each $y \in Y$, $\varphi^{-1}(y)$ is open, then $\varphi$ has a continuous selection.*

*Proof.* Clearly $\{\varphi^{-1}(y)\}_{y\in Y}$ is an open covering of $X$. Let $\{\lambda_s\}_{s\in S}$ be a partition of unity subordinated to this cover (i.e. for $s \in S$, there is $y_s \in Y$ such that $\operatorname{supp}\lambda_s \subset \varphi^{-1}(y_s)$) and define

$$f(x) := \sum_{s\in S} \lambda_s(x) y_s, \quad x \in X.$$

Then $f$ is continuous. Let $x \in X$. If $\lambda_s(x) \neq 0$, then $x \in \varphi^{-1}(y_s)$, i.e. $y_s \in \varphi(x)$; the convexity of $\varphi(x)$ shows that $f(x) \in \varphi(x)$. $\square$

The question of the existence of Lipschitz continuous selections is more complicated. Perhaps the best-known result is this direction is the following result (see [52] and strict references therein).

**Proposition 2.1.4.** *If $X$ is a metric space, $\varphi\colon X \multimap \mathbb{R}^n$ is $L$-Lipschitz with compact convex values, then it admits a Lipschitz continuous selection* (*with a constant $L'$ depending on $L$ and $n$: precisely $L' = (n!!/(n-1)!!)L$ if $n$ is odd and $L' = (n!!/(\pi(n-1)!!))L$ if $n$ is even*)*.*

The constructive proof involves the so-called Steiner points. Let us mention that this results has no infinite-dimensional counterpart: Lipschitz selections to Lipschitz set-valued maps $\varphi\colon X \multimap Y$ with values in a Banach space $Y$ exist if and only $\dim Y < \infty$.

Finally we mention an important result due to Fryszkowski (compare [14] for some generalizations).

**Theorem 2.1.5** (Fryszkowski, [40])**.** *Suppose that $X$ is a separable metric space, $\varphi\colon X \multimap L^1(T,E)$, where $(T,\mathfrak{M},\mu)$ is a complete $\sigma$-finite non-atomic measure space, $E$ is a Banach space and $L^1(T,E)$ stands for the space of* (*Bochner*) *integrable functions* $T \to E$, *such that, for each* $x \in X$, $\varphi(x)$ *is a decomposable set* (7). *Then* $\varphi$ *admits a continuous selection* $f: X \to L^1(T,E)$.

**2.2.** $\varepsilon$**-selections.** Let $X$ be a topological space and let $Y$ be a metric space. Given $\varepsilon > 0$, we say that $f: X \to Y$ is an *$\varepsilon$-selection* of a set-valued map $\varphi: X \multimap E$ if, for each $x \in X$, $d(f(x), \varphi(x)) < \varepsilon$. It is also adequate to say that $f$ is an $\varepsilon$-uniform approximation of $\varphi$.

It is possible to characterize set-valued maps that have $\varepsilon$-selections for each $\varepsilon > 0$. To this end let us pose the following definition.

**Definition 2.2.1** (comp. [34])**.** We say that $\varphi: X \multimap Y$ is *sub-lower semicontinuous* at $x \in X$, if for every $\varepsilon > 0$, there is $y_x \in \varphi(x)$ and a neighbourhood $U_x$ of $x$ such that $y_x \in B(\varphi(x'), \varepsilon)$ for all $x' \in U_x$. The map $\varphi$ is *sub-lower semicontinuous* if so it is at any $x \in X$.

It is clear that any lower semicontinuous map is sub-lower semicontinuous, but not conversely.

**Example 2.2.2.** A map $\varphi: \mathbb{R} \multimap \mathbb{R}$ given by

$$\varphi(x) = \begin{cases} 1 & \text{if } x \neq 1, \\ [1,2] & \text{if } x = 1, \end{cases}$$

is sub-lower semicontinuous but not lower semicontinuous.

**Proposition 2.2.3.** *Suppose that a map* $\varphi: X \multimap Y$ *has a continuous* $\varepsilon$*-selections for any* $\varepsilon > 0$. *Then* $\varphi$ *is sub-lower semicontinuous.*

*Proof.* Let $\varepsilon > 0$ and $x \in X$. There is a continuous $\varepsilon/3$-selection $f: X \to Y$ of $\varphi$. Hence, there is $y_x \in \varphi(x)$ such that $d(f(x), y_x) < \varepsilon/3$ and a neighbourhood $U_x$ of $x$ such that $d(f(x), f(x')) < \varepsilon/3$ if $x' \in U_x$. Hence, for $y \in U_x$,

$$d(y_x, \varphi(x')) \le d(y_x, f(x)) + d(f(x), f(x')) + d(f(x'), \varphi(x')) < \varepsilon. \qquad \square$$

As for the existence we have a result similar to the Michael theorem.

**Proposition 2.2.4** (see [34], [72])**.** *If* $X$ *is paracompact,* $\varphi: X \multimap E$*, where* $E$ *is a normed space, is sub-lower semicontinuous and has convex values, then, for each* $\varepsilon > 0$*, there is a continuous* $\varepsilon$*-selection* $f: X \to E$ *such that* $f(x) \in \operatorname{conv} \varphi(X)$ *for each* $x \in X$.

*Proof.* Let $\varepsilon > 0$. For any $x \in X$, choose $y_x \in \varphi(x)$ and a neighbourhood $U_x$ as in the above definition. Let $\{\lambda_s\}_{s \in S}$ be a partition of unity subordinated to the covering $\{U_x\}_{x \in X}$, i.e. for all $s \in S$ there is $x_s \in X$ such that $\operatorname{supp} \lambda_s \subset U_{x_s}$. Let $y_s := y_{x_s}$ and

$$f(x) := \sum_{s \in S} \lambda_s(x) y_s, \quad x \in X.$$

(7) A set $K \subset L^1(T,E)$ is *decomposable* if, for each $u, v \in K$ and $A \in \mathfrak{M}$, $\chi_A u + \chi_{T \setminus A} v \in K$, where $\chi_A$ is a characteristic function of $A$.

Then $f$ is continuous. If $x \in X$ and $\lambda_s(x) \neq 0$, then $x \in U_{x_s}$; hence $y_s \in B(\varphi(x), \varepsilon)$. Choose $y'_s \in \varphi(x)$ such that $\|y_s - y'_s\| < \varepsilon$. Then

$$d(f(x), \varphi(x)) \leq \left\| f(x) - \sum_{s \in S} \lambda_s(x) y'_s \right\| \leq \sum_{s \in S} \lambda_s(x) \|y_s - y'_s\| < \varepsilon. \qquad \square$$

The concept of sub-lower semicontinuity may be easily generalized. If $Y$ is a uniform space with the uniform structure $\mathcal{V}$, then we say that $\varphi\colon X \multimap Y$ is *sub-lower semicontinuous* if, for any $x \in X$ and any $V \in \mathcal{V}$, there is $y_x \in \varphi(x)$ and a neighbourhood $U_x$ of $x$ such that $y_x \in V(\varphi(x')) := \{y \in Y \mid \exists\, y' \in \varphi(x')\ (y, y') \in V\}$ for any $x' \in U_x$. At the same time, given $V \in \mathcal{V}$, we say that $f\colon X \to Y$ is a $V$-selection of a set-valued map $\varphi\colon X \multimap Y$ if, for each $x \in X$, $f(x) \in V(\varphi(x))$. Propositions 2.2.3 and 2.2.4 are still valid (in case of Proposition 2.2.4 one has to assume that $E$ is a locally convex space). We leave the easy proofs to the reader (one has to use the properties of uniform spaces corresponding to the triangle inequality).

The notion of $\varepsilon$-selection (or, more generally, $V$-selection where $V \in \mathcal{V}$) may not be sufficient on many occasions. For that reason assume that $X$, $Y$ are topological spaces and let $W$ be a neighbourhood (in $Y \times Y$) of the diagonal $\Delta_Y := \{(y, y') \in Y \times Y \mid y = y'\}$. We say that $f\colon X \to Y$ is a *$W$-selection* of $\varphi$, if $f(x) \in W(\varphi(x)) := \{y \in Y \mid \text{there exists } y' \in \varphi(x) \text{ such that } (y, y') \in W\}$ for any $x \in X$.

We say that $\varphi$ is *nearly selectionable* if, for any neighbourhood $W \supset \Delta_Y$, there is a continuous $W$-selection of $\varphi$.

It is clear that if $Y$ is a uniform space and a map $\varphi$ is nearly selectionable, then it has $V$-selections for any vicinity $V$ from the uniform structure $\mathcal{V}$ in $Y$. Hence we have

**Proposition 2.2.5.** *If $E$ is a linear topological space, then a sub-lower semicontinuous map $\varphi\colon X \multimap E$ with convex values is nearly selectionable.*

To see the main reason to consider $W$-selections we state

**Proposition 2.2.6.** *Let $X$, $Y$, $Z$ be topological spaces, $\varphi\colon X \multimap Y$ and let $g\colon Y \to Z$ be continuous. If $\varphi$ is nearly selectionable, then so is the composition $g \circ \varphi$.*

*Proof.* Let $U$ be an arbitrary neighbourhood of the diagonal $\Delta_Z$. Consider a map $G\colon Y \times Y \to Z \times Z$ given by $G(y, y') = (g(y), g(y'))$ for $y, y' \in Y$. The continuity of $G$ implies that $W := G^{-1}(U)$ is a neighbourhood of $\Delta_Y$. Thus if $f\colon X \to Y$ is a continuous $W$-selection of $\varphi$, then $g \circ f$ is a $U$-selection of $g \circ \varphi$. $\square$

**Remark 2.2.7.** (a) If the spaces $Y$, $Z$ are uniform and $g$ above is uniformly continuous, then $g \circ \varphi$ has $V$-selections for any vicinity $V$ from the uniform structure in $Z$.

(b) It is easy to show that if $Z$ is a uniform space with the uniform structure $\mathcal{W}$, $\psi: Y \multimap Z$ has continuous $W$-selections for any $W \in \mathcal{W}$, then the composition $\psi \circ \varphi$ is nearly selectionable provided so is $\varphi$. To see this take $W \in \mathcal{W}$ and $V \in \mathcal{W}$ such that $V \circ V \subset W$. There is a continuous $g: Y \to Z$ such that $g(y) \in V(\psi(y))$ for any $y \in Y$. Hence $g \circ \varphi(x) \subset V(\psi \circ \varphi(x))$ for all $x \in X$. By Proposition 2.2.6, there is $f: X \to Y$ such that $g \circ f(x) \in V(g \circ \varphi(x)) \subset W(\psi \circ \varphi(x))$.

**2.3. Graph-approximations.** As we saw above, selections or 'near'-selections are available for maps satisfying some sorts of lower semicontinuity assumptions. In case of upper semicontinuous maps, it may appear that neither continuous selections nor $\varepsilon$-selections exist. Hence we shall discuss a different appropriate concept of a single-valued approximation (see survey [57]).

**Definition 2.3.1.** Let $X$, $Y$ be topological spaces, $\varphi: X \multimap Y$, $\mathcal{U}$ be a neighbourhood of $\mathrm{Gr}(\varphi)$ (in $X \times Y$) and $A \subset X$. We say that $f: A \to Y$ is a $\mathcal{U}$-*approximation* (or $\mathcal{U}$-*graph-approximation*) of $\varphi$ over $A$ if $\mathrm{Gr}(f) \subset \mathcal{U}$ (8). We say that $\varphi$ is *approximable over* $A$ (resp. *approximable*) if, for each neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there are $\mathcal{U}$-approximations of $\varphi$ over $A$ (resp. over $X$).

Note that approximablity of $\varphi$ over $A$ is not the same as approximablity of the restriction $\varphi|_A$ of $\varphi$ to $A$: it is easy to provide counterexamples.

Some relations between graph-approximations and $\varepsilon$-selections are show in the following result (9).

**Proposition 2.3.2.** *Suppose that $X$ is a topological space, $A \subset X$, $Y$ is a metric space and $\varphi: X \multimap Y$.*

(a) *If $X$ is paracompact, $\varphi$ is upper semicontinuous with compact values, then for any neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a function $\varepsilon: X \to (0,\infty)$ such that any $\varepsilon(\,\cdot\,)$-selection (i.e a map $f: A \to Y$ such that $f(x) \in B(\varphi(x), \varepsilon(x))$ for $x \in A$) is a $\mathcal{U}$-approximation.*

(b) *If $\varphi$ is $H$-lower semicontinuous, the for any continuous $\varepsilon: X \to (0,\infty)$, there is a neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ such that any $\mathcal{U}$-approximation $f: A \to Y$ is an $\varepsilon(\,\cdot\,)$-selection.*

*Proof.* (a) Fix $x \in X$ and a neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$. Then $\varphi(x) \subset \mathcal{U}(x)$. There is $\varepsilon_x > 0$ and a neighbourhood $U_x$ of $x$ such that $U_x \times B(\varphi(x), 2\varepsilon_x) \subset \mathcal{U}$ since $\varphi(x)$ is compact. In particular, for any $y \in U_x$, $B(\varphi(x), 2\varepsilon_x) \subset \mathcal{U}(y)$. By the upper semicontinuity, we may assume without loss of generality that, for $y \in U_x$, $\varphi(y) \subset B(\varphi(x), \varepsilon_x)$. Consider a partition of unity $\{\lambda_j\}_{j\in J}$ subordinated to the covering $\{U_x\}_{x\in X}$, i.e. for each $j \in J$, there is $x_j \in X$ such that $\operatorname{supp}\lambda_j \subset$

(8) Observe that if $\mathrm{Gr}(f) \subset \mathcal{U}$ is and only if $f(x) \in \mathcal{U}(x) := \{y \in Y \mid (x,y) \in \mathcal{U}\}$ for any $x \in X$.

(9) Some of results given below are new (or at least they are new in the provided general context); others are taken mainly from [56].

$U_j := U_{x_j}$. For $j \in J$, let $\varepsilon_j := \varepsilon_{x_j}$ and define

$$\varepsilon(x) := \sum_{j\in} \lambda_j(x)\varepsilon_j, \quad x \in X.$$

It is clear that $\varepsilon$ is continuous and $\varepsilon(x) > 0$ for all $x \in X$.

Let $x \in X$. There is $j \in J$ such that $\lambda_j(x) > 0$ (i.e. $x \in U_j$) and $\varepsilon(x) \le \varepsilon_j$. Hence $B(\varphi(x_j), 2\varepsilon_j) \subset \mathcal{U}(x)$ and $\varphi(x) \subset B(\varphi(x_j), \varepsilon_j)$. Therefore

$$B(\varphi(x), \varepsilon(x)) \subset B(\varphi(x_j), \varepsilon(x) + \varepsilon_j) \subset B(\varphi(x_j), 2\varepsilon_j) \subset \mathcal{U}(x).$$

If $f: A \to Y$ is an $\varepsilon(\,\cdot\,)$-selection, then $f(x) \in B(\varphi(x), \varepsilon(x)) \subset \mathcal{U}(x)$ for any $x \in A$.

(b) For any $(z, y) \in \mathrm{Gr}(\varphi)$, let

$$U(z, y) = [\varepsilon^{-1}(\varepsilon(z)/2, \infty) \cap U_z] \times B(y, \varepsilon(z)/4)$$

where $U_z$ is a neighbourhood of $z$ such that $h(\varphi(z), \varphi(x)) < \varepsilon(z)/4$ for any $x \in U_z$ ($U_z$ exists in view of $H$-lower semicontinuity of $\varphi$). Clearly $U(z, y)$ is open neighbourhood of $(z, y)$. Put

$$\mathcal{U} := \bigcup_{(z,y)\in\mathrm{Gr}(\varphi)} U(z, y).$$

Then $\mathcal{U}$ is a neighbourhood of $\mathrm{Gr}(\varphi)$. Suppose that $f: A \to Y$ is a $\mathcal{U}$-approximation of $\varphi$ and let $x \in A$. Then $(x, f(x)) \in \mathcal{U}$, i.e. there is $(z, y) \in \mathrm{Gr}(\varphi)$ such that $(x, f(x)) \in U(z, y)$. This means that $\varepsilon(z)/2 < \varepsilon(x)$, $h(\varphi(z), \varphi(x)) < \varepsilon(z)/4$ and $f(x) \in B(y, \varepsilon(z)/4)$. Hence $y \in \varphi(z) \subset B(\varphi(x), \varepsilon(z)/4)$ and

$$d(f(x), \varphi(x)) \le d(f(x), y) + d(y, \varphi(x)) < 2\varepsilon(z)/4 < \varepsilon(x). \qquad \square$$

The last result implies that any approximable $H$-lower semicontinuous map (or lower semicontinuous with compact values) has $\varepsilon(\,\cdot\,)$-selections (resp. $\varepsilon$-selections) for any continuous function $\varepsilon: X \to (0, \infty)$ (resp. any $\varepsilon > 0$). This also means that in case of continuous maps with compact values the concepts $\varepsilon$-selections and $\mathcal{U}$-approximations coincide in a sense.

Apart from $\mathcal{U}$-approximations (they may be studied in case of maps acting between topological spaces), it makes sense to consider $\varepsilon(\,\cdot\,)$-approximations.

**Definition 2.3.3.** Suppose $X$, $Y$ are metric spaces, $A \subset X$ and $\varphi: X \multimap Y$. We say that $f: A \to Y$ is an *$\varepsilon(\,\cdot\,)$-approximation of $\varphi$ over $A$* if, for each $x \in A$,

$$f(x) \subset B(\varphi(B(x, \varepsilon(x))), \varepsilon(x)).$$

In particular, if $\varepsilon > 0$, then $f: A \to Y$ is an *$\varepsilon$-approximation* if, for each $x \in A$, $f(x) \in B(\varphi(B(x, \varepsilon)), \varepsilon)$. One says that $\varphi$ is *weakly approximable over $A$* if, for each $\varepsilon > 0$, $\varphi$ admits continuous $\varepsilon$-approximations.

Its is clear that if $\varphi$ is approximable, then it is weakly approximable.

To see the immediate connection of $\varepsilon(\,\cdot\,)$-approximations and $\mathcal{U}$-approximations, let $\varepsilon\colon X \to (0,\infty)$ be continuous. We define a particular neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, namely put

$$\mathcal{U}(\varepsilon) := \bigcup_{(x,y)\in\mathrm{Gr}(\varphi)} B((x,y),\varepsilon(x))$$

where on $X \times Y$ we consider the max-metric, i.e. $B((x,y),\varepsilon(x)) := \{(x',y') \in X \times Y \mid \max\{d_X(x,x'), d_Y(y,y')\} < \varepsilon(x)\}$.

**Proposition 2.3.4.**

(a) *If $f\colon A \to Y$ is an $\varepsilon(\,\cdot\,)$-approximation of $\varphi$ over $A \subset X$ if and only if $f$ is a $\mathcal{U}(\varepsilon)$-approximation over $A$.*

(b) *If $\varphi$ is upper semicontinuous with compact values, then for any neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a continuous function $\varepsilon\colon X \to (0,\infty)$ such that if $f\colon A \to Y$ is an $\varepsilon(\,\cdot\,)$-approximation over $A$, the $f$ is a $\mathcal{U}$-approximation of $\varphi$ over $A$.*

*Proof.* Fix a neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$. The compactness of values and the upper semicontinuity of $\varphi$ implies that, for each $x \in X$, there is $r_x > 0$ such that

$$B(x,r_x) \times B(\varphi(B(x,2r_x)),r_x) \subset \mathcal{U}.$$

Let $\{\lambda_j\}_{j\in J}$ be a partition of unity subordinated to the covering $\{B(x,r_x)\}_{x\in X}$, i.e. for any $j \in J$, there is $x_j \in X$ such that $\operatorname{supp}\lambda_j \subset B(x_j,r_j)$ where $r_j := r_{x_j}$. We put

$$\varepsilon(x) := \sum_{j\in J} \lambda_j(x) r_j, \quad x \in X.$$

Suppose that $f\colon A \to Y$ be an $\varepsilon(\,\cdot\,)$-approximation of $\varphi$. For $x \in X$, there is $j \in J$ such that $\lambda_j > 0$ (thus $x \in B(x_j,r_j)$ and $\varepsilon(x) \le r_j$. Since $\mathrm{Gr}(f) \subset \mathcal{O}(\mathrm{Gr}(\varphi),\varepsilon)$, there is a point $(x',y') \in \mathrm{Gr}(\varphi)$ such that $d(x,x') < \varepsilon(x)$ and $d(y',f(x)) < \varepsilon(x)$. Thus $x' \in B(x_j,2r_j)$, $y' \in \varphi(x') \subset \varphi(B(x_j,2r_j))$ and $f(x) \in B(\varphi(B(x_j,2r_j)),r_j)$. This implies that

$$(x,f(x)) \in B(x_j,r_j) \times B(\varphi(B(x_j,2r_j)),r_j) \subset \mathcal{U}. \qquad \square$$

**Remark 2.3.5.** (a) If a neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ is such that, for each $x \in A$, there is $\delta_x > 0$ such that $B(x,\delta_x) \times B(\varphi(x),\delta_x) \subset \mathcal{U}$, then the compactness of values is not necessary to obtain the existence of a continuous $\varepsilon(\,\cdot\,)$ such that $\varepsilon(\,\cdot\,)$-approximations are $\mathcal{U}$-approximations.

(b) If $X$ is a compact metric space, $A \subset X$ is closed, then to each neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ there corresponds a number $\delta > 0$ such that any $\delta$-approximation of $\varphi$ over $A$ is a $\mathcal{U}$-approximation over $A$. Indeed, if $\varepsilon\colon X \to (0,\infty)$ is a continuous function that exists in view of the above proposition, then putting $\delta := \inf_{x\in A} \varepsilon(x)$ satisfies the requirements.

In spite of that fact that $\mathcal{U}$-approximations may be studied in the context of topological spaces (not necessarily metric ones), the use of this sort of approximations also in metric spaces (in opposition to the use of $\varepsilon$-approximations or $\varepsilon(\,\cdot\,)$-approximations) stems e.g. from the following facts (comp. [56], [57]).

**Proposition 2.3.6.** *If Let $X$, $Y$ are topological spaces, $\varphi\colon X \multimap Y$ is approximable over $A \subset X$ and $g\colon Y \to Z$, where $Z$ is a topological space, is continuous, then the composition $g \circ \varphi$ is approximable over $A$.*

*Proof.* Let $\mathcal{U}$ be an arbitrary neighbourhood of $\mathrm{Gr}(g \circ \varphi)$. Let $G\colon X \times Y \to X \times Z$ be given by $G(x, y) := (x, g(y))$ for $x \in X$ and $y \in Y$. It is easy to see that $\mathcal{W} := G^{-1}(\mathcal{U})$ is open and $\mathrm{Gr}(\varphi) \subset \mathcal{W}$. Therefore, given a $\mathcal{W}$-approximation $f\colon A \to Y$ of $\varphi$, $g \circ f$ is a $\mathcal{U}$-approximation of $g \circ \varphi$. $\square$

**Theorem 2.3.7.** *Suppose that set-valued maps $\varphi\colon X \to Y$, $\psi\colon Y \multimap Z$ are approximable. If $Y$ is paracompact, $\psi$ is upper semicontinuous with compact values and $\varphi$ is perfect in the sense that, for each $y \in Y$, $\varphi^{-1}(y)$ is compact and $\varphi(B)$ is closed whenever $B \subset X$ is closed* (10)*, then $\psi \circ \varphi$ is approximable. Precisely: if $A \subset X$, then, for any neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\psi \circ \varphi)$, there are neighbourhoods $\mathcal{W}$ of $\mathrm{Gr}(\varphi)$ and $\mathcal{V}$ of $\mathrm{Gr}(\psi)$ such that $\mathcal{V}$ ; hence $g \circ f$ is a $\mathcal{U}$-approximation of $\psi \circ \varphi$ over $A$ provided $f\colon A \to Y$ is a $\mathcal{W}$-approximation over $A$ of $\varphi$ and $g\colon Y \to Z$ is a $\mathcal{V}$-approximation of $\psi$.*

*Proof.* Let $\mathcal{U}$ be a neighbourhood of $\mathrm{Gr}(\psi \circ \varphi)$. For each $y \in Y$, the set $\varphi^{-1}(y) \times \psi(y)$ is closed and

$$\varphi^{-1}(y) \times \psi(y) \subset \mathrm{Gr}(\psi \circ \varphi) \subset \mathcal{U}.$$

Since $\varphi^{-1}(y)$ and $\psi(y)$ are compact, there are neighbourhoods $M_y$ of $\varphi^{-1}(y)$ (in $X$) and $N_y$ of $\psi(y)$ (in $Z$) such that

$$M_y \times N_y \subset \mathcal{U}.$$

Observe now that $\varphi^{-1}$ is upper semicontinuous in the sense that given a closed $B \subset X$, $(\varphi^{-1})^{-1}(B) = \varphi(B)$ is closed; this, together with the upper semicontinuity of $\psi$ and the paracompactness of $Y$, implies that there is a locally finite open covering $\{L_y\}_{y \in Y}$ of $Y$ such that

$$\varphi^{-1}(\mathrm{cl}\, L_y) \subset M_y, \quad \psi(\mathrm{cl}\, L_y) \subset N_y.$$

We define relations $\mathcal{U}$ $Y \times X$ and $\mathcal{V}$ $Y \times Z$ by specifying, for $y \in Y$,

$$\begin{aligned} \mathcal{W}(y) &:= \bigcap \{M_w \mid w \in Y \text{ and } y \in \mathrm{cl}\, L_w\}, \\ \mathcal{V}(y) &:= \bigcap \{N_w \mid w \in Y \text{ and } y \in \mathrm{cl}\, L_w\}. \end{aligned}$$

(10) This terminology agrees with the usual one.

Let $y \in Y$. Then $\mathcal{W}(L_y) \subset M_y$. Indeed: if $z \in L_y$, then $\mathcal{W}(z) \subset M_y$. Analogously $\mathcal{V}(L_y) \subset N_y$. Moreover, $\varphi^{-1}(y) \subset \mathcal{W}(y)$ and $\psi(y) \subset \mathcal{V}(y)$. Indeed, for example, if $w \in Y$ and $y \in \operatorname{cl} L_w$, then $\varphi^{-1}(y) \subset \varphi^{-1}(\operatorname{cl} L_w) \subset M_w$ (analogously for $\psi$). Therefore $\varphi^{-1} \subset \mathcal{W}$ and $\mathrm{Gr}(\psi) \subset \mathcal{V}$.

We shall see that $\mathcal{W}$ and $\mathcal{V}$ are open (in $Y \times X$ and $Y \times Z$, respectively). The local finitness of $\{L_w\}_{w \in Y}$ implies that, for each $y \in Y$, $\mathcal{W}(y)$ is open in $X$ and $H_y := Y \setminus \bigcup \{\operatorname{cl} L_w \mid w \in Y \text{ and } y \notin \operatorname{cl} L_w\}$ is an open neighbourhood of $y$ (in $Y$). We claim that $H_y \times \mathcal{W}(y) \subset \mathcal{W}$. The definition of $H_y$ implies that, for any $v \in H_y$: if $w \in Y$ and $v \in \operatorname{cl} L_w$, then $y \in \operatorname{cl} L_w$. It follows that $\mathcal{W}(y) \subset \mathcal{W}(v)$ for each $v \in H_y$. This establishes our claim and implies that $\mathcal{W} = \bigcup_{y \in Y} H_y \times \mathcal{W}(y)$, i.e. $\mathcal{W}$ is open. Similarly we show that $\mathcal{V}$ is open.

We have an open covering $\{L_y\}_{y \in Y}$ of $Y$, open sets $\mathcal{W} \subset Y \times X$ and $\mathcal{V} \quad Y \times Z$ such that $\varphi^{-1} \subset \mathcal{W}$ and $\mathrm{Gr}(\psi) \subset \mathcal{V}$; $\mathcal{W}(L_y) \subset M_y$ and $\mathcal{V}(L_y) \subset N_y$ for all $y \in Y$. Thus, for each $y \in Y$, there is $w \in Y$ such that $y \in L_w$; then

$$\mathcal{W}(y) \times \mathcal{V}(y) \subset \mathcal{W}(L_w) \times \mathcal{V}(L_w) \subset M_w \times N_w \subset \mathcal{U}.$$

Hence $\mathcal{V} \qquad {}^{-1} \subset \mathcal{U}$.

Clearly $\mathrm{Gr}(\varphi) \subset \mathcal{W}^{-1}$ (obviously $\mathcal{W}^{-1}$ is open in $X \times Y$) and, again, $\mathrm{Gr}(\psi) \subset \mathcal{V}$. If $f\colon X \to Y$ is a $\mathcal{W}^{-1}$-approximation of $\varphi$ and $g\colon Y \to Z$ is a $\mathcal{V}$-approximation of $\psi$, then $\mathrm{Gr}(g \circ f) \subset \mathcal{V} \qquad {}^{-1} \subset \mathcal{U}$, i.e. $g \circ f$ is a $\mathcal{U}$-approximation of $\psi \circ \varphi$. $\square$

**Remark 2.3.8.** (a) If $X$, $Y$, $Z$ and $\psi$ are as above and $g\colon X \to Y$ is perfect, then $\psi \circ g$ is approximable. Moreover, for any neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\psi \circ g)$, there is a neighbourhood $\mathcal{V}$ of $\mathrm{Gr}(\psi)$ such that $\{(x,z) \in X \times Z \mid (g(x), z) \in \mathcal{V} \quad = \mathcal{V} \quad g \subset \mathcal{U}$.

(b) Sometimes we would like to consider a composition $\psi \circ \varphi$, where $\psi$ is upper semicontinuous with merely closed values. The fact analogous to Theorem 2.3.7 holds true provided the initial neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\psi \circ \varphi)$ is thick. Generally speaking we say that a neighbourhood $\mathcal{U}$ of the graph $\mathrm{Gr}(\varphi)$ of some set-valued map $\varphi\colon X \to Y$ between Hausdorff spaces $X$ and $Y$ is *thick* if, for each $x \in X$, there are neighbourhoods $U_x$ of $x$ and $V_x$ of $\varphi(x)$ such that $U_x \times V_x \subset \mathcal{U}$. Let us show the assertion. By inspection of the proof of Theorem 2.3.7, we see that the only instance when compactness of values of $\psi$ was used was the moment when we establish the existence of neighbourhoods $M_y$ and $N_y$ of $\varphi^{-1}$ and $\psi(y)$, $y \in Y$, respectively, such that $M_y \times N_y \subset \mathcal{U}$. Let us make it under the assumption that $\mathcal{U}$ is a thick neighbourhood of $\mathrm{Gr}(\psi \circ \varphi)$. Take $y \in Y$ and $x \in \varphi^{-1}(y)$. Then there are neighbourhoods $U_x$ of $x$ and $V_x$ of $\psi \circ \varphi(x)$ such that $U_x \times V_x \subset \mathcal{U}$. Since $y \in \varphi(x)$, $\psi(y) \subset \psi \circ \varphi(x) \subset V_x$. The compactness of $\varphi^{-1}(y)$ implies that there are points $x_1, \ldots, x_n \in \varphi^{-1}(y)$ such that $M_y := \bigcup_{i=1}^n U_{x_i} \supset \varphi^{-1}(y)$. Let $N_y := \bigcap_{i=1}^n V_{x_i}$. Then $\psi(y) \subset N_y$ and $M_y \times N_y \subset \mathcal{U}$.

**Example 2.3.9.** Let $X$, $Y$ be metric spaces, $\varphi\colon X \multimap Y$ and let $\varepsilon\colon X \to (0,\infty)$. Then

$$\mathcal{U}(\varepsilon) := \bigcup_{(x,y)\in \mathrm{Gr}(\varphi)} B((x,y),\varepsilon(x))$$

is a thick neighbourhood of $\mathrm{Gr}(\varphi)$. Indeed, for each $x \in X$,

$$B(x,\varepsilon(x)) \times B(\varphi(x),\varepsilon(x)) \subset \mathcal{U}(\varepsilon).$$

This example sheds new light onto Proposition 2.3.4 and Remark 2.3.5(a). Observe that if $\varphi$ has compact values, then each neighbourhood of $\mathrm{Gr}(\varphi)$ is thick.

**Corollary 2.3.10.** *If $X$ is compact, $Y$ is paracompact, $\varphi\colon X \multimap Y$, $\psi\colon Y \multimap Z$ are approximable and upper semicontinuous with compact values, then $\psi\circ\varphi$ is approximable. If, additionally, $X$, $Y$, $Z$ are metric spaces, then for each $\varepsilon > 0$, there is $\delta > 0$ such that given $\delta$-approximations $f\colon X \to Y$, $g\colon Y \to Z$ of $\varphi$ and $\psi$, respectively, $g \circ f$ is an $\varepsilon$-approximation of $\psi \circ \varphi$.*

This follows immediately from Theorem 2.3.7 since $\varphi^{-1}$ is upper semicontinuous in the above sense.

The reader will easily formulate and prove a corresponding result concerning the existence of $\mathcal{U}$-approximations of $\psi\circ\varphi$ when $\psi$ is upper semicontinuous with closed values and $\mathcal{U}$ is a thick neighbourhood of $\mathrm{Gr}(\psi\circ\varphi)$.

As for the existence of $\mathcal{U}$-approximations, we have the following fundamental result.

**Theorem 2.3.11** (Cellina, [18])**.** *Suppose that $X$ is a paracompact space, $A \subset X$ is closed, $E$ is a locally convex space and $\varphi\colon X \multimap E$ is upper semicontinuous. If $\mathcal{U}$ is an open neighbourhood of $\mathrm{Gr}(\varphi)$ such that, for any $x \in A$, there are a neighbourhood $U_x$ of $x$ and a convex neighbourhood $V_x$ of $\varphi(x)$ such that $U_x \times V_x \subset \mathcal{U}$ (11), then $\varphi$ admits a $\mathcal{U}$-approximation over $A$.*

*Proof.* For any $x \in A$, take $U_x$ and $V_x$ as above. Diminishing $U_x$ if necessary, we may assume that $U_x$ is open and $\varphi(U_x) \subset V_x$. Let $\mathfrak{B}$ be an open star-refinement of the cover $\mathfrak{A} := \{U_x\}_{x\in A}$ of $A$. Now let $\{\lambda_s\}_{s\in S}$ be a partition of unity subordinated to $\mathfrak{B}$, i.e. for each $s \in S$, $\operatorname{supp}\lambda_s \subset W_s$ for some $W_s \in \mathfrak{B}$. Choose $y_s \in \varphi(W_s)$ and define

$$f(x) := \sum_{s\in S} \lambda_s(x) y_s, \quad x \in A.$$

Clearly $f$ is well-defined and continuous. We shall show that $f$ is a $\mathcal{U}$-approximation of $\varphi$ over $A$. Take $x \in A$ and let $S(x) := \{s \in S \mid \lambda_s(x) \neq 0\}$, then $f(x) = \sum_{s\in S(x)} \lambda_s(x) y_s$. If $s \in S(x)$, then $x \in \operatorname{supp}\lambda_s \subset W_s \subset \mathrm{st}\,(W_s,\mathfrak{B}) \subset U_y$ for some $y \in A$; thus $y_s \in \varphi(W_s) \subset \varphi(U_y) \subset V_y$. Therefore $f(x) \subset V_y$ because $V_y$ is convex, i.e. $(x, f(x)) \in U_y \times V_y \subset \mathcal{U}$. □

(11) So again we have a thickness of $\mathcal{U}$ of sorts here.

Theorem 2.3.11 does not hold true for an arbitrary neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ even if values of $\varphi$ are convex and closed. For instance, if $\varphi\colon [0,\infty) \multimap \mathbb{R}$ is given by $\varphi(x) = \{1/x\}$ for $x > 0$ and $\varphi(0) = [0,\infty)$, then $\varphi$ has closed convex values and is upper semicontinuous, but it has no continuous $\mathcal{U}$-approximations when $\mathcal{U} = \{(x,y) \in [0,\infty) \times \mathbb{R} \mid 1/2 < xy < 2 \text{ or } 3xy < 1\}$. Observe that the constructed neighbourhood is not thick.

However, if $\varphi\colon X \multimap E$ is upper semicontinuous with compact convex values, the $\varphi$ is approximable over $A$. Indeed, let $\mathcal{U}$ be an arbitrary neighbourhood of $\mathrm{Gr}(\varphi)$; then $\mathcal{U}$ is thick, i.e. for any $x \in A$, there are open neighbourhoods $U_x$ of $x$ and $V_x'$ of $\varphi(x)$ such that $U_x \times V_x' \subset \mathcal{U}$. Since $\varphi(x)$ is compact, there is a convex neighbourhood $V$ of the origin in $E$ such that $V_x := \varphi(x) + V \subset V_x'$. Obviously $U_x \times V_x \subset \mathcal{U}$ and $V_x$ is convex. Therefore Theorem 2.3.11 applies.

In the same spirit we have

**Corollary 2.3.12.** *If $X$ is a metric space and $E$ is a locally convex metric space (e.g. a normed space), $\varphi\colon A \multimap E$ is upper semicontinuous with convex values, then for each $\varepsilon\colon X \to (0,\infty)$, there exist an $\varepsilon(\,\cdot\,)$-approximation of $\varphi$ over $A$.*

*Proof.* In $E$ we choose a translation invariant metric such that balls are convex. Let again $\mathcal{U}(\varepsilon) := \bigcup_{(x,y)\in\mathrm{Gr}(\varphi)} B((x,y),\varepsilon(x))$. If $x \in A$, then $U_x \times V_x \subset \mathcal{U}$ where $U_x := B(x,\varepsilon(x))$ and $V_x := B(\varphi(x),\varepsilon(x))$. The convexity of $\varphi(x)$ implies that $V_x$ is a convex neighbourhood of $\varphi(x)$. The assertion follows from Proposition 2.3.4(a) $\square$

**Remark 2.3.13.** Observe that in course of the proofs of our existence results 2.2.4, 2.3.11 and 2.3.12, the existing 'almost' selections and graph-approximations of a map $\varphi\colon X \multimap E$ take values in $\operatorname{conv}\varphi(X)$. Therefore, if $E$ is complete and $\varphi$ is compact, then so are these single-valued approximations.

Let us end this section mentioning that there are some other concepts of graph-approximations (see e.g. [66]). For instance, suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are open coverings of Hausdorff spaces $X$ and $Y$, respectively. We say that $f\colon X \to Y$ is $\mathfrak{A}\times\mathfrak{B}$-approximation of $\varphi\colon X \multimap Y$ if, for each $p \in \mathrm{Gr}(f)$, there is $q \in \mathrm{Gr}(\varphi)$ such that $p$ and $q$ lie in same element of the cover $\mathfrak{A}\times\mathfrak{B} := \{U \times V \mid U \in \mathfrak{A},\ V \in \mathfrak{B}\}$ of $X \times Y$. It is clear that if $\mathcal{U} := \bigcup\{W \in \mathfrak{A}\times\mathfrak{B} \mid W \cap \mathrm{Gr}(\varphi) \neq \emptyset\}$, then $\mathcal{U}$ is a neighbourhood of $\mathrm{Gr}(\varphi)$ and each $\mathfrak{A}\times\mathfrak{B}$-approximation of $\varphi$ ia a $\mathcal{U}$-approximation. Observe that the constructed neighbourhood $\mathcal{U}$ is thick. Indeed, for each $x \in X$, $U_x \times V_x \subset \mathcal{U}$, where $U_x := \mathrm{st}\,(x,\mathfrak{A})$ and $V_s := \mathrm{st}\,(\varphi(x),\mathfrak{B}) := \bigcup\{V \in \mathfrak{B} \mid V \cap \varphi(x) \neq \emptyset\}$.

As indicated in Remark 2.3.8(a), applications of this less general concept of a graph-approximation being valid for not necessarily metrizable space may prove useful when dealing with set-valued maps having noncompact values. In metrizable spaces the use of $\varepsilon(\,\cdot\,)$-approximations seems to be totally satisfactory.

**2.4. Relative selections an approximations.** Let $A \subset X$, $\varphi\colon X \multimap Y$ be a set-valued map and suppose that $f\colon X \to Y$ is a *partial selection* $f\colon A \to Y$ *of* $\varphi$, i.e. $f(x) \in \varphi(x)$ for $x \in A$. The problem is whether there is an extension $F\colon X \to Y$ of $f|_A$ such that $F(x) \in \varphi(x)$ for $x \in X$.

**Corollary 2.4.1.** *If $A$ is closed in a paracompact space $X$, $Y$ is a complete metric locally convex space, then any continuous partial selection $f\colon A \to Y$ of a lower semicontinuous set-valued map $\varphi\colon X \multimap Y$ with closed convex values admits a continuous extension $F\colon X \to Y$ such that $F$ is a selection of $\varphi$.*

*Proof.* Consider a set-valued map $\Phi\colon X \multimap Y$ given by

$$\Phi(x) = \begin{cases} \{f(x)\} & \text{if } x \in A, \\ \varphi(x) & \text{if } x \notin A. \end{cases}$$

It is easy to see that $\Phi$ is lower semicontinuous with closed convex values; hence — in view of the Michael theorem — its admits a continuous selection $F\colon X \to Y$. Evidently $F|_A = f$. □

As it is easy to see a similar result for $\varepsilon$-selections of sub-lower semicontinuous maps does not hold true. However:

**Proposition 2.4.2.** *If $X$ is a metric space, $A \subset X$ is closed, $\varphi\colon X \multimap Y$, where $Y$ is a normed space, is sub-lower semicontinuous, has convex values and $\varphi$ is $H$-lower semicontinuous at each point $a \in A$, then for any $\varepsilon > 0$ and $0 < \delta < \varepsilon$ any continuous partial $\delta$-selection $f\colon A \to Y$ of $\varphi$ admits a continuous extension $F\colon X \to Y$ being an $\varepsilon$-selection of $\varphi$.*

*Proof.* Fix $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$. Let $h\colon X \to Y$ be a continuous extension of a $\delta$-selection $f\colon A \to Y$ of $\varphi|_A$ (existing in view of the Dugundji theorem). The $H$-lower semicontinuity of $\varphi|_A$ implies that, for each $a \in A$, there is $\delta(a) > 0$ such that $\varphi(a) \subset B(\varphi(x), \rho)$, where $\rho := (\varepsilon - \delta)/2$, provided $d(x, a) < \delta(a)$. On the other hand, for each $a \in A$, there is $0 < \eta(a) \le \delta(a)$ such that $\|h(x) - f(a)\| < \rho$ for $x \in B(a, \eta(a))$. Let

$$V = \bigcup_{a \in A} B(a, \eta(a)).$$

Then $V$ is an open neighbourhood of $A$. Let $x \in V$; there is $a \in A$ such that $d(x, a) < \eta(a)$ and, hence, $\varphi(a) \subset B(\varphi(x), \rho)$, $\|h(x) - f(a)\| < \rho$ and $f(a) \in B(\varphi(a), \delta)$. Therefore

$$g(x) \in B(f(a), \rho) \subset B(\varphi(a), \rho + \delta) \subset B(\varphi(x), 2\rho + \delta) = B(\varphi(x), \varepsilon).$$

In other words, $h|_V$ is an $\varepsilon$-selection of $\varphi|_V$.

In view of the sub-lower semicontinuity, there is an $\varepsilon$-selection $g\colon X \to Y$ of $\varphi$. Take a partition of unity $\{\lambda_1, \lambda_2\}$ subordinated to the cover $\{V, X \setminus A\}$ of $X$, i.e. $\operatorname{supp} \lambda_1 \subset V$ and $\operatorname{supp} \lambda_2 \subset X \setminus A$. Define

$$F(x) = \lambda_1(x)h(x) + \lambda_2(x)g(x), \quad x \in X.$$

Then $F$ is continuous, $F|_A = h|A = f$ and, for $x \in X$, $h(x), g(x) \in B(\varphi(x), \varepsilon)$. The convexity of $B(\varphi(x), \varepsilon)$ implies that $F(x) \in B(\varphi(x), \varepsilon)$. □

It is clear that this result stays true if we replace a normed space $Y$ by an arbitrary locally convex space and $\varepsilon$-selections by $V$-selections, where $V$ is a neighbourhood of the origin in $Y$.

Let us now state a general result concerning $\varepsilon$-approximations (we restrict ourselves to $\varepsilon(\,\cdot\,)$-approximations and leave to the reader generalizations to non-metrizable case).

**Theorem 2.4.3.** *Let $X$ be a metric space, $A \subset X$ be closed, $E$ a normed space and suppose that $\varphi: X \multimap E$ is upper semicontinuous with convex values. Let a function $\varepsilon: X \to (0, \infty)$ be continuous. Then:*

(a) *For any continuous function $\delta: X \to (0, \infty)$ such that $\delta < \varepsilon$, any continuous partial $\delta(\,\cdot\,)$-selection $f: A \to E$ may be extended to a continuous $\varepsilon(\,\cdot\,)$-approximation $F: X \to E$ of $\varphi$.*

(b) *There exists a continuous function $\rho: X \to (0, \infty)$ such that any continuous $\rho(\,\cdot\,)$-selection $f: A \to E$ of $\varphi$ over $A$ may be extended to a continuous $\varepsilon(\,\cdot\,)$-approximation $F: X \to E$.*

*Proof.* Choose an arbitrary continuous function $\delta: X \to (0, \infty)$ such that $\delta(x) < \varepsilon(x)$ on $X$.

*Step 1.* There exists a continuous function $\eta: X \to (0, \infty)$ such that, for each $x \in X$, there is $x' \in X$ such that $d(x', x) < \varepsilon(x)$ and

$$B(\varphi(B(x, \eta(x))), \delta(x)) \subset B(\varphi(x'), \varepsilon(x)).$$

To see this take a continuous function $\mu: X \to (0, \infty)$ such that $2\mu(x) + \delta(x) < \varepsilon(x)$ on $X$ and, for $x \in X$, choose $r_x \in (0, 2\mu(x) + \delta(x))$ such that

$$\varphi(B(x, 2r_x)) \subset B(\varphi(x), \mu(x))$$

and

$$B(x, r_x) \subset \delta^{-1}((0, \mu(x) + \delta(x))) \cap \varepsilon^{-1}((2\mu(x) + \delta(x), \infty)).$$

Let $\{\lambda_s\}_{s \in S}$ be a partition of unity subordinated to the cover $\{B(x, r_x)\}_{x \in X}$, i.e. for each $s \in S$, there is $x_s \in X$ such that $\operatorname{supp} \lambda_s \subset B(x_s, r_s)$ where $r_s := r_{x_s}$. Put

$$\eta(x) := \sum_{s \in S} \lambda_s(x) r_s, \quad x \in X.$$

Then $\eta$ is well-defined and continuous. Let $x \in X$. There exists $s \in S$ such that $\lambda_s(x) \neq 0$ and $\eta(x) \le r_s$. Clearly $x \in B(x_s, r_s)$; hence $d(x, x_s) < r_s < 2\mu(x_s) + \delta(x_s) < \varepsilon(x)$. If $y \in B(x, \eta(x))$, then $d(y, x) < \eta(x) < r_s$ and, hence, $d(y, x_s) \le d(y, x) + d(x, x_s) < 2r_s$. Therefore $B(x, \eta(x)) \subset B(x_s, 2r_s)$ and

$$\varphi(B(x, \eta(x))) \subset \varphi(B(x_s, 2r_s)) \subset B(\varphi(x_s), \mu(x_s)).$$

Hence

$$B(\varphi(B(x,\eta(x))),\delta(x)) \subset B(\varphi(x_s),\mu(x_s)+\delta(x)) \subset B(\varphi(x_s),\varepsilon(x))$$

because $x \in B(x_s, r_s)$ and, thus, $\delta(x) < \mu(x_s) + \delta(x_s)$ and $\mu(x_s) + \delta(x) < 2\mu(x_s) + \delta(x_s) < \varepsilon(x)$. Putting $x' := x_s$ we end the proof of Step 1.

*Step 2.* For any $(x,y) \in X \times E$, let

$$U(x,y) := [\eta^{-1}((\eta(x)/2,\infty)) \cap B(x,\eta(x)/2)] \times D(y,\delta(x))$$

and

$$\mathcal{U} := \bigcup_{(x,y)\in \mathrm{Gr}(\varphi)} U(x,y).$$

Then $\mathcal{U}$ is a neighbourhood of $\mathrm{Gr}(\varphi)$.

(i) Observe that if $f\colon A \to E$ is a $\delta(\,\cdot\,)$-selection of $\varphi$, then $\mathrm{Gr}(f) \subset \mathcal{U}$, i.e. $f$ is a $\mathcal{U}$-approximation of $\varphi$ over $A$.

(ii) It is easy to see that if, for some set $W \subset X$, $f\colon W \to E$ is a $\mathcal{U}$-approximation of $\varphi$ over $W$, then, for each $x \in W$,

$$f(x) \in B(\varphi(B(x,\eta(x))),\delta(x)).$$

(iii) By Proposition 2.3.4(b), there exists a continuous function $\rho\colon X \to (0,\infty)$ such that if $f\colon A \to E$ is a $\rho(\,\cdot\,)$-approximation of $\varphi$ over $A$, then $f$ is a $\mathcal{U}$-approximation of $\varphi$ over $A$.

*Step 3.* Now let $f\colon A \to E$ be an arbitrary continuous $\delta(\,\cdot\,)$-selection (resp. $\rho(\,\cdot\,)$-approximation) of $\varphi|_A$ (resp. of $\varphi$ over $A$). By condition (i) (resp. (iii)), $f$ is a $\mathcal{U}$-approximation of $\varphi$ over $A$. By the Dugundji theorem, there is a continuous extension $g\colon X \to E$ of $f$ (i.e. $g|_A = f$). Since $\mathcal{U}$ is open in $X \times E$, there exists a neighbourhood $W$ of $A$ such that, for $x \in W$, $(x, g(x)) \in \mathcal{U}$. Hence, by condition (ii), for $x \in W$, $g(x) \in B(\varphi(B(x,\eta(x))),\delta(x))$.

Let $V$ be an open neighbourhood of $A$ such that $A \subset V \subset \mathrm{cl}\, V \subset W$ and consider a partition of unity $\{\alpha,\beta\}$ subordinated to the cover $\{W, X \setminus \mathrm{cl}\, V\}$, i.e. $\alpha,\beta\colon X \to [0,1]$ are continuous function with $\operatorname{supp}\alpha \subset W$, $\operatorname{supp}\beta \subset X \setminus \mathrm{cl}\, V$ and $\alpha(x) + \beta(x) = 1$ for all $x \in X$. Finally let $h\colon X \to E$ be a continuous $\mathcal{U}$-approximation of $\varphi$ (existing in view of Theorem 2.3.11. By (ii), for each $x \in X$, $h(x) \in B(\varphi(B(x,\eta(x))),\delta(x))$. Let $F\colon X \to E$ be given by

$$F(x) = \alpha(x)g(x) + \beta(x)h(x), \quad x \in X.$$

Then $F|_A = f$. If $x \in X$ and $\alpha(x) \neq 0$, then $x \in W$ and

$$g(x) \in B(\varphi(B(x,\eta(x))),\delta(x)).$$

Hence, for such $x$, by Step 1, $g(x), h(x) \in B(\varphi(x'),\varepsilon(x))$ for some $x'$ with $d(x',x) < \varepsilon(x)$. Thus, by convexity

$$F(x) \in B(\varphi(x'),\varepsilon(x)) \subset B(\varphi(B(x,\varepsilon(x))),\varepsilon(x)).$$

If $\alpha(x) = 0$, then $F(x) = h(x) \in B(\varphi(B(x, \varepsilon(x))), \varepsilon(x))$, too. This completes the proof. $\square$

The importance of Theorem 2.4.3 is reflected by the following corollary.

**Corollary 2.4.4.** *Let $X$ be a metric space, $E$ a normed space and $\varphi: X \multimap E$ an upper semicontinuous map with convex values. To any continuous function $\varepsilon: X \to (0, \infty)$, there corresponds a continuous function $\rho: X \to (0, \infty)$ such that any two continuous $\delta(\,\cdot\,)$-approximations $f, g: X \to E$ are homotopic through a continuous homotopy $h: X \times [0, 1] \to E$ such that, for each $t \in [0, 1]$, $h(\cdot, t): X \to E$ is an $\varepsilon(\,\cdot\,)$-approximation of $\varphi$.*

*Proof.* Let $X' := X \times [0, 1]$ and let $\pi: X' \to X$ be the projection. Define $\varphi' := \varphi \circ \pi: X' \multimap E$. It is clear that $\varphi'$ is upper semicontinuous with convex values. The set $A' := X \times \{0, 1\}$ is closed in $X'$. Given a continuous $\varepsilon: X \to (0, \infty)$, define $\varepsilon': X' \to (0, \infty)$ putting $\varepsilon'(x, t) := \varepsilon(x)$ for any $x \in X$ and $t \in [0, 1]$. There is a function $\delta': X' \to (0, \infty)$ such that any continuous $\delta'(\,\cdot\,)$-approximation of $\varphi'$ over $A'$ extends to a continuous $\varepsilon'(\,\cdot\,)$-approximation of $\varphi'$ (over $X'$). Let

$$\delta(x) := \min\{\delta'(x, 0), \delta'(x, 1)\}, \quad x \in X,$$

consider two continuous $\delta(\,\cdot\,)$-approximations $f, g: X \to E$ of $\varphi$ and let $h': A' \to E$ be given for by

$$h'(x, t) = \begin{cases} f(x) & \text{if } x \in X,\ t = 0, \\ g(x) & \text{if } x \in X,\ t = 1. \end{cases}$$

Then $h'$ is a $\delta'(\,\cdot\,)$-approximation of $\varphi'$ over $A'$. There is a continuous extension $h: X \times [0, 1] \to E$ of $h'$ such that $h$ is an $\varepsilon'(\,\cdot\,)$-approximation of $\varphi'$. It is easy to see that, for each $t \in [0, 1]$, $h(\cdot, t)$ is an $\varepsilon(\,\cdot\,)$-approximation of $\varphi$. $\square$

**Remark 2.4.5.** (a) One says that a map $\varphi: X \multimap Y$, where $X$, $Y$ are metric spaces, is *weakly homotopy approximable over* $A$ if, for any $\varepsilon > 0$, there is a continuous function $\delta: X \to (0, +\infty)$ such that any two continuous $\delta(\,\cdot\,)$-approximations $f, g: A \to Y$ of $\varphi$ over $A$ may be joined by a continuous homotopy $h: A \times [0, 1] \to Y$ such that $h(\cdot, t)$ is a $\varepsilon$-approximation of $\varphi$ over $A$ ([12]). Corollary 2.4.4 states actually that an upper semicontinuous map $\varphi: X \multimap E$, where $X$ is a metric space and $E$ is a normed space, with convex values is weakly homotopy approximable.

(b) By inspection of the proof of Theorem 2.4.3, we see that if $\varphi$ is compact, then for any continuous $\varepsilon(\,\cdot\,)$, the constructed $\rho(\,\cdot\,)$ is such that any compact continuous $\rho(\,\cdot\,)$-approximation $f: A \to E$ of $\varphi$ over $A$ extends to a continuous compact $\varepsilon(\,\cdot\,)$-approximation provided $E$ is a Banach space. This follows from the

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([12]) Analogously a map $\varphi: X \multimap Y$, between topological spaces, is *homotopy approximable over* $A$ if, for any neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a neighbourhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that any two $\mathcal{V}$-approximations of $\varphi$ over $A$ may be joined by a continuous homotopy whose fibres are $\mathcal{U}$-approximations over $A$.

fact that if $f$ is a continuous compact $\rho(\,\cdot\,)$-approximation of $\varphi$ over $A$ (such approximations exist in view of Remark 2.3.13 and $f(A) \subset K$, where $K$ is compact convex, then there is a continuous extension $g\colon X \to E$ of $f$ such that $g(X) \subset K$. On the other hand, again in view of Theorem 2.3.11 and Remark 2.3.13, there is a compact continuous $\mathcal{U}$-approximation of $\varphi$. Hence $F$, as a convex combination of compact maps is compact. In particular, if in Corollary 2.4.4 $\varphi$ is compact, then one may produce a compact homotopy.

**2.5. Strict approximations.** For simplicity assume now that $X, Y$ are metric spaces. Given $\varepsilon > 0$ we have considered the existence of continuous $\varepsilon$-approximations $f\colon X \to Y$ of $\varphi\colon X \multimap Y$, i.e. such that $\mathrm{Gr}(f) \subset B(\mathrm{Gr}(\varphi), \varepsilon)$. It is clear that in this case the Hausdorff distance

$$h(\mathrm{Gr}(f), \mathrm{Gr}(\varphi)) < \varepsilon.$$

Apart from his first result concerning the existence of $\varepsilon$-approximations of an upper semicontinuous map $\varphi\colon X \multimap E$ with convex values in a normed space $E$, Cellina has shown that if $X$ has no isolated points, then $\varphi$ admits the *strict $\varepsilon$-approximations*, i.e. continuous maps $f\colon X \to E$ such that

$$d_H(\mathrm{Gr}(f), \mathrm{Gr}(\varphi)) < \varepsilon$$

provided values of $\varphi$ are additionally compact. This very interesting result has the following generalization due to Brodsky and Semenov.

**Theorem 2.5.1** (Cellina, [19])**.** *Let $\varphi\colon X \multimap Y$ be an upper semicontinuous map such that, for $x \in X$, $\varphi(x)$ is compact and convex. If $X$ has no isolated points* (*or isolated points of $X$ are sent to singletons*) *and $Y$ is locally contractible, then for each $\varepsilon > 0$ there is a strict $\varepsilon$-approximation of $\varphi$.*

The proof is fairly complicated and will not be reproduced here.

**2.6. Approximations without convexity.** Existence results provided in the above sections concern set-valued maps with *convex* values. From the viewpoint of applications this level of generality is not sufficient. Several results weakening the usual convexity assumption rely on various notions of generalized convexity or relaxed convexity (see e.g. [65] and references therein). We shall not dwell upon these results since they have mostly theoretic interest. Here we shall discuss rather some aspect of the approximation theory for set-valued maps whose values satisfy some purely geometric and topological assumptions.

Let $Y$ be a Hausdorff topological space. We say that a closed set $A \subset Y$ has *$UV^n$-property in $Y$* (see e.g. [59]), where $n \geq 0$ is an integer, if any neighbourhood $U$ of $A$ (in $Y$) contains a neighbourhood $V$ of $A$ such that the inclusion $V \hookrightarrow U$ is homotopy $k$-trivial for $0 \leq k \leq n$, i.e. any continuous map $f\colon S^k \to V$ has a continuous extension $F\colon D^{k+1} \to U$ ([13]).

([13]) Here and in what follows, for $k \geq 0$, $B^{k+1} := \{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$ is the unit closed ball in $\mathbb{R}^{k+1}$ and $S^k := \partial D^{k+1}$ is the unit sphere in $\mathbb{R}^{k+1}$.

For instance $A$ has $UV^0$-property in $Y$ if and only if, for each neighbourhood $U$ of $A$, there is a neighbourhood $V \subset U$ of $A$ such that any two points form $V$ are joined by a path in $U$. A point $y$ has $UV^n$-property in $Y$ if and only if $Y$ is locally $k$-connected at $y$ for all $0 \leq k \leq n$.

A closed set $A \subset Y$ has a $UV^\omega$*-property in* $Y$ if it has $UV^n$-property in $Y$ for all integers $n \geq 0$.

Finally, we say that $A \subset Y$ has $UV^\infty$*-property in* $Y$ if any neighbourhood $U$ of $A$ contains a neighbourhood $V$ of $A$ such that $V$ is contractible in $U$ (i.e. there is a continuous map $h: V \times [0,1] \to U$ such that, for all $x \in V$, $h(x,0) = x$ and $h(x,1) = x_0$ where $x_0 \in U$ is a constant).

It is clear that $UV^\infty$-property implies $UV^\omega$-property which, in turn, implies $UV^n$-property for each integer $n \geq 0$ ([14]). It is also clear that properties defined above are properties of the embedding of a given set in the ambient space $Y$ rather than of the set $A$ itself. For instance a point $y \in Y$ has $UV^\infty$-property in $Y$ if and only if $Y$ is locally contractible at $y$; if $y$ is considered as a point of a different space, then it may loose this property. Therefore it makes sense to consider corresponding absolute properties. Namely we say that a space $A$ has $UV^n$*-property* (where $n \geq 0$ is an integer, $n = \omega$ or $n = \infty$), written $A \in UV^n$, if there is a closed embedding of $A$ into an absolute neighbourhood retract $Y$ ([15]), i.e. there is a homeomorphism $e: A \to B$ where $B \subset Y$ is closed, such that $B$ has $UV^n$-property in $Y$.

It is easy to prove that $A \in UV^n$ if and only if $A$ has $UV^n$-property in *any* $Z \in \mathrm{ANR}$ in which $A$ is closed. Moreover, the $UV^n$-property is a homotopy type invariant, i.e. if spaces $A$ and $B$ have the same homotopy type and $A \in UV^n$, then $B \in UV^n$, too.

In what follows we shall often speak of the so-called $R_\delta$-sets (see [53]). We say that a compact metric space $A$ is an $R_\delta$*-set*, written $A \in R_\delta$, if there is a decreasing sequence $\{A_n\}_{n=1}^\infty$ of compact contractible metric spaces such that $A = \bigcap_{n=1}^\infty A_n$. One shows that a space $A \in R_\delta$ if and only if $A$ is compact metrizable and $A \in UV^\infty$. At this point let us also mention that if $A$ is a closed subset of $Y \in \mathrm{ANR}$, then $A \in UV^\infty$ if and only if $A$ is contractible in any of its neighbourhoods (in $Y$).

The class of $R_\delta$-sets (and henceforth of sets having $UV^\infty$-property) is quite reach: for instance any compact convex subset a normed space and, more generally, a compact contractible metric space belongs to the class of $R_\delta$-sets.

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([14]) There is an intermediate property: we say that $A \subset Y$ is *proximally $\infty$-connected in* $Y$ (or $A \in PC_Y^\infty$) if each neighbourhood $U$ of $A$ contains a neighbourhood $V$ of $A$ such that the inclusion $V \hookrightarrow U$ is homotopy $k$-trivial for any integer $k \geq 0$. It is clear that if $A \in PC_Y^\infty$, then it has $UV^\omega$-property in $Y$, but — in general — not conversely.

([15]) A metric space $Y$ is an ANR if given a metric space $Z$ and a closed subset $A \subset Z$, any continuous map $f: A \to Y$ admits a continuous extension $f^*: U \to Y$ onto a neighbourhood $U$ of $A$ in $Z$. We write $Y \in \mathrm{ANR}$. For details on theory of retracts — see [13] and [46].

We say that a set-valued map $\varphi\colon X \multimap Y$, where $X, Y$ are topological spaces, is a *$UV^n$-valued map*, $n \geq 0$ is an integer, or $n = \omega, \infty$, if, for each $x \in X$, the set $\varphi(x)$ is closed and has $UV^n$-property in $Y$. If, additionally, $\varphi(x)$ is compact for any $x \in X$, then we say that $\varphi$ is *$UV^n$-compact-valued* map.

We start our discussion of the approximation results for $UV^n$-valued maps with the following result which slightly generalizes a result due to Grniewicz, Granas and Kryszewski [47] and [56].

**Theorem 2.6.1.** *Suppose that $X$ is a finite dimensional polyhedron and $\dim X = N = n + 1$, where $n \geq 0$ is an integer. Let $X_0$ be a subpolyhedron of $X$, $\dim X_0 = N_0$. Suppose that $\varphi\colon X \to Y$ is an upper semicontinuous $UV^n$-compact-valued map. Then, for any neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ there is a neighbourhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that any continuous $\mathcal{V}$-approximation $f_0\colon X_0 \to Y$ of $\varphi$ over $X_0$ extends to a continuous $\mathcal{U}$-approximation of $\varphi$. In particular $\varphi$ is approximable and weakly approximable* [16].

*Proof.* First we shall prove the following lemmata:

Lemma 1. *Let $\varphi\colon X \to Y$ be an upper semicontinuous set-valued map and let $X$ be paracompact. Suppose that, for each $x \in X$, $N_x$ is a neighbourhood of $\varphi(x)$ in $Y$ and let $\{U_x\}_{x \in X}$ be an open cover of $X$ such that $x \in U_x$ for each $x \in X$. Then there are an open cover $\{L_x\}_{x \in X}$ of $X$ and a neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ such that $L_x \subset U_x$ for all $x \in X$ and $\mathcal{U}(L_x) \subset N_x$.*

*Proof.* By upper semicontinuity, for each $x \in X$, there is a neighbourhood $x \in V_x \subset U_x$ such that $\varphi(V_x) \subset N_x$. Paracompactness of $X$ implies that there is a locally finite open covering $\{L_x\}_{x \in X}$ such that $\mathrm{cl}\, L_x \subset V_x \subset U_x$, i.e. $\varphi(\mathrm{cl}\, L_x) \subset N_x$. We define $\mathcal{U}$ $X \times Y$ by saying that, for $x \in X$,

$$\mathcal{U}(x) = \bigcap \{N_w \mid w \in X \text{ and } x \in \mathrm{cl}\, L_w\}.$$

Let $x \in X$. If $z \in L_x$, then $\mathcal{U}(z) \subset N_x$: thus $\mathcal{U}(L_x) \subset N_x$. Observe that if $w \in X$ and $x \in \mathrm{cl}\, L_w$, then $\varphi(x) \subset \varphi(\mathrm{cl}\, L_w) \subset N_w$. Thus $\varphi(x) \subset \mathcal{U}(x)$. This proves that $\mathrm{Gr}(\varphi) \subset \mathcal{U}$. Finally it is not difficult to show that $\mathcal{U}$ is open (in $X \times Y$). $\square$

Lemma 2. *Let $X$ be a paracompact space, $\varphi\colon X \multimap Y$ be an upper semicontinuous $UV^n$-compact-valued map ($n \geq 0$ is an integer) and let $\mathcal{U}$ be a neighbourhood of $\mathrm{Gr}(\varphi)$. Then there is a sequence $\{(\mathcal{U}_i, \mathfrak{A}_i)\}_{i \geq 0}$ such that:*

(a) *$\mathcal{U}_{i+1} \subset \mathcal{U}_i$, $i \geq 0$, are (open) neighbourhoods of $\mathrm{Gr}(\varphi)$ in $X \times Y$;*
(b) *$\mathfrak{A}_i$, $i \geq 0$, are open covers of $X$;*
(c) *for each $i \geq 1$ and each $U \in \mathfrak{A}_i$, there is $U^\varphi \in \mathfrak{A}_{i-1}$ such that $\mathrm{st}\,(U, \mathfrak{A}_i) \subset U^\varphi$ (i.e. $\mathfrak{A}_i$ is a star-refinement of $\mathfrak{A}_{i-1}$) and, for any $0 \leq k \leq n$, the inclusion $\mathcal{U}_i(\mathrm{st}\,(U, \mathfrak{A}_i)) \hookrightarrow \mathcal{U}_{i-1}(U^\varphi)$ is homotopy $k$-trivial;*
(d) *for any $U \in \mathfrak{A}_0$ and $x \in U$, $\mathcal{U}_0(U) \subset \mathcal{U}(x)$.*

[16] Observe that $X$ (being considered with the CW-topology, i.e. Whitehead topology), as a finite-dimensional polyhedron, is paracompact (even metrizable).

*Proof.* Since $\mathrm{Gr}(\varphi \circ \mathrm{id}_X \circ \mathrm{id}_X) = \mathrm{Gr}(\varphi)$, by Theorem 2.37, there are neighbourhoods $\mathcal{U}_0$ of $\mathrm{Gr}(\varphi)$ and $\mathcal{M}$ of $\Delta_X = \mathrm{Gr}(\mathrm{id}_x)$ in $X \times X$ such that $\mathcal{U}_0 \circ \mathcal{M} \circ \mathcal{M}^{-1} \subset \mathcal{U}$. Let $\mathfrak{A}_0 := \{\mathcal{M}(x) \mid x \in X\}$. Then $\mathfrak{A}_0$ is an open cover of $X$ and if $x \in \mathcal{M}(x')$ for some $x' \in X$, then $x' \in \mathcal{M}^{-1}(x)$. Hence

$$\mathcal{U}_0(\mathcal{M}(x')) \subset \mathcal{U}_0 \circ \mathcal{M} \circ \mathcal{M}^{-1}(x) \subset \mathcal{U}(x).$$

This shows property (d).

Now let $i \geq 1$ and suppose that a neighbourhood $\mathcal{U}_{i-1}$ of $\mathrm{Gr}(\varphi)$ and an open cover $\mathfrak{A}_{i-1}$ of $X$ have been constructed. Let $x \in X$. Choose $U_x \in \mathfrak{A}_{i-1}$ such that $x \in U_x$. Then $\varphi(x) \subset \mathcal{U}_{i-1}(U_x)$. Since $\varphi(x)$ has the $UV^n$-property in $Y$, there is a neighbourhood $N_x$ of $\varphi(x)$ in $Y$ such that $N_x \subset \mathcal{U}_{i-1}(U_x)$ and the inclusion $N_x \hookrightarrow \mathcal{U}_{i-1}(U_x)$ is homotopy $k$-trivial for any $0 \leq k \leq n$. By Lemma 1, there is an open cover $\{L_x\}$ of $X$ and a neighbourhood $\mathcal{U}_i$ of $\mathrm{Gr}(\varphi)$ such that $\mathcal{U}_i \subset \mathcal{U}_{i-1}$ and, for each $x \in X$, $L_x \subset U_x$ and $\mathcal{U}_i(L_x) \subset N_x$. Finally, let $\mathfrak{A}_i$ be the star-refinement of $\{L_x\}$ It is easy to see that this completes the proof. $\square$

We may pass to the proof of the theorem. Fix a neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ and take a sequence $\{(\mathcal{U}_i, \mathfrak{A}_i)\}_{i \geq 0}$ as in Lemma 2. Let $\mathcal{V} = \mathcal{U}_N$ and take a $\mathcal{V}$-approximation $f_0 \colon X_0 \to Y$ of $\varphi$ over $X_0$.

Suppose that $(T, T_0)$ is a triangulation of $(X, X_0)$ finer than the covering $\mathfrak{A}_{N+1}$, i.e. the bodies $|T| = X$, $|T_0| = X_0$, $T_0$ is a subcomplex of $T$ and given a simplex $\sigma$ in $X$ (i.e. $\sigma \in T$), there is $V \in \mathfrak{A}_{N+1}$ such that $\sigma \subset V$. For any $0 \leq k \leq N$, let $T^k$ be the $k$-dimensional skeleton of $T$ (in particular $T^N = T$) and let $X^k := |T^k|$. Then $X^N = X$.

Assume that, for any $k$-simplex in $X$, an element $U_\sigma \in \mathfrak{A}_{N-k+1}$ has been selected such that $\sigma \subset U_\sigma$. Then if $\tau$ is a face of $\sigma$, then $U_\tau \cap U_\sigma \neq \emptyset$.

Let $\sigma \in T_0$ be a $k$-simplex in $X_0$, $0 \leq k \leq N$. Then

$$f_0(\sigma) \subset \mathcal{U}_N(\sigma) \subset \mathcal{U}_N(U_\sigma) \subset \mathcal{U}_{N-k}(U_\sigma^\varphi).$$

We shall show now that there exists a continuous extension $f \colon X \to Y$ of $f_0$ such that, for any $k$-simplex $\sigma$ (i.e. $\sigma \in T^k$), $f(\sigma) \subset \mathcal{U}_{N-k}(U_\sigma^\varphi)$. We define $f$ inductively on skeleta of increasing dimension. Let $k \geq 0$ and assume that $f$ has been defined on the $(k-1)$-dimensional skeleton of $X$ (recall that, by definition $T^{(-1)} = \emptyset$). Let $\sigma$ be an arbitrary $k$-simplex. Thus $f$ is already defined on $\partial\sigma$. If $\sigma \subset X_0$, then we set $f|_\sigma = f_0|_\sigma$. Assume that $\sigma$ is not a simplex in $X_0$. If $k = 0$, i.e. $\sigma = \{v\}$ is a vertex, then we choose $f(v) \in \mathcal{U}_N(U_v^\varphi)$. Suppose that $k \geq 1$. Then $\sigma = \tau_0 \cup \tau_1 \cup \ldots \cup \tau_k$ where $\dim \tau_i = k-1$ and $\tau_i$ is a face of $\sigma$; hence $U_{\tau_i} \cap U_\sigma \neq \emptyset$ for all $i = 0, \ldots, k$. For all $0 \leq i \leq k$, we have $U_{\tau_i} \in \mathfrak{A}_{N-k+2}$ and $U_{\tau_i} \subset U_{\tau_i}^\varphi \in \mathfrak{A}_{N-k+1}$ ($\mathfrak{A}_{N-k+2}$ refines $\mathfrak{A}_{N-k+1}$). Hence $\bigcup_{i=0}^k U_{\tau_i}^\varphi \subset \mathrm{st}\,(U_\sigma, \mathfrak{A}_{N-k+1})$. By inductive hypothesis, $f(\tau_i) \subset \mathcal{U}_{N-k+1}(U_{\tau_i}^\varphi)$ for all $0 \leq i \leq k$. Therefore $f(\partial\sigma) \subset \mathcal{U}_{N-k+1}(\mathrm{st}\,(U_\sigma, \mathfrak{A}_{N-k+1})$ and $f$ has a continuous extension onto $\sigma$ such that $f(\sigma) \subset \mathcal{U}_{N-k}(U_\sigma^\varphi)$.

This ends the proof. Indeed, for any $x \in X$, there is a $k$-simplex in $X$ such that $x \in \sigma \subset U_\sigma \subset U_\sigma^\varphi$. Hence $f(x) \in \mathcal{U}_{N-k}(U_\sigma^\varphi)$. There is $U \in \mathfrak{A}_0$ such that $x \in U_\sigma^\varphi \subset U$. Thus $f(x) \in \mathcal{U}_{N-k}(U) \subset \mathcal{U}_0(U) \subset \mathcal{U}(x)$. □

If we do not control the dimension of the finite dimensional polyhedron $X$, then under the assumption that $\varphi: X \multimap Y$ is an upper semicontinuous $UV^\omega$-valued map we get the same conclusion as concerns the existence of graph-approximations. It is also worthwhile to note that Theorem 2.6.1 stays true if $X$ is merely a locally finite-dimensional polyhedron (with the CW-topology); the proof is similar although much more complicated.

Our next results show how approximation of $UV$-set-valued maps defined on polyhedra may be 'lifted' to more general spaces. First we formulate the simplest fact; then we shall discuss ways to generalize it.

**Theorem 2.6.2** (see [56])**.** *Let* $(X, A)$ *be a compact* ANR*-pair* (*i.e.* $X$ *and* $A$ *are compact* ANR*s and* $A \subset X$) *and let* $Y$ *be a topological space. If* $\varphi: X \multimap Y$ *is an upper semicontinuous* $UV^\omega$*-compact-valued map, then for each neighbourhood* $\mathcal{U}$ *of* $\mathrm{Gr}(\varphi)$*, there is a neighbourhood* $\mathcal{V}$ *of* $\mathrm{Gr}(\varphi)$ *such that any continuous* $\mathcal{V}$*-approximation* $f: A \to Y$ *of* $\varphi$ *over* $A$ *extends to a continuous* $\mathcal{U}$*-approximation of* $\varphi$*. In particular* $\varphi$ *is approximable and weakly approximable.*

*Proof.* We shall need the following lemma which may be of interest on its own. □

**Lemma 2.6.3.** *Let* $(X, A)$ *be an* ANR*-pair and let* $\varphi: X \multimap Y$*, where* $Y$ *is topological space, be an upper semicontinuous map with compact values. Let* $M := X \times \{0\} \cup A \times [0,1]$*. For any neighbourhood* $\mathcal{U}$ *of* $\mathrm{Gr}(\varphi)$*, there is a neighbourhood* $\mathcal{U}_0$ *of* $\mathrm{Gr}(\varphi)$ *with the following property: for every continuous map* $g: M \to Y$ *such that* $(x, g(x,t)) \in \mathcal{U}_0$ *for all* $(x,t) \in M$*, there is a continuous extension* $G: X \times [0,1] \to Y$ *of* $g$ *such that* $(x, G(x,t)) \in \mathcal{U}$ *for all* $x \in X$ *and* $t \in [0,1]$.

*Proof.* Let $\mathcal{U}' := \{(x,t,y) \in X \times [0,1] \times Y \mid (x,y) \in \mathcal{U}\ .$ It is clear that $\mathcal{U}'$ is a neighbourhood of $\mathrm{Gr}(\varphi')$ where $\varphi' := \varphi \circ \pi$ and $\pi: X \times [0,1] \to X$. In view of Theorem 2.3.7, there are a neighbourhood $\mathcal{U}_0'$ of $\mathrm{Gr}(\varphi')$ and a neighbourhood $\mathcal{N}$ of the diagonal in $(X \times [0,1])^2$ (being the graph of identity $X \times [0,1] \to X \times [0,1]$) such that $\mathcal{U}_0' \circ \mathcal{N} \qquad '$.

Obviously $M$ is a neighbourhood retract of $X \times [0,1]$; hence there is a retraction $r: U \to M$ where $U$ is a neighbourhood of $M$ in $X \times [0,1]$. It is easy to see that there is a neighbourhood $V$ of $M$ (in $X \times [0,1]$) such that $V \subset U$ and $r(x,t) \in \mathcal{N}(x,t)$ for all $(x,t) \in V$.

Again by Theorem 2.3.7 (or rather Remark 2.3.8(a)), and since $\pi$ is perfect, there is a neighbourhood $\mathcal{U}_0$ of $\mathrm{Gr}(\varphi)$ such that $\{(x,t,y) \in X \times [0,1] \times Y \mid (x,y) \in \mathcal{U}_0\} \subset \mathcal{U}_0'$. Take $g: M \to Y$ such that$(x, g(x,t)) \in \mathcal{U}_0)$ for $(x,t) \in M$ and define

$g' = g \circ r|_V \colon V \to Y$. Then, for all $(x,t) \in V$, $(x, t, g'(x,t)) \in \mathcal{U}'_0 \circ \mathcal{N}$ $'$; hence

$$(x, g'(x,t)) \in \mathcal{U}.$$

Take any neighbourhood $N$ of $A$ such that $N \times [0,1] \subset V$, an Urysohn function $\lambda\colon X \to [0,1]$ such that $\lambda|_A \equiv 1$ and $\lambda|_{X\setminus N} \equiv 0$ and define $G\colon X \times [0,1] \to Y$ putting $G(x,t) := g'(x, \lambda(x)t)$ for $x \in X$ and $t \in [0,1]$. For all $(x,t) \in X \times [0,1]$, $(x, G(x,t)) \in \mathcal{U}$ and $G|_M = g$. □

Let us pass to the proof of Theorem 2.6.2. Take a neighbourhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ and a neighbourhood $\mathcal{U}_0$ of $\mathrm{Gr}(\varphi)$ as indicated in the lemma. Since $\varphi = \varphi \circ \mathrm{id}_X$ and (obviously) $\mathrm{id}_x$ is perfect, by Theorem 2.3.7, there are neighbourhoods $\mathcal{W}$ of $\mathrm{Gr}(\varphi)$ (in $X \times Y$) and $\mathcal{N}$ of the diagonal $X$ such that $\mathcal{W} \circ \mathcal{N}$ $_0$. Moreover, let $\mathfrak{A}$ be an open cover of $X$ such that, for each $U \in \mathfrak{A}$ and $x \in U$, $U \subset \mathcal{N}(x)$ (i.e. $\mathfrak{A}$ is a refinement of the cover $\{\mathcal{N}(x)\}_{x \in X}$).

*Key Step.* Since $(X, A)$ is a compact ANR-pair, there are continuous maps $p\colon (X,A) \to (P,Q)$ and $r\colon (P,Q) \to (X,A)$, where $(P,Q)$ is a finite polyhedral pair, such that $r \circ p$ and $\mathrm{id}_X$ are joined by a homotopy $h\colon (X,A) \times [0,1] \to (P,Q)$ such that the family $\{h(\{x\} \times [0,1])\}_{x \in X}$ refines $\mathfrak{A}$. Moreover, since $P$ is compact $r$ is perfect (or, equivalently, proper).

The map $\varphi \circ r\colon P \multimap Y$ is an upper semicontinuous $UV^\omega$-valued map and $\mathrm{Gr}(\varphi \circ r) \subset \mathcal{U}' := \{(z,y) \in P \times Y \mid (r(z), y) \in \mathcal{W}\}$. One sees easily that $\mathcal{U}'$ is a neighbourhood of $\mathrm{Gr}(\varphi \circ r)$. In view of Theorem 2.6.1, there is a neighbourhood $\mathcal{V}'$ of $\mathrm{Gr}(\varphi \circ r)$ such that any continuous $\mathcal{V}'$-approximation $f'\colon Q \to Y$ of $\varphi \circ r$ over $Q$ extends to a continuous $\mathcal{U}'$-approximation $F'\colon P \to Y$ of $\varphi \circ r$.

Again by Remark 2.3.8, there is a neighbourhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$ such that $\mathcal{V}$ and $\{(z,y) \in P \times Y \mid (r(z), y) \in \mathcal{V}$ $'$.

Take any continuous $\mathcal{V}$-approximation $f\colon A \to Y$ of $\varphi$ over $A$ and let $f' := f \circ r$. Then $f'\colon Q \to Y$ is a continuous $\mathcal{V}'$-approximation of $\varphi \circ r$. Hence there is a $\mathcal{U}'$-approximation $F'\colon P \to Y$ of $\varphi \circ r$ extending $f'$. Define $g\colon M := X \times \{0\} \cup A \times [0,1] \to Y$ by

$$g(x,t) := \begin{cases} F' \circ p(x) & \text{if } x \in X,\ t = 0, \\ f \circ h(x,t) & \text{if } x \in A,\ t \in [0,1]. \end{cases}$$

Then $g$ is well-defined and continuous. Moreover, for any$(x,t) \in M$, $(x, g(x,t)) \in \mathcal{U}_0$. In view of Lemma 2.6.3, and due to the choice of $\mathcal{U}_0$, we gather that there is a continuous extension $G\colon X \times [0,1] \to Y$ of $g$ such that $(x, G(x,t)) \in \mathcal{U}$ for all $(x,t) \in X \times [0,1]$.

Finally let $F\colon = G(\cdot, 1)$. Then $F|_A = f$ and $\mathrm{Gr}(F) \in \mathcal{U}$. □

**Remark 2.6.4.** The most important ingredient of the proof is contained in the Key Step above: the compactness of $X$ implies that:

(a) *for any open cover* $\mathfrak{A}$ *of* $X$*,* $X$ *as a compact* ANR *is* $\mathfrak{A}$*-dominated by a finite polyhedron* (in the sense that there is a finite polyhedron $P$ and continuous maps $p\colon X \to P$ and $r\colon P \to X$ such that $r \circ p$ is $\mathfrak{A}$-homotopic to $\mathrm{id}_X$), and

(b) *the existing map $r$ is proper.*

It is known that *any separable or a locally compact $X \in$* ANR, is dominated in the same sense by a locally finite (hence, locally finite dimensional) polyhedron $P$. Therefore if $(X, A)$ is a separable or locally compact ANR-pair, then in view of the remark given after the proof of Theorem 2.6.1, the assertion of Theorem 2.6.2 stays true.

Analogously to Corollary 2.4.4 we get

**Corollary 2.6.5.** *If $X$ is a compact* ANR*, $\varphi\colon X \multimap Y$ i a $UV^\omega$-compact-valued upper semicontinuous map, then for each neighbourhood $\mathcal{U}$ of* $\mathrm{Gr}(\varphi)$*, there exists a neighbourhood $\mathcal{V}$ of* $\mathrm{Gr}(\varphi)$ *such that if $f, g\colon X \to Y$ are continuous $\mathcal{V}$-approximations of $\varphi$, then there is a continuous homotopy $h\colon X \times [0,1] \to Y$ such that $h(\,\cdot\,,0) = f$, $h(\,\cdot\,,1) = g$ and, for each $t \in [0,1]$, $h(\,\cdot\,,t)$ is a $\mathcal{U}$-approximation of $\varphi$. In other words $\varphi$ is homotopy approximable.*

**Remark 2.6.6.** The reader will easily show the analogues of Theorems 2.6.1, Theorem 2.6.2 and Corollary 2.6.5 for $UV^\omega$-valued maps (i.e. with values being (closed) sets having the $UV^\omega$-property) provided the initial neighbourhoods $\mathcal{U}$ are thick. In particular for such maps $\varepsilon(\,\cdot\,)$-approximations always exist; hence they are weakly approximable and weakly homotopy approximable.

Let us finally mention the following:

**Theorem 2.6.7** (see [56]). *Let $X$ be an* ANR *and $\varphi\colon X \multimap Y$, where $Y$ is a metric space, be a $UV^\omega$-valued upper semicontinuous set-valued map with compact values. Then, for any continuous $\varepsilon\colon X \to (0,\infty)$, there is an $\varepsilon(\,\cdot\,)$-approximation of $\varphi$.*

The proof involves different techniques and will not be reproduced here.

**2.7. Other types of approximations.** So far we have approximated set-valued maps by single-valued ones. Sometimes it is also convenient to approximate, in a sense, set-valued maps by more regular set-valued ones.

We shall start by a slight generalization of the result attributed to de Blasi and Deimling [32] saying that upper semicontinuous maps may be approximated from above by continuous maps.

**Theorem 2.7.1.** *Let $X$ be a metric space, $E$ a normed space and $\varepsilon > 0$. Suppose that $\varphi\colon X \multimap E$ is $H$-upper semicontinuous with closed convex values. For any integer $n \geq 1$, there is a $H$-continuous set-valued map $\varphi_n\colon X \multimap E$ with closed convex values such that:*

(a) $\varphi_n(x) \subset \operatorname{cl}\operatorname{conv}\varphi(X)$ *for every $n \geq 1$ and $x \in X$;*

(b) *for all $n \geq 1$, $\varphi(x) \subset \varphi_{n+1}(x) \subset \varphi_n(x) \subset B(\varphi(x), \varepsilon)$ for every $x \in X$;*

(c) $\lim_{n\to\infty} d_H(\varphi_n(x), \varphi(x)) = 0$ *for any $x \in X$.*

*Proof.* Take a sequence $(r_n)$ such that $3r_{n+1} < r_n$ for $n \ge 1$. For each $n \ge 1$ and $x \in X$, let $U_x^n := B(x, r_n)$; clearly $x \in U_x^n$. Let $\mathfrak{A}_n := \{U_x^n\}_{x \in X}$. It is easy to see that $\mathfrak{A}_{n+1}$ is a star-refinement of $\mathfrak{A}_n$. More precisely $\operatorname{st}(U_x^{n+1}, \mathfrak{A}_{n+1}) \subset U_x^n$ for all $n \ge 1$ and $x \in X$.

Let $\{\lambda_s^n\}_{s \in S_n}$ be a partition of unity subordinated to $\mathfrak{A}_n$, i.e. for each $s \in S_n$, there is $U_s^n \in \mathfrak{A}_n$ such that $\operatorname{supp} \lambda_s^n \subset U_s^n$. Let $F_s^n := \operatorname{cl}\operatorname{conv} \varphi(\operatorname{st}(U_s^n, \mathfrak{A}_n))$.

Define

$$\varphi_n(x) := \operatorname{cl}\left(\sum_{s \in S_n} \lambda_s^n(x) F_s^n\right) \quad \text{for } x \in X$$

It is clear that $\varphi_n$ is $H$-continuous and has closed convex values. We will show that the sequence $\{\varphi_n\}_{n \ge 1}$ has properties enlisted above.

(a) For each $x \in X$, $F_s^n \subset \operatorname{cl}\operatorname{conv} \varphi(X)$; hence $\varphi_n(x) \subset \operatorname{cl}\operatorname{conv} \varphi(X)$ for any $x \in X$.

(b) Let $n \ge 1$ and $x \in X$. If $s \in S_n$ and $\lambda_s^n(x) \ne 0$, then $x \in U_s^n$. Hence $\varphi(x) \subset F_s^n$ and $\varphi(x) \subset \varphi_n(x)$. Now we shall show that $\varphi_{n+1}(x) \subset \varphi_n(x)$. Let $y \in \sum_{s \in S_{n+1}} \lambda_s^{n+1}(x) F_s^{n+1}$. If $s \in S_{n+1}$ and $\lambda_s^{n+1}(x) \ne 0$, then $x \in U_s^{n+1}$ and there is $V \in \mathfrak{A}_n$ such that $x \in U_s^{n+1} \subset \operatorname{st}(U_s^{n+1}, \mathfrak{A}_{n+1}) \subset V$. Suppose that $t \in S_n$ and $\lambda_t^n(x) \ne 0$, i.e. $x \in U_t^n$. Thus $V \subset \operatorname{st}(U_t^n, \mathfrak{A}_n)$. This implies that $F_s^{n+1} \subset F_t^n$ and that $\varphi_{n+1}(x) \subset \varphi_n(x)$.

(c) Let $x \in X$; it is clear that $h(\varphi(x), \varphi_n(x)) = 0$ for all $n \ge 1$. Suppose to the contrary that there is $\varepsilon > 0$ such that $h(\varphi_{n_k}(x), \varphi(x)) > \varepsilon$ for some subsequence $(n_k)$. Then, for each $k \ge 1$ there must exists $y \in \varphi_{n_k}(x)$ such that $d(y, \varphi(x)) > \varepsilon$. Recall that, without loss of generality, $y = \sum_{s \in S^{n_k}} \lambda_s(x) F_s^{n_k}$. Hence there is $s \in S_{n_k}$ such that $\lambda_s(x) \ne 0$ and $y_k \in F_s^{n_k}$ such that $d(y_k, \varphi(x)) > \varepsilon$ (since otherwise, if for all $s \in S_{n_k}$, $F_s^{n_k} \subset D(\varphi(x), \varepsilon)$, then $y \in D(\varphi(x), \varepsilon)$). Again without loss of generality we may assume that $y_k \in \varphi(\operatorname{st}(U_s^{n_k}, \mathfrak{A}_{n_k}))$ where $s$ is such that $x \in \operatorname{supp} \lambda_s \subset U_s^{n_k}$. This means that $y_k \in \varphi(z_k)$ where $z_k \in \operatorname{st}(U_s^{n_k}, \mathfrak{A}_{n_k})$. There is $x_s \in X$ such that $U_s^{n_k} = B(x_s, r_{n_k})$. However it means that $z \in B(x, r_{n_k - 1})$, i.e. $z_k \to x$ as $k \to \infty$. This is a contradiction since $h(\varphi(z_k), \varphi(x)) \to 0$ as $k \to \infty$. $\square$

**Remark 2.7.2.** An interesting observation is that if $X$ is a metric space, $E$ is a normed space and $\varphi\colon X \multimap E$ is an upper semicontinuous with convex values if and only if there exists a sequence $\varphi_n\colon X \multimap E$ of maps satisfying conditions (a)–(c) from the above theorem. Moreover, it is clear that taking the Lipschitz partitions of unity subordinated to coverings $\mathfrak{A}_n$ we get that $\varphi_n$ is a locally Lipschitz map (see Remark 2.0.1.

The next result can be traced back to Michael. It says that upper semicontinuous maps may be sometimes graph-approximated by upper semicontinuous ones with compact values.

**Theorem 2.7.3.** *Let $X$ be a paracompact space, $Y$ be a Hausdorff space and let $\Phi\colon X \multimap E$ be upper semicontinuous and have closed values. For any thick* *neighbourhood* $\mathcal{U}$ *of* $\mathrm{Gr}(\Phi)$ *there are an upper semicontinuous map* $\varphi\colon X \multimap Y$ *and a lower semicontinuous map* $\psi\colon X \multimap Y$*, both with compact values, such that* $\psi(x) \subset \varphi(x)$ *for all* $x \in X$ *and* $\mathrm{Gr}(\varphi) \subset \mathcal{U}$.

*Proof.* Let $\mathcal{U}$ be a thick open neighbourhood of $\mathrm{Gr}(\Phi)$. By Remark 2.3.8(b), there are a neighbourhood $\mathcal{W}$ of $\mathrm{Gr}(\Phi)$ and a neighbourhood $\mathcal{N}$ of the diagonal in $X \times X$ such that $\mathcal{W} \circ \mathcal{N}$ . For each $x \in X$, $\Phi(x) \subset \mathcal{W}(x)$. The upper semicontinuity of $\Phi$ implies the existence, for each $x \in X$, of a neighbourhood $U_x$ of $x$ such that $\Phi(\operatorname{cl} U_x) \subset \mathcal{W}(x)$. Let $\mathfrak{B} := \{V_s\}_{s\in S}$ be a locally finite star-refinement of $\mathfrak{A} := \{U_x \cap \mathcal{N}^{-1}(x)\}_{x\in X}$. For each $s \in S$, choose $y_s \in \Phi(V_s)$ and define $\psi(x) := \{y_s \mid x \in V_s\}$ and $\varphi(x) := \{y_s \mid x \in \operatorname{cl} V_s\}$ for $x \in X$. Then clearly $\psi(x) \subset \varphi(x)$ on $X$. Observe moreover, that for each $x \in X$, $\#\psi(x), \#\varphi(x) < \infty$; hence both these sets are compact. We shall show that $\mathrm{Gr}(\varphi) \subset \mathcal{U}$. To this end take $x \in X$ and $y \in \varphi(x)$, i.e. $y = y_s$ where $s \in S$ and $x \in \operatorname{cl} V_s$. Thus

$$x \in \operatorname{cl} V_s \subset \operatorname{st}(\operatorname{cl} V_s, \mathfrak{B}) = \operatorname{st}(V_s, \mathfrak{B}) \subset U_{x_s} \cap \mathcal{N}^{-1}(x_s)$$

for some $x_s \in X$. Therefore

$$y = y_s \in \Phi(V_s) \subset \Phi(U_{x_s}) \subset \mathcal{W}(x_s)$$

and $x_s \in \mathcal{N}(x)$. Hence $(x, y) \in \mathcal{W} \circ \mathcal{N}$ .

For $B \subset Y$, let $S(B) := \{s \in S \mid y_s \in B\}$. Observe that

$$\varphi^{-1}(B) = \bigcup_{s\in S(B)} \operatorname{cl} V_s, \quad \psi^{-1}(B) = \bigcup_{s\in S(B)} V_s.$$

Therefore $\psi^{-1}(B)$ is open and, since the cover $\{\operatorname{cl} V_s\}$ is locally finite, $\varphi^{-1}(B)$ is closed. This completes the proof. □

**Remark 2.7.4.** If $Y$ is a locally convex space and $\mathcal{U}$ is a neighbourhood of $\mathrm{Gr}(\Phi)$ such that, for each $x \in X$, $\mathcal{U}(x)$ is convex, then we may take $\varphi(x) = \operatorname{conv}\{y_s \mid x \in \operatorname{cl} V_s\}$ and $\psi(x) = \operatorname{conv}\{y_s \mid x \in V_s\}$. Then $\varphi$ and $\psi$ have compact convex values, $\psi(x) \subset \varphi(x) \subset \mathcal{U}(x)$ for all $x \in X$ since, as above, $\{y_s \mid x \in \operatorname{cl} V_s\} \subset \mathcal{U}(x)$ and $\mathcal{U}(x)$ is convex. This shows that when studying upper semicontinuous map $\Phi$ with closed values in a locally convex space, we may restrict our attention to compact convex valued upper semicontinuous maps. The problem is that graphs of these 'approximations' lie in special neighbourhoods of the graph of $\Phi$, which — in general — is too large.

**2.8. Approximations with constraints.** The problem is as follows: *given spaces* $X, Y$*, set-valued maps* $\varphi, \psi\colon X \multimap Y$*, find a continuous* $\varepsilon$*-approximation* $f\colon X \to Y$ *of* $\varphi$ *such that* $f(x) \in \psi(x)$ *for all* $x \in X$.

First we shall prove a result stating the existence of a simultaneous approximation and an almost selection for upper and lower semicontinuous maps.

**Theorem 2.8.1** (see [8])**.** *Let $X$ be a metric space and $E$ a normed space. Let $\psi\colon X \multimap E$ be a lower semicontinuous map with convex values and $\varphi\colon X \multimap E$ an upper semicontinuous map with convex values such that $\varphi(x) \cap \psi(x) \neq \emptyset$ for all $x \in X$. Then for each $\varepsilon > 0$, there is a continuous function $f\colon X \to E$ such that $f$ is an $\varepsilon$-selection of $\psi$ and an $\varepsilon$-approximation of $\varphi$.*

*Proof.* Let, for $x \in X$, $U(x) := B(x,\varepsilon) \cap \varphi^{+1}(B(\varphi(x),\varepsilon))$ and let an open covering $\mathfrak{A}$ be a star-refinement of $\{U(x)\}_{x\in X}$. For any $x \in X$, choose $z_x \in \varphi(x) \cap \psi(x)$ and consider the open covering $\mathfrak{B} := \{B_V(x)\}_{V\in\mathfrak{A}, x\in V}$ of $X$, where $B_V(x) := \{y \in V \mid \psi(y) \cap B(z_x,\varepsilon) \neq \emptyset\}$. Let $\{\lambda\}_{s\in S}$ be a partition of unity subordinated to $\mathfrak{B}$, i.e. for each $s \in S$, there are $V_s \in \mathfrak{A}$, $x_s \in V_s$ such that $\operatorname{supp}\lambda_s \subset B_s := B_{V_s}(x_s)$. The map

$$f(x) := \sum_{s\in S} \lambda_s(x) z_s, \quad x \in X,$$

where $z_s := z_{x_s}$, is well-defined and continuous. If $x \in X$ and $\lambda_s(x) \neq 0$, then $x \in B_s$, i.e. there is $z'_s \in \psi(x)$ such that $\|z_s - z'_s\| < \varepsilon$. Thus, by convexity of $\psi(x)$, $y := \sum_{s\in S} \lambda_s(x) z'_s \in \psi(x)$. Moreover,

$$\|f(x) - y\| \leq \sum_{s\in S} \lambda_s(x) \|z_s - z'_s\| < \varepsilon.$$

On the other hand, given $x \in X$ and $s \in S$, $\lambda_s(x) \neq 0$, then $x \in B_s \subset V_s$ and $x_s \in V_s$. Since $\mathfrak{A}$ is a star refinement of $\{U(z)\}_{z\in X}$, $x_s \in \operatorname{st}(x,\mathfrak{A}) \subset U(z)$ for some $z \in X$. Therefore $z_s \in \phi(x_s) \subset B(\varphi(z),\varepsilon)$ and $\|x - z\| < \varepsilon$. The convexity of $\varphi(z)$ implies that

$$f(x) \in B(\varphi(z),\varepsilon) \subset B(\varphi(B(x,\varepsilon)),\varepsilon). \qquad \square$$

**Remark 2.8.2.** The reader will easily get a similar result concerning the existence of an $\varepsilon(\,\cdot\,)$-approximation of $\varphi$ being an $\varepsilon(\,\cdot\,)$-selection of $\psi$, where $\varepsilon\colon X \to (0,\infty)$ is a continuous function as well as its version valid for nonmetrizable spaces.

**Corollary 2.8.3.** *In addition to assumptions of the Theorem 2.8.1, suppose that $\psi$ has closed values and $E$ is complete. Then, for each $\varepsilon > 0$, there is a continuous selection of $\psi$ being an $\varepsilon$-approximation of $\varphi$.*

*Proof.* Fix $\varepsilon > 0$. By the above theorem there exists a continuous map $g\colon X \to E$ such that $g(x) \in B(\psi(x),\varepsilon/4)$ and $g(x) \in B(\varphi(B(x,\varepsilon)),\varepsilon/2)$. We shall construct a sequence $\{f_n\}_{n\geq 1}$ of continuous maps $f_n\colon X \to E$ such that, for $x \in X$ and $n \geq 1$:

(1) $d(f_n(x),\psi(x)) < 2^{-n-1}\varepsilon$;
(2) $\|f_{n+1}(x) - f_n(x)\| < 2^{-n-1}\varepsilon$.

Let $f_1 := g$. Suppose that continuous functions $f_i\colon X \to E$, $i = 1, \dots, n$ has been constructed. In order to define $f_{n+1}$, let

$$\psi_n(x) := \psi(x) \cap B(f_n(x), 2^{-n-2}\varepsilon), \quad x \in X.$$

Then $\psi_n$ is lower semicontinuous and has convex values. Exactly as in the proof of the Michael theorem we show that there is a continuous $f_{n+1}\colon X \to E$ such that $f_{n+1}(x) \in B(\psi_n(x), 2^{-n-2}\varepsilon)$. Then $d(f_{n+1}(x), \psi(x)) < 2^{-n-2}\varepsilon$ and $\|f_{n+1}(x) - f_n(x)\| < 2^{-n-1}\varepsilon$ for all $x \in X$. Since $E$ is complete $(f_n)$ converges uniformly to a continuous selection $f\colon X \to E$ of $\psi$. At the same time, for all $x \in X$,

$$\|g(x) - f(x)\| = \lim_{n\to\infty} \|f_1(x) - f_{n+1}(x)\| \le \lim_{n\to\infty} \sum_{k=1}^{n} \|f_k(x) - f_{k+1}(x)\| < \frac{\varepsilon}{2}.$$

Thus $f(x) \in B(\varphi(B(x,\varepsilon)), \varepsilon)$. □

Note that while in Theorem 2.8.1 the existing map $f$ may be proven to be locally Lipschitz (see Remark 2.0.1), in Corollary 2.8.3 the resulting $f$, as the uniform limit of locally Lipschitz map, is not locally Lipschitz in general.

Our next result is similar, but now $\psi$ is specified; however, we need no completeness.

**Theorem 2.8.4** (see [6]). *Let $K$ be a closed subset of a normed space $E$ Suppose that $\varphi\colon X \multimap E$ is an upper semicontinuous map with closed convex values. If $\varphi$ is weakly tangent to $K$, i.e. for each $x \in K$, $\varphi(x) \cap T_K(x) \neq \emptyset$, then, for any $\varepsilon > 0$, there exists a locally Lipschitz map $f\colon K \to E$ being an $\varepsilon$-approximation of $\varphi$ and, for all $x \in K$, $f(x) \in T_K(x)$.*

Obviously if $E$ is complete, then the existence of a *continuous* map $f\colon K \to E$ follows from Corollary 2.8.3 since the map $K \ni x \multimap T_K(x)$ is lower semicontinuous and has convex closed values (see Remark 1.3.10).

*Proof.* Take $\varepsilon > 0$ and $x \in K$. There is $v(x) \in E$ such that

$$v(x) \in B(\varphi(x), \varepsilon/4) \cap S_K(x),$$

since, by Remark 1.3.10,

$$C_K(x) = T_K(x) = \operatorname{cl} S_K(x)$$

where

$$S_K(x) := \bigcup_{h>0} \frac{K-x}{h}.$$

Hence, there is $\alpha(x) > 0$ such that

$$x + \alpha(x)v(x) \in K.$$

By the upper semicontinuity choose a number $\gamma(x)$, $0 < \gamma(x) < \varepsilon/4$ such that $\varphi(B(x, 2\gamma(x))) \subset B(\varphi(x), \varepsilon/2)$ and a number $0 < \delta(x) < \min\{\gamma(x), \gamma(x)/\alpha(x)\}$.

Let $\{\lambda_s\}_{s\in S}$ be a locally finite locally Lipschitzian partition of unity refining the open cover $\{B(x,\delta(x)\alpha(x))\}_{x\in K}$. For any $s\in S$, there is $x_s\in K$ such that $\operatorname{supp}\lambda_s\subset B(x_s,\delta_s\alpha_s)$ where we have put $\delta_s:=\delta(x_s)$ and $\alpha_s:=\alpha(x_s)$. Additionally let us set $v_s:=v(x_s)$ and $\gamma_s:=\gamma(x_s)$.

For any $s\in S$, we define a map $f_s\colon K\to E$ by the formula

$$f_s(x):=\frac{1}{\alpha_s}(x_s-x)+v_s,\quad x\in K.$$

Observe, that for $s\in S$, $x\in X$,

$$x+\alpha_s f_s(x)=x_s+\alpha_s v_s\in K.$$

Hence, for all $x\in X$,

$$f_s(x)\in S_K(x)\subset T_K(x).$$

It is clear that $f_s$, $s\in S$, is Lipschitz continuous (with the Lipschitz constant $\alpha_s^{-1}$).

Now we define $f\colon K\to E$ by the formula

$$f(x):=\sum_{s\in S}\lambda_s(x)f_s(x),\quad x\in K.$$

Observe that $f$ is locally Lipschitz because so are all functions $\lambda_s$, $f_s$ for $s\in S$, and the covering $\{\operatorname{supp}\lambda_s\}_{s\in S}$ is locally finite.

Moreover, since, for $x\in K$, $f(x)$ is a (finite) convex combination of vectors $f_s(x)\in T_K(x)$ and since $T_X(x)$ is convex, we see that $f(x)\in T_K(x)$ for all $x\in K$.

Take $x\in K$ and let $S(x)=\{s\in S\mid x\in\operatorname{supp}\lambda_s\}$. It is clear that $S(x)$ is a finite set and

$$f(x)=\sum_{s\in S(x)}\lambda_s(x)f_s(x).$$

For any $s\in S(x)$, we have $x\in\operatorname{supp}\lambda_s\subset B(x_s,\delta_s\alpha_s)$, i.e.

$$\|x-x_s\|<\delta_s\alpha_s<\gamma_s\quad\text{and}\quad\|f_s(x)-v_s\|<\delta_s<\gamma_s.$$

There is $s_0\in S(x)$ such that $\gamma_{s_0}=\max_{s\in S(x)}\gamma_s$. If $s\in S(x)$, then

$$\|x_s-x_{s_0}\|\le\|x_s-x\|+\|x_{s_0}-x\|<\gamma_s+\gamma_{s_0}\le 2\gamma_{s_0}.$$

Therefore, for any $s\in S(x)$,

$$\begin{aligned}f_s(x)\in B(v_s,\gamma_{s_0})\subset B(\varphi(x_s),\varepsilon/4+\gamma_{s_0})&\subset B(\varphi(B(x_{s_0},2\gamma_{s_0})),\varepsilon/4+\gamma_{s_0})\\&\subset B(\varphi(x_{s_0}),\varepsilon/4+\varepsilon/2+\gamma_{s_0})\subset B(\varphi(x_{s_0}),\varepsilon).\end{aligned}$$

Hence, by convexity of $B(\varphi(x_{s_0}),\varepsilon)$,

$$f(x)\in B(\varphi(x_{s_0}),\varepsilon)\subset B(\varphi(B(x,\gamma_{s_0})),\varepsilon)\subset B(\varphi(B(x,\varepsilon)),\varepsilon).$$

This concludes the proof. $\square$

Note that in the course of the proof we have not used the lower semicontinuity of $T_K(\,\cdot\,)$. Instead we have applied the following, astonishingly simple, observation: *if $K$ is a convex closed set in a normed space $E$, then for every $x_0 \in K$, $v_0 \in S_K(x_0)$ and $\alpha_0 > 0$ such that $x_0 + \alpha_0 v_0 \in K$ (existing in view of the very definition of $S_K(x_0)$), an affine mapping $g(x) = \frac{1}{\alpha_0}(x_0 - x) + v_0$, $x \in K$, provides a selection of $S_K(x)$.* This proves the lower semicontinuity of both $S_X(\,\cdot\,)$ and $T_X(\,\cdot\,)$.

In the similar spirit we have the next result. In order to state it and understand the assumption we need some more concepts.

Let $E$ be a normed space. Given an open set $U \subset E$, a locally Lipschitz continuous function $g\colon U \to \mathbb{R}$, by $g^\circ(x;u)$ we denote the Clarke *generalized directional derivative* of $g$ at $x \in U$ in the direction $u \in E$

$$g^\circ(x;u) := \limsup_{y \to x,\, h \to 0^+} \frac{g(y+hu) - g(y)}{h}.$$

It is well-known that, for each $x \in E$, the function $E \ni u \mapsto g^\circ(x;u)$ is Lipschitz, subadditive and positively homogeneous. The *generalized gradient* of $g$ at $x \in U$ is defined by

$$\partial g(x) := \{p \in E^* \mid \langle p, u\rangle \le g^\circ(x;u) \text{ for all } u \in E\}.$$

Hence $g^\circ(x;\,\cdot\,)$ is the support function $\partial g(x)$:

$$g^\circ(x;u) = \sigma_{\partial g(x)}(u) := \sup_{p \in \partial g(x)} \langle p, u\rangle, \quad u \in E,$$

and, the (negative) polar cone

$$\partial g(x)^\perp = \{u \in E \mid g^\circ(x;u) \le 0\}.$$

Hence, for all $x \in U$, the set $\partial g(x)$ is convex and weak$^*$-compact. The function $U \times E \ni (x,u) \mapsto g^\circ(x;u)$ is upper semicontinuous; in other words the set-valued map $U \ni x \mapsto \partial f(x) \subset E^*$ is upper hemicontinuous and upper demicontinuous (see Subsection 2.9).

If $K \subset E$ is closed, $x \in K$, then — by the very definition — $C_K(x) = \partial d_K(x)^\perp$ and $N_K(x) = \partial d_K(x)^{--}$ where $d_K(x) := d(x,K) := \inf_{y \in K} \|x - y\|$ for $x \in E$. Proofs for all facts mentioned above and other details can be found in e.g. [21], [3].

Let $K \subset E$ be closed. We say that $K$ is an *epi-Lipschitz set* (see [67]) if there exists a locally Lipschitz function $g\colon U \to \mathbb{R}$ (the so-called *representing function*), where $U$ is an open neighbourhood of $K$ such that

$$K = \{x \in U \mid g(x) \le 0\}$$

and, for all $x \in \operatorname{bd} K$,

$$0 \notin \partial g(x).$$

It is easy to show that if $K$ is epi-Lipschitz, then $\partial g(x)^{\perp} \subset C_K(x)$ for all $x \in \operatorname{bd} K$. Hence the normal cone

$$N_K(x) \subset \partial f(x)^{--} = \bigcup_{\lambda \geq 0} \lambda \partial g(x)$$

because $0 \notin \partial g(x)$.

**Theorem 2.8.5** (see [6]). *Assume that $K$ is an epi-Lipschitz subset of a normed space $E$ represented by a locally Lipschitz function $g: U \to \mathbb{R}$, $\varphi: T \times K \multimap E$, where $(T, \mathfrak{M}, \mu)$ is a complete measurable space, is a set-valued map such that, for each $x \in \operatorname{bd} K$, $\varphi(\,\cdot\,, x) \cap \partial g(x)^{\perp}$ has a* (*strongly*) *measurable selection and, for each $x \in K$, the set $\varphi(T \times \{x\})$ is relatively compact. If, for almost all $t \in T$, $\varphi(t, \cdot\,): K \multimap E$ is upper semicontinuous with closed convex values, then for each $\varepsilon > 0$, there is a continuous map $f: T \times K \to E$ such that, for almost all $t \in T$, $f(t, \cdot\,)$ is locally Lipschitz, for all $x \in K$, $f(\,\cdot\,, x)$ is strongly measurable, $f(t,x) \in \partial g(x)^{\perp}$ and $f(t,x) \in \operatorname{conv} \varphi(t, B(x,\varepsilon)) + B(0,\varepsilon)$ for all $t \in T$ and $x \in X$.*

*Proof.* Let $x \in \operatorname{bd} K$. There is a strongly measurable function $v_x: T \to E$ such that $v_x(t) \in \varphi(t,x) \cap \partial g(x)^{\perp}$ on $T$. Hence, for all $t \in T$, $g^{\circ}(x; v_x(t)) \leq 0$. Since $0 \notin \partial f(x)$,

$$0 < \inf_{p \in \partial g(x)} \|p\| = \inf_{p \in \partial g(x)} \sup_{\|u\| \leq 1} \langle p, u \rangle = \sup_{\|u\| \leq 1} \inf_{p \in \partial g(x)} \langle p, u \rangle$$

in view of the von Neumann–Sion minimax equality ([17]). Next, we know that, for all $u \in E$, $g^{\circ}(x;u) = \sup_{p \in \partial g(x)} \langle p, u \rangle$. Thus $\inf_{\|u\| \leq 1} g^{\circ}(x;u) < 0$. Hence there is $u_x \in E$, $\|u_x\| = \varepsilon/2$ such that $g^{\circ}(x; u_x) < 0$. The set $v_x(T)$ is relatively compact; thus there is a simple (i.e. measurable and having finite number of values) function $v'_x: T \to E$ such that $v'_x(T) \subset v_x(T)$ and $\|v_x(t) - v'_x(t)\| < \varepsilon/2$ on $T$. Hence $g^{\circ}(x; v'_x(t)) \leq 0$ for all $t \in T$. Let $w_x(t) = v'_x(t) + u_x$, $t \in T$. Then $w_x$ is a simple function and, for all $t \in T$,

$$g^{\circ}(x; w_x(t)) < 0$$

because $g^{\circ}(x; \cdot\,)$ is subadditive. Since $w_x$ admits a finite number of values and, for each $w \in E$, the function $g^{\circ}(\,\cdot\,; w)$ is upper semicontinuous, there is $0 < r_x < \varepsilon$ such that, for all $y \in B(x, r_x)$ and $t \in T$,

$$g^{\circ}(y; w_x(t)) < 0.$$

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([17]) See [69]; it states that *given a convex subset $X$ of a topological vector space, a convex compact subset $Y$ of a topological vector space, a function $F: X \times Y \to \mathbb{R}$ such that $F(\,\cdot\,, y)$ is concave and upper semicontinuous for all $y \in Y$ and $F(x, \cdot\,)$ is convex and lower semicontinuous for all $x \in X$, the min-max equality* $\sup_{x \in X} \inf_{y \in Y} F(x,y) = \inf_{y \in Y} \sup_{x \in X} F(x,y)$ *holds true.* This theorem applies in our case since $\partial g(x)$ is convex and weakly*-compact, $D := \{u \in E \mid \|u\| \leq 1\}$ is convex and the function $F \times \partial g(x) \ni (u,p) \mapsto \langle p, u \rangle \in \mathbb{R}$ is linear, weakly*-continuous with respect to $p$ and continuous with respect to $u$.

If $x \in \operatorname{int} K$, then we choose $0 < r_x < \varepsilon$ such that $B(x, r_x) \subset \operatorname{int} K$ and let $w_x: T \to E$ be ab arbitrary strongly measurable selection of $\varphi(\,\cdot\,, x)$.

We have constructed an open covering $\{B(x, r_x)\}_{x\in K}$ of $K$; let $\{\lambda_s\}_{s\in S}$ be a locally Lipschitz partition of unity subordinated to this cover. Hence, for each $s \in S$, there is $x_s \in K$ such that $\operatorname{supp} \lambda_s \subset B(x_s, r_s)$ where $r_s := r_{x_s}$. Let $w_s := w_{x_s}$ and define

$$f(t, x) = \sum_{s\in S} \lambda_s(x) w_s(t), \quad x \in K,\ t \in T.$$

It is easy to check that $f$ has all the required properties. $\square$

Both results 2.8.4, 2.8.5 have immediate applications concerning the existence and the structure of solutions of differential inclusions — see [5], [6].

**2.9. Acute-angled approximations.** Finally we discuss the existence of approximations under conditions weaker than upper semicontinuity.

Let $X$ be a topological space and let $E$ be a normed space. We say that a set-valued map $\varphi: X \multimap E$ (resp. $\varphi: X \multimap E^*$) is *upper hemicontinuous* if, for each $p \in E^*$ (resp. $y \in X$), the real (extended) function

$$X \ni x \mapsto \sigma_{\varphi(x)}(p) := \sup_{y\in\varphi(x)} \langle p, y\rangle \in \mathbb{R} \cup \{\infty\}$$

$$\Big(\text{resp. } X \ni x \mapsto \sigma_{\varphi(x)}(y) := \sup_{p\in\varphi(x)} \langle p, y\rangle \in \mathbb{R} \cup \{\infty\}\Big)$$

is upper semicontinuous (as a real-valued function). It is clear that if $\varphi: X \to E$ (resp. $\varphi: X \multimap E^*$) is upper semicontinuous or *upper demicontinuous* (in the sense that it is upper semicontinuous when $E$ (resp. $E^*$) is endowed with the weak (resp. weak$^*$) topology), then $\varphi$ is upper hemicontinuous. The converse result is not valid in general.

**Example 2.9.1.** Let $\varphi: \mathbb{R} \multimap \mathbb{R}^2$ be given by $\varphi(x) := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 = xy_1\}$, then is neither upper semicontinuous nor $H$-upper semicontinuous, but it is upper hemicontinuous.

However if $\varphi$ has convex and weakly (resp. weak$^*$) compact values, then any upper hemicontinuous map is upper demicontinuous. It is easy to show that if $\varphi$ is upper hemicontinuous with bounded values and $X$ is compact, then $\varphi(X)$ is bounded in $E$ (resp. in $E^*$). The graph $\operatorname{Gr}(\varphi)$ of a hemicontinuous set-valued map with closed (resp. weak$^*$-closed) convex values is closed in $X \times E$ (resp. $X \times E^*$) provided $E$ (resp. $E^*$) has weak (resp. weak$^*$) topology (for other results on upper hemicontinuous maps — see [3]).

Our aim is to study the existence of continuous (in the original topology of $E$ or $E^*$) approximations of a given upper hemicontinuous map. Since the question concerning the existence of the usual graph-approximations is not clear we shall establish the availability of other types of 'approximations', the so-called *acute angled* approximations.

**Theorem 2.9.2** (see [55]). *Suppose that $X$ is paracompact and $\varphi\colon X \multimap E$ (resp. $X \multimap E^*$) is upper hemicontinuous with convex closed (resp. weakly*-closed) values and such that $0 \notin \varphi(x)$ for $x \in X$. Then there are continuous maps $f\colon X \to E$ (resp. $X \to E^*$) and $g\colon X \to E^*$ (resp $X \to E$) such that $f(X) \subset \operatorname{conv}\varphi(X)$ and, for any $x \in X$, $0 \notin \operatorname{cl}\operatorname{conv}(\{f(x)\} \cup \varphi(x))$ and $\inf_{z\in\varphi(x)}\langle g(x), z\rangle > 0$ (resp. $\inf_{z\in\varphi(x)}\langle z, g(x)\rangle > 0$).*

*Proof.* For any $y \in E^*$ (resp. $y \in E$), let

$$U_y := \Big\{x \in X \;\Big|\; \inf_{z\in\varphi(x)}\langle y, z\rangle > 0 \quad \Big(\text{resp. } \inf_{z\in\varphi(x)}\langle z, y\rangle > 0\Big)\Big\}.$$

Since $\varphi$ is upper hemicontinuous, $U_y$ is open for any $y \in E^*$ (resp. $y \in E$). Moreover, $\mathfrak{B} := \{U_y\}_{y\in E^*}$ (resp. $\mathfrak{B} := \{U_y\}_{y\in E}$) is a covering of $X$: for if $x \in X$, then $0 \notin \varphi(x)$ and there is $y \in E^*$ (resp. $y \in E$) such that $\sup_{z\in\varphi(x)}\langle y, z\rangle < 0$ (resp. $\sup_{z\in\varphi(x)}\langle z, y\rangle < 0$). Let $\mathfrak{A}$ be an open point-star-refinement of $\mathfrak{B}$. Take a partition of unity $\{\lambda_s\}_{s\in S}$ subordinated to $\mathfrak{A}$, i.e. for each $s \in S$ there is $V_s \in \mathfrak{A}$ such that $\operatorname{supp}\lambda_s \subset V_s$. Take any $z_s \in \varphi(x_s)$ where $x_s \in V_s$ and let

$$f(x) := \sum_{s\in S}\lambda_s(x)z_s, \quad x \in X.$$

Then $f$ is continuous and, for $x \in X$, if $\lambda_s(x) \neq 0$, then $x \in V_s$ and there is $y_x \in E^*$ (resp. $y_x \in E$) such that $x_s \in \operatorname{st}(x, \mathfrak{A}) \subset U_{y_x}$. Therefore, for some $\varepsilon > 0$, $\langle y_x, z_s\rangle > \varepsilon$ (resp. $\langle z_s, y_x\rangle > \varepsilon$) for $s$ from (a finite set) $S(x) := \{s \in S \mid \lambda_s(x) > 0\}$ and $\inf_{z\in\varphi(x)}\langle y_x, z\rangle > \varepsilon$ (resp. $\inf_{z\in\varphi(x)}\langle z, y_x\rangle > \varepsilon$). Hence $\inf\{\langle y_x, z\rangle \mid z \in \operatorname{conv}(\{f(x)\} \cup \varphi(x))\} \geq \varepsilon$ (resp. $\inf\{\langle z, y_x\rangle \mid z \in \operatorname{conv}(\{f(x)\} \cup \varphi(x))\} \geq \varepsilon$).

Similarly, let $\{\lambda_s\}_{s\in S}$ be a partition of unity subordinated to the cover $\mathfrak{B}$, i.e. for any $s \in S$, there is $y_s \in E^*$ (resp. $y_s \in E$) such that $\operatorname{supp}\lambda_s \subset U_{y_s}$. Let

$$g(x) := \sum_{s\in S}\lambda_s(x)y_s, \quad x \in X.$$

Then it is easy to see that $\inf_{z\in\varphi(x)}\langle g(x), z\rangle > 0$ (resp. $\inf_{z\in\varphi(x)}\langle z, g(x)\rangle > 0$) for each $x \in X$. $\square$

The constructed map $f$ may have nothing to do with $\varphi$ (i.e. the distance of $f(x)$ from $\varphi(x)$ may be large). However it is easy to see that given two continuous maps $f_1, f_2\colon X \to E$ (resp. $X \to E^*$) satisfying properties described above, then they are homotopic through 0-avoiding homotopy. Hence, as we shall see later, they reflect homotopical properties of $\varphi$. The map $g\colon X \to E^*$ (resp. $X \to E$) is called an *acute-angled* approximation of $\varphi$; it seems that the origin of this terminology is clear.

Finally let us make the following observation.

**Lemma 2.9.3** (see [28]). *Let $X$ be a metric space, $E$ a normed space and $\varepsilon\colon X \to (0,\infty)$. If $\varphi\colon X \multimap E$ be a set-valued map with convex values, then $f\colon X \to \mathbb{R}$ is an $\varepsilon(\,\cdot\,)$-approximation of $\varphi$ if and only if, for any $x \in X$, there is $x' \in X$ such that $\|x - x'\| < \varepsilon(x)$ and*

$$\sup_{p\in E^*,\, \|p\|\le 1} (\langle p, f(x)\rangle - \sigma_{\varphi(x')}(p)) < \varepsilon(x).$$

*Proof.* A map $f\colon X \to E$ is an $\varepsilon(\,\cdot\,)$-approximation of $\varphi$ if and only if, for each $x \in X$, there is $x' \in X$ such that $\|x - x'\| < \varepsilon(x)$ and $\|f(x) - y'\| < \varepsilon(x)$ for some $y' \in \varphi(x')$, i.e. by the von Neumann–Sion min-max equality (recall that $\{p \in E^* \mid \|p\| \le 1\}$ is weakly$^*$-compact in view of the Banach–Alaoglu theorem),

$$\begin{aligned}\varepsilon(x) &> \|f(x) - y'\| \ge \inf_{y\in\varphi(x')} \|f(x) - y\| \\ &= \inf_{y\in\varphi(x')} \sup_{p\in E^*,\, \|p\|\le 1} \langle p, f(x) - y\rangle = \sup_{\|p\|\le 1} (\langle p, f(x)\rangle - \sigma_{\varphi(x')}(p)).\end{aligned}$$

This completes the proof. □

It is doubtful whether Lemma 2.9.3 may be a source of a proof of the existence of graph-approximations for upper-hemicontinuous set-valued maps (or a different proof of the Cellina theorem). It is, however, a starting point for the following important fact being a generalization of the celebrated Whitney theorem.

**Theorem 2.9.4** (see [28]). *Let $U \subset \mathbb{R}^n$ and let $f\colon U \to \mathbb{R}$ be a locally Lipschitz function. For any continuous $\varepsilon\colon U \to (0,+\infty)$, there is a $C^\infty$-function $g\colon U \to \mathbb{R}$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in U$ and the gradient $\nabla g$ is an $\varepsilon(\,\cdot\,)$-approximation the generalized gradient $\partial f$ (18).*

The proof of this results is fairly complicated and will not be reproduced here.

## 3. Selections, approximations and fixed points of set-valued maps

As mentioned in the introduction approximation methods may be useful in the study of the existence of fixed points of set-valued maps. Here we shall study only global results; local theories (such as the availability different homotopy invariants monitoring the existence of fixed points) will not be discussed.

The idea of the approximation approach is simple: given a set-valued map $\varphi\colon X \multimap X$ acting in a space $X$ having the fixed point property (for a certain class $\mathcal{F}$ of single-valued maps), if $\varphi$ admits arbitrarily close approximations by maps from $\mathcal{F}$, then one may expect to derive fixed points of $\varphi$ (i.e. points $x \in X$ such that $x \in \varphi(x)$) as limits of appropriate sequences of fixed points of

(18) Recall that $\partial f\colon U \to (\mathbb{R}^n)^* = \mathbb{R}^n$; $\partial f$ is upper semicontinuous with compact convex values.

approximations. The idea is simple, but in concrete situations, it requires some care.

The simplest (and, therefore, in general not interesting) is the case when $\varphi$ admits selections. For if $\varphi$ admits a selection $f\colon X \to X$, $f \in \mathcal{F}$, then there is $x_0 \in X$ such that $x_0 = f(x_0) \in \varphi(x_0)$. In the same spirit one can get fixed point results for maps admitting $\varepsilon$-selections.

For example let us prove the Browder–Fan fixed point theorem.

**Theorem 3.0.1** (Browder, [16], [38])**.** *Let $K$ be a compact convex subset of a topological vector space $E$ and let $\varphi\colon K \multimap K$ has convex values and open fibres (i.e. for each $y \in K$, $\varphi^{-1}(y)$ is open in $K$). Then $\varphi$ has a fixed point.*

*Proof.* In view of Theorem 2.1.3, $\varphi$ admits a selection which is produced via partition of unity subordinated to an open covering of the form $\{\varphi^{-1}(y)\}_{y\in K}$. Since $K$ is compact, one can choose a finite partition of unity which gives a continuous finite-dimensional selection $f\colon K \to K$ of $\varphi$ (i.e. $f(K) \subset K' := K \cap E'$ where $E'$ is a finite dimensional subspace of $E$) Thus $f\colon K' \to K'$ and, by the Brouwer fixed point theorem $f$, $f$ and, therefore, $\varphi$ has a fixed point. $\square$

Theorem of Browder–Fan gives a nice bonus. Namely it allows the following simple proof of the Tikhonov fixed point theorem: *if $K$ is a compact convex subset of a locally convex space $E$ and $f\colon K \to K$ is continuous, then* $\operatorname{Fix}(f) := \{x \in K \mid x = f(x)\} \neq \emptyset$. Indeed, for any convex open neighbourhood $V$ of the origin in $E$, let $\varphi_V(x) := f(x) + V$, $x \in K$. Then $\varphi_V^{-1}(y) = f^{-1}(y - V)$ is open. Since values of$\varphi_V$ are convex, we infer that there is $x_V \in K$ such that $x_V \in \varphi_V(x_V)$, i.e. $x_V - f(x_V) \in V$. The continuity of $f$ and the compactness of $K$, implies the existence of a fixed point of $f$.

In what follows we shall provide some other applications of the Browder–Fan theorem.

**3.1. Fixed points via graph-approximations.** Graph approximations play a similar role as selections; however the general remark is that when dealing with graph-approximations of approximable maps one has to be careful. To illustrate the situation consider the following general results.

**Theorem 3.1.1.** *Let $X$ be a compact absolute retract* ($X \in \mathrm{AR}$ — *see* [13]) *and let $\varphi\colon X \multimap X$ be a weakly approximable set-valued map with closed graph (i.e. upper semicontinuous). Then it has fixed points.*

*Proof.* For any integer $n \geq 1$, there is a continuous $n^{-1}$-approximation $f_n\colon X \to X$ of $\varphi$. By the (generalized) Schauder theorem, there is $x_n \in X$ such that $x_n = f(x_n)$. Since $x_n = f(x_n) \in B(\varphi(B(x_n, n^{-1})), n^{-1})$, there is $x_n' \in X$, $d(x_n, x_n') < n^{-1}$ and $y_n \in \varphi(x_n')$ such that $d(y_n, x_n) < n^{-1}$. Passing to a subsequence if necessary we may assume that $x_n \to x_0 \in X$. Therefore $(x_n', y_n) \to (x_0, x_0)$. Since the graph $\mathrm{Gr}(\varphi)$ is closed we see that $x_0 \in \varphi(x_0)$. $\square$

The situation is a bit more difficult if we $X$ is not compact.

**Theorem 3.1.2.** *Suppose that $X \in \mathrm{AR}$ and $\varphi\colon X \multimap X$ is a weakly approximable compact map with closed graph. Then $\varphi$ has fixed points.*

*Proof.* Without loss of generality we may assume that $X$ is a retract of a normed space $E$. Let $r\colon E \to X$ be a retraction. Let $K := \operatorname{cl}\varphi(X)$; $K$ is compact. For any integer $n \geq 1$, there is a continuous map (a so-called *Schauder projection*) $\pi_n\colon U \to E_n$, where $U$ is an open neighbourhood of $K$ (in $E$) and $E_n$ is a linear subspace of $E$, $\dim E_n < \infty$, such that $\|\pi_n(x) - x\| < n^{-1}$ for all $x \in U$. Let $f_n\colon X \to X$ be a continuous $n^{-1}$-approximation of $\varphi$. For large $n$ (for $n \geq N$ say), $f_n(X) \subset U$. For $n \geq N$, let $g_n := \pi_n \circ f_n \circ r\colon E \to E_n$. Then $g_n$ is well-defined and compact (as a bounded finite-dimensional map). By the Schauder theorem, for each $n \geq N$, there is $x_n \in E$ such that $x_n = g_n(x_n)$. Then $r(x_n) = r \circ \pi_n \circ f_n(r(x_n))$, i.e. $y_n := r(x_n)$ is a fixed point of $r \circ \pi_n \circ f_n\colon X \to X$. There is $y_n' \in X$, $d(y_n, y_n') < n^{-1}$ and $z_n \in \varphi(y_n') \subset K$ such that $d(z_n, f_n(y_n)) < n^{-1}$. The compactness of$K$ implies that (for a subsequence), $z_n \to z_0 \in K \subset X$. Then $f_n(y_n) \to z_0$. Since $\|\pi_n(f_n(y_n)) - f_n(y_n)\| < n^{-1}$, we see that $\pi_n(f_n(y_n)) \to z_0$. Thus $y_n = r(\pi_n(f_n(y_n))) \to r(z_0) = z_0$. Therefore $y_n' \to z_0$ and $z_0 \in \varphi(z_0)$. □

**Remark 3.1.3.** The above proof may be simplified. To this end let us proceed as follows (the assumptions and notation are sustained). Since $K$ is compact, we may assume that $i\colon K \hookrightarrow \mathbb{I}^\infty$, where $\mathbb{I}^\infty$ is the Hilbert cube (and, henceforth, a compact AR), is the continuous embedding. Since $X \in \mathrm{AR}$, the inclusion $j'\colon K \hookrightarrow X$ extends to a continuous map $j\colon \mathbb{I}^\infty \to X$. Therefore we have a map $\psi := i \circ \varphi' \circ j\colon \mathbb{I}^\infty \multimap \mathbb{I}^\infty$, where $\varphi'\colon X \multimap K$ is given by $\varphi'(x) = \varphi(x) \subset K$ for $x \in X$. Now the point is that $\varphi$ is approximable: is that true that $i \circ \varphi'$ is approximable? In general this might not be true. However if we assume, for instance, that, for each $x \in X$, $\varphi(x)$ has (the absolute) $UV^\omega$-property (or, in particular, is convex), then so does $i \circ \varphi'(x)$ and, since $i \circ \varphi'$ is upper semicontinuous, it is approximable in view of Theorem 2.6.2. Next, since $\mathbb{I}^\infty$ is compact, we see that $\psi$ is approximable. Thus, in view of Theorem 3.1.1, there is $x_0 \in \mathbb{I}^\infty$ such that $x_0 \in \psi(x_0)$. It is clear that $x_0 \in K$, hence $\psi(x_0) = \varphi(x_0)$.

It is also worthwhile to note that the approach presented in Theorem 3.1.2 is quite general. For instance, in place of $\varphi$ we may take the composition $\varphi = g \circ \Phi$ where $\Phi\colon X \multimap Y$, $Y$ is a topological space, is approximable and upper semicontinuous with compact values and $g\colon Y \to X$ is continuous. Then $\varphi$ is upper semicontinuous with compact values (hence its graph is closed) and weakly approximable and in view of Proposition 2.3.6.

Taking the above into account we have the following general result that follows immediately from Theorem 2.6.2.

**Theorem 3.1.4.** *Let $X \in \mathrm{AR}$ and let $\varphi\colon X \multimap X$ be compact upper semicontinuous and let $\varphi(x) \in UV^\omega$ for all $x \in X$. Then $\varphi$ has fixed points.*

Of course the same proof (with almost immediate changes) applies to the case of the following result that generalizes the Tikhonov fixed point theorem (the so-called Fan-Glicksberg [42] or Bohnenblust–Karlin theorem [11]).

**Theorem 3.1.5.** *Let $K$ be a closed convex subset of a locally convex space $E$ and let $\varphi: K \multimap E$ be an upper semicontinuous compact set-valued with closed* (*i.e. a posteriori compact*) *convex values. Then $\varphi$ admits fixed points.*

Theorem 3.1.5 may be generalized to the case of upper semicontinuous maps with closed convex values. The only difference is that the usual compactness (which, in the context of Theorem 3.1.5, says that $\operatorname{cl}\varphi(K)$ is compact) should be replaced with some less restrictive assumption.

**Corollary 3.1.6.** *Under the hypotheses of Theorem* 3.1.5, *assume that there is a compact convex set $L \subset K$ such that $\varphi(x) \cap L \neq \emptyset$, where $\varphi: K \multimap K$ is upper semicontinuous with closed convex values. Then $\varphi$ has fixed points.*

*Proof.* Consider $\varphi': K \multimap K$ given by $\varphi'(x) = \varphi(x) \cap L$ for $x \in K$. Then $\varphi'$ satisfies assumptions of Theorem 3.1.5 and, therefore, has fixed points. Clearly $\operatorname{Fix}(\varphi') \subset \operatorname{Fix}(\varphi)$. □

**3.2. Homotopy invariants via approximations.** There is a variety of homotopy invariants detecting zeros of fixed points of set-valued maps that may be defined and studied by means of approximations. We shall provide a number of sketches of such constructions in order to illustrate the relevance of different approximations techniques. The presented approach i modelled mainly after [20], [47] and in the most general case [7].

**A. The finite-dimensional degree theory.** Suppose $U$ is an open set in the Euclidean space $\mathbb{R}^n$ and let $\varphi: U \multimap \mathbb{R}^n$ be a set-valued map such that:

(A1) $\varphi$ is weakly approximable and weakly homotopy approximable over any compact polyhedron $K \subset U$;
(A2) $Z(\varphi) = \{x \in U \mid 0 \in \varphi(x)\}$ is compact;
(A3) $\varphi$ has closed graph.

**Example 3.2.1.** (a) If $\varphi$ is upper semicontinuous, has convex values (resp. $\varphi$ is $UV^m$-valued with $m \geq n-1$), then $\varphi$ is weakly approximable and weakly homotopy approximable over any subset of $U$ (resp. over any compact polyhedron $K \subset U$) in view of Corollary 2.3.12, Corollary 2.4.4 and Remark 2.4.5 (resp. Theorem 2.6.1, Corollary 2.6.5 and Remark 2.6.6).

(b) If $\varphi = g \circ \Phi$, where $\Phi: X \multimap Y$, $Y$ is a topological space, is approximable and homotopy approximable, then $\varphi$ is weakly approximable in view of Proposition 2.3.6. Moreover, we see easily that $\varphi$ is 'weakly homotopy approximable' in the following sense: given $\varepsilon > 0$, there is a neighbourhood $\mathcal{U}$ of the graph $\operatorname{Gr}(\Phi)$ such that, for any two continuous $\mathcal{U}$-approximations $f, f': X \to Y$ of $\Phi$, the maps there is a homotopy $h: X \times [0,1] \to Y$ such that $g \circ h(\,\cdot\,, t)$ is an $\varepsilon$-approximation of $\varphi$ for any $t \in [0,1]$. We shall see that this sort of 'weak homotopy approximability' is sufficient for our aims. Using Theorem 2.3.7 we consider even more complicated compositions.

(c) If $\varphi$ is upper semicontinuous, then always $Z(\varphi) = \varphi^{-1}(0)$ is closed in $U$. Hence, $Z(\varphi)$ is compact if and only if there is an open bounded $V$ such that $Z(\varphi) \subset V \subset \operatorname{cl} V \subset U$. In particular, if $U$ is bounded, an upper semicontinuous $\varphi$ is defined on $\operatorname{cl} U$ and $\varphi$ has no zeros on $\operatorname{bd} U$, then (A2) is satisfied.

(d) If $\varphi$ is upper semicontinuous with closed values, then $\operatorname{Gr}(\varphi)$ is closed.

Thus we see that the class of maps satisfying assumptions (A1)–(A3) is rich.

By (A2) (see also Example 3.2.1(b), it is easy to see that there is an open set $V$ such that $\operatorname{cl} V$ is a compact polyhedron and $Z(\varphi) \subset V \subset \operatorname{cl} V \subset U$. There is $\varepsilon > 0$ such that, if $f\colon \operatorname{cl} V \to \mathbb{R}^n$ is an $\varepsilon$-approximation of $\varphi$ over $K$, then $Z(f) \cap \operatorname{bd} V = \emptyset$, where here $Z(f) := \{x \in \operatorname{cl} V \mid f(x) = 0\}$. For otherwise, for each $n \in \mathbb{N}$, there is $x_n \in \operatorname{bd} V$, $x'_n \in U$ and $y'_n \in \varphi(x'_n)$ such that $\|x_n - x'_n\| \leq n^{-1}$, $\|y'_n\| < n^{-1}$. By the compactness of $\operatorname{bd} V$ (and passing to a subsequences if necessary), we may assume that $x_n \to x \in \operatorname{bd} V$; thus $(x'_n, y'_n) \to (x, 0)$, i.e. $0 \in \varphi(x)$ in view of (A3): a contradiction. By (A1), there is a continuous function $\delta\colon U \to (0,\infty)$ (recall that *clV* is compact; thus we may assume without loss of generality that $\delta > 0$ is a constant and $\delta < \varepsilon$) such that any two $\delta$-approximations $f, g\colon \operatorname{cl} V \to \mathbb{R}^n$ of $\varphi$ over $\operatorname{cl} V$ are homotopic through a continuous homotopy $h\colon \operatorname{cl} V \times [0,1]$ with the property that $h(\,\cdot\,, t)$ is an $\varepsilon$-approximation of $\varphi$ over $\operatorname{cl} V$. In particular $\{x \in \operatorname{cl} V \mid 0 \in h(x,t) \text{ for some } t \in [0,1]\} \cap \operatorname{bd} V = \emptyset$.

Let $f\colon \operatorname{cl} V \to \mathbb{R}^n$ be an arbitrary $\delta$-approximation of $\varphi$ over $\operatorname{cl} V$. Then $Z(f) \cap \operatorname{bd} V = \emptyset$ and, thus, the Brouwer degree $\deg_B(f, V, 0)$ is well-defined (see e.g. [36]).

Let us define

$$\operatorname{Deg}(\varphi, U, 0) := \deg_B(f, V, 0).$$

The above argument show that this definition does not depend on the choice of $f$. Let us show that it is independent of the choice of $V$, as well. To see this take another open set $V'$ such that $\operatorname{cl} V'$ is a compact polyhedron and $Z(\varphi) \subset V' \subset \operatorname{cl} V' \subset U$. Obviously we may assume that $V' \subset V$. Without loss of generality (arguing as above) we may assume that given a $\delta$-approximation $f\colon \operatorname{cl} V \to \mathbb{R}^n$ of $\varphi$ over $\operatorname{cl} V$, $Z(f) \cap (\operatorname{cl} V \setminus V') = \emptyset$. Thus, by the additivity property of the Brouwer degree, $\deg(f, V, 0) = \deg(f, V', 0)$.

The reader will easily check that the defined degree Deg satisfies *all* the usual properties of the topological degree.

Now we shall outline the construction of the topological degree for upper hemicontinuous set-valued maps with closed convex values. Suppose that $U \subset \mathbb{R}^n$ is open bounded, $\varphi\colon \operatorname{cl} U \multimap \mathbb{R}^n$ is upper hemicontinuous with closed convex values and $\varphi^{-1}(0) \cap \operatorname{bd} U = \emptyset$. By Theorem 2.9.2, there is a continuous map

$f\colon \mathrm{bd}\, U \to \mathbb{R}^n$ such that, for any $x \in \mathrm{bd}\, U$, $0 \notin \mathrm{cl}\,\mathrm{conv}\,(\{f(x)\} \cup \varphi(x))$. In particular $0 \notin f(\mathrm{bd}\, U)$. Let $f^*\colon \mathrm{cl}\, U \to \mathbb{R}^n$ be an arbitrary continuous extension of $f$ onto $\mathrm{cl}\, U$. We define

$$\mathrm{Deg}(\varphi, U, 0) := \deg_B(f, U, 0).$$

This definition is correct since it does not depend on the choice of $f$ and $f^*$. To see this suppose that a continuous $g\colon \mathrm{bd}\, U \to \mathbb{R}^n$ is such that $0 \notin \mathrm{cl}\,\mathrm{conv}\,(\{g(x)\} \cup \varphi(x))$ for $x \in \mathrm{bd}\, U$ and let $g^*\colon \mathrm{cl}\, U \to \mathbb{R}^n$ be a continuous extension of $g$ onto $\mathrm{cl}\, U$. It is easy to see that set-valued maps $\psi_1, \psi_2\colon \mathrm{bd}\, U \times [0,1] \multimap \mathbb{R}^n$ given by $\psi_1(x,t) := (1-t)f(x) + t\varphi(x)$, $\psi_2(x,t) = tg(x) + (1-t)\varphi(x)$. It is clear that $\psi_i$, $i = 1, 2$, has closed convex values, is upper hemicontinuous and $0 \notin \psi_i(x,t)$ for $x \in \mathrm{bd}\, U$ and $t \in [0,1]$. Define $\psi\colon \mathrm{bd}\, U \times [0,1] \multimap \mathbb{R}^n$ by

$$\psi(x,t) := \begin{cases} \psi_1(x, 2t) & \text{for } t \in [0, 1/2], \\ \psi_2(x, 2t-1) & \text{for } t \in [1/2, 1], \end{cases}$$

for $x \in \mathrm{bd}\, U$. Then again $\psi$ has convex closed values, is upper hemicontinuous and $0 \notin \psi(x,t)$ for $x \in \mathrm{bd}\, U$ and $t \in [0,1]$. Therefore, again by Theorem 2.9.2, there is a continuous $h\colon \mathrm{bd}\, U \times [0,1] \to \mathbb{R}^n$ such that $h(x,t) \neq 0$ on $\mathrm{bd}\, U \times [0,1]$. Let $h^*\colon \mathrm{cl}\, U \times [0,1] \to \mathbb{R}^n$ be a continuous extension of a map $h'\colon \mathrm{cl}\, U \times \{0,1\} \cup \mathrm{bd}\, U \times [0,1] \to \mathbb{R}^n$ given by

$$h'(x,t) = \begin{cases} f^*(x) & \text{for } x \in \mathrm{cl}\, U,\ t = 0, \\ h(x,t) & \text{for } x \in \mathrm{bd}\, U,\ t \in [0,1], \\ g^*(x) & \text{for } x \in \mathrm{cl}\, U,\ t = 1. \end{cases}$$

Therefore $h^*$ provides a homotopy joining $f^*$ to $g^*$ such that $h^*(x,t) \neq 0$ on $\mathrm{bd}\, U \times [0,1]$. Hence

$$\deg_B(f^*, U, 0) = \deg_B(g^*, U, 0).$$

Again it is not difficult to show that the defined degree has all the usual properties. For instance, if $\mathrm{Deg}(\varphi, U, 0) \neq 0$, then $0 \in \varphi(x)$ for some $x \in U$. Indeed if it is not the case, then there is a continuous $f\colon \mathrm{cl}\, U \to \mathbb{R}^n$ such that $0 \notin f(\mathrm{cl}\, U)$ is view of Theorem 2.9.1; hence $\mathrm{Deg}(\varphi, U, 0) = \deg_B(f, U, 0) = 0$ in virtue of the existence property of the Brouwer degree.

Both defined degrees have the infinite-dimensional versions, too.

**B. The Leray–Schauder fixed point index.** Suppose $E$ is a normed space, $U \subset E$ is open, $\varphi\colon U \multimap E$ and let $W \subseteq U$ be open. Assume that

- (A1) $\varphi$ is weakly approximable and weakly homotopy approximable over any compact ANR contained in $U$;
- (A2) $\mathrm{Gr}(\varphi)$ is closed;
- (A3) The set $\mathrm{cl}\,\varphi(U)$ is compact and contained in $U$;
- (A4) $\mathrm{Fix}\,(\varphi) \cap \mathrm{bd}\, W = \emptyset$.

As before one sees easily that, for instance, conditions (A1) and (A2) are satisfied if $\varphi$ is upper semicontinuous with closed convex values or $\varphi$ is an upper semicontinuous $UV^\omega$-valued map (see Example 3.2.1).

Let $K := \operatorname{cl}\varphi(U)$. By (A3), $K \subset U$. By a theorem due to J. Girolo [41], there is a compact absolute neighbourhood retract $X$ such that $K \subset X \subset U$. Since $X$ is an ANR in $E$, there is an open neighbourhood $V$ of $X$ in $E$ and a retraction $r: V \to X$. Since $X$ is compact, there is $\varepsilon > 0$ such that $B(X, \varepsilon) \subset V$. It is clear that (diminishing $\varepsilon$ if necessary) that if $f: X \to E$ is an $\varepsilon$-approximation of $\varphi$ over $X$, then

$$f(x) \in B(K, \varepsilon) \subset B(X, \varepsilon) \subset V$$

and, moreover, neither $f$ nor $r \circ f$ (being well-defined) have fixed points in $\operatorname{bd} W \cap X$. Indeed: otherwise, for each sufficiently large $n \in \mathbb{N}$, say $n \geq N$, take an arbitrary $n^{-1}$-approximation $f_n: X \to E$ of $\varphi$ over $X$ and suppose that $f_n$ (or $r \circ f_n$) has a fixed point $x_n \in \operatorname{bd} W \cap X$; for each $n \geq N$, there are $x_n' \in U$, $\|x_n - x_n'\| < n^{-1}$ and $y_n' \in \varphi(x_n')$ (hence $y_n' \in K \subset X$) such that $\|f_n(x_n) - y_n'\| < n^{-1}$. Since $d(f_n(x_n), X) < n^{-1}$ and $X$ is compact, we gather (passing to a subsequence if necessary) that $f_n(x_n) \to z \in X$ and $x_n \to x \in \operatorname{bd} W \cap X$. Therefore $x_n' \to x$ and $y_n' \to z$. Hence $z \in \varphi(x)$. But remember that $x_n = f_n(x_n)$, i.e. $x = z$ (or $x_n = r(f_n(x_n)) \to r(z) = z$, i.e. again $x = z$); this means that $x \in \varphi(x)$: a contradiction.

Next there is $0 < \mu < \varepsilon$ such that, for $y \in B(X, \mu)$, $\|r(y) - y\| < \varepsilon/2$.

By (A1), there is $0 < \delta < \min\{\mu, \varepsilon\}$ such that given continuous $\delta$-approximations $f, f': X \to E$ of $\varphi$ over $X$, there is a continuous homotopy $h: X \times [0,1] \to E$ joining $f$ to $f'$ and such that, for any $t \in [0,1]$, $h(\,\cdot\,, t)$ is an $\varepsilon$-approximation of $\varphi$ over $X$.

Let $f: X \to E$ be a $\delta$-approximation of $\varphi$ over $X$. Then, for each $x \in X$, $f(x) \subset B(K, \delta) \subset B(X, \varepsilon) \subset V$. Let us consider the composition $g := r \circ f: X \to X$. Our construction shows that $g$ has no fixed points on $\operatorname{bd} W \cap X$.

We may define

$$\operatorname{Ind}(\varphi, W) := \operatorname{ind}(X, g, W \cap X)$$

where $\operatorname{ind}(X, g, W \cap X)$ stands for the fixed point index of a continuous $g: X \to X$ over an open subset $W \cap X$ of a compact ANR $X$ (see [36]).

We shall check that this definition is correct, i.e. does not depend on auxiliary objects: compact ANR $X$, a retraction $r: V \to X$ and a continuous (sufficiently close) approximation $f$ of $\varphi$ over $X$. First regarding $X$ and $r$ being fixed, suppose that $f': X \to E$ is a different continuous $\delta$-approximation of $\varphi$ over $X$. Due to our choice of $\delta$, there is a continuous homotopy $h: X \times [0,1] \to E$, joining $f$ to $f'$, such that, for any $t \in [0,1]$, $h(\,\cdot\,, t)$ is an $\varepsilon$-approximation. Thus $r \circ h: X \times [0,1] \to X$ is well-defined and the set $\{x \in \operatorname{bd} W \cap X \mid x \in x = h(x,t) \text{ or } x = r \circ h(x,t) \text{ for some } t \in [0,1]\}$ is empty. The homotopy invariance of ind implies that $\operatorname{ind}(X, r \circ f, W \cap X) = \operatorname{ind}(X, r \circ f', W \cap X)$.

Let $X'$ be a compact ANR such that $K \subset X' \subset U$. Choose a neighbourhood $V'$ of $X'$ in $E$, a retraction $r'\colon V' \to X'$ and numbers $\varepsilon', \mu', \delta' > 0$ having the same properties as $\varepsilon, \mu$ and $\delta$ constructed above. We are to show that $\mathrm{ind}(X, r \circ f, W \cap X) = \mathrm{ind}(X', r' \circ f', W \cap X)$ where $f\colon X \to E$ and $f'\colon X' \to E$ are continuous $\delta$- and $\delta'$-approximations $\varphi$ over $X$ and $X'$, respectively. Without loss of generality we may suppose that $X' \subset X$, $\varepsilon' < \varepsilon$, $\mu' < \mu$ and $\delta' < \delta$. Let $f\colon X \to E$ be a continuous $\delta'$-approximation $f\colon X \to E$ of $\varphi$ over $X$; then $f' := f|_{X'}$ is $\delta'$-approximation of $\varphi$ over $X'$. For each $x \in X$, $f(x) \in B(K, \mu') \subset B(X', \mu') \subset B(X, \mu)$; hence $\|r'(f(x)) - f(x)\| < \varepsilon' < \varepsilon$ and $\|r(f(x)) - f(x)\| < \varepsilon$. Consider a map $h\colon X \times [0,1] \to E$ given by $h(x,t) = r((1-t)r(f(x)) + tr'(f(x)))$ for $x \in X$ and $t \in [0,1]$. This map is well-defined for if $x \in X$, then

$$\begin{aligned} d((1-t)r(f(x)) + tr'(f(x)), X) &\le t\|r(f(x)) - r'(f(x))\| \\ &\le \|r(f(x)) - f(x)\| + \|r'(f(x)) - f(x)\| < \varepsilon. \end{aligned}$$

Observe that $h(\,\cdot\,, 0) = r \circ f$ and $h(\,\cdot\,) = r' \circ f$. Moreover it is easy to see that, for each $t \in [0,1]$, $h(\,\cdot\,, t)$ has no fixed points in $\mathrm{bd}\, W \cap X$. Thus, in view of the homotopy invariance of ind, $\mathrm{ind}(X, r \circ f, W \cap X) = \mathrm{ind}(X, r' \circ f, W \cap X)$. But, for $x \in X$, $r'(f(x)) \in X'$. Thus, by the restriction property of ind,

$$\mathrm{ind}(X, r' \circ f, W \cap X) = \mathrm{ind}(X', r' \circ f|_{X'}, W \cap X') = \mathrm{ind}(X', r' \circ f', W \cap X').$$

**Remark 3.2.2.** (a) The part of assumption (A3) stating that $\mathrm{cl}\, \varphi(U) \subset U$ may be avoided. However this involves much more tedious technical arguments, at least in case of $\varphi$ with $UV^\omega$ values. If $E$ is a Banach space, $\varphi\colon U \multimap E$ has closed convex values, is upper semicontinuous compact, $\mathrm{Fix}\,(\varphi) := \{x \in U \mid x \in \varphi(x)\}$ is compact, then one may argue as follows. Let $X := \mathrm{cl}\,\mathrm{conv}\, \varphi(U)$. Then $X$ is compact convex and, therefore, $X$ is a compact ANR. Take an arbitrary open $V \subset U$ such that $\mathrm{Fix}\,(\varphi) \subset V \subset \mathrm{cl}\, V \subset U$. For each $\varepsilon > 0$ there is an $\varepsilon$-approximation $f\colon U \cap X \to X$ of $\varphi$ over $W \cap X$. If $\varepsilon$ is sufficiently small, then $\mathrm{Fix}\,(f) \cap \mathrm{bd}\, V = \emptyset$ and the index $\mathrm{ind}(X, f, V)$ is defined. Hence one may put $\mathrm{Ind}\,(\varphi, U) := \mathrm{ind}(X, f, V)$. It is not difficult to show that this definition is correct since it does not depend on the choice of $f$ and $V$. It is also clear that if $\varphi$ is not compact, but there is a compact convex $L$ such that $\varphi(x) \cap L \ne \emptyset$ for all $x \in U$, then replacing $\varphi$ by $\varphi \cap L$ we may also define the respective index, whose nontriviality implies the existence of fixed points of $\varphi$.

(b) Suppose that $\varphi\colon U \multimap E$ is compact upper hemicontinuous with closed (*a posteriori* compact) values. Hence $\varphi$ is upper demicontinuous and the above construction does not apply. The definition of the fixed-point index in this case is provided in [55]; it involves some modification of Theorem 2.9.2.

**C. Fixed point index on arbitrary** ANRs. Suppose that $X$ is an arbitrary ANR and let $\varphi\colon X \multimap X$ be a compact upper semicontinuous $UV^\omega$-valued map. If $W \subset X$ is open and $\mathrm{Fix}\,(\varphi) \cap \mathrm{bd}\, U = \emptyset$, then the fixed point index

$\mathrm{Ind}\,(X, \varphi, U)$ is defined. The idea is simple. There is an embedding $s: X \to E$, where $E$ is a normed space, onto a closed set $s(X)$. By the very definition of ANR, there is an open neighbourhood $U \subset E$ of $s(X)$ and a retraction $r': U \to s(X)$. Consider $r = s^{-1} \circ r': U \to X$. Then $r \circ s = \mathrm{id}_X$. Hence $\psi := s \circ \varphi \circ r: U \multimap U$, $\mathrm{cl}\,\psi(U) \subset U$ and $\mathrm{Fix}\,(\psi) \cap \mathrm{bd}\, r^{-1}(W) = \emptyset$. Using the constructions from paragraph B. we are in a position to define

$$\mathrm{Ind}\,(X, \varphi, U) = \mathrm{Ind}\,(\psi, r^{-1}(W)).$$

The justification of this definition is technically involved; the reader should consult [7] (for this and much more general approach).

**3.3. Fixed points and equilibria under constraints.** The best known equilibrium (or fixed point) under constraints result is the following pioneering result of Browder (with some modification due to Halpern [48], [50]).

**Theorem 3.3.1** (Browder, [16])**.** *Assume that $K$ is a compact convex subset of a normed space $E$ and $\varphi: K \multimap E$ is upper semicontinuous with closed convex values. If $\varphi$ satisfies the weak tangency condition with respect to $K$ (19), i.e. for all $x \in K$,*

$$\varphi(x) \cap T_K(x) \neq \emptyset,$$

*then $\varphi$ has an equilibrium: there is $x_0 \in K$ such that $0 \in \varphi(x_0)$.*

In particular, we get the following corollary yielding a generalization of the Kakutani, Bohnenblust and Karlin theorems [11].

**Corollary 3.3.2.** *Given a convex compact set $K \subset E$ and an upper semicontinuous map $vp: K \multimap E$ with closed convex values, if $\varphi$ is weakly inward, i.e. for all $x \in K$,*

$$(\varphi(x) - x) \cap T_K(x) \neq \emptyset,$$

*or weakly outward*

$$(x - \varphi(x)) \cap T_K(x) \neq \emptyset,$$

*then $\varphi$ has a fixed point.* □

Observe that weak tangency (inwardness or outwardness) conditions are in fact boundary conditions: if $x \in \mathrm{int}\, K$, then $T_K(x) = E$ and they hold automatically. It is also clear that if, for $x \in K$, $\varphi(x) \subset K$ (i.e. $\varphi: K \multimap K$), then the $\varphi$ is weakly inward since then, for each $x \in K$, $\varphi(x) \subset K - x \subset T_K(x)$.

Theorem 3.3.1 and Corollary 3.3.2 were generalized many times: Ky Fan proved that Browder's result remains true under weaker assumptions concerning regularity of $\varphi$. Finally Cornet [24] has shown that the weak tangency condition may be substantially relaxed. Below we present the result of Cornet with a different proof using the Browder–Fan fixed point theorem.

We start with a next result essentially due to Browder.

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(19) The terminology given here and below may differ from the one used elsewhere.

**Theorem 3.3.3.** *If an upper hemicontinuous map $\varphi\colon K \multimap E^*$ has weak$^*$-compact convex values, then it admits a generalized equilibrium, i.e. there is $x_0 \in K$ such that $\varphi(x_0) \cap N_K(x_0) \neq \emptyset$.*

*Proof.* Observe that $x_0 \in K$ is a generalized equilibrium of $\varphi$ if and only if

$$\sup_{y\in K} \inf_{p\in\varphi(x_0)} \langle p, y - x_0\rangle \leq 0.$$

Indeed, the necessity of the above condition is clear; if this condition holds, then by the von Neumann-Sion min-max theorem,

$$0 \geq \inf_{p\in\varphi(x_0)} \sup_{y\in K} \langle p, y - x_0\rangle = \sup_{y\in K} \langle p_0, y - x_0\rangle$$

for some $p_0 \in \varphi(x_0)$, since the function $E^* \ni p \mapsto \sup_{y\in K} \langle p, y - x_0\rangle$ is weak$^*$ lower semicontinuous and $\varphi(x_0)$ is weak$^*$-compact. Since $N_K(x_0) = \{p \in E^* \mid$ for all $y \in K$ $\langle p, y - x_0\rangle \leq 0\}$, we see that $p_0 \in N_K(x_0)$.

Suppose to the contrary that $\varphi$ has no generalized equilibria, i.e. by for any $x \in K$, there is $y \in K$ such that $\inf_{p\in\varphi(x)} \langle p, y - x\rangle > 0$, i.e.

$$S(x) := \Big\{ y \in K \;\Big|\; \sigma_{\varphi(x)}(x - y) = \sup_{p\in\varphi(x)} \langle p, x - y\rangle < 0 \Big\} \neq \emptyset.$$

It is clear that, for each $x \in K$, $S(x)$ is convex and, for any $y \in K$, $S^{-1}(y)$ is open in view of the upper hemicontinuity of $\varphi$. Hence, by the Browder–Fan Theorem 3.0.1, there is $x_0 \in K$ such that $x_0 \in S(x_0)$: a contradiction. □

A similar result holds for proximal normal cones (see the lecture by Plaskacz [64] and Remark 1.3.10).

**Proposition 3.3.4.** *Let $K \subset E$ be compact convex and let $\varphi\colon K \multimap E$ be upper semicontinuous with compact convex values. Then there is $x_0 \in K$ such that $\varphi(x_0) \cap (\pi_K^{-1}(x_0) - x_0) \neq \emptyset$; in particular $\varphi(x_0) \cap N(x_0, K) \neq \emptyset$.*

*Proof.* It is clear that, for each $y \in E$, $\pi_K(y)$ is nonempty compact convex and $\pi_K\colon E \multimap K$ is upper semicontinuous. The map $\psi\colon K \times E \multimap K \times E$, given by $\psi(x, y) = \pi_K(y) \times (\varphi(x) + x)$ for $x \in K$ and $y \in E$, is upper semicontinuous, compact and has compact convex values. Hence, by the Bohnenblust-Karlin (or Fan–Glicksberg) fixed point theorem (see Theorem 3.1.5, there is $(x_0, y_0) \in K \times E$ such that $(x_0, y_0) \in \psi(x_0, y_0)$. Hence $x_0 \in \pi_K(y_0)$ and $y_0 \in \varphi(x_0) + x_0$, i.e. $\varphi(x_0) \cap (\pi_K^{-1}(x_0) - x_0) \neq \emptyset$. □

Application of the above proposition to the map $\varphi - \mathrm{id}_K$ yields immediately the next result of Ky Fan [39].

**Theorem 3.3.5.** *For any upper semicontinuous map $\varphi: K \multimap E$ with compact convex values, where $K \subset E$ is convex compact, there are points $x_0 \in K$ and $y_0 \in \varphi(x_0)$ such that $\|y_0 - x_0\| = d_K(y_0)$.*

Our next result, being a modification of a result essentially due to Cornet, provides the above mentioned direct generalization of the Browder–Fan Theorem 3.3.1.

**Theorem 3.3.6** (Cornet, [24])**.** *Suppose that $K$ is a convex compact subset of a Banach space $E$, $F$ is a Banach space, $A \in \mathcal{L}(E,F)$ (i.e. $A: E \to F$ is a bounded linear operator) and let $M := A(K)$. If an upper hemicontinuous map $\varphi: K \multimap E$ with closed convex values satisfies the normality condition, i.e. for all $x \in K$,*

$$\sup_{p \in N_M(A(x))} \inf_{y \in \varphi(x)} \langle p, y \rangle \le 0,$$

*then there is $x_0 \in K$ such that $0 \in \varphi(x_0)$.*

Before we give a proof let us discuss the normality condition and its relation to the usual (weak) tangency condition. First observe that, for all $x \in K$,

$$A^{*-1}(N_K(x)) = N_{A(K)}(A(x)) \quad \text{and} \quad \operatorname{cl} A(T_K(x)) = T_{A(K)}(A(x)).$$

**Lemma 3.3.7.** *Let $K$, $A$ and $M = A(K)$ be as above. Consider the following conditions:*

(a) *for all $x \in K$, $\varphi(x) \cap T_M(A(x)) \neq \emptyset$;*
(b) *for all $x \in K$, $\varphi(x) \cap \operatorname{cl} A(T_K(x)) \neq \emptyset$;*
(c) *for all $x \in K$, $\sup_{p \in A^{*-1}(N_K(x))} \inf_{y \in \varphi(x)} \langle p, y \rangle \le 0$;*
(d) *for all $x \in K$, $\sup_{p \in A^{*-1}(\partial d_K(x))} \inf_{y \in \varphi(x)} \langle p, y \rangle \le 0$;*
(e) *for all $x \in K$, $\sup_{p \in \partial d_M(Ax)} \inf_{y \in \varphi(x)} \langle p, y \rangle \le 0$;*
(f) *for all $x \in K$, $\inf_{y \in \varphi(x)} d_M^{\circ}(A(x); y) \le 0$.*

*Then* (a) $\Leftrightarrow$ (b) $\Rightarrow$ (*normality*) $\Leftrightarrow$ (c) $\Rightarrow$ (d)*,* (*normality condition*) $\Rightarrow$ (e) $\Leftrightarrow$ (f)*. If values of $\varphi$ are bounded, then* (d) $\Rightarrow$ (c) *and* (e) $\Rightarrow$ (*normality condition*)*. All these conditions are equivalent if $\varphi$ has weakly compact values.*

In particular if $E = F$ and $A = I$ is the identity, then (normality) is weaker than the weak tangency condition. Hence the Cornet theorem may indeed be considered as an extension of the Browder theorem.

*Proof of Theorem* 3.3.6. Suppose to the contrary that $\varphi$ has no equilibria, i.e. for each $x \in K$, $0 \notin \varphi(x)$. Theorem 2.9.2 implies the existence of a continuous map $p: K \to F^*$ is continuous such that, for each $x \in K$,

$$\inf_{y \in \varphi(x)} \langle p(x), y \rangle > 0.$$

By Theorem 3.3.3, there is $x_0 \in K$ such that $A^* p(x_0) \in N_K(x_0)$. Hence, by Lemma 3.3.7(c),

$$\inf_{y \in \varphi(x_0)} \langle p(x_0), y \rangle \le 0,$$

a contradiction. $\square$

**Remark 3.3.8.** (a) It would be interesting to prove the Cornet theorem under assumption (f) (from Lemma 3.3.7) replacing the normality. In view of Lemma 3.3.7, it holds when values of $\varphi$ are additionally bounded. If $E = F$ and $A = I$, then condition (f) reads

$$\inf_{y\in\varphi(x)} d_K^\circ(x;y) \leq 0 \quad \text{for all } x \in K \tag{3.1}$$

and is weaker than the normality and the usual weak tangency conditions. Below we shall see that (3.1) is sufficient for the existence of equilibria.

(b) Note that in the course of the above proof we have used the upper hemicontinuity of $\varphi$ only in a very restrictive manner. It is sufficient that given $p \in F^*$, the set $\{x \in K \mid \inf_{y\in\varphi(x)}\langle p, y\rangle > 0\}$ is open; the "full" upper hemicontinuity is not necessary.

Theorem 3.3.6 yields some consequences in the theory of the constrained coincidences. Replacing $\varphi$ by $\varphi - A$ we get

**Corollary 3.3.9.** *If $K$, $M$, $A$ and $\varphi$ are as in Theorem 3.3.6, for all $x \in K$ and $p \in N_M(A(x))$,*

$$\inf_{y\in\varphi(x)} \langle p, y\rangle \leq \langle p, A(x)\rangle,$$

*then there is $\overline{x} \in K$ such that $A(\overline{x}) \in \varphi(\overline{x})$.*

Now we shall try to discuss the possibilities to relax the assumption of compactness in the above results. As we shall see this requires a more delicate treatment. Perhaps the first result in this direction is that of Aubin.

**Proposition 3.3.10** (Aubin, [3]). *Suppose that $K \subset \mathbb{R}^n$ is closed and convex, an upper hemicontinuous $\varphi: K \multimap \mathbb{R}^n$ has closed convex values, satisfies the weak tangency condition and is coercive in the following sense:*

$$\limsup_{x\in K,\, \|x\|\to\infty} \sigma_{\varphi(x)}(x) < 0. \tag{3.2}$$

*The $\varphi$ has an equilibrium in $K$.*

*Proof.* By (3.2), there is $r > 0$ such that $B(0, r) \cap K \neq \emptyset$ and

$$\sup_{x\in K,\, \|x\|\geq r} \sigma_{\varphi(x)}(x) < 0.$$

Hence, for all $x \in K$, $\|x\| = r$ and all $y \in \varphi(x)$, $\langle x, y\rangle \leq 0$, i.e. $\varphi(x) \subset T_D(x)$ where $D := D(0, r)$ is the closed ball of radius $r$ around 0. Observe that, for all $x \in K \cap D$, $\varphi(x) \cap T_{K\cap D}(x)$ since, as it is easy to see,

$$T_{K\cap D}(x) = T_D(x) \cap T_K(x). \tag{3.3}$$

Thus, by Theorem 2.1.3, there is $x_0 \in K \cap D$ such that $0 \in \varphi(x_0)$. $\square$

A general principle implying this result is the following:

**Lemma 3.3.11** (see [4]). *Given a continuous operator $A: E \to F$, closed convex sets $K \subset E$, $C \subset F$, if the transversality condition holds, i.e.* $0 \in \operatorname{int}(A(K) - C)$, *then*

$$T_{K\cap A^{-1}(C)}(x) = T_K(x) \cap A^{-1}(T_C(A(x)))$$

*for each* $x \in K \cap A^{-1}(C)$.

It immediately gives the next result.

**Proposition 3.3.12** (see [58]). *Suppose that $K \subset E$ is closed convex and $\operatorname{int} K \neq \emptyset$. If an upper hemicontinuous $\varphi: K \multimap E$ with compact convex values is compact (i.e.* $\operatorname{cl}\varphi(K)$ *is compact) weakly inward, i.e. for each $x \in K$, $\varphi(x) \cap (x + T_K(x)) \neq \emptyset$, then $\varphi$ has a fixed point.*

*Proof.* Suppose, without loss of generality, that $0 \in \operatorname{int} K$. There is a compact convex set $C \subset E$ such that $0 \in C$ and $\varphi(K) \subset C$. Hence, for any $x \in K\cap C$, $\varphi(x) \subset x + T_C(x)$. Since $C \cap \operatorname{int} K \neq \emptyset$, we see by Lemma 3.3.11 (with $A = I$) that, for each $x \in K\cap C$, $T_K(x)\cap T_C(x) = T_{K\cap C}(x)$ and $(\varphi(x)-x)\cap T_{K\cap C}(x) \neq \emptyset$. By Theorem 2.1.3 we conclude the proof. □

The above results motivate the following definition: we say that a bounded linear operator $A: E \to F$ and a closed convex set $C \subset F$ *control directions admitted by* $\varphi: K \multimap F$ if, for each $x \in K\cap A^{-1}(C)$, we have $\varphi(x)\cap T_C(A(x)) \neq \emptyset$.

For instance, if $K \subset E$, $C \subset F$ are closed convex, $\varphi: K \multimap F$ and, for each $x \in K$, $\varphi(x) \cap C \neq \emptyset$, then $C$ and $A$ control directions admitted by $\Phi = \varphi - A$.

This allows the following *non-compact* generalization of the results of Cornet and Browder–Fan 3.3.1, 3.3.6

**Theorem 3.3.13.** *Suppose that $K \subset E$ is closed convex, $F$ is a Banach space, $A: E \to F$ is a linear bounded operator which, together with a compact set $C \subset F$, controls directions admitted by an upper hemicontinuous $\varphi: K \multimap F$ having closed convex values. If the restriction $A|_K$ is proper* ([20]), $0 \in \operatorname{int}(A(K) - C)$ *and, for all $x \in K \cap A^{-1}(C)$ and all $p \in A^{*-1}(N_K(x))$,*

$$\inf_{y\in\varphi(x)\cap T_C(A(x))} \langle p, y\rangle \le 0,$$

*then $\varphi$ has an equilibrium.*

*Proof.* By Lemma 3.3.11, for any $x \in K \cap A^{-1}(C)$,

$$\begin{aligned} N_{K\cap A^{-1}(C)}(x) &= T_{K\cap A^{-1}(C)}(x)^{\perp} \\ &= (T_K(x) \cap A^{-1}T_C(A(x)))^{\perp} = N_K(x) + A^*(N_C(A(x)). \end{aligned}$$

([20]) This holds for instance if $A$ is a semi-Fredholm operator and $K$ is bounded. Recall that $A \in \mathcal{L}(E, F)$ is *semi-Fredholm* if the range $\operatorname{Im}(A)$ is closed and $\dim \operatorname{Ker}(A) < \infty$.

Let $x \in K \cap A^{-1}(C)$ and $A^*(p) \in N_{K\cap A^{-1}(C)}(x)$. Then, there are $p_1 \in N_K(x)$ and $q \in N_C(A(x))$ such that $A^*(p) = p_1 + A^*(q)$. Hence $p - q \in A^{*-1}(N_K(x))$ and, for any $y \in \varphi(x) \cap T_C(A(x))$,

$$\inf_{y\in\varphi(x)} \langle p, y\rangle \le \inf_{y\in\varphi(x)\cap T_C(A(x))} \langle p, y\rangle \le \inf_{y\in\varphi(x)\cap T_C(A(x))} \langle p - q, y\rangle \le 0.$$

The properness of $A$ implies that $A^{-1}(C)$ is compact; this, by Theorem 3.3.6, ends the proof. $\square$

The result stated above, although sufficient on many occasions, in practice requires to know that $\operatorname{int} K \neq \emptyset$. From that reason stems the necessity to establish a result which relaxes this assumption, too. To discuss and explain an approach due to Deimling, let us make the following observations.

Let, as usual, $K \subset E$ be a closed set of a Banach space $E$ and let $x \in K$. If $\varepsilon > 0$ and $u: [0, \varepsilon] \to K$ is a continuous function such that $u(0) = x$ and the right derivative $v = u'_+(0)$ exists, then it is easy to see that $v \in T_K(x)$. Given an upper semicontinuous map $\varphi: K \multimap E$ with compact convex values such that there exists a solution $u: [0, \varepsilon] \to K$ of the Cauchy problem:

$$\begin{cases} u'(t) \in \varphi(u(t)), \\ u(0) = x, \end{cases} \tag{3.4}$$

i.e. there is an (Bochner) integrable function $w: [0, \varepsilon] \to K$ such that $w(t) \in \varphi(u(t))$ and $u(t) = x + \int_0^t w(s)\, ds$ on $[0, \varepsilon]$, then $\varphi(x) \cap T_K(x) \neq \emptyset$. Indeed, for an arbitrary sequence $h_n \to 0^+$ and any $n \in \mathbb{N}$, $u(h_n) = x + h_n v_n \in K$ where

$$v_n = \frac{1}{h_n} \int_0^{h_n} w(s)\, ds.$$

It is clear that $v_n \in \operatorname{cl}\operatorname{conv}\{w(s) \mid s \in [0, h_n]\} \subset \operatorname{cl}\operatorname{conv}\varphi(\{u(s) \mid s \in [0, h_n]\})$. The upper semicontinuity of $\varphi$ and the compactness of its values implies that, passing to subsequences if necessary, $v_n \to v \in \varphi(x)$. Hence $v \in \varphi(x) \cap T_K(x)$.

This means that in order to verify the weak tangency condition for $\varphi$ it is sufficient to show that, for each $x \in K$, problem (3.4) admits a solution. The converse implication does hold under some additional assumptions. In particular one has the following result.

**Proposition 3.3.14** (see [32]). *Suppose that $K$ is closed, bounded in $E$, an upper semicontinuous $\varphi: K \multimap E$ with convex compact values is $k$-set-contractive ($k \ge 0$) with respect to the Kuratowski or Hausdorff measure of noncompactness $\gamma$. If $\varphi$ is weakly tangent to $K$, then problem* (3.4) *admits a solution.*

**Theorem 3.3.15** (Deimling, [31]). *Let $K$ be a closed bounded convex subset of a Banach space $E$ and let an upper semicontinuous map $\varphi: K \multimap E$ with compact convex values be condensing with respect to the Kuratowski or Hausdorff* *measure of noncompactness* $\gamma$ ([21]). *If* $\varphi$ *is weakly inward, i.e. for each* $x \in K$, $\varphi(x) \cap x + T_K(x) \neq \emptyset$, *then* $\varphi$ *has a fixed point.*

*Proof.* We may assume that $0 \in K$ since a translation does not destroy any of the assumptions. Moreover, we may suppose that $\varphi$ is a set-contraction. For if we take $k \in (0,1)$, then it easy to see that $k\varphi$ is a set-contraction which also satisfies the weak inwardness condition. Assuming that the result holds for set-contractions and taking a sequence $k_n \to 1^{\perp}$, for each $n \in \mathbb{N}$ we get $x_n \in K$ such that $x_n \in k_n\varphi(x_n)$. For any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $k_n^{-1} - 1 < \varepsilon$ for $n \geq N$. For such $n$, $x_n \in B(k_n^{-1}x_n, \varepsilon) \subset B(\varphi(x_n), \varepsilon)$. Hence

$$\gamma(\{x_n\}_{n=1}^{\infty}) = \gamma(\{x_n \mid n \geq N\}) \leq \varepsilon + \gamma(\varphi(\{x_n \mid n \geq N\})) \leq \varepsilon + \gamma(\varphi(\{x_n\}_{n=1}^{\infty}).$$

This show that the set $\operatorname{cl}\{x_n\}_{n=1}^{\infty}$ is compact. Passing to a subsequence if necessary, we may assume that $x_n \to x_0$ and $x_0 \in \varphi(x_0)$.

Suppose that, for a bounded set $B \subset K$, $\gamma(\varphi(B)) \leq k\gamma(B)$, where $0 \leq k < 1$. Let $\varepsilon_n = 2^{-n}$ and

$$K_0 = K, \quad K_n = C_n \cap K_{n-1} \quad \text{where } C_n := \operatorname{cl}\operatorname{conv}[B(\varphi(K_{n-1}), \varepsilon_n) \cup B(0, \varepsilon_n)]$$

for $n \in \mathbb{N}$. It is clear that, for all $n \geq 0$, $0 \in K_n$, $K_{n+1} \subset K_n \subset K$ and

$$\gamma(K_n) \leq \gamma(C_n) \leq \gamma(\varphi(K_{n-1})) + 2\varepsilon_n \leq k\gamma(K_{n-1}) + 2\varepsilon_n.$$

Thus, by induction

$$\gamma(K_n) \leq k^n\gamma(K_0) + 2\sum_{i=1}^{n} k^{n-i}\varepsilon_i.$$

This shows that $\gamma(K_n) \to 0$; hence, by the Kuratowski theorem, the set

$$C := \bigcap_{n=0}^{\infty} K_n$$

is nonempty and compact convex. The map $\varphi$ is weakly inward to $K = K_0$; suppose that so it does with respect to $K_{n-1}$ ($n \geq 1$). We shall show that $\varphi$ is weakly inward to $K_n$ as well. Let $x \in K_n$ and take $y \in \varphi(x)$ such that $y - x \in T_{K_{n-1}}(x)$. Since $0 \in \operatorname{int} C_n \cap K_{n-1}$ and $\varphi(x) \subset C_n$, we have that $y - x \in T_{C_n}(x)$ and, by Lemma 3.3.11,

$$y - x \in T_{C_n}(x) \cap T_{K_{n-1}}(x) = T_{K_n}(x).$$

Having this we shall prove that $\varphi$ is weakly inward to $C$. To see this take $x \in C$ and observe that the Cauchy problem

$$\begin{cases} u'(t) \in \varphi(u(t)) - u(t), \\ u(0) = x, \end{cases} \tag{3.5}$$

([21]) Measures of noncompactness are treated e.g. in [1].

has a solution $u_n\colon[0,1]\to K_n$ in view of Proposition 3.3.14. It is easy to see that the family $\{u_n\}_{n=1}^{\infty}$ is equicontinuous and, for each $t\in[0,1]$, the orbit $\{u_n(t)\}_{n=1}^{\infty}$ is relatively compact. Hence, by the Ascoli–Arzela theorem, we may assume that $u_n\to u\in C([0,1],C)$ uniformly on $[0,1]$. Obviously $u(0)=x$. It is also standard (see e.g. [6, Appendix] for an argument in a more general situation) to see that, for almost all $t\in[0,1]$, the orbit $\{u_n'(t)\}_{n=1}^{\infty}$ is relatively compact. By a result due to Diestel [35], passing to a subsequence if necessary, we infer that $u_n\rightharpoonup w\in L^1([0,1],E)$ weakly in $L^1$. Hence $u(t)=x+\int_0^t w(s)\,ds$, i.e. $u'=w$ almost everywhere on $[0,1]$. The application of the so-called convergence theorem [3] (or [4]) shows that $w(s)\in\varphi(u(s))-u(s)$ almost everywhere on $[0,1]$, i.e. $u$ is a solution to (3.5). By the remarks preceding Proposition 3.3.14, this implies that $(\varphi(x)-x)\cap T_C(x)\neq\emptyset$. In virtue of Theorem 3.3.1, $\varphi$ has a fixed point. □

Theorem of Deimling gives sufficient conditions for the so-called *essentiality* of set-valued maps which generalizes the so-called Leray–Schauder nonlinear alternative of Granas and a result due to Aubin (see [3], [58]).

**Proposition 3.3.16.** *Suppose that $K\subset E$ is closed convex and $\operatorname{int}K\neq\emptyset$. If a compact map $\varphi\colon K\multimap E$ has compact convex values and is weakly inward to $K$, then it is essential with respect to the boundary $\operatorname{bd}K$, i.e. any compact map $\psi\colon K\multimap E$ with compact convex values such that $\psi|_{\operatorname{bd}K}=\varphi|_{\operatorname{bd}K}$ has a fixed point. In particular, if a compact map $\Phi\colon K\times[0,1]\multimap E$ with compact convex values is such that $\Phi(\,\cdot\,,0)=\varphi$, for all $x\in\operatorname{bd}K$ and $t\in[0,1]$, $x\notin\Phi(x,t)$, then $\Phi(\,\cdot\,,1)$ has a fixed point.*

*Proof.* The first part is obvious. For each $x\in K$, $\psi(x)\cap(x+T_K(x))\neq\emptyset$: for $x\in\operatorname{bd}K$ it follows by assumption; if $x\in\operatorname{int}K$, then $T_K(x)=E$.

As concerns the second assertion, the proof uses the method of Borsuk. Let $B=\{x\in K\mid x\in\Phi(x,t)\text{ for some }t\in[0,1]\}$. The upper semicontinuity of $\Phi$ implies that $B$ is closed; moreover $\operatorname{bd}K\cap B=\emptyset$. Take an Urysohn function $t\colon K\to[0,1]$ separating $\operatorname{bd}K$ from $A$, i.e. $t$ is continuous and $t|_{\operatorname{bd}K}\equiv 1$, $t|_B\equiv 1$. It is easy to see that $\psi(x)=\Phi(x,t(x))$ defines a compact map $\psi\colon K\multimap E$ with compact convex values and $\psi_{\operatorname{bd}K}=\varphi|_{\operatorname{bd}K}$. Therefore there is $x_0\in K$ such that $x_0\in\psi(x_0)=\Phi(x_0,t(x_0))$. Hence $x_0\in B$, $t(x_0)=1$ and $x_0\in\Phi(x_0,1)$. □

The next result follows in the same spirit.

**Theorem 3.3.17** (see [58])**.** *Suppose $K$ is convex closed, $U\subset K$ is open, a set-valued $\varphi\colon K\to E$ is compact, for all $x\in K$, $\varphi(x)\cap(x+T_K(x))\neq\emptyset$ and the fixed point set $\{x\in K\mid x\in\varphi(x)\}\subset U$. If $\psi\colon\operatorname{cl}_K U\multimap E$ is compact* ([22])*, for*

([22]) In what follows $\operatorname{cl}_K U$, $\operatorname{bd}_K U$ denotes the closure and the boundary of $U$ relative to $K$.

*each $x \in U$, $\psi(x) \cap (x + T_K(x)) \neq \emptyset$, then at least one of the following properties is satisfied:*

(a) *there is $x \in \mathrm{bd}_K U$ and $t \in (0,1)$ such that $x \in (1-t)\varphi(x) + t\psi(x)$;*
(b) *there is $x_0 \in \mathrm{cl}_K U$ such that $x_0 \in \psi(x_0)$.*

*Proof.* Suppose that (a) does not hold and $x \notin \psi(x)$ for $x \in \mathrm{bd}_K U$. The homotopy $\Phi\colon \mathrm{cl}_K U \times [0,1] \multimap E$ given by $\Phi(x,t) = (1-t)\varphi(x) + t\psi(x)$ is compact and has compact convex values. The set $B = \{x \in \mathrm{cl}_K U \mid x \in \Phi(x,t)$ for some $t \in [0,1]\}$ is clearly closed and $B \cap (K \setminus U) = \emptyset$. Take an Urysohn function $t\colon K \to [0,1]$ such that $t|_{K\setminus U} \equiv 0$ and $t|_B \equiv 1$. The map $\Phi(\,\cdot\,, t(\,\cdot\,))$ is actually defined on $K$, has compact convex values, is compact and satisfies the weak tangential inwardness condition, i.e. for each $x \in K$, $\Phi(x, t(x)) \cap (x + T_K(x)) \neq \emptyset$. Indeed, for $x \in K \setminus U$, $\Phi(x,t(x)) = \varphi(x)$; for $x \in U$, there are $y_1 \in \varphi(x)$ and $y_2 \in \psi(x)$ such that $y_i \in x + T_K(x)$, hence $y = (1-t(x))y_1 + t(x)y_2 \in x + T_K(x)$ and $y \in \Phi(x,t(x))$. By Theorem 3.3.15 there is $x_0 \in K$ such that $x_0 \in \Phi(x_0, t(x_0))$. Thus $x_0 \in B$, $t(x_0) = 1$ and $x_0 \in \psi(x_0)$. □

**Remark 3.3.18.** A much more involved arguments lead to the definition of the constrained fixed-point index. In [30] the following situation was studied: let $K \subset E$ be closed convex, $U \subset E$ be open and let $\Phi\colon U \multimap E$ be a compact upper semicontinuous map with compact convex values such that, for all $x \in U$, $\Phi(x) \cap (x + T_K(x)) \neq \emptyset$ (i.e $\Phi$ is weakly inward to $K$). Then the index $\mathrm{Ind}\,(\Phi, U)$ detecting fixed points of $\Phi$ is defined provided $\mathrm{Fix}\,(\Phi) := \{x \in U \mid x \in \Phi(x)\}$ is compact. As usual this index has all the standard properties. If $U = K$, then $\mathrm{Ind}\,(\Phi, K) = 1$. This, together with the existence property of the index implies the 'compact' version of Theorem 3.3.15.

**3.4. Beyond convexity.** Let us consider the following examples showing that the weak tangency or weak inwardness is not sufficient for the existence of equilibria or fixed points om maps defined on nonconvex sets.

**Example 3.4.1.** Let

$$\begin{aligned} K &:= \Big\{x = (x_1,x_2,x_3) \in \mathbb{R}^3 \;\Big|\; |x| \le \sqrt{2} \text{ and } \sqrt{x_1^2 + x_2^2} \ge x_3\Big\}, \\ S &:= \{x \in K \mid x_1^2 + x_2^2 = 1 \text{ and } x_3 = 1\}, \\ Z &:= \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \le 1 \text{ and } x_3 = 1\}. \end{aligned}$$

It is easy to see that $K$ is a compact retract. Next, for $x \in K$, put

$$\varphi(x) = \begin{cases} Z & \text{for } x \in D \setminus S, \\ \mathrm{conv}\,\{Z \cup \{(-x_2, x_1, 0\}\} & \text{for } x \in S. \end{cases}$$

Clearly $\varphi\colon K \multimap E$ is upper semicontinuous with compact convex values and $\varphi(x) \cap T_K(x) \neq \emptyset$ on $D$. But is easy to see that $\varphi$ has no equilibria. Observe also that, for $x = 0$, the set $S(x)$ of all solutions to the Cauchy problem (3.4) is homeomorphic to the unit sphere $S^1$; hence it is not an $R_\delta$-set (this will be explained later on). Notice that, for all $x \in K$, $x \neq 0$, the Bouligand and the Clarke tangent cones $T_K(x)$ and $C_K(x)$ coincide; however $T_K(0) \neq C_K(0)$ and $\varphi(0) \cap C_K(0) = \emptyset$.

Below we shall see that if, in the above example, should we take another map $\varphi$ that satisfies the weak tangency condition with the Bouligand cones replaced by the Clarke cones, then $\varphi$ would possess equilibria. However, it is not true that such a procedure would be a general remedy.

**Example 3.4.2.** Let $K := S_1 \cup S_{-1}$ where $S_i := \{z = (x,y) \in \mathbb{R}^2 \mid (x - i)^2 + y^2 = 1\}$. It is readily seen that $K$ is a neighbourhood retract in $\mathbb{R}^2$ and $\chi(K) \neq 0$. Let

$$f(x,y) = \begin{cases} (y, 1-x) & \text{for } (x,y) \in S_1, \\ (-y, 1+x) & \text{for } (x,y) \in S_{-1}. \end{cases}$$

For all $z \in K$, $f(z) \in T_K(z) = C_K(z)$ but $f$ has no zeros. At the same time the set of all solutions to (3.4) (with $\varphi$ replaced by $f$ and $x = (0,0)$) is even not connected.

We shall see that apart form the necessity to consider Clarke cones (instead of Bouligand ones), one should impose certain topological conditions on the set $K$.

Let $(X, d)$ be a metric space. We say that a set $K \subset X$ is an *$\mathcal{L}$-retract* (of $X$) if there is a neighbourhood retraction $r: U \to K$ and a constant $L \geq 1$ such that, for all $x \in U$,

$$d(r(x), x) \leq L d_K(x). \tag{3.6}$$

Clearly any $\mathcal{L}$-retract is a neighbourhood retract and is closed. The class of $\mathcal{L}$-retracts has been introduced and studied in [8]. It is clear that an $\mathcal{L}$-retract $K$ is an ANR; hence if $K$ is compact, then the *Euler characteristic* $\chi(K)$ is well-defined (see [17]).

Before we study the existence of equilibria on $\mathcal{L}$-retracts, let us provide some examples of such sets.

**Example 3.4.3.** (a) Suppose that $K \subset X$ is closed and bi-Lipschitz homeomorphic with a closed convex set $A \subset E$ (i.e. there is a Lipschitz homeomorphism $h: K \to A$ with Lipschitz inverse $g = h^{-1}: A \to K$). Then $K$ is an $\mathcal{L}$-retract.

To see this let $f: X \to A$ be an extension of $h$ given by the Arens-Dugundji formula (see [10]) and put $r(x) = g \circ f(x)$ for $x \in X$. In [8] it is shown that (3.6) holds for all $x \in X$ and $L = 3L_g L_h + 1$ where $L_h$ and $L_g$ are the Lipschitz constants of $h$ and $g$, respectively.

(b) If $K \subset E$ is closed and convex, then for each $\varepsilon > 0$, there is $r: E \to K$ such that $\|r(x) - x\| \leq (1+\varepsilon) d_K(x)$.

To see this, for $x \in E$, let $\psi(x) := \{y \in K \mid \|y - x\| \le (1+\varepsilon)d_K(x)\}$. It is easy to see that $\psi$ has closed convex values and is lower semicontinuous and, for $x \in K$, $\psi(x) = \{x\}$. In view of the Michael selection theorem, $\psi$ has a continuous selection $r: E \to K$.

(c) Following [63] (where the finite-dimensional case was presented) we say that $K \subset E$ is a *proximate retract* if there are a neighbourhood $U$ of $K$ and a retraction $r: U \to K$ such that $\|r(x) - x\| = d_K(x)$ (we say that $r$ is a *metric retraction* or *projection*). Proximate retracts in a Hilbert space (under the name *$\varphi$-convex sets*) have been studied in detail in [23] (see also the extensive bibliography therein) and some equivalent conditions were formulated. In particular, proximate retracts are tangentially regular. For instance, sets with $C^{1,1}$-boundary are proximate retracts. Obviously each proximate retract is an $\mathcal{L}$-retract.

(d) If $K \subset E$ is a neighbourhood retract with Lipschitz continuous neighbourhood retraction $r: U \to K$. Then (3.6) holds with $L = \ell + 1$ where $\ell$ is the Lipschitz constant of $r$. In particular, if $K$ is a compact neighbourhood with a locally Lipschitz retraction $r: U \to K$, then $K$ is an $\mathcal{L}$-retract.

(e) Above after Rockafellar [67] we have introduced the class; each epi-Lipschitz set is an $\mathcal{L}$-retract.

As we see the class of $\mathcal{L}$-retracts is pretty large and, as it appears, it behaves well as concerns the constrained equilibrium theory.

The first result in this direction is due to Plaskacz [63] who proved that *if $K \subset \mathbb{R}^n$ is a compact proximate retract with nontrivial Euler characteristic, $\varphi: K \multimap \mathbb{R}^n$ is upper semicontinuous with compact convex values satisfying the weak tangency condition* (*involving Bouligand or Clarke cones: it is equivalent in view of the tangential regularity of $K$*)*, then $\varphi$ has an equilibrium.*

The next step was done by Clarke, Ledyaev and Stern [22] (see also [70]) who proved that *given $K \subset E$ such that either*

(i) *$K$ is bi-Lipschitz homeomorphic with a compact convex set, or*
(ii) *$E = \mathbb{R}^n$ and $K$ is epi-Lipschitz and homeomorphic with a compact convex set, an upper semicontinuous $\varphi: K \multimap E$ with closed convex values satisfying the weak tangency condition* (*i.e. $\varphi(x) \cap C_K(x) \neq \emptyset$ for each $x \in K$*)*, then $\varphi$ has an equilibrium* ([23]).

Observe that above, the set $K$ is a compact $\mathcal{L}$-retract and $\chi(K) = 1$. The decisive contribution to the problem was done by Ben-El-Mechaiekh and the present author in [8] and the following result was obtained.

**Theorem 3.4.4** (see [8]). *Let $K \subset E$ be a compact $\mathcal{L}$-retract with $\chi(K) \neq 0$. If $\varphi: K \multimap E$ is upper semicontinuous with closed convex values and weakly*

([23]) Epi-Lipschitz in $\mathbb{R}^n$ having nontrivial Euler characteristic have been studied in [25] in the context of generalized equilibria.

*tangent to $K$, i.e.*

$$\varphi(x) \cap C_K(x) \neq \emptyset, \quad \text{for all } x \in K \tag{3.7}$$

*then $\varphi$ has an equilibrium.*

This result constitutes a direct generalization of the Browder, Plaskacz, Clarke, Ledyaev and Stern and others. We shall prove a result even more general due to Ćwiszewski and the author [27]. We start with a generalization of Theorem 3.3.3.

**Theorem 3.4.5** (see [27]). *Suppose $K \subset E$ is a compact $\mathcal{L}$-retract with the nontrivial Euler characteristic. Any upper hemicontinuous set-valued map $\varphi: K \multimap E^*$ with convex weak* compact values admits a generalized equilibrium i.e. a point $x_0 \in K$ such that $\varphi(x_0) \cap N_K(x_0) \neq \emptyset$.*

In fact in [27] the following fact was established.

(GE) There is $x_0 \in K$ and $\delta > 0$ such that, for all $y \in E$, if $d_K^\circ(x_0; y) < \delta$, then $\inf_{p \in \varphi(x_0)} \langle p, y\rangle \leq 1$.

It is not difficult to see that (GE) implies

$$\sup_{y \in C_K(x)} \inf_{p \in \varphi(x)} \langle p, y\rangle \leq 0$$

being equivalent to the existence of a generalized equilibrium.

Now we ready for an extension of Theorem 3.4.4.

**Theorem 3.4.6** (see [27], [58]). *Suppose that $K \subset$ is a compact $\mathcal{L}$-retract with $\chi(K) \neq 0$.*

(a) *Let $F$ be a Banach space, $A \in \mathcal{L}(E, F)$ and let $\varphi: K \multimap F$ be an upper hemicontinuous mapping with closed convex values satisfying the normality condition, i.e. $\inf_{y \in \varphi(x)} \langle p, y\rangle \leq 0$ for all $x \in K$ and $p \in A^{*-1}(N_K(x))$.*

(b) *Let an upper hemicontinuous map $\varphi: K \multimap E$ have closed convex values and, for all $x \in K$,*

$$\inf_{y \in \varphi(x)} d_K^\circ(x; y) \leq 0.$$

*Then, in both cases* (a) *and* (b), *$\varphi$ admits an equilibrium.*

*Proof.* The proof of the first part is identical to that of Theorem 3.3.6 (instead of Theorem 3.3.3 one applies Theorem 3.4.5).

When (b) holds, then again suppose to the contrary that $0 \notin \varphi(x)$ on $K$. The separation theorem implies the existence of a bounded linear form (of a sufficiently large norm) $p_x \in E^*$ such that $\inf_{y \in \varphi(x)} \langle p_x, y\rangle > 1$. As in the proof of Theorem 3.3.6, one constructs a continuous $p: K \to E^*$ such that, for all $x \in K$,

$$\inf_{y \in \varphi(x)} \langle p(x), y\rangle > 1. \tag{3.8}$$

Condition (GE) (with $p$ replacing $\varphi$) is satisfied for some $x_0 \in K$ and $\delta > 0$. By the assumption, there is $y \in \varphi(x_0)$ such that $d_K^\circ(x_0; y) < \delta$; hence $\langle p(x_0), y\rangle \le 1$: a contradiction with (3.8). $\square$

In both Theorems 3.4.5 and 3.4.4 the nontriviality of $\chi(K)$ is crucial. In [25] the authors show that this topological assumption is, in a sense, necessary for the existence of equilibria. Namely, they prove that if $K$ is a compact epi-Lipschitz subset of $\mathbb{R}^n$ (hence a compact $\mathcal{L}$-retract), then there exists a nonzero single-valued continuous map tangent to $K$. The corresponding result for general $\mathcal{L}$-retracts in $E$ (or even in $\mathbb{R}^n$) is not known; however we conjecture it to be true.

Unfortunately there is still no way to proceed with the constrained fixed problem of set-valued maps defined on arbitrary (noncompact) $\mathcal{L}$-retracts. The method proposed above strongly relies on the compactness. That is why criteria for the existence of equilibria of maps defined on sets of a slightly less general nature than $\mathcal{L}$-retracts have been established in [27], [58], [26]. The authors were able to relax the compactness assumption which was yet unavoidable in above theorems.

**3.5. Homotopy invariants in the constrained case.** Let $K \subset E$ be a locally compact $\mathcal{L}$-retract, let $U \subset K$ be open and suppose that $\varphi: U \multimap E$ be an upper hemicontinuous map with closed convex values such that, for all $x \in U$, $\varphi(x) \cap C_K(x) \neq \emptyset$ and (i.e. $\varphi$ is weakly tangent) $\varphi^{-1}(0)$ is compact. Under these assumptions the integer-valued degree $\deg(\varphi, U)$ has been defined in [29]. The construction is provided in three steps.

At first one assumes that $\varphi$ has compact values and is upper semicontinuous. For any $\varepsilon > 0$, let $C_\varepsilon: K \multimap E$ be given by

$$C_\varepsilon(x) = \left\{ u \in E \;\middle|\; \limsup_{y \xrightarrow{K} x,\, h \to 0^+} \frac{d_K(y + hu)}{h} < \varepsilon \right\}, \quad x \in K.$$

It is quite simple to show that $C_\varepsilon$ is lower semicontinuous and has convex values. Moreover, for each $x \in U$,$\varphi(x) \cap C_\varepsilon(x) \neq \emptyset$. Therefore, in view of Theorem 2.8.1, there is a continuous map $f_\varepsilon: U \to E$ being and $\varepsilon$-approximation of $\varphi$ and an $\varepsilon$-selection of $C_\varepsilon$. It may be shown that if $\varepsilon$ and $t > 0$ are sufficiently small, then the map $g_\varepsilon: U \to K$ given by

$$g_\varepsilon(x) = r(x + t f_\varepsilon(x)), \quad x \in U,$$

where $r: \Omega \to K$ is the $\mathcal{L}$-retraction (defined on an open neighbourhood $\Omega$ of $K$) is well-defined; moreover the fixed-point index $\mathrm{ind}(K, g_\varepsilon, U)$ is well-defined and does not depend on $\varepsilon$, $t$ and $r$. Hence one may put

$$\mathrm{Deg}(\varphi, U) := \lim_{\varepsilon, t \to 0^+} \mathrm{ind}(K, g_\varepsilon, U).$$

In the second step one considers an upper hemicontinuous set-valued map $\Phi: U \multimap E^*$ with weak$^*$-compact convex values such that the set $Z^*(\Phi) := \mathrm{cl}\,\{x \in U \mid \Phi(x) \cap N_K(x) \neq \emptyset\}$ (i.e. the closure of the set of all generalized equilibria) is compact. In this case [29] defines the so-called *co-degree* $*\mathrm{deg}(\Phi, U)$ which detects the existence of points in $Z^*(\Phi)$.

Finally, having all these we consider the general situation: $\varphi: U \multimap E$ is upper hemicontinuous with closed convex values. Using approximation constructions similar to those from Theorems 2.1.3 and 2.9.2, the existence of a continuous map $q: U \to E^*$ such that, for $x \in U \setminus \varphi^{-1}(0)$, $\inf_{y\in\varphi(x)}\langle q(x), y\rangle > 0$ and $Z^*(q) \subset \varphi^{-1}$. Therefore, one may put

$$\mathrm{Deg}(\varphi, U) := \deg(q, U).$$

This definition is correct since, as it is shown in [29], it does not depend on the choice of $q$. The whole construction relies heavily on various approximation techniques (graph-approximations, constrained graph-approximations as well as acute-angled approximations). The constructed degree (considered for weakly inward maps instead of weakly tangent ones) can be easily converted to a well-defined constrained fixed-point index.

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# SET-VALUED ANALYSIS IN OPTIMAL CONTROL PROBLEMS

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Abstract. In the lecture differential techniques in set-valued analysis are presented. Several applications to the control theory are given.

One of the main problems in optimal control theory is to describe the value function as a solution to the corresponding Hamilton–Jacobi–Bellman (H-J-B) equation. The value function is, in general, not differentiable. So, the value function is not a classical solution to the H-J-B equation. The crucial role is played by the notion of viscosity solutions introduced by Crandall and Lions in [8]. In the definition of viscosity solution the gradient is replaced by subgradient and supergradient — two basic notions of nonsmooth analysis. The subdifferential is the set of subgradients. In this way we obtain the set valued map that associates to any element in the domain of the value function its subdifferential (superdifferential). Methods of set valued analysis are deeply used in the differential inclusion approach initiated by Frankowska [11]. Our aim is to present the self contained series of three lectures that start with some elements of convex and nonsmooth analysis and through differential inclusion finish with weak solutions of H-J-B equations corresponding to the Mayer problem.

In the first section we introduce some basic notions of convex and nonsmooth analysis. We compare normal and tangent cones to convex sets. In the nonconvex case we study the Bouligand tangent cone, the Clarke tangent cone and the proximal normal cone. We show that the lower limit of Bouligand's cones is a subset of the Clarke cone. We study the subdifferential to a convex function

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and show the connections with normals to the epigraph. A similar connection we obtain also for noncovex functions.

In the second section we consider differential inclusions. We consider the problem of the existence of a solution to the Cauchy problem, we characterize the set of solutions to the Cauchy problem and study the invariance and the viability problems. First we consider differential inclusions with Lipschitz continuous right hand side. Then we pass to differential inclusions with upper semicontinuous (u.s.c.) right hand side using the approximation theorem of u.s.c. maps.

In the last section we consider the value function in the Mayer problem. We follow the Frankowska viability approach. We show that if a lower semicontinuous function is a weak solution to the H-J-B equation and satisfies the terminal condition then it is equal to the value function. We show that the epigraph of the function is forward viable and backward invariant to a corresponding differential inclusion. It follows the desired conclusion.

## 1. Elements of convex and nonsmooth analysis

**1.1. Proximal projection.** Suppose that $K \subset \mathbb{R}^d$ is a closed set. The proximal projection $\Pi_K$ from $\mathbb{R}^d$ onto the set $K$ is defined by

$$\Pi_K(y) = \{x \in K : \operatorname{dist}(y, K) = |y - x|\}.$$

If $x \in K$ and $n \in \mathbb{R}^d \setminus \{0\}$ then

$$x \in \Pi_K(x + n) \Leftrightarrow B(x + n, |n|) \cap K = \emptyset \tag{1.1}$$

where $B(z, r)$ denote a ball with the center at the point $z \in \mathbb{R}^d$ and a radius $r > 0$.

If $x \in \Pi_K(x + n)$ and $\alpha \in (0, 1)$ then $x \in \Pi_K(x + \alpha n)$.

We have $B(x + \alpha n, |\alpha n|) \subset B(x + n, |n|)$. Thus, if $B(x + n, |n|) \cap K = \emptyset$ then $B(x + \alpha y, |\alpha y|) \cap K = \emptyset$.

In general, the proximal projection is a set-valued map. We easy see that if $K$ is a sphere and x is the center then $\Pi_K(x)$ is the whole sphere. If $K$ is convex then the proximal projection is single-valued and posses more regular properties.

**Proposition 1.1.** *Suppose that $K \subset \mathbb{R}^d$ is convex and closed. Then $\Pi_K : \mathbb{R}^d \to K$ is a single-valued nonexpansive map*

$$|\Pi_K(y_1) - \Pi_K(y_2)| \le |y_1 - y_2|.$$

*Moreover,*

$$x \in \Pi_K(x + n) \Leftrightarrow \forall z \in K, \quad \langle z - x, n \rangle \le 0. \tag{1.2}$$

*Proof.* If $x_1, x_2 \in \Pi_K(y)$ and $x_1 \ne x_2$ then $|y - (x_1 + x_2)/2| < |y - x_1|$. Thus $\Pi_K$ is a single-valued map.

Suppose that $x \in \Pi_K(x+n)$ and $z \in K$. Then
$$|z-(x+n)|^2 \geq |n|^2.$$
Thus
$$\left\langle \frac{z-x}{|z-x|}, n \right\rangle \leq \frac{1}{2}|z-x|.$$
For $h > 0$ we set $z_h = x + h(z-x)/|z-x|$. For sufficiently small $h$ we have $z_h \in K$. Thus
$$\left\langle \frac{z-x}{|z-x|}, n \right\rangle = \left\langle \frac{z_h-x}{|z_h-x|}, n \right\rangle \leq \frac{1}{2}|z_h-x|.$$
Taking the limit with $h \to 0$ we obtain $\langle z-x, n\rangle \leq 0$.

To obtain the reverse implication in (1.2) let us take an arbitrary $z \in K$. Then
$$|z-(x+n)|^2 = |z-x|^2 + 2\langle z-x, n\rangle + |n|^2 \geq |n|^2.$$
Thus $x \in \Pi_K(x+n)$.

Now, we show that the proximal projection is a nonexpansive map. Let $y_1, y_2 \in \mathbb{R}^d$ and $x_i = \Pi_K(y_i)$, $i = 1, 2$. By (1.2),we have
$$\langle x_1 - x_2, y_2 - x_2\rangle \leq 0 \quad \text{and} \quad \langle x_2 - x_1, y_1 - x_1\rangle \leq 0.$$
Thus
$$\langle x_1 - x_2, y_2 - x_2 + x_1 - y_1\rangle \leq 0.$$
By the Cauchy inequality
$$|x_1 - x_2|^2 \leq |x_1 - x_2||y_1 - y_2|. \qquad \square$$

**1.2. Normal and tangent cones to a convex set.** We assume that $K \subset \mathbb{R}^d$ is a closed convex nonempty set. We say that a $n \in \mathbb{R}^d$ is a normal vector to the set $K$ at the point $x \in K$ if $\Pi_K(x+n) = x$. The set of normal vectors we denote by $N_K(x)$. That is
$$n \in N_K(x) \Leftrightarrow \Pi_K(x+n) = x.$$

**Proposition 1.2.** *The set $N_K(x)$ is a closed convex cone. The set-valued map $N_K\colon K \multimap \mathbb{R}^d$ has a closed graph.*

*Proof.* By (1.2), $n$ is a normal vector to $K$ at $x$ if and only if
$$\langle z-x, n\rangle \leq 0 \quad \text{for all } z \in K. \tag{1.3}$$
It follows that $N_K(x)$ is a closed convex cone.

Suppose that $n_m \in N_K(x_m)$ and $n_m = n$, $\lim_{m\to\infty} x_m = x$. Thus $x = \lim_{m\to\infty} x_m = \lim_{m\to\infty} \Pi_K(x_m + n_m) = \Pi_k(x+n)$. $\square$

The polar cone $S^\perp$ to a set $S \subset \mathbb{R}^d$ is given by
$$v \in S^\perp \Leftrightarrow \forall s \in S \ \langle s, v\rangle \leq 0.$$

It is easy to see that $S^\perp = (\operatorname{cl} S)^\perp$ and $S^\perp$ is a closed convex cone. If $S_1 \subset S_2$, then $S_2^\perp \subset S_1^\perp$.

**Lemma 1.3** (Fenchel)**.** *If $S \subset \mathbb{R}^d$ is a closed convex cone then $(S^\perp)^\perp = S$.*

*Proof.* It is obvious that $S \subset (S^\perp)^\perp$.

To show the reverse inclusion assume to the contrary that there exists $v \in (S^\perp)^\perp$ such that $v \notin S$. By the separation theorem, there exists $p \in \mathbb{R}^d \setminus \{0\}$ such that

$$\langle p, v\rangle > \sup\{\langle p, w\rangle : w \in S\}.$$

Thus $\sup\{\langle p, w\rangle : w \in S\} = 0$ and $p \in S^\perp$. But $v \in (S^\perp)^\perp$, so $\langle p, v\rangle \le 0$. A contradiction. □

By Walkup–Wets formulaa (see Lemma 1.3.7 in [16]) we obtain:

**Lemma 1.4.** *If a set valued map $N: K \multimap \mathbb{R}^d$ has closed graph then the set valued map $S: K \multimap \mathbb{R}^d$ given by $S(x) = (N(x))^\perp$ is lower semicontinuous.*

We define the tangent cone $S_K(x)$ to the set at $x \in K$ by

$$S_K(x) = \operatorname{cl}\bigg(\bigcup_{h>0} \frac{K - x}{h}\bigg).$$

**Theorem 1.5.** *If $K \subset \mathbb{R}^d$ is a closed convex set then for $x \in K$*

$$S_K(x) = N_K(x)^\perp, \quad N_K(x) = S_K(x)^\perp.$$

*The set valued map $S_K\colon \multimap \mathbb{R}^d$ is lower semicontinuous.*

*Proof.* We have

$$\bigg(\bigcup_{h>0} \frac{K - x}{h}\bigg)^\perp = N_K(x). \tag{1.4}$$

Indeed,

$$\begin{aligned} p \in \bigg(\bigcup_{h>0} \frac{K - x}{h}\bigg)^\perp &\Leftrightarrow \forall z \in K\ \forall h > 0, \quad \Big\langle \frac{z - x}{h}, p\Big\rangle \le 0 \\ &\Leftrightarrow \forall z \in K, \quad \langle z - x, p\rangle \le 0 \Leftrightarrow p \in N_K(x). \end{aligned}$$

We have

$$\bigcup_{h>0} \frac{K - x}{h} \quad \text{is a convex cone.}$$

If $v = (z - x)/h$ and $\alpha > 0$ then $\alpha v = (z - x)/(h/\alpha) \in (K - x)/(h/\alpha)$.

If $z_i \in K$ and $v_i = (z_i - x)/h_i$ , $i = 1, 2$, then

$$v_1 + v_2 = \frac{h_2 z_1/(h_1 + h_2) + h_1 z_2/(h_1 + h_2) - x}{h_1 h_2/(h_1 + h_2)} \in \bigcup_{h>0} \frac{K - x}{h}.$$

By (1.4) and (1.4), we obtain $N_K(x) = S_K(x)^\perp$. By Fenchel Lemma 1.3, we have $S_K(x) = N_K(x)^\perp$. By Proposition 1.2 and Lemma 1.4, the set valued map $S_K$ is lower semicontinuous. □

**1.3. A subdifferential to a convex function.** The condition (1.3) characterising a normal vector $n \neq 0$ to a convex set $K$ at a point $x \in K$ can be formulating as

$$\sup_{z\in K} \langle n, z\rangle = \langle n, x\rangle \,(=: c).$$

It can be geometrically interpreted as follows: the set $K$ is located on one side of the hyperplane $\{y \in \mathbb{R}^d : \langle n, y\rangle = c\}$. This hyperplane is a supporting hyperplane the set $K$ at the point $x$. The same geometrical idea is used to define the subdifferential to a convex function.

A function $f\colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is convex if its epigraph

$$\mathrm{Epi}(f) = \{(x, y) : x \in \mathbb{R}^d \text{ and } y \geq f(x)\}$$

is a convex subset of $\mathbb{R}^d \times \mathbb{R}$. The domain of the extended function $f$ is $\mathrm{dom}(f) = \{x \in \mathbb{R}^d : f(x) < \infty\}$. The subdifferential $\partial f(x_0)$ of the convex function $f$ at $x_0 \in \mathrm{dom}(f)$ is defined by

$$\partial f(x_0) = \{p \in \mathbb{R}^d : f(x) \geq f(x_0) + \langle p, x - x_0\rangle \text{ for all } x \in \mathbb{R}^d\}.$$

**Proposition 1.6.** *Suppose that $f\colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is a function with nonempty closed convex epigraph and $f(x_0) \neq \infty$. Then*

$$p \in \partial f(x_0) \Leftrightarrow (p, -1) \in N_{\mathrm{Epi}(f)}(x_0, f(x_0)).$$

*Proof.* Let $p \in \partial f(x_0)$ and $w \geq f(x)$. Then

$$\langle (x, w) - (x_0, f(x_0)), (p, -1)\rangle = (f(x_0) + \langle p, x - x_0\rangle) - w \leq 0.$$

Thus $(p, -1) \in N_{\mathrm{Epi}(f)}(x_0, f(x_0))$.

Suppose that $(p, -1) \in N_{\mathrm{Epi}(f)}(x_0, f(x_0))$. Then

$$\langle (p, -1), (x, f(x)) - (x_0, f(x_0))\rangle \leq 0$$

for every $x \in \mathrm{dom}(f)$. Thus $p \in \partial f(x_0)$. □

**Example 1.7.** We define $f\colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ by

$$f(x) = \begin{cases} -\sqrt{1 - x^2} & \text{for } |x| \leq 1, \\ \infty & \text{for } |x| > 1. \end{cases}$$

If $|x| = 1$ then the subdifferential $\partial f(x)$ is an empty set. But the normal cones to the epigraph are nonempty, $N_{\mathrm{Epi}(f)}(1, 0) = \{(n_1, 0) : n_1 \geq 0\}$.

**1.4. Tangent and normal cones to a closed set.** We suppose that $K \subset \mathbb{R}^d$ is a nonempty closed set. There exists several concept of tangent cones to a nonsmooth, nonconvex set. We present Bouligant tangent cone $T_K(x)$ and Clarke tangent cone $C_K$. Systematic presentation of different concepts of tangent cones can be find in [3].

We define the Bouligand tangent cone to the set $K$ at a point $x \in K$ by

$$T_K(x) = \limsup_{h \to 0^+} \frac{K - x}{h}$$

where $\limsup A_h$ denote the set upper limit of the family $\{A_h\}$, i.e.

$$a \in \limsup_{h \to 0^+} A_h \Leftrightarrow \liminf_{h \to 0^+} \operatorname{dist}(a, A_h) = 0.$$

**Proposition 1.8.** *If $K \subset \mathbb{R}^d$ and $x \in K$ then the following conditions are equivalent:*

(a) $v \in T_K(x)$*;*
(b) $\liminf_{h \to 0^+} \operatorname{dist}(x + hv, K)/h = 0$*;*
(c) *there exists $h_n \to 0^+$ and $v_n \to v$ such that $x + h_n v_n \in K$.*

*Proof.* (a) $\Rightarrow$ (b) If $v \in T_K(X)$ then $\liminf_{h \to 0^+} \operatorname{dist}(v, (K - x)/h) = 0$. For every $h > 0$ there exists $z_h \in K$ such that $\operatorname{dist}(v, (K - x)/h) = |v - (z_h - x)/h|$ We have

$$\frac{\operatorname{dist}(x + hv, K)}{h} \leq \frac{|z_h - (x + hv)|}{h} = \left| v - \frac{z_h - x}{h} \right|.$$

Thus

$$\liminf_{h \to 0^+} \frac{\operatorname{dist}(x + hv, K)}{h} \leq \liminf_{h \to 0^+} \operatorname{dist}\left( v, \frac{K - x}{h} \right) = 0.$$

(b) $\Rightarrow$ (c) We choose $h_n \to 0^+$ such that $\lim_{n \to \infty} \operatorname{dist}(x + h_n v, K)/h_n = 0$. Let $z_n \in K$ and $\operatorname{dist}(x + h_n v, K) = |x + h_n v - z_n|$. We set $v_n = (z_n - x)/h_n$. Then

$$|v_n - v| = \frac{|z_n - x - h_n v|}{h_n} \xrightarrow{n \to \infty} 0.$$

(c) $\Rightarrow$ (a) If $x + h_n v_n \in K$ then

$$\operatorname{dist}\left( v, \frac{K - x}{h_n} \right) \leq \left| v - \frac{(x + h_n v_n) - x}{h_n} \right| = |v - v_n|.$$

If $v_n \to v$ then $\liminf_{h \to 0^+} \operatorname{dist}(v, (K - x)/h) = 0$. $\square$

The Bouligand tangent cone $T_K(x)$ is a closed cone. In general, it is not convex.

**Example 1.9.** Let $K = \{(x, y) : x \geq 0,\ y \geq 0 \text{ and } xy = 0\}$. We have $T_K(0, 0) = K$, so it is not convex.

The set valued map $T_K \colon K \multimap \mathbb{R}^d$ is not lower semicontinuous.

**Example 1.10.** Let $K = \bigcup_{n=1}^{\infty} S_n \cup \{(0,0)\}$, where $S_n$ is a sphere in $\mathbb{R}^2$ with the radius $r_n$ and the center at $(x_n, 0)$. We choose sequences $x_n \to 0^+$ and $r_n \to 0^+$ such that $x_n = x_{n+1} + r_{n+1} + r_n$ and $x_n^2 = 2r_n^2$. We have $T_K(0,0) = \{(v_1, v_2) : v_1 \geq 0 \text{ and } -v_1 \leq v_2 \leq v_1\}$. And the set valued map $T_K$ is not lower semicontinuous at $(0,0)$.

We define the proximal normal cone $PN_K(x)$ to a closed set $K \subset \mathbb{R}^d$ at a point $x \in K$ by

$$PN_K(x) = \{n \in \mathbb{R}^d : \text{there exists } \alpha > 0 \text{ such that } x \in \Pi_K(x + \alpha n)\}.$$

We have

$$T_K(x) \subset (PN_K(x))^{\perp}. \tag{1.5}$$

Let $v \in T_K(x)$ and $n \in PN_K(X)$. Then there exists $v_n \to v$, $h_n \to 0^+$ such that $x + h_n v_n \in K$. Without loss of generality we can assume that $x \in \Pi_K(x+n)$. By (1.1), we have $K \cap B(x+n, |n|) = \emptyset$. So, we obtain $|x + h_n v_n - (x+n)|^2 \geq |n|^2$ and $h_n |v_n|^2 - 2\langle v_n, n\rangle \geq 0$. Passing to the limit we obtain $-2\langle v, n\rangle \geq 0$, which follows (1.5).

**Lemma 1.11.** *Suppose that a function $x\colon (a,b) \to \mathbb{R}^d$ and the function $g(t) = \operatorname{dist}(x(t), K)$ are differentiable at $t_0 \in (a,b)$, where $K \subset \mathbb{R}^d$ is a closed set. If $g(t_0) > 0$ and $y \in \Pi_K(x(t_0)$ then*

$$g'(t) \leq \left\langle x'(t_0), \frac{n}{|n|} \right\rangle,$$

*where $n = x(t_0) - y$.*

*Proof.* We have

$$x(t_0 + h) = x(t_0) + hv + o(h),$$

where $v = x'(t_0)$ and $\lim_{h\to 0} o(h)/h = 0$. Thus

$$\begin{aligned} \frac{1}{h}(g(t_0+h) - g(t_0)) &\leq \frac{1}{h}(|x(t_0+h) - y| - |x(t_0) - y|) \\ &\leq \frac{|x(t_0) + hv + o(h) - y|^2 - |x(t_0) - y|^2}{h(|x(t_0) + hv + o(h) - y| + |x(t_0) - y|)} \end{aligned}$$

and

$$g'(t_0) \leq \lim_{h\to 0^+} \frac{2\langle hv + o(h), h\rangle + |hv + o(h)|^2}{h(|x(t_0) + hv + o(h) - y| + |x(t_0) - y|)} = \left\langle v, \frac{n}{|n|} \right\rangle. \qquad \square$$

We define the Clarke tangent cone $C_K(x)$ to the set $K$ at a point $x \in K$ by

$$C_K(x) = \operatorname*{Lim\,inf}_{h\to 0^+,\, y\to_K x} \frac{K - y}{h} \tag{1.6}$$

where $\operatorname{Lim\,inf}_{z\to z_0} A_z$ denotes the set lower limit, i.e.

$$a \in \operatorname*{Lim\,inf}_{z\to z_0} A_z \Leftrightarrow \limsup_{z\to z_0} \operatorname{dist}(a, A_z) = 0$$

and the symbol $y \to_K x$ denotes that $y$ belongs to $K$ and tends to $x$.

The definition of the Clarke tangent cone can be formulated equivalently as follows.

**Proposition 1.12.** *If $K \subset \mathbb{R}^d$ is a closed set and $x \in K$ then the following conditions are equivalent:*

(a) $v \in C_K(x)$;
(b) $\liminf_{h\to 0^+,\, y\to_K x} \operatorname{dist}(y + hv, K)/h = 0$;
(c) *for all $h_n \to 0^+$ and all $y_n \to_K x$ tehre exists $v_n \to v$ such that $y_n + h_n v_n \in K$.*

The proof of Proposition 1.12 is similar to the proof of Proposition 1.8.

**Proposition 1.13.** *If $K \subset \mathbb{R}^d$ is a closed set and $x \in K$ then the Clarke tangent cone $C_K(x)$ is a closed convex cone.*

*Proof.* By (1.6) we obtain that it is a closed cone. To show that $C_K(x)$ is a convex set we shall use the condition (c) in Proposition 1.12.

Let $v, w \in C_K(x)$ and $\lambda \in (0, 1)$. We set $z = \lambda v + (1 - \lambda)w$. Let us take an arbitrary $h_n \to 0^+$ and $y_n \to_K x$. Then, there exist $v_n \to v$ such that $y_n + \lambda h_n v_v \in K$. Since $y_n + \lambda h_n v_n \to_K x$ and $w \in C_K(x)$ then there exist $w_n \to w$ such that $(y_n + \lambda h_n v_n) + (1-\lambda)h_n w_n \in K$. Thus, we obtained a sequence $\lambda v_n + (1 - \lambda)w_n$ converging to $z$ such that $y_n + h_n(\lambda v_n + (1 - \lambda)w_n) \in K$. So, $z \in C_K(x)$. $\square$

The main result concerning the connection between the Bouligand cone and the Clarke cone is that

$$\operatorname*{Lim\,inf}_{y\to_K x} \overline{\operatorname{conv}}\, T_K(y) \subset C_K(x). \tag{1.7}$$

**Theorem 1.14.** *If $K \subset \mathbb{R}^d$ is a closed set and $x \in K$ then*

$$\operatorname*{Lim\,inf}_{y\to_K x} (PN_K(y))^{\perp} \subset C_K(x).$$

*Proof.* We show that if $v \in \operatorname{Lim\,inf}_{y\to_K x} (PN_K(y))^{\perp}$ then the condition (b) in Proposition 1.12 holds true.

Let $\varepsilon > 0$. We choose $\delta > 0$ such that

$$\forall |z - x| < \delta,\ z \in K,\ \exists v_z \in (PN_K(z))^{\perp}, \quad |v - v_z| < \varepsilon.$$

We choose $\delta_1 > 0$ such that for $|y - x| < \delta_1$ and $0 < t < \delta_1$ it holds $|y + tv - x| < \delta/2$. Fix $y$. Let us consider the function $g(t) = \operatorname{dist}(y + tv, K)$. The function $g$ is lipchitz continuous. We choose $t$ such that $g$ is differentiable at $t$. Let

$z \in \Pi_K(y+tv)$. Then $|z-x| < \delta$. We set $n := (y+tv) - z \in PN_K(z)$. By Lemma 1.11, we have

$$g'(t) \leq \left\langle v, \frac{n}{|n|} \right\rangle = \left\langle v_z, \frac{n}{|n|} \right\rangle + \left\langle v - v_z, \frac{n}{|n|} \right\rangle \leq \varepsilon.$$

Let $t_h = \sup\{t \in [0,h] : g(t) = 0\}$. Thus

$$\operatorname{dist}(y+hv, K) = \int_{t_h}^{h} g'(s)\, ds \leq \varepsilon h$$

for $y \in K$, $|y-x| < \delta_1$, $0 < h < \delta_1$. □

**Remark 1.15.** (a) Since $(PN_K(x))^{\perp}$ is a convex cone then $\overline{\operatorname{conv}}\, T_K(x) \subset (PN_K(x))^{\perp}$. By Theorem 1.14, we obtain (1.7).

(b) Suppose that $K \subset \mathbb{R}^d$ is a closed convex set. Then $PN_K(x) = N_K(x)$. Since the set-valued map $N_K(\,\cdot\,)$ has the closed graph then, by Proposition 1.4, the set-valued map $N_K(\,\cdot\,))^{\perp}$ is lower semicontinuous. Thus

$$(N_K(x))^{\perp} \subset \operatorname*{Lim\,inf}_{y \to_K x} (N_K(y))^{\perp}.$$

By (1.5) and (1.7) we obtain

$$T_K(x) \subset (N_K(x))^{\perp} = S_K(x) \subset \operatorname*{Lim\,inf}_{y \to_K x} (N_K(y))^{\perp} \subset C_K(x) \subset T_K(x).$$

Thus the Bouligand tangent cone and the Clarke tangent cone coincide if the set $K$ is convex, i.e.

$$T_K(x) = C_K(x) = S_K(x).$$

(c) If the set-valued map $T_K(\,\cdot\,)$ is lower semicontinuous then $T_K(x) = C_K(x)$ for all $x \in K$. In particular, if $K$ is a proximal neighbourhood retract then the set-valued map $T_K(\,\cdot\,)$ is lower semicontinuous (see [17]).

**1.5. A subdifferential to a lower semicontinuous function.** In the section we assume that the function $u\colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous. Equivalently we can say that the epigraph $\operatorname{Epi}(u)$ is a closed subset of $\mathbb{R}^{d+1}$.

**Definition 1.16.** We say that $p \in \mathbb{R}^d$ is a subgradient of the function $u(\,\cdot\,)$ at the point $x_0$ if there exists an open neighbourhood $V$ of $x_0$ and a $C^2$ function $\varphi\colon V \to \mathbb{R}$ such that

$$\varphi(x) \leq u(x) \quad \text{for } x \in V \text{ and } \varphi(x_0) = u(x_0)$$

and

$$p = \operatorname{grad} \varphi(x_0).$$

The subdifferential $\partial u(x_0)$ of the function $u$ at the point $x_0$ is the set of all subgradients $p$ of $u$ at $x_0$.

We use the same notation to denote the subdifferential of an arbitrary lower semicontinuous function and the subdifferential of a convex function. As one can expect the two notions coincide if the function $u$ is convex. Also Proposition 1.6 has an analog in the non-convex case.

**Proposition 1.17.** *If $u: \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous function and $u(x_0) < \infty$ then*

$$p \in \partial_- u(x_0) \Leftrightarrow (p, -1) \in PN_{\mathrm{Epi}\,(u)}(x_0, u(x_0)).$$

*Proof.* Suppose that $p \in \partial u(x_0)$ and

$$\varphi(x) = \varphi(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2}(x - x_0)^\top D^2\varphi(x_0)(x - x_0) + o(|x - x_0|^2) \leq u(x).$$

For $x$ sufficiently close to $x_0$ we have

$$\varphi(x) \geq \varphi(x_0) + \langle p, x - x_0 \rangle - \frac{1}{2}\|D^2\varphi(x_0)\||x - x_0|^2 - \varepsilon|x - x_0|^2.$$

Thus there exists $C > 0$ such that

$$u(x) \geq \varphi(x_0) + \langle p, x - x_0 \rangle - C|x - x_0|^2$$

on a neighbourhood of $x_0$.

Let $\alpha < 1/2C$. We define

$$\psi(x) := \varphi(x_0) - \alpha + \sqrt{\alpha^2(|p|^2 + 1) - |x - x_0 - \alpha p|^2}.$$

The function $\psi$ is defined on the ball centered at $x_0 + \alpha p$ with the radius $|\alpha(p, -1)|$. The graph of $\psi$ is the upper hemisphere in $\mathbb{R}^{d+1}$ centered at the point $(x_0, \varphi(x_0)) + \alpha(p, -1)$ with the radius $|\alpha(p, -1)|$.

By the Cauchy inequality we have

$$2C\langle p, x - x_0 \rangle \leq |p|^2 + c^2|x - x_0|^2.$$

Since $2C\alpha < 1$ then

$$2C(\langle p, x - x_0 \rangle |x - x_0|^2 + \alpha|x - x_0|^2) \leq |x - x_0|^2(1 + |p|^2 + C|x - x_0|^2)$$

and

$$0 \leq |x - x|^2(1 + |p|^2 - 2C\alpha + C^2|x - x_0|^2) - 2C\langle p, x - x_0 \rangle.$$

So

$$\alpha^2(|p|^2 + 1) \leq |x - x_0 - \alpha p|^2 + (\alpha + \langle p, x - x_0 \rangle - C|x - x_0|^2)^2$$

and

$$\sqrt{\alpha^2(|p|^2 + 1) - |x - (x_0 + \alpha p)|^2} \leq \alpha + \langle p, x - x_0 \rangle - C|x - x_0|^2.$$

It follows that for sufficiently small $|x - x_0|$ we have $\varphi(x) \leq \psi(x)$.

If we take $\alpha$ sufficiently small we obtain that the ball centered at $(x_0, \varphi(x_0)) + \alpha(p, -1)$ with the radius $|\alpha(p, -1)|$ has a nonempty intersection with the epigraph of the function $u$. Thus $(p, -1) \in PN_{\mathrm{Epi}\,(u)}(x_0, u(x_0))$.

Now, suppose that $(p,-1) \in PN_{\mathrm{Epi}\,(u)}(x_0, u(x_0))$ and let $\alpha > 0$ be such that

$$\mathrm{dist}((x_0, u(x_0)) + \alpha(p,-1), \mathrm{Epi}\,(u)) = \alpha|(p,-1)|.$$

Consider the function

$$\varphi(x) := u(x_0) - \alpha + \sqrt{\alpha^2(|p|^2+1) - |x - x_0 - \alpha p|^2}$$

at the ball $B(x_0 + \alpha p, \alpha|(p,-1)|)$. We have $\varphi(x_0) = u(x_0)$ and $\operatorname{grad}\varphi(x_0) = p$. Since the intersection of the ball $B((x_0 + \alpha p, u(x_0) - \alpha), \alpha|(p,-1)|)$ with the epigraph of the function $u$ is empty then $\varphi(x) \leq u(x)$. The function $\varphi$ is $C^\infty$ smooth, so $p \in \partial u(x_0)$. □

**Remark 1.18.** Usually authors take the function $\varphi$ from the class $C^1$ in the definition of the subgradient (comp. [8]). In Definition 1.16 we assumed that the funtion $\varphi$ supporting the epigraph $\mathrm{Epi}\,(u)$ from below is $C^2$ smooth. From the proof of Proposition 1.17 we conclude that replacing the class $C^2$ by $C^\infty$ in Definition 1.16 we obtain an equivalent notion of subgradient. Below we provide an example showing that taking $\varphi$ from the class $C^1$ instead of $C^2$ we essentially change the notion of subgradient.

**Example 1.19.** Let $\alpha \in (1,2)$. We define $u(x) = -|x|^\alpha$. Since the function $u$ is $C^1$ smooth then 0 is a $C^1$-subgradient of $u$ at $x_0 = 0$. But we have that the cone of proximal normals $PN_{\mathrm{Epi}\,(u)}(0,0)$ to the epigraph of $u$ at $(0,0$ consists of one element$(0,0)$. Indeed, the Bouligand tangent cone $T_{\mathrm{Epi}\,(u)}(0,0)$ is the half space $\{(v_1,v_2) : v_2 \geq 0\}$. The proximal normal cone $PN_{\mathrm{Epi}\,(u)}(0,0)$ is a subset of the polar cone to the Bouligand tangent cone. Thus

$$PN_{\mathrm{Epi}\,(u)}(0,0) \subset \{(0,n_2) : n_2 \leq 0\}.$$

If $n_2 < 0$ then $\mathrm{dist}((0,n_2), \mathrm{Epi}\,(u)) < |n_2|$. Thus $PN_{\mathrm{Epi}\,(u)}(0,0) = \{(0,0)\}$ and by Proposition 1.17, we obtain that the set of $C^2$ subgradients is an empty set, i.e. $\partial u(0) = \emptyset$.

The following example shows that the horizontal proximal normals to the epigraph have not a counterpart in the subdifferential.

**Example 1.20.** Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \mathrm{sign}(x)\sqrt{|x|}$. Then the subdifferential $\partial g(0)$ is an empty set. But the proximal normal cone to the epigraph of $g$ at $(0,0)$ is nonempty. Namely

$$PN_{\mathrm{Epi}\,(g)}(0,0) = \{(x,0) : x \geq 0\}.$$

If a function $g\colon \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz continuous then no horizontal normal to the epigraph exists and we have

$$PN_{\mathrm{Epi}\,(g)}(x, g(x)) = \{\alpha(p,-1) : p \in \partial g(x),\ \alpha \geq 0\}.$$

If $g$ is lower semicontinuous then a horizontal normal can be approximated by downward directed normals in the following way. The proof of the following lemma was communicated the author by Pierre Cardaliaguet [5]

**Lemma 1.21** ([22])**.** *Suppose that* $g\colon \mathbb{R}^d \to \mathbb{R}$ *is lower semicontinuous ,* $(p,0) \in PN_{\mathrm{Epi}\,(g)}(x, g(x))$ *and* $p \neq 0$*. Then there exist* $x_n \to x$*,* $p_n \to p$ *and* $v_n \to 0$*,* $v_n < 0$ *such that*

$$(p_n, v_n) \in PN_{\mathrm{Epi}\,(g)}(x_n, g(x_n)).$$

We procede the proof by some simple geometrical lemmas.

**Lemma 1.22.** *If* $n, b \in \mathbb{R}^d$ *and* $|n| = 1$*,* $|b| < 1$ *then*

$$\langle n + b, n\rangle \geq \sqrt{1 - |b|^2}|n + b|.$$

*Proof.* We have

$$1 = |n|^2 = |(n+b) - b|^2 \leq |n+b|^2 + |b|^2 - 2\langle n+b, b\rangle + \left\langle \frac{n+b}{|n+b|}, b\right\rangle^2.$$

Thus

$$|n+b|^2 - 2\langle n+b, b\rangle + \left\langle \frac{n+b}{|n+b|}, b\right\rangle^2 \geq 1 - |b|^2.$$

So

$$\left\langle \frac{n+b}{|n+b|}, n\right\rangle = |n+b| - \left\langle \frac{n+b}{|n+b|}, b\right\rangle \geq \sqrt{1 - |b|^2}. \qquad \square$$

If $0 < c < 1$, $n \in \mathbb{R}^d$ and $|n| = 1$ then the set $\{v \in \mathbb{R}^d : \langle v, n\rangle \geq c|v|\}$ is a revolving cone with the axis $n$ and with the angle $\alpha$ at the vertex such that $\cos\alpha = c$. From Lemma 1.22 we obtain that ball centered at $n$ ($|n| = 1$) with the radius $r < 1$ is a subset of the cone with the axis $n$ and the constant $c = \sqrt{1 - r^2}$.

**Lemma 1.23.** *If* $|n| = 1$*,* $0 < c < 1$ *and* $z\colon [0,t] \to \mathbb{R}^d$ *is an integrable function such that*

$$\langle z(s), n\rangle \geq c|z(s)| \quad \textit{for a.a. } s \in [0,t]$$

*then*

$$\left\langle \int_0^t z(s) ds, n\right\rangle \geq c\left|\int_0^t z(s)\, ds\right|.$$

**Lemma 1.24.** *Suppose that* $r, c > 0$ *and* $|n| = 1$*. If* $\langle z, n\rangle \geq c|z|$ *and* $0 < |z| < 2rc$ *then* $|z - rn| < r$*.*

The proofs of Lemmas are obvious. Geometrically, Lemma 1.24 means that the intersection of the revolving cone with the axis $n$ and the angel at the vertex $\alpha < \Pi/2$ with a sufficiently small ball centered at the origin is contained in the ball centered at $rn$ with the radius $r$.

*Proof of Lemma* 1.21. The proof base on the Viability Theorem for locally compact sets (see Theorem 2.9 and Remark 2.11(b)). Since $(p, 0)$ is a proximal normal to the epigraph $\mathrm{Epi}\,(g)$ at the point $(x_0, g(x_0))$ then there exists $r > 0$ such that

$$B((x_0 + rn, g(x_0)), r) \cap \mathrm{Epi}\,(g) = \emptyset$$

where $n = p/|p|$. It follows that

$$g(x) > g(x_0) \quad \text{for } |x - (x_0 + rn)| < r. \tag{1.8}$$

Suppose to the contrary that there exists $\varepsilon > 0$ such that if $|x - x_0| < \varepsilon$ and $(p_x, v_x) \in PN_{\mathrm{Epi}\,(g)}(x, g(x))$ then either $v_x = 0$ or

$$p_x = 0 \quad \text{or} \quad \left| \frac{p_x}{|p_x|} - n \right|^2 \geq 2\varepsilon. \tag{1.9}$$

By (1.9), we have

$$\langle p_x, n\rangle - (1 - \varepsilon)|p_x| \leq 0.$$

Thus

$$\inf_{|b|\leq 1} \langle p_x, n + (1 - \varepsilon)b\rangle \leq 0. \tag{1.10}$$

We set

$$F := \{u(n + (1 - \varepsilon)b, 0) + (1 - \varepsilon)(0, -1) : u \in [0, 1],\ |b| \leq 1\}.$$

$F$ is a compact convex set. We define

$$K := \{(x, v) : |x - x_0| < \varepsilon,\ v \geq g(x)\}.$$

$K$ is a locally compact set. ($K$ is the intersection of the epigraph of $g$ (closed) and the open set $B(x_0, \varepsilon) \times \mathbb{R}$). We claim that $K$, $F$ satisfies (2.4). Let $(p_x, p_v) \in PN_K(x, v)$. Then $(p_x, p_v) \in PN_{\mathrm{Epi}\,(g)}(x, g(x))$. If $p_v = 0$ then for $f = (0, -1) \in F$ we have

$$\langle (p_x, 0), f\rangle \leq 0.$$

If $p_v < 0$ then by (1.10) there exists $|b| \leq 1$ such that for $f = (n+(1-\varepsilon)b, 0) \in F$ (we take $u = 1$) we have

$$\langle (p_x, p_v), f\rangle \leq 0.$$

By the viability Theorem 2.9, there exists $t_0 > 0$ and a solution $(x, v)\colon [0, t_0] \to \mathbb{R}^{d+1}$ of the Cauchy problem

$$\begin{cases} (x', v') \in F, \\ (x(0), v(0)) = (x_0, g(x_0)), \end{cases}$$

such that $(x(t), v(t)) \in K$ for $t \in [0, t_0]$. There exist measurable functions $u\colon [0, t_0] \to [0, 1]$ and $b\colon [0, t_0] \to D$ ($D$ denotes the unit disk in $\mathbb{R}^d$) such that

$$(x'(t), v'(t)) = u(t)(n + (1 - \varepsilon)b(t), 0) + (1 - u(t))(0, -1).$$

If $u(s) = 0$ for almost all $t \in [0,t]$ then $x(t) \equiv x_0$ and $v(t) = g(x_0) - t$. Thus $(x(t), v(t)) \notin K$ and we have obtained a contradiction.

Now, consider the case when the Lebesgue measure of the set $\{s \in [0,t] : u(s) > 0\}$ is positive. By Lemma 1.22, we have

$$\langle n + (1-\varepsilon)b(s), n\rangle \geq c|n + (1-\varepsilon)b(s)|,$$

where $c = \sqrt{1-(1-\varepsilon)^2} > 0$.

By Lemma 1.23, we obtain

$$\biggl\langle \int_0^t u(s)(n + (1-\varepsilon)b(s))ds, n \biggr\rangle \geq c\biggl|\int_0^t u(s)(n + (1-\varepsilon)b(s))\,ds\biggr|.$$

If $t < rc$, then $|\int_0^t u(s)(n + (1-\varepsilon)b(s))\,ds| < 2rc$ and by Lemma 1.24 we obtain

$$\biggl|x_0 + \int_0^t u(s)(n + (1-\varepsilon)b(s))\,ds - (x_0 + rn)\biggr| < r.$$

Thus $x(t) - (x_0 + rn)| < r$. But $v(t) = g(x_0) + \int_0^t (1 - u(s)(-1)\,ds \leq g(x_0)$ and we have obtained a contradiction with (1.8). $\square$

## 2. Differential inclusions

This section is just a refreshed and shortened version of some parts of the books "Differential inclusions" by Aubin and Cellina and "Viability theory" by Aubin. We provide some simplification in the proof of the Filippov Theorem. We formulate the Convergence Theorem (Theorem 1.4.1 in [1]) in a different way. We consider the Cauchy problem for differential inclusions with an u.s.c. right hand side with compact convex values. The proof of the existence of a solution to the Cauchy problem makes extensive use of an approximation theorem of u.s.c. mappings. The invariance and viability problems we study first for Lipschitz continuous right hand side. Next we use an approximation theorem of u.s.c. mappings to extend viability result to differential inclusions with an u.s.c. right hand side.

Let $F\colon [a,b] \times \mathbb{R}^d \multimap \mathbb{R}^d$ be a set valued map with nonempty compact values. We shall consider the differential inclusion

$$x'(t) \in F(t,x). \tag{2.1}$$

A funtion $x\colon [a,b] \to \mathbb{R}^d$ is absolutely continuous if there exists a Lebesque integrable function $y\colon [a,b] \to \mathbb{R}^d$ such that for every $t \in [a,b]$ we have $x(t) = x(a) + \int_a^t y(s)\,ds$. An absolutely continuous function $x$ is almost everywhere differentiable and $x'(t) = y(t)$ for almost all $t \in [a,b]$. More interesting facts concerning absolutely continuous functions can be find in [14].

An absolutely continuous function $x\colon [a,b] \to \mathbb{R}^d$ is a solution to (2.1) if $x'(t) \in F(t, x(t))$ for almost all $t \in [a,b]$. We shall consider the Cauchy problem of the existence of a solution to the differential inclusion (2.1) that satisfies a given initial condition

$$x(t_0) = x_0. \tag{2.2}$$

We start with the Fillipov Theorem concerning the existence of solution to the Cauchy problem for the Lipschitz continuous right hand side . Next we show that if the right hand side $F(t,x)$ is a convex valued map then the set of solutions to the Cauchy problem is closed. Moreower we study the invariance and the viability problem for differential inclusions. Let $K \subset \mathbb{R}^d$. The Invariance Theorem states that if the right hand side $F\colon \mathbb{R}^d \multimap \mathbb{R}^d$ is Lipschitz continuous and satisfies a strong boundary condition

$$\forall x \in K\, \forall n \in PN_K(x),\ \forall f \in F(x), \quad \langle f, n\rangle \le 0 \tag{2.3}$$

then any solution to (2.1) starting at the time $t_0$ from a point $x_0 \in K$ remains in the set $K$, i.e. $x(t) \in K$ for $t > t_0$. If the right hand side is upper semicontinuous and satisfies a weak boundary condition

$$\forall x \in K,\ \forall n \in PN_K(x),\ \exists f \in F(x), \quad \langle f, n\rangle \le 0 \tag{2.4}$$

then for any $x_0 \in K$ there exist a solution to (2.1) satisfying the initial condition(2.2) such that $x(t) \in K$ for $t > t_0$. This is the Viability Theorem.

### 2.1. The Fillippov Theorem.

**Theorem 2.1.** *Suppose that a set valued map $F\colon [0,T] \times \mathbb{R}^d \multimap \mathbb{R}^d$ satisfies:*

(a) *$t \multimap F(t,x)$ is mesurable for every $x \in \mathbb{R}^d$,*
(b) *$d_H(F(t,x), F(t,y)) \le l(t)|x-y|$ for $x, y \in \mathbb{R}^d$,*
(c) *$F(t,x)$ is a closed nonempty set for every $(t,x)$,*
(d) *$|F(t,x)| \le \mu(t)$,*

*where the functions $l, \mu\colon [0,T] \to \mathbb{R}$ are integrable ($l, \mu \in L^1$). Assume that $y\colon [0,T] \to \mathbb{R}^d$ is an absolutely continuous function such that*

$$\operatorname{dist}(y'(t), F(t, y(t)) \le c(t)$$

*where $c \in L^1$. If $x_o \in \mathbb{R}^d$ then there exists a solution to*

$$\begin{cases} x'(t) \in F(t, x(t)), \\ x(0) = x_0, \end{cases} \tag{2.5}$$

*that satisfies*

$$|x(t) - y(t) \le |x_0 - y(0)| e^{\int_0^t l(s)\, ds} + \int_0^t c(s) e^{\int_s^t l(\tau) d\tau}\, ds.$$

**Lemma 2.2.** *If $l, c\colon [a,b] \to \mathbb{R}$ are integrable functions then*

$$\int_a^b l(x_1)\bigg(\int_a^{x_1} l(x_2)\dots\bigg(\int_a^{x_{n-1}} l(x_n)\,dx_n\bigg)\dots dx_2\bigg)\,dx_1 = \frac{1}{n!}\bigg(\int_a^b l(s)\,ds\bigg)^n \tag{2.6}$$

*and*

$$\int_0^t l(x_1)\bigg(\int_0^{x_1} l(x_2)\dots \bigg(\int_0^{x_{n-1}} l(x_n)\bigg(\int_0^{x_n} c(x_{n+1}\,dx_{n+1}\bigg)\,dx_n\bigg)\dots dx_2\bigg)\,dx_1 = \int_0^t c(s)\frac{1}{n!}\bigg(\int_s^t l(\tau)\,d\tau\bigg)^n\,ds. \tag{2.7}$$

*Proof.* We start with the proof of (2.6).

*Method 1.* We denote a simplex $\Delta := \{(x_1,\dots,x_n) : a < x_n < \dots < x_2 < x_1 < b\}$. The Lebesque measure $|\Delta|$ of the simplex $\Delta$ equals

$$|\Delta| = \frac{1}{n!} = \int_\Delta 1.$$

Let $\sigma \in S_n$ be a permutation of the set $\{1,\dots,n\}$. We denote

$$\Delta_\sigma := \{(x_{\sigma(1)},\dots,x_{\sigma(n)}) : (x_1,\dots,x_n) \in \Delta\}.$$

If $\sigma$, $\delta$ are permutations from $S_n$ and $\sigma \neq \delta$ then $\Delta_\delta \cap \Delta_\sigma = \emptyset$ and $|\Delta_\sigma| = |\Delta_\delta|$. We have

$$\int_a^b l(x_1)\bigg(\int_a^{x_1} l(x_2)\dots\bigg(\int_a^{x_{n-1}} l(x_n)\,dx_n\bigg)\dots dx_2\bigg)\,dx_1 = \int_\Delta l(x_1)\dots l(x_n)\,dx_1\dots dx_n.$$

The integrated function is symmetric. Thus

$$\int_{\Delta_\sigma} l(x_1)\dots l(x_n) = \int_\Delta l(x_1)\dots l(x_n).$$

Thus

$$\bigg(\int_a^b l(s)\,ds\bigg)^n = \int_{[a,b]^n} l(x_1)\dots l(x_n) = n!\int_\Delta l(x_1)\dots l(x_n).$$

*Method 2.* We set $L(x) := \int_a^x l(s)\,ds$. The function $L(\,\cdot\,)$ is an absolutely continuous primitive function to the function $l(\,\cdot\,)$. We inductively prove the formula (2.6). Indeed, we have

$$\int_a^b l(x_1)\Big(\int_a^{x_1} l(x_2)\dots\Big(\int_a^{x_n} l(x_{n+1})\,dx_{n+1}\Big)\dots dx_2\Big)\,dx_1$$
$$= \int_a^b l(x_1)\frac{1}{n!}\Big(\int_a^{x_1} l(s)\,ds\Big)^n\,dx_1 = \frac{1}{n!}\frac{1}{n+1}L^{n+1}(x)|_a^b.$$

To obtain (2.7) we use Fubini Theorem and (2.6):

$$\int_0^t l(x_1)\,\Big(\int_0^{x_1} l(x_2)\dots\Big(\int_0^{x_{n-1}} l(x_n)\Big(\int_0^{x_n} c(x_{n+1})\,dx_{n+1}\Big)\,dx_n\Big)\dots dx_2\Big)\,dx_1$$
$$= \int_0^t c(x_{n+1})\Big(\int_{x_{n+1}}^t l(x_1)\int_{x_{n+1}}^{x_1} l(x_2)\dots\int_{x_{n+1}}^{x_{n-1}} l(x_n)\,dx_n\dots dx_1\Big)\,dx_{n+1}$$
$$= \int_0^t c(x_{n+1})\frac{1}{n!}\Big(\int_{x_{n+1}}^t l(s)\,ds\Big)^n\,dx_{n+1}. \qquad \Box$$

*Proof of Theorem* 2.1. We set $v_0(t) = y'(t)$ and $x_0(t) = y(t)$. By the selection theorem (see Theorem 1.4 in [24]) there exists a measurable function $v_1(t) \in F(t, x_0(t))$ such that $|v_1(t) - y'(t)| \le c(t)$. We set

$$x_1(t) = x_0 + \int_0^t v_1(s)\,ds.$$

We inductively construct two sequences $v_k(\,\cdot\,)$, $x_k(\,\cdot\,)$ such that: $v_{i+1}\colon [0,t] \to \mathbb{R}^d$ is a measurable selection $v_{i+1}(t) \in F(t, x_i(t))$ such that

$$|v_{i+1}(t) - v_i(t)| \le d_H(F(t, x_i(t)), F(t, x_{i-1}(t)))$$

and

$$x_{i+1}(t) = x_0 + \int_0^t v_{i+1}(s)\,ds.$$

We have

$$|x_1(t) - x_0(t)| \le |x_0 - y(0)| + \int_0^t c(s)\,ds,$$
$$|x_{i+1}(t) - x_i(t)| \le \int_0^t l(s)|x_i(s) - x_{i-1}(s)|\,ds,$$

for $i \ge 1$. Thus

$$|x_{n+1}(t) - x_n(t)|$$
$$\le \int_0^t l(s_1)\int_0^{s_1} l(s_2)\dots\int_0^{s_{n-1}} l(s_n)\Big(|x_0 - y(0)| + \int_0^{s_n} c(s_{n+1})\,ds_{n+1}\Big)ds_n\dots ds_1.$$

By (2.6) and (2.7) we obtain

$$
\begin{aligned}
(2.8)\qquad |x_{n+1}(t) &- x_n(t)| \\
&\le |x_0 - y(0)| \frac{1}{n!}\bigg(\int_0^t l(s)\,ds\bigg)^n + \int_0^t c(s)\frac{1}{n!}\bigg(\int_s^t l(\tau)\,d\tau\bigg)^n ds \\
&\le \frac{1}{n!}\bigg(\int_0^t l(s)\,ds\bigg)^n \bigg(|x_0 - y(0)| + \int_0^t c(s)\,ds\bigg).
\end{aligned}
$$

We obtained that $|x_{n+1}(t) - x_n(t)| \le c_n$ for $t \in [0,T]$ and $\sum_{n=0}^{\infty} c_n < \infty$. Thus the sequence $x_n(\,\cdot\,)$ satisfies the uniform Cauchy condition on the interval $[0,T]$. So, it is uniformely convergent to a function $x\colon [0,T] \to \mathbb{R}^d$.

We have

$$|v_{n+2}(t) - v_{n+1}(t)| \le l(t)|x_{n+1}(t) - x_n(t)| \le l(t)c_n$$

for almost all $t \in [0,T]$. Thus the sequence $v_n(t)$ is convergent for almost all $t$ to a measurable function $v(t)$. We have

$$
\begin{aligned}
|v_n(t)| \le |v_n(t) - v_{n-1}(t)| + \ldots + |v_2(t) - v_1(t)| + |v_1(t) - v_0(t)| + |v_0(t)| \\
\le l(t)\sum_{n=0}^{\infty} c_n + c(t) + |y'(t)|.
\end{aligned}
$$

By the Lebesgue dominated convergence theorem

$$x_n(t) = x_0 + \int_0^t v_n(s)\,ds \xrightarrow{n\to\infty} x_0 + \int_0^t v(s)\,ds = x(t).$$

So

$$
\begin{array}{ccc}
v_{n+1}(t) & \in F(t, & x_n(t)) \\
\Big\downarrow & & \Big\downarrow \\
v(t) & & x(t)
\end{array}
$$

Since the graph of the set valued map $F(t,\,\cdot\,)$ is closed then $x'(t) = v(t) \in F(t,x(t))$ for almost all $t \in [0,T]$. By (2.8) we obtain

$$
\begin{aligned}
|x(t) - y(t)| \le \sum_{n=1}^{\infty} |x_n(t) - x_{n-1}(t)| \le |x_0 - y(0)| + \int_0^t c(s)\,ds \\
+ \sum_{n=1}^{\infty}\bigg(|x_0 - y(0)|\frac{1}{n!}\bigg(\int_0^t l(s)\,ds\bigg)^n + \int_0^t c(s)\frac{1}{n!}\bigg(\int_s^t l(\tau)\,d\tau\bigg)^n ds\bigg) \\
= |x_0 - y(0)| e^{\int_0^t l(s)\,ds} + \int_0^t c(s) e^{\int_s^t l(\tau)\,d\tau}\,ds.
\end{aligned}
$$

□

**2.2. Closedness of the solution set.** In the section we show that the set of solutions to the Cauchy problem is a closed subset of the space of continuous function with the supremum norm $C([a,b],\mathbb{R}^d)$. Convexity of the right hand sides $F(t,x)$ is an essential assumption to obtain closedness of the solution set.

We start with a convergence lemma

**Lemma 2.3.** *Suppose that $x_n\colon[a,b]\to\mathbb{R}^d$ are absolutely continuous and their derivatives are bounded by an integrable function $c\in L^1(a,b)$, i.e. $x'(t)|\le c(t)$ for $n\in\mathbb{N}$. If the sequence $(x_n)$ converge uniformely to a function $x\colon[a,b]\to\mathbb{R}$*

$$x_n \rightrightarrows x \quad \text{on } [a,b]$$

*then $x(\,\cdot\,)$ is absolutely continuous, $|x'(t)|\le c(t)$ for almost all $t\in[a,b]$ and*

$$x'(t)\in\bigcap_{n=1}^{\infty}\overline{\operatorname{conv}}\,\{x'_m(t): m\ge n\}\quad \text{for almost all } t. \tag{2.9}$$

*Proof.* The sequence $y_n=x'_n/c$ is bounded in $L^\infty$. By the Alaoglu Theorem, there exists a subsequence (denoted again by) $y_n$ that weakly converge in $L^\infty$ to a $y\in L^\infty$. By the Mazur Theorem, the strong and the weak closure of the set $\{y_k : k\ge n\}$ coincide. Thus, there exists a convex combination

$$z_n=a_{0n}y_n+a_{1n}y_{n+1}+\ldots+a_{s_n n}y_{n+s_n}$$

such that $\|z_n-y\|_{L^\infty}<1/n$. Thus the sequence $w_n$

$$w_n:=cz_n=a_0x'_n+\ldots+a_sx'_{n+s}$$

converges in the $L^1$ norm to $cy\in L^1$. There exists a subsequence (denoted again by) $w_n$ that converges to $cy$ almost everywhere. By the Lebesque Dominated Convergence Theorem we have

$$\int_a^t w_n \xrightarrow{n\to\infty} \int_a^t cy.$$

Moreover,

$$\int_a^t w_n = a_{0n}(x_n(t)-x_n(a))+\ldots+a_{k_n n}(x_{k_n+n}(t)-x_{k_n+n}(a)) \xrightarrow{n\to\infty} x(t)-x(a).$$

Thus $x(t)=x(a)+\int_a^t cy$ and $x(\,\cdot\,)$ is an absolutely continuous function satisfying $|x'(t)|\le c(t)$.

Fix $t$ such that $\lim_{n\to\infty}w_n(t)=x'(t)$ and take an arbitrary $k\in\mathbb{N}$. For $n\ge k$ we have

$$w_n(t)\in\operatorname{conv}\{x'_m(t): m\ge k\}.$$

Thus

$$x'(t)\in\overline{\operatorname{conv}}\,\{x'_m(t): m\ge k\}. \qquad\square$$

**Proposition 2.4.** *Suppose that a set valued map $F: [0, T] \times \mathbb{R}^d \multimap \mathbb{R}^d$ satisfies:*

(a) *$F(t, \cdot\,)$ is an u.s.c. map for every $t$,*
(b) *$F(t, x)$ is a compact convex nonempty set for every $(t, x)$,*
(c) *$F$ is integrably bounded, i.e. $|F(t, x)| \le \mu(t)$, $\mu \in L^1$.*

*If $x_n: [a, b] \to \mathbb{R}^d$ are solutions of the differential inclusion* (2.1) *and $x_n \rightrightarrows x$ on $[a, b]$ then $x$ is a solution to* (2.1).

*Proof.* We choose $t$ such that (2.9) holds true and $x_n'(t) \in F(t, x_n(t))$ for every $n$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$F(t, y) \subset F(t, x(t)) + \varepsilon B \quad \text{for } |y - x(t)| < \delta.$$

For sufficiently large $m$ $(m > n)$

$$x_m'(t) \in F(t, x_m(t)) \subset F(t, x(t)) + \varepsilon B.$$

Thus

$$x'(t) \in \overline{\operatorname{conv}}\,\{x_m'(t) : m \ge n\} \subset F(t, x(t)) + \varepsilon B.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $x'(t) \in F(t, x(t))$. □

If $F(t, x)$ is integrably bounded then, by the Arzeli–Ascoli Theorem, the set of solutions to the Cauchy problem (2.5) is precompact in the space of continuous functions $C([a, b], R^d)$. By Theorems 2.1 and 2.4 we obtain the following

**Corollary 2.5.** *If $F: \mathbb{R}^d \multimap \mathbb{R}^d$ is a bounded Lipschitz continuous map with compact convex nonempty values then the set of solutions to the Cauchy problem*

$$\begin{cases} x' \in F(x), \\ x(0) = x_0, \end{cases} \tag{2.10}$$

*is compact in $C([0, T], \mathbb{R}^d)$.*

**Theorem 2.6.** *If $F: \mathbb{R}^d \multimap \mathbb{R}^d$ is a bounded u.s.c. map with compact convex nonempty values then the set of solutions to the Cauchy problem* (2.10) *is compact in $C([0, T], \mathbb{R}^d)$.*

*Proof.* By the Cellina Theorem (see Theorem 2.3.11 in [16]) we find a sequence of set valued maps $F_n: \mathbb{R}^d \multimap \mathbb{R}^d$ such that

- $F_n$ satisfies assumptions of Corollary 2.5,
- $F(x) \subset F_{n+1}(x) \subset F_n(x)$ for $x \in \mathbb{R}^d$,
- $F(x) = \bigcap_{n=1}^{\infty} F_n(x)$ for $x \in \mathbb{R}^d$.

Then the set $S$ of solutions of the Cauchy problem (2.10) is the intersection of the set $S_n$, where $S_n$ is the set of solutions to (2.10) with the right hand side $F_n$. The sequence $S_n$ is a decreasing sequence of compact sets. So, $S$ is a nonempty compact set. □

**Proposition 2.7.** *Suppose that $F\colon \mathbb{R}^d \multimap \mathbb{R}^d$ is a bounded Lipschitz continuous map with nonempty closed values. If $v \in F(x_0)$ then the problem*

$$\begin{cases} x' \in F(x), \\ x(0) = x_0, \\ x'(0) = v, \end{cases}$$

*has an absolutely continuous solutions that is differentiable at $t = 0$.*

*Proof.* We set $y(t) = x_0 + tv$. By Theorem 2.1, there exists a solution $x$ to (2.10) such that

$$|x(t) - y(t)| \leq \int_0^t 2ce^{(t-s)l}\, ds$$

where $l$ is the Lipschitz constant for $F$ and $c$ is an upper bound of $F$.Thus

$$\frac{|x(t) - x_0 - tv|}{t} \leq 2c\left(\frac{1}{t}\int_0^t se^{(t-s)l}\, ds\right) \xrightarrow{t\to 0^+} 0$$

which follows that the right derivative $x'(0) = v$. □

## 2.3. Invariance and viability.

**Theorem 2.8** (Invariance Theorem)**.** *Suppose that $K \subset \mathbb{R}^d$ is closed and $F\colon \mathbb{R}^d \multimap \mathbb{R}^d$ is a nonempty compact valued Lipschitz continuous map. If the strong boundary condition* (2.3) *holds true and $x\colon [0,T] \to \mathbb{R}^d$ is a solution to the differential inclusion $x' \in F(x)$ and $x(0) \in K$ then $x(t) \in K$ for $t \in [0,T]$.*

*Proof.* We set $g(t) := \operatorname{dist}(x(t), K)$. The function $g$ is the composition of the absolutely continuous function $x(\,\cdot\,)$ and a Lipschitz continuous function $\operatorname{dist}(\,\cdot\,, K)$. So, it is absolutely continuous. Suppose that $x(t) \notin K$. We set $t_0 = \inf\{s < t : x(\tau) \notin K \text{ for all } \tau \in (s,t)\}$. We have $g(t_0) = 0$ and $g(\tau) > 0$ for $\tau \in (t_0, t]$. Fix $\tau \in (t_0, t)$ such that the derivatives $g'(\tau)$ and $x'(\tau)$ exists and choose an $y \in \Pi_K(x(\tau))$. By Lemma 1.11, we have

$$g'(\tau) \leq \left\langle x'(\tau), \frac{n}{|n|} \right\rangle$$

where $n := x(\tau) - y$. There exists $f \in F(y)$ such that $|f - x'(\tau)| \leq lg(\tau)$, where $l$ is the Lipschitz constant of the set valued map $F$. Thus

$$g'(\tau) \leq \left\langle x'(\tau) - f, \frac{n}{|n|} \right\rangle + \left\langle f, \frac{n}{|n|} \right\rangle \leq lg(\tau).$$

By the Gronwall Lemma we obtain that $g \equiv 0$. □

**Theorem 2.9** (Viability Theorem). *Suppose that $K \subset \mathbb{R}^d$ is closed and $F:\mathbb{R}^d \multimap \mathbb{R}^d$ is a nonempty compact convex valued Lipschitz continuous map with a constat $l$. If the weak boundary condition* (2.4) *holds true and $x_0 \in K$ then there exists a solution $x:[0,T] \to \mathbb{R}^d$ to the differential inclusion $x' \in F(x)$ such that $x(t) \in K$ for $t \in [0,T]$.*

*Proof.* The set $S$ of solutions of Cauchy Problem (2.10) is compact in the space $C([0,T],\mathbb{R}^d)$. Since $F(x_0)$ is compact then there is $C > 0$ such that $|f| \le C$ for $f \in F(x_0)$. If $x \in S$ then $x(t) - x(0)| \le \int_0^t l|x(s) - x(0)|ds$. By Gronwall Lemma, we have $|x(t) - x_0| \le Cte^{lt}$. Thus we obtain $|x'(t) \le C + lCTe^{lT} := L$. Thus $S$ is a family of Lipschitz continuous functions with a common Lipschitz constant $L$.

We define $g(t) = \inf\{\operatorname{dist}(x(t),K) : x \in S\}$. The function $g$ is Lipschtz continuous with the constant $L$. We choose $t$ such that the derivative $g'(t)$ exists. Suppose that $g(t) > 0$. Since $S$ is compact then there exists $\overline{x} \in S$ such that $\operatorname{dist}(\overline{x}(t) = g(t)$. We choose $y \in \Pi_K(\overline{x}(t))$. By (2.4), there exists $f \in F(y)$ such that $\langle f, \overline{x}(t) - y\rangle \le 0$. Since $F$ is $l$-Lipschitz continuous then there exists $f_1 \in F(\overline{x}(t)$ such that $|f_1 - f| \le lg(t)$. By Proposition 2.7, there exists $\widetilde{x}:[t,T] \to \mathbb{R}^d$ that is a solution to

$$\begin{cases} x' \in F(x), \\ x(t) = \overline{x}_0, \\ x'(t) = f_1. \end{cases}$$

By Lemma 1.11

$$g'(t) \le \left\langle f_1, \frac{n}{|n|} \right\rangle \le lg(t)$$

where $n := \overline{x}(t) - y$. By Gronwall Lemma we obtain that $g \equiv 0$.

The above consideration imply that for every $t_1 < t_2$ and $x_1 \in K$ there exists a solution to $x' \in F(x)$ satisfying the initial condition $x(t_1) = x_1$ such that $x(t_2) \in K$. Thus, for every $n$ we can construct $x_n \in S$ such that $x_n(kT/2^n \in K$ for $k = 1,\dots,2^n$. Since $S$ is compact there exists a subsequence (denoted again by) $x_n$ that converge uniformely to $x \in S$. Obviousily, the obtained $x$ is a desired viable solution, i.e. $x(t) \in K$ for $t \in [o,T]$. $\square$

Using the approximation theorem of u.s.c. maps we can easy generalize the Viability Theorem. We repeat the same arguments as in the proof of Theorem 2.6.

**Corollary 2.10.** *Suppose that $K \subset \mathbb{R}^d$ is closed and $F:\mathbb{R}^d \multimap \mathbb{R}^d$ is a nonempty compact convex valued upper semicontinuous bounded map. If the weak boundary condition* (2.4) *holds true and $x_0 \in K$ then there exists a solution $x:[0,T] \to \mathbb{R}^d$ to the differential inclusion $x' \in F(x)$ such that $x(t) \in K$ for $t \in [0,T]$.*

*Proof.* By the Antosiewicz–Cellina Theorem we find a sequence of set valued maps $F_n: \mathbb{R}^d \multimap \mathbb{R}^d$ such that

- $F_n$ satisfies assumptions of Theorem 2.9,
- $F(x) \subset F_{n+1}(x) \subset F_n(x)$ for $x \in \mathbb{R}^d$,
- $F(x) = \bigcap_{n=1}^{\infty} F_n(x)$ for $x \in \mathbb{R}^d$.

Every $F_n$ satisfies the weak boundary condition (2.4). By Theorem 2.9, the set $S_n(K)$ of $K$-viable solutions to the differential inclusion $x' \in F_n(x)$ satisfying the initial condition $x(0) = x_0$ is nonempty. The set $S_n(K)$ is compact as a closed subset of the compact set $S_n$. The set $S(K)$ of $K$-invariant solutions to the Cauchy problem (2.10) is the intersection of decreasing sequence of sets $S_n(K)$. So, $S(K)$ is a nonempty compact set. □

**Remark 2.11.** (a) The boundary conditions (2.4) and (2.3) in many papers appear in a stronger form using the Bouligand tangent cone (or its convexification):

$$F(x) \cap T_K(x) \neq \emptyset, \quad F(x) \subset T_K(x).$$

The possibility of use of proximal normals in the boundary conditions was observed by Cardaliaguet and appear in [4] in the framework of differential games.

(b) Theorems 2.9 and 2.8 remain true if we replace the assumption that $K$ is closed by the assumption that $K$ is locally compact, i.e. for every $x \in K$ there exists $r > 0$ such that the intersection of $K$ with the closed ball centered at $x$ with the radius $r$ is closed. If $K$ is locally compact then the viable (invariant) solution $x(\cdot)$ satisfies $x(t) \in K$ for $t \in [0, \varepsilon]$, where $\varepsilon > 0$ depends to the initial condition $x(0) = x_0$.

Now we consider the viability and invariance problem in the case when the set of state constraints $K$ varies in time. Let $P: [0, T] \multimap \mathbb{R}^d$ be a set valued map. We say that $P$ is a tube. The invariance or/and viability problem can be considered forward in time

$$\begin{cases} x' \in F(t, x), \\ x(t_0) = x_0, \\ x(t) \in P(t) \quad \text{for } t \in [t_0, T), \end{cases} \tag{2.11}$$

or backward in time

$$\begin{cases} x' \in F(t, x), \\ x(t_0) = x_0, \\ x(t) \in P(t) \quad \text{for } t \in (0, t_0], \end{cases}$$

for an initial condition $x_0 \in P(t_0)$, $t_0 \in (0, T)$.

**Theorem 2.12** (Backward Invariance Theorem)**.** *Suppose that* $F: [0, T] \times \mathbb{R}^d \multimap \mathbb{R}^d$ *is a nonempty compact valued Lipschitz continuous map and* $P: [0, T] \multimap$

$\mathbb{R}^d$ *is a tube with nonempty values and a closed graph. If the strong boundary condition*

$$\forall t \in (0,T),\ \forall x \in P(t),\ \forall (n_t,n_x) \in PN_{\mathrm{Graph}(P)}(t,x),\ \forall f \in F(x),\qquad \langle(-1,-f),(n_t,n_x)\rangle \leq 0$$

*holds true and* $x\colon [0,t_0] \to \mathbb{R}^d$ *is a solution to the differential inclusion* $x' \in F(x)$ *and* $x(t_0) \in P(t_0)$ *then* $x(t) \in P(t)$ *for* $t \in (0,t_0]$.

To obtain the forward invariance result we have to assume that

$$\langle(1,f),(n_t,n_x)\rangle \leq 0$$

holds true. To prove Theorem 2.12 we reduce the nonautonomous case to the autonomous one by treating the variable $t$ as a state variable. We set $K = \{(t,x) : t \in (0,T),\ x \in P(t)\}$. The set $K$ is locally compact (see Remark 2.11). Extending maximally backward in time a solution $x(\,\cdot\,)$ we easy see that it remains in the tube $P$ on the whole interval $(0,t_0)$. It can leave the tube at the earliest at the time 0.

Using similar arguments we obtain from the autonomous viability Theorem 2.9. the nonautonomous one for tubes.

**Theorem 2.13** (Forward Viability Theorem)**.** *Suppose that* $F\colon [0,T]\times\mathbb{R}^d \multimap \mathbb{R}^d$ *is a nonempty compact convex valued Lipschitz continuous map and* $P\colon [0,T] \multimap \mathbb{R}^d$ *is a tube with nonempty values and a closed graph. If the weak boundary condition*

$$\forall t \in (0,T),\ \forall x \in P(t),\ \forall (n_t,n_x) \in PN_{\mathrm{Graph}(P)}(t,x),\ \exists f \in F(x)\qquad \langle(1,f),(n_t,n_x)\rangle \leq 0$$

*holds true and* $x_0 \in P(t_))$, $t_0 \in (0,T)$ *then there exist a solution to* (2.11).

## 3. Hamilton–Jacobi equations

**3.1. The value function in the Mayer problem.** Viability approach to the problem of the description of a discontinuous value function was initiated by Frankowska in [11]. This method is based on the fact that the value function is uniquely determined by invariance properties of its epigraph with respect to an appropriate dynamical system. In the Mayer problem for control systems the epigraph of value function is forward (in time) viable and backward invariant. These two properties of the epigraph and a terminal condition uniquely characterize the value function. Viability theory provides geometric conditions which are equivalent to viability or invariance properties. These conditions can be expressed with contingent cones or with normal cones.

Denote by $S_F(t_0, x_0)$ the set of solutions to the Cauchy problem

$$\begin{cases} x'(t) \in F(t, x(t)), \\ x(t_0) = x_0, \end{cases}$$

that are defined on the interval $[t_0, T]$. Let $g\colon \mathbb{R}^d \to \mathbb{R}$ be a function. We consider the following Mayer problem

$$\text{minimize } g(x(T)) \text{ over } x \in S_F(t_0, x_0).$$

The value function in the Mayer problem is a function $V\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$V(t_0, x_0) = \inf_{x \in S_F(t_0, x_0)} g(x(T)). \tag{3.1}$$

A solution $\overline{x} \in S_F(t_0, x_0)$ is optimal if

$$g(\overline{x}(T)) = V(t_0, x_0).$$

The set valued map $R\colon [0,T] \times \mathbb{R}^d \multimap \mathbb{R}^d$ is given by

$$R(t_0, x_0) := \{x(T) : x \in S_F(t_0, x_0)\}.$$

The regularity of the value function $V$ is inherited from the regularity of $F$ and $g$.

**Proposition 3.1.** *Suppose that $F(\,\cdot\,,\,\cdot\,)$ is a convex valued map satisfing conditions* (a)–(d) *of Theorem* 2.1. *If $g$ is lower semicontinuous* (*continuous, Lipschitz continuous*) *then $V$ is lower semicontinuous* (*continuous or Lipschitz continuous, respectively*).

By Theorems 2.1 and 2.4, the set valued map $R$ is Lipschitz continuous and has nonempty compact values. We have

$$V(t_0, x_0) = \inf\{g(x) : x \in R(t_0, x_0)\}.$$

The conclusion of Proposition 3.1 can be easy obtained by standard methods.

**Example 3.2.** Suppose that $d = 1$, $F(t,x) = [-1,1]$ and $g(x) = -x^2$. By easy calculation we obtain

$$V(t,x) = \begin{cases} -(x + (T-t))^2 & \text{if } x \geq 0, \\ -(x - (T-t))^2 & \text{if } x < 0. \end{cases}$$

The value function $V$ is not differentiable at points $(t, 0)$.

If the value function is differentiable then it is a classical solution to the following first order partial differential equation

$$\frac{\partial V}{\partial t} + H\left(t, x, \frac{\partial V}{\partial x}\right) = 0 \tag{3.2}$$

where

$$H(t,x,p) = \sup_{f\in F(t,x)} \langle f,p\rangle.$$

Usually the value function is not differentiable as we see in the above Example. The natural problem that appear is, "How define a solution to (3.2)". A "good" definition should imply that a solution $W(t,x)$ to (3.2) satisfying an additional terminal condition

$$W(T,\,\cdot\,) = g(\,\cdot\,)$$

is unique and equels to the value function. Lions and Souganidis introduced in [8] the notion of viscosity solutions. This notion became very usefull for a large class of first and secod order PDE. In particular, the value function $V$ in the Mayer problem is a viscosity solution to the Hamilton–Jacobi–Bellman equation (3.2) if $g$ is a continuous function. We consider the case when $g$ is only lower semicontinuous. We follow the viability approach that was initiated by Frankowska [11].

**Theorem 3.3.** *Suppose that $F(\,\cdot\,,\,\cdot\,)$ is a convex valued map satisfying conditions* (a)–(d) *of Theorem* 2.1 *and $g\colon\mathbb{R}^d\to\mathbb{R}$ is a lower semicontinuous function. If $W\colon(0,T]\times\mathbb{R}^d\to\mathbb{R}$ is a lower semicontinuous function such that*

$$\forall (t,x)\in(0,T)\times\mathbb{R}^d,\ \forall (p_t,p_x)\in\partial_- W(t,x),\quad -p_t + H(t,x,-p_x) = 0$$

*and*

$$\liminf_{t\to T^-,\,y\to x} W(t,y) = W(T,x) = g(x)\quad \textit{for all } x\in\mathbb{R}^d$$

*then we have $W\equiv V$ when $V$ is the value function given by* (3.1).

*Proof.* The proof of Theorem 3.3 base on the Viability Theorem and the Invariance Theorem. First we have to deduce from the assumption (3.3) the weak and strong boundary condition for the tube $P_W\colon[0,T]\multimap\mathbb{R}^d$ given by

$$P_W(t) = \operatorname{Epi} W(t,\,\cdot\,)$$

and for the differential inclusion $(x',v')\in\widetilde{F}(t,x,v)$, when

$$\widetilde{F}(t,x,v) = F(t,x)\times\{0\}.$$

Let $(n_t,n_x,n_v)\in PN_{\operatorname{Epi}(W)}(t,x,v)$, where $v\ge W(t,x)$. Then $(n_t,n_x,n_v)\in PN_{\operatorname{Epi}(W)}(t,x,W(t,x))$. We claim that

$$-n_t + H(t,x,-n_x) = 0. \tag{3.5}$$

First consider the case $n_v<0$. By Proposition 1.17, we have $(n_t/-n_v, n_x/-n_v)\in\partial W(t,x)$. By (3.3),

$$-\frac{n_t}{-n_v} + H\left(t,x,-\frac{n_x}{-n_v}\right) = 0.$$

The hamiltonian $H$ is positively homogenous with respect to the last variable and it follows (3.5).

Now, consider the case $n_v = 0$. By Lemma 1.21, there exists sequences $(t_k, x_k) \to (t, x)$ and $(n_{t,k}, n_{x,k}, n_{v,k}) \in PN_{\mathrm{Epi}\,(W)}(t_k, x_k, W(t_k, x_k))$ such that $n_{v,k} < 0$ and $(n_{t,k}, n_{x,k}, n_{v,k}) \to (n_t, n_x, 0)$. As above we conclude that

$$-n_{t,k} + H(t_k, x_h, -n_{x,k}) = 0.$$

The hamiltonian $H$ is continuous and it follows (3.5).

From (3.5), in particular, we obtain

$$-n_t + \sup_{f \in F(t,x)} \langle f, -n_x \rangle \geq 0.$$

The set $F(t, x)$ is compact. So there exists $f \in F(t, x)$ such that

$$\langle (n_t, n_x, n_v), (1, f, 0) \rangle \leq 0.$$

Thus we have obtained that the epitube $P_W$ is forward viable for the differential inclusion with the right hand side $\widetilde{F}$. By Theorem 2.13, there exists a solution $(\overline{x}, \overline{v})$ to the Cauchy problem

$$\begin{cases} (x', v') \in F(t, x) \times \{0\}, \\ (x(t_0), v(t_0)) = (x_0, W(t_0, x_0)), \end{cases}$$

such that $(\overline{x}(t), \overline{v}(t)) \in P_W(t)$ for $t \in [t_0, T)$. We have $\overline{v}(t) \equiv W(t_0, x_0)$ and $\overline{v}(t) \geq W(t, x(t))$ for $t \in (t_0, T)$. By the assumption (3.4) and the definition (3.1), we obtain

$$\begin{aligned} V(t_0, x_0) &\leq g(\overline{x}(T)) = \liminf_{t \to T^-,\, x \to \overline{x}(T)} W(t, x) \\ &\leq \lim_{t \to T^-} W(t, \overline{x}(t)) \leq \lim_{t \to T^-} \overline{v}(t) = W(t_0, x_0). \end{aligned}$$

Since the reachable set $R(t_0, x_0)$ is compact and the function $g$ is lower semicontinuous then there exists an optimal solution $\overline{x} \in S_F(t_0, x_0)$ to the Mayer problem, i.e.

$$V(t_0, x_0) = g(\overline{x}(T)).$$

By (3.4), there exists sequences $x_n \to \overline{x}(T)$ and $t_n \to T^-$ such that

$$\lim_{n \to \infty} W(t_n, x_n) = g(\overline{x}(T).$$

From (3.5), we conclude that for every $(t, x) \in (0, T) \times \mathbb{R}^d$ and every $(n_t, n_x, n_v) \in PN_{\mathrm{Epi}\,(W)}(t, x, W(t, x))$ we have

$$\langle (n_t, n_x, n_v), (-1, -f, 0) \rangle \leq 0 \quad \text{for all } f \in F(t, x).$$

Thus the tube $P_W$ is backward invariant for $\widetilde{F}$. By Theorem 2.12, every solution to the Cauchy problem

$$\begin{cases} (x', v') \in \widetilde{F}(t, x, v), \\ (x(t_n), v(t_n)) = (x_n, W(t_n, x_n)), \end{cases}$$

satisfies $(x(t), v(t)) \in P_W(t)$ for $t \in [t_0, t_n]$. By Theorem 2.1, there exists a solution $x_n\colon [t_0, t_n] \to \mathbb{R}^d$ to

$$\begin{cases} x' \in F(t, x), \\ x(t_n) = x_n, \end{cases}$$

such that

$$|x_n(t) - \overline{x}(t)| \le |x_n - \overline{x}(t_n)| e^{L(t_n - t)}.$$

The function $(x_n, v)$ is a solution to (3.6), where $v \equiv W(t_n, x_n)$.

Since $(x_n(t_0), v(t_0)) \in P_W(t_0)$, then $W(t_n, x_n) \ge W(t_0, x_0)$. The function $W$ is lower semicontinuous. So

$$W(t_0, x_0) \le \liminf_{n\to\infty} W(t_0, x_n(t_0)) \le \lim_{n\to\infty} W(t_n, x_n) = g(\overline{x}(T)) = V(t_0, x_0).$$

Thus we obtained $W = V$. $\square$

Some generalizations of results presented in the section can be find in [12], [13], [20], [21] and in [18].

**3.2. Value functions in differential games.** In this section we only sketch one possible generalization of the results that has been presented in the previous section. We skip all proofs. We consider zero-sum differential games with dynamics given by $x'(t) = f(t, x(t), y, z)$. By $x(\,\cdot\,; t_0, x_0, y(\,\cdot\,), z(\,\cdot\,))$ we denote the solution of the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t), y(t), z(t)) \quad \text{for a.e. } \in [0, T], \\ x(t_0) = x_0, \end{cases}$$

where $y\colon [0, T] \to Y$, $z\colon [0, T] \to Z$ are measurable controls (open loops) of player I and II, respectively and $Y$, $Z$ are compact metric spaces.

Let $M_t = \{y\colon [t, T] \to Y : y \text{ is measurable}\}$ and $N_t = \{z\colon [t, T] \to Z : z \text{ is measurable}\}$. We say that a map $\alpha\colon N_t \to M_t$ is a nonanticipative strategy of the first player if for every control $z_1$, $z_2 \in N_t$ such that

$$z_1(s) = z_2(s) \quad \text{for almost all } s \in [t, \tau]$$

we have

$$\alpha(z_1)(s) = \alpha(z_2)(s) \quad \text{for almost all } s \in [t, \tau].$$

We say that a map $\beta\colon M_t \to N_t$ is a nonanticipative strategy of the second player if for every control $y_1$, $y_2 \in M_t$ such that

$$y_1(s) = y_2(s) \quad \text{for almost all } s \in [t, \tau]$$

we have

$$\beta(y_1)(s) = \beta(y_2)(s) \quad \text{for almost all } s \in [t, \tau].$$

Let $\Gamma_t$, $\Delta_t$ denote the set of all nonanticipative strategies of the first and of the second player, respectively.

We shall consider a terminal time payoff functional

$$Q(y, z) = Q_{t_0 x_0}(y, z) = g(x(T\ , t_0, x_0, y, z)),$$

where $g\colon R^n \to R$ is a terminal cost function, $y \in M_{t_0}$, $z \in N_{t_0}$. The aim of the first player is to maximize the payoff, the aim of the second player is to minimize it.

The value function of the first player is given by

$$U^+(t_0, x_0) = \sup_{\alpha \in \Gamma_{t_0}} \inf_{z \in N_{t_0}} Q_{t_0 x_0}(\alpha(z), z).$$

The value function of the second player is given by

$$U^-(t_0, x_0) = \inf_{\beta \in \Delta_{t_0}} \sup_{y \in M_{t_0}} Q_{t_0, x_0}(y, \beta(y)).$$

The value of the first player $U^+$ is also called an upper value and $U^-$ is called a lower value. If the upper value is equal to the lower value then we say the game has a value. The main problem in zero sum differential games is the existence of value. It has been considered by many authors. A pioneering work was that of Isaacs [15]. He introduced condition (3.7) which provides the existence of the value in the case where both values are smooth,

$$\max_{y \in Y} \min_{z \in Z} \langle f(t, x, y, z), p \rangle = \min_{z \in Z} \max_{y \in Y} \langle f(t, x, y, z), p \rangle \quad \text{for every } t,\ x \text{ and } p \in \mathbb{R}^n. \tag{3.7}$$

Evans and Souganidis in [10] proved that if $g$ is Lipschitz continuous and $f$ is continuous and Lipschitz continuous with respect to $x$ then the upper value $U^+$ is the viscosity solution of the upper Isaacs equation

$$\begin{cases} U_t + H^+(t, x, U_x) = 0 & \text{for } 0 \le t \le T,\ x \in \mathbb{R}^n, \\ U(T, x) = g(x) & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{3.8}$$

where the upper Hamiltonian $H^+$ is given by

$$H^+(t, x, p) = \min_{z \in Z} \max_{y \in Y} \langle f(t, x, y, z), p \rangle$$

and the lower value $U^-$ is the viscosity solution to the lower Isaacs equation

$$\begin{cases} U_t + H^-(t, x, U_x) = 0 & \text{for } 0 \le t \le T,\ x \in \mathbb{R}^n, \\ U(T, x) = g(x) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{3.9}$$

where the lower Hamiltonian $H^-$ is defined by

$$H^-(t,x,p) = \max_{y\in Y}\min_{z\in Z}\langle f(t,x,y,z),p\rangle.$$

The Isaacs condition (3.7) says that $H^- = H^+$. Thus the upper and the lower Isaacs equations are the same. A direct conclusion from uniqueness of viscosity solutions to (3.8) and to (3.9) is that the value of the game exists.

The proof of the existence of value for a game with dynamics given by a right-hand side $f(t,x,u,v)$ is based on the notions of discriminating and leadership domains. Briefly speaking, we say that a tube $P(t)$ has discriminating property for the first player if for every initial condition at the tube there exists a strategy of the first player such that whatever control is chosen by the second player the corresponding trajectory remains in the tube $P$.

Detailed results concerning value functions for differential can be find in [6] and [19]

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# MEASURABLE AND CONTINUOUS SELECTIONS

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Abstract. We present several selected methods of applying classical selection theorems: Aumann measurable selection theorem, Kuratowski–Ryll-Nardzewski measurable selection theorem and Michael continuous selection theorem (or the ideas from the proofs of these results) in order to get:

- the results on Carathéodory selections (i.e. selections that are measurable in first and continuous in second varaiable),
- simple random fixed point principle,
- retractive representation for fixed point map associated with contractive set valued map,
- continuous selection results for set valued maps which are not lower semicontinuous.

Some applications are outlined.

## 1. Preliminaries

For a relation $F \subseteq X \times Y$, let us denote:

$$
\begin{aligned}
F(x) &= \{y \in Y : (x, y) \in F\},\\
F(W) &= \bigcup_{x \in W} F(x),\\
\operatorname{Dom} F &= \{x \in X : F(x) \neq \emptyset\},\\
F^{-}(V) &= \{x \in X : F(x) \cap V \neq \emptyset\},\\
F^{+}(V) &= \{x \in X : F(x) \subseteq V\}.
\end{aligned}
$$

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A relation $F \subseteq X \times Y$, such that $\operatorname{Dom} F = X$ we call *a multimap* and denote by $F\colon X \multimap Y$. We use symbol $\operatorname{Gr} F$ when we refer to a multimap as a subset of Cartesian product $X \times Y$.

A map $f\colon X \to Y$ such that $f(x) \in F(x)$ for $x \in X$ is called *a selection* of a multimap $F\colon X \multimap Y$.

We presume the Axiom of Choice.

Let $(\Omega, \mathcal{F})$ be a measurable space, $(Y,\tau)$-topological space, by $\mathcal{B}(Y)$ we denote the $\sigma$-algebra of Borel sets in $Y$, and by $\mathcal{F}\otimes\mathcal{B}(Y)$ — the product $\sigma$-algebra in $\Omega\times Y$, i.e. $\sigma$-algebra generated by rectangles $A \times B$, where $A \in \Omega$, $B \in \mathcal{B}(Y)$.

We say that a multimap $F\colon \Omega \multimap Y$:

- is *measurable* if $F^{-}(V) \in \mathcal{F}$ for $V \in \tau$,
- has *measurable graph* if $\operatorname{Gr} F \in \mathcal{F}\otimes\mathcal{B}(Y)$.

A measurable space $(\Omega, \mathcal{F})$ is *complete* if $\mathcal{F} = \widehat{\mathcal{F}}$, where $\widehat{\mathcal{F}} = \bigcap_{\mu\in\mathrm{prob}(\Omega,\mathcal{F})} \mathcal{F}_\mu$, and $\mathcal{F}_\mu$ denotes $\mu$-complement of $\sigma$-algebra $\mathcal{F}$. Let us observe that $\mathcal{F} = \widehat{\mathcal{F}}$, whenever $\mathcal{F} = \mathcal{F}_\mu$ for some $\sigma$-finite measure, hence every complete $\sigma$-finite measure space is a complete measurable space. The following lemma is a final conclusion when one compares various notions of measurability of multimaps. The details can be found e.g. in [30] or [32]. Recall that a topological space $Y$ is called a *Polish space* if $Y$ can be a metrized so that it becomes a separable and complete metric space.

**Lemma 1.1.** *Any measurable multimap $F\colon \Omega \multimap Y$ with closed values has measurable graph. If the measurable space $(\Omega, \mathcal{F})$ is complete and $Y$ is a Polish space, then a multimap $F\colon \Omega \multimap Y$ with measurable graph is measurable.*

Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and $F\colon X \multimap Y$ be a multimap. We say that $F$:

- is *lower semicontinuous* (l.s.c.) if $F^{-}(V) \in \tau_X$ for $V \in \tau_Y$,
- is *upper semicontinuous* (u.s.c.) if $F^{+}(V) \in \tau_X$ for $V \in \tau_Y$,
- has *closed graph* if $\operatorname{Gr} F$ is closed in product topology on $X \times Y$.

**Lemma 1.2.** *An u.s.c. multimap $F\colon X \multimap Y$ with closed values has closed graph. If $F$ has closed graph and every point $x \in X$ has a neighbourhood $U$ such that the set $\overline{F(U)}$ is compact, then $F$ is u.s.c..*

When $(Y,\rho)$ is a metric space, then for $r > 0$, $y \in Y$ and nonempty sets $A, C \subseteq Y$, we denote:

$$\begin{aligned}
B(y,r) &= \{v \in Y : \rho(v,y) < r\},\\
d(y,A) &= \inf_{a\in A} \rho(y,a),\\
B(A,r) &= \{v \in Y : d(v,A) < r\},\\
d(C,A) &= \sup_{c\in C} d(c,A),
\end{aligned}$$

$$D(C,A) = \max\{d(C,A), d(A,C)\} = \sup_{y\in Y} |d(y,C) - d(y,A)|.$$

In this case, a multimap $F\colon X \multimap Y$ is lsc if and only if all real-valued functions $x \mapsto d(v, F(x)), v \in Y$ are upper semicontinuous in classical sense, i.e. for every $v \in Y$, $x \in X$, $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho(x,x') < \delta \Rightarrow d(v, F(x')) < d(v, F(x)) + \varepsilon.$$

We say that $F$ is *$H$-l.s.c.*, if the functions $x \mapsto d(v, F(x)), v \in Y$ are equi-upper semicontinuous, i.e. for every $x \in X$, $\varepsilon > 0$ there exists $\delta > 0$ such that for all $v \in Y$

$$\rho(x,x') < \delta \Rightarrow d(v, F(x')) < d(v, F(x)) + \varepsilon.$$

When the functions $x \mapsto d(v, F(x)), v \in Y$ are lower semicontinuous (resp. equi-lower semicontinuous) in classical sense, we say that a multimap $F$ is *$W$-u.s.c.* (resp.: *$H$-u.s.c.*).

It is easy to verify that if $F$ is u.s.c. then it is $H$-u.s.c., hence also $W$-u.s.c. and if $F$ is $W$-u.s.c. then has closed graph.

We say that a multimap $F$ is *$W$-continuous* if it is l.s.c. and $W$-u.s.c. and we say that $F$ is *$H$-continuous* if it is $H$-l.s.c. and $H$-u.s.c.

Let us recall now two fundamental measurable selection theorems.

**Theorem 1.3** (Aumann–Yankov–von Neumann)**.** *Let $(\Omega, \mathcal{F})$ be a complete measurable space, $Y$ be a Polish space. If a multimap $F\colon \Omega \multimap Y$ has measurable graph, then it has a measurable selection. Moreover, there exist a countable family of measurable selections $f_n\colon \Omega \to Y$ for $F$, such that $F(\omega) \subseteq \overline{\{f_n(\omega) : n \in \mathbb{N}\}}$ for every $\omega \in \Omega$.*

**Theorem 1.4** (Kuratowski–Ryll-Nardzewski [39])**.** *Let $(\Omega, \mathcal{F})$ be a measurable space, $Y$ be a Polish space. If a multimap $F\colon \Omega \multimap Y$ has closed values and is measurable, then it has a measurable selection. Moreover, a multimap $F\colon \Omega \multimap Y$ with closed values is measurable if and only if there exist a countable family of measurable selections $f_n\colon \Omega \to Y$ for $F$, such that $F(\omega) = \overline{\{f_n(\omega) : n \in \mathbb{N}\}}$ for every $\omega \in \Omega$.*

The family of measurable selections $f_n\colon \Omega \to Y$, such that

$$F(\omega) = \overline{\{f_n(\omega) : n \in \mathbb{N}\}}$$

for every $\omega \in \Omega$ is called *Castaing representation* for $F$,

The range of applications of measurable selection theorems is very wide. At the moment, following [30], we recall only an illustrative example of short proof that can be given to famous Filipov Lemma connecting control theory with differential inclusions.

**Theorem 1.5** (Filipow, 1959). *Let $(\Omega, \mathcal{F})$ be a measurable space, $X$ be a metric space, and $Y$ be a separable metric space. If $h\colon \Omega \times X \to Y$ is a Carathéodory map, $F\colon \Omega \multimap Y$ is a measurable multimap with compact values and $g\colon \Omega \to Y$ is a measurable map such that $g(\omega) \in h(\omega, F(\omega))$ for $\omega \in \Omega$, then there exist a measurable selection $f$ for $F$ such that $g(\omega) = h(\omega, f(\omega))$ for $\omega \in \Omega$.*

*Proof.* We define a multimap $H\colon \Omega \multimap X$ by letting

$$H(\omega) = F(\omega) \cap \bigcap_{n=1}^{\infty} \overline{\{x \in X : \rho_Y(h(\omega, x), g(\omega)) < 1/n\}}.$$

One can check that $H$ is measurable, hence by virtue of Kuratowski–Ryll-Nardzewski Theorem, has a measurable selection $f$. This selection has required properties. $\square$

Paracompact (compact) Hausdorff topological space will be called *paracompact* (*compact*) space. We recall now (50 years old) theorem on continuous selections of lower semicontinuous multimaps.

**Theorem 1.6** (Michael, [44], 1956). *Let $X$ be a paracompact space, $Y$ be a Banach space. Lower semicontinuous multimap $F\colon X \multimap Y$ with closed and convex values has a continuous selection. Moreover, if $X$ is a metric space and $Y$ is separable, then a multimap $F\colon X \multimap Y$ with closed and convex values is lsc if and only if there exists a countable family of continuous selections $f_n\colon X \to Y$ of multimap $F$, such that $F(x) = \overline{\{f_n(x) : n \in \mathbb{N}\}}$.*

By applying Michael Theorem, an easy proof of Bartle–Graves result can be given.

**Theorem 1.7** (Bartle–Graves, 1952). *Let $X$ and $Y$ be Banach spaces. If $u\colon Y \to X$ is a continuous and surjective linear operator, then there exists a continuous map $f\colon X \to Y$ such that $u(f(x)) = x$ for $x \in X$ and $f(0) = 0$.*

*Proof.* Consider a multimap $F\colon X \multimap Y$ defined by $F(x) = u^{-1}(x)$. It has closed, convex values and for every set $U \subseteq Y$ we have $F^{-}(U) = \{x \in X : u^{-1}(x) \cap U \neq \emptyset\} = u(U)$. It follows from Banach Open Mapping Theorem, that for open set $U$ also $u(U)$ is an open set. Hence $F$ is l.s.c. By Michael Theorem, $F$ has a continuous selection $\varphi$. By letting $f(x) = \varphi(x) - \varphi(0)$ we define a mapping satisfying the statement of theorem. $\square$

Finally let us mention a different, continuous selection (or approximation) theorem for upper semicontinuous multimaps.

**Theorem 1.8** (Cellina, [17]). *Let $X$ be metric space, $Y$ be a Banach space. If a multimap $F\colon X \multimap Y$ with closed and convex values is u.s.c. then for every $\varepsilon > 0$ there exists continuous* (*locally Lipschitz*) *map $f_\varepsilon\colon X \to Y$ such that* $\operatorname{Gr} f_\varepsilon \subset B(\operatorname{Gr} F; \varepsilon)$.

Basing on that result, a short proof of Kakutani (Bochnenblust–Karlin) Fixed Point Theorem can be given.

**Theorem 1.9** ([17], [5]). *Let $K$ be a compact convex subset of a Banach space. If a multimap $F: K \multimap K$ with closed and convex values is u.s.c. then there exists $x \in K$ such that $x \in F(x)$.*

*Proof.* For $\varepsilon_n = 1/n$ choose continuous maps $f_n: X \to Y$ such that $\operatorname{Gr} f_n \subset B(\operatorname{Gr} F; \varepsilon_n)$. By Schauder Theorem, there are $x_n \in K$, $n \in \mathbb{N}$, such that $x_n = f_n(x_n)$. A subsequence of the sequence $(x_n)$ is convergent to some $x \in K$, and we have $x \in F(x)$. $\square$

Let us mention an obvious extension of Kakutani (Fan–Glicksberg) Fixed Point Theorem. We define

$$(\operatorname{Ls} F)(x) = \operatorname{Ls}_{x' \to x} F(x') = \bigcap_{U \in \mathcal{U}(x)} \overline{F(U)} = \bigcap_{U \in \mathcal{U}(x)} \overline{\bigcup_{x' \in U} F(x')}.$$

**Theorem 1.10** [59]). *Let $K$ be a compact convex subset of a locally convex Hausdorff topological vector space. If a multimap $F: K \multimap K$ satisfies condition:*

$$\textit{for every } x \in K \textit{ there hold: } x \notin F(x) \Rightarrow x \notin \overline{\operatorname{co}}(\operatorname{Ls} F)(x),$$

*then there exists $\widetilde{x} \in K$ such that $\widetilde{x} \in F(\widetilde{x})$.*

*Proof.* Define a multimap $G: K \multimap K$ by

$$G(x) = \overline{\operatorname{co}}(\operatorname{Ls} F)(x).$$

Clearly, $G$ has convex compact values and maps $K$ into $K$. We will verify that $G$ is u.s.c. First observe that $\operatorname{Ls}(\operatorname{Ls} F)) = (\operatorname{Ls} F)$, since $\operatorname{Graph} \operatorname{Ls}(\operatorname{Ls} F) = \overline{\operatorname{Graph}(\operatorname{Ls} F)} = \overline{\operatorname{Graph} F}$. Then, since $(\operatorname{Ls} F)(x) \subseteq K$, the mapping $(\operatorname{Ls} F)$ is u.s.c. (see e.g. [8, Proposition 6.3.2]), i.e. continuous as a single valued map into the space $\operatorname{Comp}(Y)$ of nonempty compact subsets of $Y$ equipped with the upper Vietoris topology (generated by the sets $V^+ = \{A : A \subseteq V\}$ for $V \in \tau$). It remains to recall that the mapping $A \mapsto \overline{\operatorname{co}} A$ is a continuous self mapping on $\operatorname{Comp}(Y)$, (to see that, assume that $W$ is an open set such that $\overline{\operatorname{co}} A \subset W$, $V \in U(0)$ is chosen so that $\overline{\operatorname{co}} A + V \subset W$, and choose convex $U \in U(0)$ such that $\overline{U} \subset V$; then, if $B \subset A + U$ it follows that $\overline{\operatorname{co}} B \subset \overline{\operatorname{co}}(A + U) \subset \overline{\operatorname{co}} A + \overline{U} \subset \overline{\operatorname{co}} A + V \subset W$). Now, being upper semicontinuous and convex compact valued, $G$ has a fixed point, by virtue of Kakutani–Fan–Glicksberg Theorem. But by the hypothesis, any fixed point of $G$ is a fixed point of $F$. $\square$

Obviously, for any u.s.c. multimap $F: K \multimap K$ with convex closed values we have $\overline{\operatorname{co}}(\operatorname{Ls} F)(x) \subseteq F(x)$, hence the hypothesis is satisfied. When $K$ is a compact convex subset of a normed space and $F$ has closed values, above theorem may be rephrased in the following equivalent form (compare with [60, Theorem 4]):

**Corollary 1.11.** *$K$ be a compact convex subset of a normed space $(X, \|\cdot\|)$. If a multimap $F\colon K \multimap K$ with closed values satisfies the following condition:*

*for every $x \in K$ there exists $p \in X^*$ such that*

$$\sup_{U\in\mathcal{U}(x)} \inf\langle F(U) - x, p\rangle \geq \inf \|F(x) - x\|,$$

*then there exists $\widetilde{x} \in K$ such that $\widetilde{x} \in F(\widetilde{x})$.*

*Proof.* We will show that the condition from preceding theorem is actually equivalent to above condition. First, suppose that the condition

$$\text{for every } x \in K \text{ there hold: } x \notin F(x) \Rightarrow x \notin \overline{\mathrm{co}}(\mathrm{Ls}\, F)(x),$$

is satisfied and there exists some $x \in K$ such that for all $p$

$$\sup_{U\in\mathcal{U}(x)} \inf\langle F(U) - x, p\rangle < \inf \|F(x) - x\|.$$

Since $\sup_{U\in\mathcal{U}(x)} \inf\langle F(U) - x, 0\rangle = 0$, then $\alpha = \inf \|F(x) - x\| > 0$, i.e. $x \notin F(x)$. By the hypothesis and convex separation theorem, there exists some $q$ such that $\inf\langle\overline{\mathrm{co}}\ (\mathrm{Ls}\, F)(x) - x, q\rangle = c > 0$. Then for $r = 2\alpha q/c$ we have $\inf\langle\overline{\mathrm{co}}\ (\mathrm{Ls}\, F)(x) - x, r\rangle \geq 2\alpha$, i.e.

$$\inf\left\langle \bigcap_{U\in\mathcal{U}(x)} \overline{F(U)} - x, r\right\rangle \geq 2\alpha.$$

But since $\sup_{U\in\mathcal{U}(x)} \inf\langle F(U) - x, p\rangle < \alpha$ for all $p$, then for each $U \in \mathcal{U}(x)$ there exists some $z \in F(U)$ such that $\langle z - x, r\rangle < \alpha$ and therefore

$$A_U = (\overline{F(U)} - x) \cap \{w : \langle w, r\rangle \leq \alpha\} \neq \emptyset.$$

Since $K - x \supset A_{U_1} \cap \ldots \cap A_{U_n} \supset A_{U_1\cap\ldots\cap U_n} \neq \emptyset$, for every finite family $\{U_1, \ldots, U_n\} \subset \mathcal{U}(x)$, by the finite intersection property we have $\bigcap_{U\in\mathcal{U}(x)} A_U \neq \emptyset$, and then it follows that

$$\inf\left\langle \bigcap_{U\in\mathcal{U}(x)} \overline{F(U)} - x, r\right\rangle \leq \alpha,$$

a contradiction. Suppose now that the condition in Corollary is satisfied and $x \notin F(x)$, so $\alpha = \inf \|F(x) - x\| > 0$. Then there exist $p$ and $U \in \mathcal{U}(x)$ such that $\inf\langle F(U) - x, p\rangle \geq \alpha/2$. This implies

$$\inf\left\langle \overline{\mathrm{co}} \bigcap_{U\in\mathcal{U}(x)} F(U) - x, p\right\rangle \geq \frac{\alpha}{2},$$

consequently $x \notin \overline{\mathrm{co}}(\mathrm{Ls}\, F)(x)$. $\square$

## 2. Carathéodory selections

Common idea for both the proofs of Michael Theorem and Kuratowski– Ryll-Nardzewski Theorem can be described as follows. Given a multimap $F\colon W \multimap Y$, then:

(1) construct continuous/measurable $\varepsilon$-selection, i.e. a map $f_\varepsilon\colon W \to Y$ such that $F(w) \cap B(f_\varepsilon(w);\varepsilon) \neq \emptyset$ for $w \in W$ and verify the lower semicontinuity/measurability of the multimap $w \mapsto F(w) \cap B(f_\varepsilon(w);\varepsilon)$,
(2) construct a sequence of continuous/measurable $1/2^n$-selections, which is uniformly Cauchy, and get a required continuous/measurable selection as a limit.

One can ask for each of unified statement of measurable and continuous selection results, covering each the two theorems as particular case. An approach unifying the notion of measurable space and paracompact topological space on one side, and the notion of convex sets and closed convex sets on the second side has been proposed e.g. in [43]. However, it does seem that such unification gain much interest. We will consider quite different way of unifying selection theorems. This approach is strongly supported by applications.

Let $(\Omega,\mathcal{F})$ be a measurable space, $X$, $Y$ topological spaces. A selection $f\colon \Omega\times X \to Y$ of a multimap $F\colon \Omega\times X \multimap Y$ will be called *Carathéodory selection*, if the following Carathéodory's conditions are fulfilled:

(1) a map $\omega \mapsto f(\omega,x)$ is measurable for every $x \in X$,
(2) a map $x \mapsto f(\omega,x)$ is continuous for every $\omega \in \Omega$.

When $X$ is a separable metric space and $Y$ is a metric space, then a map satisfying Carathéodory's conditions is $\mathcal{F}\otimes\mathcal{B}(X)$-measurable.

The desired "unified-extended" statement of Michael Theorem and Kuratowski–Ryll-Nardzewski Theorem, would be the result on the existence of Carathéodory selections and countable Michael–Castaing representation:

$$F(\omega,x) = \overline{\{f_n(\omega,x) : n \in \mathbb{N}\}},$$

for a multimap $F\colon \Omega\times X \multimap Y$, where a measurable space $\Omega$ is arbitrary, $X$ is a separable metric space, $F$ has convex closed values in separable Banach space$Y$ and is of Carathéodory type, i.e. a multimap $\omega \mapsto F(\omega,x)$ is measurable for every $x \in X$, as well as a multimap $x \mapsto F(\omega,x)$ is lsc for every $\omega \in \Omega$. Such conditions however does not imply $\mathcal{F}\otimes\mathcal{B}(X)$-measurability of a multimap (see e.g. [32], [53]), while a countable representation does. So we accept the following definition. We say that a multimap $F\colon \Omega\times X \multimap Y$ is *a Carathéodory multimap* if:

(1*) $F$ is $\mathcal{F}\otimes\mathcal{B}(X)$-measurable,
(2*) a multimap $x \mapsto F(\omega,x)$ is lsc for every $\omega \in \Omega$.

When $\Omega$ is assumed to be a locally compact metric space with $\mathcal{F}$ a $\sigma$-algebra of the sets measurable for Radon measure on $\Omega$, the existence of countable representation by Carathéodory selections can be derived from Scorza–Dragoni property (see [15], [16] and [18], see also [3]).

When $(\Omega, \mathcal{F})$ is an abstract measurable space, two simple and natural approaches to derive Carathéodory selection results from continuous selection theorem and measurable selection theorem are:

(A) define a multimap $\omega \mapsto C_F(\omega) = \{f: X \to Y : f$ continuous selection of $x \mapsto F(\omega, x)\} \subset C(X;Y)$, with nonempty (by Michael Theorem) closed values, then apply Kuratowski–Ryll-Nardzewski Theorem to this multimap, or

(B) define a multimap $x \mapsto M_F(x) = \{f: \Omega \to Y : f$ measurable selection of $\omega \mapsto F(\omega, x)\} \subset M(\Omega; Y)$ (this should be a Banach space), with nonempty (by Kuratowski–Ryll-Nardzewski Theorem) closed and convex values, then apply Michael Theorem to this multimap.

Scheme (A) has been used e.g. by Castaing [15], Fryszkowski [24], Kucia [40]. We recall Fryszkowski's result.

**Theorem 2.1** (Fryszkowski, [24]). *Let $Y$ be a separable Banach space, $X$ be a separable, locally compact metric space, and let a measurable space $(\Omega, \mathcal{F})$ be such that $\mathrm{Pr}_\Omega(A) \in \mathcal{F}$ for every $A \in \mathcal{F} \otimes \mathcal{B}(Y)$. A Carathéodory multimap $F: \Omega \times X \multimap Y$ with closed and convex values has a Carathéodory selection. Moreover, any multimap $F: \Omega \times X \multimap Y$ with closed and convex values is Carathéodory if and only if there exists a countable family of Carathéodory selections $f_n: \Omega \times X \to Y$ of $F$, such that $F(\omega, x) = \overline{\{f_n(\omega, x) : n \in \mathbb{N}\}}$.*

In order to obtain Carathéodory selections proceeding along scheme (B), one has to equip the space $M(\Omega; Y)$ with the norm such that $\|\phi_n\| \to 0$ implies $\phi_n(\omega) \to 0$ for every $\omega \in \Omega$, for instance with $L^\infty$-norm $\|\phi\|_\infty = \sup\{\|\phi(\omega)\| : \omega \in \Omega\}$.

Scheme (B) has been used by Ricceri [50] who obtained two Carathéodory selection results. In the first result he assumed that multimaps $x \mapsto F(\omega, x)$, $\omega \in \Omega$, are almost equi-uniformly lower semicontinuous, the hypothesis essentially stronger than lower semicontinuity of all these multimaps. In the second result these multimaps are assumed to be lower semicontinuous, however a measurable space $\Omega$ is a countable set. A simplified statement of this result reads as follows.

**Theorem 2.2** ([50]). *Let $Y$ be a Banach space, $X$ be a paracompact space, and let a measurable space $\Omega$ be countable. Then Carathéodory multimap $F: \Omega \times X \multimap Y$ with closed and convex values and such that the set $F(\Omega, x)$ is separable for every $x \in X$, has a Carathéodory selection.*

Let us mention that continuous selections of lower semicontinuous multimaps of the form $x \mapsto M_F(x) = \{f \in L^p(\Omega; Y) : f$ is a selection of $\omega \mapsto F(\omega, x)\}$, with $1 \leq p < \infty$, do not give rise to Carathéodory selections of $F$. However, they are important and useful in the theory of differential inclusions. Multimaps $x \mapsto M_F(x)$ have *decomposable* values, i.e. if $A \in \mathcal{F}$, $f, g \in M_F(x)$, then $\chi_A f + (1 - \chi_A) g \in M_F(x)$. The result on continuous selections of lower semicontinuous multimaps with decomposable (not convex) values is due to Antosiewicz and Cellina [2], and its subsequent extensions — to compact space $X$ is due to Fryszkowski [25], and to separable metric space $X$ is due to Bressan and Colombo [9].

Yet another natural idea of "measurable parametrized" proof of Michael Theorem give rise to Carathéodory selections (see [53]). We state the result with the sketch of the proof.

**Theorem 2.3** ([53]). *Let $Y$ be a separable Banach space, $X$ be a Polish space, and let a measurable space $(\Omega, \mathcal{F})$ be complete. A Carathéodory multimap $F: \Omega \times X \multimap Y$ with closed and convex values has a Carathéodory selection. Moreover, any multimap $F: \Omega \times X \multimap Y$ with closed and convex values is Carathéodory if and only if there exists a countable family of Carathéodory selections $f_n: \Omega \times X \to Y$ of $F$, such that $F(\omega, x) = \overline{\{f_n(\omega, x) : n \in \mathbb{N}\}}$ for $(\omega, x) \in \Omega \times X$.*

First we extract *a random partition of unity*, which is the basic tool in the proof.

**Lemma 2.4** ([54]). *If $U_k: \Omega \multimap X$, $k \in \mathbb{N}$, is a countable family of multimaps with measurable graphs and such that for every $\omega \in \Omega$ the family $\{U_k(\omega)\}_{k \in \mathbb{N}}$ is an open cover of $X$, then there exists countable family of multimaps with measurable graphs $V_k^m: \Omega \multimap X, k, m \in \mathbb{N}$, such that the family $\{V_k^m(\omega)\}_{k,m \in \mathbb{N}}$ is a locally finite refinement of $\{U_k(\omega)\}_{k \in \mathbb{N}}$ (with $V_k^m(\omega) \subseteq U_k(\omega)$ for $k \leq m$, $\omega \in \Omega$). Consequently, there exists a random partition of unity, i.e. a countable family of functions $p_k^m: \Omega \times X \to [0;1]$, $k \leq m$, such that*

- (a) *the functions $\omega \mapsto p_k^m(\omega, x)$, $x \in X$, $k, m \in \mathbb{N}$ are measurable,*
- (b) *for every $\omega \in \Omega$ the family of functions $\{x \mapsto p_k^m(\omega, x): k, m \in \mathbb{N}\}$ forms a locally finite partition of unity on $X$ and*

$$\{x \in X : p_k^m(\omega, x) > 0\} \subset U_k(\omega)\}.$$

*Proof.* Let us define functions $f_k: \Omega \times X \to [0;1]$ by

$$f_k(\omega, x) = \frac{d(x, X \setminus U_k(\omega))}{1 + d(x, X \setminus U_k(\omega))}.$$

Since the multimaps $\omega \mapsto X \setminus U_k(\omega)$ have measurable graphs, the functions $\omega \mapsto f_k(\omega, x)$ are measurable. The functions $x \mapsto f_k(\omega, x)$ are continuous. By

letting
$$f(\omega,x)=\sum_{k=1}^{\infty}2^{-k}f_k(\omega,x)$$
we define a Carathéodory function. Let us define multimaps $V^m, W^m, V_k^m\colon \Omega \multimap X$ for $k\le m$:
$$\begin{aligned} V^m(\omega)\ &= \{x\in X : f(\omega,x)>1/m\},\\ W^0(\omega)\ &=\emptyset,\quad W^m(\omega)=\{x\in X: f(\omega,x)\ge 1/m\},\\ V_k^m(\omega)\ &= U_k(\omega)\cap (V^{m+1}(\omega)\setminus W^{m-1}(\omega)).\end{aligned}$$
For every $\omega\in\Omega$ the family $\{V_k^m(\omega)\}_{k\le m}$ is a locally finite open cover of $X$ such that $V_k^m(\omega)\subseteq U_k(\omega)$ for $k\le m$. Since $f$ is $\mathcal{F}\otimes\mathcal{B}(X)$-measurable, then multimaps $V^m, W^m, W_k^m$ have measurable graphs. We define now a random partition of unity $p_k^m\colon\Omega\times X\to[0;1]$ by
$$f_k^m(\omega,x)=\frac{d(x,X\setminus V_k^m(\omega))}{1+d(x,X\setminus V_k^m(\omega))},\qquad p_k^m(\omega,x)=\frac{f_k^m(\omega,x)}{\sum\limits_{l=1}^{\infty}\sum\limits_{i\le l}f_i^l(\omega,x)}.\qquad\square$$

*Proof of Theorem2.3.* (1) $\varepsilon$-selection.

Let $\{y_k\}_{k\in\mathbb{N}}$ be a countable dense subset of $Y$ and let $\varepsilon>0$. We define multimaps $U_k\colon\Omega\multimap X$, $k\in\mathbb{N}$, by letting $U_k(\omega)=\{x: F(\omega,x)\cap B(y_k,\varepsilon)\ne\emptyset\}$. Since $F$ is $\mathcal{F}\otimes\mathcal{B}(Y)$-measurable, it follows that $U_k, k\in\mathbb{N}$, have measurable graphs. Since $x\mapsto F(\omega,x)$ is lsc, the family $\{U_k(\omega)\}_{k\in\mathbb{N}}$ is an open cover of $X$. Choose a random partition of unity $p_k^m\colon\Omega\times X\to[0;1]$, $k\le m$ inscribed into $\{U_k(\omega)\}_{k\in\mathbb{N}}$ and define Carathéodory $\varepsilon$-selection $f_\varepsilon\colon\Omega\times X\to Y$ by
$$f_\varepsilon(\omega,x)=\sum_{m=1}^{\infty}\sum_{k\le m}p_k^m(\omega,x)y_k.$$
As in the proof of Michael Theorem one can verify that for every $\omega\in\Omega$ the map $x\mapsto f_\varepsilon(\omega,x)$ is continuous, the sets $G_\varepsilon(\omega,x)=F(\omega,x)\cap B(f_\varepsilon(\omega,x);\varepsilon)$ are nonempty and the multimap $x\mapsto\overline{G_\varepsilon(\omega,x)}$ is l.s.c. Moreover, the map $f_\varepsilon$ is $\mathcal{F}\otimes\mathcal{B}(Y)$-measurable as a limit of a sequence of measurable maps given by the formula $S_n(\omega,x)=\sum_{m=1}^{n}\sum_{k\le m}p_k^m(\omega,x)y_k$. This implies $\mathcal{F}\otimes\mathcal{B}(Y)$-measurability of the multimap $(\omega,x)\mapsto\overline{G_\varepsilon(\omega,x)}$.

(2) Uniformly convergent sequence of $1/2^m$-selections.

Basing on the first part of the proof, just like in the proof of Michael Theorem, we define inductively a sequence $(f_m)_{m\in\mathbb{N}}$ of Carathéodory maps $f_m\colon\Omega\times X\to Y$ such that
$$\begin{aligned}\|f_{m+1}(\omega,x)-f_m(\omega,x)\|\ &<2^{-m+1},\\ F(\omega,x)\cap B(f_m(\omega,x);2^{-m})\ &\ne\emptyset.\end{aligned}$$

Since this sequence is uniformly Cauchy, its limit $f$ is a Carathéodory map, and since $F$ has closed values the map $f$ is a selection of $F$.

(3) Michael–Castaing representation.

If $F(\omega,x)=\overline{\{f_n(\omega,x):n\in\mathbb{N}\}}$ for $(\omega,x)\in\Omega\times X$, then it is easy to check that $F$ is a Carathéodory multimap. Assume that $F$ is a Carathéodory multimap. Let $\{y_k\}_{k\in\mathbb{N}}$ be a separable dense subset of $Y$ and let $B_k^m=B(y_k;2^{-m})$ for $k,m\in\mathbb{N}$. By letting

$$U_k^m(\omega)=\{x\in X:F(\omega,x)\cap B_k^m\neq\emptyset\},$$

we define multimaps $U_k^m\colon\Omega\multimap X$ with open values and $\mathcal{F}\otimes\mathcal{B}(Y)$-measurable graphs. Next, by letting

$$W_{kj}^m(\omega)=\{x\in X:d(x,X\setminus U_k^m(\omega))\geq j^{-1}\}$$

we define multimaps $W_{kj}^m\colon\Omega\multimap X$ with closed values and $\mathcal{F}\otimes\mathcal{B}(Y)$-measurable graphs, such that $\bigcup_j \operatorname{Gr}W_{kj}^m=\operatorname{Gr}U_k^m$. Now let

$$\begin{aligned} G_{kj}^m(\omega,x)&=F(\omega,x)\cap B_k^m &&\text{for } (\omega,x)\in\operatorname{Gr}W_{kj}^m,\\ G_{kj}^m(\omega,x)&=F(\omega,x) &&\text{for } (\omega,x)\notin\operatorname{Gr}W_{kj}^m,\end{aligned}$$

and finally

$$F_{kj}^m(\omega,x)=\overline{G_{kj}^m(\omega,x)}.$$

One can check that every multimap $F_{kj}^m$ is a Carathéodory multimap with closed and convex values, hence has a Carathéodory selection $f_{kj}^m$. By the definition of $F_{kj}^m$ we obtain

$$F(\omega,x)=\overline{\{f_{kj}^m(\omega,x):m,k,j\in\mathbb{N}\}}$$

for every $(\omega,x)\in\Omega\times X$. $\square$

The same result with slightly different proof has been given by Kim, Prikry and Yannelis [38] (see also [37], [32]).

## 3. Random fixed points

Let $(\Omega,\mathcal{F})$ be a measurable space, $X$ be a topological space, $F\colon\Omega\times X\multimap X$ be a multimap. *A random fixed point* of $F$ is a measurable map $f\colon\Omega\to X$ such that

$$f(\omega)\in F(\omega,f(\omega))$$

for every $\omega\in\Omega$, i.e. a measurable selection of the multimap

$$\omega\mapsto\operatorname{Fix}_F(\omega)=\operatorname{Fix}_{F(\omega,\,\cdot\,)}=\{x\in X:x\in F(\omega,x)\}.$$

Obviously, if $F$ has a random fixed point, then $\operatorname{Fix}_F(\omega)\neq\emptyset$ for every $\omega\in\Omega$, i.e. every multimap $x\mapsto F(\omega,x)$ has a fixed point.

There are numerous papers, where the proofs of classical fixed point theorems are parametrized step by step in order to get random fixed point results. We do not recommend such approach, whenever the existence of random fixed points can be deduced easily from the existence of fixed points, via some universal measurable selection rule. Such idea had appeared e.g. in the proofs at [22]. First, following [32], we formulate a rule based on Kuratowski–Ryll-Nardzewski Theorem.

**Theorem 3.1** ([32]). *Let $X$ be a $\sigma$-compact Polish space and $(\Omega, \mathcal{F})$ be a measurable space. If a multimap $F\colon \Omega \times X \multimap Y$ has closed values,* $\mathrm{Fix}_F(\omega) \neq \emptyset$ *for $\omega \in \Omega$ and*

(a) *the multimap $\omega \mapsto F(\omega, x)$ is measurable for every $x \in X$,*
(b) *the multimap $x \mapsto F(\omega, x)$ is $H$-continuous for every $\omega \in \Omega$,*

*then there exists a measurable map $f\colon \Omega \to X$ such that $f(\omega) \in F(\omega, f(\omega))$ for every $\omega \in \Omega$.*

If $(\Omega, \mathcal{F})$ is a complete measurable space, Aumann–Yankov–von Neumann Theorem leads to the following rule.

**Theorem 3.2** ([55]). *Let $X$ be a Polish space and $(\Omega, \mathcal{F})$ be a complete measurable space. If a multimap $F\colon \Omega \times X \multimap Y$ is $\mathcal{F} \otimes \mathcal{B}(X)$-measurable and has closed values, a multimap $K\colon \Omega \multimap X$ has measurable graph and* $\mathrm{Fix}_F(\omega) \cap K(\omega) \neq \emptyset$ *for $\omega \in \Omega$, then there exists a measurable map $f\colon \Omega \to X$ such that $f(\omega) \in F(\omega, f(\omega)) \cap K(\omega)$ for $\omega \in \Omega$.*

*Proof.* Let us define the function $f\colon \Omega \times X \to [0; \infty)$ by letting

$$f(\omega, x) = d(x, F(\omega, x)).$$

Since a multimap $F$ is $\mathcal{F} \otimes \mathcal{B}(Y)$-measurable, then the map $f$ is $\mathcal{F} \otimes \mathcal{B}(Y)$-measurable. Since $F$ has closed values, then $x \in \mathrm{Fix}_{F(\omega,\,\cdot\,)}$ if and only if $f(\omega, x) \leq 0$. Therefore the multimap $\omega \mapsto \mathrm{Fix}_F(\omega) \cap K(\omega)$ has measurable graph. By virtue of Aumann–Yankov–von Neumann Theorem, this multimap has a measurable selection, which is a random fixed point of $F$ in a random set $K$. $\square$

## 4. Retractive representation of a multimap

For a multimap $F\colon X \multimap Y$, where $X$ is a measurable space or topological space or the Cartesian product of measurable and topological space, and $Y$ is a topological space, together with countable representations $F(x) = \overline{\{f_n(x) \colon n \in \mathbb{N}\}}$ (Michael representation — for l.s.c. multimap, Castaing representation — for measurable multimap), there were considered representations of the form $F(x) = f(x, Z)$, where $f\colon X \times Z \to Y$ is a map, which is continuous with respect to variable $z$ from some topological space $Z$. Such representations for continuous multimaps as well as Carathéodory multimaps with closed values in compact subset of separable normed space has been studied by Ekeland and Valadier ([23]). Their idea was to embed compact set into Hilbert space $l^2$, and then use continuous representation of convex sets in Hilbert space, by means of metric projections.

For measurable or Carathéodory multimaps, systematic study of similar representations has been done by Ioffe ([33], [34]).

We outline some simple idea, which helps one to identify topological properties of the values of a multimap on one side, and gives continuous selections on the other side.

Let $X$, $Y$ be topological (or metric) spaces. We say that a multimap $F\colon X \multimap Y$ has *a retractive representation*, if there exists a set $Z \subseteq Y$ such that $F(X) \subseteq Z$ and continuous map $f\colon X \times Z \to Y$ such that:

(i) $f(x, y) \in F(x)$,
(ii) $f(x, y) = y$ if and only if $y \in F(x)$,

for $(x, y) \in X \times Z$. Retractive representation $f\colon X \times Z \to Y$ is equicontinuous, respectively: uniformly, equi-uniformly continuous, if the maps $x \mapsto f(x, y)$, $y \in Z$, are equicontinuous, respectively: uniformly, equi-uniformly continuous.

Since for a multimap $F$ with retractive representation $(Z, f)$ there holds

$$F(x) = f(x, Z) = \{y \in Y : f(x, y) = y\},$$

we then have:

(1) for every $(x_0, y_0) \in \operatorname{Gr} F$, the multimap $F$ has a continuous selection $f_0\colon x \mapsto f(x, y_0)$ such that $f_0(x_0) = y_0$,
(2) $F$ is l.s.c.,
(3) $\operatorname{Gr} F$ is closed,
(4) for every $x \in X$ the set $F(x)$ is a retract of $Z$,
(5) if $(Y, \rho)$ is metric space, then for Hausdorff distance we have

$$D(F(x_1), F(x_2)) \leq \sup_{z \in Z} \rho(f(x_1, z), f(x_2, z)),$$

hence equicontinuity (equi-uniform continuity) of the representation $f$ implies $H$-continuity (uniform $H$-continuity) of the multimap $F$.

Straightforward application of Michael Theorem gives general sufficient conditions for the existence of a retractive representation.

**Lemma 4.1** ([56]). *Let $Y$ be a Banach space, $X$ be a paracompact and perfectly normal topological space (e.g. metric). If a multimap $F\colon X \multimap Y$ has closed convex values and is $W$-continuous (i.e. functions $x \mapsto d(y, F(x)), y \in Y$ are continuous), then for every $L > 1$ a multimap $F$ has a retractive representation $(Y, f_L)$ such that*

$$\|f_L(x, y) - y\| \leq L d(y, F(x)).$$

*for all $(x, y) \in X \times Y$.*

*Proof.* Let us define $H: X \times Y \multimap Y$ and $m_L: X \times Y \to [0;\infty)$ by letting

$$\begin{aligned} H(x,y) &= F(x) - y, \\ m_L(x,y) &= Ld(y,F(x)) = Ld(0,H(x,y)). \end{aligned}$$

Since the functions $x \mapsto d(y,F(x)), y \in Y$ are continuous and $\|d(y,A)-d(v,A)\| \leq \|y-v\|$, then the function

$$(x,y) \mapsto Ld(y,F(x)) = m_L(x,y)$$

is continuous. Since the space $X \times Y$ is paracompact, a multimap

$$(x,y) \mapsto H(x,y) \cap \overline{B(0;m_L(x,y))}$$

is l.s.c. and has closed convex values, then by virtue of Michael Theorem there exists a continuous map $h: X \times Y \to Y$ such that

$$h(x,y) \in H(x,y), \quad \|h(x,y)\| \leq m_L(x,y).$$

By letting $f_L(x,y) = h(x,y) - y$ we define required retractive representation. □

For every nonempty closed and convex set $A \subseteq Y$, $L > 1$ and $m_L(A) = Ld(0,A)$, the intersection $A \cap \overline{B(0;m_L(A))}$ is the subset of $A$ consisting of the elements with "almost" minimal norm, while $A \cap \overline{B(0;m_1(A))}$ consists of the elements of minimal norm (this set is the *metric projection* of 0 onto $A$). When $Y$ is reflexive and strictly convex, this set is a singleton. In this case we have a unique minimal selection $A \mapsto s(A) \in A$, where $\|s(A)\| = \min_{a\in A}\|a\|$. Therefore a multimap $F: X \multimap Y$ has in this case unique retractive representation $(Y, f_1)$

$$f_1(x,y) = s(F(x)-y) + y,$$

satisfying (i), (ii) and inequality $\|f_1(x,y) - y\| \leq d(y,F(x))$.

However, since the metric projection onto infinite dimensional convex set in reflexive and strictly convex space need not to be continuous, a map $f_1$ may be not continuous, even if a multimap $F$ is $H$-continuous. Let us mention that this cannot happen if the metric projection is taken with respect to the norm in $Y$ having Kadec–Klee property (see e.g. [31]):

(KK) if $\|y_n\| \to \|y\|$ and $y_n \rightharpoonup y$ (weakly), then $\|y_n - y\| \to 0$.

By virtue of Kadec–Klee Theorem, every separable Banach space admits an equivalent strictly convex norm with the property (KK).

In [52] and [21] an elementary proof of almost uniform continuity of minimal selection in uniformly convex Banach space has been given. Let us recall that a normed space $(Y, \|\cdot\|)$ is uniformly convex, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x,y \in \overline{B(0;1)}$ and $\|x-y\| > \varepsilon$, implies $\|(x+y)/2\| < 1-\delta$. Recall that the spaces $L^p$ and $l^p$ are uniformly convex, whenever $1 < p < \infty$.

We then have the following refinement of the Lemma given above for $L = 1$, basing on the norm properties, instead of Michael Theorem.

**Lemma 4.2** ([52]). *Let $Y$ be a uniformly convex Banach space, $X$ be a topological* (*metric*) *space. If a multimap $F\colon X \multimap Y$ has closed convex values and is $H$-continuous, then $(Y, f_1)$, (where $f_1(x,y) = s(F(x) - y) + y$), is a retractive representation for $F$ such that*

$$\|f_1(x,y) - y\| = d(y, F(x))$$

*for all $(x,y) \in X \times Y$. If $F$ is uniformly $H$-continuous, then $f_1$ is uniformly continuous on sets $W_r = \{(x,y) \in X \times Y : d(y, F(x)) \le r\}$, $r \ge 0$.*

We will apply now above results to contractive multimaps with convex values and obtain retractive representations for fixed point multimaps associated with them. Notice that fixed point multimaps, resulting from contractive multimaps with convex values, need not have convex values. Simple examples can be given in $\mathbb{R}^2$ (see e.g. [56]).

**Theorem 4.3** ([54], [56]). *Let $Y$ be Banach space, $X$ be a paracompact and perfectly normal topological space. If a multimap $H\colon X \times Y \multimap Y$ has closed convex values and:*

(a) *for every $y \in Y$ the multimap $x \mapsto H(x,y)$ is $W$-continuous,*
(b) *there exists $K > 1$ such that $D(H(x,y_1), H(x,y_2)) \le K\|y_1 - y_2\|$ for every $x \in X$,*

*then the multimap $\mathrm{Fix}_H\colon x \mapsto \mathrm{Fix}_{H(x,\,\cdot\,)}$ has a retractive representation $(Y,h)$.*

*Proof.* Cartesian product $X \times Y$ is paracompact and perfectly normal. From (a), (b) and the inequality

$$\begin{aligned}|d(z, H(x,y) - d(z, H(x_0,y_0)| \\ \le D(H(x,y), H(x,y_0)) + |d(z, H(x,y_0) - d(z, H(x_0,y_0)|\end{aligned}$$

it follows that a multimap $H$ is $W$-continuous with respect to $(x,y)$. Let us fix $L \in (1; 1/K)$. By virtue of the first retractive representation lemma, there exists a continuous map $h_L\colon X \times Y \times Y \to Y$ satisfying (i), (ii) and the inequality

$$\|h_L(x,y,z) - z\| \le Ld(z, H(x,y))$$

for $(x,y,z) \in X \times Y \times Y$. Let us define $g^n\colon X \times Y \to Y$, $n \in \mathbb{N}$, by letting

$$\begin{aligned} g^1(x,y) &= h(x,y,y), \\ g^{n+1}(x,y) &= g^1(x, g^n(x,y)). \end{aligned}$$

By induction, one can prove that all maps $g^n$, $n \in \mathbb{N}$, are continuous and there hold

$$\begin{aligned} g^{n+1}(x,y) &\in H(x, g^n(x,y)), \\ \big\|g^{n+1}(x,y) - g^n(x,y)\big\| &\le Ld(g^n(x,y), H(x, g^n(x,y))). \end{aligned}$$

The last inequality and the assumptions (a), (b) give

$$\begin{aligned}\|g^{n+1}(x,y)-g^n(x,y)\| &\leq LD(H(x,g^{n-1}(x,y)),H(x,g^n(x,y))\\ &\leq LK\|g^{n-1}(x,y)-g^n(x,y)\|\\ &\leq (LK)^n d(y,H(x,y)).\end{aligned}$$

Hence, $(g^n(x,y))_{n\in\mathbb{N}}$ is a Cauchy sequence for every $(x,y)\in X\times Y$. We define $h\colon X\times Y\to Y$ by letting

$$h(x,y)=\lim_{n\to\infty} g^n(x,y).$$

By standard reasoning we obtain

$$\|h(x,y)-g^n(x,y)\|\leq \sum_{n=m}^{\infty}(LK)^n d(y,H(x,y)),$$

and then $d(h(x,y),H(x,y))\leq 0$, which in turn implies that $h(x,y)\in H(x,y)$, i.e. $y\in \mathrm{Fix}_H(x)$ for every $(x,y)\in X\times Y$. If $h(x,y)=y$, then certainly $y\in\mathrm{Fix}_H(x)$. If $y\in\mathrm{Fix}_H(x)$, then $g^n(x,y)=y$ for $n=1,2,\ldots$, hence $h(x,y)=y$.

It remains to show the continuity of the map $h$. For $(x_0,y_0)\in X\times Y$, $\varepsilon>0$ let us choose $m\in\mathbb{N}$ such that

$$\sum_{n=m}^{\infty}(LK)^n(2d(y_0,H(x_0,y_0))+1)<\frac{\varepsilon}{2}$$

and a neighbourhood $W$ of $(x_0,y_0)$ such that $d(y,H(x,y))<d(y_0,H(x_0,y_0))+1$ and $\|g^m(x,y)-g^m(x_0,y_0)\|<\varepsilon/2$ for $(x,y)\in W$. Then for $(x,y)\in W$ we have

$$\begin{aligned}\|h(x,y)-h(x_0,y_0)\| &\leq \|h(x,y)-g^n(x,y)\|\\ &\quad+\|g^n(x,y)-g^n(x_0,y_0)\|+\|g^n(x_0,y_0)-h(x_0,y_0)\|\\ &\leq \sum_{n=m}^{\infty}(LK)^n d(y,H(x,y))+\frac{\varepsilon}{2}+\sum_{n=m}^{\infty}(LK)^n d(y_0,H(x_0,y_0))<\varepsilon. \qquad \square\end{aligned}$$

Using the retractive representation lemma for uniformly convex Banach space, one can repeat above proof to obtain the following refinement of retractive representation result for fixed point set valued multimap.

**Theorem 4.4** ([21]). *Let $Y$ be a uniformly convex Banach space, $X$ be a topological* (*metric*) *space. If a multimap $H\colon X\times Y\multimap Y$ has closed convex values and:*

(a) $r(x)=\sup_{y\in Y} d(y,H(x,y))<\infty$ *for every $x\in X$,*
(b) *the multimaps $x\mapsto H(x,y), y\in Y$ are equi-$H$-continuous* (*equi-uniformly $H$-continuous on the sets $X_r=\{x\in X:r(x)\leq r\}$, $r>0$*)*,*
(c) *there exists $K>1$ such that $D(H(x,y_1),H(x,y_2))\leq K\|y_1-y_2\|$ for every $x\in X$,*

*then a multimap* $\mathrm{Fix}_H \colon x \mapsto \mathrm{Fix}_{H(x,\,\cdot\,)}$ *has equicontinuous* (*equi-uniformly* $H$*-continuous on the sets* $X_r$) *retractive representation* $(Y,h)$.

From above retractive representation theorems it follows in particular that the fixed point sets of contractive multimaps with convex values are the absolute retracts. This property has been proved by Ricceri [51].

On the other hand, retractive representation provides an information on continuity of the multimap $\mathrm{Fix}_H \colon x \mapsto \mathrm{Fix}_{H(x,\,\cdot\,)}$. The distance between fixed point sets of contractive multimaps with closed values in complete metric spaces can be estimated as follows

$$D(\mathrm{Fix}_H(x_1), \mathrm{Fix}_H(x_2)) \leq \frac{1}{1-K} \sup_{y\in Y} D(H(x_1,y), H(x_2,y))$$

(see Lim [42]).

The same idea as in above retractive representation result, has been used by Bressan, Cellina and Fryszkowski [10], for the fixed point sets of contractive multimaps with decomposable values. The selection theorem for lsc multimaps with decomposable values due to Bressan–Colombo has been used there, instead of Michael Theorem. Further extensions, covering both convex and decomposable valued contractive multimaps are due to Górniewicz and Marano [27], Górniewicz, Marano, Ślosarski [28], as well as Andres and Górniewicz [1].

## 5. Continuous selections of non l.s.c. multimaps

We assume further that $X$ is a paracompact space and $Y$ is a normed space. If a multimap $F \colon X \multimap Y$ has *a continuous selection representation* $F(x) = f(x, W)$, where $W$ is an arbitrary set and $f \colon X \times W \to Y$ is a mapping such that for every $w \in W$ the map $x \mapsto f(x,w)$ is continuous, then $F$ is l.s.c. The existence of some continuous selections does not imply the lower semicontinuity of a multimap. The set of continuous selections

$$C_F = \{f \colon X \to Y \text{ continuous} : f(x) \in F(x) \text{ for } x \in X\}$$

defines in natural way lower semicontinuous *submultimap (multiselection)*

$$x \mapsto C_F(x) = \{f(x) : f \in C_F\}$$

of $F$. Under the assumptions of Michael Theorem we have $F(x) = C_F(x)$ for every $x \in X$.

The study of continuous selections of convex valued, but not l.s.c., multimaps in approximation (optimization), has been motivated by the stability problems in algorithms of best approximation, when normed space is not strictly convex. Recall that for a convex subset $A$ of a normed space $Y$ and $y \in Y$ the set

$$P_A(y) = A \cap \overline{B(y; d(y,A))} = P(0; A-y) + y$$

is *a metric projection* of $y$ onto $A$.

Elementary examples of metric projections onto one dimensional linear subspace $A$ in three dimensional space show that the multimap $y \mapsto P_A(y)$:

(a) may be not l.s.c. and admit continuous selections:

**Example 5.1.** Let $\|y\| = \sqrt{y_1^2 + y_2^2} + |y_3| = 1$, $A = \mathrm{Lin}\{(0,1,-1)\}$, $y(t) = (\cos t, \sin t, 0)$, $t \in [0,\pi]$. Then

$$P_A(y(t)) = \begin{cases} \{(0,0,0)\} & \text{for } t \neq \pi/2, \\ \mathrm{co}\{(0,0,0),(0,-1,1)\} & \text{for } t = \pi/2, \end{cases}$$

hence, the multimap $t \mapsto P_A(y(t))$, has a continuous selection $t \mapsto (0,0,0)$.

(b) may not admit continuous selections

**Example 5.2.** Let

$$\begin{aligned} B &= \mathrm{co}\{y : \max\{|y_2|,|y_3|\} = 1 \\ &\qquad \vee (y_1^2 + y_2^2 = 1, y_1 \le 0, y_3 = 1) \vee (y_1^2 + y_2^2 = 1, y_1 \ge 0, y_3 = -1)\}, \\ \|y\| &= \mu_B(y) = \inf\{\lambda : (1/\lambda)y \in B\}, \\ A &= \mathrm{Lin}\{(0,0,1)\}, \\ y(t) &= (\cos t, \sin t, 0), t \in [0,\pi]. \end{aligned}$$

Then

$$P_A(y(t)) = \begin{cases} \{(0,0,-1)\} & \text{for } t \in [0,\pi/2), \\ \mathrm{co}\{(0,0,-1),(0,0,1)\} & \text{for } t = \pi/2, \\ \{(0,0,1)\} & \text{for } t \in (\pi/2,\pi], \end{cases}$$

hence the multimap $t \mapsto P_A(y(t))$ has no continuous selection.

In contrast to above examples, where unit balls are hybrids of polyhedral and Euclidean balls of lower dimension, the following property of the norm introduced in [11], should be mentioned:

(P) for every $y, d \in Y$ such that $\|y+d\| \le \|y\|$ there exists $\delta > 0$, $\alpha > 0$ such that $\|z + \alpha d\| \le \|z\|$ for all $z \in B(y;\delta)$.

Brown proved that the property (P) implies the lower semicontinuity of metric projections onto finite dimensional subspaces, hence the existence of continuous selections. Every polyhedral norm in finite dimensional space as well as any strictly convex norm has the property (P).

We will discuss some concepts weaker than lower semicontinuity, but still sufficient for the existence of continuous selections. These concepts are inspired by the work of Brown [12], [13] and Gel'man [26], as well as by the papers due to Deutsch and Kenderov [20], Beer [7], De Blasi and Myjak [19].

Brown on one side and Deutsch and Kenderov on the other side, had revised carefully different aspects Michael Theorem, with regard to its relevance in the study of continuous selections of metric projections.

For a multimap $F\colon X \multimap Y$ and $x \in X$ let

$$F_0(x) = \operatorname*{Li}_{x' \to x} F(x') = \{y \in F(x) \colon d(y, F(x')) \to 0 \text{ for } x' \to x\}.$$

By the very definition, we have $F_0(x) \subseteq F(x)$ for all $x \in X$. A multimap $F$ is lsc if and only if $F = F_0$. The sets of continuous selections of the multimap $F$ and its submultimap $F_0$ always are identical (see [12]):

$$C_{F_0} = C_F.$$

By iterating an operator $F \mapsto F_0$ one can generate a transfinite sequence of relations ($F^{(\lambda)}$ : $\lambda$ — an ordinal)

$$F^{(0)}(x) = F(x), \quad F^{(\lambda+1)}(x) = (F^{(\lambda)})_0(x),$$
$$F^{(\mu)}(x) = \bigcap_{\lambda < \mu} F^{(\lambda)}(x), \quad \text{when } \mu \text{ is a limit ordinal}$$

The sequence ($\operatorname{Gr} F^{(\lambda)}$ : $\lambda$ — an ordinal) of subsets of the Cartesian product $X \times Y$ is decreasing, until it becomes constant. Then, a relation $F^*$ defined by the condition

$$F^* = F^{(\lambda)} \text{ if } F^{(\lambda+1)}(x) = F^{(\lambda)}(x) \quad \text{for } x \in X$$

is such that $C_{F^*} = C_F$ (see [13], [14]).

Brown has proved that for $Y = \mathbb{R}^n$, there holds $F^* = F^{(n+1)}$, and if moreover $F^{(n)}(x) \neq \emptyset$ for every $x \in X$, then $F^* = F^{(n)}$.

Consequently, Michael Theorem admits the following restatement.

**Theorem 5.3** ([13]). *Let $X$ be a paracompact space, $Y$ be a Banach space. A multimap $F\colon X \multimap Y$ with closed and convex values has a continuous selection if and only if $F^*(x) \neq \emptyset$ for every $x \in X$ (hence $F^*(x) = C_{F^*}(x) = C_F(x)$ for every $x \in X$). If $Y = \mathbb{R}^n$, then $C_F \neq \emptyset$ if and only if $F^{(n)}(x) \neq \emptyset$ for every $x \in X$.*

Brown ([13]) has also constructed an u.s.c. multimap $F\colon [0,1]^n \multimap \mathbb{R}^n$ with compact and convex values such that $F^{(n)}(x) \neq \emptyset$ for $x \in X$, but $F^{(k)} \neq F^{(k-1)}$ for $k \leq n$. Using this multimap, he has shown that for any $n$, there is a real normed space $Y$ of dimension $2n+1$ having a subspace $M$ of dimension $n$, such that $P_M^{(n-1)}(x) \neq \emptyset$ for every $x \in X$, but $P_M^{(n)} \neq P_M^{(n+1)}$, (hence $P_M$ has no continuous selection).

A modification of Brown's example has been given in [57]. A multimap $F\colon [0,1] \multimap [0,1]^n$ with the above mentioned properties can be defined as follows.

**Example 5.4.** For $x \in (0,1]$ let

$$x = \sum_{i=1}^{\infty} 2^{-i} \delta_i, \quad \delta_i \in \{0,1\},$$

be a unique infinite binary expansion, and let

$$i_1(x) = \min\{i : \delta_i \neq 0\}, \quad i_k(x) = \min\{i > i_{k-1}(x) : \delta_i \neq 0\},$$

for $k = 1, 2, \ldots$ and $\{e_1, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^n$. Let us put

$$\begin{aligned} H_n(0) &= \{0\}, & H_n(x) &= \{2^{-i_1(x)}e_1, \ldots, 2^{-i_n(x)}e_n\}, \\ F_n(x) &= \mathrm{co}(H_n(x) \cup H_n(x+)). \end{aligned}$$

Then we have $F_n^{(n)}(0) = \{0\}$ as well as $F_n^{(n+1)}(0) = \emptyset$.

Using this multimap and idea from [13] and [61], we obtain the following information about possible continuous selection properties of metric projections onto subspaces of codimension 2 in finite dimensional spaces.

**Corollary 5.5** ([57]). *For every $n \in \mathbb{N}$, a linear space $M$, of dimension $n$ is a subspace of some normed space $Y$ of dimension $n+2$, with the norm such that the metric projection $P_M: Y \multimap M$ satisfies condition $P_M^{(n-1)}(y) \neq \emptyset$ for every $y \in Y$, but has no continuous selection (since $P_M^{(n)} \neq P_M^{(n+1)}$).*

*Proof.* Let a multimap $F_n$ be as in above example and let us define a multimap $F: [-\pi, \pi] \multimap [0,1]^n$ by letting

$$\begin{aligned} F(t) &= F_n(2t/\pi) && \text{for } t \in [0, \pi/2], \\ F(t) &= F(\pi - t) && \text{for } t \in (\pi/2, \pi], \\ F(t) &= -F(-t) && \text{for } t < 0. \end{aligned}$$

Parametrize one dimensional sphere $S^1 \subset \mathbb{R}^2$ by letting $s = s(t) = (\cos t, \sin t)$, $t \in (-\pi, \pi]$ for $s \in S^1$ and define $\Phi: S^1 \multimap [0,1]^n \subset M$, by letting $\Phi(s) = F(t)$ for $s = s(t)$. Let $Y = \mathbb{R}^2 \oplus M$ and let $D$ be the closed unit Euclidean ball in $Y$. Define compact convex symmetric neighbourhood of 0 as the convex hull of $\frac{1}{2}D$ and the graph $\Phi$:

$$K = \mathrm{co}\bigg(\frac{1}{2}D \cup \bigcup_{s \in S^1} (s - \Phi(s))\bigg).$$

Norm $\|y\| = \mu_K(y)$ is such that for $s \in S^1$ we have $P_M(s) = \Phi(s)$. Therefore the properties of $P_M$ follow immediately from the properties of $F_n$. $\square$

Considering Brown's result:

- *If $Y = R^n$, then for a multimap $F: X \multimap Y$* we have $C_F \neq \emptyset$ if and only if $F^{(n)}(x) \neq \emptyset$ for every $x \in X$,

one can ask whether for some multimaps $F: X \multimap Y$ with values in infinite dimensional spaces the same property hold true. The answer is yes for $Y = C(S, \mathbb{R}^n)$ and $Y = L^\infty(\Omega, \mathbb{R}^n)$ provided that $F$ has "rectangular" values.

A subset $D \subseteq C(S, \mathbb{R}^n)$ (respectively, $D \subseteq L^p(\Omega, \mathbb{R}^n)$) will be called *$C$-convex* (respectively, *$L$-convex*), if $pf + (1-p)g \in D$ for $f, g \in D$ and any continuous (resp. measurable) function $p\colon S \to [0,1]$. Closed subset $D \subset L^\infty(\Omega,\mathbb{R}^n)$ is $L$-convex if and only if it is convex and decomposable ([35], [45], [46]). The following selection theorem of Michael–Brown type is due to A. Kisielewicz.

**Theorem 5.7** ([35], [36])**.** *Let $X$ be a paracompact space, $S$ be a compact space, $(\Omega,\mu)$ be a measure space and let $Y = C(S,\mathbb{R}^n)$ or $Y = L^\infty(\Omega,\mathbb{R}^n)$. If a multimap $F\colon X \multimap Y$ has closed and $C$-convex or respectively $L$-convex values, then admits a continuous selection if and only if $F^{(n)}(x) \neq \emptyset$ for every $x \in X$.*

*Proof.* The necessity of the condition $F^{(n)}(x) \neq \emptyset$ for every $x \in X$ is obvious. We sketch the proof of sufficiency. First consider the case $F\colon X \multimap C(S,\mathbb{R}^n)$. By hypothesis, the sets $F^{(n)}(x)$ are nonempty. It can be proved inductively that these sets are closed and $C$-convex, in particular convex. Hence, it suffices to show the lower semicontinuity of a multimap $F^{(n)}$, and then apply Michael Theorem. To see that $F^{(n)}$ is l.s.c., it suffices to know that for every $x_0 \in X$ and $y_0 \in F^{(n)}(x_0)$ there exists a continuous selection $f$ of a multimap $F^{(n)}$ such that $f(x_0) = y_0$. For fixed $x_0, y_0$ define a multimap $G\colon X \multimap C(S,\mathbb{R}^n)$ by letting $G(x_0) = \{y_0\}$ and $G(x) = F(x)$ if $x \neq x_0$. We have $G^{(n)}(x_0) = \{y_0\}$ and $G^{(n)}(x) = F^{(n)}(x)$ if $x \neq x_0$. Next we define a multimap $H\colon X \times S \multimap \mathbb{R}^n$ by letting $H(x,s) = G(x)(s)$. Since the sets $G(x)$ are $C$-convex, then $y \in G(x)$ if and only if $y(s) \in H(x,s)$ for $s \in S$. We therefore know that $G$ has a continuous selection if and only if $H$ has a continuous selection. The multimap $H$ closed and convex values in $\mathbb{R}^n$, and it can be proved inductively that $G^{(n)}(x)(s) \subseteq H^{(n)}(x,s)$ for $(x,s) \in X \times S$, hence $H^{(n)}(x,s) \neq \emptyset$. By the result of Brown, $H^{(n)}$ has a continuous selection $g\colon X \times S \to \mathbb{R}^n$. Since $g(x,s) \in H^{(n)}(x,s) \subseteq G(x)(s)$, then $f\colon X \to C(S,\mathbb{R}^n)$ defined by $f\colon x \mapsto g(x,\cdot\,)$, is a required continuous selection.

For $Y = L^\infty(\Omega,\mathbb{R}^n)$ the result follow from the case $Y = C(S,\mathbb{R}^n)$, by applying Gelfand's transformation, which is used to define an isomorphism of the space $L^\infty(\Omega,\mathbb{R}^n)$ with the space $C(\Delta,\mathbb{R}^n)$, where $\Delta$ is the set of complex homomorphism of commutative Banach algebra $L^\infty(\Omega,\mathbb{C})$, equipped with Gelfand topology. □

Let us mention that for every $k \in \mathbb{N}$ one can construct a multimap $F\colon [0,1] \multimap L^1([0,1],\mathbb{R})$, with closed $L$-convex values and such that $F^{(k)}(x) \neq \emptyset$ for every $x \in [0,1]$, but $F^{(k+1)}(0) = \emptyset$, hence $F$ does not admit a continuous selection (see [35]).

Now, we discuss briefly some applications of Michael–Brown Theorem to multimaps, which assign to each variable $x \in X$ an epigraph of real valued function. Much exhaustive exposition can be found in [57].

Assume that $X$ is a paracompact space. For a *variational system $g$*, i.e. for a family of functions $g_x\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $x \in X$, let us define a multimap $E(g)\colon X \multimap$

$\mathbb{R}^n \times \mathbb{R}$, by letting

$$E(g)(x) = \mathrm{epi} g_x = \{(y,r) \in \mathbb{R}^n \times \mathbb{R} : g_x(y) \le r\}.$$

We assume that the functions $g_x$ are *proper*, i.e. $E(g)(x) \neq \emptyset$ and lower semicontinuous (in classical sense), i.e. the sets $E(g)(x)$ are closed We define epigraphical upper limit $(\mathrm{epils}\, g)_x \colon \mathbb{R} \to \overline{\mathbb{R}}$ of the system $g$ at $x \in X$ by:

$$(\mathrm{epils}\, g)_x(y) = \sup_{V \in U(y)} \inf_{W \in \mathcal{U}(x)} \sup_{x' \in W} \inf_{y' \in V} g_{x'}(y'),$$

or equivalently

$$E(\mathrm{epils}\, g)(x) = \operatorname*{Li}_{x' \to x} F(x') = E(g)^{(1)}(x).$$

Epigraphical lower limit $(\mathrm{epili}\, g)_x \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$(\mathrm{epili}\, g)_x(y) = \sup_{V \in U(y)} \sup_{W \in \mathcal{U}(x)} \inf_{x' \in W} \inf_{y' \in V} g_{x'}(y'),$$

equivalently

$$E(\mathrm{epili}\, g)(x) = \mathrm{Ls}_{x' \to x} F(x') = \bigcap_{U \in \mathcal{U}(x)} \overline{F(U)}.$$

By the definition we have

$$(\mathrm{epili}\, g)_x(y) \le \sup_{W \in \mathcal{U}(x)} \inf_{x' \in W} g_{x'}(y) \le g_x \le (\mathrm{epils}\, g)_x(y) \le \inf_{W \in \mathcal{U}(x)} \sup_{x' \in W} g_{x'}(y).$$

The motivation for studying epigraphical limits comes from the basic variational property (see [4]):

- *Assume that* $(\mathrm{epili}\, g)_x = g_x = (\mathrm{epils}\, g)_x$. *Then for* $\varepsilon_m \to 0$, $(x_m, y_m) \to (x, y)$ *and* $g_{x_m}(y_m) < \inf g_{x_m} + \varepsilon_m$, *there hold* $g_{x_m}(y_m) \to \inf g_x = g_x(y)$.

The iterates $g^{(k)}$, $k = 1, 2, \ldots$ of epigraphical upper limit are defined by the equality

$$E(g^{(k)})(x) = (E(g))^{(k)}(x)$$

for every $x \in X$. Certainly

$$g_x(y) \le g_x^{(1)}(y) \le \ldots \le g_x^{(k)}(y) \le \inf_{W \in \mathcal{U}(x)} \sup_{x' \in W} g_{x'}(y)$$

for $y \in \mathbb{R}^n, k \in \mathbb{N}$. Moreover, if $g^{(k)} = g^{(k+1)}$, then $g^{(k)} = g^{(l)}$ for $l > k$.

An example of a multimap $F \colon [0,1] \multimap [0,1]^n$ such that $F^{(n)} \neq F^{(n+1)}$, which we have considered above, allows us to define a system $g_x \colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $x \in [0,1]$, with the property:

- $(E(g))^{(n-1)}(x) \neq \emptyset$ *for every* $x \in X$, *but* $g^{(n)} \neq g^{(n+1)}$.

Indeed, one can define $g_x$ by letting $g_x(y) = 0$ if $y \in F(x)$, and $g_x(y) = +\infty$ if $y \notin F(x)$.

When the functions $g_x$ are convex, i.e. the sets $E(g)(x)$ are convex, then Brown's result applied to a multimap $x \mapsto E(g)(x)$ says that $g^{(n+2)} = g^{(n+3)}$, and if $(E(g))^{(n+1)}(x) \neq \emptyset$ for every $x \in X$, then $g^{(n+1)} = g^{(n+2)}$. However, one may expect the equality $g^{(n+1)} = g^{(n+2)}$ should hold true, since the epigraphs are not arbitrary, but rather special convex sets in $\mathbb{R}^n \times \mathbb{R}$. This equality is true indeed, even for the system of *quasi-convex* functions, i.e. for functions with convex *sublevel sets* $L(g;\alpha)(x) = \{y \in \mathbb{R}^n : g_x(y) \le \alpha\}$. The proof can be done by applying Brown's result to the iterates of a multimap $L(g;\alpha)\colon x \mapsto L(g;\alpha)(x)$, and with the help of generalized Wets formula for sublevel sets of epigraphical limits:

$$L(g;\alpha)^{(k)}(x) = L(g^{(k)};\alpha)(x) = \bigcap_{\beta>\alpha} L(g;\beta)^{(k)}(x).$$

Let us give an elementary example showing that even for convex functions we can have $(E(g))^{(n+1)}(x) \neq \emptyset$ for every $x \in X$, as well as $g^{(n)} \neq g^{(n+1)}$.

**Example 5.8.** For $x \in [0,1]$ we define $g_x\colon \mathbb{R}^1 \to \mathbb{R}$ by letting

$$g_x(y) = \begin{cases} \max\left\{\dfrac{y}{x}, 0\right\} & \text{if } \dfrac{1}{x} \in \mathbb{N}, \\ -1 & \text{if } x = 0, \\ \max\left\{2 - \dfrac{y}{x}, 0\right\} & \text{if } \dfrac{1}{x} \notin \mathbb{N}. \end{cases}$$

We have

$$g_0(0) = -1 < g_0^{(1)}(0) = 0 < g_0^{(2)}(0) = 1 < \inf_{W\in U(0)} \sup_{x'\in W} g_{x'}(0) = 2.$$

In contrast to this example, for convex functions $g_x\colon S \to \overline{\mathbb{R}}$, which are uniformly bounded on convex set with nonempty relative interior, there must hold $g^{(1)} = g^{(2)}$ ([57]).

The properties we have just discussed together with Michael–Brown Theorem lead in particular to the following result (that can be easily restated as a result on continuous selection of "almost minimizers").

**Theorem 5.9** ([58])**.** *Let $X$ be a paracompact space, $S$ be a convex subset of $\mathbb{R}^n$, and assume that the functions $g_x\colon S \to \overline{\mathbb{R}}$ are proper, lower semicontinuous and quasi-convex. There exists a continuous map $f\colon X \to S$ such that $g_x(f(x)) < 0$ for $x \in X$ if and only if $\inf g_x^{(n)} < 0$ for $x \in X$. If the functions $g_x$ are uniformly bounded and convex the same statement is true with the condition $\inf g_x^{(1)} < 0$ for $x \in X$.*

Let us turn back to lower limit iterates $F^{(k)}$, $k \in \mathbb{N}$, of arbitrary convex valued multimap and discuss general conditions under which they are constant starting from $k = 1$. Recall that the equality $F = F^{(1)} = F_0$ is equivalent to the lower semicontinuity of a multimap $F\colon X \multimap Y$.

Assume further that $F$ has closed values and for $\varepsilon > 0$, $x \in X$, define:

$$F_\varepsilon(x) = \bigcup_{U\in\mathcal{U}(x)} \bigcap_{x'\in U} B(F(x');\varepsilon).$$

Then

$$F_0(x) = \bigcap_{\varepsilon>0} F_\varepsilon(x),$$

and the lower semicontinuity of $F$ is also equivalent to the condition: $F_\varepsilon(x) = B(F(x);\varepsilon)$ for every $x \in X$, $\varepsilon > 0$.

The set of continuous $\varepsilon$-selections of $F$ is the set

$$C_F^\varepsilon = \{f\colon X \to Y \text{ continuous} : f(x) \in B(F(x);\varepsilon) \text{ for } x \in X\}.$$

A multimap $F$ will be called *almost lower semicontinuous* (a.l.s.c.) (see [20]) if $F_\varepsilon(x) \neq \emptyset$ for all $\varepsilon > 0$, $x \in X$. Certainly almost lower semicontinuity is necessary for the existence of continuous selection. But almost lower semicontinuity is not sufficient even for the existence of Borel selection for a multimap $F\colon [0,1] \multimap \mathbb{R}^2$ with convex closed values (see [7, Example 2]). Deutsch and Kenderov extracted from Michael Theorem the following property of a multimap with convex values:

- $C_F^\varepsilon \neq \emptyset$ *for every* $\varepsilon > 0$ *if and only if* $F$ *is a.l.s.c.*

Thanks to that the relations $F_\varepsilon$ have open lower sections, i.e. the sets $F_\varepsilon^-(y)$, $y \in Y$, are open, it is not hard to establish the following properties hold true for $x \in X$, $\varepsilon, \delta > 0$ and arbitrary continuous map $f\colon X \to Y$ (see [48], [57]):

(1) $C_{F_\varepsilon} \subseteq C_F^\varepsilon \subseteq C_{F_\delta}$ for $\varepsilon < \delta$,
(2) $\bigcap_{\varepsilon>0} C_{F_\varepsilon} = \bigcap_{\varepsilon>0} C_F^\varepsilon = C_F$,
(3) $C_{F_\varepsilon}(x) = F_\varepsilon(x)$,
(4) $d(f(x), C_F^\varepsilon(x)) = d(f(x), F_\varepsilon(x))$,
(5) $d_{\sup}(f, C_F^\varepsilon) = \sup_x d(f(x), F_\varepsilon(x))$,
(6) $D(C_F^\delta(x), C_F^\varepsilon(x)) = D(F_\delta(x), F_\varepsilon(x))$,
(7) $D_{\sup}(C_F^\delta, C_F^\varepsilon) = \sup_x D(F_\delta(x), F_\varepsilon(x))$.

The idea to give a sufficient condition for $C_F \neq \emptyset$, by means of convergence of the sets $(F_\varepsilon(x))_{\varepsilon\searrow 0}$ in Hausdorff metric, locally uniformly with respect to $x \in X$, has appeared in [7]. Let us mention several notions of relaxed lower semicontinuity implying that "$F_0$ has nonempty values and is l.s.c." These notions are expressed by the properties of a multimap $F_\varepsilon$, and weaker than notions of lower semicontinuity considered by De Blasi and Myjak [19] Gutev [29] (see also Przesławski and Rybiński [47] and the book by Repovš and Semenov [49]). We list below three conditions guaranteeing that $C_F \neq \emptyset$ and determining the mode of convergence of the sets of continuous $\varepsilon$-selections $(C_F^\varepsilon\colon \varepsilon \searrow 0)$ to the set of continuous selections $C_F$:

(I) $\emptyset \neq F_\delta(x) \subseteq B(F_\mu(x); K\delta)$ for $\delta, \mu > 0$, $x \in X$ and some $K \geq 1$,
(II) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\emptyset \neq F_\delta(x) \subseteq B(F_\mu(x);\varepsilon)$ for $\mu > 0$, $x \in X$,

(III) for every $x \in X$, $y \in Y$, $\varepsilon > 0$ there exist a neighbourhood $U$ of $x$ and $\delta > 0$ such that for every $x' \in U$ there exists $r' > 0$, such that $\emptyset \neq B(y; r') \cap F_\delta(x') \subset B(F_\mu(x'); \varepsilon)$.

Certainly, if $F$ is l.s.c. then satisfies condition (I), since in this case $F_\delta(x) = B(F(x); \delta) \neq \emptyset$. Moreover, (I) $\Rightarrow$ (II) $\Rightarrow$ (III) and there are simple examples of multimaps $F: \mathbb{R} \multimap \mathbb{R}^2$ showing that in general (III) $\not\Rightarrow$ (II) $\not\Rightarrow$ (I). We give below only a sample of such multimaps.

**Example 5.10.** (a) A multimap $F_\alpha$ for $\alpha \in (0, \pi]$ defined by

$$F_\alpha(x) = \begin{cases} \{(s, 0) : 0 \leq s \leq 1\} & \text{if } x \text{ is irrational}, \\ \{(s, s \operatorname{tg} \alpha) : 0 \leq s \leq 1\} & \text{if } x \text{ is rational}, \end{cases}$$

satisfies condition (I) for $K \geq (\sin \frac{\alpha}{2})^{-1}$.

(b) A multimap $F$ defined by

$$F(x) = \begin{cases} \{(s, 0) : 0 \leq s \leq 1\} & \text{if } x \text{ is irrational}, \\ \{(s, t) : s^2 \leq t \leq 1\} & \text{if } x \text{ is rational}, \end{cases}$$

satisfies condition (II) and does not satisfies condition (I).

If a multimap $F$ satisfies condition (III), then it has the following property:

- *for every $x \in X$, $y \in Y$, $\varepsilon > 0$ there exist a neighbourhood $U$ of $x$ and $\delta > 0$ such that*

$$d(y, F_\delta(x')) \leq d(y, F_0(x')) < d(y, F_\delta(x')) + \varepsilon,$$

  *for $x' \in U$.*

Therefore we have the following conclusion.

**Corollary 5.11** ([57])**.** *Let $X$ be a topological space and $Y$ be a Banach space. If a multimap $F: X \multimap Y$ has closed values and satisfies condition* (III)*, then $F_0(x) \neq \emptyset$ for every $x \in X$ and the multimap $F_0$ is l.s.c.* (*hence $F^{(1)} = F^{(2)}$*)*.*

Invoking Michael Theorem we can extend retractive representation lemma.

**Lemma 5.12** ([57])**.** *Let $Y$ be a Banach space, $X$ be a paracompact and perfectly normal* (*e.g. metric*) *topological space. If a multimap $F: X \multimap Y$ has closed convex values, satisfies condition* (III) *and the multimaps $x \mapsto F_\varepsilon(x)$, $\varepsilon \geq 0$, are $W$-u.s.c. then for every $L > 1$ the multimap $(x, \varepsilon) \mapsto F_\varepsilon(x)$ has a retractive representation* $(Y \times [0, \infty), f_L)$ *such that*

$$\|f_L(x, y, \varepsilon) - y\| \leq L d(y, F_\varepsilon(x)).$$

Let us finally characterize the convergence of the sets of continuous $\varepsilon$-selections $C_F^\varepsilon$ under each of the hypothesis (I)–(III).

**Theorem 5.13** ([57]). *Let $X$ be a paracompact space,$Y$ be a Banach space and $F\colon X \multimap Y$ be a multimap with closed convex values such that $F(x) \neq Y$ for some $x \in X$. If $F$ satisfies condition* (III)*, then $C_F \neq \emptyset$ and for every continuous map $f\colon X \to Y$, $x \in X$, $\gamma > 0$, there exist a neighbourhood $U$ of $x$ and $\varepsilon > 0$ such that for $x' \in U$ there hold*

$$d(f(x'), C^{\varepsilon}_F(x')) \leq d(f(x'), C_F(x')) < d(f(x'), C^{\varepsilon}_F(x')) + \gamma.$$

*If $F$ satisfies condition* (II)*, then $\varepsilon \leq D(C^{\varepsilon}_F, C_F) \to 0$ as $\varepsilon \to 0$ and the function $\varepsilon \to D(C^{\varepsilon}_F, C_F)$ is locally Lipschitz in $(0,\infty)$. If $F$ satisfies condition* (I) *for $K \geq 1$, then $\varepsilon \leq D(C^{\varepsilon}_F, C_F) \leq K\varepsilon$.*

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# PART II

# COMMUNICATIONS

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# NONLINEAR EVOLUTION INCLUSIONS WITH CONSTRAINTS

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Abstract. The paper is aimed to introduce the reader into nonlinear evolutions equations governed by perturbations of accretive operators and applications of topological tools to that sort of problems. We consider differential inclusions of the form

$$(P) \qquad \begin{cases} \dot{u}(t) \in -Au(t) + F(t, u(t)), \\ u(t) \in M, \end{cases}$$

where $A\colon D(A) \multimap E$ is a $m$-accretive operator on a Banach space $E$, $F\colon [t_0, T] \times M \multimap E$ is an upper semicontinuous set-valued map with compact convex values, where $M \subset E$ is a closed subset of constraints. For the initial value problem associated to $(P)$ existence, continuity with respect to initial data and topological structure of solution set problems shall be studied. The results on the solution set structure shall be used in the construction of the topological degree for maps of the form $-A+F$ and applications of the degree to continuation and bifurcation of equilibria as well as branching of periodic points associated to $(P)$ shall be provided. Moreover, the obtained results shall be applied to a class of partial differential equations involving nonlinear diffusion operator.

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## 1. Introduction

**1.1. Presentation of main problems.** Inclusions with constraints of the form

$$(P) \qquad \begin{cases} 0 \in -Au + F(u), \\ u \in M, \end{cases}$$

where $M$ is a closed subset of a Banach space $E$, $A: D(A) \multimap E$ a $m$-accretive operator and $F: M \multimap E$ an upper semicontinuous map, are abstract formulations of many partial differential equations, inclusions and systems in which additional conditions on the state are imposed. One of typical examples is the reaction-diffusion equation

$$-\Delta\rho(u(x)) = f(x, u(x)), \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^N$, $\rho: \mathbb{R} \to \mathbb{R}$ and $f: \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$. And if, in this equation, $u(x)$ is interpreted as the concentration of some chemical at the point $x \in \Omega$, then it is natural to expect that $0 \le u(x) \le c_{\max}$ for a.e. $x \in \Omega$ where $c_{\max}$ is some maximal concentration of the chemical, in other words $u \in M$ where the constraint set is given by $M := \{v \in L^1(\Omega) \mid 0 \le v(x) \le c_{\max}$ for a.e. $x \in \Omega\}$. Observe that the set of constraints $M$ has empty interior (in the $L^1$-topology).

The existence of solutions for the constrained problem in the form $(P)$ has been studied by many authors who used various methods (see e.g. [34], [24], [8] or [3]). To address continuation and bifurcation problems for $(P)$ or the existence of periodic solutions and other related phenomena for differential equation (or inclusion) governed by $-A + F$, one usually has to employ proper (local) homotopy invariants such as topological degree. But, in the constrained problems of the form $(P)$, as we could see, constraint sets often have empty interiors, which does not allow to apply directly classical topological degrees. Topological degrees for nonlinear maps of this kind, in situations with no constraints, are usually defined as the Leray–Schauder degree of $I - (I + \lambda A)^{-1}(I + \lambda F)$ or $I - (I + \lambda)^{-1}F$, for $\lambda > 0$, as well as by an approximation scheme of the Galerkin type (e.g. [10], [22], [27], [26] and [20]). However, these methods fail in the presence of the constraint set $M$, even if $M$ is convex.

We construct a suitable homotopy invariant of the topological degree type for the class of maps having the form $-A+F$ on $M \cap D(A)$ where $A: D(A) \multimap E$ is a $m$-accretive operator such that $\dot{u} \in -Au$ generates a compact semigroup of contractions, $M \subset E$ is a closed set being resolvent invariant (i.e. $(I+\lambda A)^{-1}(M) \subset M$ for $\lambda > 0$) with some regularity property and $F: M \multimap E$ is compact convex valued, upper semicontinuous and tangent to $M$ in the Clarke sense, i.e. $F(x) \cap C_M(x) \neq \emptyset$ for $x \in M$, where $C_M(x)$ is the Clarke tangent cone.

In the paper we overcome geometric difficulties by using a different approach. This idea is consistent with the Krasnosel'skiĭ formula for the finite dimensional equation $\dot{u} = f(u)$, stating that the Brouwer degree of $-f$ is equal to the fixed point index of the translation along trajectories operator. Namely, roughly speaking, we shall consider the semiflow $\{\Phi_t\colon M\cap\overline{D(A)}\to M\cap\overline{D(A)}\}_{t\geq 0}$ generated by $\dot u\in -Au+F(u)$ (under our assumptions it is well-defined). It appears that if $U\subset M$ is open and bounded such that $-A+F$ has no zeros on the boundary $\partial_M U$ (of $U$ with respect to $M$), then, for sufficiently small $t>0$, the map $\Phi_t$ has no fixed points on the boundary $\partial_M U$. Then as the topological degree of $-A+F$ with respect to $U\subset M$ we take the fixed point index of $\Phi_t$. It has all the expected properties and it can be effectively applied to differential inclusions of the type $\dot u\in -Au+F(t,u)$ in studying such problems as continuation and bifurcation of equilibria as well as branching of periodic points.

Nevertheless, in order to carry out this program of studying periodic problem and construction of the topological degree, we have to use a version of the fixed point index for set-valued maps $\Phi_t$, which admit a decomposition $\Phi_t=e_t\circ L$, where $L$ is the solution operator for $(P)$, i.e. assigning to each $x\in M$ the set $L(x)$ of solutions for $(P)$ starting from $x$, and $e_t$ is the evaluation at time $t$. But, usually theories of fixed point indices require that $L$ should have properly regular values. If $M=E$, then it is sufficient that $L(x)$ is acyclic for each $x\in E$, but, if $M$ is an ANR, then some additional regularity is needed. The proper assumption on $L$ is to require that $L(x)$ is a *cell-like* set (see the definition in Section 4). Therefore, it is of great importance to assure the proper regularity of the set of solutions for $(P)$.

It should be stressed that the main intention of the author is to present the results as well as ideas and techniques used while studying nonlinear inclusions (with or without constraints). So as to fit the material in a reasonable capacity, some proofs, which are either classical or not related with our approach or exceed the scope of the paper, have been omitted (nevertheless in such situations proper references to proofs are indicated).

**1.2. Notation.** Let $(X,d)$ be a metric space. If $B\subset A\subset X$, then $\partial_A B$ denotes the boundary of $B$ in $(A,d_{|A})$ where $d_{|A}$ is the metric induced from $(X,d)$. *The distance function* $d_A\colon X\to[0,\infty)$ from the set $A\subset X$ is defined by $d_A(x)=d(x,A):=\inf\{d(x,y)\mid y\in A\}$. If $x\in X$ and $r>0$, then we put $B_A(x,r):=\{y\in A\mid d(x,y)<r\}$, $D_A(x,r):=\{y\in A\mid d(x,y)\leq r\}$, $B(A,r):=\{y\in X\mid d_A(y)<r\}$ and $D(A,r):=\{y\in X\mid d_A(y)\leq r\}$. If $(X_n)$ is a sequence of subsets of $X$, then we consider the inferior and superior limits, given by

$$\operatorname*{Liminf}_{n\to\infty} X_n:=\Big\{x\in X\ \Big|\ \lim_{n\to\infty} d(x,X_n)=0\Big\}$$

and

$$\operatorname*{Limsup}_{n\to\infty} X_n:=\Big\{x\in X\ \Big|\ \liminf_{n\to\infty} d(x,X_n)=0\Big\},$$

respectively.

Let $E$ be a normed space and $M \subset E$. By $\operatorname{conv} M$ and $\overline{\operatorname{conv}} M$ one denotes the convex envelope and the closed convex envelope of $M$, respectively. In problems with constraints various tangent cones enable to express conditions assuring existence of solutions in a set of constraint. Here we shall use the Bouligand cone and the Clarke one. The Bouligand tangent cone to $M$ at a point $x \in M$ is given by

$$T_M(x) := \left\{ u \in E \;\middle|\; \liminf_{h \to 0^+} \frac{d_M(x+hu)}{h} = 0 \right\}.$$

It is clearly a closed cone (but in general nonconvex). The Clarke tangent cone to $M$ at $x \in M$ is given by

$$C_M(x) := \left\{ u \in E \;\middle|\; \lim_{y \xrightarrow{M} x,\, h \to 0^+} \frac{d_M(y+hu)}{h} = 0 \right\}.$$

$C_M(x)$ is a closed convex cone and, in general, $C_M(x) \subset T_M(x)$ and

$$\operatorname*{Liminf}_{y \xrightarrow{M} x} T_M(x) \subset C_M(x). \tag{1.1}$$

For more details and information concerning tangent cones we refer to [2].

## 2. Accretive operators, differential inclusions and semigroups of contractions

**2.1. Accretive operators.** A set-valued map $A\colon D(A) \multimap E$, where $E$ is a Banach space and $D(A) \subset E$, is called *an accretive operator* if, for any $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ and $\lambda > 0$,

$$\|x - y\| \le \|x - y + \lambda(u - v)\|.$$

If, additionally, $R(I + A) = E$, where $R(I + A) := \bigcup_{x \in D(A)} (I + A)x$, then $A$ is called *$m$-accretive*.

**Remark 2.1.** (a) It is easy to observe that if $E$ is a Hilbert space with the scalar product $\langle \cdot\,, \cdot \rangle$, then $A$ is accretive if, and only if, it is monotone, i.e. $\langle x - y, u - v \rangle \ge 0$ for any $x, y \in D(A)$, $u \in Ax$ and $v \in Ay$.

(b) If $E$ is a Hilbert space and $A$ is linear, then the accretivity coincides with the positivity of $A$.

(c) If $R(I + \lambda_0 A) = E$ for some $\lambda_0 > 0$, then $R(I + \lambda A) = E$ for all $\lambda > 0$ (see e.g. [36]).

(d) If $E$ is reflexive and $A\colon D(A) \multimap E$ is $m$-accretive, then the set $\overline{D(A)}$ is convex (see e.g. [23]).

If $A\colon D(A) \multimap E$ is an accretive operator, then, for $\lambda > 0$, one defines an operator $J_\lambda\colon R(I + \lambda A) \to E$ by $J_\lambda u = J_\lambda^A u := (I + \lambda A)^{-1} u$. It is called *the resolvent of* $A$. It follows straightforward from the definition of accretive operators that $J_\lambda$ is a well-defined single-valued operator.

**Proposition 2.2** (e.g. [37], [23]). *If $A: D(A) \multimap E$ is $m$-accretive operator, then:*

(a) $\|J_\lambda u - J_\lambda v\| \le \|u - v\|$ *for* $u, v \in E$ *and* $\lambda > 0$;
(b) $\lim_{\lambda \to 0^+} J_\lambda u = u$ *for* $u \in \overline{D(A)}$;
(d) $J_\mu = J_\lambda((\lambda/\mu)I + (1 - \lambda/\mu)J_\mu)$ *for* $\mu, \lambda > 0$;
(d) $A_\lambda u \in AJ_\lambda u$, *for any* $\lambda > 0$ *and* $u \in E$, *where* $A_\lambda := \lambda^{-1}(I - J_\lambda)$.

To get the better feeling of accretivity, we provide some facts on scalar semiproducts in arbitrary Banach spaces. It appears useful in handling accretive operators. The *duality* $\langle\,\cdot\,,\,\cdot\,\rangle: E^* \times E \to \mathbb{R}$ in a Banach space $E$ is given by $\langle p, u\rangle := p(u)$ for $p \in E^*$ and $u \in E$, and *the duality map* $J: E \multimap E^*$ is given by $J(x) := \{p \in E^* \mid \langle p, x\rangle = \|p\|^2 = \|x\|^2\}$, for $x \in E$. The *semiproducts* $\langle\,\cdot\,,\,\cdot\,\rangle_+: E \times E \to \mathbb{R}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_-: E \times E \to \mathbb{R}$ are defined by the formulae $\langle x, y\rangle_+ := \sup_{p \in J(x)} \langle p, y\rangle$, $\langle x, y\rangle_- := \inf_{p \in J(x)} \langle p, y\rangle$, $x, y \in E$. They admit the following properties:

**Proposition 2.3** ([34]). *If $x, y, z \in E$, then:*

(a) $\langle x, y\rangle_+ \le \|x\|\|y\|$;
(b) $\langle x, y\rangle_+ = -\langle x, -y\rangle_- = -\langle -x, y\rangle_-$, $\langle x, x\rangle_+ = \langle x, x\rangle_- = \|x\|^2$;
(c) $\langle ax, by\rangle_+ = ab\langle x, y\rangle_+$ *for* $a, b > 0$;
(d) $\langle x, y + ax\rangle_+ = \langle x, y\rangle_+ + a\|x\|^2$ *for* $a \in \mathbb{R}$;
(e) $\langle x, y + z\rangle_+ \le \langle x, y\rangle_+ + \langle x, z\rangle_+$;
(f) $\langle x, y + z\rangle_- \le \langle x, y\rangle_- + \langle x, z\rangle_+$;
(g) *The function* $\langle\,\cdot\,,\,\cdot\,\rangle_+: E \times E \to \mathbb{R}$ (*resp.* $\langle\,\cdot\,,\,\cdot\,\rangle_-: E \times E \to \mathbb{R}$) *is upper semicontinuous* (*resp. lower semicontinuous*).

**Lemma 2.4.** *If $u_n \to u$ in $C([t_0, T], E)$ and $w_n \to w$ in $L^1([t_0, T], E)$, then*

$$\limsup_{n \to \infty} \int_{t_0}^T \langle u_n(\tau), w_n(\tau)\rangle_+ \, d\tau \le \int_{t_0}^T \langle u(\tau), w(\tau)\rangle_+ \, d\tau.$$

The semiproducts allow to express accretivity in a sometimes more convenient form, for instance, one may prove that $A: D(A) \multimap E$ is accretive if, and only if,

$$\langle y_1 - y_2, v_1 - v_2\rangle_+ \ge 0,$$

for any $(y_1, v_1), (y_2, v_2) \in \mathrm{Gr}(A) := \{(y, v) \in E \times E \mid y \in D(A),\ v \in Ay\}$ (see e.g. [36] or [23]).

Moreover, it appears that $m$-accretive operators are maximal one among all accretive operators in the sense of their graphs.

**Proposition 2.5** (see [36] or [23]). *Let $A: D(A) \multimap E$ be a $m$-accretive operator. If for some pair $(x, u) \in E \times E$ one has $\langle x - y, u - v\rangle_+ \ge 0$ for any $(y, v) \in \mathrm{Gr}(A)$, then $(x, u) \in \mathrm{Gr}(A)$.*

Observe that, in view of Proposition 2.5 and Proposition 2.3(g), for any $m$-accretive operator $A: D(A) \multimap E$, the graph $\mathrm{Gr}(A)$ is closed in $E \times E$.

For more on accretive operators the reader is referred to [4], [32] or [36].

**2.2. Semigroups and solution operator.** A family of maps $\{S(t)\colon D \to D\}_{t\geq 0}$, where $D \subset E$, is called a *semigroup of contractions* provided that

- $S(0) = \mathrm{id}_D$;
- $S(t)S(s) = S(t+s)$ for any $t, s \geq 0$;
- $\|S(t)x - S(t)y\| \leq \|x - y\|$ for any $x, y \in D$ and $t \geq 0$;
- $\lim_{t\to 0^+} S(t)x = x$.

The notion of semigroup is strictly related to accretive operators, which is subject to our further study. Namely, consider the following differential inclusion

$$\begin{cases} \dot u(t) \in -Au(t), & t > 0, \\ u(0) = x, \end{cases} \tag{2.1}$$

where $A\colon D(A) \multimap E$ is a $m$-accretive operator and $x \in \overline{D(A)}$. In general, without additional assumptions, the inclusion (2.1) may possess no pointwise (or almost everywhere pointwise) solutions. Therefore, another notion of solution is introduced (in more general setting).

**Definition 2.6** (see [4] or [36]). Let $A\colon D(A) \multimap E$ be a $m$-accretive operator, $w \in L^1([t_0, T], E)$ and $x_0 \in \overline{D(A)}$. A continuous function $u\colon [t_0, T] \to E$ is said to be an *integral solution* of the problem

$$\begin{cases} \dot u(t) \in -Au(t) + w(t), & t \in [t_0, T], \\ u(t_0) = x_0, \end{cases}$$

if and only if $u(t_0) = x_0$, $u(t) \in \overline{D(A)}$, for all $t \in [t_0, T]$, and

$$\|u(t) - y\|^2 \leq \|u(s) - y\|^2 + 2\int_s^t \langle u(\tau) - y, w(\tau) - v\rangle_+ \, d\tau$$

for any $t_0 \leq s < t \leq T$ and $(y, v) \in \mathrm{Gr}\,(A)$.

**Remark 2.7.** (a) If a linear operator $A\colon D(A) \to E$ is such that $-A$ is a generator of a $C_0$ semigroup of contractions, then every mild solution of $\dot u(t) = -Au(t) + w(t)$ is an integral solution, in the sense of Definition 2.6.

(b) One may show that if $u$ is an integral solution of $\dot u(t) \in -Au(t) + w(t)$, then, for any $(y, u) \in \mathrm{Gr}(A)$ and $t_0 \leq s < t \leq T$, one has

$$\|u(t) - y\| \leq \|u(s) - y\| + \int_s^t \|w(\tau) - v\| \, d\tau.$$

**Proposition 2.8** (see e.g. [37] or [23]). *Let $A\colon D(A) \multimap E$ be such that $A - \omega I$ is $m$-accretive for some $\omega \geq 0$. Then for any $x \in \overline{D(A)}$ and $w \in L^1([t_0, T], E)$ the problem*

$$\begin{cases} \dot u(t) \in -Au(t) + w(t) & \textit{on } [t_0, T], \\ u(t_0) = x, \end{cases} \tag{$P_{A,w,x}$}$$

*has a unique integral solution* $u = \Sigma_A(x,w)\colon [t_0,T] \to E$. *Moreover, for any* $x_1, x_2 \in \overline{D(A)}$, $w_1, w_2 \in L^1([t_0,T],E)$ *and all* $t_0 \le s < t \le T$,

$$\begin{aligned}\|\Sigma_A(x_1,w_1)(t) - \Sigma_A(x_2,w_2)(t)\| \\ \le e^{-\omega(t-s)}\|\Sigma_A(x_1,w_1)(s) - \Sigma_A(x_2,w_2)(s)\| \\ + \int_s^t e^{-\omega(t-\tau)}\|w_1(\tau) - w_2(\tau)\|\,d\tau.\end{aligned}$$

In particular, if $w := 0$, $t_0 := 0$ and $T := +\infty$, by Proposition 2.8, for every operator $A\colon D(A) \multimap E$ with $A - wI$ $m$-accertive, the differential problem $\dot u \in -Au$ determines the semigroup of contractions $S_A(t)\colon \overline{D(A)} \to \overline{D(A)}$ (with the constants $e^{-\omega t}$). This semigroup (generated by $A$) is denoted by $\{S_A(t)\}_{t\ge 0}$ or $S_A$.

**Remark 2.9.** If a constant map $u \equiv x_0$ is an integral solution of $\dot u \in -Au + v_0$ on $[t_0,T]$, for some $v_0 \in E$, then $x_0 \in D(A)$ and $0 \in -Ax_0 + v_0$. Indeed, since $u \equiv x_0$ is an integral solution, one has, for $t \in [t_0,T]$,

$$0 \le \int_{t_0}^t \langle x_0 - y, v_0 - v\rangle_+\,d\tau = (t-t_0)\langle x_0 - y, v_0 - v\rangle_+$$

for all $(y,v) \in \mathrm{Gr}(A)$ and $t \in [t_0,T]$, which, in view of Proposition 2.5, gives $(x_0,v_0) \in \mathrm{Gr}(A)$.

Below the notion convergence for $m$-accretive operators is briefly introduced. It enables to consider homotopy in the family of perturbations of $m$-accretive operators with the accretive part changing.

**Definition 2.10** (see e.g.[36] or [24])**.** A sequence $(A_n\colon D(A_n) \multimap E)_{n\ge1}$ of $m$-accretive operators is said to be *$G$-convergent* (or *graph convergent*) to a $m$-accretive operator $A\colon D(A) \multimap E$, which is written as $A_n \xrightarrow{G} A$, if

$$\mathrm{Gr}\,(A) \subset \mathrm{Liminf}_{n\to\infty}\mathrm{Gr}\,(A_n).$$

**Remark 2.11** (see [24])**.** The graph-convergence can be also characterized as follows: $A_n \xrightarrow{G} A$ *if, and only if, for each* $\mu > 0$ $J_\mu^{A_n}u \to J_\mu^A u$ *for all* $u \in E$. Moreover, one may show that if there exists at least one $\mu_0 > 0$ such that $J_{\mu_0}^{A_n}u \to J_{\mu_0}^A u$, for any $u \in E$, then $A_n \xrightarrow{G} A$.

**Example 2.12** (see [13])**.** (a) Let $A\colon D(A) \multimap E$ be a $m$-accretive operator. If $q \ge 0$ is a fixed parameter and $\lambda_n \to \lambda_0 > 0$, then $qI + \lambda_n A \xrightarrow{G} qI + \lambda_0 A$.

(b) Let $A$ and $B$ be $m$-accretive operators such that $D(A) = D(B)$ and the operator $C(\lambda) := \lambda A + (1-\lambda)B$ is $m$-accretive, for all $\lambda \in [0,1]$, then $C(\lambda_n) \xrightarrow{G} C(\lambda_0)$ if $\lambda_n \to \lambda_0$.

The solution operator has the following continuity properties.

**Proposition 2.13** (see [36]). *If $A_n \xrightarrow{G} A$, $x_n \to x$, where $x_n \in \overline{D(A_n)}$, for $n \geq 1$, $x \in \overline{D(A)}$, and $w_n \to w$ in $L^1([t_0,T],E)$, then $\Sigma_{A_n}(x_n,w_n)(t) \to \Sigma_A(x,w)(t)$ uniformly on $[t_0,T]$.*

**Proposition 2.14.** *If the dual space $E^*$ is uniformly convex, $A_n \xrightarrow{G} A$ and $x_n \to x$, where $x_n \in \overline{D(A_n)}$, for $n \geq 1$, $x \in \overline{D(A)}$, $w_n \rightharpoonup w$ weakly in $L^1([t_0,T],E)$, and $\Sigma_{A_n}(x_n,w_n) \to u$ in $C([t_0,T],E)$, then $u = \Sigma_A(x,w)$.*

**Remark 2.15.** If $E^*$ is uniformly convex, then $E$ is reflexive and the duality mapping $J$ is single-valued and uniformly continuous on bounded sets (Kato's theorem, see [4]). It is mainly these properties that make us assume the uniform convexity of $E^*$.

In the proof of Proposition 2.14, we need the following property of the semiproduct, which is a consequence of properties mentioned in Remark 2.15 (for the detailed proof see e.g. [37], also [13]).

**Lemma 2.16.** *If $E^*$ is uniformly convex, $u_n \to u$ in $C([t_0,T],E)$ and $w_n \rightharpoonup w$ weakly in $L^1([t_0,T],E)$, then*

$$\lim_{n\to\infty} \int_{t_0}^{T} \langle u_n(\tau), w_n(\tau)\rangle_+ \, d\tau = \int_{t_0}^{T} \langle u(\tau), w(\tau)\rangle_+ \, d\tau.$$

*Proof of Proposition* 2.14. Set $u_n := \Sigma_{A_n}(x_n,w_n)$ and take any $t_0 \leq s < t \leq T$ and $(y,v) \in \operatorname{Gr}(A)$. Then, by the definition of the $G$-convergence, there exist sequences $y_n \to y$ and $v_n \to v$ such that $v_n \in A_n y_n$ for $n \geq 1$. By the definition of integral solution

$$\|u_n(t) - y_n\|^2 \leq \|u_n(s) - y_n\|^2 + 2\int_s^t \langle u_n(\tau) - y_n, w_n(\tau) - v_n\rangle_+ \, d\tau.$$

Passing to the limits and using Lemma 2.16, one obtains

$$\|u_0(t) - y\|^2 \leq \|u_0(s) - y\|^2 + 2\int_s^t \langle u_0(\tau) - y, w_0(\tau) - v\rangle_+ \, d\tau,$$

which, by the uniqueness of integral solutions, implies $u = \Sigma_A(x,w)$. □

**2.3. Compactness properties of solution operator.** A family (or a sequence) $\mathcal{A}$ of $m$-accretive operators on a Banach space $E$ is called *relatively $G$-compact*, if any sequence of operators in $\mathcal{A}$ contains a subsequence $G$-convergent to some $m$-accretive operator. The following criterion is a general tool for checking compactness of solution sets for $(P_{A,w,x})$.

**Theorem 2.17** (see [13]). *Let a family $\mathcal{A}$ of $m$-accretive operators be relatively $G$-compact and such that, for any sequence $(A_n) \subset \mathcal{A}$ with $A_n \xrightarrow{G} A_0$, the* *equality holds*

$$\overline{D(A_0)} = \operatorname*{Limsup}_{n\to\infty} \overline{D(A_n)}. \ (^1) \tag{2.2}$$

*If the set $K \subset E$ is relatively compact and $W \subset L^1([t_0,T],E)$ is uniformly integrable* (2)*, then the following conditions are equivalent:*

(a) *the set*

$$\Sigma_{\mathcal{A}}(K\times W) := \bigcup_{A\in\mathcal{A}} \{\Sigma_A(x,w) \mid x \in K\cap \overline{D(A)}, w\in W\}$$

*is relatively compact in $C([t_0,T],E)$;*

(b) *there exists a dense set $P\subset [t_0,T]$ such that, for all $t\in P$,*

$$\Sigma_{\mathcal{A}}(K\times W)(t) := \bigcup_{A\in\mathcal{A}} \{\Sigma_A(x,w)(t) \mid x \in K\cap \overline{D(A)},\ w\in W\}$$

*is relatively compact in $E$.*

A semigroup $\{S(t)\colon D\to D\}_{t\ge 0}$, where $D$ is a closed subsets of a Banach space $E$, is called *compact*, if the set $S(t)(\Omega\cap D)$ is relatively compact for any $t>0$ and any bounded $\Omega\subset E$. A family (or a sequence) of semigroups $\mathcal{S} = \{S(t) : D_S \to D_S\}_{S\in\mathcal{S}}$, where $D_S$ are closed subsets of $E$, is called *compact*, if, for any $t>0$ and any bounded set $\Omega\subset E$, the set $\bigcup_{S\in\mathcal{S}} S(t)(\Omega\cap D_S)$ is relatively compact.

**Example 2.18.** Let $A\colon D(A) \multimap E$ be a $m$-accretive operator such that the semigroup $S_A$ is compact. The family $\mathcal{A} := \{\lambda A\}_{\lambda\in[\alpha,\beta]}$, where $0<\alpha<\beta$, is relatively $G$-compact and the family of semigroups $\mathcal{S} := \{S_{\lambda A}\colon \overline{D(A)}\to\overline{D(A)} \mid \lambda\in[\alpha,\beta]\}$ is compact. To see the $G$-compactness, observe that if $\lambda_n\to\lambda\in[\alpha,\beta]$, then, by the resolvent's properties, $J_\mu^{\lambda_n A}u = J^A_{\mu\lambda_n}u \to J^A_{\mu\lambda}u = J^{\lambda A}_\mu u$ for any $\mu>0$ and $u\in E$, which, in view of Remark 2.11, implies $\lambda_n A \xrightarrow{G} \lambda A$. To show the compactness of the family $\mathcal{S}$, observe that $S_{\lambda A}(t)x = S_A(\lambda t)x$ for any $\lambda>0$, $t\ge 0$ and $x\in\overline{D(A)}$. Hence, for any bounded $\Omega\subset E$ and $t>0$, one gets

$$\bigcup_{\lambda\in[\alpha,\beta]} S_{\lambda A}(t)(\Omega\cap\overline{D(A)}) = \bigcup_{\lambda\in[\alpha,\beta]} S_A(\lambda t)(\Omega\cap\overline{D(A)}) \subset S_A(\alpha t)(\Omega_0\cap\overline{D(A)}), \tag{2.3}$$

(1) Note that the inclusion $\overline{D(A_0)}\subset \operatorname{Liminf}_{n\to\infty}\overline{D(A_n)} \subset \operatorname{Limsup}_{n\to\infty}\overline{D(A_n)}$ follows from the definition of $G$-convergence. Hence, to verify (2.2) just the converse inclusion must be checked.

(2) A subset (or a sequence) $W\subset L^1([t_0,T],E)$ is said to be *uniformly integrable*, if for every $\varepsilon>0$ there is $\delta>0$ such that $\int_J \|w(\tau)\|\,d\tau<\varepsilon$ for any $J\subset[t_0,T]$ with the Lebesgue measure $\mu(J)<\delta$ and any $w\in W$. Moreover, $W\subset L^1([t_0,T],E)$ is said to be *integrally bounded*, if there exists $q\in L^1([t_0,T])$ such that, for all $w\in W$, $\|w(t)\|\le q(t)$ for a.e. $t\in[t_0,T]$. Observe that if $W$ is integrally bounded, then it is uniformly integrable.

where $\Omega_0 := \{S_A(s)x \mid x \in \Omega \cap \overline{D(A)},\ s \in [0,(\beta-\alpha)t]\}$. It is clear that $\Omega_0$ is bounded, and note that the relative compactness in (2.3) follows from the compactness of $S_A$.

The following compactness criterion is a direct consequence of Theorem 2.17.

**Proposition 2.19** (see [12])**.** *Let $(A_n)$ be a relatively $G$-compact sequence of $m$-accretive operators such that the corresponding sequence of semigroups $(S_{A_n})$ is compact and $\overline{D(A_n)} = \overline{D(A_m)}$ for any $n, m \geq 1$. Then:*

(a) *If $(x_n) \subset \overline{D(A_1)}$ is a bounded sequence, $(w_n) \subset L^1([t_0,T],E)$ is uniformly integrable, then, for any $t \in (t_0,T]$, the sequence $(\Sigma_{A_n}(x_n,w_n)(t))$ is relatively compact.*

(b) *If, additionally, $(x_n)$ is relatively compact, then the sequence of functions $(\Sigma_{A_n}(x_n,w_n))$ is relatively compact in $C([t_0,T],E)$.*

**2.4. Resolvent invariant sets.** A set $M \subset E$ is called *resolvent invariant* with respect to a $m$-accretive operator $A\colon D(A) \multimap E$ if, and only if,

$$J_\lambda^A(M) \subset M, \quad \text{for each } \lambda > 0.$$

**Remark 2.20.** (a) In general, if a closed $M \subset E$ is resolvent invariant, then $M$ is *invariant* with respect to the semigroup $S_A$, i.e.

$$S_A(t)(M \cap \overline{D(A)}) \subset M \cap \overline{D(A)} \quad \text{for any } t > 0.$$

It follows immediately from the Crandall–Liggett exponential formula saying that $S_A(t)x = \lim_{n\to\infty}(J_{t/n}^A)^n x$ for $x \in \overline{D(A)}$ and $t > 0$. Hence, the resolvent invariance, in the case when $w \equiv 0$ is sufficient, for existence of integral solutions of $\dot{u} \in -Au$ staying in $M$.

(b) If $A$ is such that $-A$ is a $m$-accretive generator of a $C_0$ semigroup of bounded linear operators and $M$ is closed and convex, then the converse implication is true, that is invariance with respect to semigroups implies resolvent invariance (see [32, Proposition VII.5.3]). For nonlinear $A$ some additional condition on the position of $M$ with respect to $\overline{D(A)}$ is needed (see [4, Chapter IV, Theorem 1.7]).

The following proposition collects basic facts concerning resolvent invariant sets.

**Proposition 2.21** (see [13])**.** *Let $A$ be a $m$-accretive operator.*

(a) *If $M \subset E$ is resolvent invariant, then $M_A := M \cap \overline{D(A)}$ is a retract of $M$.*

(b) *If $M$ is a resolvent invariant neighbourhood retract in $E$, then $M_A$ is a neighbourhood retract in $E$.*

(c) *If $M$ is a resolvent invariant neighbourhood retract in $E$, $M_A$ is bounded and the semigroup $\{S_A(t)\colon M_A \to M_A\}_{t\geq 0}$ is compact, then $\chi(M_A)$ is* *well-defined and* $\Lambda(S_A(t)|M_A) = \chi(M_A)$ *for any* $t > 0$, *where* $\Lambda(\,\cdot\,)$ *stands for the generalized Lefschetz number of a compact self-map of an absolute neighbourhood retract.*

**Example 2.22.** Suppose $H$ is a Hilbert space and $\varphi\colon H \to \mathbb{R}\cup\{\infty\}$ is a lower semicontinuous convex functional with the proper domain $D(\varphi) := \{x \in H \mid \varphi(x) < +\infty\} \neq \emptyset$. Define $Ax := \partial\,\varphi(x)$ for $x \in D(A) := \{x \in D(\varphi) \mid \partial\,\varphi(x) \neq \emptyset\}$ (see [37]). The operator $A$ is $m$-accretive and $\overline{D(A)} = D(\varphi)$ (see [4]). For any $a \in \mathbb{R}$, put $M_a := \{x \in D(\varphi) \mid \varphi(x) \leq a\}$. It is clear that $M_a$ are closed convex and invariant with resolvent invariant. Indeed, the convexity is immediate and the closeness is a consequence of lower semicontinuity. By the definition of subgradient and the inclusion $A_\lambda x \in AJ_\lambda x$ (see Proposition 2.2(d)), for any $x \in D(\varphi)$ and $\lambda > 0$, one has $\varphi(x) - \varphi(J_\lambda x) \geq \langle A_\lambda x, x - J_\lambda x\rangle = \lambda^{-1}\|x - J_\lambda x\|^2 \geq 0$. Hence, $\varphi(J_\lambda x) \leq \varphi(x)$, i.e. $J_\lambda(M_a) \subset M_a$, for any $\lambda > 0$ and $a \in \mathbb{R}$.

An example for the nonlinear diffusion operator is provided in Section 7.

## 3. Existence for constrained differential inclusions and properties of solution operators

Consider the inclusion

$$(P_{A,F,x}) \qquad \begin{cases} \dot u(t) \in -Au(t) + F(t,u(t)), \\ u(t) \in M_A := M \cap \overline{D(A)}, \\ u(t_0) = x, \end{cases}$$

where $A\colon D(A) \to E$ is $m$-accretive, $M \subset E$ is closed and $F\colon [t_0,T] \times M \multimap E$ is a set-valued map. A continuous function $u\colon [t_0,T] \to E$ is said to be *an integral solution* of $(P_{A,F,x})$ if, and only if, $u(t_0) = x$ and there exists a measurable a.e. selection $w\colon [t_0,T] \to E$ of the set-valued map $F(\,\cdot\,,u(\,\cdot\,))$ (i.e. $w(t) \in F(t,u(t))$ for a.e. $t \in [t_0,T]$) such that $\Sigma_A(x,w) = u$. The set of all integral solutions shall be denoted by $L(x,-A+F)$.

We divide this section into two parts, in the first one we consider the case when $F$ is a single-valued locally Lipschitz map with some other standard properties and in the second one, the case with set-valued upper semicontinuous $F$ with other necessary properties.

**3.1. Solution operator for inclusions with single-valued perturbations.** Due to our further needs, we shall deal from the very beginning with parameterized differential problem

$$(P_\lambda) \qquad \begin{cases} \dot u \in -A(\lambda)u + F(t,u,\lambda) \quad \text{for } \lambda \in \Lambda, \\ u(t) \in M, \end{cases}$$

where

- ($M_1$) the family $\{A(\lambda)\}_{\lambda\in\Lambda}$, with a compact metric space $\Lambda$, is $G$-continuous, i.e. $A(\lambda_n) \xrightarrow{G} A(\lambda_0)$ as $\lambda_n \to \lambda_0$;
- ($M_2$) $M \subset E$ is closed and resolvent invariant with respect to each $A(\lambda)$, $\lambda \in \Lambda$, i.e. $J_\mu^{A(\lambda)}(M) \subset M$ for any $\mu > 0$ and $\lambda \in \Lambda$;
- ($M_3$) the family of semigroups $\{S_{A(\lambda)}\}_{\lambda\in\Lambda}$ is compact,

and a continuous map $F: [t_0, T] \times M \times \Lambda \to E$ satisfies the following conditions:

- ($F_1$) for any $x \in M$ there are $\delta_x > 0$ and $L_x > 0$ such that
$$\|F(t, x_1, \lambda) - F(t, x_2, \lambda)\| \le L_x\|x_1 - x_2\|,$$
for any $x_1, x_2 \in B(x, \delta_x)$, $t \in [t_0, T]$ and $\lambda \in \Lambda$;
- ($F_2$) there exists $c \in L^1([t_0, T])$ such that $\|F(t, x, \lambda)\| \le c(t)(1 + \|x\|)$, for any $(t, x, \lambda) \in [t_0, T] \times M \times \Lambda$;
- ($F_3$) $F(t, x, \lambda) \in T_M(x)$ for any $(t, x, \lambda) \in [t_0, T] \times M \times \Lambda$.

**Theorem 3.1.**

- (P1) (Existence) *For any $\lambda \in \Lambda$ and $x \in M_\lambda := M \cap \overline{D(A(\lambda))}$, there exists a unique integral solution $u: [t_0, T] \to E$ of $(P_\lambda)$ with $u(t_0) = x$.*
- (P2) (Continuity) *The map $L: \bigcup_{\lambda\in\Lambda} M_\lambda \times \{\lambda\} \to C([t_0, T], E)$, given by*
$$L(x, \lambda) = L(x, -A(\lambda) + F(\,\cdot\,, \cdot\,, \lambda)) := u,$$
*where $u$ is the unique solution of $(P_\lambda)$ on $[t_0, T]$ with $u(a) = x$, is continuous. Moreover, for $s_n \to s_0$ in $[t_0, T]$, $x_n \to x_0$ in $M$ and $\lambda_n \to \lambda_0$ in $\Lambda$, if $u_n$ is the solution of $(P_{\lambda_n})$ on $[s_n, T]$ with $u_n(s_n) = x_n$, $n \ge 0$, then*
$$\sup_{t\in[\max\{s_n, s_0\}, T]} \|u_n(t) - u_0(t)\| \to 0 \textit{ as } n \to \infty.$$
- (P3) (Compactness) *Suppose the family $\{A(\lambda)\}_{\lambda\in\Lambda}$ has the additional property $\overline{D(A(\lambda_1))} = \overline{D(A(\lambda_2))}$ for any $\lambda_1, \lambda_2 \in \Lambda$. Then, for any $t \in (t_0, T]$, the translation along trajectories operator $\Phi_t: M_D \times \Lambda \to M_D$, with $M_D := M \cap \overline{D(A(\lambda))}$ for $\lambda \in \Lambda$, given by $\Phi_t(x, \lambda) := e_t(L(x, \lambda))$, where $e_t: C([t_0, T], M_D) \to M_D$ is the evaluation map, is compact, i.e. for any bounded $\Omega \subset M_D$ the set $\Phi_t(\Omega \times \Lambda)$ is relatively compact.*

**Remark 3.2.** (a) Fix $\lambda \in \Lambda$ and $x \in M_\lambda$. Suppose that $u \in C([t_0, T], E)$ is an integral solution of $(P_\lambda)$ with $u(t_0) = x$. Observe that, by Proposition 2.8, $\|u(t) - \Sigma_{A(\lambda)}(x, 0)(t)\| \le \int_{t_0}^t \|F(\tau, u(\tau), \lambda)\|\,d\tau$ and, consequently, by ($F_2$),
$$\begin{aligned}\|u(t)\| &\le \|S_{A(\lambda)}(t - t_0)x\| + \int_{t_0}^t c(\tau)(1 + \|u(\tau)\|)\,d\tau \\ &\le K_{x,\lambda} + \|c\|_{L^1} + \int_{t_0}^t c(\tau)\|u(\tau)\|\,d\tau\end{aligned}$$

where $K_{x,\lambda} := \max_{\tau\in[0,T-t_0]} \|S_{A(\lambda)}(\tau)x\|$. Hence, the Gronwall inequality provides the estimate

$$\|u(t)\| \le (K_{x,\lambda} + \|c\|_{L^1}) \exp\bigg(\int_{t_0}^{t} c(\tau)\,d\tau\bigg).$$

(b) We claim that, for any $R > 0$, $K_R := \sup\{K_{x,\lambda} \mid \lambda\in\Lambda,\ x\in\overline{D(A(\lambda))}\cap B(0,R)\} < \infty$. Indeed, fix $\lambda\in\Lambda$ and take any $x_\lambda\in\overline{D(A(\lambda))}$. By the definition of $G$-convergence and Proposition 2.13, there exists $0<\delta<1$ such that, for all $\mu\in B(\lambda,\delta)$, $B(x_\lambda,\delta)\cap\overline{D(A(\mu))}\ne\emptyset$ and $\max_{\tau\in[0,T-t_0]}\|S_{A(\mu)}(\tau)y - S_{A(\lambda)}(\tau)x_\lambda\|\le 1$ for any $y\in B(x_\lambda,\delta)\cap\overline{D(A(\mu))}$. Further, for any $\mu\in B(\lambda,\delta)$, $z\in\overline{D(A(\mu))}\cap B(0,R)$ and $\tau\in[0,T-t_0]$, one has

$$\begin{aligned}\|S_{A(\mu)}(\tau)z\| &\le \|S_{A(\mu)}(\tau)z - S_{A(\mu)}(\tau)y\| \\ &\qquad + \|S_{A(\mu)}(\tau)y - S_{A(\lambda)}(\tau)x_\lambda\| + \|S_{A(\lambda)}(\tau)x_\lambda\| \\ &\le \|z-y\| + 1 + \|S_{A(\lambda)}(\tau)x_\lambda\|\end{aligned}$$

where $y\in B(x_\lambda,\delta)\cap\overline{D(A(\mu))}$ is arbitrary. This implies

$$\begin{aligned}\sup\{K_{z,\mu} \mid \mu\in B(\lambda,\delta),\ z\in\overline{D(A(\mu))}\cap B(0,R)\}& \\ \le 2R+\delta+1+K_{x_\lambda,\lambda} < 2(R+1)+K_{x_\lambda,\lambda}&.\end{aligned}$$

Since $\Lambda$ is compact, we see that $K_R<\infty$.

(c) If one combines (a) and (b), then it is clear that the set of solutions of $(P_\lambda)$ starting from a bounded set is bounded.

*Proof of Theorem* 3.1. The part (P1) is proved in [8].

As for the part (P2), we show the short prove in the case $s_n\equiv t_0$ (in general case the prove is similar but more technical). Take any sequence of parameters $\lambda_n\to\lambda_0$ in $\Lambda$ and $(x_n)$ such that $x_n\in M_{\lambda_n}$, for $n\ge 1$, and $x_n\to x_0\in M_{\lambda_0}$. Let $u_n := L(x_n,\lambda_n)$, for $n\ge1$. In view of Remark3.2(c), $u_n$ is bounded in $C([t_0,T],E)$ and, by ($\mathrm{F}_2$), the sequence $(F(\,\cdot\,,u_n(\,\cdot\,),\lambda_n))$ is integrally bounded. Hence, in view of Proposition 2.19(b), each subsequence of $(u_n)$ contains a subsequence $(u_{k_l})$ convergent to some $u_0\in C([t_0,T],E)$. Then $w_{k_l} := F(\,\cdot\,,u_{k_l}(\,\cdot\,),\lambda_{k_l})\to w_0 := F(\,\cdot\,,u_0(\,\cdot\,),\lambda_0)$ in $L^1([t_0,T],E)$, which, due to Proposition 2.13, implies $u_{k_l}=\Sigma_{A(\lambda_{k_l})}(x_{k_l},w_{k_l})\to\Sigma_{A(\lambda_0)}(x_0,w_0)=u_0$. Summing up, it has been showed that, each subsequence of $(u_n)$ contains a subsequence convergent to $u_0=L(x_0,\lambda_0)$, which proves that $L(x_n,\lambda_n)\to L(x_0,\lambda_0)$.

To prove the part (P3), take a bounded sequence $(x_n)\subset\Omega$ and $(\lambda_n)\subset\Lambda$. Let, for any $n\ge1$, $u_n$ be the solution of $(P_{\lambda_n})$ with $u_n(t_0)=x_n$. In view of Remark 3.2(c),the sequence $(u_n)$ is bounded in $C([t_0,T],E)$ and, by ($\mathrm{F}_2$), the functions $F(\,\cdot\,,u_n(\,\cdot\,),\lambda_n)$ are integrally bounded, and, by Proposition 2.19(a), the sequence $(u_n(t))$ is relatively compact for any $t\in(t_0,T]$. □

**Remark 3.3.** (a) Under the above assumptions, the existence part (P1) is true without the assumption of the compactness of the semigroup (see [8]).

(b) Theorem 3.1 holds, if the assumptions ($\mathrm{M}_2$) and ($\mathrm{F}_3$) is replaced by the following weaker condition

$$F(t,x,\lambda) \in T_M^{A(\lambda)}(x) \quad \text{for any } (t,x,\lambda) \in [t_0,T] \times M_\lambda \times \Lambda$$

where for a $m$-accretive operator $A\colon D(A) \multimap E$

$$T_M^A(x) := \left\{ u \in E \;\middle|\; \liminf_{h\to 0^+} \frac{d_{M_A}(\Sigma_A(x,u)(h))}{h} = 0 \right\}$$

and $\Sigma_A(x,u)$ stands for the solution of $\dot{v} \in -Av + u$ with $v(0) = x$ (see [8]). Hence, some of the results in the sequel also remain true under this weaker condition.

**3.2. Solution operator for inclusions with set-valued perturbations.** We settle a set-valued version of Theorem 3.1. Consider $(P_\lambda)$ with $\{A(\lambda)\}_{\lambda\in\Lambda}$ and $M \subset E$ satisfying ($\mathrm{M}_1$)–($\mathrm{M}_3$) and a set-valued map $F\colon [t_0,T] \times M \times \Lambda \multimap E$ satisfying the following conditions:

- ($\widetilde{\mathrm{F}}_1$) $F$ is compact convex valued and upper semicontinuous,
- ($\widetilde{\mathrm{F}}_2$) there exists $c \in L^1([t_0,T])$ such that $\sup_{u\in F(t,x,\lambda)} \|u\| \le c(t)(1+\|x\|)$ for any $(t,x,\lambda) \in [t_0,T] \times M \times \Lambda$,
- ($\widetilde{\mathrm{F}}_3$) $F(t,x,\lambda) \cap T_M(x) \ne \emptyset$ for any $(t,x,\lambda) \in [t_0,T] \times M \times \Lambda$.

Moreover, if $F$ is single-valued, then there is no additional assumptions on $E$, otherwise, if $F$ is set-valued, then we assume that the dual $E^*$ is uniformly convex.

**Theorem 3.4.**

- (P1) (Existence) *For any $\lambda \in \Lambda$ and $x \in M_\lambda := M \cap \overline{D(A(\lambda))}$ there exists an integral solution $u\colon [t_0,T] \to E$ of $(P_\lambda)$ with $u(t_0) = x$.*
- (P2) (Continuity) *The map $L\colon \bigcup_{\lambda\in\Lambda} M_\lambda \times \{\lambda\} \multimap C([t_0,T],E)$, given by*

$$\begin{aligned} L(x,\lambda) &= L(x, -A(\lambda) + F(\,\cdot\,,\,\cdot\,,\lambda)) \\ &:= \{u \in C([t_0,T],E) \mid u \text{ is an integral solution of } (P_\lambda) \text{ and } u(t_0) = x\} \end{aligned}$$

  *is upper semicontinuous and has compact values.*
- (P3) (Compactness) *Suppose the family $\{A(\lambda)\}_{\lambda\in\Lambda}$ has the additional property $\overline{D(A(\lambda_1))} = \overline{D(A(\lambda_2))}$ for any $\lambda_1, \lambda_2 \in \Lambda$. Then, for any $t \in (t_0,T]$, the translation along trajectories operator $\Phi_t\colon M_D \times \Lambda \multimap M_D$, where $M_D := M \cap \overline{D(A(\lambda))}$, for $\lambda \in \Lambda$, given by $\Phi_t(x,\lambda) := e_t(L(x,\lambda))$, is compact, i.e. for any bounded $\Omega \subset M_D$, the set $\Phi_t(\Omega\times\Lambda)$ is relatively compact.*

**Remark 3.5.** (a) The assumption of the uniform convexity of $E^*$ allows to use Proposition 2.14. Moreover, if $E^*$ is uniformly convex, then, in particular, $E$ is reflexive, which means that bounded subsets of $E$ are relatively weakly compact.

(b) The conclusion of Remark 3.2(c) remains valid in the set-valued case with suitable adjustments.

(c) Observe that a set-valued map $\varphi: X \multimap Y$ is upper semicontinuous if and only if *for any* $x_n \to x_0$ *in* $X$ *and any* $(u_n) \subset Y$ *such that* $u_n \in \varphi(x_n)$, *for any* $n \geq 1$, *there is a subsequence* $u_{n_k} \to u_0 \in \varphi(x_0)$.

The proof of Theorem 3.4 is very similar to those for inclusions with bounded right hand sides. The following general rules are crucial for the proof of Theorem 3.4.

**Lemma 3.6** ($L^1$-compactness criterion, see [18])**.** *Let* $W \subset L^1([t_0,T],E)$ *be a uniformly integrable set such that there exists a family* $\{C(t)\}_{t\in[t_0,T]}$ *of relatively compact subsets of* $E$ *such that, for each* $w \in W$,

$$w(t) \in C(t) \text{ for a.e. } t \in [t_0,T].$$

*Then* $W$ *is relatively weakly compact in* $L^1([t_0,T],E)$.

**Lemma 3.7** (Convergence theorem, see [1])**.** *Suppose that:*

(a) $u_n \to u$ *in* $C([t_0,T],X)$ *where* $X$ *is a metric space,*
(b) $w_n \rightharpoonup w$ *weakly in* $L^1([t_0,T],E)$,
(c) $G: [t_0,T] \times X \multimap E$ *is an upper semicontinuous map with closed convex values,*
(d) *for any* $\varepsilon > 0$ *there exists* $n_0 \geq 1$ *such that, for each* $n \geq n_0$ *and a.e.* $t \in [t_0,T]$,

$$w_n(t) \in \overline{\mathrm{conv}}[G(([t-\varepsilon,t+\varepsilon] \cap [t_0,T]) \times B(u_n(t),\varepsilon)) + B(0,\varepsilon)].$$

*Then* $w(t) \in F(t,u(t))$ *for a.e.* $t \in [t_0,T]$.

*Proof of Theorem* 3.4. The part (P1) is proved in [8] (even under weaker assumptions).

To prove (P2) we shall show that, for any $x_n \to x_0 \in M_{\lambda_0}$, $\lambda_n \to \lambda_0 \in \Lambda$ and $u_n \in L(x_n,\lambda_n)$, $n \geq 1$, $(u_n)$ contains a subsequence convergent to some $u_0 \in L(x_0,\lambda_0)$ (see Remark 3.5(c)).

Consider the case when $F$ is set-valued (then $E^*$ is assumed to be uniformly convex). Since, $u_n$ are integral solutions, there exist integrable selections $w_n: [t_0,T] \to E$ of $F(\,\cdot\,,u_n(\,\cdot\,),\lambda_n)$. By $(\widetilde{F}_2)$ and the fact that there is $R > 0$ such that $\|u_n\| \leq R$ for $n \geq 1$ (see Remark 3.5), we see that $(w_n)$ is integrally bounded by $c(\,\cdot\,)(1+R)$. Hence, in view of Proposition 2.19(b), $(u_n)$ is relatively compact, i.e. it contains a subsequence convergent to some $u_0 \in C([t_0,T],E)$ with $u_0(t_0) = x_0$; furthermore, by Lemma 3.6, $(w_n)$ contains a subsequence weakly convergent in $L^1([t_0,T],E)$ to some $w_0$. Therefore passing to subsequences, if necessary, $u_n \to u_0$ and $w_n \rightharpoonup w_0$ weakly in $L^1([t_0,T],E)$. Finally, by use of Lemma 3.7 (applied to $(u_n,\lambda_n) \in C([t_0,T], M \times \Lambda)$, $(w_n,0) \in L^1([t_0,T], E \times \mathbb{R})$

and $G(t,(x,\lambda)) := (F(t,x,\lambda),0))$, we get $w_0 \in F(\,\cdot\,,u_0(\,\cdot\,),\lambda_0)$, which together with Proposition 2.14 implies that $u_0 \in L(x_0,\lambda_0)$.

If $F$ is single-valued, then the proof is analogical to the proof of the part (P2) of Theorem 3.1.

To see (P3) one needs to proceed exactly like in the proof of the part (P3) of Theorem 3.1. □

## 4. Topological structure of solution sets for constrained evolution problems

We shall study the structure of the set of all solutions of the inclusion

$$(P_{A,F,x_0}) \qquad \begin{cases} \dot u \in -Au + F(t,u), \\ u(t) \in M_A := M \cap \overline{D(A)}, \\ u(t_0) = x_0, \end{cases}$$

where $A\colon D(A) \multimap E$ is a $m$-accretive operator, $F\colon [t_0,T] \times M \multimap E$ is an upper semicontinuous with compact convex values and $x_0 \in \overline{D(A)}$. We intend to show that, under some general conditions, the set of solutions for $(P_{A,F,x_0})$ is a cell-like set (see for the definition below).

The regularity of solution set is a problem involving the geometry of both the mapping $F$ and the constraint set $M$. It appears that even in some finite dimensional problems with $A := 0$ solution sets of $(P_{A,F,x_0})$ are not connected or acyclic, that is neither cell-like (see examples in [29]). It is either the lack of proper regularity of the shape of $M$ or too weak tangency of $F$ with respect to $M$, which is the reason of that phenomenon. However, if $M$ is of a proper regularity and $F$ is tangent to $M$ in the sense of Clarke's cones (that is $F$ satisfies a more restrictive condition), then solution sets have the required structure.

**Definition 4.1.** A (nonempty) compact metric space $L$ is called *a cell-like set*, if there exists a metrizable ANR $X$ and an embedding $i\colon L \to X$ such that $i(L)$ is contractible in any neighbourhood of $i(L)$ in $X$ (or, equivalently, for any neighbourhood $U$ of $i(L)$, there exists a neighbourhood $V \subset U$ of the set $i(L)$ such that the set $V$ is contractible in $U$).

**Remark 4.2.** (a) The property of being cell-like is an absolute property, that is if $L$ is a cell-like set and $i'\colon L \to X'$ is an embedding into an $ANR$ $X'$, then $i'(L)$ is contractible in any its neighbourhood in $X'$.

(b) Being a cell-like set is a topological invariant (cf. [31]). One may also show that this property is even a homotopic invariant.

(c) A cell-like set is acyclic with respect to the Čech cohomology with integer coefficients (by the continuity property of the Čech cohomology functor).

**Example 4.3.** (a) By the definition, it is obvious, that contractible compact metric spaces are cell-like sets. Hence, in particular, compact convex subsets of normed spaces are cell-like.

(b) Let $X \subset Y$ be a compact subset of a metric space $Y$. $X$ is said to be of $R_\delta$ type if $X = \bigcap_{n\ge 1} X_n$ where $\{X_n\}_{n\ge 1}$ is a descending family of compact $AR$'s (absolute retracts). It is easy to see that sets of $R_\delta$ type are cell-like. And conversely, one may show that each compact cell-like set is of $R_\delta$ type (see [25]). Hence, for compact sets these two properties are equivalent.

(c) In practice, the following characterization of $R_\delta$ type sets is useful: If $(L_n)_{n\ge 1}$ is a sequence of closed and bounded subsets of a metric space $X$ such that $L_{n+1} \subset L_n$ for $n \ge 1$, $L_n$ is contractible, for $n \ge 1$, and $\beta(L_n) \to 0$, as $n \to \infty$, where $\beta$ denotes the Hausdorff measure of noncompactness, then $L := \bigcap_{n\ge 1} L_n$ is of $R_\delta$ type.

(d) If $(L_n)$ is a descending sequence of $R_\delta$ sets, then $\bigcap_{n\ge 1} L_n$ is also of $R_\delta$ type.

**4.1. Structure of solution set — convex constraints.** This is the ability of approximating $F$ with locally Lipschitz mappings inheriting the tangency to $M$, which plays the key rule in studying the structure of solutions sets. As one encounters different difficulties connected with approximations, each of the situations when $M$ is convex, $M$ is epi-Lipschitz and $M$ is a proximate retract are considered separately. Here, we assume that $F$ is upper semicontinous (jointly, i.e. with respect to both variables). The more general case when $F$ is just upper Caratheodory or $M$ is strictly regular sets, can be obtained by use of the techniques due to Bader and Kryszewski (see [3]) where $A$ was assumed to be a linear generator of $C_0$ semigroup. The adaptation to the nonlinear case is straightforward and the arguments from this section apply.

In the convex case we use the following approximation result .

**Lemma 4.4** ([28], [3]). *Let $M \subset E$ be a closed convex set in a Banach space $E$ and $\Omega$ be a metric space. If $F\colon \Omega \times M \multimap E$ is upper semicontinuous with closed convex values such that*

$$F(\omega, x) \cap T_M(x) \neq \emptyset \quad \text{for } (\omega, x) \in \Omega \times M,$$

*then, for any $\varepsilon > 0$, there exists a locally Lipschitz map $f_\varepsilon\colon \Omega \times M \to E$ such that, for any $\omega \in \Omega$ and $x \in M$,*

$$\begin{gathered} f_\varepsilon(\omega, x) \in \overline{\operatorname{conv}}[F(B(\omega,\varepsilon) \times B_M(x,\varepsilon)) + B(0,\varepsilon)], \\ f_\varepsilon(\omega, x) \in T_M(x). \end{gathered}$$

**Theorem 4.5.** *Let $A\colon D(A) \multimap E$ be a $m$-accretive operator such that the semigroup $S_A$ is compact and $M \subset E$ be a resolvent invariant closed convex set. If $F\colon [t_0, T] \times M \multimap E$ is an upper semicontinuous map with compact convex values and sublinear growth such that $F(t, x) \cap T_M(x) \neq \emptyset$, for each $(t, x) \in$*

*$[t_0,T]\times M$, then, for any $x_0\in M_A$, the set $L(x_0,-A+F)$ of solutions for $(P_{A,F,x_0})$ is cell-like in each of the following cases:*

(a) *$F$ is a single-valued map;*
(b) *$E^*$ is uniformly convex.*

*Proof. Step 1.* For any $n\geq 1$, by Lemma 4.4, there exists a locally Lipschitz $f_n\colon [t_0,T]\times M\to E$ such that, for each $t\in[t_0,T]$ and $x\in M$,

$$f_n(t,x)\in T_M(x), \tag{4.1}$$

$$f_n(t,x)\in F_n(t,x), \tag{4.2}$$

where $F_n(t,x):=\overline{\operatorname{conv}}[F(((t-1/n,t+1/n)\cap[t_0,T])\times B_M(x,1/n))+B(0,1/n)]$.

*Step 2.* We shall show that, for any $n\geq 1$, the set $\overline{L(x_0,-A+F_n)}$ is contractible.

By Theorem 3.1 (P1) and (4.1), for any $(s,x)\in[t_0,T]\times M_A$ the problem

$$(P_n)\qquad \begin{cases} \dot u\in -Au+f_n(t,u),\\ u(t)\in M_A,\\ u(s)=x,\end{cases}$$

has a unique solution $u(\,\cdot\,;s,x)\colon[s,b]\to M_A$. In view of (4.2), it is clear that the solution $u(\,\cdot\,;s,x_0)$ of $(P_n)$ belongs to $L(x_0,-A+F_n)$. Define $\widehat M:=\{u\in C([t_0,T],E)\mid u(t)\in M_A$ for any $t\in[t_0,T]\}$ and $H\colon\widehat M\times[t_0,T]\to\widehat M$ by

$$H(v,s)(t):=\begin{cases} v(t) & \text{for } t\in[t_0,s),\\ u(t;s,v(s)) & \text{for } t\in[s,T].\end{cases}$$

The map $H$ is continuous. Indeed, suppose that $v_m\to v$ and $s_m\to s$ as $m\to\infty$. Without loss of generality one may assume that either $s_m>s$ for all $m\geq 1$ or $s_m<s$ for all $m\geq 1$. In the first case, if $s_m>s$, then,

- for $t\in[t_0,s)$, one has $\|H(v_m,s_m)(t)-H(v,s)(t)\|=\|v_m(t)-v(t)\|\leq\|v_m-v\|\to 0$;
- for $t\in[s,s_m)$, by continuity,

$$\begin{aligned}\|H(v_m,s_m)(t)-H(v,s)(t)\|&=\|v_m(t)-u(t;s,v(s))\|\\ &\leq\|v_m-v\|+\sup_{\tau\in[s,s_m]}\|v(\tau)-v(s)\|+\sup_{\tau\in[s,s_m]}\|v(s)-u(\tau;s,v(s))\|\to 0;\end{aligned}$$

- for $t\in[s_m,T]$, by use of Theorem 3.1(P2),

$$\|H(v_m,s_m)(t)-H(v,s)(t)\|=\|u(t;s_m,v_m(s_m))-u(t;s,v(s))\|\to 0,$$

  since $v_m(s_m)\to v(s)$.

In the other case when $s_m<s$ for all $m\geq 1$, one has:

- for $t\in[t_0,s_m)$, $\|H(v_m,s_m)(t)-H(v,s)(t)\|=\|v_m(t)-v(t)\|\leq\|v_m-v\|\to 0$;

- for $t \in [s_m, s)$,

$$\begin{aligned}\|H(v_m,s_m)(t)-H(v,s)(t)\| &= \|u(t;s_m,v_m(s_m))-v(t)\| \\ &\le \sup_{t\in[s_m,s)} \|u(\tau;s_m,v_m(s_m))-v_m(s_m)\| \\ &\quad + \|v_m-v\| + \sup_{\tau\in[s_m,s)} \|v(s_m)-v(\tau)\|;\end{aligned}$$

- for $t \in [s,T]$, by Theorem 3.1(P2),

$$\|H(v_m,s_m)(t)-H(v,s)(t)\| \le \sup_{\tau\in[s,T]} \|u(\tau;s_m,v_m(s_m))-u(\tau;s,v(s))\| \to 0.$$

Hence $H$ is continuous.

Further observe that

(4.3) if $v \in L(x_0,-A+F_n)$, then $H(v,s) \in L(x_0,-A+F_n)$ for each $s \in [t_0,T]$.

Indeed, if $w(\,\cdot\,) \in F_n(\,\cdot\,,v(\,\cdot\,))$ is a measurable selection such that $\Sigma_A(x_0,w)=v$, then $H(s,v)=\Sigma_A(x_0,\overline{w})$ with $\overline{w} \in L^1([t_0,T],E)$ given by

$$\overline{w}(\tau) := \begin{cases} w(\tau) & \text{for } \tau \in [t_0,s) \\ f_n(\tau,u(\tau;s,v(s))), & \text{for } \tau \in [s,T]. \end{cases}$$

Since $\overline{w}$ is a selection of $F_n(\,\cdot\,,H(v,s)(\,\cdot\,))$, one has $H(v,s) \in L(x_0,-A+F_n)$.

It follows, by the continuity of $H$ and (4.3), that if $v \in \overline{L(x_0,-A+F_n)}$, then $H(v,s) \in \overline{L(x_0,-A+F_n)}$. Hence, the homotopy $H$ shows that $\overline{L(x_0,-A+F_n)}$ is contractible to $\{u(\,\cdot\,;t_0,x_0)\}$.

*Step 3.* We shall prove that, any sequence $(u_n)$ of integral solutions such that $u_n \in L(x_0,-A+F_n)$, $n \ge 1$, contains a subsequence convergent to some $u_0 \in L(x_0,-A+F)$.

First, we show that $(u_n)$ is relatively compact. Let $w_n\colon [t_0,T] \to E$ be a measurable selection $w_n(\,\cdot\,) \in F_n(\,\cdot\,,u_n(\,\cdot\,))$ such that $u_n = \Sigma_A(x_0,w_n)$ for $n \ge 1$. By (4.2) and Remark 3.5, there exists a constant $R>0$ such that $\|u_n(t)\| \le R$ for any $t \in [t_0,T]$ and $n \ge 1$. This and the sublinear growth condition implies that $(w_n)$ is integrally bounded. Hence, in view of Proposition 2.19(b) $(u_n)$ is relatively compact and, without loss of generality, we may assume that $(u_n)$ converges in $C([t_0,T],E)$ to some $u_0$. One needs to show that $u_0 \in L(x_0,-A+F)$.

In the case (a). It is easy to see that $w_n := f_n(\,\cdot\,,u_n(\,\cdot\,))$ converges in $C([t_0,T],E)$ to $w_0 := F(\,\cdot\,,u_0(\,\cdot\,))$, and, by the growth condition on $F$, $w_n \to w_0$ in $L^1([t_0,T],E)$. By Proposition 2.13, $u_0 = \Sigma_A(x_0,w_0)$, i.e. $u_0 \in L(x_0,-A+F)$.

In the case (b), in view of Lemma 3.6, one may assume that (passing to a subsequence) $w_n \rightharpoonup w_0$ (weakly in $L^1([t_0,T],E)$). Hence, by Lemma 3.7, one has

$$w_0(t) \in F(t,u_0(t)) \quad \text{for a.e. } t \in [t_0,T]$$

and, finally, in view of Proposition 2.14, one obtains $\Sigma_A(x_0,w_0)=u_0$, i.e. $u_0 \in L(x_0,-A+F)$.

*Step 4.* For any $n \geq 1$, $L(x_0, -A + F_{n+1}) \subset L(x_0, -A + F_n)$ and

$$L(x_0, -A + F) = \bigcap_{n\geq 1} \overline{L(x_0, -A + F_n)}.$$

Indeed, it follows from the definition of $F_n$ that, for $n \geq 1$, $F(t,x) \subset F_{n+1}(t,x) \subset F_n(t,x)$ and

$$L(x_0, -A + F) \subset L(x_0, -A + F_{n+1}) \subset L(x_0, -A + F_n)$$

and, consequently, $L(x_0, -A + F) \subset \bigcap_{n\geq 1} \overline{L(x_0, -A + F_n)}$. In order to prove the converse inclusion, take any $u \in \bigcap_{n\geq 1} \overline{L(x_0, -A + F_n)}$. Then there exists a sequence $(u_n)$ such that $u_n \in L(x_0, -A + F_n)$ and $\|u - u_n\| \leq 1/n$ for $n \geq 1$. By Step 3, we infer that $u \in L(x_0, -A + F)$, which proves the claim of Step 4.

*Step 5.* Finally, we shall show that

$$\beta(\overline{L(x_0, -A + F_n)}) \to 0 \quad \text{as } n \to \infty,$$

which, by Example 4.3(c), Steps 2 and 4, will imply that $L(x_0, -A + F)$ is cell-like.

By Step 3, $L(x_0, -A + F)$ is a nonempty compact set, therefore

$$\beta(\overline{L(x_0, -A + F_n)}) = \beta(L(x_0, -A + F_n)) \leq \sup_{u \in L(x_0, -A + F_n)} d(u, L(x_0, -A + F)).$$

For $n \geq 1$, choose $u_n \in L(x_0, -A + F_n)$ such that

$$\sup_{u \in L(x_0, -A + F_n)} d(u, L(x_0, -A + F)) \leq d(u_n, L(x_0, -A + F)) + 1/n.$$

By Step 3, without loss of generality, one may assume that the sequence $(u_n)$ converges to some $u_0 \in L(x_0, -A + F)$, which gives $\beta(L(x_0, -A + F_n)) \to 0$ as $n \to \infty$. $\square$

**Remark 4.6.** (a) Theorem 4.5 is, in particular, an existence theorem for set-valued $F$ (obtained independently of Theorem 3.4(P1).

(b) Theorem 4.5 is a generalization of results from [7], where some additional assumption on the convex set $M$ was imposed.

**4.2. Structure of solution set — epi-Lipschitz constraints.** We shall consider the case when $M$ is a set given by functional constraints satisfying some regularity assumption.

**Definition 4.7.** A closed set $M$ is said to be *epi-Lipschitz*, if there exist an open $V \subset E$ and a locally Lipschitz function $f\colon \overline{V} \to \mathbb{R}$ such that $M = \{x \in \overline{V} \mid f(x) \leq 0\}$, $\inf_{\partial V} f > 0$ and $0 \notin \partial f(x)$ (3) for any $x \in f^{-1}(0)$. The function $f$ is called a *representing function* for the set $M$.

(3) $\partial f(x) \subset E^*$ is a generalized gradient given by $\partial f(x) := \{p \in E^* \mid \langle p, u\rangle \leq f^\circ(x;u)$ for all $u \in E\}$ where $f^\circ(x;u) := \limsup_{y\to x,\, h\to 0^+} (f(y + hu) - f(y))/h$ is the directional derivative of $f$ at $x$ in the direction $u \in E$. For these notions and other basic elements of nonsmooth analysis the reader is referred to textbooks [11] or [2].

**Remark 4.8.** (a) For a closed set $M$ represented by locally Lipschitz constraint $f\colon \overline{V} \to \mathbb{R}$ it is natural to consider a tangent cone at $x \in M$ being the polar cone of the generalized gradient $\partial f(x)$, i.e. the cone given by

$$\partial f(x)^\circ = \{u \in E \mid \langle p, u\rangle \leq 0 \text{ for all } p \in \partial f(x)\} = \{u \in E \mid f^\circ(x;u) \leq 0\}.$$

In particular, if $f := d_M$, then $\partial d_M(x)^\circ = C_M(x)$ where $C_M(x)$ is the Clarke cone (see [11]).

(b) If $x \in M$, $f(x) = 0$ and $0 \notin \partial f(x)$, then $\partial f(x)^\circ \subset C_M(x)$ (see [11]).

A set-valued map $F\colon [t_0, T] \times M \multimap E$, where $M$ is an epi-Lipschitz set represented by $f$, is said to be tangent to $M$ with respect to $f$ if

$$F(t,x) \cap \partial f(x)^\circ \neq \emptyset \quad \text{for } (t,x) \in [t_0,T] \times \partial M. \tag{4.4}$$

As it was mentioned, to address the solution set structure problem one needs to approximate $F\colon [t_0,T] \times M \multimap E$ with locally Lipschitz maps satisfying the tangency condition. For mappings on epi-Lipschitz sets we use the following approximation method.

**Lemma 4.9** (see [3] and [28])**.** *If $M \subset E$ is epi-Lipschitz with a representing function $f\colon V \to \mathbb{R}$ and $F\colon [t_0,T] \times M \multimap E$ is an upper semicontinuous map with compact convex values, satisfying* (4.4)*, then, for any $\varepsilon > 0$, there exists a locally Lipschitz map $f_\varepsilon\colon [t_0,T] \times M \to E$ such that*

$$f_\varepsilon(t,x) \in \operatorname{conv}[F(I(t,\varepsilon) \times B_M(x,\varepsilon)) + B(0,\varepsilon)] \quad \textit{for } (t,x) \in [t_0,T] \times M,$$
$$f^\circ(x; f_\varepsilon(t,x)) < 0 \quad \textit{for } (t,x) \in [t_0,T] \times \partial M$$

*where $I(t,\varepsilon) := (t-\varepsilon, t+\varepsilon) \cap [t_0,T]$.*

**Theorem 4.10.** *Let $A\colon D(A) \multimap E$ an $m$-accretive operator, $M \subset E$ be a resolvent invariant epi-Lipschitz set represented by $f$. If an upper semicontinuous map $F\colon [t_0,T] \times M \multimap E$ with compact convex values, has sublinear growth and is tangent to $M$ in the sense of* (4.4)*, then for any $x_0 \in M_A$, the solution set $L(x_0, -A+F)$ for $(P_{A,F,x_0})$ is a cell-like set provided one of the conditions:* (a) *or* (b) *from Theorem* 4.5 *holds.*

*Proof.* can be carried out along the lines of the proof of Theorem 4.5, but instead of Lemma 4.4 one uses Lemma 4.9. Moreover, to show that the inclusion $(P_n)$ has solutions it is sufficient to observe that $f_n(t,x) \in \partial f(x)^\circ \subset C_M(x)$ for $t \in [t_0,T]$, $x \in \partial M$ (see Remark 4.8). □

**4.3. Structure of solution set — proximate retract constraints.** It was observed in [35] that so-called *proximate retracts* in $\mathbb{R}^n$ are sets, in which viable solutions of inclusions make a set of proper topological structure. This observation can be extended to inclusions in Banach spaces.

**Definition 4.11.** A closed set $M \subset E$ is *a proximate retract* provided there exists a continuous function $r\colon B(M,\rho) \to M$ with $\rho > 0$ such that

$$\|x - r(x)\| = d_M(x) \quad \text{for any } x \in B(M,\rho).$$

**Theorem 4.12.** *Let $A\colon D(A) \multimap E$ be a $m$-accretive operator and $M \subset E$ be a resolvent invariant proximate retract. If an upper semicontinuous map $F\colon [t_0,T] \times M \multimap E$ with compact convex values has sublinear growth and*

$$F(t,x) \cap C_M(x) \neq \emptyset \quad \textit{for } (t,x) \in [t_0,T] \times M, \ (^4)$$

*then, for any $x_0 \in M_A$, the solution set $L(x_0, -A+F)$ of $(P_{A,F,x_0})$ is a cell-like set provided one of the conditions* (a) *or* (b) *from Theorem* 4.5 *holds.*

**Lemma 4.13.** *The map $\overline{F}\colon [t_0,T] \times B(M,\rho) \multimap E$ defined by $\overline{F}(t,x) := F(t,r(x))$ has the following properties:*

(a) *$\overline{F}$ is upper semicontinuous;*
(b) *if $c \in L^1([t_0,T])$ is such that $\sup_{u\in F(t,x)} \|u\| \le c(t)(1+\|x\|)$, for any $(t,x) \in [t_0,T] \times M$, then*

$$\sup_{u\in \overline{F}(t,x)} \|u\| \le (1+\rho)c(t)(1+\|x\|)$$

*for any $t \in [t_0,T] \times B(M,\rho)$;*
(c) *for any $x \in B(M,\rho) \setminus M$, $\overline{F}(t,x) \cap \partial\, d_M(x)^\circ \neq \emptyset$.*

*Proof.* (a) The upper semicontinuity follows immediately from the decomposition $\overline{F} = F \circ (\mathrm{id}_{[t_0,T]} \times r)\colon [t_0,T] \times B(M,\rho) \multimap E$.

(b) For any $(t,x) \in [t_0,T] \times B(M,\rho)$,

$$\begin{aligned}\sup_{u\in \overline{F}(t,x)} \|u\| &\le c(t)(1+\|r(x)\|) \le c(t)(1+\|r(x)-x\|+\|x\|) \\ &= c(t)(1+d_M(x)+\|x\|) \le (1+\rho)c(t)(1+\|x\|).\end{aligned}$$

(c) By assumption, there is $u \in F(t,r(x)) \cap C_M(r(x))$. Observe that, by the definition of proximate retract and the Lipschitz property of $d_M$,

$$\begin{aligned} d_M^\circ(x;u) &= \limsup_{y\to x,\, h\to 0^+} \frac{d_M(y+hu) - d_M(y)}{h} \\ &= \limsup_{y\to x,\, h\to 0^+} \frac{d_M(y+hu) - \|r(y)-y\|}{h} \\ &\le \limsup_{y\to x,\, h\to 0^+} \frac{d_M(r(y)+hu)}{h} \le \limsup_{y\to r(x),\, h\to 0^+} \frac{d_M(y+hu)}{h} = 0, \end{aligned}$$

which implies $u \in \overline{F}(t,x) \cap \partial\, d_M(x)^\circ$ (see Remark 4.8(a)). □

(4) If $M$ is a proximate retract, then it can be shown that $T_M(x) = C_M(x)$ for any $x \in M$.

**Lemma 4.14** (see [14]). *For any $x \in B(M,\rho) \setminus M$, $0 \notin \partial\, d_M(x)$.*

*Proof of Theorem* 4.12. Let $n_0 \ge 1$ be such that $1/n_0 \le \rho$. And put $M_n := B(M, 1/n)$ for $n \ge n_0$. The sets $M_n$ are represented by $f_n := d_M - 1/n$ and, in view of Lemma 4.14, are epi-Lipschitz with $f_n$ as the representing functional. Moreover, Lemma 4.13 implies that $\overline{F}|_{[t_0,T]\times M_n}$ are tangent to $M_n$, i.e.

$$\overline{F}(t,x) \cap \partial\, f_n(x)^\circ \neq \emptyset \quad \text{for } (t,x) \in [t_0,T] \times \partial\, M_n = [t_0,T] \times f_n^{-1}(0).$$

By Theorem 4.10, the sets $L(x_0, -A + \overline{F}|_{[t_0,T]\times M_n})$ are cell-like. Since it is clear that

$$L(x_0, -A + F) = \bigcap_{n \ge n_0} L(x_0, -A + \overline{F}|_{[t_0,T]\times M_n}),$$

it follows, by Example 4.3(d), that $L(x_0, -A + F)$ is cell-like. □

## 5. Topological degree for perturbations of $m$-accretive operators

**5.1. Fixed point index for $c$-admissible set-valued maps.** To perform the construction of degree by use of our approach, one needs an adequate fixed point index for set-valued maps. Therefore we briefly present such a version of fixed point index coming from [15], which is an extension of the earlier fixed point indices (see [21] and [30]).

A map $\Phi: X \multimap Z$, where $X$ and $Z$ are metric spaces, is called *$c$-admissible* if it admits a decomposition

$$\Phi(x) = f(\varphi(x)) \quad \text{for any } x \in X,$$

where $\varphi: X \multimap Y$ takes cell-like values in a metric space $Y$ and is upper semicontinuous, and $f: Y \to Z$ is a continuous map. If $\Phi$ is $c$-admissible, then the corresponding diagram

$$D: X \overset{\varphi}{\multimap} Y \xrightarrow{f} Z$$

is called *a $c$-decomposition* of $\Phi$. Obviously, in general, one map may admit many decompositions. For that reason a $c$-admissible map is referred to as a pair $(\Phi, D)$ where $D$ is a given $c$-decomposition of $\Phi$.

**Definition 5.1.** Two $c$-admissible maps $(\Phi_k, D_k)$ with decompositions

$$D_k: X \overset{\varphi_k}{\multimap} Y_k \xrightarrow{f_k} Z, \quad k = 0, 1,$$

are said to be *homotopic* if there exist a $c$-admissible map $\Psi: X \times [0,1] \multimap Z$ with the decomposition

$$D: X \times [0,1] \overset{\psi}{\multimap} Y \xrightarrow{g} Z$$

and continuous maps $j_k \colon Y_k \to Y$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, multimap, "\varphi_0"] \arrow[d, "i_0"'] & Y_0 \arrow[d, "j_0"'] \arrow[dr, "f_0"] & \\
X \times [0,1] \arrow[r, multimap, "\psi"] & Y \arrow[r, "g"] & Z \\
X \arrow[u, "i_1"] \arrow[r, multimap, "\varphi_1"'] & Y_1 \arrow[u, "j_1"'] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

where $i_k \colon X \to X \times [0,1]$, for $k = 0,1$, are given by $i_k(x) := (x,k)$, commutes. It is written as $(\Phi_0, D_0) \simeq (\Phi_1, D_1)$.

In practice, the following method of establishing a homotopy relationship is useful.

**Proposition 5.2.** *Let $\Phi \colon X \times [0,1] \multimap Z$ be such that*

$$\Phi(x,t) = f_t(\varphi_t(x)) \quad \textit{for each } (x,t) \in X \times [0,1],$$

*where $\varphi \colon X \times [0,1] \multimap Y$ is an upper semicontinuous map with cell-like values and $f \colon Y \times [0,1] \to Z$ is continuous (and $\varphi_t := \varphi(\,\cdot\,, t)$, $f_t := f(\,\cdot\,, t)$). Then the pairs $(\Phi_k, D_k)$ with $D_k \colon X \overset{\varphi_k}{\multimap} Y \xrightarrow{f_k} Z$, for $k = 0,1$, are homotopic.*

*Proof.* Define $\widetilde{Y} := \{(t,y) \in [0,1] \times Y \mid y \in \varphi(x,t),\ x \in X\}$, $\psi(x,t) \colon X \times [0,1] \multimap \widetilde{Y}$ by $\psi(x,t) := \{t\} \times \varphi(x,t)$, $g \colon \widetilde{Y} \to Z$ by $g(t,y) := f(y,t)$ and $j_k \colon Y \to \widetilde{Y}$ by $j_k(y) := (k,y)$, $k = 0,1$. It is easy to verify that $\psi i_k = j_k \varphi_k$ and $g j_k = f_k$, $k = 0,1$. $\square$

Denote by $\mathcal{C}$ the class of pairs $(\Phi, D)$, where $\Phi \colon U \multimap X$, with an ANR $X$ and an open subset $U \subset X$, is a locally compact $c$-admissible map with a $c$-decomposition

$$D \colon U \overset{\varphi}{\multimap} Y \xrightarrow{f} X$$

and such that $\mathrm{Fix}(\Phi, U) := \{x \in U \mid x \in \Phi(x)\}$ is compact.

**Theorem 5.3.** *There is a correspondence assigning to any $(\Phi, D) \in \mathcal{C}$ an integer $\mathrm{Ind}_X((\Phi, D), U)$ — the fixed point index of $(\Phi, D)$ with respect to $U$ — having the following properties:*

(IND1) (Existence) *If $\mathrm{Ind}((\Phi, D), U) \neq 0$, then there exists $x \in U$ such that $x \in \Phi(x)$.*

(IND2) (Additivity) *If $\mathrm{Fix}(\Phi, U) \subset U_1 \cup U_2 \setminus (U_1 \cap U_2)$, where $U_1, U_2 \subset U$ are open, then*

$$\mathrm{Ind}_X((\Phi, D), U) = \mathrm{Ind}_X((\Phi|U_1, D_{U_1}), U_1) + \mathrm{Ind}_X((\Phi|U_2, D_{U_2}), U_2)$$

*where $D_{U_1}$ and $D_{U_2}$ have obvious meaning.*

(IND3) (Homotopy invariance) *If* $(\Phi_0, D_0)$ *and* $(\Phi_1, D_1)$ *are homotopic via a compact homotopy* $(\Psi, D)$, *with the decomposition* $D\colon U \times [0,1] \overset{\psi}{\multimap} Y \xrightarrow{g} X$, *such that the set* $\bigcup_{t\in[0,1]} \mathrm{Fix}(\Psi(\,\cdot\,,t),U)$ *is compact, then*

$$\mathrm{Ind}_X((\Phi_0, D_0), U) = \mathrm{Ind}_X((\Phi_1, D_1), U).$$

(IND4) (Normalization) *If* $U = X$ *and* $\Phi$ *is compact, then*

$$\mathrm{Ind}_X((\Phi, D), X) = \Lambda((\Phi, D))$$

*where* $\Lambda((\Phi, D))$ *is the Lefschetz number of the pair* $(\Phi, D)$.

**Remark 5.4.** (a) For more information on the Lefschetz number we refer to [21] and in this particular case to [15].

(b) At this point it should mentioned that if a compact $c$-admissible map $(\Phi, D)$ is homotopic to the identity map $\mathrm{id}_X$ (with the trivial decomposition), then $\Lambda((\Phi, D)) = \chi(X)$.

**5.2. Construction of topological degree.** By $\mathcal{A}(M,E)$ denote the class of maps $-A + F\colon M \cap D(A) \multimap E$ where $A\colon D(A) \multimap E$ is a $m$-accretive operator and $F\colon M \multimap E$ is such that:

- $(\mathrm{A}_1)$ a closed set $M \subset E$ is either convex, epi-Lipschitz or a proximate retract;
- $(\mathrm{A}_2)$ $J_\lambda^A(M) \subset M$ for any $\lambda > 0$;
- $(\mathrm{A}_3)$ the semigroup $\{S_A(t)\colon \overline{D(A)} \to \overline{D(A)}\}_{t\geq 0}$ is compact;
- $(\mathrm{A}_4)$ $F\colon M \multimap E$ is an upper semicontinuous with compact convex values and sublinear growth, i.e. such that there exists $c > 0$ such that

$$\sup_{u\in F(x)} \|u\| \leq c(1 + \|x\|) \quad \text{for } x \in M;$$

- $(\mathrm{A}_5)$ for any $x \in M$

$$\begin{aligned} F(x) \cap C_M(x) &\neq \emptyset \quad \text{if } M \text{ is a convex set or a proximate retract;} \\ F(x) \cap \partial f(x)^\circ &\neq \emptyset \quad \text{if } M \text{ is an epi-Lipshitz set represented by } f; \end{aligned}$$

- $(\mathrm{A}_6)$ if $F$ is not single-valued, then the dual $E^*$ is assumed to be uniformly convex.

**Remark 5.5.** If $M$ is either closed convex, epi-Lipschitz or a proximate retract, then $M$ is strictly regular in the sense of [14] and, therefore, it is a neighbourhood retract. Furthermore, if additionally, $M$ is a resolvent invariant with respect to some $m$-accretive $A$, then, in view of Proposition 2.21, $M_A := M \cap \overline{D(A)}$ is a neighbourhood retract, too.

If $U \subset M$ is an open (in $M$) and bounded set, then one defines the class

$$\mathcal{A}_U(M,E) := \{-A + F \in \mathcal{A}(M,E) \mid 0 \notin (-A + F)(\partial_M U \cap D(A))\}.$$

*A homotopy* in the class $\mathcal{A}_U(M,E)$ (or in $\mathcal{A}(M,E)$) is a mapping $(x,\lambda)\mapsto -A(\lambda)x+F(x,\lambda)$ where $F\colon M\times[0,1]\multimap E$ is upper semicontinuous, with sublinear growth uniform with respect to $\lambda$ and $\{A(\lambda)\}_{\lambda\in[0,1]}$ is a family of operators such that

- $-A(\lambda)+F(\,\cdot\,,\lambda)\in\mathcal{A}_U(M,E)$ (resp. $\mathcal{A}(M,E)$) for each $\lambda\in\Lambda$;
- $\overline{D(A(\lambda_1))}=\overline{D(A(\lambda_2))}$ for any $\lambda_1,\lambda_2\in[0,1]$;
- $A(\lambda_n)\xrightarrow{G}A(\lambda_0)$ as $\lambda_n\to\lambda_0$;
- the family of semigroups $\{S_{A(\lambda)}\}_{\lambda\in[0,1]}$ is compact.

The following lemmata are the main steps in the construction of the degree.

**Lemma 5.6.** *Suppose* $(x,\lambda)\mapsto -A(\lambda)x+F(x,\lambda)$ *is a homotopy in* $\mathcal{A}(M,E)$. *Then:*

(a) *the set* $\{(x,\lambda)\in M\times[0,1]\mid x\in D(A(\lambda)),\ 0\in -A(\lambda)x+F(x,\lambda)\}$ *is closed in* $M\times[0,1]$*;*

(b) *if* $U\subset M$ *is a bounded open subset of* $M$*, then the set*

$$\mathcal{Z}:=\bigcup_{\lambda\in[0,1]}\{x\in\overline{U}\cap D(A(\lambda))\mid 0\in -A(\lambda)x+F(x,\lambda)\}$$

*is compact.*

*Proof.* (a) Suppose $(x_n,\lambda_n)\to(x_0,\lambda_0)$ and $0\in -A(\lambda_n)x_n+F(x_n,\lambda_n)$ for $n\geq 1$. The constant functions $u_n\equiv x_n$ on $[0,1]$, for $n\geq 1$, are integral solutions of the problem $\dot u\in -A(\lambda_n)u+F(u,\lambda_n)$. Therefore, by Theorems 3.1(P2) and 3.4(P2), $(u_n)$ converges to a solution of $\dot u\in -A(\lambda_0)u+F(u,\lambda_0)$, i.e. $u_0\equiv x_0$ is an integral solution, which, in view of Remark 2.9, means that $0\in -A(\lambda_0)x_0+F(x_0,\lambda_0)$.

(b) Let $(x_n)\subset\overline{U}$ and $(\lambda_n)\subset[0,1]$ be sequences such that $0\in A(\lambda_n)x_n+F(x_n,\lambda_n)$, for $n\geq 1$. One may assume that $\lambda_n\to\lambda_0\in[0,1]$. The constant functions $v_n\equiv x_n$, $n\geq 1$, are integral solutions of the inclusions $\dot u\in -A(\lambda_n)u+F(u,\lambda_n)$. This, in particular, means that $\{x_n\}_{n\geq 1}\subset\Phi_1([M_D\cap\overline{U}]\times[0,1])$ where $M_D:=M\cap A(\lambda)$ (independently of $\lambda\in[0,1]$). By Theorems 3.1(P3) and 3.4(P3), the sequence $(x_n)$ is relatively compact. Hence, by (a), we gather that $\mathcal{Z}$ is compact. $\square$

**Remark 5.7.** In view of Lemma 5.6, it is clear that the set-valued mapping $\lambda\mapsto\{x\in\overline{U}\cap D(A(\lambda))\mid 0\in -A(\lambda)x+F(x,\lambda)\}$ is upper semicontinuous (as it has closed graph and all values are contained in the compact set $\mathcal{Z}$).

**Lemma 5.8.** *Under the assumptions of Lemma* 5.6(b)*, for any* $\varepsilon>0$ *there exists* $\overline{t}>0$ *such that for* $t\in(0,\overline{t}]$

$$\bigcup_{\lambda\in[0,1]}\{x\in\overline{U}\cap\overline{D(A)}\mid x\in\Phi_t(x,\lambda)\}\subset\mathcal{Z}+B(0,\varepsilon).$$

*Proof.* We proceed by contradiction. Suppose that there exist $\overline{\varepsilon} > 0$ and sequences $(\lambda_n) \subset [0,1]$, $t_n \to 0^+$ and $(x_n) \subset \overline{U} \cap M_D$ such that, for any $n \geq 1$, $x_n \in \Phi_{t_n}(x_n, \lambda_n)$ and

$$d(x_n, \mathcal{Z}) \geq \overline{\varepsilon}. \tag{5.1}$$

Observe that $x_n \in \Phi_{kt_n}(x_n, \lambda_n)$ for any integers $k \geq 1$ and $n \geq 1$, which implies that

$$\{x_n\}_{n\geq 1} \subset \Phi_{1/2}(\Omega_0 \times [0,1]),$$

with $\Omega_0 := \{\Phi_t(x,\lambda) \mid x \in \overline{U} \cap M_D,\ t \in [0,1/2],\ \lambda \in [0,1]\}$. Since, in view of Remarks 3.2 and 3.5(b), the set $\Omega_0$ is bounded, by Theorems 3.1(P3) and 3.4(P3), the set $\{x_n\}_{n\geq 1}$ is relatively compact. Without loss of generality, one may assume that $x_n \to x_0 \in \overline{U}$ and $\lambda_n \to \lambda_0$.

Clearly, for any $n \geq 1$, there exists a $t_n$-periodic solution $u_n \colon [0,1] \to E$ of $\dot{u} \in -A(\lambda_n)u + F(u, \lambda_n)$ with $u_n(0) = x_n$ and $u_n(t) \in M$ for $t \in [0,1]$. By Theorem 3.4(P2), $(u_n)$ is convergent in $C([0,1], E)$ to some solution $u_0$ with $u_0(0) = x_0$. Therefore, for any $t \in (0,1]$ and $n \geq 1$, one has

$$\begin{aligned}\|u_0(0) - u_0(t)\| \leq \|u_0(0) &- u_n([t/t_n]t_n)\| \\ &+ \|u_n([t/t_n]t_n) - u_n(t)\| + \|u_n(t) - u_0(t)\|.\end{aligned}$$

Since $u_n([t/t_n]t_n) = x_n = u_n(0)$, $u_n \to u_0$ in $C([0,1], E)$ and $\{u_n\}$ is equicontinuous, we infer that $u_0 \equiv x_0$. Finally, in view of Remark 2.9, $x_0 \in \overline{U} \cap D(A(\lambda_0))$ and $0 \in -A(\lambda_0)x_0 + F(x_0, \lambda_0)$, i.e. $x_0 \in \mathcal{Z}$. This is a contradiction to (5.1). $\square$

In order to define the degree take any $-A + F \in \mathcal{A}_U(M, E)$. In view of Theorems 3.1(P3) and 3.4(P3), $\Phi_t$ is compact for $t > 0$ and admits a decomposition

$$D_t \colon U \cap \overline{D(A)} \xrightarrow{L(\,\cdot\,, -A+F)} C([t_0, T], M_A) \xrightarrow{\ e_t\ } M_A.$$

$L(\,\cdot\,, -A + F)$ has cell-like values, in view of Theorem either 4.5, 4.10 or 4.12 and, by Remark 5.5, $M_A$ is ANR. By the compactness of $\mathcal{Z}$ (see Lemma 5.6(b)) there exists $\overline{\varepsilon} > 0$ such that $[\mathcal{Z} + B(0, \overline{\varepsilon})] \cap M \subset U$. By Lemma 5.8, there is $\overline{t} > 0$ such that $\{x \in \overline{U} \mid x \in \Phi_t(x)\} \subset U$ for $t \in (0, \overline{t}]$, which means that $(\Phi_t, D_t)$ is admissible in the fixed point index theory for $c$-admissible maps. Moreover, observe that, in virtue of Proposition 5.2 and Theorem 5.3, for each $t_1, t_2 \in [0, \overline{t}]$, the pairs $(\Phi_{t_1}, D_{t_1})$ and $(\Phi_{t_2}, D_{t_2})$ are homotopic and $\operatorname{Ind}_{M_A}((\Phi_{t_1}, D_{t_1}), U \cap \overline{D(A)}) = \operatorname{Ind}_{M_A}((\Phi_{t_2}, D_{t_2}), U \cap \overline{D(A)})$.

Thus, the following definition is correct

$$\deg_M(-A + F, U) := \lim_{t \to 0^+} \operatorname{Ind}_{M_A}((\Phi_t, D_t), U \cap \overline{D(A)}). \tag{5.2}$$

**Theorem 5.9.** *The correspondence defined by* (5.2) *has all the properties of the topological degree, i.e.*

(DEG1) (Existence) *If* $\deg_M(-A+F,U)\neq 0$*, then there exists* $x\in U\cap D(A)$ *such that* $0\in -Ax+F(x)$.

(DEG2) (Additivity) *If* $U_1,U_2$ *are open disjoint subsets of* $U$ *and* $0\notin(-A+F)([\overline{U}\setminus(U_1\cup U_2)]\cap D(A))$*, then*

$$\deg_M(-A+F,U)=\deg_M(-A+F,U_1)+\deg_M(-A+F,U_2).$$

(DEG3) (Homotopy invariance) *If* $(x,\lambda)\mapsto A(\lambda)x+F(x,\lambda)$ *is an admissible homotopy in* $\mathcal{A}_U(M,E)$*, then* $\deg_M(-A(\lambda)+F(\,\cdot\,,\lambda),U)$ *does not depend on* $\lambda\in[0,1]$.

(DEG4) (Normalization) *If* $M_A$ *is bounded, then* $\deg_M(-A+F,M)=\chi(M_A)$.

*Proof.* (DEG1) If $\deg_M(-A+F,U)\neq 0$, then, by definition, there is an integer $n_0\geq 1$ such that, for $t_n:=2^{-n}$ with $n\geq n_0$, $\operatorname{Ind}_{M_A}((\Phi_{t_n},D_{t_n}),U\cap\overline{D(A)})\neq 0$, which, by Theorem 5.3(IND1), gives the existence of $x_n\in U$ such that $x_n\in\Phi_{t_n}(x_n)$, for each $n\geq n_0$. Hence, by use of Lemmas 5.6 and 5.8, the sequence $(x_n)$ contains a subsequence convergent to some $x_0\in\mathcal{Z}$, i.e. satisfying $0\in -Ax_0+F(x_0)$.

(DEG2) Since $\{x\in\overline{U}\cap D(A)\mid 0\in -Ax+F(x)\}=\mathcal{Z}\subset U_1\cup U_2$, using Lemmata 5.6 and 5.8, one gets $\overline{t}>0$ such that, for $t\in(0,\overline{t}]$,

$$\{x\in\overline{U}\cap\overline{D(A)}\mid x\in\Phi_t(x)\}\subset U_1\cup U_2.$$

Hence, by the additivity property of the fixed point index (Theorem 5.3(IND2)), one obtains, for $t\in(0,\overline{t}]$,

$$\begin{aligned}&\operatorname{Ind}_{M_A}((\Phi_t,D_t),U\cap\overline{D(A)})\\&\qquad=\operatorname{Ind}_{M_A}((\Phi_t,D_t),U_1\cap\overline{D(A)})+\operatorname{Ind}_{M_A}((\Phi_t,D_t),U_2\cap\overline{D(A)}),\end{aligned}$$

which, by the definition of the degree, implies (DEG2).

(DEG3) By Theorems 3.1(P2), (P3) and 3.4(P2), (P3), the maps $\Phi_t\colon[\overline{U}\cap M_D]\times[0,1]\multimap M_D$ with $M_D:=M\cap\overline{D(A(\lambda))}$, where $\lambda\in[0,1]$, are $c$-admissible and compact for $t>0$. By Lemmas 5.6 and 5.8, there is $\overline{t}>0$ such that, for $t\in(0,\overline{t}]$, $\bigcup_{\lambda\in[0,1]}\{x\in\overline{U}\mid x\in\Phi_t(x,\lambda)\}\subset U$. Hence, by use of the homotopy invariance of the fixed point index (Theorem 5.3(IND3)) and the definition of the degree, one obtains (DEG3).

(DEG4) Note that $\deg_M(-A+F,M)=\operatorname{Ind}_{M_A}((\Phi_t,D_t),M_A)$ for $t>0$. Since $M_A$ is bounded $\Phi_t$ are compact and homotopic to the identity $\mathrm{id}_{M_A}$ (via the homotopy $(x,\lambda)\mapsto\Phi_{\lambda t}(x)$). Hence, by the normalization property of the fixed point index and Remark 5.4, $\deg_M(-A+F,M)=\chi(M_A)$. $\square$

An immediate implication of the existence and normalization property of the degree is the following existence criterion being an extension of the result obtained in [5] where $A := 0$ and $M$ is a compact $\mathcal{L}$-retract.

**Corollary 5.10.** *If $-A + F \in \mathcal{A}(M, E)$, $M_A$ is bounded and $\chi(M_A) \neq 0$, then there exists $x_0 \in M_A$ such that $0 \in -Ax_0 + F(x_0)$.*

## 6. Applications of topological degree

**6.1. Continuation and bifurcation of equilibria.** Let the family of $m$-accretive operators $\{A(\lambda)\}_{\lambda\in[a,b]}$, a neighbourhood retract $M \subset E$ and $F: M \times [a, b] \multimap E$ be such that the map $(x, \lambda) \mapsto -A(\lambda)x + F(x, \lambda)$ is a homotopy in the class $\mathcal{A}(M, E)$. We are concerned with the following continuation problem

$$(C_{M,\lambda}) \qquad \begin{cases} 0 \in -A(\lambda)x + F(x, \lambda), \\ x \in M \cap D(A(\lambda)), \\ \lambda \in [a, b]. \end{cases}$$

Let $W$ be an open and bounded subset of $M \times [a, b]$ and let, as before,

$$\mathcal{Z} := \{(x, \lambda) \in \overline{W} \mid 0 \in -A(\lambda)x + F(x, \lambda)\}.$$

Observe that, in view of Lemma 5.6, the set $\mathcal{Z}$ is compact. By use of the topological degree and the proper topological lemma (the so-called separation lemma), one derives the following criterion for continuation.

**Theorem 6.1** (Continuation of equilibria)**.** *If $\mathcal{Z}_a \cap [\partial_{M\times[a,b]} W]_a = \emptyset$ (5) and $\deg_M(-A(a) + F_a, W_a) \neq 0$, then there exists a connected component $\Sigma$ of $\mathcal{Z}$ such that*

$$\Sigma \cap [W_a \times \{a\}] \neq \emptyset$$

*and either $\Sigma \cap \partial_{M\times[a,b]} W \neq \emptyset$ or $\Sigma \cap [W_b \times \{b\}] \neq \emptyset$.*

For the proof see [13]. It is an adaptation of the classical arguments (e.g. [33]).

To pass to bifurcation, suppose, additionally, that a branch $\Sigma^0$ of ("trivial") solutions of $(C_{M,\lambda})$ is given, and $\Sigma^0$ is a closed connected subset of $\mathcal{Z}$ such that $\Sigma^0_a \neq \emptyset$ and $\Sigma^0_b \neq \emptyset$. One says that a point $\lambda_0 \in [a, b]$ is a *bifurcation value*, if

$$[\Sigma^0_{\lambda_0} \times \{\lambda_0\}] \cap \overline{\mathcal{Z} \setminus \Sigma^0} \neq \emptyset,$$

i.e. there is $(x_0, \lambda_0) \in \Sigma^0$ being a cluster point of a sequence of ("nontrivial") solutions $((x_n, \lambda_n)) \subset \mathcal{Z} \setminus \Sigma^0$. The set of all bifurcation values of $(C_{M,\lambda})$ is denoted by $\mathcal{B}$.

Suppose that $a, b \notin \mathcal{B}$. Then $\Sigma_a \times \{a\}$ and $\Sigma_b \times \{b\}$ are isolated in $\mathcal{Z}$. Hence, there are open $V^{(a)} \subset W_a$ and $V^{(b)} \subset W_b$ such that $V^{(a)} \cap \mathcal{Z}_a = \Sigma^0_a$,

(5) If $A \subset X \times Y$, $y \in X$, then $A_y := \{x \in X \mid (x, y) \in A\}$.

$V^{(b)} \cap \mathcal{Z}_b = \Sigma_b^0$ and the degrees $\deg_M(-A(a)+F_a, V^{(a)})$, $\deg_M(-A(b)+F_b, V^{(b)})$ are well-defined.

**Theorem 6.2** (Bifurcation of equilibria)**.** *Under the above assumptions, if*

$$\deg_M(-A(a) + F_a, V_a) \neq \deg_M(-A(b) + F_b, V_b),$$

*then $\mathcal{B} \neq \emptyset$ and there exists a connected set $\Sigma \subset \mathcal{Z} \setminus \Sigma^0$ such that $\overline{\Sigma} \cap \Sigma^0 \neq \emptyset$ and*

(a) *either $\overline{\Sigma} \cap \partial_{M\times[a,b]} W \neq \emptyset$, if $\partial_{M\times[a,b]} W \neq \emptyset$,*
(b) *or $\overline{\Sigma} \cap [(W_a \setminus \Sigma_a^0) \cup (W_b \setminus \Sigma_b^0)] \neq \emptyset$.*

**6.2. Periodic solutions — existence and branching.** Start with the global criterion for the existence of periodic solutions for problems

$$\begin{cases} \dot{u} \in -Au + F(t,u), \\ u(t) \in M, \\ u(0) = u(T), \end{cases} \tag{6.1}$$

where $A\colon D(A) \multimap E$ and $F\colon [0,T] \times M \multimap E$ satisfy $(A_1), (A_2), (A_3)$ and the following conditions hold:

($\mathrm{A}_4'$) $F$ is upper semicontinuous with compact convex values and of sublinear growth, i.e. there exists $c \in L^1([0,T])$ such that

$$\sup_{u \in F(t,x)} \|u\| \le c(t)(1 + \|x\|)$$

for any $x \in M$ and almost all $t \in [0,T]$;

($\mathrm{A}_5'$) $F$ is $T$-periodic in $t$, i.e. $F(0,x) = F(T,x)$ for $x \in M$, and tangent to $M$, i.e. for any $(t,x) \in [0,T] \times M$,

$$\begin{aligned} F(t,x) \cap C_M(x) &\neq \emptyset \quad \text{if } M \text{ is a convex set or a proximate retract;} \\ F(t,x) \cap \partial f(x)^\circ &\neq \emptyset \quad \text{if } M \text{ is an epi-Lipschitz set represented by } f. \end{aligned}$$

($\mathrm{A}_6'$) One the following conditions is satisfied:
(a) $E^*$ is uniformly convex;
(b) $F$ is single-valued and does not depend on $t$.

**Theorem 6.3.** *If $A$, $F$ and $M$ are as above and $M_A$ is bounded with $\chi(M_A) \neq 0$, then the periodic problem* (6.1) *admits at least one solution.*

*Proof.* Since $\Phi_T\colon M_A \multimap M_A$ is homotopic to the identity map $\mathrm{id}_{M_A}$ via a homotopy $M_A \times [0,1] \ni (x,s) \mapsto \Phi_{sT}(x)$, by Theorem 5.3, $\mathrm{Ind}_{M_A}(\Phi_T, M_A) = \chi(M_A) \neq 0$. This, by the existence property of the fixed point index, implies the existence of $x \in M_A$ such that $x \in \Phi_T(x)$, which means that there exists a corresponding solution $u\colon [0,T] \to E$ of (6.1) with $u(T) = x = u(0)$. $\square$

**Remark 6.4.** A version of the existence criterion for convex $M$ with some additional property was obtained in [7]. For the case when $A$ is a generator of a $C_0$ semigroup and $F$ is an upper Carathéodory map see [3].

Before passing to branching of periodic points, we state a branching result for fixed points, which makes an abstract setting for studying periodic points of differential inclusions. Suppose that $\Phi: \overline{U} \times [0,\infty) \multimap M$ is a set-valued map where is $U$ is a bounded open subset of the metric ANR $M$ (see Remark 5,5). A point $(x,\lambda) \in \overline{U} \times [0,\infty)$ is said to be a *resting point*, if $x \in \Phi(x,\lambda)$. The set of all resting points for $\Phi$ is denoted by $\mathcal{R}(\Phi)$. A point $x \in \overline{U}$ is called a *branching point*, if $(x,0) \in \overline{\mathcal{R}(\Phi) \setminus [M \times \{0\}]}$; the set of all branching points of $\Phi$ is denoted by $\mathcal{B}(\Phi)$.

**Proposition 6.5** (see [12]). *Let an upper semicontinuous map $\Phi: \overline{U} \times [0,\infty) \multimap M$ be such that*

(H1) *there are an upper semicontinuous cell-like valued $\varphi: \overline{U} \times [0,\infty) \multimap Y$, where $Y$ is a metric space, and a continuous $g: Y \to M$ such that $\Phi = g \circ \varphi$;*
(H2) *for any $\lambda_1, \lambda_2 \in (0,\infty)$, the set $\Phi(\overline{U} \times [\lambda_1,\lambda_2])$ is relatively compact;*
(H3) *$\Phi$ has the property: if $\lambda_n \to 0^+$ and $x_n \in \Phi(x_n,\lambda_n)$, for $n \geq 1$, then $(x_n)$ is relatively compact;*
(H4) *$\mathcal{B}(\Phi) \cap \operatorname{bd}_M U = \emptyset$ and*

$$\operatorname{Ind}_M(\Phi(\,\cdot\,,\lambda), U) \neq 0, \ (^6)$$

*for sufficiently small $\lambda > 0$.*

*Then there exists a connected set $\Sigma \subset \mathcal{R}(\Phi) \cap [U \times (0,\infty)]$ such that $\overline{\Sigma} \cap [\mathcal{B}(\Phi) \times \{0\}] \neq \emptyset$ (in particular $\mathcal{B}(\Phi) \neq \emptyset$) and $\Sigma$ is not contained in any compact subset of $[U \times (0,\infty)] \cup [\mathcal{B}(\Phi) \times \{0\}]$.*

The proof uses the basic properties of the fixed point index and a proper topological lemma allowing to obtain the existence of a branch of resting points (see [12] for the single-valued version).

Now we are be concerned with periodic points of the parameterized problems of the form

$$(B_\lambda) \qquad \begin{cases} \dot{u} \in -\lambda A u + \lambda F(t,u) & \text{for } \lambda \geq 0, \\ u(t) \in M_A, \end{cases}$$

where a $m$-accretive operator $A: D(A) \multimap E$, a closed set $M \subset E$ and an upper semicontinuous compact convex valued map $F: [0,T] \times M \multimap E$ $(T > 0)$ satisfy $(\mathrm{A}_1)$–$(\mathrm{A}_3)$ and $(\mathrm{A}_4')$–$(\mathrm{A}_6')$.

(6) The condition $\mathcal{B}(\Phi) \cap \operatorname{bd}_M U = \emptyset$ together with (H3) implies that there is $\lambda_0 > 0$ such that $x \notin \Phi(x,\lambda)$ for any $x \in \operatorname{bd}_M U$ and $\lambda \in (0,\lambda_0]$.

A point $(x,\lambda) \in M_A \times [0,\infty)$ is called a *$T$-periodic point* if there exists an integral solution $u\colon[0,T] \to E$ of $(B_\lambda)$ such that $u(0) = u(T) = x$. Note that any $(x,0)$ with $x \in M_A$ is a $T$-periodic point. For a given $K \subset M$, by $\mathcal{P}_T(K)$ denote the set of all $T$-periodic points in $K \times [0,\infty)$. A point $x_0 \in K$ is a *branching point* if $(x_0,0) \in \overline{\mathcal{P}_T(K) \setminus M_A \times \{0\}}$. The set of all branching points in $K$ is denoted by $\mathcal{B}_T(K)$.

The necessary condition for branching is provided below.

**Theorem 6.6.** *Under the above assumptions, if $x_0 \in \mathcal{B}_T(M)$, then $0 \in -Ax_0 + \widehat{F}(x_0)$ where $\widehat{F}\colon M \multimap E$ is defined by*

$$\widehat{F}(x) := \frac{1}{T}\int_0^T F(t,x)\,dt.\ (^7)$$

**Lemma 6.7.** *Let $A\colon D(A) \multimap E$ be a $m$-accretive operator. If $x_n \to x_0$ in $\overline{D(A)}$, $(w_n) \subset L^1([0,T],E)$ is bounded and $\lambda_n \to 0^+$, then $\Sigma_{\lambda_n A}(x_n,\lambda_n w_n) \to x_0$ (in $C([0,T],E)$).*

*Proof.* By Remark 2.7, for any $(y,v) \in \mathrm{Gr}\,(A)$,

$$\max_{t\in[0,T]} \|\Sigma_{\lambda_n A}(x_n,\lambda_n w_n)(t) - y\| \le \|x_n - y\| + \lambda_n \int_0^T \|w_n(\tau) - v\|\,d\tau.$$

Further, let $\varepsilon > 0$ and $\overline{x}_\varepsilon \in D(A)$ be such that $\|x_0 - \overline{x}_\varepsilon\| < \varepsilon$ and take $v_\varepsilon \in E$ with $(\overline{x}_\varepsilon, v_\varepsilon) \in \mathrm{Gr}\,(A)$. Then, passing to the limit,

$$\begin{aligned}\limsup_{n\to\infty}\Big(\max_{t\in[0,T]} &\|\Sigma_{\lambda_n A}(x_n,\lambda_n w_n)(t) - \overline{x}_\varepsilon\|\Big)\\ &\le \|x_0 - \overline{x}_\varepsilon\| + \lim_{n\to\infty}(\lambda_n\|w_n - v_\varepsilon\|_{L^1}) = \|x_0 - \overline{x}_\varepsilon\| < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we gather that $\Sigma_{\lambda_n A}(x_n,\lambda_n w_n) \to x_0$ in $C([a,b],E)$. $\square$

*Proof of Theorem* 6.6. Since $x_0 \in \mathcal{B}_T(M)$, there exist a sequence of $T$-periodic points $(x_n,\lambda_n) \to (x_0,0)$. Then there are a sequence $(u_n) \subset C([0,T],E)$ and $(w_n) \subset L^1([0,T],E)$ such that $u_n = \Sigma_{\lambda_n A}(x_n,\lambda_n w_n)$ and $w_n$ is an a.e. selection of $F(\,\cdot\,,u_n(\,\cdot\,))$. By Example 2.12 and Remark 3.5, the sequence $(u_n)$ is bounded and, by the $(\mathrm{A}_4')$, the sequence $(w_n)$ is bounded in $L^1([0,T],E)$. Hence, in view of Lemma 6.7, $u_n \to x_0$. Further, by the $T$-periodicity of $u_n$ and the definition of integral solution, one has, for any $(y,v) \in \mathrm{Gr}\,(A)$,

$$\int_0^T \langle u_n(t) - y, w_n(t) - v\rangle_+\,dt \ge 0 \quad \text{for } n \ge 1. \tag{6.2}$$

($^7$) The integral is defined as $\int_0^T F(t,x)\,dt := \big\{\int_0^T w(t)\,dt \mid w$ is a Bochner integrable selection of $F(\,\cdot\,,x)\big\}$.

If $F$ is single-valued and does not depend on $t$, then $w_n = F(u_n(\,\cdot\,)) \to F(x_0)$ and, in view of (6.2) and Lemma 2.4, one has $\int_0^T \langle x_0 - y, F(x_0) - v\rangle_+ \,dt \geq 0$ for any $(y,v) \in \mathrm{Gr}\,(A)$, i.e. by Proposition 2.5, $0 \in Ax_0 + F(x_0)$.

In the case when $E^*$ is uniformly convex, then putting $C(t) := \overline{\mathrm{conv}}\, F(\{t\} \times \{u_n(t)\}_{n\geq 1})$ and using Lemma 3.6, we may assume that $w_n \rightharpoonup w_0$ weakly in $L^1([0,T],E)$. By Lemma 2.16, and Remark 2.15, for any $(y,v) \in \mathrm{Gr}\,(A)$,

$$0 \leq \int_0^T \langle x_0 - y, w_0(t) - v\rangle_+ \,dt = \int_0^T J(x_0 - y)(w_0(t) - v)\,dt.$$

By the continuity and linearity of $J(x_0 - y)$ one has

$$\left\langle x_0 - y, \frac{1}{T}\int_0^T w_0(t)\,dt - v\right\rangle_+ \geq 0,$$

and, in view of Proposition 2.5, $0 \in -Ax_0 + \int_0^T w_0(t)\,dt$. Finally, by Lemma 3.7, $w_0$ is a selection of $F(\,\cdot\,, x_0)$ and, in consequence, one gets $0 \in -Ax_0 + \widehat{F}(x_0)$.□

To prove a sufficient criterion for branching of periodic points, we need the following formula.

**Proposition 6.8.** *Suppose $A$, $M$ and $F$ satisfy conditions* $(\mathrm{A}_1)$–$(\mathrm{A}_3)$, $(\mathrm{A}_4')$–$(\mathrm{A}_6')$. *And let, for $\lambda \geq 0$, $\Phi_T^\lambda\colon M_A \times [0,\infty) \multimap M_A$ be given by*

$$\Phi_T^\lambda(x) := e_T(L(x, -\lambda A + \lambda F)).$$

*If $U$ is a bounded open subset of $M$ such that $0 \notin (-A + \widehat{F})(\partial_M U \cap D(A))$, then there exists $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0]$,*

$$\mathrm{Ind}_{M_A}((\Phi_T^\lambda, D_T^\lambda), U \cap \overline{D(A)}) = \deg_M(-A + \widehat{F}, U), \tag{6.3}$$

*where $D_T^\lambda$ has the obvious meaning.*

*Proof. Step 1.* For $\lambda > 0$, define the map $\Psi^\lambda\colon M_A \times [0,1] \multimap M_A$ by

$$\Psi^\lambda(x,s) := e_T(L(x, -\lambda A + \lambda F_s))$$

where $F_s(t,x) := (1-s)\widehat{F}(x) + sF(t,x)$. Since $t \mapsto F(t,x) \cap C_M(x)$ is upper semicontinuous with compact values, it is also measurable with the image in a compact set, therefore it admits a measurable a.e. selection $w_x\colon [0,T] \to E$ due to the Kuratowski-Ryll-Nardzewski Selection Theorem. This implies that $w_{x,s}(t) := (1-s)\frac{1}{T}\int_0^T w_x(\tau)\,d\tau + sw_x(t) \in F_s(t,x) \cap C_M(x)$ for any $s \in [0,1]$, $t \in [0,T]$ and $x \in M$. Note also that $F_s$ are $T$-periodic and, by $(A_4')$,

$$\sup_{u \in F_s(t,x)} \|u\| \leq ((1-s)\|c\|_{L^1}/T + sc(t))(1 + \|x\|) = \widehat{c}(t)(1 + \|x\|) \tag{6.4}$$

where $\widehat{c}(t) := \max\{\|c\|_{L^1}/T, c(t)\}$. Therefore, by Theorem 3.1(P1), the map $\Psi^\lambda$, for $\lambda \geq 0$ is well-defined and, by Theorem 3.1(P2) and (P3), it is upper semicontinuous and compact. And, finally, due to Theorems 4.5, 4.10 and 4.12, $\Phi^\lambda$ is $c$-admissible.

*Step 2.* We shall prove that there exists $\lambda_1 > 0$ such that, for $\lambda \in (0, \lambda_1]$, the homotopy $\Psi^\lambda$ is admissible on $U \cap \overline{D(A)}$ in the fixed point index theory sense, i.e.

$$x \notin \Psi^\lambda(x,s), \quad \text{for } (x,s) \in (\partial_M U \times \overline{D(A)}) \times [0,1]. \tag{6.5}$$

Suppose, to the contrary, that there are $\lambda_n \to 0^+$, $s_n \to s_0 \in [0,1]$ and $(x_n) \subset \partial_M U$ such that $x_n \in \Psi^{\lambda_n}(x_n, s_n)$ for $n \geq 1$. Then it is clear that, for any integer $k \geq 1$, $x_n \in e_{kT}(L(x_n, -\lambda_n A + \lambda_n \overline{F}_{s_n}))$ where $\overline{F}_{s_n}\colon [0,\infty) \times M \multimap E$ is given by $\overline{F}_{s_n}(t,x) := F_{s_n}(t - [t/T]T, x)$. Let $u_n\colon [0,\infty) \to E$, $n \geq 1$, be $T$-periodic functions such that $u_n = \Sigma_{\lambda_n A}(x_n, \lambda_n w_n)(T)$ with $w_n$ being a.e. selection of $\overline{F}_{s_n}(\,\cdot\,, u_n(\,\cdot\,))$, and $u_n(0) = x_n$.

First, we prove that the sequence $(u_n)$ contains a subsequence convergent in $C([0,T],E)$ to some point from $\partial_M U$. Note that, by the growth condition (6.4) and Remark 3.5, there exists a constant $R > 0$ such that $\|u_n\|_{C([0,T],E)} \leq R$, which, in particular, means that $(w_n)$ is integrally bounded by $\widehat{c}(\,\cdot\,)(1+R)$. And since $w_n$ are $T$-periodic, for any integer $k \geq 1$, $\Sigma_{\lambda_n A}(x_n, \lambda_n w_n)(kT) = x_n$. In particular, putting $T_n := \lambda_n(1 + [1/\lambda_n])T$ and $\widehat{w}_n(\tau) := w_n(\tau/\lambda_n)$ and changing the variables, one obtains

$$\Sigma_A(x_n, \widehat{w}_n)(T_n) = \Sigma_{\lambda_n A}(x_n, \lambda_n w_n)((1 + [1/\lambda_n])T) = x_n. \tag{6.6}$$

By Proposition 2.8 and the periodicity of $w_n$, one has the estimate

$$\begin{aligned}
\|\Sigma_A(x_n, \widehat{w}_n)(T_n) &- S_A(T_n - T)\Sigma_A(x_n, \widehat{w}_n)(T)\| \\
&\leq \int_T^{T_n} \|\widehat{w}_n(\tau)\|\, d\tau = \lambda_n \int_{T/\lambda_n}^{(1+[1/\lambda_n])T} \|w_n(\xi)\|\, d\xi \\
&\leq \lambda_n \int_0^T \|w_n(\xi)\|\, d\xi \leq \lambda_n \|\widehat{c}\|_{L^1([0,T])}(1+R).
\end{aligned}$$

Therefore, keeping in mind (6.6), for any $N \geq 1$,

$$\{x_n\}_{n \geq N} \subset \{S_A(T_n - T)(\Sigma_A(x_n, \widehat{w}_n)(T))\} + B(0, \varepsilon_N)$$

where $\varepsilon_N := \max_{n \geq N} \lambda_n \|\widehat{c}\|_{L^1([0,T])}(1+R)$.

Further, since, by Proposition 2.19(a), the set $\{\Sigma_A(x_n, \widehat{w}_n)(T)\}_{n \geq 1}$ is relatively compact, $(t,x) \mapsto S_A(t)x$ is continuous and $\varepsilon_N \to 0$ as $N \to \infty$, we conclude that $\{x_n\}_{n \geq 1}$ contains a subsequence convergent to some $x_0 \in \partial_M U \cap \overline{D(A)}$. Without loss of generality one may assume that $x_n \to x_0$. By Lemma 6.7, we see that $u_n|[0,T] = \Sigma_{\lambda_n A}(x_n, \lambda_n w_n)$ converges to $x_0$.

By the periodicity of $u_n$ and the definition of integral solution, for any $(y,v)\in \operatorname{Gr}(A)$ and $n\geq 1$, one has

$$\int_0^T \langle u_n(\tau)-y, w_n(\tau)-v\rangle_+ \geq 0. \tag{6.7}$$

If (a) holds, i.e. when $E^*$ is uniformly convex, then by use of Lemma 3.6, we may assume that $w_n \rightharpoonup w_0$ weakly in $L^1([0,T],E)$. Then, by (6.7), Lemma 2.16 and Remark 2.15, for any $(y,v)\in \operatorname{Gr}(A)$,

$$0\leq \int_0^T \langle x_0-y, w_0(t)-v\rangle_+\,dt = \int_0^T J(x_0-y)(w_0(t)-v)\,dt,$$

which, by the continuity and linearity of $J(x_0-y)$, gives

$$\left\langle x_0-y, \frac{1}{T}\int_0^T w_0(t)\,dt - v\right\rangle_+ \geq 0,$$

and, in view of Proposition 2.5, $0\in -Ax_0+\int_0^T w_0(t)\,dt$. Since, in view of Lemma 3.7, $w_0$ is a selection of $F_{s_0}(\,\cdot\,,x_0)$, one has $0\in -Ax_0+\widehat{F}(x_0)$, a contradiction.

If (b) holds, i.e. if $F$ is single-valued and does not depend on $t$, then $w_n = F_{s_n}(u_n(\,\cdot\,))\to F(x_0)$ and, in view of (6.7) and Lemma 2.4, one has $\int_0^T\langle x_0 - y, F(x_0)-v\rangle_+\,dt\geq 0$ for any $(y,v)\in\operatorname{Gr}(A)$, i.e. by Proposition 2.5, $0\in Ax_0 + F(x_0)$, a contradiction.

Thus, we have proved the existence of $\lambda_1>0$ such that (6.5) holds for $\lambda\in(0,\lambda_1]$.

*Step 3.* Finally, due to the admissibility of $\Psi^\lambda$ on $U\cap\overline{D(A)}$, for $\lambda\in(0,\lambda_1]$, one has

$$\begin{aligned}\operatorname{Ind}_{M_A}((\Phi_T^\lambda, D_T^\lambda), U\cap\overline{D(A)}) &= \operatorname{Ind}_{M_A}((\Psi^\lambda(\,\cdot\,,1), D^{\lambda,1}), U\cap\overline{D(A)})\\ &= \operatorname{Ind}_{M_A}(\Psi^\lambda(\,\cdot\,,0), D^{\lambda,0}), U\cap\overline{D(A)}) = \operatorname{Ind}_{M_A}((\widehat{\Phi}_T^\lambda, \widehat{D}_T^\lambda), U\cap\overline{D(A)})\end{aligned}$$

where $\widehat{\Phi}_T^\lambda(x) := e_T(L(x,-\lambda A+\lambda\widehat{F}))$ and $D^{\lambda,1}$, $D^{\lambda,0}$ and $\widehat{D}_T^\lambda$ are natural decompositions. On the other hand, by the definition of the topological degree for $-A+\widehat{F}$, there is $\lambda_0\in(0,\lambda_1]$ such that, for $\lambda\in(0,\lambda_0]$, one has

$$\deg_M(-A+\widehat{F},U) = \operatorname{Ind}_{M_A}((\widehat{\Phi}_{\lambda T}^1, \widehat{D}_{\lambda T}^1), U\cap\overline{D(A)}).$$

Since, $\widehat{\Phi}_T^\lambda$ is the operator of translation along trajectories for an autonomous inclusion, one has $\widehat{\Phi}_{\lambda T}^1 = \widehat{\Phi}_T^\lambda$, which finally implies (6.3). □

**Proposition 6.9.** *Under the assumptions of Proposition* 6.8*, if* $\lambda_n\to 0^+$ *and* $(x_n)$ *is a bounded sequence such that* $x_n\in\Phi_T^{\lambda_n}(x_n)$*, then* $(x_n)$ *is relatively compact.*

To prove it one needs proceed like in the proof of Proposition 6.8.

Now we state the sufficient condition for branching of periodic points.

**Theorem 6.10.** *Suppose that $A$, $M$ and $F$ satisfy conditions* $(\mathrm{A}_1)$–$(\mathrm{A}_3)$, $(\mathrm{A}_4')$–$(\mathrm{A}_6')$. *If* $\deg_M(-A+\widehat{F},U)\neq 0$, *then there exists a connected set* $\Sigma\subset\mathcal{P}_T(\overline{U})$ *such that*

$$\overline{\Sigma}\cap\mathcal{P}_T(\overline{U})\neq\emptyset$$

*and either* $\Sigma\subset U\times(0,\infty)$ *is unbounded or* $\overline{\Sigma}\cap[\partial_M U\times(0,\infty)]\neq\emptyset$.

*Proof.* We shall apply Proposition 6.5. Define $\Phi\colon\overline{U}_A\times[0,\infty)\multimap M_A$, where $U_A:=U\cap\overline{D(A)}$, by the formula

$$\Phi(x,\lambda):=\Phi_T^\lambda(x)=e_T(L(x,-\lambda A+\lambda F)).$$

Clearly, by Example 2.12(a) and Theorems 3.1(P2) and 3.4(P2), $\Phi$ is upper continuous on $\overline{U}_A\times(0,\infty)$. The continuity on $\overline{U}_A\times\{0\}$ follows directly from Lemma 6.7. In view of Theorems 3.1(P3) and 3.4(P3), the map $\Phi$ satisfies the assumption (H2) of Proposition 6.5. By Proposition 6.9, $\Phi$ satisfies also (H3). Observe also that, in view of Theorem 6.6 and the assumption $0\notin(-A+\widehat{F})(\partial_M U\cap D(A))$, one obtains $\mathcal{B}_T(\overline{U})\cap\mathrm{bd}_{M_A}U_A\subset\mathcal{B}_T(\overline{U})\cap[\mathrm{bd}_M U\cap\overline{D(A)}]=\emptyset$. Observe that $\mathcal{R}(\Phi)=\mathcal{P}_T(\overline{U})$ and $\mathcal{B}(\Phi)=\mathcal{B}_T(\overline{U})$.

By Proposition 6.8 we get for sufficiently small $\lambda>0$

$$x\notin\Phi(x,\lambda)\ \text{for any}\ x\in\mathrm{bd}_{M_A}U_A\subset\mathrm{bd}_M U\cap\overline{D(A)}$$

and

$$\mathrm{Ind}_{M_A}((\Phi(\,\cdot\,,\lambda),D_T^\lambda),U_A)=\deg_M(-A+\widehat{F},U)\neq 0.$$

Hence, in view of Proposition 6.5, there exists a connected set $\Sigma\subset\mathcal{P}_T(\overline{U})\cap[U\times(0,\infty)]$ such that $\overline{\Sigma}\cap[\mathcal{B}_T(\overline{U})\times\{0\}]\neq\emptyset$ and $\Sigma$ is not contained in any compact subset of $\mathcal{P}_T(\overline{U})\cap[U\times(0,\infty)]$. This implies that either $\Sigma\subset U\times(0,\infty)$ is unbounded or $\overline{\Sigma}\cap[\partial_M U\times(0,\infty)]\neq\emptyset$. □

**Corollary 6.11.** *Let $A$, $M$ and $F$ satisfy conditions* $(\mathrm{A}_1)$–$(\mathrm{A}_3)$, $(\mathrm{A}_4')$–$(\mathrm{A}_6')$. *If $M_A$ is bounded and $\chi(M_A)\neq 0$, then there exists a connected and unbounded set of $T$-periodic points $\Sigma$ such that $\overline{\Sigma}\cap\mathcal{B}_T(M)\neq\emptyset$.*

*Proof.* By the definition of the topological degree and Proposition 2.21, for $t>0$,

$$\begin{aligned}\deg_M(-A+\widehat{F},M)&=\mathrm{Ind}_{M_A}((e_t(L(\,\cdot\,,-A+\widehat{F}),\widehat{D}_t),M_A)\\&=\mathrm{Ind}_{M_A}(\mathrm{id}_{M_A},M_A)=\chi(M_A)\neq 0.\end{aligned}$$

Hence, it suffices to apply Theorem 6.10. □

**Corollary 6.12** (Continuation principle)**.** *Suppose that $A$, $M$ and $F$ satisfy conditions* $(\mathrm{A}_1)$–$(\mathrm{A}_3)$, $(\mathrm{A}_4')$–$(\mathrm{A}_6')$. *If* $\deg_M(-A+\widehat{F},U)\neq 0$ *and* $(B_\lambda)$ *has no periodic points in* $\partial_M U\times(0,1)$, *then there exists a solution to the periodic problem* (6.1) *Moreover, the corresponding periodic point* $(x,1)$ *is connected with* *the set* $\{(x,0) \mid x \in U \cap D(A),\ 0 \in -Ax_0 + \widehat{F}(x_0)\}$ *by a closed connected set* $\Sigma \subset \overline{U} \times [0,1]$ *of* $T$*-periodic points.*

*Proof.* By Theorem 6.10, one obtains a connected set $\Sigma \subset \mathcal{P}_T(\overline{U})$ such that $\overline{\Sigma} \cap \mathcal{B}_T(\overline{U}) \neq \emptyset$ and, since $\Sigma \cap [\partial_M U \times (0,1)] = \emptyset$, then either $\Sigma \subset U \times (0,\infty)$ is unbounded or $\overline{\Sigma} \cap [\partial_M U \times [1,+\infty)] \neq \emptyset$. Since $\Sigma$ is connected, in both cases one has the existence of a periodic point in $\overline{U} \times \{1\}$. The other part of the assertion is clear. □

## 7. Example of application to PDEs

In this section, we discuss just one relatively simple example, in order to indicate the area of possible applications of the abstract setting. But it is clear that results can be applied to a broad class of partial differential equations and systems. Other problems, to which our setting is applicable, the reader can find e.g. in [6], [16], [37] and [17].

Consider the following constrained nonlinear problem

$$\text{(R)} \qquad \begin{cases} u_t = \Delta\rho(u) + f(t,x,u) & \text{on } (0,T) \times \Omega, \\ u_{|[0,T] \times \partial\Omega} = 0, \\ 0 \le u(t,x) \le m & \text{on } [0,T] \times \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ is a bounded domain with the smooth boundary $\partial\Omega$, $\rho\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable on $\mathbb{R} \setminus \{0\}$ and there exist $c > 0$ and $\alpha > \max\{0, (N-2)/N\}$ such that $\dot{\rho}(t) \le c|t|^{\alpha-1}$, for any $t \in \mathbb{R} \setminus \{0\}$ and such that $\rho(0) = 0$, $f\colon [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(t,x,0) \ge 0$ and $f(t,x,m) \le 0$ on $[0,T] \times \Omega$ and $m > 0$.

Let us put $(R)$ into an abstract setting. Let $E := L^1(\Omega)$ and define an operator $A\colon D(A) \multimap E$ by

$$Au := -\Delta\rho(u), \quad u \in D(A),$$

where $D(A) := \{u \in L^1(\Omega) \mid \rho(u) \in W_0^{1,1}(\Omega),\ \Delta\rho(u) \in L^1(\Omega)\}$.

It can be shown that $A$ is $m$-accretive and that the semigroup $S_A$ is compact ([37, Example 1.5.5 and Lemma 2.6.2]). Let $M := \{v \in L^1(\Omega) \mid 0 \le v(x) \le m\}$. Clearly $M$ is convex and closed; moreover, it can be shown that $J_\lambda^A(M) \subset M$ for $\lambda > 0$ (see [6]). Further, it is easy to verify that the mapping $F\colon [0,T] \times M \to E$ given by $F(t,u)(x) := f(t,x,u(x))$, for $(t,x) \in [0,T] \times M$ and $x \in \Omega$ is well-defined and continuous. It is also tangent to $M$. Indeed, for any $(t,u) \in [0,T] \times M$ and $h > 0$, one has

$$\begin{aligned} &d_M(u + hF(t,u)) \\ &\quad = \inf\{\|v - u - hF(t,u)\|_{L^1} \mid v \in L^1(\Omega),\ 0 \le v(x) \le m \text{ a.e. on } \Omega\} \end{aligned}$$

$$= \inf \left\{ \int_\Omega |v(x) - u(x) - hf(t,x,u(x))|\, dx \;\middle|\; v \in L^1(\Omega),\ v(x) \in [0,m] \text{ a.e. on } \Omega \right\}$$

$$\le \int_\Omega d_{[0,m]}(u(x) + hf(t,x,u(x)))\, dx,$$

which follows from the fact that the function $\overline{v}: \Omega \to [0,m]$ defined by

$$|\overline{v}(x) - u(x) - hf(t,x,u(x))| = d_{[0,m]}(u(x) + hf(t,x,u(x)))$$

is measurable and $\overline{v} \in M$. Further fix $x \in \Omega$, and consider the following cases:

- if $0 < u(x) < m$, then, for small sufficiently small $h > 0$, $u(x) + hf(t,x,u(x)) \in [0,m]$;
- if $u(x) = 0$, then $u(x) + hf(t,x,u(x)) = hf(t,x,u(x)) \ge 0$ and, for sufficiently small $h > 0$, $hf(t,x,u(x)) \le m$;
- if $u(x) = m$, then $u(x) + hf(t,x,u(x)) = m + hf(x,u(x)) \le m$ and, for sufficiently small $h > 0$, $u(x) + hf(x,u(x)) = m + hf(t,x,u(x)) \ge 0$.

Therefore, by use of the Lebesgue convergence theorem,

$$\lim_{h\to 0^+} \frac{d_M(u + hF(u))}{h} = 0,$$

i.e. $F(u) \in T_M(u)$.

Thus the problem $(R)$ has been transformed into the following one

$$\begin{cases} \dot{u} = -Au + F(t,u), \\ u \in M, \end{cases}$$

and $\chi(M_A) = \chi(M) = 1$, since $\overline{D(A)} = E$ and $M$ is convex. Hence, we can apply the results of the previous sections.

If $f$ does not depend on $t$, then, by Corollary 5.10, there exists a solution of

$$\begin{cases} -\Delta\rho(u) = f(x,u) & \text{on } \Omega, \\ u_{|\partial\Omega} = 0, \\ 0 \le u(x) \le m & \text{on } \Omega. \end{cases}$$

If $f(0,x,u) = f(T,x,u)$ for any $x \in \Omega$, $u \in \mathbb{R}$, then, by Corollary 6.11, there exists a connected and unbounded set $\Sigma \subset M \times (0,1)$ such that each $(u,\lambda) \in \Sigma$ is an integral solution of

$$\begin{cases} u_t = \lambda\Delta\rho(u) + \lambda f(t,x,u) & \text{on } (0,T) \times \Omega, \\ u_{|[0,T]\times\partial\Omega} = 0, \\ u(0,x) = u(T,x) & \text{on } \Omega, \\ 0 \le u(t,x) \le m & \text{on } \Omega, \end{cases}$$

and $\overline{\Sigma}$ contains $u_0\colon \Omega \to \mathbb{R}$ being a solution of

$$\begin{cases} 0 = \Delta\rho(u) + \widehat{f}(x,u) & \text{on } \Omega, \\ u_{|\partial\Omega} = 0, \\ 0 \le u(x) \le m, \end{cases}$$

where $\widehat{f}(x,u) := (1/T)\int_0^T f(t,x,u)\,dt$ for $(x,u) \in \Omega \times \mathbb{R}$.

The above applications show also that there is a further need for formulae allowing to compute the topological degree in concrete situations, e.g. degrees of isolated zeros, isolated zeros at cones, etc.

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# GENERALIZED DEGREES FOR COINCIDENCE PROBLEMS INVOLVING FREDHOLM OPERATORS

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Abstract. In the pper we collect nd briefly describe the most important homotopy invarints concerning coincidence problems with Fredholm operator.

## 1. Introduction

The coincidence problem

$$L(x) = f(x), \quad \text{(or, more general, } L(x) \in \phi(x)),$$

where $L$ is a linear Fredholm operator and $f$ is a continuous map (resp. $\phi$ is a multivalued map), seems to be a natural generalization of the fixed point problem. On the other hand many differential equations and inclusions may be rewritten in this form. There are various methods to deal with the coincidence problem, but here we restrict considerations to homotopy invariants often called a "generalized degree" or a "coincidence degree".

The coincidence degree theory for single-valued perturbations of a linear Fredholm operator of index zero was started by Mawhin (see e.g. [26], [27]) and next developed and applied by many authors (e.g. [16], [20], [33], [34], [29], [13]). If one wants to admit a nonnegative index of Fredholm operator, the situation becomes much more complicated and needs different tools. (see [37], [25], [19])

The main aim of this paper is to introduce briefly generalized degrees with some important properties and to mention about directions of generalization.

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Therefore we have to omit some technical details, which make definitions and results more general, but also more complicated. Many examples of applications are included in cited papers.

The paper is organized as follows. In the next section, after some preliminary remarks, we introduce the problem in the simplest situation. Section 3 contains a few examples of possible more general considerations strictly connected with the Mawhin degree. In Section 4 we compare three homotopy invariants defined for the coincidence problems with Fredholm operator of nonnegative index.

## 2. Mawhin coincidence degree

We start with some preliminary notation. If $V$ is a subset of a metric space, then we denote the *closure* and *boundary* of $V$ by $\operatorname{cl} V$ and $\operatorname{bd} V$, respectively. If $z$ belongs to a Banach space $E$, then $B^E(z,\varepsilon) = \{x \in E \mid \|x-z\| < \varepsilon\}$.

All single-valued maps considered in the paper are *continuous*. If $g: X \to Y$ is a map, $A$, $B$ are closed subsets of $X$ and $Y$, respectively, and $g(A) \subset B$, then we write $g: (X,A) \to (Y,B)$. By $I_X$ we denote the identity map of the space $X$.

As usual, by the homotopy between two single-valued maps $f_0, f_1: X \to Y$ we understand a map $H: X \times [0,1] \to Y$ such that $H(\,\cdot\,,0) = f_0$ and $H(\,\cdot\,,1) = f_1$.

Let $E$, $E'$ be Banach spaces. We denote by $L(E,E')$ the Banach space of bounded linear maps from $E$ to $E'$. An operator $L \in L(E,E')$ is called *Fredholm* if dimensions of its kernel $\operatorname{Ker} L$ and cokernel $\operatorname{Coker} L := E'/\operatorname{Im} L$ (where $\operatorname{Im} L$ is the image of $L$) are finite. Remind that by the index of $L$ one understands the integer

$$i(L) = \dim \operatorname{Ker} L - \dim \operatorname{Coker} L.$$

The set $\Phi_n(E,E')$ of all Fredholm operators of index $n$ is an open subset of $L(E,E')$. Both $\operatorname{Ker} L$ and $\operatorname{Im} L$ are direct summands in $E$ and $E'$, respectively. Therefore we may consider continuous linear projections $P: E \to E$ and $Q: E' \to E'$, such that $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Clearly $E$, $E'$ split into (topological) direct sums

$$\operatorname{Ker} P \oplus \operatorname{Ker} L = E, \quad \operatorname{Im} L \oplus \operatorname{Im} Q = E'. \tag{2.1}$$

Moreover, $L|_{\operatorname{Ker} P}$ is a linear homeomorphism onto $\operatorname{Im} L$. By $K_P: \operatorname{Im} L \to \operatorname{Ker} P$ we denote the inverse operator to $L|_{\operatorname{Ker} P}$.

Assume now, that $i(L) = 0$. Then, of course, $\dim \operatorname{Ker} L = \dim \operatorname{Coker} L = \dim \operatorname{Im} Q$. Let us fix the orientations in $\operatorname{Ker} L$ and in $\operatorname{Coker} L$, and take in $\operatorname{Im} Q$ the orientation which is preserved by $z \circ i$, where $i: \operatorname{Im} Q \to E'$ is the inclusion and $z: E' \to \operatorname{Coker} L$ is the quotient map. Denote by $J: \operatorname{Ker} L \to \operatorname{Im} Q$ the isomorphism preserving these orientations.

Observe that then $L + J \circ P$ is an isomorphism and $(L + J \circ P)^{-1} = K_P \circ (I_{E'} - Q) + J^{-1} \circ Q$. Moreover, the problem

$$0 = L(x) - f(x) \tag{2.2}$$

is equivalent to the following one:

$$\begin{aligned}0 &= (L + J\circ P)^{-1}\circ(L(x) - f(x))\\ &= (I_E - P)(x) - (K_P\circ(I_{E'} - Q) + J^{-1}\circ Q)\circ f(x)\\ &= x - (L + J\circ P)^{-1}\circ(f + J\circ P)(x).\end{aligned}$$

Below we introduce the definition of generalized degree. It is a bit simplified version of the one due to Mawhin (see [26], [28]).

Assume that $f\colon \mathrm{cl}\, U \to E'$ is a compact map, where $U$ is an open subset of $E$ and $L\colon E \to E'$ is a Fredholm operator of index 0.

**Definition 2.1.** If $F := L - f$ is such that $0 \notin F(\mathrm{bd}\, U)$, then the degree of $F$ in $U$ with respect to $L$ is defined by

$$D_L(F, U) := \deg(I_E - (L + J\circ P)^{-1}\circ(f + J\circ P), U, 0) \in \mathbb{Z},$$

where 'deg' is the Leray–Schauder degree.

This definition does not depend on the choice of projections $P$, $Q$ and an isomorphism $J$ (see [28]). Moreover, if $F$ has another representation of the form $F = L_1 + f_1$, then (under suitable assumptions), its degree does not depend on this representation (see [29]) In the next section we describe also larger classes of perturbations $f$ for which it is valid. The degree $D_L(F, U)$ has usual properties collected in the following theorem.

**Theorem 2.2.** *Under the previous assumptions:*

(a) (existence property) *if $D_L(F, U) \neq 0$, then $F$ has at last one zero in $U$ (i.e. there is a solution of $L(x) = f(x)$);*

(b) (excision property) *if $U_1 \subset U$ is an open set such that $0 \notin F(\mathrm{cl}\, U \setminus U_1)$, then*
$$D_L(F, U) = D_L(F|_{U_1}, U_1);$$

(c) (addition property) *if $U_1$, $U_2$ are disjoint open subsets of $U$ such that $0 \notin F(\mathrm{cl}\, U \setminus (U_1 \cup U_2))$, then*
$$D_L(F, U) = D_L(F|_{U_1}, U_1) + D_L(F|_{U_2}, U_2);$$

(d) (homotopy invariance property) *if $H\colon \mathrm{cl}\, U \times [0, 1] \to E'$ is a compact homotopy such that $L(x) \neq H(x, \lambda)$ for $x \in \mathrm{bd}\, U$ and each $\lambda \in [0, 1]$, then the map $\lambda \mapsto D_L(L|_{\mathrm{cl}\, U} - H(\cdot, \lambda), U)$ is constant on $[0, 1]$. In particular*
$$D_L(L|_{\mathrm{cl}\, U} - H(\cdot, 0), U) = D_L(L|_{\mathrm{cl}\, U} - H(\,\cdot\,, 1), U).$$

Many further consequences of these properties for the degree defined above and its generalizations may be applied in various differential problems (see e.g. [13]–[15], [33], [35], [31]).

## 3. Some possible generalizations

As we have mentioned in Introduction, this section is devoted to a few directions of generalization of the coincidence problem, strictly connected with the generalized degree. The whole section concerns the situation when $i(L) = 0$.

**Perturbations.** The generalized degree from Definition 2.1 was in fact defined for so-called $L$-compact perturbations $f$ (see [28]). Let us remind that $f$ is $L$-compact if $(L+J\circ P)^{-1}\circ f$ is a compact map (what implies that $(L+A)^{-1}\circ f$ is compact provided that $A\colon E\to E'$ is an arbitrary linear operator such that $L+A$ is invertible and $\dim \operatorname{Im} A<\infty$). If one assumes that $L$ is a bounded operator (as we have done for simplicity), then an $L$-compact map is simply a compact one. But in general situation, Definition 2.1 is valid also for unbounded Fredholm operators.

It is also natural to consider $L$-condensing maps (see [20], [33]). Namely, let $\mu$ be a measure of noncompactness in $E$ (see e.g. [2]), then $f$ is an $L$-condensing map provided that for any bounded set $V\subset \operatorname{cl} U$ if $\mu(V)>0$, then $\mu((L+J\circ P)^{-1}\circ f(V))<\mu(V)$. The generalized degree for such maps can be defined by replacing in Definition 2.1 the Leray–Schauder degree by the Nussbaum–Sadovskii one.

Sometimes another special forms of perturbations are needed in applications and the respective degrees are defined in particular cases. ([12], [8]).

If we replace a single-valued map $f$ by a multivalued one $\phi\colon \operatorname{cl} U\multimap E'$ and assume that $\phi$ is $L$-compact with compact convex values, then the respective degree may be obtained by using the Leray-Schauder degree for multivalued vector fields (see [34]). As well one can consider $L$-condensing multivalued maps (see [36]). All above invariants have properties mentioned in Theorem 2.2.

Some examples of applications one can find also in [28].

**Continuous deformations of the linear part.** The homotopy property of the degree is a very important one, especially in applications, since it often allows to simplify the problem. Observe that in Theorem 2.2(d) the homotopy concerns only a perturbation, while $L$ is constant, what means that the role of $L$ is similar to the one of the identity map in a fixed point problem. But it is not necessary. However, continuous deformations of the Fredholm operator need some concept of saving an orientation along homotopies. Below we introduce very briefly two possible approaches to this problem.

Denote by $GL(E,E')$ and $\mathcal{K}(E,E')$ the subsets of $L(E,E')$ consisting of all isomorphisms and of all compact maps, respectively, and by $GL_c(E,E')$ the group of all isomorphisms of the form $I_E-K$, where $K\in\mathcal{K}(E,E')$. The operator $S\in GL(E',E)$ is said to be a *parametrix* of $L$ if $L\circ S\in GL_c(E',E')$. By a *corrector* of $L\in\Phi_0(E,E')$ we understand a linear map $A\colon E\to E'$ such that $L+A\in GL(E,E')$ and $\dim\operatorname{Im} A<\infty$. The set of all correctors of $L$ is denoted by $C(L)$. Observe that, e.g. $J\circ P\in C(L)$. Moreover, for any two

$A, B \in C(L)$, $(L+A)^{-1} \circ (L+B) \in GL_c(E,E')$, and for any $A \in C(L)$, $L \circ (L+A)^{-1} \in GL_c(E,E')$, i.e. $(L+A)^{-1}$ is a parametrix for $L$.

Fitzpatrick and Pejsachowicz (see [14], [15]) define the orientation as a function $\varepsilon\colon Z \to \{-1,1\}$, where $Z$ is a suitable subset of $GL(E,E')$, satisfying two following properties:

(i) $\varepsilon(I_E) = 1$.

(ii) If $\dim \operatorname{Im}(M_1 - M_2) < \infty$, then $\varepsilon(M_1) = \varepsilon(M_2)$ if and only if $\deg((M_1^{-1} \circ M_2), B^E(0,1), 0) = 1$, where, as earlier, deg denotes the Leray–Schauder degree.

It allows to define the generalized degree for problem (2.2) as follows:

**Definition 3.1.** If $F := L - f$ is such that $0 \notin F(\operatorname{bd} U)$, then the degree of $F$ in $U$ with respect to $L$ is defined by

$$D_{FP}(F,U) := \varepsilon(S) \deg(I_E - S \circ (f + A), U, 0) \in \mathbb{Z},$$

where $S$ is an arbitrary parametrix of $L$.

It was shown in [14], [15] that $D_{FP}$ is well-defined and satisfies properties of Theorem 2.2. One can easily compare it with Definition 2.1, replacing $S$ by a particular parametrix $(L + J \circ P)^{-1}$. Moreover, the homotopy invariance property can be extended to the following one.

**Proposition 3.2.** *If* $[0,1] \mapsto L_\lambda \in \Phi_0(E,E')$ *is a continuous map and* $H\colon \operatorname{cl} U \times [0,1] \to E'$ *is a compact homotopy such that* $L_\lambda(x) \neq H(x,\lambda)$ *for all* $(x,\lambda) \in \operatorname{bd} U \times [0,1]$*, then*

$$D_{FP}(L_0 - H(\,\cdot\,,0), U) = \varepsilon(L_0 + A_0)\varepsilon(L_1 + A_1) D_{FP}(L_1 - H(\,\cdot\,,1), U),$$

*where the map* $[0,1] \ni \lambda \mapsto A_\lambda \in \mathcal{K}(E,E')$ *is continuous and such that* $L_\lambda + A_\lambda \in GL(E,E')$ *for all* $\lambda \in [0,1]$.

The degree defined above has been developed and applied in many ways (see e.g. [13] for not bounded Fredholm operators of index 0, [35] for $L$-contractive perturbations).

Benevieri and Furi proposed another approach to the orientation based on the notion of correctors (see [3]). The set $C(L)$ is divided into two classes by the following equivalence relation: $A \sim B \Leftrightarrow \det(L+A)^{-1} \circ (L+B) > 0$. Since $(L+A)^{-1} \circ (L+B) = I_E - K$, where $\dim \operatorname{Im} K < \infty$, this determinant is well defined ([1]). An orientation of $L$ is simply one of the two classes of correctors. It determines the choice of such orientation in some neighborhood of $L$ in $\Phi_0(E,E')$ (and then also along the homotopy) thanks to the following fact.

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([1]) $\det(I_E - K) := \det(I_E - K)|_{E_1}$, where $E_1$ is an arbitrary finite dimensional subspace of $E$ containing $\operatorname{Im} K$; the definition does not depend on the choice of $E$.

**Theorem 3.3** ([4]). *Let $A$, $B$ be two $L$-equivalent correctors of an operator $L$ (i.e. determining the same orientations of $L$). Then there exist two neighborhoods $U_A$ and $U_B$ of $A$ and $B$ in the set of all operators with finite dimensional range, and a neighborhood $V_L$ of $L$ in $\Phi_0(E, E')$ such that $A'$ and $B'$ are $L'$-equivalent correctors of $L'$ for any $A' \in U_A$ and $B' \in U_B$.*

A definition of the respective degree can be similar to Definition 3.1, but in fact Benevieri and Furi use their orientation in another way (see the next subsection). Connections and differences between these notions are described in [4]. The authors also compare them with the earlier concepts due to Elworthy and Tromba ([10], [11]).

**Nonlinear Fredholm operators.** A continuously differentiable map $f: U \to E'$ is called Fredholm if at each $x \in U$ its Fréchet derivative $Df(x)$ is a linear Fredholm operator (of index 0). By the orientation of $f$ one often understands the orientation of the family of its derivatives in some sense. A degree, which nontriviality implies the existence of a solution to the problem

$$f(x) = y,$$

is strictly connected with a respective concept of this orientation. Roughly speaking its construction is the following: Let $y$ be a regular value of $f$ (i.e. $f^{-1}(y)$ is empty or, for each $x \in F^{-1}(y)$, $Df(x)$ is an isomorphism (since $i(L) = 0$)). Then

$$\deg(g, U, y) = \sum_{x \in f^{-1}(y)} \eta(Df(x)),$$

where $\eta$ may be understood as the orientation $\varepsilon$ of the family $\{Df(x)\}$ (see [16] for details), or, in the Benevieri and Furi approach, $\eta(Df(x)) = \operatorname{sign} Df(x)$, i.e. $\eta(Df(x) = 1$, if a trivial operator determines the orientation of $Df(x)$ and $\eta(Df(x)) = -1$ otherwise. For not necessarily regular value $y$ of $f$, $\deg(f, U, y) = \deg(f, W, z)$, where $z$ is a regular value of $f$ sufficiently close to $y$, and $W$ is a respective subset of $U$ (such that $f|_{\operatorname{cl} W}$ is a proper map).

More details and applications one can find in, e.g. [16], [4]–[6].

Such invariants are also defined for maps acting between Banach manifolds (see [4], [3], [32]).

## 4. Invariants admitting a dimensional defect

If we admit that the index of $L$ is nonnegative, then previous methods are not sufficient (comp. [26]), since, roughly speaking, degree of Mawhin's type is trivial. Neverthless there are some ways to deal with such situation. We consider below a multivalued situation, i.e.

$$L(x) \in \phi(x). \tag{4.1}$$

with usual assumption: $\{x \in \operatorname{bd} U \mid L(x) \in \phi(x)\} = \emptyset$.

The first approach develops ideas described above, namely, the degree is defined by

$$D((L,\phi),U) := \{\deg(I_E - M_{V,J},U,0) \mid V \text{ is a subspace of } \operatorname{Ker} L, \\ \dim V = \dim \operatorname{Coker} L\},$$

where $M_{J,V} = R_V \circ P + [J \circ Q + K_P \circ (I_{E'} - Q)] \circ \phi$ and $J\colon \operatorname{Coker} L \to \operatorname{Ker} L$ is a one to one linear map, $V = \operatorname{Im} J$, $R_V\colon \operatorname{Ker} L \to \operatorname{Ker} L$ is a projector such that $V = \operatorname{Im} R_V$, $\phi$ is a $L$-condensing multivalued map with compact convex values (see [1]). As one can see, the degree is a subset of $\mathbb{Z}$. It has usual properties mentioned in Theorem 2.2, in particular its existence property says, that if it is different from $\{0\}$, then the problem (4.1) has a solution.

Quite different degree is due to Borisovich and Zvyagin (see [7], [37]). It takes values in Rohlin–Thom ring of bordisms and may be generalized in the same directions as Mawhin's degree (see [9], [30]), but still for multivalued maps with convex values.

The last invariant was constructed by Kryszewski (see [25]) for multivalued maps with not necessarily convex values. Remind that the wide class of maps, for which the degree or the fixed point index can be defined is the one of maps admissible in the sense of Górniewicz, i.e. determined by a pair of continuous maps

$$\operatorname{cl} U \xleftarrow{p} \Gamma \xrightarrow{q} E'$$

such that $F(x) = q(p^{-1}(x))$ for $x \in \operatorname{cl} U$, where $p$ is a Vietoris map, i.e. a proper surjection with acyclic fibers (with respect to the Čech cohomology with integer coefficients).

But a possible dimensional defect does not allow previous (co)homological methods (see [21], [22]), because they lead to a trivial invariant. The same reason cuts the class of admissible maps to one of its following subsets:

- $c$-admissible maps, i.e. such that $p$ is a cell-like map, i.e. a proper surjection with cell-like fibers (2) (this class contains the maps with convex values);
- admissible in sense of Górniewicz maps with additional assumption: $\dim p := \sup \dim p^{-1}(x) < \infty$.

In both situations the construction is the same: in finite dimensional situation, i.e. for $E = \mathbb{R}^m$, $E' = \mathbb{R}^n$, $m \geq n$ we consider the sequence of maps:

$$(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) \xleftarrow{L\circ p - q} (\Gamma, \Gamma') \xrightarrow{\ p\ } (\operatorname{cl} U, \operatorname{bd} U) \\ \xrightarrow{i_1} (\mathbb{R}^m, \mathbb{R}^m \setminus U) \xleftarrow{i_2} (\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)),$$

(2) Compact space $A$ is *cell-like* if there exists an absolute neighborhood retract $Y$ and an embedding $i\colon A \to Y$ such that the set $i(A)$ is contractible in any of its neighborhoods $V \subset Y$.

where $\rho$ is such that $(L - q(p^{-1}))(\mathrm{bd}\, U) \subset \mathbb{R}^n \setminus B^n(0,\rho)$, and the maps induced on the level of cohomotopy sets (groups) (see [23]):

$$i_2^{\#} \circ (i_1^{\#})^{-1} \circ (p^{\#})^{-1} \circ q^{\#} \colon \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) \to \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,\varepsilon)).$$

Since $\pi^n(S^n) \cong \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho))$ and $\pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,\varepsilon)) \cong \pi^n(S^m)$, the degree can be defined by

$$\deg((p,q),U,0) := \mathcal{K}(\mathbf{1}) \in \pi^n(S^m),$$

where $\mathcal{K} := i_2^{\#} \circ (i_1^{\#})^{-1} \circ (p^{\#})^{-1} \circ q^{\#}$ and $\mathbf{1}$ is a homotopy class of $id\colon S^n \to S^n$ in $\pi^n(S^n) \cong \mathbb{Z}$.

In infinite dimensional situation for compact maps one can use a standard idea of respective Schauder approximations (for details see [25] or [19]) and next generalize the degree to noncompact maps, called fundamentally restrictible containing, among others, $L$-compact and $L$-condensing ones (see [17], [19]). It is worth mention, that this approach needs some non trivial algebraic results, especially the cohomotopy version of Vietoris–Begle Theorem due to Kryszewski (see [24], [25]).

While this invariant was defined for the largest class of perturbations, the Fredholm operator $L$ had to be fixed. The recent results concern the possibility of continuous deformation of $L$ along the homotopy (see [18]). The problem for nonlinear Fredholm maps is still open.

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# MULTIVALUED GENERALIZATIONS OF THE WAŻEWSKI RETRACT THEOREM

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Abstract. This note is a short survey on the famous Ważewski's retract method and its subsequent developments, especially those appropriate for differential inclusions. The corresponding multivalued problems are discussed with some former as well as present results on the existence of viable trajectories.

## 1. Ważewski's retract method

As a starting point let us state the following problem:

Assume that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is such that the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(x(t)) & \text{for } t \geq 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

has a unique solution for every $x_0 \in \mathbb{R}^n$, which depends continuously on the initial condition (we can think about $f$ as a Lipschitz continuous map), and $K \subset \mathbb{R}^n$ is a closed subset.

(P) Is there any solution $x$ to problem (1.1) such that $x(t) \in K$ for every $t \geq 0$?

Each solution to problem (P) is called a *viable trajectory* in $K$ when taking into account motivations from the mathematical biology.

In the literature one can find several sufficient conditions for positive invariance of the set $K$, i.e. to assure that solutions starting in $K$ remain there forever.

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One of the best known is the Nagumo condition introduced in 1942 (see [30]) in terms of tangent cones. If $K$ is positively invariant, each point of $K$ is a starting point of a trajectory solving problem (P). Otherwise, some trajectories leave the set $K$ or, in other words, from some points in $K$ there start trajectories going immediately outside the set. Then, these points may be called the points of egress. When they appear, problem (P) becomes nontrivial.

It occurs that one can study topological properties of the set of egress points to obtain sufficient conditions for the existence of viable trajectories. In 1947 (see [36, Theorem 2]) Ważewski proved his famous theorem where such condition was described in terms of retracts. Let us recall it briefly.

Denote by $\pi: \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ a dynamical system corresponding to our equation $\dot{x}(t) = f(x(t))$, i.e. for each $x_0 \in \mathbb{R}^n$ and $t \geq 0$ we put $\pi(x_0, t)$, the value of the solution to problem (1.1) at time $t$.

Assume that $K = \overline{V}$, where $V$ is open in $\mathbb{R}^n$. We define the set of *egress points* of $V$

$$V^{\mathrm{e}} := \{x_0 \in \partial V \mid \exists \delta > 0 : \pi(\{x_0\} \times [-\delta, 0)) \subset V\},$$

and the set of *strict egress points* of $V$

$$V^{se} := \{x_0 \in V^{\mathrm{e}} \mid \exists \delta > 0 : \pi(\{x_0\} \times (0, \delta]) \subset \mathbb{R}^n \setminus K\}.$$

Following Ważewski we assume that

(W) $V^{se} = V^{\mathrm{e}}$.

**Theorem 1.1** ([36, Theorem 2]). *If $Z \subset V \cup V^{\mathrm{e}}$, $Z \cap V^{\mathrm{e}}$ is a retract* (1) *of $S$, and $Z \cap V^{\mathrm{e}}$ is not a retract of $Z$, then there exists a viable trajectory in $V$ which starts from a point of $Z \setminus V^{\mathrm{e}}$.*

A simple and brilliant idea of proof will be presented after the next theorem below. Notice that, if $Z = V \cup V^{\mathrm{e}}$, we simply get: if $V^{\mathrm{e}}$ is not a retract (in fact: a strong deformation retract) of $Z$, then there exists a viable trajectory in $V$.

In 1976 Conley (see [9]) formulated a new statement of the Ważewski theorem where condition (W) was replaced by the following one:

(C) the set $K^- := \{x_0 \in K \mid \forall \varepsilon > 0 : \pi(\{x_0\} \times (0, \varepsilon)) \not\subset K\}$ is closed in $K^* := \{x \in K \mid \exists t > 0 : \pi(x, t) \notin K\}$.

**Theorem 1.2** ([9, Theorem 1.3]). *For a closed set $K \subset \mathbb{R}^n$, if condition* (C) *is satisfied, then $K^-$ is a strong deformation retract of $K^*$ and $K \setminus K^*$ is closed in $K$.*

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(1) A closed subset $M$ of a space $X$ is said to be a *retract* of $X$ provided there exists a map $r: X \to M$ such that $r(x) = x$, for every $x \in M$, and *strong deformation retract* of $X$ if there is a homotopy $h: X \times [0,1] \to X$ such that $h(x, 0) = x$, $h(x, 1) \in M$, for all $x \in X$, and $h(x, t) = x$, for each $x \in M$ and $t \in [0, 1]$.

*Sketch of proof.* We define the *exit function* $\tau: K^* \to \mathbb{R}$,

$$\tau(x) := \sup\{t \geq 0 \mid \pi(\{x\} \times [0, t]) \subset K\}.$$

Condition (C) (in the original Ważewski theorem condition (W)) implies that $\tau$ is continuous.

Define a homotopy $h: K^* \times [0, 1] \to K^*$ by $h(x, \lambda) := \pi(x, \lambda\tau(x))$. It is easy to check that $h(x, 0) = x$ and $h(x, 1) \in K^-$ for every $x \in K^*$, and $h(x, \lambda) = x$ for every $x \in K^-$. So, $K^-$ is a strong deformation retract of $K^*$. From the continuity of $\pi$ it follows that $K^*$ is open in $K$, and hence, $K \setminus K^*$ is closed in $K$. □

Notice that the above theorem immediately implies:

**Corollary 1.3.** *If $K^-$ is closed and is not a strong deformation retract of $K$, then there exists a viable trajectory in $K$.*

Let us remark that Conley's formulation of the retract theorem allows us to consider sets with empty interior. Moreover, the set $K^-$ can be localized and its closedness can be verified in practice.

## 2. Briefly on the homotopy index

It is easy to check that, if $K$ is compact, then

$$\exists x \in K \ \pi(\{x\} \times [0, \infty)) \subset K \Leftrightarrow \exists x \in K \ \pi(\{x\} \times \mathbb{R}) \subset K.$$

Thus in this case, problem (P) is equivalent to the problem of nonemptiness of the set

$$\operatorname{inv}(K, f) := \{x \in K \mid \pi(\{x\} \times \mathbb{R}) \subset K\}$$

which is called a *maximal invariant subset of* $K$ (one easily checks that it is really invariant).

The idea of defining a suitable homotopy invariant to study the existence of invariant sets appeared in 1971 in Conley and Easton's paper [11] on isolated invariant sets and isolated blocks, and it started what is now known as the Conley index theory. The celebrated monograph [10] contains main ideas of the theory. Later on many proofs were simplified [34] and the ideas were developed in many directions such as discrete dynamical systems (see [31], [29]), infinite dimensional flows (see [33], [4], [21], [23]) and multivalued systems (see [28], [25], [26]). Since the beginning the Conley approach has been successfully applied to study shock waves and periodic traveling waves [12]–[14], [35] as well as to study qualitative properties of solutions to pendulum-type equations with friction [26]. The list of references concerning the Conley index theory and its applications may be enlarged. We refer the reader to survey-type articles in [27] and to [1] for more information.

We recall basic notions of the Conley index theory in the simple case of $\mathbb{R}^n$. If $K = \overline{\operatorname{int} K} \subset \mathbb{R}^n$ is compact and $S = \operatorname{inv}(K, f) \subset \operatorname{int} K$, then $K$ is called an *isolating neighbourhood*, and $S$ an *isolated invariant set*.

By an *index pair* for an isolated invariant set $S \subset \mathbb{R}^n$ we mean a compact pair $(P_1, P_2)$ such that

(i) the set $\overline{P_1 \setminus P_2}$ is an isolating neighbourhood for $S$;
(ii) (positive invariance of $P_2$ in $P_1$) if $x \in P_2$ with $\pi(x,t) \in P_1$ for every $t \in [0, t_0]$, then $\pi(x,t) \in P_2$ for every $t \in [0, t_0]$;
(iii) if $x \in P_1$ and there is $t \geq 0$ with $\pi(x,t) \notin P_1$, then there exists $0 \leq t_0 < t$ such that $\pi(x, t_0) \in P_2$.

Notice that $P_2$ in the index pair $(P_1, P_2)$ plays a role of an exit set for $P_1$, and $P_2$ need not be contained in a boundary of $P_1$. The situation where $P_2 = (P_1)^-$ is a particular case.

Two main theorems in the Conley index theory are as follows.

**Theorem 2.1** ([34, Theorem 4.3])**.** *If $K$ is an isolating neighbourhood for $S$, then there exists an index pair $(P_1, P_2)$ for $S$ with $P_1 \subset K$ and both $P_1$ and $P_2$ positively invariant in $K$.*

**Theorem 2.2** ([34, Theorem 4.10, Corollary 4.11])**.** *The Conley homotopy index of an isolated invariant set $S$,*

$$I(S, f) := [P_1/P_2, [P_2]]$$

*does not depend on the choice of an index pair $(P_1, P_2)$ for $S$, where $[P_1/P_2, [P_2]]$ stands for a homotopy type of the pointed space $(P_1/P_2, [P_2])$. If $I(S, f) \neq \overline{0}$ (is not a trivial homotopy type $[\{*\}, *]$), then $S \neq \emptyset$.*

A power of the above topological tool lies in its homotopy invariance (see [27, p. 24], for collected properties of the index), and this gives a kind of its superiority over the Ważewski method. But, on the other hand, there are examples where $[K/K^-, [K^-]] = \overline{0}$ while $K^-$ is not a strong deformation retract of $K$ and the Ważewski theorem implies the existence of a viable trajectory in $K$ (see [10, Chapter II]).

## 3. When multivalued problems appear

When a map $f$ in (1.1) is less regular, or we have to study a multivalued problem

$$\begin{cases} \dot{x}(t) \in F(x(t)) & \text{for a.e. } t \in \mathbb{R}, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

then we meet a difficulty that from a point there can start a lot of solutions, and the differential equation (inclusion) does not generate a dynamical system.

Instead, a so-called *multivalued dynamical system* appears which can be described as below.

Let $S_F(x_0)$ denote the set of solutions to problem (3.1). If $F$ satisfies

(F1) $F$ is upper semicontinuous (u.s.c.) with compact convex values,

(F2) $F$ has a sublinear growth, i.e. there is a constant $c \geq 0$ such that $|y| \leq c(1+|x|)$ for each $x \in \mathbb{R}^n$ and $y \in F(x)$,

then the map $x_0 \mapsto S_F(x_0) \subset C(\mathbb{R},\mathbb{R}^n)$ is u.s.c. with compact $R_\delta$ values (2). We define a multivalued Poincaré operator $P\colon \mathbb{R}^n \times \mathbb{R} \multimap \mathbb{R}^n$, $P(x_0,t) := e_t \circ S_F(x_0)$, where $e_t\colon C(\mathbb{R},\mathbb{R}^n) \to \mathbb{R}^n$, $e_t(x) := x(t)$ is an evaluation map. The operator satisfies the obvious conditions $P(x,0) = \{x\}$, $y \in P(x,t)$ if and only if $x \in P(y,-t)$, and $P(x,t+s) = P(P(x,t)\times\{s\}), ts \geq 0$. Moreover, $P$ is admissible in the sense of Górniewicz (see, e.g. [22]).

The second difficulty in the case of differential inclusions is that there are two exit sets, different in general,

$$\begin{aligned} K^-(F) &:= \{x_0 \in \partial K \mid \forall x \in S_F(x_0)\ \forall t > 0 : x([0,t]) \not\subset K\},\\ K_e(F) &:= \{x_0 \in \partial K \mid \exists x \in S_F(x_0)\ \forall t > 0 : x([0,t]) \not\subset K\},\end{aligned}$$

with $K^-(F) \subset K_e(F)$, and it is natural that from points in $K_e(F) \setminus K^-(F)$ there can start trajectories going into $K$ for both positive and negative times. This implies that the *isolation assumption*

$$\operatorname{inv}(K,F) := \{x_0 \in K \mid \exists x \in S_F(x_0)\ \forall t \in \mathbb{R} : x(t) \in K\} \subset \operatorname{int} K$$

is hard to check.

## 4. Isolation assumption and the homotopy index

In spite of what has been noted in the last lines above, let us *a priori* assume that $\operatorname{inv}(K,F) \subset \operatorname{int} K$, and that the set $K$ is compact. Then the Conley index theory can be adopted. We refer to Mrozek's paper [28] where a cohomological index has been constructed. An alternative approximation technique can be applied to construct a homotopy index (see [26]) which is closer to Ważewski's retract method we deal with. Let us briefly recall a sketch of this construction.

Let $F\colon \mathbb{R}^n \multimap \mathbb{R}^n$ satisfy assumptions (F1)–(F2). By the Cellina approximation theorem ([7, Theorem 1]; see also results in [8]) it follows that for any $\varepsilon > 0$ there exists $\varepsilon_0 > 0$ such that, for each $0 < \delta \leq \varepsilon_0$, there is a single-valued Lipschitz $\delta$-approximation (3) of $F$, and each level $h(\cdot,t)$ of a linear homotopy joining any two such approximations is an $\varepsilon$-approximation of $F$. This easily leads to the following:

(2) A space $X$ is a compact $R_\delta$-set provided it is homeomorphic to an intersection of a decreasing sequence of compact contractible spaces. In particular, it is acyclic.

(3) We say that $f\colon X \to \mathbb{R}^n$ is a $\delta$-approximation of $F\colon X \multimap \mathbb{R}^n$ if $f(x) \in F(B(x,\delta)) + B(0,\delta)$ for every $x \in X$.

**Proposition 4.1** (comp. [26, pp. 150–152]). *Let* $K = \overline{\operatorname{int} K} \subset \mathbb{R}^n$ *be a compact set,* $F$ *satisfies* (F1)–(F2), *and* $\operatorname{inv}(K, F) \subset \operatorname{int} K$. *Then there is* $\varepsilon > 0$ *such that* $\operatorname{inv}(K, f) \subset \operatorname{int} K$ *for every* $\varepsilon$*-approximation* $f$ *of* $F$, *and* $I(\operatorname{inv}(K, f_1), f_1) = I(\operatorname{inv}(K, f_2), f_2)$ *for each Lipschitz* $\varepsilon$*-approximations* $f_1$ *and* $f_2$.

Now, we define the homotopy index for a multivalued flow generated by differential inclusion (3.1) as $I(\operatorname{inv}(K, F), F) := I(\operatorname{inv}(K, f), f)$, where $f$ is as in the above proposition. The index satisfies standard properties ([26, Theorems 5.3.1–5.3.4]). In particular, if $I(\operatorname{inv}(K, F), F) \neq \overline{0}$, then there exists a viable trajectory in $K$.

As we can see, the index is known implicitly as an index of sufficiently near approximations. An important question is how to describe it in terms of a given right-hand side $F$ of a differential inclusion, namely, examining behaviour of $F$ on the boundary of $K$. It occurs that in some situations it is possible. We will come back to it at the end of the last section.

## 5. Without a priori isolation assumption: from connectedness to deformation retracts conditions

Now we do not assume that $\operatorname{inv}(K, F) \subset \operatorname{int} K$. As at the beginning we look for sufficient conditions for the existence of viable trajectories in $K$ in terms of exit sets to obtain multivalued generalizations of the Ważewski theorem. But we have two exit sets $K^-(F)$ and $K_e(F)$, and it is not clear which one is more useful.

In the first papers dealing with differential problems without uniqueness ([2], [3], [24]) the authors did not use the sets $K^-(F)$ and $K_e(F)$ but followed Ważewski and, instead, considered sets of egress and strict egress points ([4]). They assumed that these sets were equal which implied that the set of egress points was relatively invariant in $\partial K$. In particular, there was no point in $\partial K$ from which there started trajectories going into $\operatorname{int} K$ for both positive and negative times. The Ważewski type result was the following:

- If the set of egress points is not a multivalued retract ([5]) of $K$, then there is a viable trajectory in $K$.

From a topological point of view this result is very weak. Even a sphere $\partial B(0, 1) \in \mathbb{R}^n$ is a multivalued retract of the unit ball $\overline{B(0, 1)}$. What we know is that $S \subset \partial K$ is not a multivalued retract of $K$ if, for instance, $K$ is connected and $S$ is disconnected. Obviously, a connectedness criterion is far from the strong

([4]) A point $x_0 \in \partial K$ is an *egress point* (see [3]), if there is a solution $x$ such that $x([0, t_1)) \subset \operatorname{int} K, x([t_1, t_2]) \subset \partial K$ and $x(t_2) = x_0$ for some $0 < t_1 \leq t_2$. An egress point $x_0$ is a *strict egress point*, if for every solution $x \in S_F(x_0)$, $c(x) := \sup\{t \geq 0 \mid x([0, t]) \subset K\} < \infty$, and $x([c(x), c(x) + \varepsilon]) \not\subset K$ for any $\varepsilon > 0$.

([5]) We say that $A \subset X$ is a *multivalued retract* of $X$ if there exists an u.s.c. map $\Phi: X \multimap A$ with compact values such that $x \in \Phi(x)$, for every $x \in A$.

deformation retract approach proposed by Ważewski. Some more general results have been proved in [20] and [16], where the set $K_e(F)$ is used allowing us to consider sets of constraints with empty interior (like with Conley's condition (C)). Let us state, for example:

**Theorem 5.1** (comp. [16, Theorem 2.1 and Corollary 2.2]). *Assume that the set $K_e := K_e(F)$ is closed and*

- *there is a subset $A \subset K$, $K_e \subset A$, and there exists a retraction $r: A \to K_e$ such that $x([0, \tau_K(x)]) \subset A$ for every $x_0 \in K_e$ and every $x \in S_F(x_0)$.*

*If there is no multivalued admissible deformation* (6) *of $K$ onto $K_e$, then there is a viable trajectory in $K$.*

It occurs that, to obtain a sufficient condition for the existence of viable trajectories in terms of strong deformation retracts, the smaller exit set $K^-(F)$ is more appropriate. It is worth adding that $K^-(F)$ can be characterized by Bouligand tangent cones (see, e.g. [6, Lemma 5.2]). This characterization is due to Cardaliaguet who has proved in [5] that there exists a viable trajectory in a convex set (connected $C^{1,1}$-manifold) $K$ whenever $K^-(F)$ is closed and disconnected. This was the first Ważewski type result without paying any attention to the set $K_e(F)$.

Recall that the deformation retract approach consists in possibility of continuous deformation along trajectories of a dynamical system. Therefore, the idea has arisen to find a Lipschitz selection, or a sequence of sufficiently near Lipschitz approximations of $F$, generating a dynamical system with the same exit set $K^-(f) = K^-(F)$. This selection technique is possible under rather strong assumptions on regularity of the map $F$ (see [20, Theorem 3.16]). An approximation Lemma 3.3 in [6] allows us to obtain the following more general result.

**Theorem 5.2.** *Let $F: \mathbb{R}^n \multimap \mathbb{R}^n$ satisfying* (F1)–(F2) *be continuous. Assume that $K \subset \mathbb{R}^n$ is a compact $C^{1,1}$ $n$-manifold with a boundary, $K^-(F)$ is closed and, if it is nonempty, it is a $C^{1,1}$ $(n-1)$-submanifold of $\partial K$ with a boundary. If $K^-(F)$ is not a strong deformation retract of $K$, then there is a viable trajectory in $K$.*

In the proof we find, following Lemma 3.3 in [6], a sequence of Lipschitz $(1/n)$-approximations $(f_n)$ of $F$ with $K^-(f_n) = K^-(F)$, and, by the Ważewski theorem, a sequence of viable solutions $x_n$ corresponding to $f_n$. Since $K$ as well as the graph of $F$ are closed, we can go with $n$ to infinity, and obtain a viable trajectory $x$ for $F$ in $K$.

(6) A *multivalued admissible deformation* of $X$ onto $A \subset X$ is a map $H: X \times [0,1] \multimap X$ admissible in the sense of Górniewicz, and such that $H(x,0) = x$, $H(x,1) \subset A$ for every $x \in X$, and $x \in H(x,t)$ for every $x \in A$. It is seen that $H(\,\cdot\,,1)$ is a multivalued (admissible) retraction.

An approximation technique will be even more effective if we do not insist that approximating single-valued problems induce the same exit set. The main result in this direction has been proved in [16], and is as follows:

**Theorem 5.3.** *Let* $K = \overline{\operatorname{Int} K}$ *be a sleek* (7) *subset of* $\mathbb{R}^n$ *and* $F\colon \mathbb{R}^n \multimap \mathbb{R}^n$ *be a map satisfying* (F1)–(F2) *and such that* $K^-(F)$ *is a closed strong deformation retract of some its open neighbourhood* $V$ *in* $K$. *Assume that* $\operatorname{Int} T_K(x) \neq \emptyset$ *for every* $x \in K \setminus K^-(F)$. *If* $K^-(F)$ *is not a strong deformation retract of* $K$, *then there is a viable trajectory in* $K$.

Let us give some comments. Sleekness we assume above is an essentially weaker condition than $C^{1,1}$ regularity which means lipschitzeanity of the map $T_K(\,\cdot\,)$, as required in Theorem 5.2. We have also dropped the continuity assumption on $F$. Note that assumption $\operatorname{Int} T_K(x) \neq \emptyset$ eliminates "too sharp corners" of the set $K$, and means, in other words, that $K$ is epi-lipschitz in points of $K \setminus K^-(F)$ (comp. [32]).

The method of proof of Theorem 5.3 allowed to add new essential information to the homotopy index theory for multivalued flows generated by differential inclusions (see Section 4). The paper [18] concerns the matter. We recall one of the conclusions of considerations therein.

**Theorem 5.4.** *Under the assumptions of Theorem* 5.3, *if*

$$[K/K^-(F), [K^-(F)]] \neq \overline{0},$$

*then there is a viable trajectory in* $K$. *If* $\operatorname{inv}(K,F) \subset \operatorname{int} K$, *then*

$$I(\operatorname{inv}(K,F),F) = [K/K^-(F), [K^-(F)]],$$

*where* $I(\operatorname{inv}(K,F),F)$ *is a homotopy index defined in Section* 4.

**Final remarks.** (1) The Ważewski retract method can be applied to multivalued problems in infinite dimensional spaces for various classes of maps (see e.g. [15], [19]).

(2) The open problem is whether $I(\operatorname{inv}(K,F),F) = [K/K^-(F), [K^-(F)]]$ for every isolating neighbourhood $K$ and a closed exit set $K^-(F)$.

(3) After the existence problem for viable trajectories, some further steps are natural and interesting, namely, we can study qualitative properties of viable trajectories such as stationarity or periodicity. We refer to [6], [18], [17] and references therein to acquaint oneself with the subject.

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# MULTIVALUED FRACTALS

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Abstract. We communicate that under suitable assumptions the system of multivalued maps generates a fractal.

Let $(X,d)$ be a complete metric space. A family of multifunctions $\{\varphi_i\colon X \multimap X\}_{i\in I}$ is called *multivalued iterated function system*. This system induces the so-called *Barnsley–Hutchinson operator* defined as

$$\Phi\colon 2^X \to 2^X, \quad \Phi(A) \doteq \overline{\bigcup_{i\in I} \varphi_i(A)}$$

for $A \subset X$. Our main interest is to investigate the sequence of successive images:

$$A \mapsto \Phi(A) \mapsto \Phi(\Phi(A)) = \Phi^2(A) \mapsto \cdots \mapsto \Phi^n(A).$$

**Theorem.** *Let $\{\varphi_i\colon X \multimap X\}_{i\in I}$ be a finite system of multivalued contractions* (*with nonempty bounded values*)*. Then there exists nonempty closed bounded subset $A_* \subset X$ with the following properties:*

(a) (invariance) $\Phi(A_*) = A_*$,
(b) (uniqueness) *for any nonempty bounded set $A \subset X$ with $\Phi(A) = A$ we have $A = A_*$,*
(c) (attractor) *for any nonempty bounded set $A \subset X$ we have $\Phi^n(A) \overset{n\to\infty}{\longrightarrow} A_*$, where the convergence is in the Hausdorff metric,*
(d) (compactness) *if $\varphi_i$ have compact values, then $A_*$ is compact.*

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The set $A_*$ is called a *multivalued fractal*. This theorem can be extended on systems of weak contractions. Its proof relies on application of the Banach fixed point theorem. We only need to verify that $\Phi$ is contraction with respect to the Hausdorff metric on the family of nonempty closed bounded subsets of $X$. Thus conditions (a)–(c) follow immediately. To see that (d) holds also, first restrict $\Phi$ to the family of nonempty closed bounded sets, and then to the narrower family of nonempty compact sets. In both cases $\Phi$ possess unique fixed point. Since it is the same fixed point, it must be compact.

**Example.** Let $X$ be a Banach space, $D$ a closed unit ball at 0 and $\varphi_1\colon X \multimap X$, $\varphi_1(x) = D$. Then $\Phi^n(A) = D$ for every nonempty $A \subset X$. Hence $A_* = D$ is infinite dimensional fractal for the system $\{\phi_i\}_{i=1}$. Note that $\varphi_1$ is a multivalued contraction with Lipschitz constant 0, so the above theorem is applicable. This is "truely" multivaled fractal in the sense that a finite system of single-valued contractions always yields compact attractor. On the other hand one can always get any nonempty closed set $P$ as a fractal for infinite system of single-valued contractions $\{f_p\colon X \to X\}_{p\in P}$, $f_p(x) = p$.

We can also formulate similar theorem for more general systems, although the notion of attractor needs careful revision. But then we have to replace Banach Principle for other fixed point theorem. Unfortunately hyperspaces (i.e. appropriately topologized families of sets) hardly ever possess fixed point property which makes the application of the Schauder Principle almost impossible. The other fruitful technique provides the Knaster–Tarski Principle. It allows us to consider infinite iterated function systems consisting of condensing multifunctions (joint generalization of compact map and compact-valued contraction). Research along these lines was done by the author e.g. in [7]. More details can be found in the author's Ph.D. thesis (Nicolaus Copernicus University, Toruń, June 2005).

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# INWARD CONTRACTIONS ON METRIC SPACES

Mateusz Maciejewski

Abstract. We study the existence of constrained fixed points of contractions in arbitrary complete metric spaces from a global and local point of view. In particular, we provide generalizations of results due to Lim, Downing–Kirk and others. Some aspects of the topological transversality in the spirit of Frigon–Granas of contractions under constraints are also considered.

## 1. Introduction

This paper contains some results related to the existence of fixed points of contractions of any complete metric space $(X,d)$. The main theorem in this direction is due to S. Banach. He proved, that any contraction $f\colon X \to X$ of a space $X$ has a unique fixed point. Contractivity means that there exists a constant $k<1$, that for any $x,y \in X$

$$d(f(x),f(y)) \leq kd(x,y).$$

The multi-valued version of above result due to Nadler states that any multi-valued contraction $F\colon X \multimap X$ with closed values has a fixed point (i.e. a point $x_0 \in F(x_0)$). Multi-valued contractivity is in the sense of the Hausdorff distance, i.e. there exists a constant $k<1$ that for any $x,y \in X$

$$d_H(F(x),F(y)) \leq kd(x,y). \tag{1.1}$$

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*Key words and phrases.* Fixed point, contraction, presence of constraints, inwardness condition, continuation under constraints, complete metric space.

We shall concentrate on the case of contractions which are not defined on a whole space. Let $F: K \multimap X$ be a multi-valued contraction with closed values defined on a closed subset $K$ of a complete metric space $X$. It is clear that – in general – $F$ has no fixed points.

**Example 1.1.** Let $X := \mathbb{R}$, $K := [-1,1]$. Consider a single-valued mapping $F: K \to X$ defined by $F(x) := x/2 + 100$. Then $F$ is a contraction, but does not has fixed points.

Of course we could assume that the mapping $F$ admits values in the set $K$. It turns out that it suffices to control a behavior of $F$ on the boundary $\partial K$ of $K$.

Let us start with a definition of an inward set. Consider a Banach space $E$ and its closed subset $K$. A set

$$I_K + (x) := x + \overline{\{h(y-x) \mid h \geq 1,\ y \in K\}}$$

is called an *inward set* of the subset $K$ in a point $x$.

It can be shown (see e.g. [6]) that

$$I_K(x) = \left\{ y \in E \;\middle|\; \inf_{h \in (0,1]} \frac{d_K(x + h(y-x))}{h} = 0 \right\},$$

where $d_K(z)$ is the distance between $z$ and $K$, i.e. $d_K(z) := \inf\{d(k,z) \mid k \in K\}$.

**Example 1.2.** Let $X := \mathbb{R}^2$ and $X \ni K := \{(x,y) \mid x^2 + y^2 = 1\}$ be a unit sphere. Then for $(x_0, y_0) \in K$

$$I_K(x_0, y_0) := \{(x,y) \mid x^2 + y^2 \geq 1 \text{ and } \langle (x_0,y_0), (x - x_0, y - y_0) \rangle \leq 0\},$$

where $\langle \cdot\,, \cdot \rangle$ is the standard scalar product.

Now, we are prepared to formulate a result that seems to be the most interesting in this direction

**Lemma 1.3** (Lim, [5]). *Let $K$ be a closed subset of a Banach space $E$. Consider a contraction $F: K \multimap E$ satisfying an inwardness condition:*

$$F(x) \subset I_K(x) \quad \textit{for every } x \in K. \tag{1.2}$$

*Then $F$ has a fixed point.*

**Remark 1.4.** Assumption (1.2) is indeed the boundary condition, because if a point $x$ lies in the interior of $K$, then $I_K(x) = E$.

Notice that the above theorem generalize the Banach Principle only in the case of Banach spaces. Our aim is to state a theorem that generalizes the Lim Theorem to the context of an arbitrary complete metric space. To this end, given a complete metric space $X$, a closed set $K \subset X$ and a point $x \in K$ we shall introduce a *generalized inward set* $\widetilde{I}_K(x)$. It appears that if $X$ is a Banach space then $I_K(x) \subset \widetilde{I}_K(x)$, in general. Moreover, the proof of our result (see

Theorem 2.6 below) is simpler than the technical and long proof of the Lim Theorem.

## 2. Generalization of the Lim Theorem

We shall use the following well-known fixed-point theorem due to Caristi.

**Theorem 2.1** (Caristi, [1]). *Assume that $F\colon X \multimap X$ is a multi-valued mapping and, for each $x \in X$, there is $y \in F(x)$ such that*

$$d(x,y) \le e(x) - e(y), \tag{2.1}$$

*where $e\colon X \to \mathbb{R}_+$ is a lower semicontinuous function. Then $F$ has a fixed point.*

**Remark 2.2.** (a) Lower semicontinuity of $e$ means that for any $c \in \mathbb{R}$ the set $\{x \in X \mid e(x) \le c\}$ is closed.

(b) If $F\colon X \to X$ is a contraction, then satisfies assumption (2.1) with a function $e(x) := d(x, F(x))/(l-k)$, where a constant $k$ is from the definition of contractions (1.1), and $l \in (k,1)$ is any number. This proves the Nadler Theorem.

Theorem of Caristi is equivalent to the following

**Theorem** (Ekeland, [2]). *Let $e\colon X \to [0,\infty]$ be a lower semicontinuous function, $x_0 \in \mathrm{Dom}(e) := \{x \mid e(x) < \infty\}$ and $\varepsilon > 0$. Then there exists $\overline{x} \in X$ that*

(a) *$e(\overline{x}) + \varepsilon d(x_0, \overline{x}) \le e(x_0)$ (hence $d(x_0, \overline{x}) \le e(x_0)/\varepsilon$),*
(b) *for every $x \ne \overline{x}$ one has $e(\overline{x}) < e(x) + \varepsilon d(x, \overline{x})$.*

Let us return to the notion of the inward set. It works only in linear spaces. However, one can introduce an analogous set in any metric space. First, we start with definitions of segments.

By a *linear segment* in a vector space $E$, joining $x, y \in E$ we mean the set

$$\mathrm{lin}[x,y] := \{(1-h)x + hy \in E \mid h \in [0,1]\} = \mathrm{conv}\{x,y\},$$

and by a *linear left-open segment* $\mathrm{lin}(x,y] := \mathrm{lin}[x,y] \setminus \{x\}$. In any metric space $(X,d)$ one can consider a *metric segment* defined by the formula:

$$[x,y] := \{z \in X \mid d(x,z) + d(z,y) = d(x,y)\} \quad \text{for } x,y \in X$$

and a *metric left-open segment* $(x,y] := [x,y] \setminus \{x\}$.

It is clear that in normed spaces metric segments always contain linear segments and, in strictly convex normed spaces ([1]), these two concepts coincide.

([1]) A space $E$ is strictly convex, if for any two different points $x$ and $y$ with $\|x\|, \|y\| \le 1$, the inequality holds: $\|(x+y)/2\| < 1$.

**Example 2.4.** Let $X := \mathbb{R}^2$ and consider three norms on $X$:

(a) $\|(x,y)\|_1 := |x| + |y|$,
(b) $\|(x,y)\|_2 := \sqrt{x^2+y^2}$,
(c) $\|(x,y)\|_\infty := \max\{|x|,|y|\}$.

Then $(X, \|\cdot\|_2)$ is strictly convex; therefore for any $z, z' \in X$ one has $\operatorname{lin}[z,z'] = [z,z']$. However such equality does not hold in two other cases. For example, $[(0,1),(1,0)] = [0,1]\times[0,1]$ in case (b) and $[(-1,0),(1,0)] = \{(x,y) \mid |x|+|y| \le 1\}$ in case (c).

Notice that the set $I_K(x)$ can be written by the following formula:

$$I_K(x) := \{x\} \cup \left\{ y \in E,\ y \neq x \;\middle|\; \inf_{z \in \operatorname{lin}(x,y]} \frac{d_K(z)}{\|z-x\|} = 0 \right\}. \tag{2.2}$$

If we replace a Banach space $E$ by any complete metric space $X$ and a segment $\operatorname{lin}(x,y]$ by $(x,y]$ in an equation (2.2), we shall obtain the definition of the *generalized inward set*: for $x \in X$ let

$$\widetilde{I}_K(x) := \{x\} \cup \left\{ y \in X,\ y \neq x \;\middle|\; \inf_{z \in (x,y]} \frac{1}{d(z,x)} d_K(z) = 0 \right\}.$$

For a given real number $0 < \alpha < 1$, let us define an auxiliary set $J_K^\alpha(x)$ (2) by the formula:

$$J_K^\alpha(x) := \{y \in E \mid \text{there exists } x' \in K,\ x' \neq x \text{ such that } \alpha d(x,x') + d(x',y) \le d(x,y)\}.$$

Relations between sets defined above are collected in the following lemma.

**Lemma 2.5.** *If $K$ is a closed subset of a metric space $X$ and $0 < \alpha < 1$, then $\widetilde{I}_K(x) \subset \{x\} \cup J_K^\alpha(x)$. If, in addition, $X = E$ is a normed space, then $I_K(x) \subset \widetilde{I}_K(x)$. Moreover, if $E$ is strictly convex, then $I_K(x) = \widetilde{I}_K(x)$.*

Now, we are prepared to formulate and prove a generalization of Lim's Theorem.

**Theorem 2.6.** *Let $K$ be a closed subset of a complete metric space $X$. Let us consider a multi-valued contraction $F\colon K \multimap X$ with closed values. If $F$ satisfies an inwardness condition:*

$$F(x) \subset \widetilde{I}_K(x) \quad \text{for all } x \in K$$

*then $F$ has a fixed point.*

*Proof.* Assume that the conclusion is false. Choose $l$ and $\beta$ such that $0 < k < l < l + \beta < 1$, and define a metric $D$ on the graph $G := \operatorname{Gr}(F) = \{(x,y) \in$

(2) This set is introduced in [4].

$K \times X \mid y \in F(x)\}$ of $F$:

$$D((x,y),(x',y')) := \max\left\{d(x',x), \frac{1}{l}d(y',y)\right\}, \quad (x,y),(x',y') \in G.$$

Obviously, $(G,D)$ is a complete metric space.

Now, we shall construct a map $\varphi\colon G \to G$ without fixed points, but satisfying the assumptions of the Caristi Theorem. To this end, let $(x,y) \in G$, i.e. $y \in F(x) \subset \widetilde{I}_K(x) \setminus \{x\} \subset J_K^{l+\beta}(x)$. By Lemma 2.5, $y \in F(x) \subset \widetilde{I}_K(x) \setminus \{x\} \subset J_K^{l+\beta}(x)$. Hence, there is $x' \in K$, $x' \neq x$ such that

$$(l+\beta)d(x,x') + d(x',y) \le d(x,y). \tag{2.3}$$

On the other hand, $d_H(F(x),F(x')) < ld(x,x')$; therefore there is a point $y' \in F(x')$ such that

$$d(y,y') < ld(x,x'). \tag{2.4}$$

Let $\varphi(x,y) := (x',y')$. It is clear that $\varphi$ has no fixed points. However, in virtue of (2.3) and (2.4),

$$\begin{aligned} d(y',x') + \beta d(x,x') &\le d(y',y) + d(y,x') + \beta d(x,x') \\ &< (l+\beta)d(x,x') + d(y,x') \le d(x,y). \end{aligned}$$

Hence

$$\beta d(x,x') \le d(x,y) - d(x',y').$$

Similarly, thanks to (2.4),

$$\beta\frac{1}{l}d(y',y) \le \beta d(x',x) \le d(x,y) - d(x',y').$$

It means that

$$\beta D((x,y),\varphi(x,y)) \le d(x,y) - d(\varphi(x,y)).$$

Therefore

$$D((x,y),\varphi(x,y)) \le e(x,y) - e(f(x,y)),$$

where $e(x,y) := d(x,y)/\beta$. From Caristi's theorem it follows that $\varphi$ has a fixed point: a contradiction. $\square$

## 3. Local version of the Lim Theorem

Let us recall the local version of the Nadler Theorem.

**Theorem 3.1.** *Consider a multi-valued $k$-contraction $F\colon B(x_0,r) \multimap X$, where $B(x_0,r) := \{x \in X \mid d(x,x_0) < r\}$ is an open ball. Suppose that*

$$d(x_0,F(x_0)) < (1-k)r.$$

*Then $F$ has a fixed point.*

It turns out that an analogous version of Theorem 2.6 is also true.

**Theorem 3.2.** *Let $K$ be a closed subset of a complete metric space $X$ and $x_0 \in K$. Let $F: B(x_0, r) \cap K \multimap X$ be a multi-valued $k$-contraction such that*

$$F(x) \subset \widetilde{I}_K(x)$$

*for all $x \in B_K(x_0, r)$, and $d(x_0, F(x_0)) < (1-k)r$. Then $F$ admits a fixed point.*

The proof of above theorem repeats some arguments from the proof of Theorem 2.6, but instead of the Caristi Theorem, the following result is used:

**Theorem 3.3.** *Let $F: B(x_0, r) \multimap X$ be a multi-valued mapping. Assume that, for each $x \in B(x_0, r)$, there exists $y \in F(x)$ such that*

$$d(x, y) \leq e(x) - e(y),$$

*where a function $e: X \to \mathbb{R}_+$ is lower semicontinuous. Moreover, let $e(x_0) < r$. Then $F$ has a fixed point.*

## 4. Topological transversality under constraints

By using the local version of the Lim Theorem (Theorem 3.2) we can prove a continuation principle for inward contractions.

**Theorem 4.1.** *Let $K$ be a closed subset of a complete space $X$ and $V$ be an open (in $K$) subset of $K$. Consider a homotopy $H: I \times \overline{V} \multimap X$ with closed bounded nonempty values satisfying the following conditions:*

- (a) $d_H(H(t, x), H(t, y)) \leq kd(x, y)$, *for every points $x, y \in \overline{V}$ and a number $t \in I$,*
- (b) $d_H(H(t, x), H(s, x)) \leq |\varphi(t) - \varphi(s)|$, *where $\varphi: I \to \mathbb{R}$ is a continuous and increasing function, for every point $x \in \overline{V}$ and numbers $t, s \in I$,*
- (c) $\mathrm{Fix} H(t, \cdot) \cap \partial_K V = \emptyset$, *for any $t \in I$, where $\partial_K V$ denotes the boundary of $V$ relatively to $K$,*
- (d) $H(t, x) \subset \widetilde{I}_K(x)$ *for any $x \in \overline{V}$ and $t \in I$.*

*Then* $\mathrm{Fix} H(1, \cdot) \neq \emptyset$ *provided* $\mathrm{Fix} H(0, \cdot) \neq \emptyset$.

The proof of Theorem 4.1 is similar to this of Theorem 4.3 from [3], but instead of Theorem 3.1, Theorem 3.2 is used.

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# THE ROLE OF VARIOUS KINDS OF CONTINUITY OF SET-VALUED MAPPINGS. A SURVEY

Dariusz Miklaszewski

Abstract. We present some generalizations of the Brouwer Fixed Point Theorem for the set-valued mappings. The related open problems (e.g. on the Stiefel–Whitney classes) are indicated.

## 1. Introduction

Let $X$ be a disc $B^n$ and $f$ — a mapping assigning a nonempty compact set $f(x) \subset X$ to every point $x \in X$. We study the assumptions on the values $f(x)$ and on the continuity of $f$ which guarantee that $f$ has at least one fixed point $x \in f(x)$. The more complicated values are taken, the stronger continuity is required. We consider 4 kinds of continuity: (1) the upper semicontinuity (u.s.c.), the continuities with respect to (2) the Hausdorff metric $\rho_s$, (3) the Borsuk metric of continuity $\rho_c$ and (4) the Borsuk metric of homotopy $\rho_h$. Here the $j$th continuity implies the $i$th one for $j > i$. Taking into considerations the Borsuk metrics is a beautiful idea of Górniewicz.

Let us recall, that $f$ is u.s.c., if its graph is closed in $X \times X$;

$$\rho_s(A, B) = \inf\{\varepsilon > 0 : A \subset O_\varepsilon B \text{ and } B \subset O_\varepsilon A\},$$

where $O_\varepsilon C = \{x \in X : \inf\{d(x, c) : c \in C\} < \varepsilon\}$ for $A, B, C \subset X$ and $d$ — a metric in $X$;

$$\rho_c(A, B) < \varepsilon$$

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if and only if there are continuous functions $g\colon A \to B$, $h\colon B \to A$ which locate each value at a point closer than $\varepsilon$ to the argument. The Hausdorff metric could be defined similarly, but with any, not necessarily continuous $g$ and $h$.

The Borsuk metric of homotopy $\rho_h$ is defined on the set of all compact ANRs in $X$. Let us fix $t \geq 0$ and a locally contractible set $A$, which is a compact subset of $X$. We define $\phi_A(t)$ to be the lower bound of the set, which is composed of 1 and all $s \geq t$ such that every set $T \subset A$ with the diameter $\mathrm{diam}(T) \leq t$ is contractible in a set $S \subset A$ with $\mathrm{diam}(S) \leq s$.

We say, that sets from the class $\Theta \subset 2^X$ are equally locally contractible (e.l.c.), if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall A \in \Theta \quad \phi_A(\delta) < \varepsilon.$$

If we replace the contractibility of $T$ in $S$ by the assumption that each mapping from the $j$-sphere into $T$ is homotopy trivial in $S$ for $j = 0, \dots, m$, then the above condition makes the sets from $\Theta$ equally locally connected in the dimension $m$ ($eLC^m$).

The explicit formula for $\rho_h$ (see [1]) can be replaced by the observation, that the mapping $f\colon X \to 2^X$ is $\rho_h$-continuous if and only if $f$ is $\rho_s$-continuous and has e.l.c. values.

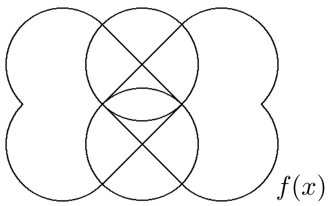
We now recall some classical results of the fixed point theory to create the context of this paper.

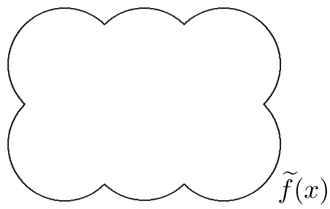
- In 1912 Brouwer proved, that every continuous map $f\colon B^n \to B^n$ has a fixed point, [2]. The Brouwer Fixed Point Theorem was generalized by Schauder for compact maps of infinite-dimensional normed spaces, [35]. The Schauder Theorem is basic for many applications to differential equations, [8]. Another excellent generalization is the Lefschetz Fixed Point Theorem, [24], [25], [3], [7], [16].
- In 1941 Kakutani proved the Brouwer Fixed Point Theorem for *u.s.c.* mappings with convex compact values, [21].
- In 1946 Eilenberg and Montgomery proved the Lefschetz Fixed Point Theorem for u.s.c. set-valued mappings with $Q$-acyclic values, [10]. The set is $Q$-acyclic, if its Čech cohomology groups with rational coefficients are isomorphic to these of the one-point space. The star-shaped sets, known from the Poincaré Lemma, are $Q$-acyclic. Another example is the real projective space $RP^{2n}$. The Górniewicz generalization of the Eilenberg–Montgomery theorem, [13], opened the doors to many applications of set-valued mappings to differential equations and inclusions, [12].
- In 1947 O'Neill gave an example of the fixed point free mapping $f\colon B^2 \to 2^{B^2}$, which is $\rho_s$-continuous and takes values homeomorphic to $S^1$, [33]. Set $\eta(x) = 1 - \|x\| + \|x\|^2$ for $x \in B^2$. Then

  $$f(x) = \{y \in S^1 : \|y - x\| \geq \eta(x)\} \cup \{y \in B^2 : \|y - x\| = \eta(x)\}.$$

**Exercise.** Show that $\lim_{x\to 0}\rho_c(f(x),f(0))=2$.

- In 1987 Jezierski proved, that there is a fixed point free mapping $B^2 \to 2^{B^2}$, which is $\rho_s$-continuous and takes the finite values, which have 1, 2 or 3 elements, [20].
- In 1990 Schirmer defined the fixed point index for bimaps, [36]. The bimap is the $\rho_s$-continuous mapping, which takes values having one or two elements.
- In 1977 Górniewicz defined the spheric mappings, [15]. This notion was developed by Górniewicz's student Dawidowicz in [5] and [6]. The most general definition comes from [14]. Denote by $Bf(x)$ the set, which is the sum of all bounded components of the complement of $f(x)$ in $\mathbb{R}^n$. Set $\widetilde{f}(x)=f(x)\cup Bf(x)$.




The above figure shows a 1-dimensional continuum $f(x)$ in $\mathbb{R}^2$ shaped as two hearts joined by two wedding rings and a 2-dimensional continuum $\widetilde{f}(x)$ which has a form of the gingerbread "katarzynka" baked in the town Toruń as a souvenir connected with a beautiful ancient legend.

The map $f\colon B^n \to 2^{B^n}$ is called the spheric mapping, if it satisfies the following conditions:

- $f$ is u.s.c.;
- the graph $\Gamma(Bf)$ of the mapping $Bf$ is an open subset of $B^n\times\mathbb{R}^n$;
- the mapping $\widetilde{f}$ has a fixed point.

Let us recall two properties of spheric mappings:

- every spheric mapping has a fixed point, [14];
- every $\rho_c$-continuous mapping $f\colon B^2\to 2^{B^2}$ with compact connected values is a spheric mapping, [14].

The following question (called in [27] the Górniewicz Conjecture) was the main source of inspiration for the author of this paper:

Is the Brouwer Fixed Point Theorem true for $\rho_c$-continuous mappings with compact connected values?

We now formulate our results:

**Theorem 1.1** ([29]). *There is a fixed point free mapping $f\colon B^4\to 2^{B^4}$, which is $\rho_c$-continuous and has compact connected values.*

**Theorem 1.2** ([29]). *Every $\rho_h$-continuous mapping $f\colon B^n \to 2^{B^n}$ has a continuous selector and a fixed point.*

**Theorem 1.3** (see [29]). *Every $\rho_s$-continuous mapping $f\colon B^n \to 2^{B^n}$ with $eLC^{n-2}$ values, such that the mapping $\widetilde{f} = f \cup Bf$ has $eLC^{n-1}$ values, is a spheric mapping and has a fixed point.*

**Theorem 1.4** ([30]). *Every $\rho_c$-continuous mapping $f\colon B^n \to 2^{B^n}$ such that for every $x \in B^n$, $f(x)$ is homeomorphic to either a point or the $n-2$-sphere, has a fixed point, $n \neq 6$.*

In the next paragraphs we comment these results and give examples.

## 2. Borsuk continuities and fibrations

Let $X$, $Y$ be two compact finite-dimensional metric spaces, $f\colon X \to 2^Y$ — a mapping and $\Gamma(f)$ — its graph. We formulate here two results, which connect the continuity types of $f$ to some fibre properties of the projection $p\colon \Gamma(f) \to X$.

**Theorem 2.1** ([31]). *Let $f$ be a $\rho_h$-continuous mapping. Then the projection $p$ is a Hurewicz fibration.*

**Theorem 2.2** ([31]). *Let $f$ be a $\rho_c$-continuous mapping with values, which are compact connected topological $n$-manifolds without boundary, $n \neq 4$. Suppose that for $n = 3$ the values contain no "fake 3-cell". Then the projection $p$ is a locally trivial bundle.*

The proofs of these theorems are based on strongly regular mappings [11], completely regular mappings [9] and approximating homotopy equivalences by homeomorphisms [4], [19]. Another theorem in this spirit, though not connected with the Borsuk metrics, is the famous Vietoris Theorem explored in [13] and [23].

## 3. Mappings with e.l.c. or $eLC^m$ values

Since every Hurewicz fibration over the contractible base has a section, the Theorem 1.2 follows immediately from Theorem 2.1. Let us note that the fact that $\widetilde{f}$ in Theorem 1.3 has a fixed point (which is a necessary condition for $f$ to be spheric) follows from Theorem 1.2 and from the equivalence of $eLC^{n-1}$ and e.l.c. in $\mathbb{R}^n$.

**Problem 3.1.** Could we drop in Theorem 1.3 the assumption on $\widetilde{f}$?

The positive answer is known for $n = 2$ only, [29]. The assumption on $f$ in Theorem 1.3 can not be weakened, because there is a fixed point free $\rho_s$-continuous mapping $f\colon B^n \to 2^{B^n}$ with $eLC^{n-3}$ values:

$$f(x) = \{y \in S^{n-1} : y \cdot x \leq (1 - \|x\|)\|x\|\}.$$

A special kind of mappings with $eLC^{n-3}$ values which have fixed points are mappings from the Theorem 1.4. More generally, we have the following

**Definition 3.2.** We call $f\colon B^n \to 2^{B^n}$ an $1 - S^k$-mapping if $f$ is $\rho_c$-continuous and for every $x$ in $B^n$, $f(x)$ is homeomorphic to either a point or the $k$-sphere.

One can check that $1 - S^k$-mappings have $eLC^{k-1}$ values. We know that $1 - S^{n-1}$-mappings of $B^n$ are spheric, $1 - S^0$-mappings are Schimer's bimaps and $1 - S^{n-2}$-mappings are "heroes" of Theorem 1.4. All these mappings have fixed points. The following problem is still open.

**Problem 3.3.** Has every $1 - S^k$-mapping $f\colon B^n \to 2^{B^n}$ a fixed point for $1 \le k \le n-3$?

In the next paragraph we tell the story, how this problem was attacked.

## 4. $1 - S^k$-mappings

Let us start with the following

**Definition 4.1.** Let $F$ be a field and $\check{H}_\star$ denote the Čech homology functor. The u.s.c. compact-valued map $\varphi\colon B^n \to 2^{B^n}$ is called an $F$-Brouwer mapping if and only if

$$\check{H}_n(\Gamma(\varphi), \Gamma(\varphi|S^{n-1}); F) \xrightarrow{i_\star} \check{H}_n(B^n \times B^n, S^{n-1} \times B^n; F)$$

induced by inclusion is a non-zero homomorphism.

**Theorem 4.2** ([26], [34, Corollary 5.1])**.** *Every $F$-Brouwer mapping has a fixed point.*

The class of $F$-Brouwer mappings is rich enough to contain the single-valued mappings, the maps equipped with the continuous single-valued selector, mappings graph-approximable by the single-valued continuous functions and the $F$-acyclic mappings. Moreover, this class is closed under compositions with the single-valued mappings.

Theorem 1.4 is a corollary from the following

**Theorem 4.3** ([30])**.** *Every $1 - S^{n-2}$-mapping of $B^n$ is a $Z_2$-Brouwer mapping, $1 \le n \ne 6$.*

The next example shows that $Z_2$ in the above theorem can not be replaced by any field $F$ of the characteristic $\mathrm{char}(F) \ne 2$.

**Example 4.4.** Fix $\varepsilon \in (0,1)$. We define $f\colon B^3 \to 2^{B^3}$ with $f(0) = \{0\}$ and $f(x)$ — a circle on $S^2(0; \|x\|)$ such that $d(y,x) = \varepsilon \cdot \|x\|$ for $y \in f(x)$, $x \ne 0$. Of course, $f$ is the $1 - S^1$-mapping of $B^3$.

Indeed, in the above example $\Gamma(f)$ is contractible and $\Gamma(f|S^2)$ is homeomorphic to the real projective space $RP^3$. Thus

$$H_3(\Gamma(f), \Gamma(f|S^2); F) = H_2(RP^3; F) = \mathrm{Tor}(Z_2, F) = \{y \in F : 2y = 0\}$$

which is 0 when $\mathrm{char}(F) \neq 2$, so in this case $f$ is not the $F$-Brouwer mapping.

Theorem 4.3 is a (nontrivial) consequence of the following

**Theorem 4.5** ([26], [31]). *Let $f: B^n \to 2^{B^n}$ be a $1 - S^k$-mapping and $1 \leq k < n$. Let $U(f) \subset B^n$ be the set of these points $x$, where $f$ is really multi-valued, i.e. $f(x) \cong S^k$. Roughly speaking, we now approximate the boundary of the set $U(f)$ by some closed connected $(n-1)$-manifolds $M$ in $U(f)$. By Theorem 2.2, the graph $\Gamma(f|M)$ is a space of the locally trivial bundle over $M$ with fibre $S^k$, $(k \neq 4)$. If each of these bundles (for a sufficiently close approximation) satisfies the inequality*

$$\dim H_k(\Gamma(f|M); Z_2) > \dim H_k(M; Z_2),$$

*then $f$ is a $Z_2$-Brouwer mapping and has a fixed point.*

**Remark 4.6** ([27]). The homology inequality in the above theorem is equivalent to the vanishing of the last Stiefel–Whitney class $w_{k+1}$ of the bundle

$$S^k \to \Gamma(f|M) \to M.$$

The discussion of the Stiefel–Whitney classes we postpone until the last paragraph.

We end this story with

**Theorem 4.7** ([28]). *There is an $1 - S^1$-mapping $f$ of $B^4$, which for every field $F$ is not an $F$-Brouwer mapping.*

We now describe the example from the above theorem (in general lines only). Let $\xi_H$ denote the Hopf fibration $p_H: \partial(B^4) \to S^2$. We embed $M = S^2 \times S^1$ into $Int(B^4)$ as well as its neighborhood $M \times [-1,1]$. Consider the projection $\pi: M \to S^2$ and take $\Gamma(f|M)$ equal to the space of the bundle $\pi^\star(\xi_H)$, i.e. the pullback $\{(x,y) \in M \times \partial(B^4) : \pi(x) = p_H(y)\}$. In other words, $f(x) = (p_H)^{-1}(\pi(x))$ for $x \in M = M \times 0$. Define $f(z) = (1 - |t|) \cdot f(x)$ if $z = (x,t) \in M \times [-1,1]$ and $f(z) = 0$ otherwise. The bundles from Theorem 4.5 are two copies of $\pi^\star(\xi_H)$ (over $M \times (1-\varepsilon)$ and over $M \times (-1+\varepsilon)$). Since

$$\dim H_1(\partial(B^4); Z_2) = \dim H_1(S^2; Z_2) = 0,$$

we have $w_2(\xi_H) \neq 0$. Then

$$w_2(\pi^\star(\xi_H)) = \pi^\star(w_2(\xi_H)) \neq 0,$$

because the retraction $\pi$ induces a monomorphism on cohomology groups. Of course, this calculus does not prove Theorem 4.7 but shows that $f$ is a good candidate — not excluded by Theorem 4.5. Author does not know if we could modify the above construction of $f$ to get a fixed point free $1 - S^1$-mapping. Theorem 1.1 provides with an example of a fixed point free $\rho_c$-continuous mapping of $B^4$ such that each value is the topological join of two finite sets with 1, 2 or 3 elements. This example is based on [20].

## 5. Stiefel–Whitney classes

Let $p\colon E \to M$ be a locally trivial bundle with the fibre $S^k$ and the Stiefel–Whitney classes $w_1, \dots, w_{k+1}$ (defined with help of the Steenrod squares and the Thom isomorphism [32]). The proof of the Theorem 4.3 applies Theorem 4.5, a version of the Borsuk–Ulam antipodal theorem [31, Theorem 5.3], [28, Fact 2], and the well-known equality:

$$(*) \qquad c^{k+1} = q^\star(w_{k+1}) + \sum_{j=1}^{k} q^\star(w_{k+1-j}) \cup c^j,$$

(see [18]), which holds in the $Z_2$-cohomology algebra of the space $E/Z_2$, if this projective bundle does exist, ($c$ is the first Stiefel–Whitney class of the 0-sphere bundle $E \to E/Z_2$ and $q\colon E/Z_2 \to M$ is the projection induced by $p$). Surely, the projective bundle $E/Z_2$ exists, if the antipodal actions of the group $Z_2$ on the separate fibres of $p\colon E \to M$ could be "glued together" to form the free fiber-wise $Z_2$-action on the space $E$. Such a gluing makes no difficulties when the structural group of the fibration is linear (orthogonal), or more generally, when all homeomorphisms in the structural group are odd mappings. This was the reason of the fact that author's considerations in [28] have been restricted to the case of $k = 1, 2, 3$. In this case, by theorems of Kneser [22], Smale [37] and Hatcher [17], the structural group of any locally trivial $k$-sphere bundle reduces to the orthogonal group $O(k+1)$.

Finally, the above problem of the projective bundle existence was solved in [30]. The main idea of this solution is to replace the $k$-sphere bundle $E$ with the bundle

$$E^\triangle = \{(x,y) \in E \times E\colon x \neq y \text{ and } p(x) = p(y)\}$$

which has fibres $(S^k \times S^k) \setminus \Delta$ homotopy equivalent to $S^k$ and the same Stiefel–Whitney classes as $E$. The action of $Z_2$ on these new fibres is a transposition of coordinates in $S^k \times S^k$. The same transposition defines the $Z_2$-action on $E^\triangle$.

By the Leray–Hirsch Theorem, there is a version of the formula $(*)$ for $E^\triangle/Z_2$ with coefficients $w_i$ replaced with (a priori — new) coefficients $w_i^\triangle$ for $i = 1, \dots, k+1$. We know that

- $w_{k+1}^\triangle = w_{k+1}$, [30, p. 184].

The question if $w_i^\triangle$ is the Stiefel–Whitney class of the bundle $E^\triangle$ for $i \leq k$ remains (for the author) still open.

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