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PREFACE

The “Winter School on Topological Methods in Nonlinear Analysis” took place at the Nicolaus Copernicus University in Toruń in February 9–13, 2009 and was organized by the Juliusz Schauder Center for Nonlinear Studies. The idea of the meeting comes from Prof. Lech Górniewicz — the Head of this Center.

The aim of the Organizers was to bring together young Polish researchers and to present them some of the most interesting subjects in modern nonlinear analysis. More than 50 young mathematicians participated in the 5 series of lectures given by professors: Grzegorz Gabor (Nicolaus Copernicus University in Toruń), Bogusław Hajduk (University of Wrocław), Marek Izydorek (Gdańsk University of Technology), Wacław Marzantowicz (Adam Mickiewicz University in Poznań), José M. R. Sanjurjo (Complutense University of Madrid) and Klaudiusz Wójcik (Jagiellonian University in Kraków).

At the beginning we were planning the 6 courses. Unfortunately, our colleague Prof. Andrei Borisovich (University of Gdańsk), who prepared the lectures on nonlinear Fredholm analysis and its applications to elastic mechanics, died three months before the “Winter School”. Therefore we decided for the lecture to his memory given by his first Ph.D. student in Poland — dr Joanna Janczewska (Gdańsk University of Technology).

The papers collected in this volume reflect an ample spectrum of subjects discussed during the lectures: Ważewski method, symplectic topology, Morse theory, Conley theory and many others.

We thank all our main speakers for their stimulating lectures and all participants for creating and friendly atmosphere during the meeting.

Marek Izydorek

Toruń, September 2010

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ON THE WAŻEWSKI RETRACT METHOD

Grzegorz Gabor

Abstract. The original version of the Ważewski theorem as well as its newer formulations are presented. Several applications of the Ważewski method in differential equations are given. In the second part some multivalued Ważewski type theorems are provided with open problems finishing this note.

1. Ważewski's retract method in dynamical systems

Consider the following Cauchy problem:

$$\begin{cases} \dot{x}(t) = f(x(t)) & \text{for } t \geq 0, \\ x(0) = x_0 \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ and $f\colon \Omega \to \mathbb{R}^n$ is so regular that the problem has a unique local solution for every $x_0 \in \Omega$, which depends continuously on the initial condition (we can think about a locally Lipschitz continuous map).

This implies that a *local semiflow* $\pi\colon D \to \mathbb{R}^n$, where D is an open subset of $\Omega \times [0,\infty)$ containing 0, is given. It means that, for every $x \in \Omega$, the set $\{t \geq 0 \mid (x,t) \in D\}$ is an interval $[0,\omega_x)$ for some $0 < \omega_x \leq \infty$ and

(i) $\omega_{\pi(x,t)} = \omega_x - t$ for each $x \in \Omega$ and $t \in [0,\omega_x)$,
(ii) $\pi(x,0) = x$ for every $x \in \Omega$,
(iii) $\pi(x,s+t) = \pi(\pi(x,t),s)$ for each $x \in \Omega$ and $s,t > 0$ such that $s+t < \omega_x$.

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Remain that, if $D = \Omega \times [0, \infty)$, then we say that simply a *semiflow* is given. If $D \subset \Omega \times \mathbb{R}$ and $\{t \in \mathbb{R} \mid (x, t) \in D\} = (\alpha_x, \omega_x)$, where $\alpha_x < 0, \omega_x > 0$, then we say about a *local flow*. By a *trajectory* (*positive trajectory, negative trajectory*) of x, denoted by $\pi(x)$ (resp. $\pi_+(x)$, $\pi_-(x)$), we mean the set of all points $\pi(x, t)$ (resp. with $t \geq 0$, $t \leq 0$).

Now, for a given subset W of Ω *we ask for the existence of a point $x \in W$ and a positive trajectory* (*or, even, trajectory*) *of x in W*. Is it possible to get a positive answer studying a behavior of the flow, or the right-hand side of equation (1.1), on the boundary of W? If we treat W as a set of constraints for some species to survive, we will justify the name *viable trajectory in V* for any solution of the problem (viable=vi+able).

If W is closed, and the vector field f is *tangent* to the set W, i.e. $f(x) \in T_W(x)$, where

$$T_W(x) := \left\{ v \in \mathbb{R}^n \;\middle|\; \liminf_{h \to 0^+} \frac{\operatorname{dist}(x + hv, W)}{h} = 0 \right\}$$

is the *Bouligand tangent cone*, then all trajectories remain in W. The idea to join the problem with appropriate tangency conditions was originated by Nagumo in 1942 (see [16]), and still is fruitful (see, e.g. [1]). If the above Nagumo tangency condition is satisfied, the set is *positively invariant* with respect to the local semiflow generated by the Cauchy problem (1.1).

We are interested in the case where $f(x) \notin T_W(x)$ in some points on the boundary of W. In these points the vector field is directed outside the set and trajectories leave it through such points, which are justly called the exit (or egress) points. More precisely, following Ważewski (see [19]), we say that a point $x \in \partial V$, where $V \subset \Omega$ is an open subset and ∂V denotes the boundary of V, is an *egress point* of V, if there are $\varepsilon > 0$ and a point $y \in V$ such that $\pi(\{y\} \times [0, \varepsilon)) \subset V$ and $\pi(y, \varepsilon) = x$. An egress point x is a *strict egress point*, if there exists $\delta > 0$ such that $\pi(\{x\} \times (0, \delta)) \cap \overline{V} = \emptyset$, where $\overline{V}$ is a closure of V.

Let us denote the set of egress points of V by E and the set of strict egress points by E^s. The fundamental Ważewski assumption was:

$$E = E^s. \tag{1.2}$$

Now, let S and Z be two sets such that $S \subset E$ and $Z \subset V \cup S$.

Theorem 1.1 (Ważewski Theorem, [19, Theorem 2])**.** *If $Z \cap S$ is a retract* (1) *of S, and $Z \cap S$ is not a retract of Z, then there exists a point x in $Z \setminus S$ such that the trajectory of x is viable in V or leaves V outside S.*

There are two immediate corollaries of this theorem.

(1) A closed subset M of $Y \subset X$ in a space X is said to be a *retract* of Y provided there exists a map $r: Y \to M$ such that $r(x) = x$, for every $x \in M$, and a *strong deformation retract of Y in X*, if there is a homotopy (called a *strong deformation*) $h: Y \times [0, 1] \to X$ such that $h(x, 0) = x$, $h(x, 1) \in M$, for all $x \in Y$, and $h(x, t) = x$, for each $x \in M$ and $t \in [0, 1]$.

Corollary 1.2 ($S := E$). *If $Z \cap E$ is a retract of E, and $Z \cap E$ is not a retract of Z, then there exists a viable trajectory in V.*

Corollary 1.3 ($Z = V \cup E$). *If E is not a retract of $V \cup E$, then there exists a viable trajectory in V.*

Denote $W := V \cup E$. The key point of the proof of Theorem 1.1 is the following observation:

Lemma 1.4. *If $E = E^s$, then the map $x \mapsto e(x) := \pi(x, \tau(x))$ is continuous on the set $W^* := \{x \in W \mid \pi_+(x) \not\subset V\}$, where $\tau(x) := \sup\{t \geq 0 \mid \pi(\{x\} \times [0,t]) \subset W\}$ is the exit time of the trajectory of x.*

At the moment we omit the proof since it will come back in a more general case. Let us only see how the lemma implies Theorem 1.1.

Proof of Theorem 1.1. Assume the contrary. Then, for every $x \in Z \setminus S$, there is a point $e(x)$ in S. Here we put $e(x) = x$ for $x \in S$. Denote by $\rho\colon S \to S \cap Z$ the retraction, which exists by assumptions. Then the map $r\colon Z \to S \cap Z$, $r := \rho \circ e$ is continuous, and $r(x) = \rho(e(x)) = \rho(x) = x$ for $x \in S \cap Z$, hence, r is a retraction; a contradiction. $\square$

Notice that, for $Z = V \cup E$, we can get in the proof something more than a retraction, namely, a *strong deformation* of Z onto E. Indeed, we can put $h(x, \lambda) := \pi(x, \lambda\tau(x))$, for every $x \in Z$ and $\lambda \in [0,1]$.

In the following example we illustrate an importance of Ważewski's assumptions.

Example 1.5. (a) Let $V := (0,1) \times (0,1)$ and $\pi((x,y),t) := (x+t, y)$. Then $E = \{1\} \times (0,1)$ is a retract of $Z = V \cup E$. Obviously, there is no viable trajectory in V.

(b) Let the flow be as above, and $V := [(0,1) \times (0,1)] \setminus \{(x,y) \mid 0 \leq x \leq 1/2$ and $y \leq x\}$. Then $E = (\{1\} \times (0,1)) \cup \{(x,y) \mid 0 < x < 1/2$ and $y = x\}$ being disconnected is not a retract of $Z = V \cup E$ while there is no viable trajectory in V. But we can see that the egress point $(1/2, 1/2)$ is not a strict egress point.

Some simple situations, where the existence of viable trajectories can be implied are listed below.

Example 1.6. (a) Let $V := (-1,1) \times (-1,1)$ and π be a flow generated by the hyperbolic system $\dot{x} = x, \dot{y} = -y$. Then $E = \{-1,1\} \times (0,1)$ is not a retract of $V \cup E$ as a disconnected set.

(b) Let V be as above, and π be induced by the system $\dot{x} = x$, $\dot{y} = y$. Then E is a boundary of V, so, by the equivalent theorem to the Brouwer fixed point theorem, there is no retraction from $V \cup E$ onto E. Note that there is no retraction from a closure of any open set in a finite dimensional space onto its boundary (see [10, p. 341]).

(c) Let V be the ring $B(0,2) \setminus \overline{B(0,1)}$, and a flow be such that the exit set E is a half of the bigger circle. Then E is a retract of $Z = V \cup E$ but it is not a strong deformation retract of Z. Indeed, otherwise, the homology groups of the ring (equal to $H(S^1)$) would be equal to $H(\mathrm{pt})$.

(d) Let V be a unit ball $B(0,1)$ in $\mathbb{R}^3$. It is easy to give a flow such that the exit set E is a part of a unit sphere bounded by parallels $-45°$ and $45°$. Once again, the homology argument works.

(e) Let V be a torus (let us illustrate it by $B(0,1) \times S^1$) and π be such that E is the ring $S^1 \times A$, where A is a connected subset of S^1 with $A \subsetneq S^1$. Then homology groups of $Z = V \cup E$ and E are the same, while still there is no strong deformation of Z onto E.

Using the Ważewski retract method some interesting boundary value problems can be solved. The reader can find details in [18]. Here we only formulate them and give some comments.

Example 1.7. (a) We consider the following problem in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}(t) = f(t,x,y), \\ \dot{y} = g(t,x,y), \\ y(0) = 0, \\ (x(t), y(t)) \in (-1,1)^2, \end{cases}$$

where f and g are smooth, and the following conditions are satisfied:

$$\begin{aligned} xf(t,x,y) &> 0 \quad \text{for every } t \in \mathbb{R},\ x \in \{-1,1\} \text{ and } y \in [-1,1], \\ yg(t,x,y) &< 0 \quad \text{for every } t \in \mathbb{R},\ x \in [-1,1] \text{ and } y \in \{-1,1\}. \end{aligned}$$

We can see that the problem is not autonomous, but we can extend it by adding the equation $\dot{t} = 1$. In this modified autonomous case we take $V = \mathbb{R} \times (-1,1)^2$, and then, taking $Z = \{0\} \times [-1,1] \times \{0\}$ we get $E = \mathbb{R} \times \{-1,1\} \times (-1,1)$. Thus $Z \cap E$ is a retract of E while $Z \cap E$ is not a retract of Z (one can check this as a simple exercise). A viable in V trajectory induces a solution of the problem.

(b) The problem is as above but we ask for solutions satisfying additionally the condition $x(1) = 0$. To solve the problem we proceed as before but as V we take the set $(-\infty, 1) \times (-1,1)^2$. As we can easily see, the exit set E is connected and consists of three faces $t = 1, |x| = 1$. Let Z be as before, while $S := E \setminus \{(1,0,y) \mid |y| < 1\}$. One can check that $Z \cap S$ is a retract of S and is not a retract of Z. Obviously, no trajectory is viable, but, by the Ważewski theorem, at least one trajectory must start in Z and leave the set through $E \setminus S$, that is, in a point $(1,0,y)$ for some $y \in (-1,1)$.

(c) In the third example we look for solutions of the linear equation $\dot{x}(t) = -A(t)x(t)$ in $\mathbb{R}^n$ such that $x(t) \geq 0, x(t) \neq 0$ for $t \geq 0$, where $A(\cdot)$ is a continuous map with values being matrices with nonnegative coefficients. Here

$x = (x_1, \dots, x_n) \geq 0$ means that $x_i \geq 0$ for every $1 \leq i \leq n$. We add one equation $\dot{t} = 1$ obtaining an autonomous system in $\mathbb{R}^{n+1}$. Take $V = (0, \infty) \times \Omega$, where $\Omega = \{(x_1, \dots, x_n) \mid x_i > 0 \text{ for every } i\}$. We associate with our system a family of systems $\dot{x}(t) = -(A(t) + \varepsilon_k N)x(t)$, where N is a matrix with all coefficients equal to 1, and $\varepsilon_k > 0$. These new systems generate flows with the exit set $E_k = E := (0, \infty) \times \{x \in \partial\Omega \mid x \neq 0 \text{ and } \prod_{i=1}^n x_i = 0\}$, since $\dot{x}(t) < 0$. Taking $Z := V \cup ((0, \infty) \times \Sigma)$, where $\Sigma = \{(x_1, \dots, x_n) \mid x_i \geq 0 \text{ for every } i \text{ and } \sum_{i=1}^n x_i = 1\}$, we obtain that $Z \cap E$ is a retract of E and it is not a retract of Z. Hence, for every $k \geq 1$, there is a trajectory in V for the system with ε_k which starts from Σ. Now, taking a sequence $\varepsilon_k \searrow 0$, we can obtain a subsequence of solutions x_k converging to a solution of the original system. This solution is nonnegative, starts from Σ and never meets the origin.

It appears that a definition of strict egress points can be less restrictive (see Bielecki, [4]). In fact, we can prove the Ważewski theorem assuming that every egress point x is a *strong egress point*, i.e.

$$\pi(\{x\} \times (0, \varepsilon)) \not\subset \overline{V} \quad \text{for every } \varepsilon > 0. \tag{1.3}$$

This condition is quite natural, since it means that x does not satisfy the following local viability: $\pi(\{x\} \times (0, \varepsilon)) \subset \overline{V}$ for small $\varepsilon > 0$.

We use Bielecki's condition and define, following Charles Conley's idea from [8] (see also [9]), for any subset $W \subset \Omega$ of the phase space, the *exit set* $W^- := \{x \in W \mid x \text{ satisfies (1.3)}\}$. In other words, in each point of W^- trajectory *immediately leaves* the set W. We define also $W^* := \{x \in W \mid \pi_+(x) \not\subset W\}$.

Now we can say that the Ważewski retract method allows us to answer the question: *when* $W \setminus W^* \neq \emptyset$ or, in terms of the viability theory (see [1]), $\mathrm{Viab}_f(W) \neq \emptyset$ or, in terms of dynamical systems, *when the positively invariant part* $\mathrm{inv}^+(W)$ *of* W *is nonempty.*

Conley formulated the Ważewski theorem in a more general and convenient form using the following notion.

Definition 1.8. We say that $W \subset \Omega$ is a *Ważewski set*, if

(W1) $x \in W, t > 0$ and $\pi(\{x\} \times [0, t]) \subset \overline{W}$ implies $\pi(\{x\} \times [0, t]) \subset W$,
(W2) W^- is closed in W^*.

Remark 1.9. (a) If W and W^- are closed in Ω, then W is a Ważewski set.

(b) If $W := V \cup E$ and (1.2) is satisfied, then W is a Ważewski set. Indeed, then $W^- = E$ and $W \setminus W^- = V$ is open, and hence, W^- is closed in W. Thus, W^- is closed in W^*.

(c) Assumption (W2) is more general than (1.2). To see this, let us consider $V := (-\infty, 0) \cup \bigcup_{n=1}^\infty (1/(2n+1), 1/(2n))$ and the flow $\pi(x, t) = x + t$. Then W^- is closed, $0 \in W^-$, and 0 is an egress but not a strict egress point.

(d) Ważewski sets can have empty interiors in contrast to those considered by Ważewski.

The following general version of the Ważewski theorem can be found in [18] (see Theorem 2.1).

Theorem 1.10. *If W is a Ważewski set for a local semi-flow π, $Z \subset W$, $S \subset W^-$ are such that $Z \cap W^- \subset S$, and S is not a strong deformation retract of $Z \cup S$ in $(W \setminus W^-) \cup S$, then there exists $x \in Z$ such that*

(a) *$\pi_+(x) \subset W \setminus W^-$, or*
(b) *$x \in W^*$ and $\pi_+(x)$ leaves W through $W^- \setminus S$.*

Before a proof let us observe:

Remark 1.11. (a) If we replace a strong deformation by a retraction in assumptions, then the theorem holds true as a simple corollary.

(b) If $Z \cap S$ is a retract of S, and it is not a retract of Z, then hypotheses (a), (b) are satisfied. Indeed, it is sufficient to verify assumptions of Theorem 1.10, more precisely, of the remark above. Assuming that there is a retraction $r\colon Z \cup S \to S$, we can define a retraction $\rho r|_Z\colon Z \xrightarrow{r|_Z} S \xrightarrow{\rho} Z \cap S$; a contradiction. Hence, in view of Remark 1.9(b) we have obtained a generalization of the Ważewski theorem.

(c) Assume that $S := W^-$ and $Z := W$ (Conley, [8]). If W^- is not a strong deformation retract of W, then there exists a trajectory viable in $W \setminus W^-$.

Proof of Theorem 1.10. We define the exit time function $\tau\colon W^* \to \mathbb{R}$, $\tau(x) := \sup\{t \geq 0 \mid \pi(\{x\} \times [0,t]) \subset W\}$.

Now, we notice that, if a trajectory leaves W, it does this through W^-. Indeed, let $x \in W^*$. Then $\pi(\{x\} \times [0, \tau(x) + \varepsilon)) \not\subset W$ for any $\varepsilon > 0$, and hence, $\pi(\{\pi(x, \tau(x))\} \times [0, \varepsilon)) \not\subset W$. This means that $\pi(x, \tau(x)) \in W^-$. By (W1) $\pi(\{x\} \times [0, \tau(x)]) \subset W$.

The second easy observation is that $x \in W^-$ if and only if $x \in W^*$ and $\tau(x) = 0$. Now we prove that the exit time function τ is continuous.

Step 1. τ is upper semicontinuous. Indeed, take any $p \in W^*$ and $\varepsilon > 0$. There is $t \in (\tau(p), \tau(p) + \varepsilon)$ such that $\pi(p, t) \notin \overline{W}$. By the continuity of π, there exists an open neighbourhood U of p such that $\pi(x, t) \notin \overline{W}$ for every $x \in U \cap W^*$. Hence, $\tau(x) < \tau(p) + \varepsilon$.

Step 2. τ is lower semicontinuous. To prove it, take any $p \in W^*$. If $\tau(p) = 0$, then, obviously, τ is lsc in p. Assume that $\tau(p) > 0$. Take $\varepsilon > 0$ so small that $\tau(p) - \varepsilon > 0$. Fix $t \in (\tau(p) - \varepsilon, \tau(p))$. Then $\pi(\{p\} \times [0, t]) \subset W^* \setminus W^-$ and this set is open in W^* by (W2). It implies that there exists an open neighbourhood U of p such that $\pi(\{x\} \times [0, t]) \subset W^* \setminus W^-$ for every $x \in U \cap W^*$. Hence, $\tau(x) > t > \tau(p) - \varepsilon$.

Now, assuming that the hypothesis of the theorem is false, $x \in W^*$ for every $x \in Z$. We define $h\colon (Z\cup S)\times[0,1] \to (W\setminus W^-)\cup S$, $h(x,\lambda) := \pi(x,\lambda\tau(x))$. It easy to check that h strongly deforms $Z\cup S$ onto S in $(W\setminus W^-)\cup S$; a contradiction.□

For some other Ważewski type results and applications to polyfacial sets we refer the reader to [18].

2. Multivalued generalizations of the Ważewski retract method

When a map f in (1.1) is less regular, or we have to study a multivalued problem

$$\begin{cases} \dot{x}(t) \in F(x(t)) & \text{for a.e. } t \in \mathbb{R}, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

then we meet a so-called *multivalued dynamical system.* From a point there can start a lot of solutions.

Throughout this section we will assume that F is usc (2), and

(H$_F$) F has nonempty compact convex values, and F has a sublinear growth, i.e. there exists a constant $c \geq 0$ such that

$$|F(x)| := \sup\{|y|;\ y \in F(x)\} \leq c(1+|x|) \quad \text{for every } x \in \mathbb{R}^n.$$

It is known that, under these assumptions, for every $x_0 \in \mathbb{R}^n$ the set of solutions $S_F(x_0)$ of problem (2.1) is compact R_δ (3), and the map $x_0 \mapsto S_F(x_0) \subset C(\mathbb{R},\mathbb{R}^n)$ is usc.

When the problem is multivalued, then there can appear two exit sets, different in general,

$$\begin{aligned} W^-(F) &:= \{x_0 \in \partial W \mid \forall x \in S_F(x_0) \forall t > 0 : x([0,t]) \not\subset W\}, \\ W_e(F) &:= \{x_0 \in \partial W \mid \exists x \in S_F(x_0) \forall t > 0 : x([0,t]) \not\subset W\}, \end{aligned}$$

with $W^-(F) \subset W_e(F)$, and it is natural that from points in $W_e(F) \setminus W^-(F)$ there can start trajectories going into W for both positive and negative times. For brevity we denote $W^- := W^-(F)$ and $W_e := W_e(F)$.

There is a question, *which of these two exit sets is suitable to obtain analogs to the Ważewski theorem?* Let us look at two following examples.

Example 2.1. Consider two sets (Figures 1 and 2). In the first example $W := \{(x,y) \in \mathbb{R}^2 \mid |x|-1 \leq y \leq |x| \text{ and } y \leq 1\}$ and $F(x) := [-1,1]\times\{1\}$. The set W is connected and $W^- = \{(x,y) \in W \mid y = 1\}$ is closed and disconnected, so W^- is not a retract of W and it is as good as possible. It is seen that there

(2) $F\colon X \multimap Y$ is *upper semicontinuous* (usc) if the set $F^{-1}(U) := \{x \in X \mid F(x) \subset U\}$ is open for every open set U in Y.

(3) A space X is a compact R_δ-set provided it is homeomorphic to an intersection of a decreasing sequence of compact contractible spaces. In particular, it is acyclic.

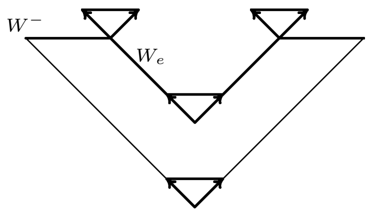

FIGURE 1

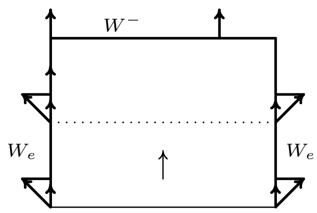

FIGURE 2

is no viable trajectory in W. Notice that $W_e = W^- \cup \{(x,y) \in W \mid y = |x|\}$ is a strong deformation retract of W. Maybe W_e is suitable?

In the second example $W := [-1,1] \times [-1,1]$ and $F(x) := [-\alpha^-(x), \alpha^+(x)] \times \{1\}$ for each $x = (x_1, x_2)$ with $x_2 \le 0$, where $\alpha^-(x) := \min\{x, 0\}$ and $\alpha^+(x) := \max\{x, 0\}$. Moreover, $F(x) := \{0, 1\}$ for every x with $x_2 > 0$ and $x_2 > |x_1| - 1$, and $F(x) = [-\alpha^-(x), \alpha^+(x)] \times \{1\}$ for every x with $x_2 > 0$ and $x_2 \le |x_1| - 1$. It is easy to check that F is a bounded usc map with compact convex values, and $W_e = \{(x,y) \in W \mid y = 1\} \cup (\{-1,1\} \times [-1,0])$ is closed disconnected. So, it is as good as possible, while, as before, there is no viable trajectory in W. Notice that now $W^-\{(x,y) \in W \mid y = 1\}$ is a strong deformation retract of W.

It appears that a choice of the exit set (W^- or W_e) in results on the existence of viable trajectories depends on methods we want to apply.

As examples of results which use W_e we present the following ones proved in [11]. We need the following notation:

$$\tau_W : S_F(W) \to [0,\infty], \quad \tau_W(x) = \sup\{t \ge 0 \mid x([0,t]) \subset W\},$$
$$\rho_W : S_F(W) \to [0,\infty], \quad \rho_W(x) := \inf\{t \ge 0 \mid x(t) \in W_e\}.$$

Theorem 2.2 ([11, Corollary 2.3]). *Let W be a closed subset of $\mathbb{R}^n$ and $Z \subset W$ be an arbitrary subset. Assume that $F : \mathbb{R}^n \multimap \mathbb{R}^n$ satisfies:*

(2.2) *for each $x_0 \in \overline{W_e} \setminus W_e$ and $x \in S_F(x_0)$, $x([0,\infty)) \cap W_e = \emptyset$*

and

(2.3) *there is a subset $A \subset W$, $W_e \subset A$ and there exists a retraction $r : A \to W_e$ such that $x([\rho_W(x), \tau_W(x)]) \subset A$ for every $x_0 \in Z$ and every $x \in S_F(x_0)$.*

If there is an admissible (4) *multivalued retraction* (5) *of W_e onto $Z \cap W_e$ and there is no admissible multivalued retraction of Z onto $Z \cap W_e$, then there is a trajectory starting from $Z \setminus W_e$ and viable in W.*

---

(4) For the definition and properties of *admissible* (in the sense of Górniewicz) multivalued maps, see e.g. [14]. In particular, compositions of compact and acyclic-valued usc maps are admissible.

(5) We say that $A \subset X$ is a *multivalued retract* of X if there exists an usc map $\Phi : X \multimap A$ with compact values such that $x \in \Phi(x)$, for every $x \in A$.

Theorem 2.3 (comp. [11, Theorem 2.1 and Corollary 2.2]). *Assume that the set W_e is closed and*

(2.4) *there is a subset $A \subset W$, $W_e \subset A$, and there exists a retraction $r: A \to W_e$ such that $x([0, \tau_W(x)]) \subset A$ for every $x_0 \in W_e$ and every $x \in S_F(x_0)$.*

If there is no multivalued admissible deformation ([6]) *of W onto W_e, then there is a viable trajectory in W.*

It is easy to see that assumption (2.2) is satisfied if W_e is closed, and that (2.4) implies (2.3). Our assumption (2.4) excludes, roughly speaking, the situation where some trajectories starting from one component of W_e leave W through another one. An example showing an importance of assumption (2.4) can be found in [11].

Note also that in early papers [2], [3], [15] multivalued versions of the Ważewski theorem were presented in terms of multivalued retracts, without an admissibility assumption. Such results were not sufficient from a topological point of view since even a sphere $\partial B(0,1) \in \mathbb{R}^n$ is a multivalued retract of the unit ball $\overline{B(0,1)}$. What we know is that $S \subset \partial W$ is not a multivalued retract of W if, for instance, W is connected and S is disconnected. Obviously, a connectedness criterion is far from the strong deformation retract approach proposed by Ważewski.

Sketch of proof of Theorem 2.3. Assuming that there is no viable trajectory in W, we define a map $H: W \times [0,1] \multimap W$ as a composition:

$$H: W \times [0,1] \xrightarrow{S_F \times \mathrm{id}} S_F(W) \times [0,1] \xrightarrow{J \times \mathrm{id}} S_F(W) \times [0,\infty) \times [0,1] \xrightarrow{k} W,$$

where

$$\begin{aligned} (S_F \times \mathrm{id})(x_0, \lambda) &:= S_F(x_0) \times \{\lambda\}, \\ (J \times \mathrm{id})(x, \lambda) &:= \{x\} \times [\rho_W(x), \tau_W(x)] \times \{\lambda\} \end{aligned}$$

and

$$k(x,t,\lambda) := \begin{cases} x(\lambda t) & \text{if } \lambda t \notin [\rho_W(x), \tau_W(x)], \\ r(x(\lambda t)) & \text{if } \lambda t \in [\rho_W(x), \tau_W(x)]. \end{cases}$$

By assumption (2.2) the map ρ_W is lsc (see [13, Lemma 1.9]). Furthermore, τ_W is usc, since W is closed (see [1, Lemma 4.2.2]). Hence, the map H, as a composition of admissible maps, is admissible. One can easily check that H is a multivalued admissible deformation of W onto W_e; a contradiction. □

It appears that, to obtain a sufficient condition for the existence of viable trajectories in terms of strong deformation retracts, the smaller exit set $W^-(F)$

([6]) A *multivalued admissible deformation* of X onto $A \subset X$ is a map $H: X \times [0,1] \multimap X$ admissible in the sense of Górniewicz, and such that $H(x,0) = x$, $H(x,1) \subset A$ for every $x \in X$, and $x \in H(x,t)$ for every $x \in A$. It is seen that $H(\,\cdot\,,1)$ is a multivalued (admissible) retraction.

is more appropriate. It is worth adding that $W^{-}(F)$ can be characterized by Bouligand tangent cones (see e.g. [7, Lemma 5.2]). This characterization is due to Cardaliaguet who has proved in [6] that there exists a viable trajectory in a convex set (or connected $C^{1,1}$-manifold) W whenever $W^{-}(F)$ is closed and disconnected. This was the first Ważewski type result for multivalued maps without paying any attention to the set $W_e(F)$.

Another idea is to find a Lipschitz selection or a sequence of sufficiently near Lipschitz approximations of the right-hand side F in such a way that a topological relation between W and $W^{-}(F)$ is the same as for approximations. The first question is: *Can we approximate F by Lipschitz maps with $W^{-}(F) = W^{-}(f)$ for sufficiently near approximations?* An approximation Lemma 3.3 in [7] allows us to obtain the following result.

Theorem 2.4. *Let $F:\mathbb{R}^n \multimap \mathbb{R}^n$ be continuous. Assume that $W \subset \mathbb{R}^n$ is a compact $C^{1,1}$ n-manifold with a boundary, $W^{-}(F)$ is closed and, if it is nonempty, it is a $C^{1,1}$ $(n-1)$-submanifold of ∂W with a boundary. If $W^{-}(F)$ is not a strong deformation retract of W, then there is a viable trajectory in W.*

In the proof we find, following Lemma 3.3 in [7], a sequence of Lipschitz $1/n$-approximations ([7]) (f_n) of F with $W^{-}(f_n) = W^{-}(F)$, and, by the Ważewski theorem, a sequence of viable solutions x_n corresponding to f_n. Since W as well as the graph of F are closed, we can go with n to infinity, and obtain a viable trajectory x for F in W.

In general, it is hard to find such good single-valued approximations as above. The problem is in a neighbourhood of the boundary of the exit set W^{-} in ∂W. An approximation technique can be applied for a larger class of sets and maps if we do not insist that approximating single-valued problems induce the same exit set. The main result in this direction has been proved in [11], and is as follows:

Theorem 2.5. *Let $W = \overline{\operatorname{Int} W}$ be a sleek ([8]) subset of $\mathbb{R}^n$ and $F:\mathbb{R}^n \multimap \mathbb{R}^n$ be a map (usc and not necessarily continuous) such that $W^{-}(F)$ is a compact strong deformation retract of its certain open neighbourhood V in W. Assume that $\operatorname{Int} T_W(x) \neq \emptyset$ for every $x \in W \setminus W^{-}(F)$. If $W^{-}(F)$ is not a strong deformation retract of W, then there is a viable trajectory in W.*

Let us give some comments. Sleekness we assume above is an essentially weaker condition than $C^{1,1}$ regularity which means lipschitzeanity of the map $T_W(\,\cdot\,)$, as required in Theorem 2.4. For instance, each closed convex set is sleek. Note that we need a lower semicontinuity of $T_W(\,\cdot\,)$ only on $W\setminus W^{-}(F)$. We have also dropped the continuity assumption on F. The assumption $\operatorname{Int} T_W(x) \neq \emptyset$

([7]) By an ε-approximation of F we mean a single-valued map f such that $d_{\operatorname{conv} F(B(x,\varepsilon))} f(x) < \varepsilon$ for every x. This condition is slightly weaker than the usual one considered in approximation techniques ("conv" is added, comp. [14]).

([8]) We say that a set W is *sleek*, if the Bouligand cone map $T_W(\,\cdot\,)$ is lower semicontinuous.

eliminates "too sharp corners" of the set W, and means, in other words, that W is epi-lipschitz in points of $W \setminus W^-(F)$ (comp. [17]).

The following lemmas are used in the proof of Theorem 2.5.

Lemma 2.6 (comp. [12, Lemma 3.3])**.** *Let $W \subset \mathbb{R}^n$ be a closed set and F be such that W^- is compact. Then, for any open neighbourhood V_0 of W^- in $\mathbb{R}^n$, there exist an open neighbourhood V_F of W^- in $\mathbb{R}^n$ and $\varepsilon_0 > 0$ such that, for every $p \in V_F \cap W$, $0 < \varepsilon \le \varepsilon_0$ and every locally Lipschitz ε-approximation f of F, there is $p \notin Viab_f(W)$ (i.e. $p \in W^*$) and $S_f(p)([0, \tau_W(p)]) \subset V_0 \cap W$, where τ_W is the exit function for f.*

Lemma 2.7 ([11, Lemma 3.5])**.** *Let A be a closed subset of $\mathbb{R}^n$. Assume that $F: \mathbb{R}^n \multimap \mathbb{R}^n$, $\Psi: A \multimap \mathbb{R}^n$ are convex valued, F is usc, and Ψ satisfies the following condition:*

(2.5) *For every $x \in A$ there exist $y_x \in F(x) \cap \operatorname{Int} \Psi(x)$ and an open neighbourhood $V(x)$ of x in X such that $y_x \in \operatorname{Int} \Psi(z)$ for each $z \in V(x) \cap A$.*

Then, for every $\varepsilon > 0$, there exists a locally Lipschitz map $f: \mathbb{R}^n \to \mathbb{R}^n$ such that

(a) *f is an ε-approximation of F,*
(b) *f is a selection of $\operatorname{Int} \psi(\,\cdot\,)$ on A.*

Let us note that assumption (2.5) is satisfied if, e.g. Ψ is lsc ([9]) and

$$F(x) \cap \operatorname{Int} \Psi(x) \neq \emptyset \quad \text{for every } x \in A.$$

Lemma 2.8 ([11, Lemma 3.6])**.** *Let $X \subset \mathbb{R}^n$ and $A \subset X$ be a closed subset. Assume that $\Psi: X \multimap \mathbb{R}^n$ is convex valued, and satisfies the following condition:*

(2.6) *For every $x \in X$ there exist $y_x \in \Psi(x)$ and an open neighbourhood $V(x)$ of x in X such that $y_x \in \Psi(z)$ for each $z \in V(x)$ with $y_x = 0$ for every $x \in A$.*

Then there exists a locally Lipschitz selection $f: \mathbb{R}^n \to \mathbb{R}^n$ of Ψ such that $f(x) = 0$ for every $x \in A$.

Sketch of proof of Theorem 2.5. We assume that there is no viable trajectory in W.

Step 1. For an open set V we find an open neighbourhood V' of $W^- := W^-(F)$ in W such that $\overline{V'} \subset V$, and each trajectory x starting from V' leaves W and up to time $\tau_W(x)$ it remains in V (see Lemma 2.6).

Step 2. Take an open neighbourhood Ω_0 of W^- in W such that $\overline{\Omega_0} \subset V'$ and, for an arbitrary small $\varepsilon > 0$, define the following auxiliary map $F_\varepsilon: \mathbb{R}^n \multimap \mathbb{R}^n$,

$$F_\varepsilon(x) := F(x) + \delta_\varepsilon(x)\overline{B_1},$$

([9]) $F: X \multimap Y$ is *lower semicontinuous* (usc) if the set $F_+^{-1}(U) := \{x \in X \mid F(x) \cap U \neq \emptyset\}$ is open for every open set U in Y.

where $\delta_\varepsilon(x) := \min\{d_{\overline{\Omega_0}}(x), \varepsilon\}$ and B_1 denotes the unit ball in $\mathbb{R}^n$. Then $W^-(F_\varepsilon) = W^-(F)$ and

$$F_\varepsilon(x) \cap \operatorname{Int} T_W(x) \neq \emptyset \quad \text{for every } x \in W \setminus \overline{\Omega_0}.$$

Step 3. Let $\Omega \supset \overline{\Omega_0}$ be an open subset in W such that $\overline{\Omega} \subset V'$. From Lemma 2.7 it follows that there exists a locally Lipschitz ε-approximation f of F_ε such that $f(x) \in \operatorname{Int} T_W(x)$ for every $x \in W \setminus \Omega$. Therefore, $W^-(f) \subset \Omega$.

Step 4. Take an open set U in W such that $\overline{\Omega} \subset U \subset \overline{U} \subset V'$. Consider the map $\Gamma\colon W \multimap [0,\infty)$,

$$\Gamma(x) := [\tau_{W\setminus U}(x), \tau_W(x)].$$

This map does not have to be lsc. Nevertheless, it satisfies the following condition:

- For every $x \in W$, there exist $\gamma_x \in \Gamma(x)$ and an open neighbourhood $V(x)$ of x in W such that $\gamma_x \in \Gamma(z)$ for any $z \in V(x)$.

Indeed, it is sufficient to take $\gamma_x \in \Gamma(x)$ such that $S_f(x)(\gamma_x) \in U \setminus \overline{W^-(f)}$ if $x \notin \overline{W^-(f)} \cup W^-$, and $\gamma_x = 0$ if $x \in \overline{W^-(f)} \cup W^-$, and use regularity of f. From Lemma 2.8 it follows that there exists a continuous selection $\gamma\colon W \to [0,\infty)$ of Γ with $\gamma(x) = 0$ for every $x \in \overline{W^-(f)} \cup W^-$. Notice that $S_f(x)(\gamma(x)) \in V$ and $\gamma(x) \le \tau_W(x)$ for every $x \in W$, where $S_f(x)$ denotes the unique solution of the Cauchy problem for f and an initial point x.

Step 5. Define the homotopy $h\colon W \times [0,1] \to W$,

$$h(x,t) := \begin{cases} S_f(x)(2t\gamma(x)) & \text{if } 0 \le t \le 1/2, \\ k(S_f(x)(\gamma(x)), 2t-1) & \text{if } 1/2 < t \le 1, \end{cases}$$

where $k\colon V \times [0,1] \to W$ is a strong deformation of V onto W^- in W. One can see that h is continuous, $h(\,\cdot\,,0) = \mathrm{id}_W$ and $h(x,1) \in W^-$ for every $x \in W$. Moreover, for every $x \in W^-$, there is $\gamma(x) = 0$ and hence $h(x,t) = k(x,t) = x$ for any $t \in [0,1]$. We conclude that W^- is a strong deformation retract of W; a contradiction. □

The assumptions of Theorem 2.5 can be slightly modified, and the result can obtain the following form.

Theorem 2.9. *Let $W = \overline{\operatorname{Int} W} \subset \mathbb{R}^n$ be a sleek* ANR*, and let $F\colon \mathbb{R}^n \multimap \mathbb{R}^n$ satisfy* (H_F) *and*

(2.7) *$W^-(F)$ is a compact* ANR *and* $\operatorname{Int} T_W(x) \neq \emptyset$ *for every $x \in W \setminus W^-(F)$.*

If $W^-(F)$ is not a strong deformation retract of W, then there is a viable trajectory in W.

Proof. Indeed, we show that we can arrive at the previous situation. From the homotopy extension theorem for compact ANRs (see [5, Corollary V.3.3]) it follows that there exists an open neighbourhood V of $W^-(F)$ in W with a strong deformation onto $W^-(F)$. $\square$

We finish with some open questions.

Open problems.

1. Suppose that F is continuous. Is it true that there exists a viable trajectory in W, if $W_e(F)$ is not a strong deformation retract of W?
2. Is Theorem 2.5 true for sets which are not sleek? Pay attention to Example 2.1, Figure 1.

The list of references below is far from completeness. They are chosen for the purpose of this note. We recommend [18] and references therein for further information on the Ważewski retract method.

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AN INTRODUCTION TO SOME PROBLEMS OF SYMPLECTIC TOPOLOGY

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Abstract. We give a short introduction to some open problems in symplectic topology, like existence of symplectic structures on $M \times S^1$ or on exotic tori and existence of symplectic circle actions on symplectic manifolds which admit smooth circle actions. Some relations between these problems are also explained.

1. Introduction

The principal aim of this note is to explain some open questions on symplectic manifolds in a way accessible to non-specialists and students. For this purpose we include an extensive preliminary part where basic notions and facts are described. Last three sections contain a discussion of:

- existence of symplectic forms on closed manifolds;
- existence of symplectic circle actions;
- existence of symplectic structures on exotic tori and a related question on symplectomorphisms of tori.

I omit most of technical details, to enable the reader to follow main route to those problems. Hopefully, this can be read by anybody familiar with main notions and facts of differential topology and the elementary part of de Rham theory of differential forms on manifolds. For further reading, detailed proofs, enlightening comments and more I recommend a beautiful book by Dusa McDuff and Dietmar Salamon [19].

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This article is based on lectures delivered during Winter School on Topological Methods in Nonlinear Analysis organized by Juliusz Schauder Center for Nonlinear Studies at Copernicus University, Toruń, in February 2009. Here I skip most of the introductory part of the lectures, which contained an elementary review of notions which were used later. The background material can be found in many textbooks and it would not be very useful to include it here. Some possible sources are [4], [6] for introduction to geometry of differential forms and de Rham complex and [18], [15] for a comprehensible introduction to differential topology.

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2. Preliminaries

Bilinear symplectic forms. A bilinear skew-symmetric form ω on $\mathbb{R}^k$ is called symplectic if it is nondegenerate, i.e. if for some $X \in \mathbb{R}^k$ we have $\omega(X,Y) = 0$ for any Y, then $X = 0$. Any bilinear skew symmmetric form in a base $e_1^*, \ldots, e_k^*$ of the dual space $(\mathbb{R}^{2n})^*$ is equal to $e_1^* \wedge e_2^* + \ldots + e_{2n-1}^* \wedge e_{2n}^*$, where $2n \leq k$. If such a form ω is non-degenerate, then $2n = k$ and there exists a base $e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*$ such that

$$\omega = e_1^* \wedge f_1^* + \ldots + e_n^* \wedge f_n^*.$$

In other words, for the dual base $e_1, \ldots, e_n, f_1, \ldots, f_n$ we get

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \quad \omega(e_i, f_j) = \delta_j^i \quad \text{for any } i, j \in \{1, \ldots, n\}.$$

Such a base is called symplectic. Any symplectic bilinear form admits many symplectic bases. and existence of symplectic bases implies that ω on $\mathbb{R}^{2n}$ is non-degenerate if and only if $\omega^n = \omega \wedge \ldots \wedge \omega$ is nonzero.

If we define J by $Je_i = f_i$, $Jf_i = -e_i$, $i = 1, \ldots, n$, then $J^2 = -\mathrm{id}$. Thus we have on $\mathbb{R}^{2n}$ a structure of complex vector space. Moreover, for any $v, w \in \mathbb{R}^{2n}$, $\omega(Jv, Jw) = \omega(v, w)$ and $\omega(v, Jv) > 0$ if $v \neq 0$. The formula $\langle v, w\rangle = \omega(v, Jw)$ defines a scalar product in $\mathbb{R}^{2n}$. We say that J is *compatible* with ω.

In the other direction, if J is a complex structure on $\mathbb{R}^{2n}$ and $\langle \cdot, \cdot \rangle$ is a J-invariant scalar product, then $\omega(v, w) = -\langle v, Jv\rangle$ is a symplectic bilinear form on $\mathbb{R}^{2n}$ and J is compatible with ω. For any given J there exists a J-invariant scalar product given for example as the averaged form $\frac{1}{2}(\langle v, w\rangle + \langle Jv, Jw\rangle)$, where $\langle \cdot, \cdot \rangle$ is arbitrary.

Using symplectic bases, it is easy to see that if J_0 is the standard complex structure, then any other J is induced from J_0 by a linear isomorphism T, $J = T^{-1}J_0T$. Since J_0 is preserved by T if and only if T is a complex isomorphism, we can identify J with an element of the quotient $\mathrm{GL}(\mathbb{R}, 2n)/\mathrm{GL}(\mathbb{C}, n)$. Up to homotopy type this is the quotient of maximal compact subgroups. For the proofs of the following two facts see [19].

Proposition 2.1. *Both the space of all linear symplectic forms on $\mathbb{R}^{2n}$ and the space of complex structures on $\mathbb{R}^{2n}$ have the homotopy type of $O(2n)/U(n)$. If $n = 2$, then it is homotopically equivalent to $S^2 \cup S^2$.*

Note also that the space of all complex structures compatible with a given symplectic form is large, since one can change a symplectic base by a symplectic isomorphism to get another complex structure compatible with the same form. As above one gets a homeomorphism of that space with $\mathrm{Sp}(n)/\mathrm{GL}(\mathbb{C}, n) \cap \mathrm{Sp}(n)$, where $\mathrm{Sp}(n)$ denotes the space of linear isomorphisms preserving the standard symplectic form on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

Proposition 2.2. *The space of all complex structures on $\mathbb{R}^{2n}$ compatible with a given symplectic form is contractible.*

Symplectic differential forms. Now we will consider exterior differential 2-forms on smooth manifolds, i.e. smooth sections of the second exterior power of the cotangent bundle. Such a form ω is called nondegenerate if at any point $x \in M$ we have a nondegenerate bilinear form ω_x on T_xM. Thus a 2-form ω is non-degenerate at $x \in M$ if for any nonzero vector $X \in T_xM$, the 1-form $\iota_X\omega$ on T_xM does not vanish, where $(\iota_X\omega)(Y) = \omega(X, Y)$.

Definition 2.3. A differential 2-form is called *symplectic* if it is *closed* and *non-degenerate*.

A smooth complex structure on the tangent vector bundle of a manifold M is called *almost complex structure* on M. This means that there is a bundle endomorphism $J\colon TM \to TM$ such that

(a) $J^2 = -\mathrm{id}$.

We say that J is compatible with ω if additionally the following two conditions hold:

(b) $\omega(JV, JW) = \omega(V, W)$ for all U, V;
(c) the symmetric form defined as $g(U, V) = \omega(U, JV)$ is positive definite, so that it defines a Riemannian metric on M.

Contractibility of the space of complex structures on T_xM compatible with ω_x implies that there exist almost complex structures compatible with any symplectic form. Namely, construct J first locally using symplectic bases and then combine local structures to a global almost structure using the contractibility to deform one local J to another. In terms of bundles, a ω-compatible J is a section of a bundle with contractible fibre and the argument describes how to construct a section of such bundle. The space of such sections is also contractible, thus we have

Proposition 2.4. *If a manifold has a symplectic structure, then it admits an almost complex structure. The space of all almost complex structures compatible with a given symplectic form is contractible.*

However, only nondegeneracy is used to construct J. Thus existence of an almost complex structure is equivalent to existence of a differential form, not necessarily closed, which is non-degenerate at each point.

Examples 2.5.

(a) In $\mathbb{R}^{2n}$ consider standard coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. The formula $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$ defines a symplectic form. Since ω is invariant with respect to translations, it defines also a symplectic form on the torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$.

(b) The volume form of any oriented Riemannian surface is symplectic.

(c) If ω_1, ω_2 are symplectic forms on manifolds M_1, M_2, then $\omega_1 \times \omega_2 = p_1^*\omega_M + p_2^*\omega_N$, where p_1, p_2 are projections, is a symplectic form on $M_1 \times M_2$.

(d) For any manifold M the cotangent bundle T^*M is a (noncompact) symplectic manifold. A symplectic form is $d\lambda$, where λ is the tautological 1-form on T^*M given by

$$\lambda_{v^*} = v^* d\pi.$$

Here $\pi\colon T^*M \to M$ jest the projection of the cotangent bundle and $v^* \in T^*M$ is a point in T^*M. In local coordinates $x_1, \dots, x_n$ on M we have the formula

$$\lambda_{v^*}\left(\frac{\partial}{\partial x_j}\right) = y_j, \quad \text{if } v^* = \sum_j y_j dx_j.$$

The following theorem shows some rigidity of symplectic structures.

Theorem 2.6 (Moser)**.** *If ω_t is a smooth path of symplectic forms on a closed manifold M such that the cohomology class $[\omega_t]$ is constant, then there exists a smooth isotopy $\psi_t \in \mathrm{Diff}(M)$ satisfying $\psi_0 = \mathrm{id}_M$ and $\psi_t^*\omega_t = \omega_0$.*

The proof is based on so called Moser's trick. Since $[\omega_t] = \text{const}$ thus there exists σ_t such that $\frac{d}{dt}\omega_t = d\sigma_t$. Consider the 1-parameter family of vector fields X_t defined (uniquely, since ω_t are non-degenerate) by the equation $\sigma_t = -\iota(X_t)\omega_t$. On a closed manifold this family defines a path ψ_t of diffeomorphisms by

$$\frac{d}{dt}\psi_t = X_t(\psi_t).$$

Differentiating the equality $\psi_t^*\omega_t = \omega_0$ with respect to t we get

$$\psi_t^* d(\sigma_t + \iota(X_t)\omega_t) = 0,$$

thus the isotopy ψ_t given by X_t has the required property.

Moser's theorem can be used to prove the following property which shows that there is no local symplectic invariants. This is in contrast with Riemannian geometry, where curvature invariants play a prominent role.

Theorem 2.7 (Darboux)**.** *For any symplectic form ω on M and any point $x \in M$ there exists a local coefficient system $x_1, \dots, x_n, y_1, \dots, y_n$ around x such that $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$.*

3. Symplectic forms on closed manifolds

The problem of existence of symplectic structures has a simple answer in the case of open (i.e. non-compact or with non-empty boundary) manifolds. However, the proof of the following theorem is quite difficult (see [19]).

Theorem 3.1 (Gromov). *If M is an open almost complex manifold, then it admits a symplectic form.*

For closed manifolds the problem whether there is a symplectic form on a given manifold is simple only in dimension 2, where orientability is necessary and sufficient. In dimension 4 there are some answers (see Section 6), but in higher dimensions essentially no general existence theorems are known.

Consider a closed symplectic manifold M of dimension $2n$. By Proposition 2.4 M is almost complex. There are non-trivial obstructions to impose an almost complex structure on M. For $n = 2$ a characterization of closed almost complex manifolds was given by Ehresman and Wu.

Theorem 3.2. *A closed oriented 4-manifold M admits an almost complex structure compatible with the orientation if and only if there exists a class $c \in H^2(M, \mathbb{Z})$ such that its reduction mod 2 is equal to the second Stiefel–Whitney class $w_2(M)$ and $c^2 = 2\chi(M) + 3\sigma(M)$. Here χ, σ denote respectively the Euler characteristic and the signature and $H^4(M, \mathbb{Z})$ is identified with the integers using the given orientation.*

Using Theorem 3.2 one can check that the connected sum $\#^k\mathbb{C}P^2$ of k copies of the complex projective space $\mathbb{C}P^2$ is almost complex if and only if k is odd. In particular, $\mathbb{C}P^2\#\mathbb{C}P^2$ admits no symplectic structure. To calculate this, let us recall that for $\mathbb{C}P^2$ we have $w_2 \neq 0$, $\chi = 3$, $\sigma = 1$. It is not difficult to calculate that $H^2(\#^k\mathbb{C}P^2) \cong \oplus^k H^2(\mathbb{C}P^2)$ and that $(a_1, \dots, a_k)^2 = a_1^2 + \dots + a_k^2 \in H^4(M, \mathbb{Z}) \cong \mathbb{Z}$. We have $\chi = k + 2$, $\sigma = k$, $w_2 = (1, \dots, 1) \mod 2$. Thus for a class $c = (a_1, \dots, a_k)$ required by the Ehresmann–Wu theorem all entries should be odd integers. An elementary argument shows that for $a_1, \dots, a_k$ odd, $a_1^2 + \dots + a_k^2$ can not be equal to $5k + 4$ if k is even. For $k = 2$ this boils down to a simple fact that the equation $a^2 + b^2 = 14$ has no integer solutions. For $k = 3$ a solution is $c = (3, 3, 1)$.

Another obstruction to existence of symplectic forms arises from the fact that ω^n is a volume form. Thus any symplectic form determines an orientation of the underlying manifold. However, this can be obtained from an almost complex structure as well, since any complex structure on a vector space V defines uniquely an orientation of V. But for a closed manifold it gives more. Namely, on a closed $2n$-dimensional manifold we have $\int_M \omega^n \neq 0$, thus the cohomology classes $[\omega]^n = [\omega] \cup \dots \cup [\omega]$ and $[\omega]$ are nonzero.

Thus we have two basic obstructions to get a symplectic structure.

Proposition 3.3. *If a closed $2n$-dimensional manifold M admits a symplectic structure, then M is almost complex and there is a class $u \in H^2(M;\mathbb{Z})$ such that $u^n \neq 0$. In particular, $H^2(M) \neq 0$.*

Examples 3.4.

(a) Complex projective space $\mathbb{C}P^n$ is symplectic and the following Fubini–Study form τ gives a symplectic structure.

$$\tau = \frac{1}{2\sum_\mu \overline{z}_\mu z_\mu} \sum_k \sum_{j\neq k} \overline{z}_j z_j dz_k \wedge d\overline{z}_k - \overline{z}_j z + k dz_j \wedge d\overline{z}_k$$

where $z_j = x_j + iy_j, j = 0, \dots, n$ are complex homogeneous coordinates and we denote $dz_j = dx_j + i\,dy_j$, $d\overline{z}_j = dx_j - i\,dy_j$. This is an example of a *Kähler* manifold, i.e. a *complex* manifold with a Riemannian metric g such that $\omega(V,W) = g(V,JW)$ is a closed form, where J is the almost complex structure on M provided by its complex structure.

(b) The sphere S^{2n} of dimension $2n$ does not admit any symplectic form for $n > 2$, since $H^2(S^{2n}) = 0$. For the same reason $S^3 \times S^1$ is not symplectic. Moreover, $S^2 \times S^4$ is not symplectic because for any $x \in H^2(S^2 \times S^4;\mathbb{Z})$ we have $x^3 = 0$.

4. Constructions of symplectic manifolds

The product $M \times N$ of two symplectic manifolds M, N is symplectic. Hence, it is natural to ask whether a fibre bundle with symplectic base and symplectic fibre is symplectic. In general this fail to be true as the following example shows.

Example 4.1. Let $S^3 \to S^2$ be the Hopf fibre bundle. It is a bundle with fibre S^1 given by the natural action, by multiplication, of unit (of module 1) complex numbers on unit quaternions. Then $S^3 \times S^1 \to S^3 \to S^2$ is a fibre bundle map with fibre T^2, hence both base and fibre are symplectic, while the total space is not. Moreover, the structure group of this fibre bundle is the symplectomorphism group of the fibre, which is (since we are in dimension 2) the group of area preserving diffeomorphisms. In fact, the structure group of Hopf fibration is the isometry group of S^1, thus the structure group of $S^3 \times S^1 \to S^2$ is the isometry group of T^2.

Sufficient conditions for a fibre bundle $p\colon M \to B$ with both base and fibre symplectic to have a symplectic total space were given by W. Thurston [25]. First condition imposed on the bundle was that the structure group is the symplectomorphism group of the fibre. We say that such a bundle is *symplectic*. This is a natural assumption if one expects on M a symplectic form which restricts to a symplectic form on all fibres.

For a point b in the base let i_b denote the inclusion of the fibre

$$F_b = p^{-1}(b) \subset M.$$

In a symplectic fibration each fibre has a well defined symplectic form ω_b symplectomorphic to ω_F. But, as Example 4.1 shows, a further assumption is needed.

Theorem 4.2 (Thurston)**.** *Consider a symplectic fibre bundle $p\colon M \to B$ with closed symplectic base and fibre. If there exists a cohomology class $u \in H^2(M, \mathbb{R})$ such that $i_b^* u = [\omega_b]$, then there exists a symplectic form ω_M on M which is compatible with the fibration, i.e. $i_b^* \omega_M = \omega_b$.*

However, if a symplectic manifolds fibres with a symplectic fibre then one can not in general expect the base to be symplectic. An example is $\mathbb{C}P^3$ which is fibred over S^4 with fibre S^2. Some other examples can be deduced from [17].

We describe now two interesting examples of a fibre bundle with symplectic base and fibre.

Example 4.3. Let $\mathrm{Diff}(D^{2n}, S^{2n-1})$ denote the group of diffeomorphisms of the $2n$-disk equal to the identity in a neighbourhood of the boundary sphere S^{2n-1}. Then f extends by the identity to a diffeomorphism of any $2n$-manifold X if an embedding of D^{2n} into X is given. Consider $f \in \mathrm{Diff}(D^{2n}, S^{2n-1})$ not in the identity component. For $X = S^{2n}$ we get again a diffeomorphism f_S which is not isotopic to the identity and it is a classical fact that f_S corresponds to an exotic $(2n+1)$-sphere $\Sigma_f = D^{2n+1} \cup_{f_S} D^{2n+1}$ (a smooth manifold homeomorphic but not diffeomorphic to the sphere S^{2n+1} with the standard differential structure). If X is the 2n-torus $\mathbb{T}^{2n} = S^1 \times \ldots \times S^1$, denote the resulting diffeomorphism by f_T. In this case we will get also an exotic manifold in the following way. Take $\mathbb{T}^{2n} \times [0,1]$ and glue the ends according to $(x, 0) \sim (f_T(x), 1)$. The resulting manifold $\mathbb{T}(f_T)$, the *mapping torus* of f_T, depends, up to diffeomorphism, only on the isotopy class of f_T. From the fact that f_T is supported in a disk (i.e. is equal to id outside a disk) it is not difficult to argue that $\mathbb{T}(f_T)$ is obtained from the standard torus $\mathbb{T}^{2n+1}$ by a connected sum with the homotopy sphere Σ_f. It is known that $\mathbb{T}(f_T)$ is homeomorphic but not diffeomorphic to $\mathbb{T}^{2n+1}$, cf. “Fake Tori” chapter in [28]. Note also that, by construction, $\mathbb{T}(f_T)$ fibers over S^1 with fibre $\mathbb{T}^{2n}$.

Now $M = \mathbb{T}(f_T) \times S^1$ fibres over $\mathbb{T}^2$ with fiber $\mathbb{T}^{2n}$. The fibration is symplectic if and only if f_T is isotopic to a symplectomorphism. Moreover, if this is the case, then the other assumption of Thurston's theorem is satisfied. This is because f_T is homotopic (even topologically isotopic) to the identity, thus the fibration is equivalent, up to fibrewise homotopy equivalence, to the product $\mathbb{T}^2 \times \mathbb{T}^{2n}$. So the required cohomology class exists and M is symplectic provided f_T is isotopic to a symplectomorphism. See Chapter **8** for further remarks on this example.

Example 4.4. Let A be a linear map of the torus $\mathbb{T}^2$ given by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Consider the mapping torus $\mathbb{T}_A = \mathbb{T}^2 \times [0,1]/(x,0) \sim (A(x),1)$ and a fibration $M = \mathbb{T}_A \times S^1 \to \mathbb{T}^2$. By Theorem 4.2 there exists a symplectic structure on M. However, direct calculations show that first Betti number $b_1 M = \dim H_1(M,\mathbb{R}) = 3$. This implies that there is no Kähler structure on M, since odd Betti numbers of closed Kähler manifolds are always even.

It was the first example of a closed symplectic manifold with no Kähler structure. Later many other such examples were constructed and one can say that all known bounds for the topology of closed Kähler manifolds fail in symplectic case (see [27]).

Let us sketch two other useful constructions of symplectic manifolds.

First one is the blow up of a manifold. By blow up of a point x of a $2n$-manifold M we mean a compactification of $M - \{x\}$ by $\mathbb{C}P^{n-1}$, where in a chart $U \cong \mathbb{C}^n$ around x (with x corresponding to $(0,\dots,0)$) any complex plane is compactified by the point of $\mathbb{C}P^{n-1}$ which represents this plane. A direct generalization gives a blow up along a submanifold, given by compactifying each normal space of a submanifold as above. Topologically, a manifold obtained by blow up of a point in M is diffeomorphic to a connected sum $M\#\overline{\mathbb{C}P^n}$, where $\overline{\mathbb{C}P^n}$ denotes $\mathbb{C}P^n$ with the reversed orientation.

Theorem 4.5. *If M is a symplectic $2n$-manifold, then a blow up of M is also symplectic.*

However, there is no canonical choice of a symplectic structure on a blow up of a symplectic manifold. For various data used to perform the operation one can get different (non symplectomorphic) symplectic structures. This is in contrast with the case of differential manifolds, where on a connected sum one can construct a unique, up to a diffeomorphism, differential structure.

Another operation on symplectic manifolds was introduced by R. Gompf [9] and it is often called Gompf's surgery. Consider two symplectic manifolds M, N and symplectic submanifolds M_0, N_0 of codimension two. Assume that M_0 is symplectomorphic to N_0 and the normal (complex linear) bundle $\nu(M_0)$ is inverse to the normal bundle $\nu(N_0)$. This means that there is a orientation changing linear isomorphism of that bundles, covering the given symplectomorphism $\Phi_0\colon M_0 \to N_0$. In terms of Chern classes, $c_1\nu(M_0) = -c_1\nu(N_0)$ (first Chern classes classify complex bundles of complex dimension 1). Then one can find a symplectic structure on $(M - \nu_\epsilon(M_0)) \cup_f (N - \nu_\epsilon(N_0)$, where ν_ϵ denotes the open ϵ-disc normal bundle and f is a diffeomorphism of the boundary sphere bundles covering Φ_0.

As an application, Gompf has shown that in any even dimension greater than 2 any finitely presented group is the fundamental group of a closed symplectic manifold. Compare also [16], where some restrictions on the fundamental group were found under assumption that the symplectic structure is *symplectically aspherical*, i.e. the symplectic form vanishes on all spherical homology 2-classes.

5. Symplectic group actions

Isomorphisms in the category of symplectic manifolds are those diffeomorphisms which preserve symplectic forms. Thus a *symplectomorphism* of (M, ω) is a diffeomorphism $f\colon M \to M$ such that $f^*\omega = \omega$. The group of all symplectomorphism will be denoted by $\mathrm{Symp}(M, \omega)$. For a compact manifold we consider the C^1 topology on the group. This is always an infinitely dimensional space, since for any path H_t of smooth functions and the path X_t of vector fields defined by $\iota_{X_t}\omega = dH_t$, the associated path of diffeomorphisms preserves the form ω (compare the proof of Theorem 2.6).

If a group G acts smoothly on a symplectic manifold (M, ω), then we say that the action is *symplectic* if ω is G-invariant or, equivalently, $g \in Symp(M, \omega)$ for any $g \in G$. In this note we restrict to the case $G = S^1$.

For a smooth action of S^1 there is a vector field V on M which generate the action, i.e. the action is the flow of V. It is the image of the unit invariant vector field tangent to S^1 under the differential of the action. The field V is tangent to orbits of the action and its zero set is equal to the fixed point set. If it is a symplectic action, then the form $\iota_V\omega$ is a closed 1-form, as it follows from the formula $L_V\omega = \iota_V d\omega + d\iota_V\omega = 0$ for the Lie derivative $L_V\omega$. If the cohomology class $[\iota_V\omega]$ vanishes, then the action is called *hamiltonian* and its *moment map* is defined as a map $H\colon M \to \mathbb{R}$ such that $dH = \iota_V\omega$. More generally, if we assume that $[\iota_V\omega]$ is an integer class (it is in the image of $H^2(M; \mathbb{Z})$), then there exists a generalized moment map $H\colon M \to S^1$ such that $H^*\theta = \iota_V\omega$, where θ is the standard invariant 1-form on S^1. Moment maps have nice properties:

• the set of critical points is equal to the zero set of V, hence to the fixed point set of the action,

• they are Morse–Bott functions, i.e. any component of the critical point set is transversally nondegenerate.

Certainly, a hamiltonian action on a closed manifold must have fixed points, since in this case the moment map is a real valued map. In dimension 4 a symplectic action on a closed manifold is hamiltonian if and only if it has fixed points. In dimension 6 a non-hamiltonian symplectic action with non-empty set of fixed points was constructed by McDuff.

It is well-known that the fixed point set of a symplectic action is a symplectic submanifold, cf. [12, Lemma 27.1].

Lemma 5.1. *Let G be a compact Lie group. If G acts symplectically on a symplectic manifold M, then the fixed point set M^G is a symplectic submanifold.*

Proof. Let $x \in M^G$. Then, when an invariant Riemannian metric is chosen, G acts on a transversal to M^G via an orthogonal representation without trivial G-subspaces. Thus $U \in T_x(M)$ belongs to $T_x(M^G)$ if and only if $g_* U = U$ for every $g \in G$. This implies that vectors of the form $V - g_* V$ span the transversal to $T_x(M^G)$ in $T_x M$. Hence for $U \in T_x(M^G)$ we have $\omega(U, V) = \omega(g_* U, g_* V) = \omega(U, g_* V)$, and therefore $\omega(U, V - g_* V) = 0$ for any $g \in G$ and $V \in T_x M$. So, if $\omega(U, W) = 0$ for all $W \in T_x M^G$, then also $\omega(U, W') = 0$ for all $W' \in T_x M$ and this implies $U = 0$. So $\omega | M^G$ is nondegenerate, thus symplectic. $\square$

Corollary 5.2. *Let G be a compact Lie group and let H be a closed subgroup of G. If G acts symplectically on a symplectic manifold M, then the set of points with isotropy equal to H is a symplectic submanifold.*

An analogous property for almost complex manifolds and actions is straightforward.

Lemma 5.3. *If a compact Lie group G acts smoothly on an almost complex manifold M preserving an almost complex structure J, then the fixed point set M^G is a J-holomorphic submanifold of M.*

Proof. If J is G-invariant, then for $U \in T_x(M^G)$ and any $g \in G$ we have $g_*(JU) = g_* J g_*^{-1} g_* U = JU$. $\square$

Let us assume now that S^1 acts freely and symplectically on (M, ω), V generates the action and $X = M/S^1$. The 1-form $\iota_V \omega$ is closed and descends to X to a closed nowhere vanishing 1-form. This implies that X fibres over a circle [26] and in fact one can prove that this is a symplectic fibration.

Conversely, if X admits a symplectic fibration over the circle with fibre (F, ω_F), then $X \times S^1$ admits a symplectic structure. A symplectic fibration over S^1 is the torus $T(f)$ of a symplectomorphism $f \colon F \to F$. To apply Thurston's theorem to $S^1 \times T(f)$, which is a symplectic fibration over $S^1 \times S^1$ with fibre F, it suffices to check that the cohomology class of the symplectic form on F is in the image of the cohomology homomorphism i^*, where $i \colon F \to T(f)$ is the inclusion. This claim follows from the Mayer–Vietoris exact sequence resulting from a decomposition of S^1 into two intervals, since elements in cohomology which are invariant under the gluing map all are in the image of i^*. No other examples of symplectic manifolds of the form $X \times S^1$ are known. See also Section 7.

The discussion above applies also to the case of a circle action with no fixed points, but then X is in general an orbifold.

6. Existence questions

As we have seen above, there are two basic obstructions to impose a symplectic structure on a closed manifold.

Definition 6.1. A closed manifold M of dimension $2n$ which is almost complex and has a class $u \in H^2(M,\mathbb{R})$ such that $u^n \neq 0$ is called *homotopically symplectic.*

The name *cohomologically symplectic*, or *c-symplectic* is used for a manifold with a class $u \in H^2(M;\mathbb{Z})$ such that $u^n \neq 0$, see e.g. [1]. We assume additionally that M is almost complex and the word "homotopically" refers to the homotopy class of the classifying map of the tangent bundle of M, which we consider as a part of the differential structure of M. An oriented manifold M is almost complex if and only if the classifying map $\tau\colon M \to \mathrm{BGL}(2n,\mathbb{R})$ of its tangent bundle lifts to a map $\widetilde{\tau}\colon M \to \mathrm{BGL}(n,\mathbb{C})$. It means that $\tau = P\widetilde{\tau}$, where $P\colon \mathrm{BGL}(n,\mathbb{C}) \to \mathrm{BGL}(2n,\mathbb{R})$ is the forgetful map and this property depends only on the homotopy class of the classifying map τ.

Question 6.2. Does any closed, homotopically symplectic manifold admit a symplectic form?

Obviously the problem depends only on the diffeomorphism type of M. There is a description of symplectic manifolds in topological terms as those manifolds which admit so called topological Lefschetz pencils [10], but to decide whether a manifold has such a structure is as difficult as to construct a symplectic form.

In dimension 4 the answer to 6.2 is negative.

Example 6.3. $\#^3\mathbb{C}P^2$ is homotopically symplectic and has no symplectic structure.

That it is a homotopically symplectic manifold we have seen in Section 3. Nonexistence of symplectic structure was proved using Seiberg–Witten invariants of diffeomorphism type. They are defined for closed 4-manifolds via moduli spaces of a differential equation related to the Dirac operator. The invariant is given by a function $SW_M\colon H^2(M,\mathbb{Z}) \to \mathbb{Z}$ with finite support, see [22]. A powerful theorem providing a necessary condition which is much more delicate than homotopical symplecticness to existence of a symplectic structure was proved by C. H. Taubes [24].

Theorem 6.4. *For any closed symplectic* 4*-manifold there exists a class* $u \in H^2(M,\mathbb{Z})$ *such that* $SW(u) = \pm 1$.

The fact that $\#^3\mathbb{C}P^2$ does not satisfy the above condition follows from properties of Seiberg–Witten invariant. Namely, for a connected sum of two closed 4-manifolds with positive b_2^+, Seiberg–Witten invariant vanishes. Here b_2^+ is the dimension of positive defined part of $H^2(M,\mathbb{R})$ with respect to the intersection form $(u,u') \mapsto (u \cup u')[M]$.

In higher dimensions there is no known example of a non-symplectic but homotopically symplectic manifold.

Seiberg–Witten invariants are defined only in dimension 4. In this dimension they are equivalent to so called Gromov–Witten invariants. The latter was defined, using moduli spaces of pseudoholomorphic curves, by Michel Gromov in his seminal paper [11]. For an exposition of the theory of pseudoholomorphic curves see [20].

We describe now some related examples.

Example 6.5. Let M and N be two closed simply connected 4-dimensional smooth manifolds such that the following conditions hold:

(a) M and N are homeomorphic, but not diffeomorphic,
(b) only one of these manifolds admits a symplectic structure,
(c) the second Stiefel–Whitney class $w_2(M)$ vanishes.

Then $M \times S^2$ is diffeomorphic to $N \times S^2$, hence both are symplectic.

We refer to [29]. Indeed, under our assumptions the diffeomorphism type is completely determined by the multiplicative structure of the cohomology ring with integer coefficients and the first Pontriagin class. It follows from Theorem 3 in [29] which can be stated as follows:

The diffeomorphism classes of closed simply connected 6*-manifolds M with torsion free integral cohomology, whose second Stiefel–Whitney class vanishes, correspond bijectively to the isomorphism classes of an algebraic invariant consisting of:*

(a) *two free abelian groups $H = H^2(M;\mathbb{Z})$ and $G = H^3(M;\mathbb{Z})$,*
(b) *a symmetric trilinear map $\mu\colon H \times H \times H \to \mathbb{Z}$ given by the cup product,*
(c) *a homomorphism $p_1\colon H \to \mathbb{Z}$ determined by the first Pontriagin class p_1.*

Note that $p_1(M\times S^2)$ is inherited from M and p_1 is a topological invariant for closed 4-manifolds. Thus $p_1(M\times S^2) = p_1(N\times S^2)$, $w_2(M\times S^2) = w_2(M\times S^2) = 0$ and thus $M \times S^2$ is diffeomorphic to $N \times S^2$.

Some examples of pairs (M, N) as required above are obtained by applying to symplectic 4-manifolds constructions such as logarithmic transformation or knot surgery. To detect both non-diffeomorphism and non-symplecticness one uses Taubes' theorem (see 12.4 in [23] or [21]). An explicit example is given by a non-symplectic smooth manifold homeomorphic but not diffeomorphic to K3 surface.

7. Circle actions: smooth versus symplectic

A natural specific existence question is when does exist a symplectic structure on the product of a manifold by the circle. As it was explained in Section 5, if a closed manifold M with a free S^1 action admits a invariant symplectic form, then $X = M/S^1$ fibres over the circle. In the other direction, for the product $X \times S^1$, a symplectic fibration of X over the circle enables us to construct a symplectic form on M.

Question 7.1. Let X be a closed manifold. Is it true that if $X \times S^1$ is symplectic, then X fibres over S^1?

For X of dimension 3 this question was posed by C. H. Taubes and answered positively, after a series of partial results of many authors, by S. Friedl and S. Vidussi [8]. Their proof uses Seiberg–Witten invariants and it does not extend to higher dimensions.

Remark 7.2. Questions 6.2 and 7.1 can not simultaneously have positive answers in higher dimensions. Namely, there are manifolds which do not fibre over the circle, but their products with the circle are homotopically symplectic, e.g. the connected sum of two copies of tori $T^{2k+1} \# T^{2k+1}$.

A more general conjecture for dimension 4 was stated by Scott Baldridge in [2].

Conjecture 7.3. Every closed 4-manifold that admits a symplectic form and a smooth circle action also admits a symplectic circle action (with respect to a possibly different symplectic form).

In the same paper Baldridge gave a partial answer.

Theorem 7.4 ([2]). *If M is a closed symplectic 4-manifold with a circle action such that the fixed point set is non-empty, then there exists a symplectic circle action on M.*

It seems unlikely that this continue to be true in higher dimensions, but one can ask the following question: *under what condition a closed symplectic manifold with a smooth circle action does admit a symplectic circle action*?

There are examples of smooth circle actions on symplectic manifolds which have non-symplectic sets of fixed points or non-symplectic sets of points with a given isotropy. By Lemma 5.1 any such action is not symplectic with respect to any symplectic structure.

Example 7.5. Let M, N be a pair of 4-manifolds described in Example 6.5. Then $M \times S^2 \times \ldots \times S^2$ is symplectic (since it is diffeomorphic to $N \times S^2 \times \ldots \times S^2$) and the action given by the standard action on each copy of S^2, has a sum of disjoint copies of M as the fixed point set.

More examples of this kind can be found in [13].

8. Symplectomorphisms and exotic tori

It is known that for any $m \geq 5$, there exist *exotic tori,* i.e. smooth manifolds $\mathcal{T}^m$ which are homeomorphic but not diffeomorphic to the standard torus $\mathbb{T}^m$.

Question 8.1. Given a symplectic manifold $\mathcal{T}^{2n}$ homeomorphic to $\mathbb{T}^{2n}$, $n > 2$, is $\mathcal{T}^{2n}$ diffeomorphic to $\mathcal{T}^{2n}$?

This is motivated by the same question posed by C. Benson and C. Gordon in [5] for Kähler manifolds. It has positive answer, a proof that there are no Kähler structures on exotic tori can be obtained from the Albanese map $M \to T^k$ by showing that for a manifold homeomorphic to a torus the map is a homotopy equivalence. This implies that it is in fact a diffeomorphism. More general results are given in [3], [7].

Let us look on Example 4.3 from that point of view. This leads to the following.

Question 8.2. Given an exotic sphere Σ_f of dimension $2n-1$, is there a symplectic structure on $\mathcal{T} = (\mathbb{T}^{2n-1} \# \Sigma_f) \times S^1$?

As we have seen in Section 4, the answer were positive when there exists a diffeomorphism $f \in \operatorname{Diff}(D^{2n-2}, S^{2n-3})$ such that Σ_f is exotic and the diffeomorphism f_T obtained from f is isotopic to a symplectomorphism. Thus we come to the following question.

Question 8.3. Given a symplectomorphism $f\colon \mathbb{T}^{2n-2} \to \mathbb{T}^{2n-2}$ supported in an embedded disc, can f be smoothly isotopic to the identity?

A similar problem whether a symplectomorphism of a torus which acts trivially on homology is isotopic to the identity was mentioned in [19, p. 328].

One can also ask under what assumptions a diffeomorphism of $\mathbb{T}^{2n}$ is isotopic to a symplectomorphism. Examples such that there is no symplectomorphisms in some isotopy classes [14].

Let $\pi_0(\operatorname{Diff}_+(M))$ denote the group of isotopy classes of orientation preserving diffeomorphisms of a smooth oriented manifold M. Assume now that M is $2n$-dimensional and admits almost complex structures, and let $\mathbb{J}M$ denote the set of homotopy classes of such structures, compatible with the given orientation. Any diffeomorphism f acts on the set of all almost complex structures by the rule

$$f_* J = df J df^{-1},$$

where $df\colon TM \to TM$ denotes the differential of f. This action clearly descends to the action of $\pi_0(\operatorname{Diff}_+(M))$ on $\mathbb{J}M$.

We show that there exist diffeomorphisms $f\colon \mathbb{T}^{8k} \to \mathbb{T}^{8k}$ supported in a disc which do not preserve the homotopy class $[J_0] \in \mathbb{J}M$ of the standard complex structure. Therefore, they cannot be isotopic to symplectomorphisms with respect to the standard symplectic structure ω_0. Indeed, conjugation by a symplectomorphism sends any almost complex structure compatible with a symplectic form to another almost complex structure compatible with the same symplectic form. Thus they are homotopic, since the space of all such almost complex structures is contractible.

Let us a sketch the proof [14] that such f exist. There is a necessary homotopic condition on a diffeomorphism to preserve the homotopy class of J_0.

Theorem 8.4. *Let $f \in \mathrm{Diff}(\mathbb{T}^{4n})$ be supported in a disc $D^{4n} \subset \mathbb{T}^{4n}$. If f preserves J_0, then the differential df restricted to its support disc D^{4n} gives in $\pi_{4n}\mathrm{GL}(4n, \mathbb{R})$ the trivial homotopy class.*

To detect nontriviality of df we apply the generalized $\widehat{a}$ genus (with values in $KO(*) \cong \mathbb{Z}_2$. It is well known that there are exotic spheres such that $\widehat{a}(\mathbb{T}(f_T)) \neq 0$. We prove that for such f_T we have $[df] \neq 0$. Thus there are f which do not preserve the homotopy class of J_0.

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MORSE INEQUALITIES VIA CONLEY INDEX THEORY

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Abstract. The relation known as the Morse inequalities can be extended to a more general setting of flows on a locally compact metric spaces (Conley index) as well as dynamical systems on Hilbert spaces ($\mathcal{L}S$-index). This paper is a discourse around this extension. Except the part concerning the $\mathcal{L}S$-index the material is self-contained and has a character of a survey.

1. Introduction

There is a deep link between the critical point theory of a smooth function and the topology of the underlying space. This is a subject of the classical but still highly celebrated Morse theory that originated in the thirties of the last century. A number of papers and textbooks concerned the Morse theory has been published. Let us only mention magnificent papers of R. Bott [3], [4]. Probably the most common association with the Morse theory is the *attaching handlebodies* procedure, the way of rebuilding the underlying manifold M up to the homotopy type. Roughly speaking, it is realized by passing through the critical levels $c \in \mathbb{R}$ of certain function $f\colon M \to \mathbb{R}$ and attaching a k-cell to the set $f^{-1}((-\infty, c-\varepsilon])$. Here k is equal to the dimension of the unstable manifold $W^u(x_0)$ of the negative gradient vector field of f at critical point $x_0 \in f^{-1}(c)$, for details see Theorems A and B of [4]. Thus the obtained CW-complex is non-distinguishable from M from the point of view of homotopy theory.

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Another way of thinking about the Morse theory is closely related with so-called *Morse inequalities*, that give estimates from below for the number of critical points of $f\colon M \to \mathbb{R}$ by the Betti numbers of M, cf. Corollary 3.2. Furthermore, these "inequalities" may be presented in a slightly more sophisticated way as follows. Let

$$\mathcal{M}_t(f) = \sum_{x \in \mathrm{Crit}(f)} t^{\mathrm{ind}_f(x)}$$

be the Morse polynomial of f and $\mathcal{P}_t(M)$ be the Poincaré polynomial of the manifold M. Then

$$(\mathcal{ME}) \qquad \mathcal{M}_t(f) = \mathcal{P}_t(M) + (1+t)\mathcal{Q}(t),$$

where all the coefficients of the polynomial $\mathcal{Q}$ are nonnegative integers. Henceforth we will interchangeably refer to the relation $(\mathcal{ME})$ as to *the Morse equation* or *the Morse inequalities*.

In this paper we are going to deal with a generalization of the Morse theoretical methods provided by Conley's theory of homotopy index. The Morse equation $(\mathcal{ME})$ can be placed in the context of flows on locally compact metric spaces. For a given dynamical system on X one can consider a compact isolated invariant set S, i.e. an invariant set (with respect to the action of the flow) that is a maximal invariant subset of some neighbourhood of itself. Such a subset S possesses a so-called index pair (N, L) (Definition 2.1), where L is roughly speaking an exit set and $N \setminus L$ isolates S. The homotopy type of a pointed quotient space $(N/L, [L])$ defines the Conley (homotopy) index $h(S)$ of S. Whilst in the classical setting the manifold is decomposed into the rest points of a gradient flow, one can decompose an isolated invariant set S into a finite collection of isolated invariant subsets $M_j \subset S$ called the Morse sets. This decomposition carries an ordering that somehow reminds an admissible ordering of rest points of the negative gradient flow of f (f is the Lyapunov function). The equation $(\mathcal{ME})$ is a particular case of more general result due to C. Conley and E. Zehnder [6] that asserts that if $\{M_\pi : \pi \in \mathcal{D}\}$ ($\mathcal{D}$ is finite) is a Morse decomposition of an isolated invariant set S then

$$(\mathcal{CZ}) \qquad \sum_{\pi \in \mathcal{D}} \mathcal{P}(t, h(M_\pi)) = \mathcal{P}(t, h(S)) + (1+t)\mathcal{Q}(t).$$

Here $\mathcal{P}(\,\cdot\,, h(I))$ stands for the Poincaré polynomial of index $h(I)$, and $\mathcal{Q}(t) = \sum_{k \geq 0} a_k t^k$ with integers $a_k \geq 0$ (cf. Theorem 2.14).

Our aim is to give a self-contained exposition of the above-mentioned aspect of the Conley theory and some of its consequences.

Recently an extension of the equation $(\mathcal{CZ})$ has been obtained by the first author [11]. We also would like to display this result and point out its efficiency in the problems of searching for periodic solutions of Hamiltonian ODE's.

The paper is organized as follows. In the first section the notion of homotopy index is introduced and some elementary examples are given. For the sake of completeness we present a proof of Morse equation for flows. The second section goes back to classical results in Morse theory. Certain consequences of the theory are discussed in the third section. The last part of this note is devoted to the infinite dimensional generalization of the Conley index and its applications.

2. Isolated invariant sets and the index

Let X be a locally compact metric space. A continuous map $\phi\colon D \to X$ is called a *local flow* on X if the following properties are satisfied:

(1) D is an open neighbourhood of $\{0\} \times X$ in $\mathbb{R} \times X$;
(2) for each $x \in X$ there exist $\alpha_x, \omega_x \in \mathbb{R} \cup \{\pm\infty\}$ such that $(\alpha_x, \omega_x) = \{t \in \mathbb{R} : (t,x) \in D\}$;
(3) $\phi(0,x) = x$ and $\phi(s, \phi(t,x)) = \phi(s+t, x)$ for all $x \in X$ and $s, t \in (\alpha_x, \omega_x)$ such that $s + t \in (\alpha_x, \omega_x)$.

In the case of $D = \mathbb{R} \times X$ we call ϕ the *flow* on X. A (local) flow provides a convenient description of solutions of differential equations that emphasize the dependence of an initial state. To be more precise, let Ω be an open subset of $\mathbb{R}^n$. Assuming that a vector-field $v\colon \Omega \to \mathbb{R}^n$ is locally Lipschtz continuous, the theorem of Picard–Lindelöf guarantees the existence and uniqueness of solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) = v(x(t)), \\ x(0) = x_0. \end{cases} \tag{2.1}$$

If $t \mapsto \phi(t, x_0)$ is a solution of (2.1) and $(\alpha_{x_0}, \omega_{x_0})$ is the maximal interval of existence of u, the map ϕ is a local flow. In general, one cannot expect that ϕ is a flow as the example of equation $\dot{x} = x^2$ shows. The notation $\phi(t, \cdot\,)$ and ϕ^t for flows will be used interchangeably.

Let ϕ be a local flow on X. A subset $S \subset X$ is called *ϕ-invariant* if $x \in S$ implies $\phi(t,x) \in S$ for all $t \in (\alpha_x, \omega_x)$. If a flow is clear from context the letter ϕ is dropped out and we call S an invariant set. For an arbitrary $N \subset X$ the set

$$\operatorname{inv}(N, \phi) = \{x \in N : \phi(t,x) \in N \text{ for all } t \in (\alpha_x, \omega_x)\}$$

is a maximal invariant subset of N; if N is closed, so is $\operatorname{inv}(N,\phi)$. A compact subset N of X is called an *isolating neighbourhood* (of ϕ) provided that

$$\operatorname{inv}(N, \phi) \subset \operatorname{int}(N).$$

If N is an isolating neighbourhood, then $S = \operatorname{inv}(N)$ is said to be an *isolated invariant set*. Hence, by the definition, isolated invariant sets are compact subsets of X.

Isolated invariant sets are objects of primary importance from the point of view of dynamical systems. Unfortunately, they are extremely unstable objects, which means that they are very sensitive with respect to perturbations. They might change their stability and even disappear. On the other hand, isolating neighbourhoods are robust, i.e. they stay isolating neighbourhoods after small perturbation of the system.

Let N be a compact subset of X. We say that $L \subset N$ is *positively invariant relative to* N if for any $x \in L$ the inclusion $\phi^{[0,t]}(x) \subset N$ implies that $\phi^{[0,t]}(x) \subset L$. Let S be an isolated invariant set.

Definition 2.1. A compact pair (N, L) is called an *index pair* for S, if:

(a) $S = \operatorname{inv}(\overline{N \setminus L}) \subset \operatorname{int}(\overline{N \setminus L})$;
(b) L is positively invariant relative to N;
(c) if $x \in N$ and there exists $t > 0$, such that $\phi^t(x) \notin N$, then there exists $s \in [0, t]$, such that $\phi^{[0,s]}(x) \subset N$ and $\phi^s(x) \in L$.

Theorem 2.2 ([18]). *Every isolated invariant set S admits an index pair (N, L).*

If (N, L) is a compact pair, then the quotient N/L is obtained from N by collapsing L to a single point denoted by $[L]$, the base point of N/L. A set $X \subset N/L$ is open if either X is open in N and $X \cap L = \emptyset$ or the set $(X \cap N \setminus L) \cup L$ is open in N. We set $N/\emptyset$ to be $N \cup \{*\}$, the disjoint union of X and a distinguished point.

Generally, we will be working in the category of compact spaces with a basepoint. The notion $f\colon (X, x_0) \to (Y, y_0)$ means that f is a continuous map preserving basepoints, i.e. $f(x_0) = y_0$.

Recall that two maps $f, g\colon (X, x_0) \to (Y, y_0)$ are *homotopic relatively* x_0, if there is a continuous map $h\colon (X \times [0, 1], \{x_0\} \times [0, 1]) \to (Y, y_0)$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. The map $f\colon (X, x_0) \to (Y, y_0)$ is a *homotopy equivalence* if there exists a map $g\colon (Y, y_0) \to (X, x_0)$ such that $g \circ f$ is homotopic to id_X rel. x_0 and $f \circ g$ is homotopic to id_Y rel. y_0. If there is a homotopy equivalence $f\colon (X, x_0) \to (Y, y_0)$ then the pairs (X, x_0) and (Y, y_0) are *homotopy equivalent* or they have the same *homotopy type*. The homotopy type of (X, x_0) is denoted by $[X, x_0]$.

Notice that the requirement for maps of being point-preserving is essential. Homotopy equivalent spaces without basepoints may have different homotopy types if they are considered as a pointed spaces. For example, let S^0 be a pointed 0-sphere and Y be a pointed interval $[0, 1]$. Let Z be a space without distinguished point that is homeomorphic to Y. Then the wedge $Y \vee S^0$ is homeomorphic to the quotient space $Z/\emptyset$, and hence, these spaces represent the same homotopy type. However, as a pointed spaces, they do not have the same homotopy type.

Theorem 2.3 ([18]). *Let (N_0, L_0) and (N_1, L_1) be two index pairs for an isolated invariant set S. Then the pointed topological spaces N_0/L_0 and N_1/L_1 are homotopy equivalent.*

Definition 2.4. Let (N, L) be an index pair for an isolated invariant set S. The homotopy type $h(S, \phi) = [N/L]$ is said to be the *Conley* (*homotopy*) *index* of S. When the flow is clear from context we just write $h(S)$ for short.

Example 2.5. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The smoothness of f implies that ∇f is a locally Lipschtz continuous map, hence the equation $\dot{u}(t) - \nabla f(u(t)) = 0$ defines a local flow on $\mathbb{R}^n$: $\phi^t_f(x) = u(t, x)$, where $u\colon (\alpha_x, \omega_x) \to \mathbb{R}^n$ is a solution curve of the above equation passing through x at $t = 0$, and defined on its maximal interval of the existence. The rest points of ϕ^t_f are the critical points of f. An equilibrium point x_0 is *hyperbolic* if the Hessian of f at x_0 is nonsingular. In this case the number

$$\mathrm{ind}_f(x_0) = \#\{\text{negative eigenvalues of the Hessian } \nabla^2 f(x_0)\}$$

is defined (cf. Definition 3.1). Then the Conley index of isolated invariant set $S = \{x_0\}$ is the homotopy type of a pointed k-sphere, where $k = n - \mathrm{ind}_f(x_0)$.

Let $\phi\colon \mathbb{R} \times X \times [0, 1] \to X$ be a continuous family of flows on X, i.e. $\phi^t_\lambda := \phi(t, \cdot, \lambda)\colon X \to X$ is a flow on X. Suppose that S_0, S_1 are isolated invariant sets of ϕ^t_0 and ϕ^t_1, respectively. These sets are said to be *related by continuation*, or S_0 *continues to* S_1 if there is a compact $N \subset X$ that is an isolating neighbourhood for ϕ^t_λ, $\lambda \in [0, 1]$ and $S_i = \mathrm{inv}(N, \phi^t_i)$, $i = 0, 1$. The notion of continuation is essential in the Conley index theory due to the following theorem.

Theorem 2.6 ([5]). *If S_0 and S_1 are related by continuation, then their Conley indices coincide.*

Later on we will use the continuation property of the index in context of Reineck's theorem, cf. Theorem 4.1.

For $x \in X$ its α-limit and ω-limit sets are defined as follows:

$$\alpha(x) := \bigcap_{t \geq 0} \overline{\phi((-\infty, -t], x)}, \qquad \omega(x) := \bigcap_{t \geq 0} \overline{\phi([t, +\infty), x)}.$$

Definition 2.7 (Morse decomposition). Assume S is a compact and invariant subset of X. A finite collection $\{M_\pi : \pi \in \mathcal{D}\}$ of compact invariant sets in S is said to be *a Morse decomposition of* S, if there exists an ordering $(\pi_1, \dots, \pi_n)$ of $\mathcal{D}$ such that for every $x \in S \setminus \bigcup_{\pi \in \mathcal{D}} M_\pi$ there are indices $i, j \in \{1, \dots, n\}$, such that $i < j$, $\omega(x) \subset M_{\pi_i}$ and $\alpha(x) \subset M_{\pi_j}$. Every ordering of $\mathcal{D}$ with this property is said to be *admissible*. The sets M_π for $\pi \in \mathcal{D}$ are called *Morse sets*.

It is worth noting that an admissible ordering of Morse decomposition is not unique. Clearly, for a given collection of compact invariant sets in S may not exist any admissible ordering. There is, however, a particular class of flows for which a Morse decomposition always exists, i.e. the gradient flows or more general gradient-like flows, cf. [5].

We are going to formulate the so-called Morse inequalities, that compare the topological-algebraic invariant of an isolated invariant set with invariants of its Morse decomposition. Therefore, one needs that the Morse sets are isolated as well.

Proposition 2.8 ([6]). *If S is an isolated invariant set of ϕ, $\{M_\pi : \pi \in \mathcal{D}\}$ is a Morse decomposition of S, then M_π are also isolated invariant sets.*

The *q-th Betti number* of a compact pair (A, B) is a number $\beta^q(A, B) := \operatorname{rank} H^q(A, B)$. Assuming that groups $H^q(A, B)$ have finite rank for all $q \geq 0$, define the formal power series

$$\mathcal{P}(t, A, B) = \sum_{q=0}^{\infty} \beta^q(A, B) t^q$$

called the *Poincaré series of pair* (A, B). If the pair (A, B) is of finite type, i.e. $H^q(A, B) = 0$ for $q \geq q_0$, then we say that $\mathcal{P}(t, A, B)$ is a *Poincaré polynomial of* (A, B). To proceed further we need the following technical lemma.

Lemma 2.9. *Let $N_0 \subset N_1 \subset \ldots \subset N_n$ be an increasing sequence of compact subsets of X. Then there is a polynomial $\mathcal{Q}$ with nonnegative integer coefficients, such that*

$$\sum_{j=1}^{n} \mathcal{P}(t, N_j, N_{j-1}) = \mathcal{P}(t, N_n, N_0) + (1 + t)\mathcal{Q}(t).$$

Proof. Let $Z \subset Y \subset X$ be a compact triple. Consider the long exact sequence of relative cohomology groups

$$\cdots \xrightarrow{\delta^{q-1}} H^q(X, Y) \xrightarrow{\imath^q} H^q(X, Z) \xrightarrow{\jmath^q} H^q(Y, Z) \xrightarrow{\delta^q} \cdots,$$

where $\imath^q$ i $\jmath^q$ are homomorphisms induced by inclusions $\imath\colon (X, Z) \hookrightarrow (X, Y)$ and $\jmath\colon (Y, Z) \hookrightarrow (X, Z)$, respectively. Let $d^q_{(X,Y,Z)}$ be the rank of the image of δ^q. The exactness implies, that

$$\begin{aligned}\beta^q(Y, Z) &= d^q_{(X,Y,Z)} + \operatorname{rank} \operatorname{Im} \jmath^q = d^q_{(X,Y,Z)} + \beta^q(X, Z) - \operatorname{rank} \operatorname{Im} \imath^q \\ &= d^q_{(X,Y,Z)} + \beta^q(X, Z) - \beta^q(X, Y) + d^{q-1}_{(X,Y,Z)}.\end{aligned}$$

Consequently

$$\beta^q(Y, Z) + \beta^q(X, Y) = \beta^q(X, Z) + d^q_{(X,Y,Z)} + d^{q-1}_{(X,Y,Z)}.$$

Notice that $d^{-1} = 0$. Multiplying the above equality by t^q and summing over $q \geq 0$ one obtains

$$\mathcal{P}(t, Y, Z) + \mathcal{P}(t, X, Y) = \mathcal{P}(t, X, Z) + (1+t) \sum_{q \geq 0} d^q_{(X,Y,Z)} t^q.$$

Applying the obtained formula to the triple $N_0 \subset N_{j-1} \subset N_j$ and summing over $2 \leq j \leq n$ one gets the desired result, where

$$\mathcal{Q}(t) = \sum_{j=2}^{n} \sum_{q \geq 0} d^q_{(N_j, N_{j-1}, N_0)} t^q. \qquad \square$$

One can prove that for an isolated invariant set there is an index pair (N, L) such that $H^*(N, L) \cong H^*(N/L)$ (recall that N/L is a pointed space). Such an index pair is called *regular*. In particular, there is no needs to use a cohomology theory satisfying the strong excision axiom.

Definition 2.10. Let S be an isolated invariant set of a flow ϕ. The Poincaré polynomial of the Conley index of S is defined by the formula

$$\mathcal{P}(t, h(S)) := \mathcal{P}(t, N, L),$$

where (N, L) is any regular index pair for S.

Remark 2.11. Since the Conley index of S is the homotopy type of a finite CW-complex the pair (N, L) is of finite type and consequently the definition of $\mathcal{P}(t, h(S))$ is correct.

Definition 2.12. Let S be an isolated invariant set with a Morse decomposition $\{M_\pi : \pi \in \mathcal{D}\}$, $\#\mathcal{D} = n$. An *index filtration* is a sequence $N_0 \subset N_1 \subset \ldots \subset N_n$ of compact subsets of X such that (N_j, N_{j-1}) is an index pair for M_{π_j} and (N_n, N_0) is an index pair for S. The above filtration is called *a regular index filtration* if (N_j, N_{j-1}) is regular for all j.

Theorem 2.13. *Every Morse decomposition of an isolated invariant set admits an index filtration.*

Proof. For proof see Corollary 3.4 in [16]. $\square$

Theorem 2.14 (Morse equation; C. Conley and E. Zehnder [6])**.** *Let S be an isolated invariant set of a flow ϕ and $\{M_\pi : \pi \in \mathcal{D}\}$ be its Morse decomposition. Then there is a polynomial $\mathcal{Q}(t)$, whose all coefficients are nonnegative integers. such that:*

$$\sum_{\pi \in \mathcal{D}} \mathcal{P}(t, h(M_\pi)) = \mathcal{P}(t, h(S)) + (1+t)\mathcal{Q}(t). \tag{2.2}$$

Proof. This is a consequence of Lemma 2.9 and Theorem 2.13. $\square$

3. Backward to the classical Morse theory

Theorem 2.14 can be seen as a generalization of the classical result of Marston Morse in the following way. Let M be a compact closed ([1]) Riemannian manifold, $\dim M = n$. Consider a C^2-smooth function $f\colon M \to \mathbb{R}$. Recall that a point $p \in M$ is called a *critical point* of f if the derivative $Df(p)\colon T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is a zero-map. In local coordinates $(x^1, \dots, x^n)$ near p this condition can be expressed as a system of equalities

$$\frac{\partial f}{\partial x^1}(p) = \dots = \frac{\partial f}{\partial x^m}(p) = 0.$$

The Hessian of f at p is the matrix of the second partial derivatives

$$\nabla^2 f(p) = \frac{\partial^2 f}{\partial x^i \partial x^j}(p).$$

The rank of $\nabla^2 f(p)$ does not depend on a particular choice of local coordinates, and therefore its nullity (the dimension of $\ker \nabla^2 f(p)$) $n - \operatorname{rank} Hf(p)$ does not depend on it as well.

Definition 3.1. Let p be a critical point of C^2-function $f\colon M \to \mathbb{R}$.

(a) We say that p is a *nondegenerate* critical point if $Hf(p)$ is nonsingular, i.e. the nullity of $\nabla^2 f(p)$ is zero.
(b) The *index of f at p*, denoted by $\operatorname{ind}_f(p)$, is the dimension of the maximal subspace of T_pM on which the Hessian $\nabla^2 f(p)$ is negative definite. In other words, this is the number of negative eigenvalues of $\nabla^2 f(p)$, counting with multiplicity.
(c) f is called a *Morse function*, if all critical points of f are nondegenerate.

The classical Morse inequalities can be obtained from Theorem 2.14 applied to the gradient flow of f. The differentiable function gives rise to a vector field $\nabla f\colon M \to TM$. Since M is compact, the equation

$$\dot{x} + \nabla f(x) = 0$$

defines a flow on M, called the gradient flow of f. In what follows we will write ϕ_f for the gradient flow of f. The rest points of ϕ_f are the critical points of f. If f is a Morse function then the equilibria of ϕ_f are hyperbolic $(\sigma(\nabla^2 f(p)) \cap i\mathbb{R} = \emptyset)$ and the set $\operatorname{Crit}(f) = \{p_i \in M : \nabla f(p_i) = 0,\ 1 \le i \le m\}$ forms a Morse decomposition of an isolated invariant set $S = M$. Suppose that there are m critical points. As we saw in Example 2.5 one has ([2])

$$\mathcal{P}(t, h(p_i)) = t^{\operatorname{ind}_f(p_i)}.$$

([1]) It means that the boundary ∂M is empty.
([2]) Notice that $\operatorname{ind}_f(p_i) + \operatorname{ind}_{(-f)}(p_i) = n$.

Also $(M, \emptyset)$ is an index pair for S and in this case $\mathcal{P}(t, h(S))$ is the classical Poincaré polynomial of a manifold

$$\mathcal{P}_t(M) = \sum_{q=0}^{n} \beta^q(M) t^q,$$

where $\beta^q(M)$ is the q-th Betti number of M. Therefore we obtain the equality

$$\sum_{i=1}^{m} t^{\mathrm{ind}_f(p_i)} = \sum_{q=0}^{n} \beta^q(M) t^q + (1+t)\mathcal{Q}(t).$$

Set $c_k := \#\{x \in \mathrm{Crit}(f) : \mathrm{ind}_f(p) = k\}$ and

$$\mathcal{M}_t(f) := \sum_{i=1}^{m} t^{\mathrm{ind}_f(p_i)} = \sum_{i=0}^{n} c_i t^i.$$

Morse inequalities. *If $f\colon M \to \mathbb{R}$ is a Morse function, then*

$$\mathcal{M}_t(f) = \mathcal{P}_t(M) + (1+t)\mathcal{Q}(t). \tag{3.1}$$

The equation (3.1) says that the coefficients of $\mathcal{M}_t(f)$ majorizes the corresponding Betti numbers of M. The factor $(1+t)$ gives an extra information. It is contained in the following statement.

Corollary 3.2. *From the above equality one can read off:*

(a) *weak Morse inequalities $c_k \geq \beta^k$;*
(b) *strong Morse inequalities*

$$c_k - c_{k-1} + \ldots \pm c_0 \geq \beta^k - \beta^{k-1} + \ldots \pm \beta^0$$

for $k = 0, \ldots, n$ and the equality holds for $k = n$;
(c) *for every Morse function f the minimal number of critical points is equal to $\mathcal{M}_1(f)$;*
(d) *the Euler characteristic $\chi(M)$ is equal to $\mathcal{M}_{-1}(f)$, where f is any Morse function defined on M.*

It is worthy to be pointed out that the strong Morse inequalities imply the Morse inequalities (3.1), cf. Lemma 3.43 in [2]. R. Bott, in [3], called $\mathcal{M}_t(f)$ the *Morse polynomial of f*. Although the polynomial $\mathcal{Q}$ has nonnegative coefficients it is not true that $\mathcal{Q}(t) \geq 0$ nor $\mathcal{M}_t(f) \geq \mathcal{P}_t(M)$. The following example nicely illustrates these nuances.

Example 3.3 ([2, Example 3.38])**.** Consider the height function defined on a outstretched 2-sphere M having two horns as a maxima (of index 2), resting on a south pole as a minimum (of index 0). By Corollary 3.2 there is another critical point of index 1. The locus of critical points is clear if we say that the manifold M is reminiscent of a hart. The Morse polynomial of f is of the form $\mathcal{M}_t(f) = 2t^2 + t + 1$, while the Poincaré polynomial of M is $\mathcal{P}_t(M) = t^2 + 1$

(M is a homological 2-sphere). We easily deduce from (3.1) that $\mathcal{Q}(t) = t$. That is $\mathcal{Q}(t) < 0$ for $t < 0$. Moreover, for $t \in (-1, 0)$ one has $\mathcal{M}_t(f) < \mathcal{P}_t(M)$.

A Morse function $f\colon M \to \mathbb{R}$ is called a *perfect Morse function*, provided that $\mathcal{M}_t(f) = \mathcal{P}_t(M)$. Notice that the notion of perfect Morse function depends of particular choice of coefficients used to calculate the cohomology groups. For this reason we should call it $\mathcal{Z}$-perfect Morse function, where $\mathcal{Z}$ stands for the coefficients ring. Note that if a manifold admits a perfect Morse function for every coefficients ring, then M is torsion-free. Precisely, $H^k(M; \mathcal{Z})$ is a free abelian group isomorphic to $\mathcal{Z}^{c_k}$.

From the equality (3.1) one gets another useful corollary, namely

Morse's lacunary principle ([3]). *If the Morse polynomial of $f\colon M \to \mathbb{R}$ has no consecutive exponents, then $\mathcal{Q} \equiv 0$ and consequently f is a perfect Morse function* (*for every coefficient ring*).

Proof. Rewrite (3.1) as

$$\sum_{i=0}^{n} \xi_i t^i = (1+t)\mathcal{Q}(t),$$

where $\xi_i = c_i - \beta^i(M)$ for $1 \leq i \leq n$ and suppose that $Q \not\equiv 0$. Assume that ξ_k is the first nonzero coefficient on the left-hand side. This implies that the right-hand side contains exponent $k+1$, and hence also $\xi_{k+1} \neq 0$. Since the Betti numbers are nonnegative one has $c_{k+1} \neq 0$. $\square$

Example 3.4 (Milnor's perfect Morse function, [3], [15]). Consider the sphere $S^{2n+1} = \{z \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} |z_j|^2\}$ and the function $f\colon S^{2n+1} \to \mathbb{R}$ defined as

$$f(z) := \sum_{j=1}^{n+1} j|z_j|^2, \quad z = (z_1, \ldots, z_{n+1}).$$

Since f is invariant with respect to the free S^1 action given by $(\gamma, z) \mapsto \gamma z = (\gamma z_0, \ldots, \gamma z_n)$, i.e. $f(\gamma z) = f(z)$, it factors through $\mathbb{C}P^n$ giving rise to $\widehat{f}\colon \mathbb{C}P^n \to \mathbb{R}$. To find the critical points of f we proceed using Lagrange multipliers principle. Let us view $\mathbb{C}^{n+1}$ as $\mathbb{R}^{2n+2}$ where points are denoted by $(x, y) = (x_1, y_1, \ldots, x_{n+1}, y_{n+1})$. Introducing the function $g\colon (x, y) \mapsto \sum_{j=1}^{n+1}(x_j^2 + y_j^2) - 1$ we need to solve the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ under constraints given by equality $g(x, y) = 0$. Explicitly

$$jx_j = \lambda x_j, \quad jy_j = \lambda y_j, \quad j = 1, \ldots, n+1, \quad \sum_{j=1}^{n+1}(x_j^2 + y_j^2) = 1.$$

One obtains $n+1$ solutions, namely for each $1 \leq j \leq n+1$, $\lambda = j$, $x_k = y_k = 0$ for $k \neq j$ and $x_j^2 + y_j^2 = 1$. Each circle $S_j^1 = \{x_j^2 + y_j^2 = 1\}$ is a critical orbit of f lying on the sphere, and it represents a critical point of $\widehat{f}$ in $\mathbb{C}P^n$. Denote it $p_j =$

$[0 : \ldots : 0 : 1 : 0 : \ldots : 0]$ with 1 at the j-th place. Fix $\lambda = k$ and a corresponding critical orbit. Obviously, the Hessian $\nabla^2 f(q)\colon T_q S^{2n+1} \to T_q S^{2n+1}$ at a point $q = (x, y) \in S^1_k$ has a nontrivial, one-dimensional kernel, namely $T_q S^1_k$, but the restriction to $T_q S^{2n+1} \ominus T_q S^1_k$ is of the form

$$\nabla^2 f(q) = \begin{bmatrix} 1-\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1-\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & n+1-\lambda & 0 \\ 0 & 0 & \cdots & 0 & n+1-\lambda \end{bmatrix},$$

where two rows and two columns with entry $(k-\lambda)$ are omitted. Since the above matrix also represents $\nabla^2 \widehat{f}(p_k)$, the Morse index of p_k is $\mathrm{ind}_{\widehat{f}}(p_k) = 2(k-1)$, $1 \leq k \leq n+1$, and consequently one has

$$\mathcal{M}_t(\widehat{f}) = 1 + t^2 + \ldots + t^{2n}.$$

By the Morse's Lacunary Principle one gets $\mathcal{P}_t(\mathbb{C}P^n) = 1 + t^2 + \ldots + t^{2n}$ for each coefficient ring. At last one can conclude that

$$H^q(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } q = 2i,\ 1 \leq i \leq n, \\ 0 & \text{else.} \end{cases}$$

The next example shows that the equality $\mathcal{Q}(t) = 0$ for all t does not imply the lack of any exponent in the Morse polynomial.

Example 3.5. Consider the function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by the formula

$$f(x) := \sum_{k=1}^{n} \cos(2\pi x_k), \quad x = (x_1, x_2, \ldots, x_n).$$

Since f is invariant with respect to the integer-vector shifts it descends to the function $\widehat{f}\colon \mathbb{T}^n \to \mathbb{R}$ on the n-dimensional torus. It is easily seen that $\nabla f(x) = -2\pi \sum_{k=1}^{n} \sin(2\pi x_k) e_k$ (e_k stands for the k-th vector of the standard basis of $\mathbb{R}^n$). The critical points of $\widehat{f}$ are 2^n in number, which are all the n-tuples with entries 0 or 1/2. The Hessian is of the form

$$\nabla^2 f(x) = -4\pi^2 \begin{bmatrix} \cos(2\pi x_1) & 0 & \cdots & 0 \\ 0 & \cos(2\pi x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cos(2\pi x_n) \end{bmatrix}$$

and one gets that $\mathrm{ind}_f(x) = n - j$, where j is the number of coordinates of x that are equal to 1/2. For instance, $\mathrm{ind}_f(0, \ldots, 0) = n$ and $\mathrm{ind}_f(1/2, \ldots, 1/2) = 0$. The Morse polynomial of f is

$$\mathcal{M}_t(f) = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} = (1+t)^n.$$

In particular, as an immediate consequence of the above formula we have $\chi(\mathbb{T}^n) = \mathcal{M}_{-1}(f) = 0$. Since k-th Betti number of $\mathbb{T}^n$ is equal to $\binom{n}{k}$ (see [10, Example 3.11, p. 210]), we see that $\mathcal{M}_t(f) = \mathcal{P}_t(\mathbb{T}^n)$, thus f is a perfect Morse function.

4. Consequences of the Morse equation

4.1. Conley index and Brouwer degree. Connections between the Conley index and the topological degree are noticeable at first glance. The homotopy invariance of the Brouwer degree corresponds to the continuation property of the Conley index. The existence axiom refers to nontriviality property which says that nontrivial index implies nonempty isolated invariant set. The next common feature of both invariants is that they are determined by a behaviour of a vector field (flow) on a boundary of the set under investigation.

In this section we shall show, following M. Fotouhi and M. R. Razvan [8], how to figure out this relation using Morse equation (2.2) and the continuation theorem due to Reineck [17]. Namely, under certain assumptions we will prove the equality

$$\chi(h(S)) = \deg(F, N),$$

where S is an isolated invariant set of a flow induced by $-F$; $S = \mathrm{inv} N$ and χ stands for the Euler characteristic. This result was obtained by R. Srzednicki for dynamical systems in $\mathbb{R}^n$, cf. [19]. Later on, C. K. McCord in [14] proved it in a slightly more general setting. Namely, he studied relation between the number $\chi(h(S))$ and the intersection number of a vector field (with the zero section of the tangent bundle) generating the flow in question on a compact manifold. Special case was obtained by E. .N. Dancer in [7]. He showed that $\chi(h(\{x\})) = \deg(f, U_x)$, where x is a degenerate rest point of a gradient flow of $-f$ and U_x stands for its neighbourhood.

We will briefly remind the reader notion of the degree. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. If $f\colon \overline{\Omega} \to \mathbb{R}^n$ is a continuous map and does not vanish on the boundary $\partial\Omega$, then there is an integer $\deg(f, \Omega) \in \mathbb{Z}$ called the Brouwer degree. It satisfies the following axioms:

- (Nontriviality) If $0 \in \Omega$ then $\deg(I, \Omega) = 1$, where I is the identity map;
- (Existence) If $\deg(f, \Omega) \neq 0$ then $f^{-1}(0) \cap \Omega$ is nonempty;
- (Additivity) If Ω_1, Ω_2 are open, disjoint subsets of Ω and there is no zeros of f in the complement $\Omega \setminus (\Omega_1 \cup \Omega_2)$, then

$$\deg(f, \Omega) = \deg(f, \Omega_1) + \deg(f, \Omega_2);$$

- (Homotopy invariance) If $h\colon \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ is a continuous map such that $h(x, t) \neq 0$ for all $(x, t) \in \partial\Omega \times [0, 1]$, then

$$\deg(h(\,\cdot\,, 0), \Omega) = \deg(h(\,\cdot\,, 1), \Omega)$$

If $\varphi\colon \overline{\Omega} \to \mathbb{R}$ is a Morse function such that $\deg(\nabla\varphi, \Omega)$ is defined, then

$$\deg(\nabla\varphi, \Omega) = \sum_{x \in (\nabla\varphi)^{-1}(0) \cap \Omega} (-1)^{\mathrm{ind}_\varphi(x)}.$$

Recall that a Morse–Smale gradient flow satisfies:

(i) all bounded orbits are either critical points of the potential function or orbits connecting two critical points;
(ii) stable and unstable manifolds of the rest points intersect transversally.

Let $\Omega \subset \mathbb{R}^n$ be an open set, $F\colon \Omega \to \mathbb{R}^n$ a smooth vector field and let $\phi_F^t\colon \Omega \to \Omega$ be a flow generated by $\dot{x}(t) = -F(x(t))$. Assume that N is an isolating neighbourhood and $S = \mathrm{inv}(N)$.

Theorem 4.1 ([17])**.** *Set S can be continued to an isolated invariant set of a positive gradient flow of certain function f defined on the open set U containing N and without changing F on $\Omega \setminus N$. Moreover, this can be done in such a way that the new flow is Morse–Smale.*

The *Euler characteristic* of a topological pair (X, A) is defined as

$$\chi(X, A) = \sum_{q=0}^{\infty} (-1)^q \mathrm{rank}\, H^q(X, A),$$

provided that pair (X, A) is of a finite type. Notice that $\chi(X, A)$ is equal to $\mathcal{P}(-1, X, A)$. In particular, the Euler characteristic is well defined for the Conley index, cf. Definition 2.10 and Remark 2.11.

Theorem 4.2 ([8])**.** *Let $F\colon \Omega \to \mathbb{R}^n$ be a locally Lipschitz map and denote by ϕ_F^t the local flow generated by $\dot{x} = -F(x)$. If N is an ϕ_F^t-isolating neighbourhood and $S = \mathrm{inv}(N)$ then*

$$\chi(h(S)) = \deg(F, \mathrm{int}(N)). \tag{4.1}$$

In what follows we will write $\deg(F, N)$ instead of $\deg(F, \mathrm{int}(N))$.

Proof. By the Reineck continuation theorem S continues to an isolated invariant set of a Morse–Smale gradient flow ϕ_f^t, that consists of non-degenerate critical points of f and connecting orbits between them. Denote this set by S'. By the continuation property of the Conley index $h(S) = h(S')$. The set of critical points $\{x_1, \ldots, x_m\}$ forms a Morse decomposition of S' and, as we saw in Example 2.5, one has that $h(\{x_i\}, \phi_f^t)$ is the homotopy type of a pointed k-sphere, where $k = n - \mathrm{ind}_f(x_i)$. Hence, the Poincaré polynomial of $h(\{x_i\}, \phi_f^t)$ is of the form

$$\mathcal{P}(t, h(\{x_i\}, \phi_f^t)) = t^{n - \mathrm{ind}_f(x_i)}.$$

Applying Theorem 2.14 one obtains

$$\begin{aligned}\chi(h(S)) &= \chi(h(S')) = \mathcal{P}(-1,h(S')) \\ &= \sum_{i=1}^{m}\mathcal{P}(-1,h(\{x_i\},\phi_f^t)) = (-1)^n\sum_{i=1}^{m}(-1)^{\operatorname{ind}_f(x_i)}.\end{aligned} \tag{4.2}$$

For $1 \le i \le m$, let Ω_i be a neighbourhood of x_i in N such that $\Omega_i \cap \Omega_j = \emptyset$. Using the homotopy invariance of the Brouwer degree and the additivity property one has

$$\deg(-F,N) = \deg(\nabla f,N) = \sum_{i=1}^{m}\deg(\nabla f,\Omega_i). \tag{4.3}$$

Since f is a Morse function, the hessian $\nabla^2 f(x_i)$ is a non-degenerate linear operator. The degree of ∇f with respect to Ω_i is equal to $(-1)^\mu$, where μ is the number of negative eigenvalues of $\nabla^2 f(x_i)$. That is, $\deg(\nabla f,\Omega_i) = (-1)^{\operatorname{ind}_f(x_i)}$. By (4.3) one obtains

$$\deg(F,N) = (-1)^n\deg(-F,N) = (-1)^n\sum_{i=1}^{m}(-1)^{\operatorname{ind}_f(x_i)}. \tag{4.4}$$

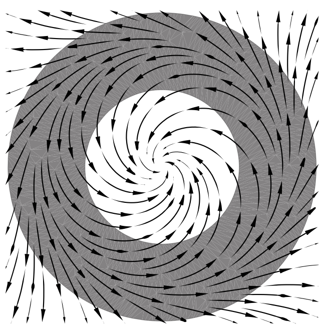
Combining (4.2) and (4.4) we get the formula (4.1). □

Example 4.3. The simplest example for Theorem 4.2 is given by the equation $\dot{x} = x$ on $\mathbb{R}^n$. Here the vector field is $-\mathrm{id}\colon\mathbb{R}^n \to \mathbb{R}^n$ and its degree with respect to the unit ball depends on the dimension and equals $(-1)^n$. The origin is an isolated equilibrium with an index pair (D^n, S^{n-1}). The Euler characteristic of index is $\chi(D^n/S^{n-1}) = (-1)^n$.

Example 4.4. The map $F\colon\mathbb{R}^2 \to \mathbb{R}^2$,

$$F(x,y) := (-x-y+x(x^2+y^2), x-y+y(x^2+y^2))$$

gives us a bit more refined illustration.



Integral curves of the vector field F and an isolating neighbourhood

The annulus $A = \{(x,y) \in \mathbb{R}^2 : r \le x^2 + y^2 \le R\}$, $0 < r < 1 < R$, is an isolating neighbourhood. Indeed, the inner product $\langle F(x,y),(x,y)\rangle = (x^2+y^2)^2 - (x^2+y^2)$ shows that for $x^2+y^2 < 1$ the vector field points inside the annulus, while for $x^2+y^2 > 1$ the vectors point outside of it. The exit set is a disjoint union of the boundary circles. The index is a homotopy type of the wedge $S^2 \vee S^1$. It is easily seen that $S^2 \vee S^1$ is composed of 0-, 1- and 2-dimensional cells. Hence the Euler characteristic modulo a basepoint equals zero. The additivity property of the Brouwer degree implies that $\deg(F,A) = 0$.

4.2. Critical point theory in finite-dimensional domains. Assume that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a function of class C^k, $k \ge 1$. The particular order of smoothness k will be specified if needed. Let f satisfies the following asymptotical condition: there exists a number $C > 0$ such that $|x| > C$ implies that f is of the form

$$f(x) = -\frac{1}{2}\langle A_\infty x, x\rangle + \varphi_\infty(x)$$

where A_∞ is linear symmetric map and $\nabla\varphi_\infty = o(|x|)$ as $|x| \to \infty$. Function satisfying the above condition is called *asymptotically quadratic*.

For a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ denote by $m^0(A)$ the nullity of A, i.e. the dimension of its kernel and by $m^-(A)$ the number of negative eigenvalues of A, counting with their multiplicity. The number $m^-(A)$ will be sometimes called the *Morse index of A* (compare Definition 3.1).

We say that f has *no resonance at the infinity*, if the map A_∞ is an isomorphism. Shortly, if $m^0(A_\infty) = 0$.

Recall that a *critical point of f* is a solution of the equation $\nabla f(x) = 0$. The critical point x_0 of f is said to be *nondegenerate*, if the bilinear form $f''(x_0)$ is nondegenerate, i.e. the equality $f''(x_0)[u,v] = 0$ satisfied for an arbitrary vector $u \in \mathbb{R}^n$ implies $v = 0$. In what follows we will identify the form $f''(x_0)$ with its matrix.

Theorem 4.5. *Every asymptotically quadratic function of class C^k, $k \ge 1$, without resonance at the infinity has a critical point.*

The above result may be obtained using the topological degree as well as the Conley index, cf. [13]. What we are going to show now is that the behaviour of a function near the infinity determines the Morse index of some critical point (which exists, by the previous theorem). This result is an immediate consequence of the Morse equation (2.2).

Theorem 4.6. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be an asymptotically quadratic function of class C^2 without resonance at the infinity. If all critical points of f are nondegenerate, then at least one of them, say x_0, satisfies $m^-(f''(x_0)) = m^-(A_\infty)$.*

Proof. Consider the negative gradient flow ϕ_f^t of function f. Since f has no resonance at the infinity, there exists a maximal compact isolated ϕ_f^t-invariant set X. By the compactness of X there is only a finite number of critical points of f, say $\{x_1, \ldots, x_m\}$. This set forms a Morse decomposition of X. The most natural ordering is given by f, i.e. $i < j$ if and only if $f(x_i) < f(x_j)$. If x is a nondegenerate critical point of f, then by Example 2.5 one has $\mathcal{P}(t, h(\{x\})) = t^{m^-(f''(x))}$. On the other hand the Conley index of X is a homotopy type of pointed sphere $S^{m^-(A_\infty)}$ and consequently $\mathcal{P}(t, h(X)) = t^{m^-(A_\infty)}$. Now, the Morse equation (2.2) has the form

$$t^{m^-(f''(x_1))} + \ldots + t^{m^-(f''(x_m))} = t^{m^-(A_\infty)} + (1+t)\mathcal{Q}(t)$$

The above equality holds for $t \in \mathbb{R}$, hence one exponent among $m^-(f''(x_i))$, $1 \le i \le m$ has to be equal to $m^-(A_\infty)$. □

5. Conley index for flows in a Hilbert space

We have already seen that the Conley index theory can be applied to solve multiple problems in critical point theory (vide Morse equation). However, those methods are strictly finite-dimensional, i.e. the local compactness property of a phase space plays a crucial role. Now we turn to the case where the presented theory does not work properly. Let us consider the Hamiltonian system of ODE's, i.e. the equations of the form

$$\dot{p}(t) = -\frac{\partial H}{\partial q}(p, q, t), \quad \dot{q}(t) = \frac{\partial H}{\partial p}(p, q, t), \tag{5.1}$$

where $H: \mathbb{R}^{2m} \times \mathbb{R} \ni (z, t) \mapsto H(z, t) \in \mathbb{R}$, $z = (p, q)$, $p, q \in \mathbb{R}^m$, is C^1-smooth and 2π-periodic in t.

Searching for periodic solutions of (5.1) can be reformulated to the variational setting as follows. Let $H := H^{1/2}(S^1, \mathbb{R}^{2m})$ be a Sobolev space of $\mathbb{R}^{2m}$-valued loops belonging to $L^2(S^1)$

$$z(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt), \quad a_0, a_n, b_n \in \mathbb{R}^{2m},$$

whose Fourier coefficients satisfy the condition

$$\sum_{n=1}^{\infty} n(a_n^2 + b_n^2) < \infty.$$

The space H becomes a Hilbert space with an inner product

$$\langle z, z' \rangle_H := 2\pi \langle a_0, a_0' \rangle + \pi \sum_{n=1}^{\infty} n(\langle a_n, a_n' \rangle + \langle b_n, b_n' \rangle). \tag{5.2}$$

Let $\Phi: H \to \mathbb{R}$ be an action functional defined by

$$\Phi(z) := -\frac{1}{2}\langle Lz, z \rangle_H - \psi(z), \tag{5.3}$$

where

$$\langle Lz, z\rangle_H = \int_0^{2\pi} \langle J\dot{z}(t), z(t)\rangle \, dt, \tag{5.4}$$

$$\psi(z) = \int_0^{2\pi} H(z(t), t) \, dt. \tag{5.5}$$

Here $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ stands for the standard symplectic matrix.

Under certain growth conditions on H (for instance, there are constants $c_1, c_2, s > 0$ such that for all $z \in \mathbb{R}^{2m}$, $\|H'_z(z,t)\| \le c_1 + c_2\|z\|^s$) a map z is a 2π-periodic solution of (5.1) if and only if it is a critical point of Φ. In Morse–Conley theory approach the critical points are obtained by analysis of gradient flow of a given functional. Since the local compactness property fails in the case of the space H, the straightforward application of these methods is out of question. Moreover, the critical points of Φ are saddle points having infinite-dimensional stable and unstable invariant manifolds, therefore the Morse index is not defined. However, the map $\nabla\phi$ turns out to be completely continuous and the gradient of Φ is of the form

bounded linear operator + completely continuous map.

This particular form of a vector field gives a hint, that the difficulties related to infinite dimension of the domain can be overcome by means of the Leray–Schauder method of finite dimensional approximation. This leads us to the concept of the $\mathcal{LS}$-index, an object that generalizes the Conley index in the same way as the Leray–Schauder degree generalizes the classical Brouwer one.

5.1. $\mathcal{LS}$-index. Let H be a real, separable Hilbert space, and $L\colon H \to H$ be a linear bounded operator which satisfies the following assumptions:

- L gives a splitting $H = \bigoplus_{n=0}^{\infty} H_n$ onto finite dimensional, mutually orthogonal L-invariant subspaces;
- $L(H_n) = H_n$ for $n > 0$ and $L(H_0) \subset H_0$, where H_0 is a subspace corresponding to the part of spectrum on the imaginary axis, i.e. $\sigma_0(L) := \sigma(L_{|H_0}) = \sigma(L) \cap i\mathbb{R}$;
- $\sigma_0(L)$ is isolated in $\sigma(L)$.

Definition 5.1. We say that a map $f\colon H \times \Lambda \to H$, where Λ is a compact metric space, is a *family of $\mathcal{LS}$-vector fields*, if f is of the form

$$f(x,\lambda) = Lx + K(x,\lambda), \quad (x,\lambda) \in H \times \Lambda,$$

where $K\colon H \times \Lambda \to H$ is a completely continuous and locally Lipschitz map. The map $f(\,\cdot\,,\lambda_0)$ is called an $\mathcal{LS}$-vector-field.

We will be concerned with the flows generated by the equation $\dot{u} = -f(u)$, where f is an $\mathcal{LS}$-vector field. In order to guarantee that such equation generates a flow it suffices to assume that f is subquadratic, i.e. the completely continuous part of f satisfies the condition $|\langle K(z), z\rangle| \leq a\|z\|^2 + b$, for some $a, b > 0$; see [9]. Without loss of generality we may assume f is a subquadratic $\mathcal{LS}$-vector field. Notice that it is also well known that the flow generated by such a field is of the form $\phi^t(x) = \exp(tL) + U(t, x)$, where $U: \mathbb{R} \times H \to H$ is a completely continuous map. In what follows we will call it an $\mathcal{LS}$-flow.

Turning to the definition of the Conley-type invariant for $\mathcal{LS}$-flows we define the *isolating neighbourhood for* ϕ^t to be a closed and bounded subset X of H such that the invariant part of it $\operatorname{inv}(X) = \{x \in X : \phi^{\mathbb{R}}(x) \subset X\}$ lies strictly in the interior of X. We say that S is a ϕ^t*-isolated* (abbrev. *isolated*) invariant set if there exists an isolating neighbourhood X for ϕ^t such that $S = \operatorname{inv}(X)$. Of course X does not need to be compact, but some compactness property holds. It is a key feature of the class of $\mathcal{LS}$-flows and it turns out to be crucial in the definition of the $\mathcal{LS}$-index. Namely, we have the following theorem:

Compactness property. *Let $\phi^{(\,\cdot\,)}: \mathbb{R} \times H \times \Lambda \to H$ be a family of $\mathcal{LS}$-flows. If $X \subset H$ is closed and bounded, then $S := \operatorname{inv}(S, \phi^t)$ is a compact subset of $H \times \Lambda$.*

Let $f: H \to H$ be an $\mathcal{LS}$-vector field, $f(x) = Lx + K(x)$, $\phi^t: H \to H$ be an $\mathcal{LS}$-flow generated by f and assume that $X \subset H$ is an isolating neighbourhood for ϕ^t. Denote by $P_n: H \to H$ the orthogonal projection onto $H^n = \bigoplus_{i=1}^n H_i$. Set $H_n^- := H^- \cap H_n$ and $H_n^+ := H^+ \cap H_n$, where H^- (resp. H^+) denotes the L-invariant subspace of H corresponding to the part of spectrum with negative (resp. positive) real part. Define $f_n: H^n \to H^n$ by

$$f_n(x) := Lx + P_nK(x)$$

and denote by ϕ_n^t a flow on H^n induced by the vector field f_n. The compactness property of the isolated invariant sets implies that the compact set $X^n := X \cap H^n$ is an isolating neighbourhood for ϕ_n^t provided that n is sufficiently large. Consequently, $S_n := \operatorname{inv}(X^n, \phi_n^t)$ is an isolated invariant set and thus, in view of Theorem 2.2, it admits an index pair (Y_n, Z_n). Let $[Y_n/Z_n]$ be the Conley index of S_n. Since K is completely continuous, it turns out that for a large n the index of S_n is determined by the spectral properties of the operator L in the following way. For each non-negative integer n let $\nu(n)$ be the dimension of H_{n+1}^-, the unstable subspace of the $(n+1)$th block of H. Using the construction presented in [9] one can prove that for n large enough, say $n \geq n_0$, the pointed space Y_{n+1}/Z_{n+1} is homotopy equivalent to the $\nu(n)$-fold suspension of Y_n/Z_n, i.e.

$$[Y_{n+1}/Z_{n+1}] = [S^{\nu(n)}(Y_n/Z_n)], \quad n \geq n_0.$$

Thus, we have constructed a sequence of pointed spaces

$$\{E_n\} = \{Y_n/Z_n\}, \quad n \geq n_0.$$

Definition 5.2. Let ϕ be an $\mathcal{L}S$-flow generated by an $\mathcal{L}S$-vector field and let X be an isolating neighbourhood for ϕ. The *$\mathcal{L}S$-index* of X is the homotopy type of E_n, where $n \geq n_0$. It is denoted by $h_{\mathcal{L}S}(X, \phi)$ or $h_{\mathcal{L}S}(X)$ for short.

The above definition is independent of n up to suspension, i.e. for $n \geq n_0$ and for any integer $k \geq 0$, one has

$$[E_{n+k}] = [S^{\nu(n)+\nu(n+1)+\ldots+\nu(n+k-1)} E_n].$$

Let θ denote the homotopy type of a pointed one-point space.

Proposition 5.3 (K. Gęba et al. [9]). *The $\mathcal{L}S$-index has the following properties:*

(a) (Nontriviality) *Let $\phi^t \colon H \to H$ be an $\mathcal{L}S$-flow and $X \subset H$ be an isolating neighbourhood for ϕ^t with $S := \mathrm{inv}(X)$. If $h_{\mathcal{L}S}(X) \neq \theta$, then $S \neq \emptyset$.*

(b) (Continuation) *Let Λ be a compact, connected and locally contractible metric space. Assume that $\phi^t \colon H \times \Lambda \to H$ is a family of $\mathcal{L}S$-flows. Let X be an isolating neighbourhood for a flow ϕ^t_λ for some $\lambda \in \Lambda$. Then there is a compact neighbourhood $\mathcal{U}_\lambda \subset \Lambda$ such that*

$$h_{\mathcal{L}S}(X, \phi^t_\mu) = h_{\mathcal{L}S}(X, \phi^t_\nu) \quad \text{for all } \mu, \nu \in \mathcal{U}_\lambda.$$

5.2. Cohomological $\mathcal{L}S$-index. Let H^* denotes the Alexander-Spanier cohomology functor with coefficients in some fixed ring $\mathcal{Z}$. To define the cohomology of the $\mathcal{L}S$-index represented by the sequence $E = \{E_n\}_{n=n_0}^{\infty}$ consider the function $\rho \colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ defined to be 0 for $n = 0$ and $\rho(n) := \sum_{i=0}^{n-1} \nu(i)$ for $n > 0$. For a fixed $q \in \mathbb{Z}$ consider a sequence of cohomology groups

$$H^{q+\rho(n)}(E_n), \quad n \geq n_0,$$

together with homomorphisms

$$h_n \colon H^{q+\rho(n+1)}(E_{n+1}) \xrightarrow{\varepsilon_n^{q+\rho(n+1)}} H^{q+\rho(n+1)}(S^{\nu(n)} E_n) \xrightarrow{(S^*)^{-\nu(n)}} H^{q+\rho(n)}(E_n)$$

that is, an inverse system $(H^{q+\rho(n)}(E_n), h_n)$. Here

$$\varepsilon_n^k \colon H^k(E_{n+1}) \to H^k(S^{\nu(n)} E_n)$$

is an isomorphism induced by the homotopy equivalence $\varepsilon_n \colon S^{\nu(n)} E_n \to E_{n+1}$ and S^* denotes the suspension isomorphism $H^k(E_n) \cong H^{k+1}(SE_n)$.

Definition 5.4. The *q-th cohomology group of E* is the inverse limit group $\varprojlim \{H^{q+\rho(n)}(E_n), h_n\}$. It will cause no confusion if we denote it $H^q(E)$ declaring earlier that E stands for a sequence of spaces.

These groups are the topological-algebraic invariants of an isolated invariant set of an $\mathcal{L}S$-flow. We call $H^*(E)$ the *cohomological $\mathcal{L}S$-index*. It is worth pointing out that $H^q(E) \cong H^{q+\rho(n)}(E_n)$ for $n \geq n_0$ due to the fact that h_n is an isomorphism. The groups $H^q(E)$ may be nonzero both for positive and negative q's. To see this let $E_n = S^{2n-1} \vee S^{2n+1}$ and $\nu(n) = 2$. Then $\rho(n) = 2n$ and

$$H^q(E) = H^{q+\rho(n)}(E_n) = \begin{cases} \mathcal{Z} & \text{for } q = -1, 1, \\ 0 & \text{else.} \end{cases}$$

Because of the remarks above we need to amend a bit definition of the Poincaré polynomial that we have made earlier, cf. Definition 2.10. Let E be a sequence that we get in the definition of an $\mathcal{LS}$-index. We define the following generalized formal power series:

$$\mathcal{P}(t, E) := \sum_{q \in \mathbb{Z}} r^q(E) t^q,$$

where $r^q(E)$ stands for the rank of $H^q(E)$. If $r^q(E)$ are 0 for all q less than some fixed $q_0 \in \mathbb{Z}$ then $\mathcal{P}(t, E)$ is called the *generalized Poincaré series.*

Let ϕ be an $\mathcal{LS}$-flow with an isolating neighbourhood X and assume that $\{M_\pi : \pi \in \mathcal{D}\}$ is a Morse decomposition of $S = \operatorname{inv}(X)$. We have the following analogue of the Morse equation. The proof can be found in the first author's paper [11].

Theorem 5.5. *Under the above assumptions one has*

$$\sum_{\pi \in \mathcal{D}} \mathcal{P}(t, h_{\mathcal{LS}}(M_\pi)) = \mathcal{P}(t, h_{\mathcal{LS}}(S)) + (1 + t)\mathcal{Q}(t),$$

where the coefficients of the generalized power series $\mathcal{Q}(t)$ are nonnegative integers.

5.3. Applications. Let us turn to the problem of searching for 2π-periodic orbits of a Hamiltonian ODE's. The system (5.1) can be written in a shortened form

$$\dot{z} = J\nabla H(z, t), \tag{5.6}$$

where the gradient is taken with respect to $z \in \mathbb{R}^{2m}$, and J stands for the symplectic linear map $\mathbb{R}^{2m} \ni (x, y) \mapsto (-y, x)$. Recall that in order to find periodic solutions one has to find critical points of the corresponding functional (5.3), whose gradient is of the form

$$-\nabla\Phi(z) = Lz + \nabla\psi(z),$$

hence, it is an $\mathcal{LS}$-vector field. Using the formulae (5.2) and (5.4) one can easily verify that

$$(Lz)(t) = \sum_{n=1}^{\infty} Jb_n \cos nt - Ja_n \sin nt. \tag{5.7}$$

Let $e_1, \dots, e_{2m}$ be the standard basis in $\mathbb{R}^{2m}$. Introduce the following functional vector spaces:

$$
\begin{aligned}
H_0 &:= \operatorname{span}\{e_1, \dots, e_{2m}\},\\
H_n^+ &:= \operatorname{span}\{\cos(nt)e_j + \sin(nt)Je_j : j = 1, \dots, 2m\},\\
H_n^- &:= \operatorname{span}\{\cos(nt)e_j - \sin(nt)Je_j : j = 1, \dots, 2m\}.
\end{aligned}
$$

Setting $H_n := H_n^+ \oplus H_n^-$ we obtain the family of finite dimensional, mutually orthogonal and L-invariant subspaces of H such that $H = \bigoplus_{n=0}^{\infty} H_n$. It is evidently seen from (5.7) that $H_0 = \ker L$ and

$$
Lz = \begin{cases} z & \text{for } z \in \bigoplus_{n=1}^{\infty} H_n^+ \\ -z & \text{for } z \in \bigoplus_{n=1}^{\infty} H_n^-. \end{cases}
$$

If the function H is a quadratic form, i.e. $H(z) = (1/2)\langle Az, z\rangle$ for a symmetric $(2m \times 2m)$-matrix A, the case is fairly easy. The equation (5.6) becomes a linear Hamiltonian system

$$\dot{z} = JAz,$$

and the corresponding functional Φ is of the form $\Phi(z) = -(1/2)\langle (L+K)z, z\rangle_H$, where L is given by (5.7) and

$$(Kz)(t) = Aa_0 + \sum_{n=1}^{\infty} \frac{1}{n} Aa_n \cos nt + \frac{1}{n} Ab_n \sin nt. \tag{5.8}$$

The vector field $\nabla\Phi\colon H \to H$ preserves all spaces H_n and by (5.7) and (5.8) one can show that its restriction to H_n, $n \geq 1$ can be identified with a linear map given by the $(4m \times 4m)$-matrix

$$T_n(A) = \begin{bmatrix} -\frac{1}{n}A & -J \\ J & -\frac{1}{n}A \end{bmatrix} \tag{5.9}$$

and with $-A$ on $\mathbb{R}^{2m}$ if $n = 0$. Following H. Amann and E. Zehnder [1] we introduce numbers generalizing the standard Morse index and nullity

$$
\begin{aligned}
i^-(A) &:= m^-(-A) + \sum_{n=1}^{\infty} (m^-(T_n(A)) - 2m),\\
i^0(A) &:= m^0(-A) + \sum_{n=1}^{\infty} m^0(T_n(A)).
\end{aligned}
$$

If $i^0(A) = 0$ then $\nabla\Phi$ is an isomorphism and for $r > 0$ the disc $D(r) = \{z \in H : \|z\| \leq r\}$ is an isolating neighbourhood of a flow ϕ given by $\dot{u} = -\nabla\Phi(u)$ with $S = \{0\} = \operatorname{inv} D(r)$. The $\mathcal{L}S$-index is a homotopy type of pointed sphere $E_n = S^{p(n)}$, where $p(n) = i^-(A) + n \cdot 2m$ for a sufficiently large n.

Let $S(r,H)=\{z\in H:\|z\|=r\}$. Notice that the pair $(D(r),S(r,H^-))$ would be an index pair in the classical Conley's theory. Since the sphere in a Hilbert space is homotopically trivial the quotient $D(r)/S(r,H^-)$ does not carry any essential information about the dynamics.

Using the above consideration about linear Hamiltonian systems one comes to the following:

Remark 5.6. (a) Assume that $H(z,t)=(1/2)\langle A_0z,z\rangle+h(z,t)$, where A_0 is a symmetric $(2n\times 2n)$-matrix and $\nabla h(z,t)=o(|z|)$ uniformly in t as $z\to 0$. If $i^0(A_0)=0$, then for r sufficiently small $D(r)$ is an isolating neighbourhood of $S=\{0\}$ and $h_{\mathcal{LS}}(D(r))=[(S^{p(n)},*)]$ where $p(n)=i^-(A_0)+n\cdot 2m$ and n is sufficiently large.

(b) Assume that $H(z,t)=(1/2)\langle A_\infty z,z\rangle+h(z,t)$, where A_∞ is a symmetric $(2n\times 2n)$-matrix and $\nabla h(z,t)$ is bounded. If $i^0(A_\infty)=0$, then for R sufficiently large $D(R)$ is an isolating neighbourhood and $h_{\mathcal{LS}}(D(R))=[(S^{p(n)},*)]$ where $p(n)=i^-(A_\infty)+n\cdot 2m$ and n is sufficiently large.

If $i^0(A)\neq 0$ this is no longer the case and detailed discussion is needed.

Example 5.7 ([11, Example 5.1])**.** Let $H:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ be a function of class C^2 such that:

- (H1) $H(z,t)=(1/2)|z|^2+|z|^4+h(z,t)$ for $|z|\le\alpha_1$, $\alpha_1>0$, where h is a perturbation of order higher than 4;
- (H2) $H(z,t)=(1/2)|z-z_0|^2+((x-x_0)^3-3(x-x_0)(y-y_0)^2)\cos(3t)$ if $|z-z_0|\le\alpha_2$ for some $z_0=(x_0,y_0)\neq(0,0)$ and $\alpha_2>0$;
- (H3) $H(z,t)=(1/2)d|z|^2+q(z,t)$ if $|z|\ge\alpha_3$, $\alpha_3>0$, $d>0$ is not an integer and $\nabla q(z,t)$ is bounded.

The equation (5.6) has two trivial solutions, namely $z(t)=0$ and $z(t)=z_0$. The derivatives $\mathcal{A}_0$ and $\mathcal{A}_{z_0}$ of the $\mathcal{LS}$-vector field $-\nabla\Phi$ at 0 and z_0 respectively have kernels of dimension 2. That is these points are degenerate critical points of $\Phi: H\to\mathbb{R}$ and thus the Morse index is not defined. Nevertheless, one can still find isolating neighbourhoods X_0 and X_{z_0} for the $\mathcal{LS}$-flow ϕ given by $\dot u=-\nabla\Phi(u)$ such that $\{0\}=\operatorname{inv}(X_0)$ and $\{z_0\}=\operatorname{inv}(X_{z_0})$. The $\mathcal{LS}$-index of X_{z_0} is equal to the homotopy type of pointed wedge of two spheres of dimension $2n+3$, i.e. $E_n=S^{2n+3}\vee S^{2n+3}$ for $n\ge 1$, whilst $h_{\mathcal{LS}}(X_0)=[E_n']$, where E_n' is a pointed sphere S^{2n+2}, $n\ge 1$.

Since d is not an integer it is easily seen from (5.9) that the derivative of $-\nabla\Phi$ at the infinity is an isomorphism. Therefore there is an isolating neighbourhood X_∞ for a flow ϕ such that $S:=\operatorname{inv}(X_\infty)$ is a maximal compact isolated invariant set of ϕ in H. By the Remark 5.6 we conclude that $h_{\mathcal{LS}}(X_\infty)$ is a homotopy type of pointed sphere $E_n''=S^{2n+2a}$ for sufficiently large n and positive $a\in\mathbb{Z}$ such that $d\in(a-1,a)$.

Since for all n one has $[E_n\vee E_n']\neq[E_n'']$ we get inequality $S\neq\{0,z_0\}$. We will show that S contains another equilibrium point except the trivial ones 0 and z_0. If this is not the case sets $M_1 = \{0\}$ and $M_2 = \{z_0\}$ form a Morse decomposition of S. The admissible ordering exists since ϕ is a gradient flow. The calculations of the cohomological $\mathcal{LS}$-indices with integer coefficients give us the following:

$$H^q(E) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{for } q=3, \\ 0 & \text{for } q\neq 3, \end{cases}$$
$$H^q(E') \cong \begin{cases} \mathbb{Z} & \text{for } q=4, \\ 0 & \text{for } q\neq 4, \end{cases}$$
$$H^q(E'') \cong \begin{cases} \mathbb{Z} & \text{for } q=2a, \\ 0 & \text{for } q\neq 2a. \end{cases}$$

By Theorem 5.5 we get the equality

$$2t^3 + t^4 = t^{2a} + (1+t)\mathcal{Q}(t)$$

which is false due to the fact that all coefficients of $\mathcal{Q}(t)$ are nonnegative integers. This proves the existence of third critical point of Φ. Consequently, the Hamiltonian system (5.6) satisfying conditions (H1)–(H3) possesses at least three 2π-periodic solutions.

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MINIMIZATION OF INTEGRAL FUNCTIONALS IN SOBOLEV SPACES

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Abstract. In this work we will be interested in the problem of minimization of integral functionals in Sobolev spaces $W^{1,p}[a,b]$, where $p \in [1,\infty]$. The survey is based on the author's lectures delivered at the Winter School on Topological Methods in Nonlinear Analysis 2009 at the Juliusz Schauder Center in Toruń.

1. Preface

In this work we will be concerned with a class of integral functionals of the form

$$I_{f,p}(u) = \int_a^b f(x, u(x), u'(x))\, dx$$

determined on the Sobolev spaces $W^{1,p}[a,b]$, where $p \in [1,\infty]$ and a function $f\colon [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. We will be interested in the problem of minimization of such functionals. The results we present are known in much more general settings (see for instance [1]) and they are applicable in the theory of ordinary and partial differential equations (see for instance [5] and [2], respectively).

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The work is organized as follows. In Section 2 we discuss direct methods of the calculus of variations for weakly lower semi-continuous and lower semi-continuous functionals in Banach spaces and for the Gâteaux differentiable ones. In Section 3 we review some of the standard facts on the Banach spaces $L^\infty(a,b)$ and $L^p(a,b)$ for $p \geq 1$. Section 4 contains some remarks on absolutely continuous functions. In Section 5 main results on the Sobolev spaces $W^{1,p}[a,b]$ for $p \geq 1$ and $W^{1,\infty}[a,b]$ are stated and proved. Finally, in Sections 6 and 7 we indicate how direct methods of the calculus of variations may be used to minimize the integral functionals $I_{f,p}\colon W^{1,p}[a,b] \to \mathbb{R}$.

2. Some direct methods of the calculus of variations

Let $(X, \|\cdot\|)$ be a real Banach space. We will say that $I\colon X \to \mathbb{R}$ possesses (has or achieves) a minimum on X if there exists a point $u \in X$ such that

$$I(u) = \inf\{I(v) : v \in X\}. \tag{2.1}$$

The number $I(u) = \inf\{I(v) : v \in X\}$ is called the minimum of I and each point u satisfying (2.1) is said to be a point of minimum.

The problem of existence of minimum of a functional $I\colon X \to \mathbb{R}$ is composed of three basic questions:

(1) the question about existence: Does I possess a minimum?
(2) the question about properties: What are properties of $u \in X$ satisfying (2.1)?
(3) the question about uniqueness: How many points in X satisfy (2.1)?

2.1. Lower semi-continuous functionals.

Definition 2.1. A sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ is called a minimizing sequence for $I\colon X \to \mathbb{R}$ if

$$\lim_{n\to\infty} I(u_n) = \inf\{I(u) : u \in X\}.$$

We show that for a certain class of functionals the questions about the existence of minimum and the existence of a minimizing sequence are equivalent.

Let X^* denote the space of linear continuous functionals on X. We will say that a sequence $\{u_n\}_{n\in\mathbb{N}}$ is weakly convergent (resp. convergent) to $u \in X$ if for every $F \in X^*$, $F(u_n) \to F(u)$ in $\mathbb{R}$ (resp. $\|u_n - u\| \to 0$ in $\mathbb{R}$) and we will write $u_n \rightharpoonup u$ in X (resp. $u_n \to u$ in X). Then u is called a weak limit (resp. a strong limit) and it is determined in a unique way.

Definition 2.2. A functional $I\colon X \to \mathbb{R}$ is weakly lower semi-continuous (resp. lower semi-continuous) at a point $u \in X$ if for every $\{u_n\}_{n\in\mathbb{N}} \subset X$,

$$u_n \rightharpoonup u \text{ in } X \Rightarrow \liminf_{n\to\infty} I(u_n) \geq I(u),$$
$$(\text{resp. } u_n \to u \text{ in } X \Rightarrow \liminf_{n\to\infty} I(u_n) \geq I(u)).$$

We will say that I is weakly lower semi-continuous (resp. lower semi-continuous) if I is weakly lower semi-continuous (resp. lower semi-continuous) in every $u \in X$. For abbreviation, we will write I is wlsc (resp. I is lsc).

Since every convergent sequence in X is weakly convergent to the same element, every weakly lower semi-continuous functional is lower semi-continuous. The inverse is not true.

Theorem 2.3. *If a functional $I\colon X \to \mathbb{R}$ is lsc, convex and bounded from below then I is wlsc.*

Proof. Let $u_n \rightharpoonup u$ in X. Since I is bounded from below,

$$\liminf_{n\to\infty} I(u_n) > -\infty.$$

If $\liminf_{n\to\infty} I(u_n) = \infty$ then $\liminf_{n\to\infty} I(u_n) \geq I(u)$. Let us assume that $\liminf_{n\to\infty} I(u_n)$ is finite. Fix $c \in \mathbb{R}$ such that $c > \liminf_{n\to\infty} I(u_n)$. Let $\{u_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence such that

$$\lim_{k\to\infty} I(u_{n_k}) = \liminf_{n\to\infty} I(u_n)$$

and $c > I(u_{n_k})$ for each $k \in \mathbb{N}$.

By the Mazur theorem (see for instance [6]), there exists a sequence of convex combinations

$$v_{n_k} = \sum_{j=1}^{k} \alpha_j^{n_k} u_{n_j}, \quad \sum_{j=1}^{k} \alpha_j^{n_k} = 1, \quad \alpha_j^{n_k} \geq 0$$

for $j = 1, \dots, k$ and $k \in \mathbb{N}$ such that $v_{n_k} \to u$ in X. By assumptions, we get

$$\begin{aligned} I(u) &\leq \liminf_{k\to\infty} I(v_{n_k}) = \liminf_{k\to\infty} I\left(\sum_{j=1}^{k} \alpha_j^{n_k} u_{n_j}\right) \\ &\leq \liminf_{k\to\infty} \left(\sum_{j=1}^{k} \alpha_j^{n_k} I(u_{n_j})\right) \leq \liminf_{k\to\infty} \left(c\sum_{j=1}^{k} \alpha_j^{n_k}\right) = c. \end{aligned}$$

In particular, for every $m \in \mathbb{N}$, we have

$$I(u) \leq \liminf_{n\to\infty} I(u_n) + \frac{1}{m}.$$

Hence

$$I(u) \leq \liminf_{n\to\infty} I(u_n),$$

which completes the proof. □

Theorem 2.4. *Assume that a functional $I\colon X \to \mathbb{R}$ is wlsc* (*resp. lsc*)*. Then I possesses a minimum on X if and only if there exists a weakly convergent* (*resp. convergent*) *minimizing sequence for I.*

Proof. ($\Rightarrow$) By assumption, there is $u \in X$ such that $I(u) = \inf\{I(v) : v \in X\}$. Set $u_n = u$ for each $n \in \mathbb{N}$. Then $u_n \rightharpoonup u$ in X and $\lim_{n\to\infty} I(u_n) = I(u)$.

($\Leftarrow$) There are $\{u_n\}_{n\in\mathbb{N}} \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ in X and $\lim_{n\to\infty} I(u_n) = \inf\{I(v) : v \in X\}$. Since I is wlsc, we get

$$I(u) \geq \inf\{I(v) : v \in X\} = \lim_{n\to\infty} I(u_n) = \liminf_{n\to\infty} I(u_n) \geq I(u).$$

Thus $I(u) = \inf\{I(v) : v \in X\}$.

For I lsc the proof is similar. Therefore we omit it. □

Combining Theorem 2.4 with Theorem 2.3 we get the following conclusion.

Conclusion 2.5. If a functional $I\colon X \to \mathbb{R}$ is lsc, convex and bounded from below, then I possesses a minimum on X if and only if there exists a weakly convergent minimizing sequence for I.

In general, the problem of existence of a weakly convergent minimizing sequence for a wlsc functional $I\colon X \to \mathbb{R}$ may be difficult to solve. It makes a little easier if X is a reflexive Banach space. Namely, it is equivalent to the problem of existence of a bounded minimizing sequence for I. This is a consequence of the following characterization of reflexive Banach spaces.

Theorem 2.6 (Eberlein's theorem, see [6])**.** *A Banach space $(X, \|\cdot\|)$ is reflexive if and only if every bounded sequence in X has a weakly convergent subsequence.*

Theorem 2.7. *Assume that $(X, \|\cdot\|)$ is a reflexive Banach space and $I\colon X \to \mathbb{R}$ is wlsc. Then I possesses a minimum on X if and only if there exists a bounded minimizing sequence for I.*

Proof. ($\Rightarrow$) By assumption, there is $u \in X$ such that $I(u) = \inf\{I(v) : v \in X\}$. Set $u_n = u$ for each $n \in \mathbb{N}$. The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X and $\lim_{n\to\infty} I(u_n) = I(u)$.

($\Leftarrow$) There is a bounded sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\lim_{n\to\infty} I(u_n) = \inf\{I(v) : v \in X\}$. By reflexivity of X, there are a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$ and $u \in X$ such that $u_{n_k} \rightharpoonup u$ in X. The subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ is a weakly convergent minimizing sequence for I. From Theorem 2.4 it follows that I possesses a minimum on X. □

In particular, a functional I, defined on a reflexive Banach space X, has a minimum if all its minimizing sequences are bounded. This takes place if I is coercive, i.e.

$$I(u) \to \infty, \quad \text{as } \|u\| \to \infty.$$

Theorem 2.8. *Assume that $I\colon X \to \mathbb{R}$ is coercive in a reflexive Banach space X. Then the following hypotheses are true:*

(H1) *Every minimizing sequence for I is bounded in X.*
(H2) *If I is wlsc then I has a minimum on X.*
(H3) *If I is lsc, convex and bounded from below then I has a minimum on X.*

Proof. (H1) Conversely, suppose that I has an unbounded minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$. Then there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$ such that $\|u_{n_k}\| \to \infty$. Since I is coercive, we get $I(u_{n_k}) \to \infty$. On the other hand,

$$I(u_{n_k}) \to \inf\{I(v) : v \in X\} < \infty,$$

a contradiction.

(H2) Let $\{u_n\}_{n\in\mathbb{N}} \subset X$ be a minimizing sequence for I. From (H1) it follows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. By Theorem 2.7, I possesses a minimum on X.

(H3) By Theorem 2.3, I is wlsc. Consequently, by (H2), I has a minimum on X. □

2.2. G-differentiable functionals. A functional $I: X \to \mathbb{R}$ is G-differentiable at a point $u \in X$ if it satisfies two conditions:

(1) for every $v \in X$, a function $\varphi_v: \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_v(t) = I(u + tv)$$

is differentiable at $t = 0$;

(2) a functional $I'(u): X \to \mathbb{R}$ defined by

$$I'(u)v = \varphi_v'(0), \quad v \in X$$

is linear and continuous.

Then $I'(u): X \to \mathbb{R}$ is called the Gäteaux derivative of a functional I at a point u. We will say that $I: X \to \mathbb{R}$ is G-differentiable if it is G-differentiable in every $u \in X$. Moreover, a point $u \in X$ such that

$$I'(u) = 0$$

is called a critical point of a G-differentiable functional I.

Theorem 2.9. *Assume that $I: X \to \mathbb{R}$ is a G-differentiable functional. If I achieves a minimum at a point $u \in X$ then u is a critical point of I.*

Proof. By assumption, $I(u) = \inf\{I(v) : v \in X\}$. Since I is G-differentiable, for each $v \in X$ a function $\varphi_v: \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_v(t) = I(u + tv), \quad t \in \mathbb{R}$$

is differentiable at $t = 0$ and $I'(u)v = \varphi_v'(0)$. Moreover, we have

$$\varphi_v(0) = I(u) \le I(u + tv) = \varphi_v(t)$$

for all $t \in \mathbb{R}$. Hence

$$\varphi_v(0) = \inf\{\varphi_v(t) : t \in \mathbb{R}\},$$

and, in consequence, $\varphi_v'(0) = 0$ for all $v \in X$. Thus $I'(u) = 0$. □

Theorem 2.10. *Assume that a functional $I: X \to \mathbb{R}$ is convex and G-differentiable. Then I achieves a minimum at a point $u \in X$ if and only if u is a critical point of I.*

Proof. From Theorem 2.9 it follows ($\Rightarrow$).

($\Leftarrow$) We show that

$$I(w) \geq I(v) + I'(v)(w - v) \tag{2.2}$$

for all $v, w \in X$.

Fix $v, w \in X$. If $v = w$ then $I'(v)(w - v) = I'(v)0 = 0$. Hence

$$I(w) = I(v) + I'(v)(w - v).$$

Consider the case $v \neq w$. Since I is convex, we receive

$$I(tw + (1 - t)v) \leq tI(w) + (1 - t)I(v)$$

for all $t \in (0, 1)$. From this

$$\frac{I(v + t(w - v)) - I(v)}{t} \leq I(w) - I(v)$$

for all $t \in (0, 1)$. Letting $t \to 0^+$, we have

$$I'(v)(w - v) \leq I(w) - I(v).$$

Thus the inequality (2.2) holds. In particular, $I(w) \geq I(u) + I'(u)(w - u)$ for all $w \in X$. Since u is a critical point of I, we get $I(w) \geq I(u)$ for all $w \in X$, which completes the proof. $\square$

3. The spaces $L^p(a, b)$ for $p \geq 1$ and $L^\infty(a, b)$

In Section 3 we review the standard facts on the spaces of p-integrable functions and the space of essentially bounded measurable functions on (a, b). The proofs we omit can be found in [6, Section II, § 6, Section III, § 18 and Section IV, § 26].

Let X be the space of all Lebesgue measurable functions from $(a, b) \subset \mathbb{R}$ into $\mathbb{R}$. Here and subsequently, the Lebesgue measure of $A \subset \mathbb{R}$ will be denoted by $\mu(A)$.

Definition 3.1. Let $u, v \in X$. We will say that $u = v$ almost everywhere on (a, b) if $\mu(\{x \in (a, b) : u(x) \neq v(x)\}) = 0$. For abbreviation, we will write $u = v$ a.e. on (a, b).

Definition 3.2. If $u, v \in X$ then $u \sim v$ if and only if $u = v$ a.e. on (a, b).

Fact 3.3. *$\sim$ is an equivalence relation in X.*

The equivalence class of $u \in X$, with respect to $\sim$, will be denoted by $[u]$. Let $\widetilde{X} = \{[u] : u \in X\}$. We define the addition of equivalence classes and multiplication by scalars as follows. If $u, v \in X$ and $\alpha \in \mathbb{R}$ then

$$[u] + [v] := [u + v] \quad \text{and} \quad \alpha[u] := [\alpha u].$$

These two definitions are independent of the choice of members of equivalence classes.

Fact 3.4. *$\widetilde{X}$ with the addition and multiplication by scalars determined above is a linear space over $\mathbb{R}$.*

3.1. Basic properties of $L^\infty(a, b)$ and $L^p(a, b)$.

Definition 3.5. A function $u \in X$ is essentially bounded if there exists $M \geq 0$ such that $\mu(\{x \in (a, b) : |u(x)| > M\}) = 0$.

Lemma 3.6. *Let $u \in X$ and*

$$A = \{M \geq 0 : \mu(\{x \in (a, b) : |u(x)| > M\}) = 0\}.$$

If u is essentially bounded then $\inf A \in A$.

Proof. By assumption, A is non-empty. By the definition of A, it follows that 0 is a lower bound of this set. By the definition of infimum, for each $n \in \mathbb{N}$ there is $M_n \in A$ such that

$$\inf A \leq M_n < \inf A + \frac{1}{n}.$$

For each $n \in \mathbb{N}$, let $A_n = \{x \in (a, b) : |u(x)| > M_n\}$. Since $M_n \in A$, $\mu(A_n) = 0$. Set

$$Z = \bigcup_{n=1}^{\infty} A_n.$$

By the subadditivity of measure, we have $\mu(Z) = 0$.

Take $x \in (a, b) \setminus Z$. Then $|u(x)| \leq M_n$ for all $n \in \mathbb{N}$, and consequently, $|u(x)| < \inf A + 1/n$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get $|u(x)| \leq \inf A$.

By the above, $\{x \in (a, b) : |u(x)| > \inf A\} \subset Z$. Thus

$$\mu(\{x \in (a, b) : |u(x)| > \inf A\}) = 0,$$

and hence $\inf A \in A$. $\square$

We will denote by $L^\infty(a, b)$ the space of equivalence classes of essentially bounded measurable functions from (a, b) into $\mathbb{R}$ with the norm

$$\|u\|_{L^\infty} = \inf\{M \geq 0 : \mu(\{x \in (a, b) : |u(x)| > M\}) = 0\}.$$

By abuse of notation, we write u instead of $[u]$.

Theorem 3.7. *$L^\infty(a,b)$ is a Banach space.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty(a,b)$ be a Cauchy sequence. For $n,m\in\mathbb{N}$, set

$$B_{n,m} = \{x \in (a,b) : |u_n(x) - u_m(x)| > \|u_n - u_m\|_{L^\infty}\}.$$

Define

$$B = \bigcup_{n,m\in\mathbb{N}} B_{n,m}.$$

By Lemma 3.6, $\mu(B_{n,m}) = 0$ for all $n,m\in\mathbb{N}$, and hence $\mu(B)=0$.

Fix $\varepsilon > 0$. There is $N\in\mathbb{N}$ such that $\|u_n - u_m\|_{L^\infty} < \varepsilon/2$ for all $n,m\ge N$. From this it follows that $|u_n(x)-u_m(x)| < \varepsilon/2$ for all $x\in(a,b)\setminus B$ and $n,m\ge N$. Thus $\{u_n(x)\}_{n\in\mathbb{N}}$ for each $x\in(a,b)\setminus B$ is a Cauchy sequence in $\mathbb{R}$. Consequently, a function $u\colon(a,b)\to\mathbb{R}$ given by

$$u(x) = \begin{cases} 0 & \text{if } x\in B, \\ \lim\limits_{n\to\infty} u_n(x) & \text{if } \ x \in (a,b)\setminus B \end{cases}$$

is measurable.

Remark that u is also essentially bounded.

There exists $K>0$ such that $\|u_n\|_{L^\infty} \le K$ for $n\in\mathbb{N}$, because $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. We check at once that there exists a subset $C\subset(a,b)$ of measure 0 such that $|u_n(x)| \le K$ for all $x\in(a,b)\setminus C$ and $n\in\mathbb{N}$. If $x\in(a,b)\setminus(B\cup C)$ then $|u_n(x)|\le K$ for each $n\in\mathbb{N}$ and, in consequence, $|u(x)|\le K$. By the above, $\mu(\{x\in(a,b) : |u(x)|>K\}) = 0$, and so u is essentially bounded. By definition, $u\in L^\infty(a,b)$.

Finally, we will show that u is a limit of $\{u_n\}_{n\in\mathbb{N}}$ in $L^\infty(a,b)$.

For every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|u_n(x)-u_m(x)|<\varepsilon/2$ for all $x\in(a,b)\setminus B$ and $n,m\ge N$. Letting $m\to\infty$, we get $|u_n(x)-u(x)|\le\varepsilon/2$ for all $x\in(a,b)\setminus B$ and $n\ge N$. From this $\|u_n-u\|_{L^\infty}<\varepsilon$ for each $n\ge N$, which implies $u_n\to u$ in $L^\infty(a,b)$. □

Theorem 3.8. *$u_n\to u$ in $L^\infty(a,b)$ if and only if there exists a subset $B\subset(a,b)$ of measure 0 such that $u_n\to u$ uniformly on $(a,b)\setminus B$.*

Proof. ($\Rightarrow$) Let $B_n = \{x\in(a,b) : |u_n(x)-u(x)| > \|u_n-u\|_{L^\infty}\}$, where $n\in\mathbb{N}$. By Lemma 3.6, $\mu(B_n)=0$ for $n\in\mathbb{N}$. Define $B=\bigcup_{n\in\mathbb{N}} B_n$. Then $\mu(B)=0$.

Fix $\varepsilon>0$. By assumption, there is $N\in\mathbb{N}$ such that $\|u_n-u\|_{L^\infty}<\varepsilon$ for all $n\ge N$. If $x\in(a,b)\setminus B$ and $n\ge N$ we have $|u_n(x)-u(x)|\le\|u_n-u\|_{L^\infty}<\varepsilon$.

($\Leftarrow$) Let $\varepsilon>0$. By assumption, there is $N\in\mathbb{N}$ such that $|u_n(x)-u(x)|<\varepsilon/2$ for all $x\in(a,b)\setminus B$ and $n\ge N$. Hence $\mu(\{x\in(a,b) : |u_n(x)-u(x)|>\varepsilon/2\})=0$ for all $n\ge N$. By Lemma 3.6, for all $n\ge N$ we have $\|u_n-u\|_{L^\infty}\le\varepsilon/2<\varepsilon$. □

For each $p \geq 1$, let $L^p(a,b)$ denote the space of equivalence classes of functions $u\colon (a,b) \to \mathbb{R}$ such that $\int_a^b |u(x)|^p dx < \infty$ with the norm

$$\|u\|_{L^p} = \left(\int_a^b |u(x)|^p \, dx \right)^{1/p}.$$

Theorem 3.9 (see [6, Section II, § 6])**.** *$L^p(a,b)$ for $p \geq 1$ is a Banach space.*

Theorem 3.10 (see [6, Section IV, § 23])**.** *$L^p(a,b)$ for $p > 1$ is a reflexive space.*

Theorem 3.11 (see [6, Section II, § 6])**.** *If $p, q \in [1,\infty]$ and $1/p + 1/q = 1$ then*

$$\int_a^b |u(x)v(x)| \, dx \leq \|u\|_{L^p} \|v\|_{L^q} \quad \text{(Hölder's inequality)}$$

for all $u \in L^p(a,b)$ and $v \in L^q(a,b)$.

Lemma 3.12. *$L^\infty(a,b) \subset L^p(a,b)$ for $p \geq 1$. Moreover, if $u \in L^\infty(a,b)$ then*

$$\|u\|_{L^p} \leq (b-a)^{1/p} \|u\|_{L^\infty}.$$

Proof. Fix $p \geq 1$. Assume that $u \in L^\infty(a,b)$. By Lemma 3.6,

$$\mu(\{x \in (a,b) : |u(x)| > \|u\|_{L^\infty}\}) = 0.$$

Hence

$$\int_a^b |u(x)|^p \, dx \leq \int_a^b \|u\|_{L^\infty}^p \, dx = (b-a)\|u\|_{L^\infty}^p,$$

which completes the proof. □

Conclusion 3.13. If $u_n \to u$ in $L^\infty(a,b)$ then $u_n \to u$ in $L^p(a,b)$ for $p \geq 1$.

Definition 3.14. A sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ is said to be convergent almost everywhere on (a,b) to a function $u \in X$ if

$$\mu(\{x \in (a,b) : u_n(x) \nrightarrow u(x)\}) = 0.$$

To shorten notation, we will write $u_n \to u$ a.e. on (a,b) or $u_n(x) \to u(x)$ for a.e. $x \in (a,b)$.

We can formulate now an immediate consequence of Theorem 3.8.

Conclusion 3.15. If $u_n \to u$ in $L^\infty(a,b)$ then $u_n \to u$ a.e. on (a,b).

Theorem 3.16 (Riesz theorem, see [3, § 5.9])**.** *If $p \in [1,\infty)$ and $u_n \to u$ in $L^p(a,b)$ then $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence that converges to u a.e. on (a,b).*

Theorem 3.17 (Riesz representation theorem, see [6]).

(a) *Assume that* $p, q > 1$ *and* $1/p+1/q = 1$. *Then a functional* $F\colon L^p(a,b) \to \mathbb{R}$ *is linear and continuous if and only if there exists* $v \in L^q(a,b)$ *such that*

$$F(u) = \int_a^b u(x)v(x)\,dx.$$

Moreover, $\|F\| = \|v\|_{L^q}$.

(b) *A functional* $F\colon L^1(a,b) \to \mathbb{R}$ *is linear and continuous if and only if there exists* $v \in L^\infty(a,b)$ *such that*

$$F(u) = \int_a^b u(x)v(x)\,dx.$$

Moreover, $\|F\| = \|v\|_{L^\infty}$.

3.2. $(*)$-weakly convergence in $L^\infty(a,b)$.

Theorem 3.18. *For every* $v \in L^1(a,b)$, *a functional* $F\colon L^\infty(a,b) \to \mathbb{R}$ *given by*

$$F(u) = \int_a^b u(x)v(x)\,dx$$

is linear and continuous. Moreover, $\|F\| = \|v\|_{L^1}$.

Proof. It is easy to check that F is linear. By Hölder's inequality, for every $u \in L^\infty(a,b)$, we get

$$\left| \int_a^b u(x)v(x)\,dx \right| \leq \int_a^b |u(x)v(x)|\,dx \leq \|u\|_{L^\infty}\|v\|_{L^1}.$$

In consequence, F is continuous and $\|F\| \leq \|v\|_{L^1}$.

To finish the proof, it is sufficient to find $u_0 \in L^\infty(a,b)$ such that $\|u_0\|_{L^\infty} \leq 1$ and $F(u_0) = \|v\|_{L^1}$. Define

$$u_0(x) = \begin{cases} 1 & \text{if } x \in v^{-1}([0,\infty)), \\ -1 & \text{if } x \in v^{-1}((-\infty,0)). \end{cases}$$

We see at once that u_0 possesses the above properties. $\square$

We will say that a sequence $\{u_n\}_{n\in\mathbb{N}} \subset L^\infty(a,b)$ is $(*)$-weakly convergent to $u \in L^\infty(a,b)$ if for every $v \in L^1(a,b)$,

$$\lim_{n\to\infty} \int_a^b u_n(x)v(x)\,dx = \int_a^b u(x)v(x)\,dx$$

and we will write $u_n \stackrel{*}{\rightharpoonup} u$ in $L^\infty(a,b)$. Then u is called a $(*)$-weak limit.

By the use of Theorem 3.18 one can easily check that a $(*)$-weak limit is well defined.

Combining the definition of weak convergence with Theorem 3.18 we receive the following conclusion.

Conclusion 3.19. If $u_n \rightharpoonup u$ in $L^\infty(a,b)$ then $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(a,b)$.

Fact 3.20. *If $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(a,b)$ then $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(a,b)$. Moreover,*

$$\|u\|_{L^\infty} \le \liminf_{n\to\infty} \|u_n\|_{L^\infty}.$$

Proof. Let $F, F_n: L^1(a,b) \to \mathbb{R}$ be given by

$$F(v) = \int_a^b v(x)u(x)\,dx \quad \text{and} \quad F_n(v) = \int_a^b v(x)u_n(x)\,dx,$$

where $n \in \mathbb{N}$. By the Riesz representation theorem, the functionals defined above are linear and continuous. Moreover, $\|F\| = \|u\|_{L^\infty}$ and $\|F_n\| = \|u_n\|_{L^\infty}$ for all $n \in \mathbb{N}$.

By assumption, $F_n(v) \to F(v)$ for every $v \in L^1(a,b)$. From the Banach-Steinhaus theorem (see for instance [6]) it follows the existence of $K > 0$ such that $\|F_n\| \le K$.

Fix $v \in L^1(a,b)$ such that $\|v\|_{L^1} \le 1$. We have $|F_n(v)| \le \|F_n\|\|v\|_{L^1} \le \|F_n\|$ for $n \in \mathbb{N}$, and hence $|F(v)| \le \liminf_{n\to\infty} \|F_n\|$. Consequently,

$$\|F\| = \sup_{\|v\|_{L^1}\le 1} |F(v)| \le \liminf_{n\to\infty} \|F_n\|.$$

By the above, we get our claim. □

4. Some remarks on absolutely continuous functions

Definition 4.1. A function $u: [a,b] \to \mathbb{R}$ is said to be absolutely continuous if for every finite sequence of subintervals $\{[a_i,b_i]\}_{i=1}^n$ of the interval $[a,b]$ such that $[a_i,b_i] \cap [a_j,b_j] = \emptyset$ for $i \ne j$ it holds

$$\sum_{i=1}^n (b_i - a_i) < \delta \;\Rightarrow\; \sum_{i=1}^n |u(b_i) - u(a_i)| < \varepsilon.$$

We will denote by $AC[a,b]$ the set of all absolutely continuous functions from $[a,b]$ into $\mathbb{R}$.

Proposition 4.2. *The sentences below are true.*

(a) *$AC[a,b]$ is a linear space over $\mathbb{R}$.*
(b) *If $u \in AC[a,b]$ then u is uniformly continuous.*
(c) *If $u \in AC[a,b]$ is strictly monotone, $v \in AC[\alpha,\beta]$ and $u([a,b]) \subset [\alpha,\beta]$ then $v \circ u \in AC[a,b]$.*
(d) *If $u \in AC[a,b]$ then its variation $\bigvee_a^b u$ is finite, i.e.*

$$\bigvee_a^b u = \sup_{a=x_0<\ldots<x_n=b} \sum_{i=1}^n |u(x_i) - u(x_{i-1})| < \infty.$$

(e) *If* $u\colon[a,b]\to\mathbb{R}$ *is continuous,* $c\in[a,b]$ *and* $u|_{[a,c]}$, $u|_{[c,b]}$ *are absolutely continuous then* u *is absolutely continuous.*

The proof of Proposition 4.2 is easy and therefore it is left to the reader.

Since every absolutely continuous function $u\colon[a,b]\to\mathbb{R}$ has a finite variation, by the Lebesgue theorem (see for instance [4, Section VII, § 1]), u is differentiable almost everywhere on (a,b), i.e. $\mu(\{x\in(a,b) : u'(x) \text{ does not exist}\})=0$. It is also known (see [4, Section VII, § 4]) that $u'\in L^1(a,b)$ and for each $x\in[a,b]$,

$$u(x)=u(a)+\int_a^x u'(t)\,dt.$$

Theorem 4.3 (see [4, Section VII, § 4, Theorem 3] and [7, p. 104])**.** *If* $v\in L^1(a,b)$ *then* $u\colon[a,b]\to\mathbb{R}$ *given by*

$$u(x)=u(a)+\int_a^x v(t)\,dt$$

is absolutely continuous and $u'=v$ *a.e. on* (a,b).

Conclusion 4.4. $u\in AC[a,b]$ if and only if u is differentiable a.e. on (a,b), $u'\in L^1(a,b)$ and for each $x\in[a,b]$ it holds

$$u(x)=u(a)+\int_a^x u'(t)\,dt.$$

5. The Sobolev spaces $W^{1,p}[a,b]$ for $p\geq 1$ and $W^{1,\infty}[a,b]$

5.1. Basic properties of the Sobolev spaces. For each $p\geq 1$, we will denote by $W^{1,p}[a,b]$ the space of all functions $u\colon[a,b]\to\mathbb{R}$ such that $u\in AC[a,b]$ and $u'\in L^p(a,b)$ with the norm

$$\|u\|_{W^{1,p}}=\left(\int_a^b(|u(x)|^p+|u'(x)|^p)\,dx\right)^{1/p}.$$

Furthermore, let $W^{1,\infty}[a,b]$ denote the space of all functions $u\colon[a,b]\to\mathbb{R}$ such that $u\in AC[a,b]$ and $u'\in L^\infty(a,b)$ with the norm

$$\|u\|_{W^{1,\infty}}=\max\{\|u\|_{L^\infty},\|u'\|_{L^\infty}\}.$$

We call $W^{1,p}[a,b]$ and $W^{1,\infty}[a,b]$ the Sobolev spaces.

Let $C[a,b]$ be the space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the standard norm

$$\|u\|_C=\sup_{x\in[a,b]}|u(x)|.$$

Of course, if $u\colon[a,b]\to\mathbb{R}$ belongs to a Sobolev space then $\|u\|_{L^\infty}=\|u\|_C$.

Fact 5.1. *The sentences below are true.*

(a) *If $u \in W^{1,p}[a,b]$, $p > 1$ and $1/p + 1/q = 1$ then*
$$\|u - u(a)\|_C \le (b-a)^{1/q}\|u\|_{W^{1,p}}.$$

(b) *If $u \in W^{1,1}[a,b]$ then*
$$\|u - u(a)\|_C \le \|u\|_{W^{1,1}}.$$

(c) *If $u \in W^{1,\infty}[a,b]$ then*
$$\|u - u(a)\|_C \le (b-a)\|u\|_{W^{1,\infty}}.$$

Proof. Let $u \in W^{1,p}[a,b]$, $p > 1$ and $1/p + 1/q = 1$. Then
$$\begin{aligned}\|u - u(a)\|_C &= \sup_{x\in[a,b]} |u(x) - u(a)| = \sup_{x\in[a,b]} \left|\int_a^x u'(t)\,dt\right| \\ &\le \int_a^b |u'(t)|\,dt \le (b-a)^{1/q}\|u'\|_{L^p} \le (b-a)^{1/q}\|u\|_{W^{1,p}}.\end{aligned}$$
Let $u \in W^{1,1}[a,b]$. Then
$$\|u - u(a)\|_C \le \int_a^b |u'(t)|\,dt = \|u'\|_{L^1} \le \|u\|_{W^{1,1}}.$$
Let $u \in W^{1,\infty}[a,b]$. Then
$$\|u - u(a)\|_C \le \int_a^b |u'(t)|\,dt \le (b-a)\|u'\|_{L^\infty} \le (b-a)\|u\|_{W^{1,\infty}}. \qquad \square$$

Theorem 5.2. *$W^{1,p}[a,b]$ for $p \ge 1$ and $W^{1,\infty}[a,b]$ are Banach spaces.*

Proof. We examine the case where $p \ge 1$. For $p = \infty$, the proof is similar.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{1,p}[a,b]$. From Fact 5.1 it follows that $\{u_n - u_n(a)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C[a,b]$. Since $C[a,b]$ is a Banach space, there is $u_0 \in C[a,b]$ such that $u_n - u_n(a) \to u_0$ uniformly on $[a,b]$. In particular, $u_n(a) - u_n(a) = 0 \to u_0(a)$ in $\mathbb{R}$, and so $u_0(a) = 0$.

By the definition of norm in $W^{1,p}[a,b]$, $\{u_n\}_{n\in\mathbb{N}}$ and $\{u_n'\}_{n\in\mathbb{N}}$ are Cauchy sequences in $L^p(a,b)$. By Theorem 3.9, there exist $w, v \in L^p(a,b)$ such that $u_n \to w$ and $u_n' \to v$ in $L^p(a,b)$.

Let $x \in [a,b]$. For each $n \in \mathbb{N}$, we have
$$\left|\int_a^x u_n'(t)\,dt - \int_a^x v(t)\,dt\right| \le \int_a^b |u_n'(t) - v(t)|\,dt = \|u_n' - v\|_{L^1}$$
if $p = 1$ and
$$\left|\int_a^x u_n'(t)\,dt - \int_a^x v(t)\,dt\right| \le (b-a)^{1/q}\|u_n' - v\|_{L^p}$$

if $p > 1$ and $1/p + 1/q = 1$. Hence

$$\lim_{n\to\infty} \int_a^x u_n'(t)\,dt = \int_a^x v(t)\,dt.$$

On the other hand,

$$\lim_{n\to\infty} \int_a^x u_n'(t)\,dt = \lim_{n\to\infty} (u_n(x) - u_n(a)) = u_0(x).$$

In consequence,

$$u_0(x) = \int_a^x v(t)\,dt$$

for each $x \in [a,b]$. Applying Theorem 4.3, we get $u_0 \in AC[a,b]$ and $u_0'(x) = v(x)$ for a.e. $x \in (a,b)$. Since $v \in L^p(a,b)$, $u_0 \in W^{1,p}[a,b]$.

For each $n \in \mathbb{N}$, it holds

$$\|(u_n - u_n(a)) - u_0\|_{W^{1,p}}^p = \int_a^b (|(u_n(x) - u_n(a)) - u_0(x)|^p + |u_n'(x) - u_0'(x)|^p)\,dx.$$

From this $u_n - u_n(a) \to u_0$ in $W^{1,p}[a,b]$, because $u_n - u_n(a) \to u_0$ uniformly on $[a,b]$ and $u_n' \to u_0'$ in $L^p(a,b)$.

By the above, $u_n(a) = u_n - (u_n - u_n(a)) \to w - u_0$ in $L^p(a,b)$, which implies that $\{u_n(a)\}_{n\in\mathbb{N}}$ is bounded in $L^p(a,b)$, and consequently, $\{u_n(a)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. By the Bolzano–Weierstrass theorem, $\{u_n(a)\}_{n\in\mathbb{N}}$ possesses a convergent subsequence $\{u_{n_k}(a)\}_{k\in\mathbb{N}}$ in $\mathbb{R}$. Set $\lim_{k\to\infty} u_{n_k}(a) = c$. Then $u_{n_k}(a) \to c$ in $L^p(a,b)$, and so $w - u_0 = c$. Moreover, since $\|u_n(a) - c\|_{W^{1,p}} = \|u_n(a) - c\|_{L^p}$ for each $n \in \mathbb{N}$ and $u_n(a) \to c$ in $L^p(a,b)$, we get $u_n(a) \to c$ in $W^{1,p}[a,b]$. Finally, we get $u_n = (u_n - u_n(a)) + u_n(a) \to u_0 + c$ in $W^{1,p}[a,b]$. □

Lemma 5.3. *$W^{1,\infty}[a,b] \subset W^{1,p}[a,b]$ for $p \geq 1$. Moreover, if $u \in W^{1,\infty}[a,b]$ then*

$$\|u\|_{W^{1,p}} \leq 2^{1/p}(b-a)^{1/p}\|u\|_{W^{1,\infty}}.$$

Proof. Fix $p \geq 1$. By Lemma 3.12, we conclude that $W^{1,\infty}[a,b] \subset W^{1,p}[a,b]$. If $u \in W^{1,\infty}[a,b]$ then

$$\|u\|_{L^p} \leq (b-a)^{1/p}\|u\|_{L^\infty} \quad \text{and} \quad \|u'\|_{L^p} \leq (b-a)^{1/p}\|u'\|_{L^\infty}.$$

Hence

$$\|u\|_{W^{1,p}}^p \leq (b-a)\|u\|_{L^\infty}^p + (b-a)\|u'\|_{L^\infty}^p \leq 2(b-a)\|u\|_{W^{1,\infty}}^p,$$

which finishes the proof. □

Fact 5.4. *For each $p \geq 1$, if $u_n \to u$ in $W^{1,p}[a,b]$ then $u_n \to u$ uniformly on $[a,b]$.*

Proof. Assume that $u_n \to u$ in $W^{1,p}[a,b]$. Then $u_n \to u$ in $L^p(a,b)$. From Fact 5.1 it follows that for each $n \in \mathbb{N}$,

$$\|u_n - u_n(a) - (u - u(a))\|_C \leq \begin{cases} (b-a)^{1/q}\|u_n - u\|_{W^{1,p}} & \text{if } p > 1,\ 1/p + 1/q = 1, \\ \|u_n - u\|_{W^{1,1}} & \text{if } p = 1. \end{cases}$$

By this $u_n - u_n(a) \to u - u(a)$ uniformly on $[a,b]$, which implies $u_n - u_n(a) \to u - u(a)$ in $L^p(a,b)$. Since $u_n \to u$ and $u_n - u_n(a) \to u - u(a)$ in $L^p(a,b)$, we have $u_n(a) \to u(a)$ in $L^p(a,b)$. Consequently, $u_n(a) \to u(a)$ in $\mathbb{R}$, and hence $u_n(a) \to u(a)$ uniformly on $[a,b]$. By the above, $u_n \to u$ uniformly on $[a,b]$. $\square$

Fact 5.5. *If $u_n \to u$ in $W^{1,\infty}[a,b]$ then $u_n \to u$ uniformly on $[a,b]$.*

Proof. Let $u_n \to u$ in $W^{1,\infty}[a,b]$. By Lemma 5.3, $u_n \to u$ in $W^{1,p}[a,b]$ for $p \geq 1$. From Fact 5.4, $u_n \to u$ uniformly on $[a,b]$. $\square$

For every $p \geq 1$, let us denote by $L^p(a,b) \times L^p(a,b)$ the inner product of $L^p(a,b)$ by itself with the norm

$$\|(u,v)\|_{L^p \times L^p} = \left(\int_a^b (|u(x)|^p + |v(x)|^p) \right)^{1/p} dx.$$

Applying Theorems 3.9 and 3.10, respectively, we get the following ones.

Theorem 5.6. *$L^p(a,b) \times L^p(a,b)$ for $p \geq 1$ is a Banach space.*

Theorem 5.7. *$L^p(a,b) \times L^p(a,b)$ for $p > 1$ is a reflexive space.*

For every $p \geq 1$, let $i_p \colon W^{1,p}[a,b] \to L^p(a,b) \times L^p(a,b)$ be given by

$$i_p(u) = (u, u').$$

It is easy to check that i_p is linear and $\|i_p(u)\|_{L^p \times L^p} = \|u\|_{W^{1,p}}$. From this we conclude that the spaces $W^{1,p}[a,b]$ and $i_p(W^{1,p}[a,b])$ are isometric. Moreover, $i_p(W^{1,p}[a,b])$ is a closed subspace of $L^p(a,b) \times L^p(a,b)$. Since a closed subspace of a reflexive space is reflexive (see [6, Section IV, § 23, Theorem 23.7]), $i_p(W^{1,p}[a,b])$ for each $p > 1$ is reflexive.

One can now prove the following theorems.

Theorem 5.8. *$W^{1,p}[a,b]$ for $p > 1$ is a reflexive space.*

Theorem 5.9 (Riesz representation theorem)**.**

(a) *Assume that* $p, q > 1$ *and* $1/p + 1/q = 1$*. Then a functional*

$$F: W^{1,p}[a,b] \to \mathbb{R}$$

is linear and continuous if and only if there exist $f, g \in L^q(a,b)$ *such that*

$$F(u) = \int_a^b u(x)f(x)\,dx + \int_a^b u'(x)g(x)\,dx.$$

(b) *A functional* $F: W^{1,1}[a,b] \to \mathbb{R}$ *is linear and continuous if and only if there exist* $f, g \in L^\infty(a,b)$ *such that*

$$F(u) = \int_a^b u(x)f(x)\,dx + \int_a^b u'(x)g(x)\,dx.$$

Conclusion 5.10. For every $p \geq 1$, $u_n \rightharpoonup u$ in $W^{1,p}[a,b]$ if and only if $u_n \rightharpoonup u$ and $u_n' \rightharpoonup u'$ in $L^p(a,b)$.

5.2. $(*)$**-weakly convergence in** $W^{1,\infty}[a,b]$**.**

Theorem 5.11. *For every* $f, g \in L^1(a,b)$*, a functional* $F: W^{1,\infty}[a,b] \to \mathbb{R}$ *given by*

$$F(u) = \int_a^b u(x)f(x)\,dx + \int_a^b u'(x)g(x)\,dx$$

is linear and continuous. Moreover, $\|F\| \leq \|(f,g)\|_{L^1 \times L^1}$*.*

Proof. It is easily seen that F is linear. For each $u \in W^{1,\infty}[a,b]$, we get

$$\begin{aligned} |F(u)| \leq & \|u\|_{L^\infty}\|f\|_{L^1} + \|u'\|_{L^\infty}\|g\|_{L^1} \\ \leq & \|u\|_{W^{1,\infty}}(\|f\|_{L^1} + \|g\|_{L^1}) = \|(f,g)\|_{L^1 \times L^1}\|u\|_{W^{1,\infty}}. \end{aligned}$$

In consequence, F is continuous and $\|F\| \leq \|(f,g)\|_{L^1 \times L^1}$. □

We will say that a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\infty}[a,b]$ is $(*)$-weakly convergent to $u \in W^{1,\infty}[a,b]$ if for every $f, g \in L^1(a,b)$,

$$\int_a^b u_n(x)f(x)\,dx + \int_a^b u_n'(x)g(x)\,dx \to \int_a^b u(x)f(x)\,dx + \int_a^b u'(x)g(x)\,dx$$

and we will write $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$. Then u is called a $(*)$-weak limit. Applying Theorem 5.11 we check at once that a $(*)$-weak limit is well defined.

Conclusion 5.12. If $u_n \rightharpoonup u$ in $W^{1,\infty}[a,b]$ then $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$.

Conclusion 5.13. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\infty}[a,b]$ and $u \in W^{1,\infty}[a,b]$. Then $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$ if and only if $u_n \overset{*}{\rightharpoonup} u$ and $u_n' \overset{*}{\rightharpoonup} u'$ in $L^\infty(a,b)$.

The conclusions above are direct consequences of Theorem 5.11.

Conclusion 5.14. If $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$ then $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,\infty}[a,b]$. Moreover,

$$\|u\|_{W^{1,\infty}} \le \liminf_{n\to\infty} \|u_n\|_{W^{1,\infty}}.$$

Proof. From Conclusion 5.13 it follows that $u_n \overset{*}{\rightharpoonup} u$ and $u_n' \overset{*}{\rightharpoonup} u'$ in $L^\infty(a,b)$. By Fact 3.20, there is $K>0$ such that $\|u_n\|_{L^\infty} \le K$ and $\|u_n'\|_{L^\infty} \le K$ for all $n \in \mathbb{N}$. Furthermore,

$$\|u\|_{L^\infty} \le \liminf_{n\to\infty} \|u_n\|_{L^\infty} \le \liminf_{n\to\infty} \|u_n\|_{W^{1,\infty}}$$

and

$$\|u'\|_{L^\infty} \le \liminf_{n\to\infty} \|u_n'\|_{L^\infty} \le \liminf_{n\to\infty} \|u_n\|_{W^{1,\infty}}.$$

Hence $\|u_n\|_{W^{1,\infty}} \le K$ for all $n \in \mathbb{N}$ and

$$\|u\|_{W^{1,\infty}} \le \liminf_{n\to\infty} \|u_n\|_{W^{1,\infty}}. \qquad \square$$

5.3. Embedding lemmas. Let $u_n\colon [a,b] \to \mathbb{R}$ for $n \in \mathbb{N}$. We will say that a sequence $\{u_n\}_{n\in\mathbb{N}}$ is equi-bounded almost everywhere on $[a,b]$ if there is $K>0$ such that, for each $n \in \mathbb{N}$,

$$\mu(\{x \in [a,b] : |u_n(x)| \le K\}) = 0.$$

Proposition 5.15. *Assume that for each $n \in \mathbb{N}$ a function $u_n\colon [a,b] \to \mathbb{R}$ is continuous. If $\{u_n\}_{n\in\mathbb{N}}$ is equi-bounded a.e. on $[a,b]$ then $\{u_n\}_{n\in\mathbb{N}}$ is equi-bounded on $[a,b]$.*

Proof. By assumption, there is a constant $K>0$ and a subset $B \subset [a,b]$ of measure 0 such that $|u_n(x)| \le K$ for all $n \in \mathbb{N}$ and $x \in [a,b] \setminus B$.

Let $y \in B$. Since $\mu(B) = 0$, the set $[a,b] \setminus B$ is dense in $[a,b]$. Hence there is $\{x_k\}_{k\in\mathbb{N}} \subset [a,b] \setminus B$ such that $x_k \to y$ in $\mathbb{R}$. By the continuity of u_n, we have

$$\lim_{k\to\infty} |u_n(x_k)| = |u_n(y)|$$

for each $n \in \mathbb{N}$. Moreover, $|u_n(x_k)| \le K$ for all $n,k \in \mathbb{N}$, and from this $|u_n(y)| \le K$ for all $n \in \mathbb{N}$. $\square$

Lemma 5.16. *If $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$ then $u_n \to u$ in $L^\infty(a,b)$.*

Proof. At the beginning assume that $u_n \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}[a,b]$. From Conclusion 5.14 it follows that there is $L>0$ such that $\|u_n\|_{W^{1,\infty}} \le L$ for all $n \in \mathbb{N}$. Hence $\|u_n\|_{L^\infty} \le L$ and $\|u_n'\|_{L^\infty} \le L$ for all $n \in \mathbb{N}$.

It is easy to check that there is a subset $B \subset (a,b)$ of measure 0 such that for all $n \in \mathbb{N}$ and $x \in (a,b) \setminus B$ we have $|u_n(x)| \le L$ and $|u_n'(x)| \le L$.

Fix $n \in \mathbb{N}$ and $x, y \in [a,b]$. Assume that $x < y$. We get

$$\begin{aligned}|u_n(x) - u_n(y)| &= \left| \int_a^x u_n'(t)\,dt - \int_a^y u_n'(t)\,dt \right| = \left| \int_x^y u_n'(t)\,dt \right| \\ &\le \int_x^y |u_n'(t)|\,dt \le L|x-y|.\end{aligned}$$

Hence $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions. In consequence, by Proposition 5.15, $\{u_n\}_{n\in\mathbb{N}}$ is equi-bounded on $[a,b]$.

Let $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$. By the Arzelà–Ascoli lemma, there exist a subsequence $\{u_{n_{k_j}}\}_{j\in\mathbb{N}} \subset \{u_{n_k}\}_{k\in\mathbb{N}}$ and a continuous function $u_0\colon [a,b] \to \mathbb{R}$ such that $u_{n_{k_j}} \to u_0$ uniformly on $[a,b]$. From Theorem 3.8, $u_{n_{k_j}} \to u_0$ in $L^\infty(a,b)$.

Remark that $u_0 = 0$. By assumption, $u_{n_{k_j}} \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}[a,b]$. Conclusion 5.13 implies $u_{n_{k_j}} \overset{*}{\rightharpoonup} 0$ in $L^\infty(a,b)$. On the other hand, $u_{n_{k_j}} \to u_0$ in $L^\infty(a,b)$. Therefore, by Conclusion 3.19, $u_{n_{k_j}} \overset{*}{\rightharpoonup} u_0$ in $L^\infty(a,b)$. Hence $u_0 = 0$ a.e. on (a,b). As $u_0\colon [a,b] \to \mathbb{R}$ is continuous we have $u_0 = 0$. By the above, $u_{n_{k_j}} \to 0$ in $L^\infty(a,b)$.

Summarizing. We have just proved that every subsequence of $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence convergent to 0 in $L^\infty(a,b)$. Therefore $u_n \to 0$ in $L^\infty(a,b)$.

If $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$ and $u \neq 0$ then $u_n - u \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}[a,b]$. Thus $u_n - u \to 0$ in $L^\infty(a,b)$, and consequently $u_n \to u$ in $L^\infty(a,b)$. □

Theorem 5.17. *Let $u, u_n\colon [a,b] \to \mathbb{R}$ be continuous functions for $n \in \mathbb{N}$. Then $u_n \to u$ uniformly on $[a,b]$ if and only if there exists a subset $C \subset [a,b]$ of measure 0 such that $u_n \to u$ uniformly on $[a,b] \setminus C$.*

Proof. ($\Rightarrow$) It is evident.

($\Leftarrow$) Fix $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that $|u_n(x) - u(x)| < \varepsilon/2$ for all $n \ge N$ and $x \in [a,b] \setminus C$.

Assume that $y \in C$. Since $\mu(C) = 0$, the set $[a,b] \setminus C$ is dense in $[a,b]$. Therefore there is a sequence $\{x_k\}_{k\in\mathbb{N}} \subset [a,b] \setminus C$ such that $x_k \to y$ in $\mathbb{R}$. By the continuity of u and u_n, we have

$$|u_n(y) - u(y)| = \lim_{k\to\infty} |u_n(x_k) - u(x_k)|$$

for each $n \in \mathbb{N}$. Since $|u_n(x_k) - u(x_k)| < \varepsilon/2$ for $n \ge N$ and $k \in \mathbb{N}$, we get

$$\lim_{k\to\infty} |u_n(x_k) - u(x_k)| \le \frac{\varepsilon}{2}$$

for $n \ge N$, and so $|u_n(y) - u(y)| < \varepsilon$ for $n \ge N$. Consequently, $|u_n(x) - u(x)| < \varepsilon$ for all $x \in [a,b]$ and $n \ge N$. □

Conclusion 5.18. If $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$ then $u_n \to u$ uniformly on $[a,b]$.

Proof. From Lemma 5.16 it follows that $u_n \to u$ in $L^\infty(a,b)$. Applying Theorem 3.8, we conclude that there is a subset $B \subset (a,b)$ of measure 0 such that $u_n \to u$ uniformly on $(a,b) \setminus B$. Set $C = B \cup \{a,b\}$. Then $C \subset [a,b]$, $\mu(C) = 0$ and $[a,b] \setminus C = (a,b) \setminus B$. Thus $u_n \to u$ uniformly on $[a,b] \setminus C$. By Theorem 5.17, we get $u_n \to u$ uniformly on $[a,b]$. □

Lemma 5.19. *If $p > 1$ and $u_n \rightharpoonup u$ in $W^{1,p}[a,b]$ then $u_n \to u$ in $L^\infty(a,b)$.*

Proof. Since every weakly convergent sequence in a norm space is bounded (see [6, Section IV, § 21, Theorem 21.5]), there is $L > 0$ such that $\|u_n\|_{W^{1,p}} \le L$ for all $n \in \mathbb{N}$, and hence $\|u_n\|_{L^p} \le L$ and $\|u_n'\|_{L^p} \le L$ for $n \in \mathbb{N}$. Let us remark that $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions.

Fix $n \in \mathbb{N}$ and $x, y \in [a,b]$. Assume that $x < y$. As in the proof of Lemma 5.16, we get

$$|u_n(x) - u_n(y)| \le \int_x^y |u_n'(t)|\,dt.$$

Using Hölder's inequality, we have

$$\begin{aligned}\int_x^y |u_n'(t)|\,dt &\le |x-y|^{1/q}\left(\int_x^y |u_n'(t)|^p\,dt\right)^{1/p}\\ &\le |x-y|^{1/q}\left(\int_a^b |u_n'(t)|^p\,dt\right)^{1/p} \le L|x-y|^{1/q},\end{aligned}$$

where $1/p + 1/q = 1$. Combining these inequalities, we receive

$$|u_n(x) - u_n(y)| \le L|x-y|^{1/q}.$$

By the above, we see that all functions $\{u_n\}_{n\in\mathbb{N}}$ satisfy the Hölder condition with power q and the same constant L. Therefore they are equicontinuous.

In particular, for all $n \in \mathbb{N}$ and $x \in [a,b]$ we get

$$|u_n(x) - u_n(a)| \le L|x-a|^{1/q},$$

and consequently,

$$|u_n(a)| \le L(b-a)^{1/q} + |u_n(x)|.$$

We may now integrate this inequality over (a,b) to conclude that

$$(b-a)|u_n(a)| \le L(b-a)^{1+1/q} + (b-a)^{1/q}\left(\int_a^b |u_n(x)|^p\,dx\right)^{1/p},$$

and finally that

$$|u_n(a)| \le L(b-a)^{1/q} + L(b-a)^{-1/p}$$

for all $n \in \mathbb{N}$. From this and the Hölder condition, we get

$$|u_n(x)| \le L(b-a)^{1/q} + |u_n(a)| \le 2L(b-a)^{1/q} + L(b-a)^{-1/p}$$

for all $n \in \mathbb{N}$ and $x \in [a,b]$. In consequence, $\{u_n\}_{n\in\mathbb{N}}$ is equi-bounded.

Let $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$. By the Arzelà–Ascoli lemma, there exist a subsequence $\{u_{n_{k_j}}\}_{j\in\mathbb{N}} \subset \{u_{n_k}\}_{k\in\mathbb{N}}$ and a continuous function $v\colon [a,b] \to \mathbb{R}$ such that $u_{n_{k_j}} \to v$ uniformly on $[a,b]$. Hence $u_{n_{k_j}} \to v$ in $L^p(a,b)$, which implies $u_{n_{k_j}} \rightharpoonup v$ in $L^p(a,b)$. On the other hand, $u_{n_{k_j}} \rightharpoonup u$ in $W^{1,p}[a,b]$, therefore $u_{n_{k_j}} \rightharpoonup u$ in $L^p(a,b)$ by Conclusion 5.10. It follows that $u = v$ a.e. on (a,b). As $u, v\colon [a,b] \to \mathbb{R}$ are continuous we have $u = v$. Consequently, $u_{n_{k_j}} \to u$ uniformly on $[a,b]$, and by Theorem 3.8, $u_{n_{k_j}} \to u$ in $L^\infty(a,b)$.

Summarizing. We have just proved that every subsequence of the sequence $\{u_n\}_{n\in\mathbb{N}}$ possesses a subsequence that converges to u in $L^\infty(a,b)$. Therefore $u_n \to u$ in $L^\infty(a,b)$. $\square$

Conclusion 5.20. If $p > 1$ and $u_n \rightharpoonup u$ in $W^{1,p}[a,b]$ then $u_n \to u$ uniformly on $[a,b]$.

The proof of Conclusion 5.20 is similar to the proof of Conclusion 5.18. We have to use Lemma 5.19 instead of Lemma 5.16. The details are left to the reader.

6. Minimization of integral functionals in the Sobolev spaces

In this section we will be concerned with a class of functionals of the form

$$I_{f,p}(u) = \int_a^b f(x, u(x), u'(x))\,dx, \tag{6.1}$$

where $u \in W^{1,p}[a,b]$, $p \in [1,\infty)$ or $p = \infty$ and a function $f\colon [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

One may ask whether a functional $I_{f,p}\colon W^{1,p}[a,b] \to \mathbb{R}$ possesses a minimum. In order to prove the existence of minimum for $I_{f,p}$, we have to assume something more about a function f.

Our purpose now is to indicate how the techniques described in Section 2 may be used to minimize $I_{f,p}$.

Theorem 6.1. *Assume that $g\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $p \in (1,\infty)$. Let $I_{g,p}\colon W^{1,p}[a,b] \to \mathbb{R}$ be given by*

$$I_{g,p}(u) = \int_a^b g(x, u(x))\,dx.$$

Then $I_{g,p}$ possesses a minimum on $W^{1,p}[a,b]$ if and only if there exists a bounded minimizing sequence for $I_{g,p}$.

Proof. By Theorem 2.7, it suffices to show that $I_{g,p}$ is wlsc. Let $u_n \rightharpoonup u$ in $W^{1,p}[a,b]$. Applying Conclusion 5.20, we receive $u_n \to u$ uniformly on $[a,b]$. Hence there is a constant $K > 0$ such that $|u_n(x)| \le K$ for all $x \in [a,b]$ and $n \in \mathbb{N}$. A function g restricted to $[a,b] \times [-K,K]$ is uniformly continuous. Thus there is $L > 0$ such that $|g(x,y)| \le L$ for all $x \in [a,b]$ and $y \in [-K,K]$. In consequence, $|g(x,u_n(x))| \le L$ for all $x \in [a,b]$ and $n \in \mathbb{N}$. Moreover, $g(x,u_n(x)) \to g(x,u(x))$

for each $x \in [a,b]$. Using the Lebesgue theorem on dominated convergence, we get

$$\lim_{n\to\infty} I_{g,p}(u_n) = I_{g,p}(u),$$

which completes the proof. $\square$

Theorem 6.2. *Let $p \in [1,\infty)$ or $p = \infty$. Suppose that $f\colon [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\varphi\colon [a,b]\times\mathbb{R}\to[0,\infty)$ are continuous functions such that*

$$|f(x,y,z)| \le \varphi(x,y)$$

for all $x \in [a,b]$ and $y,z \in \mathbb{R}$. Then

(a) *$I_{f,p}\colon W^{1,p}[a,b]\to\mathbb{R}$ given by (6.1) is continuous,*
(b) *$I_{f,p}$ possesses a minimum on $W^{1,p}[a,b]$ if and only if there exists a convergent minimizing sequence for $I_{f,p}$.*

Proof. (a) Let $u_n \to u$ in $W^{1,p}[a,b]$. Then $u_n' \to u'$ in $L^p(a,b)$. By Theorem 3.16 and Conclusion 3.15, for every subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{u_{n_{k_j}}\}_{j\in\mathbb{N}}$ such that $u'_{n_{k_j}}(x) \to u'(x)$ for a.e. $x \in (a,b)$. From Facts 5.4 and 5.5 it follows that $u_n \to u$ uniformly on $[a,b]$. From this, there is $K > 0$ such that $|u_n(x)| \le K$ for all $x \in [a,b]$ and $n \in \mathbb{N}$. A function φ restricted to $[a,b]\times[-K,K]$ is uniformly continuous, which implies the existence of a constant $L > 0$ such that $\varphi(x,y) \le L$ for all $x \in [a,b]$ and $y \in [-K,K]$.

We have

$$|f(x,u_n(x),u_n'(x))| \le \varphi(x,u_n(x)) \le L$$

for all $x \in [a,b]$ and $n \in \mathbb{N}$. Moreover,

$$f(x,u_{n_{k_j}}(x),u'_{n_{k_j}}(x)) \to f(x,u(x),u'(x))$$

a.e. on (a,b). Using the Lebesgue theorem on dominated convergence, we get

$$\lim_{j\to\infty} I_{f,p}(u_{n_{k_j}}) = I_{f,p}(u).$$

Summarizing. For every subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$, there is a subsequence $\{u_{n_{k_j}}\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} I_{f,p}(u_{n_{k_j}}) = I_{f,p}(u)$. Therefore

$$\lim_{n\to\infty} I_{f,p}(u_n) = I_{f,p}(u).$$

Thesis (b) is now an immediate consequence of Theorem 2.4. $\square$

Theorem 6.3. *Let $p \in (1,\infty)$. If $f\colon [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\varphi\colon [a,b]\times\mathbb{R}\to[0,\infty)$ are continuous functions such that*

(a) *$|f(x,y,z)| \le \varphi(x,y)$ for all $x \in [a,b]$ and $y,z \in \mathbb{R}$,*
(b) *there is $\alpha > 0$ such that $f(x,y,z) \ge \alpha(|y|^p + |z|^p)$ for all $x \in [a,b]$ and $y,z \in \mathbb{R}$,*
(c) *for all $x \in [a,b]$, $f(x,\cdot,\cdot)\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a convex function,*

then $I_{f,p}\colon W^{1,p}[a,b]\to\mathbb{R}$ given by (6.1) *has a minimum.*

Proof. From Theorem 6.2 it follows that $I_{f,p}$ is continuous. By assumption, for $u\in W^{1,p}[a,b]$ we get

$$I_{f,p}(u)\geq \alpha\int_a^b(|u(x)|^p+|u'(x)|^p)\,dx=\alpha\|u\|^p_{W^{1,p}},$$

which implies that $I_{f,p}$ is bounded from below and coercive. Furthermore, for all $u,v\in W^{1,p}[a,b]$ and $t\in(0,1)$, we have

$$\begin{aligned}I_{f,p}(tu+(1-t)v)&=\int_a^b f(x,tu(x)+(1-t)v(x),tu'(x)+(1-t)v'(x))\,dx\\&\leq t\int_a^b f(x,u(x),u'(x))dx+(1-t)\int_a^b f(x,v(x),v'(x))\,dx\\&=tI_{f,p}(u)+(1-t)I_{f,p}(v).\end{aligned}$$

Thus $I_{f,p}$ is convex. Using Theorem 2.8, more precisely (H3), we receive the claim. □

Theorem 6.4. *Assume that $g\colon[a,b]\times\mathbb{R}\to\mathbb{R}$ and $\varphi\colon[a,b]\times\mathbb{R}\to[0,\infty)$ are continuous functions that satisfy the following conditions:*

(a) *$g(x,\cdot\,)\colon\mathbb{R}\to\mathbb{R}$ is convex and differentiable for each $x\in[a,b]$,*
(b) *$|(\partial g/\partial y)(x,y)|\leq\varphi(x,y)$ for all $x\in[a,b]$ and $y\in\mathbb{R}$.*

Then

(i) *$I_{g,\infty}\colon W^{1,\infty}[a,b]\to\mathbb{R}$ given by*

$$I_{g,\infty}(u)=\int_a^b g(x,u'(x))dx$$

is G-differentiable,

(ii) *$I_{g,\infty}$ achieves a minimum at a point $u\in W^{1,\infty}[a,b]$ if and only if u is a critical point of $I_{g,\infty}$.*

Proof. (i) Fix $u,v\in W^{1,\infty}[a,b]$. Set $K=\|u'\|_{L^\infty}$ and $M=\|v'\|_{L^\infty}$. Define $Z=\{x\in(a,b)\colon|u'(x)|>K$ or $|v'(x)|>M\}$. The measure of Z is equal to 0. Let $\{t_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ be a sequence such that $t_n\neq 0$ for every $n\in\mathbb{N}$ and $t_n\to 0$. Then there is $L>0$ such that $|t_n|\leq L$ for every $n\in\mathbb{N}$. A function φ restricted to $[a,b]\times[-K-LM,K+LM]$ is uniformly continuous. Therefore there exists $C>0$ such that $\varphi(x,y)\leq C$ for all $x\in[a,b]$ and $y\in[-K-LM,K+LM]$.

Take $x\in(a,b)\setminus Z$. If $v'(x)=0$, then

$$\lim_{n\to\infty}\frac{g(x,u'(x)+t_nv'(x))-g(x,u'(x))}{t_n}=0$$

and

$$\frac{g(x,u'(x)+t_nv'(x))-g(x,u'(x))}{t_n}=\frac{\partial g}{\partial y}(x,u'(x)+cv'(x))v'(x)$$

for all $n \in \mathbb{N}$ and $c \in \mathbb{R}$. Consider now the case $v'(x) \neq 0$. Since $g(x, \cdot\,)$ is differentiable, we have

$$\lim_{n\to\infty} \frac{g(x, u'(x) + t_n v'(x)) - g(x, u'(x))}{t_n} = \frac{\partial g}{\partial y}(x, u'(x))v'(x).$$

By the Lagrange theorem, for $n \in \mathbb{N}$ there is $c_n \in \mathbb{R}$ such that $0 < |c_n| < |t_n|$ and

$$\frac{g(x, u'(x) + t_n v'(x)) - g(x, u'(x))}{t_n} = \frac{\partial g}{\partial y}(x, u'(x) + c_n v'(x))v'(x).$$

Moreover, $|u'(x)+c_n v'(x)| \leq |u'(x)|+|c_n||v'(x)| \leq K+LM$, and in consequence,

$$\begin{aligned}\left|\frac{g(x, u'(x) + t_n v'(x)) - g(x, u'(x))}{t_n}\right| &\leq \left|\frac{\partial g}{\partial y}(x, u'(x) + c_n v'(x))\right||v'(x)| \\ &\leq \varphi(x, u'(x) + c_n v'(x)) \cdot M \leq CM\end{aligned}$$

for all $n \in \mathbb{N}$. Using the Lebesgue theorem on dominated convergence, we get

$$\lim_{n\to\infty} \frac{I_{g,\infty}(u + t_n v) - I_{g,\infty}(u)}{t_n} = \int_a^b \frac{\partial g}{\partial y}(x, u'(x))v'(x)\,dx.$$

Thus, by definition,

$$\lim_{t\to 0} \frac{I_{g,\infty}(u + tv) - I_{g,\infty}(u)}{t} = \int_a^b \frac{\partial g}{\partial y}(x, u'(x))v'(x)\,dx.$$

For every $u \in W^{1,\infty}[a, b]$, let $I'_{g,\infty}(u)\colon W^{1,\infty}[a, b] \to \mathbb{R}$ be given by

$$I'_{g,\infty}(u)v = \int_a^b \frac{\partial g}{\partial y}(x, u'(x))v'(x)\,dx.$$

We have to show that $I'_{g,\infty}$ is linear and continuous.

For each $v \in W^{1,\infty}[a, b]$, we obtain

$$\begin{aligned}|I'_{g,\infty}(u)v| &\leq \int_a^b \left|\frac{\partial g}{\partial y}(x, u'(x))\right||v'(x)|\,dx \leq \int_a^b \varphi(x, u'(x))|v'(x)|\,dx \\ &\leq \int_a^b C_1\|v'\|_{L^\infty}\,dx \leq C_1(b-a)\|v\|_{W^{1,\infty}},\end{aligned}$$

where $C_1 = \max\{\varphi(x, y) : x \in [a, b],\ |y| \leq \|u'\|_{L^\infty}\}$. Furthermore,

$$\begin{aligned}I'_{g,\infty}(u)(\alpha w + \beta v) &= \int_a^b \frac{\partial g}{\partial y}(x, u'(x))(\alpha w'(x) + \beta v'(x))\,dx \\ &= \alpha I'_{g,\infty}(u)w + \beta I'_{g,\infty}(u)v\end{aligned}$$

for all $w, v \in W^{1,\infty}[a, b]$ and $\alpha, \beta \in \mathbb{R}$. Hence $I'_{g,\infty}(u)$ is the Gâteaux derivative. Since u is arbitrary in $W^{1,\infty}[a, b]$, $I_{g,\infty}$ is G-differentiable.

By the convexity of g with respect to the second variable, we deduce that $I_{g,\infty}$ is convex. From Theorem 2.10, (ii) holds. $\square$

7. Minimization of $(*)$-weakly lower semi-continuous functionals in $W^{1,\infty}[a,b]$

Definition 7.1. A functional $I: W^{1,\infty}[a,b] \to \mathbb{R}$ is said to be $(*)$-weakly lower semi-continuous at a point $u \in W^{1,\infty}[a,b]$ if for each $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\infty}[a,b]$,

$$u_n \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}[a,b] \Rightarrow \liminf_{n\to\infty} I(u_n) \geq I(u).$$

We will say that $I: W^{1,\infty}[a,b] \to \mathbb{R}$ is $(*)$-weakly lower semi-continuous if it is $(*)$-weakly lower semi-continuous in every $u \in W^{1,\infty}[a,b]$ and for abbreviation, we will write I is $(*)$-wlsc.

Let us remark that if $I: W^{1,\infty}[a,b] \to \mathbb{R}$ is $(*)$-wlsc then it is wlsc. It is an immediate consequence of Conclusion 5.12.

Theorem 7.2. *Assume that $I: W^{1,\infty}[a,b] \to \mathbb{R}$ is $(*)$-wlsc. Then I possesses a minimum on $W^{1,\infty}[a,b]$ if and only if there exists a $(*)$-weakly convergent minimizing sequence for I.*

Proof. $(\Rightarrow)$ By assumption, there is $u \in W^{1,\infty}[a,b]$ such that

$$I(u) = \inf\{I(v) : v \in W^{1,\infty}[a,b]\}.$$

Let $u_n = u$ for each $n \in \mathbb{N}$. Then $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$ and $\lim_{n\to\infty} I(u_n) = I(u)$.

$(\Leftarrow)$ Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,\infty}[a,b]$ be a sequence such that $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$ and

$$\lim_{n\to\infty} I(u_n) = \inf\{I(v) : v \in W^{1,\infty}[a,b]\}.$$

Since I is $(*)$-wlsc, we get

$$\liminf_{n\to\infty} I(u_n) \geq I(u).$$

Consequently,

$$I(u) \geq \inf\{I(v) : v \in W^{1,\infty}[a,b]\} = \lim_{n\to\infty} I(u_n) = \liminf_{n\to\infty} I(u_n) \geq I(u),$$

and hence $I(u) = \inf\{I(v) : v \in W^{1,\infty}[a,b]\}$. □

Let $I_{g,\infty}: W^{1,\infty}[a,b] \to \mathbb{R}$ be given by

$$I_{g,\infty}(u) = \int_a^b g(x, u'(x))\, dx, \tag{7.1}$$

where $g: [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Theorem 7.3. *$I_{g,\infty}$ given by* (7.1) *is a $(*)$-wlsc functional if and only if g is convex with respect to the second variable.*

The proof of this theorem will be divided into a sequence of lemmas.

Lemma 7.4. *Let $\Delta \subset \mathbb{R}$ be an interval and $h\colon \Delta \to \mathbb{R}$ be a continuous function. If $\{A_n\}_{n\in\mathbb{N}}$ is a monotonically decreasing sequence of bounded subintervals of Δ and there exists $x_0 \in \Delta$ such that $\bigcap_{n=1}^{\infty} A_n = \{x_0\}$, then*

$$\lim_{n\to\infty} \frac{\int_{A_n} h(x)\,dx}{\mu(A_n)} = h(x_0).$$

Proof. Fix $\varepsilon > 0$. By the continuity of h, there is $\delta > 0$ such that for every $x \in \Delta$, if $|x - x_0| < \delta$ then $|h(x) - h(x_0)| < \varepsilon$. Since $A_{n+1} \subset A_n$ for $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} A_n = \{x_0\}$, there is $N \in \mathbb{N}$ such that $\mu(A_N) < \delta$. Take $n \geq N$. Then $\mu(A_n) \leq \mu(A_N) < \delta$, and hence $|x - x_0| < \delta$ for $x \in A_n$. In consequence,

$$\begin{aligned}\left|\frac{\int_{A_n} h(x)\,dx}{\mu(A_n)} - h(x_0)\right| &= \left|\frac{\int_{A_n} h(x)\,dx - h(x_0)\mu(A_n)}{\mu(A_n)}\right| \\ &= \left|\frac{\int_{A_n} (h(x) - h(x_0))\,dx}{\mu(A_n)}\right| \leq \frac{\int_{A_n} |h(x) - h(x_0)|dx}{\mu(A_n)} < \varepsilon,\end{aligned}$$

which completes the proof. □

Put $W_0^{1,\infty}[c,d] = \{u \in W^{1,\infty}[c,d] : u(c) = u(d) = 0\}$. By Fact 5.5 we conclude that $W_0^{1,\infty}[c,d]$ is a closed subspace of $W^{1,\infty}[c,d]$, and so it is a Banach space.

Lemma 7.5. *If $I_{g,\infty}\colon W^{1,\infty}[a,b] \to \mathbb{R}$ given by* (7.1) *is a* $(*)$*-wlsc functional then for every $[c,d] \subset [a,b]$, $x_0 \in [a,b]$, $z_0 \in \mathbb{R}$ and $\varphi \in W_0^{1,\infty}[c,d]$ it holds*

$$\frac{1}{d-c}\int_c^d g(x_0, z_0 + \varphi'(y))\,dy \geq g(x_0, z_0).$$

The Outline of the Proof. Let $[c,d] \subset [a,b]$. Take $x_0 \in [a,b]$, $z_0 \in \mathbb{R}$ and $\varphi \in W_0^{1,\infty}[c,d]$.

Step 1. Set $T = d - c$. We define $\widetilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$ to be a T-periodic extension of φ over $\mathbb{R}$. Set

$$m = \begin{cases} \min\{k \in \mathbb{N} : [x_0, x_0 + T/k] \subset [a,b]\} & \text{if } x_0 \in [a,b), \\ \min\{k \in \mathbb{N} : [b - T/k, b] \subset [a,b]\} & \text{if } x_0 = b. \end{cases}$$

We will restrict our attention to the case where $x_0 \in [a,b)$. For $x_0 = b$ the proof is similar.

Fix $k \geq m$. Define

$$\varphi_{n,k}(x) = \begin{cases} \dfrac{1}{nk}(\widetilde{\varphi}(nk(x - x_0)) - \widetilde{\varphi}(0)) & \text{if } x \in [x_0, x_0 + T/k], \\ 0 & \text{if } x \in [a,b] \setminus [x_0, x_0 + T/k], \end{cases}$$

where $n \in \mathbb{N}$. Then $\varphi_{n,k} \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}[a,b]$, as $n \to \infty$.

Step 2. Let $v\colon[a,b]\to\mathbb{R}$ be a function given by $v(x)=z_0(x-x_0)$. Set $v_{n,k}=v+\varphi_{n,k}$, where $n\in\mathbb{N}$. Since $\varphi_{n,k}\overset{*}{\rightharpoonup}0$ in $W^{1,\infty}[a,b]$, we get $v_{n,k}\overset{*}{\rightharpoonup}v$ in $W^{1,\infty}[a,b]$. By assumption,

$$\liminf_{n\to\infty} I_{g,\infty}(v_{n,k})\geq I_{g,\infty}(v). \tag{7.2}$$

Let $x_j=x_0+jT/nk$ for $j=0,\dots,n$ and $n\in\mathbb{N}$. Then

$$[x_0,x_0+T/k]=\bigcup_{j=0}^{n-1}[x_j,x_{j+1}]$$

and

$$\begin{aligned}I_{g,\infty}(v_{n,k})=\int_{[a,b]\setminus[x_0,x_0+T/k]} g(x,z_0)\,dx+\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} g(x_j,v'_{n,k}(x))\,dx\\+\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}(g(x,v'_{n,k}(x))-g(x_j,v'_{n,k}(x)))\,dx.\end{aligned}$$

One can check that

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} g(x_j,v'_{n,k}(x))\,dx=\frac{1}{T}\int_{x_0}^{x_0+T/k}\int_c^d g(x,z_0+\varphi'(y))\,dy\,dx$$

and

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}(g(x,v'_{n,k}(x))-g(x_j,v'_{n,k}(x)))\,dx=0.$$

In consequence, we receive

$$\begin{aligned}\lim_{n\to\infty} I_{g,\infty}(v_{n,k})=\int_{[a,b]\setminus[x_0,x_0+T/k]} g(x,z_0)\,dx\\+\frac{1}{T}\int_{x_0}^{x_0+T/k}\int_c^d g(x,z_0+\varphi'(y))\,dy\,dx.\end{aligned}$$

Combining this with (7.2), we get

$$\frac{1}{T}\int_c^d\int_{x_0}^{x_0+T/k} g(x,z_0+\varphi'(y))\,dx\,dy\geq\int_{x_0}^{x_0+T/k} g(x,z_0)\,dx.$$

By the arbitrary of $k\geq m$, we have

$$\frac{1}{T}\int_c^d\frac{k}{T}\int_{x_0}^{x_0+T/k} g(x,z_0+\varphi'(y))\,dx\,dy\geq\frac{k}{T}\int_{x_0}^{x_0+T/k} g(x,z_0)\,dx \tag{7.3}$$

for each $k\geq m$.

Step 3. For each $k \geq m$, we have $[x_0, x_0 + T/(k+1)] \subset [x_0, x_0 + T/k] \subset [a, b]$ and $\bigcap_{k=m}^{\infty} [x_0, x_0 + T/k] = \{x_0\}$. By assumption, $g(\,\cdot\,, z_0 + \varphi'(y))$ for a.e. $y \in [c, d]$ and $g(\,\cdot\,, z_0)$ are continuous. From Lemma 7.4 it follows that

$$\lim_{k\to\infty} \frac{k}{T} \int_{x_0}^{x_0+T/k} g(x, z_0 + \varphi'(y))\, dx = g(x_0, z_0 + \varphi'(y))$$

for a.e. $y \in [c, d]$ and

$$\lim_{k\to\infty} \frac{k}{T} \int_{x_0}^{x_0+T/k} g(x, z_0)\, dx = g(x_0, z_0).$$

There is $L > 0$ such that for all $x \in [a, b]$ and $|z| < |z_0| + \|\varphi'\|_{L^\infty}$ we have $|g(x, z)| \leq L$. Hence

$$\left| \frac{k}{T} \int_{x_0}^{x_0+T/k} g(x, z_0 + \varphi'(y))\, dx \right| \leq L$$

for all $k \geq m$ and for a.e. $y \in [c, d]$. In consequence, applying the Lebesgue theorem on dominated convergence to (7.3), we get

$$\frac{1}{T} \int_c^d g(x_0, z_0 + \varphi'(y))\, dy \geq g(x_0, z_0),$$

which finishes the proof. $\square$

Lemma 7.6. *If $I_{g,\infty}\colon W^{1,\infty}[a, b] \to \mathbb{R}$ given by* (7.1) *is a* $(*)$*-wlsc functional then $g\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ is convex with respect to the second variable.*

Proof. Fix $\overline{x} \in [a, b]$. Take $t \in (0, 1)$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq \beta$. Let $[c, d] \subset [a, b]$. We divide the integral $[c, d]$ into 2^N subintervals as follows: $[c, d] = \bigcup_{j=0}^{2^N-1} [x_j, x_{j+1}]$, where $x_0 = c$ and $x_j = x_0 + j(d - c)/2^N$ for $j = 1, \dots, 2^N$. Next, we divide each interval (x_j, x_{j+1}) into two. Namely, $(x_j, x_{j+1}) = (x_j, t_j] \cup (t_j, x_{j+1})$, where $t_j - x_j = t(d - c)/2^N$ and $x_{j+1} - t_j = (1 - t)(d - c)/2^N$. Define

$$I_N = \bigcup_{j=0}^{2^N-1} (x_j, t_j) \quad \text{and} \quad J_N = \bigcup_{j=0}^{2^N-1} (t_j, x_{j+1}).$$

Then $\mu(I_N) = t(d - c)$ and $\mu(J_N) = (1 - t)(d - c)$.

Let $\varphi\colon [c, d] \to \mathbb{R}$ be given by

$$\varphi(x) = \begin{cases} (1 - t)(\alpha - \beta)(x - x_j) & \text{if } x \in [x_j, t_j), \\ -t(\alpha - \beta)(x - x_{j+1}) & \text{if } x \in [t_j, x_{j+1}], \end{cases}$$

where $j = 0, \dots, 2^N - 1$. We have

$$\begin{aligned}\lim_{x \to t_j^-} \varphi(x) &= \lim_{x \to t_j^-} (1-t)(\alpha-\beta)(x-x_j) = (1-t)(\alpha-\beta)(t_j - x_j) \\ &= \frac{(1-t)(\alpha-\beta)t(d-c)}{2^N} = t(\alpha-\beta)(x_{j+1}-t_j) \\ &= -t(\alpha-\beta)(t_j - x_{j+1}) = \varphi(t_j).\end{aligned}$$

Thus φ is continuous at t_j. Since $\varphi|_{[x_j,t_j]}$ and $\varphi|_{[t_j,x_{j+1}]}$ are linear, we conclude that φ is absolutely continuous. Moreover, φ is differentiable a.e. on (c,d),

$$\varphi'(x) = \begin{cases} (1-t)(\alpha-\beta) & \text{if } x \in I_N, \\ -t(\alpha-\beta) & \text{if } x \in J_N \end{cases}$$

and $|\varphi'(x)| \le |\alpha - \beta|$ for $x \in I_N \cup J_N$. Hence $\varphi \in W^{1,\infty}[c,d]$. Finally, by definition, $\varphi(c) = \varphi(x_0) = (1-t)(\alpha-\beta)(x_0 - x_0) = 0$ and $\varphi(d) = \varphi(x_{2^N}) = -t(\alpha-\beta)(x_{2^N} - x_{2^N}) = 0$. In consequence, $\varphi \in W_0^{1,\infty}[c,d]$.

Set $z_0 = t\alpha + (1-t)\beta$. By Lemma 7.5, we get

$$\frac{1}{d-c}\int_c^d g(\overline{x}, z_0 + \varphi'(y))\,dy \ge g(\overline{x}, z_0).$$

From this,

$$\begin{aligned}\int_{I_N} g(\overline{x},\alpha)\,dy + \int_{J_N} g(\overline{x},\beta)\,dy &\ge (d-c)g(\overline{x}, z_0), \\ \mu(I_N)g(\overline{x},\alpha) + \mu(J_N)g(\overline{x},\beta) &\ge (d-c)g(\overline{x}, z_0), \\ t(d-c)g(\overline{x},\alpha) + (1-t)(d-c)g(\overline{x},\beta) &\ge (d-c)g(\overline{x}, z_0), \\ tg(\overline{x},\alpha) + (1-t)g(\overline{x},\beta) &\ge g(\overline{x}, t\alpha + (1-t)\beta).\end{aligned}$$

Thus $g(\overline{x}, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is a convex function. By the arbitrary of $\overline{x} \in [a,b]$, g is convex with respect to the second variable. □

Lemma 7.7. *Let $u_n \overset{*}{\rightharpoonup} u_0$ in $W^{1,\infty}[a,b]$. Then*

(a) *for each $p > 1$ there exists a sequence of non-negative numbers $\{\lambda_i^n\}_{i \le n}^{n \in \mathbb{N}}$ such that*

$$\sum_{i=1}^n \lambda_i^n = 1 \quad \textit{for } n \in \mathbb{N}, \quad v_n = \sum_{i=1}^n \lambda_i^n u_i \to u_0 \quad \textit{and} \quad v_n' \to u_0' \quad \textit{in } L^p(a,b),$$

(b) *there is a sequence of non-negative numbers $\{\gamma_i^n\}_{i \le n}^{n \in \mathbb{N}}$ such that*

$$\sum_{i=1}^n \gamma_i^n = 1 \quad \textit{for } n \in \mathbb{N}, \quad w_n = \sum_{i=1}^n \gamma_i^n u_i \to u_0 \quad \textit{and} \quad w_n' \to u_0' \quad \textit{a.e. on } (a,b).$$

Proof. (a) Fix $p > 1$. Let q be a real number such that $1/p + 1/q = 1$. Then $q > 1$. Let $F\colon L^p(a,b) \times L^p(a,b) \to \mathbb{R}$ be given by

$$F(u,v) = \int_a^b g_1(x)u(x)\,dx + \int_a^b g_2(x)v(x)\,dx,$$

where $g_1, g_2 \in L^q(a,b)$. Since $u_n \overset{*}{\rightharpoonup} u_0$ in $W^{1,\infty}[a,b]$, we have $F(u_n, u_n') \to F(u_0, u_0')$ in $\mathbb{R}$, and consequently, $(u_n, u_n') \rightharpoonup (u_0, u_0')$ in $L^p(a,b) \times L^p(a,b)$. By the Mazur theorem, there exists a sequence of non-negative numbers $\{\lambda_i^n\}_{i\le n}^{n\in\mathbb{N}}$ such that $\sum_{i=1}^n \lambda_i^n = 1$ for $n \in \mathbb{N}$, $v_n = \sum_{i=1}^n \lambda_i^n u_i \to u_0$ and $v_n' \to u_0'$ in $L^p(a,b)$.

(b) From the above it follows that there exists a sequence of non-negative numbers $\{\lambda_i^n\}_{i\le n}^{n\in\mathbb{N}}$ such that $\sum_{i=1}^n \lambda_i^n = 1$ for $n \in \mathbb{N}$, $v_n = \sum_{i=1}^n \lambda_i^n u_i \to u_0$ and $v_n' \to u_0'$ in $L^2(a,b)$. By Theorem 3.16, there are subsequences $\{v_{n_k}\}_{k\in\mathbb{N}} \subset \{v_n\}_{n\in\mathbb{N}}$ and $\{v'_{n_{k_l}}\}_{l\in\mathbb{N}} \subset \{v'_{n_k}\}_{k\in\mathbb{N}}$ such that $v_{n_k} \to u_0$ and $v'_{n_{k_l}} \to u_0'$ a.e. on (a,b). Define

$$\gamma_i^n = \begin{cases} 0 & \text{if } 1 \le i < n, \\ 1 & \text{if } i = n, \end{cases} \qquad \text{for } n < n_{k_1}$$

and

$$\gamma_i^n = \begin{cases} 0 & \text{if } i > n_{k_l}, \\ \lambda_i^{n_{k_l}} & \text{if } 1 \le i \le n_{k_l}, \end{cases} \qquad \text{for } n_{k_l} \le n < n_{k_{l+1}}.$$

Then $\sum_{i=1}^n \gamma_i^n = \gamma_n^n = 1$ for $n < n_{k_1}$ and $\sum_{i=1}^n \gamma_i^n = \sum_{i=1}^{n_{k_l}} \lambda_i^{n_{k_l}} = 1$ for $n_{k_l} \le n < n_{k_{l+1}}$. Furthermore, $w_n = \sum_{i=1}^n \gamma_i^n u_i \to u_0$ and $w_n' \to u_0'$ a.e. on (a,b), because $w_n = \sum_{i=1}^n \gamma_i^n u_i = \sum_{i=1}^{n_{k_l}} \lambda_i^{n_{k_l}} u_i = v_{n_{k_l}}$ for $n_{k_l} \le n < n_{k_{l+1}}$. ☐

Lemma 7.8. *If $I_{g,\infty}\colon W^{1,\infty}[a,b] \to \mathbb{R}$ defined by (7.1) is convex then it is also $(*)$-wlsc.*

Proof. Let $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$. Without loss of generality we can assume that $\liminf_{n\to\infty} I_{g,\infty}(u_n) = \lim_{n\to\infty} I_{g,\infty}(u_n)$. Set

$$L = \lim_{n\to\infty} I_{g,\infty}(u_n).$$

We have to consider three cases.

Case 1. $L = \infty$. Then $L \ge I_{g,\infty}(u)$.

Case 2. $L = -\infty$. Fix $m \in \mathbb{N}$. Then there exists $N \in \mathbb{N}$ such that

$$I_{g,\infty}(u_n) < -m$$

for all $n \ge N$. By Lemma 7.7, there exists a sequence of non-negative numbers $\{\gamma_i^n\}_{i\le n}^{n\in\mathbb{N}}$ such that $\sum_{i=1}^n \gamma_i^n = 1$ for all $n \in \mathbb{N}$, $w_n = \sum_{i=1}^n \gamma_i^n u_{i+N} \to u$ and $w_n' \to u'$ a.e. on (a,b). By the continuity of g, we get

$$g(x, w_n'(x)) \to g(x, u'(x))$$

for a.e. $x \in (a,b)$. Since $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}[a,b]$, we have $u_n' \overset{*}{\rightharpoonup} u'$ in $L^\infty(a,b)$. From Fact 3.20 it follows that there exist a subset $B \subset (a,b)$ of measure 0 and a constant $M > 0$ such that $|u_n'(x)| \le M$ for all $x \in (a,b) \setminus B$ and $n \in \mathbb{N}$. In consequence, we get

$$|w_n'(x)| \le \sum_{i=1}^{n} \gamma_i^n |u_{i+N}'(x)| \le \sum_{i=1}^{n} \gamma_i^n M = M$$

for all $x \in (a,b) \setminus B$ and $n \in \mathbb{N}$. A function g restricted to $[a,b] \times [-M,M]$ is uniformly continuous. Therefore there is $K > 0$ such that $|g(x,y)| \le K$ for all $x \in [a,b]$ and $y \in [-M,M]$. Hence

$$|g(x,w_n'(x))| \le K$$

for all $x \in (a,b) \setminus B$ and $n \in \mathbb{N}$. Applying the Lebesgue theorem on dominated convergence, we receive

$$\lim_{n\to\infty} I_{g,\infty}(w_n) = \lim_{n\to\infty} \int_a^b g(x,w_n'(x))\,dx = \int_a^b g(x,u'(x))\,dx = I_{g,\infty}(u).$$

By the convexity of $I_{g,\infty}$, we have

$$I_{g,\infty}(w_n) = I_{g,\infty}\biggl(\sum_{i=1}^{n} \gamma_i^n u_{i+N}\biggr) \le \sum_{i=1}^{n} \gamma_i^n I_{g,\infty}(u_{i+N}) \le \sum_{i=1}^{n} \gamma_i^n (-m) = -m$$

for all $n \in \mathbb{N}$. Hence $I_{g,\infty}(u) \le -m$ for each $m \in \mathbb{N}$. Letting $m \to \infty$, we get $I_{g,\infty}(u) = -\infty$, a contradiction.

Case 3. $-\infty < L < \infty$. Fix $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that

$$L - \varepsilon < I_{g,\infty}(u_n) < L + \varepsilon$$

for all $n \ge N$. By Lemma 7.7, there exists a sequence of non-negative numbers $\{\gamma_i^n\}_{i\le n}^{n\in\mathbb{N}}$ such that $\sum_{i=1}^n \gamma_i^n = 1$ for all $n \in \mathbb{N}$, $w_n = \sum_{i=1}^n \gamma_i^n u_{i+N} \to u$ and $w_n' \to u'$ a.e. on (a,b). As in the 2nd case, we show that

$$\lim_{n\to\infty} I_{g,\infty}(w_n) = I_{g,\infty}(u).$$

Moreover, by the convexity of $I_{g,\infty}$, we have

$$I_{g,\infty}(w_n) \le L + \varepsilon$$

for all $n \in \mathbb{N}$. Hence $I_{g,\infty}(u) \le L + \varepsilon$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0^+$, we get $I_{g,\infty}(u) \le L$. $\square$

The Proof of Theorem 7.3. ($\Rightarrow$) It follows from Lemma 7.6.

($\Leftarrow$) We check at once that $I_{g,\infty}$ is convex. By Lemma 7.8, $I_{g,\infty}$ is $(*)$-wlsc. $\square$

Combining Lemma 7.8 with Lemma 7.6 we get the following conclusion.

Conclusion 7.9. $I_{g,\infty}\colon W^{1,\infty}[a,b] \to \mathbb{R}$ defined by (7.1) is convex if and only if g is convex with respect to the second variable.

The next conclusion is a consequence of Theorems 7.2 and 7.3.

Conclusion 7.10. Let $I_{g,\infty}\colon W^{1,\infty}[a,b] \to \mathbb{R}$ be given by (7.1). If $g\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is convex with respect to the second variable, then $I_{g,\infty}$ possesses a minimum on $W^{1,\infty}[a,b]$ if and only if there exists a $(*)$-weakly convergent minimizing sequence for $I_{g,\infty}$.

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STABILITY, ATTRACTION AND SHAPE: A TOPOLOGICAL STUDY OF FLOWS

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Abstract. This survey is an introduction to some methods from geometric topology which can be effectively applied to the study of dynamical systems. In particular, we consider applications to stable and unstable attractors, nonsaddle sets and bifurcations. We also discuss some recent developments and open problems

1. Introduction

The following is an exposition of several topics which lie at the heart of the topological theory of attractors. Historically, Algebraic Topology has been used in connection with this subject, and a relevant role has been played by the Čech and Alexander–Spanier cohomology theories. This is complemented by Borsuk's Shape Theory, which gives a more geometrical vision of the subject and is specially suited to transfer to the topological field many of the dynamical ideas related to stability and attraction. In the realm of continuous dynamical systems, the notion of attractor plays a very significant role because it captures the long term evolution of the system in question, and therefore it seems important to study the structure, both dynamical and topological, of these objects.

These notes originate from some lectures delivered by the author at a Winter School which took place at the Schauder Center for Nonlinear Studies in 2009. The content of the lectures and a considerable amount of the author's research

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owes much to the work of two outstanding Polish mathematicians, namely Karol Borsuk and Tadeusz Ważewski, which explains his feeling of obligation towards the organizers of the Winter School for their invitation.

Karol Borsuk is the creator of shape theory, which we shall use throughout the paper. There are several books and articles, listed below, that we recommend to get some acquaintance with this theory. However we include here, for the benefit of the reader, a very short and schematic presentation (as exposed by Kapitanski and Rodnianski in [44]).

Let X be a closed subset of an ANR M and Y a closed subset of an ANR N. Denote by $\mathbb{U}(X;M)$ (resp. $\mathbb{U}(Y;N)$) the set of all open neighbourhoods of X in M (resp. Y in N).

Let $\mathbf{f} = \{f: U \to V\}$ be a collection of continuous maps from the neighbourhoods $U \in \mathbb{U}(X;M)$ to $V \in \mathbb{U}(Y;N)$. We call $\mathbf{f}$ a mutation from X to Y if the following conditions are fulfilled:

(1) For every $V \in \mathbb{U}(Y;N)$ there exists (at least) a map $f: U \to V$ in $\mathbf{f}$.
(2) If $f: U \to V$ is in $\mathbf{f}$ then the restriction $f|U_1: U_1 \to V_1$ is also in $\mathbf{f}$ for every neighbourhood $U_1 \subset U$ and every neighbourhood $V_1 \supset V$.
(3) If the two maps $f, f': U \to V$ are in $\mathbf{f}$ then there exists a neighbourhood $U_1 \subset U$ such that the restrictions $f|U_1$ and $f'|U_1$ are homotopic.

An example of mutation is the *identity mutation* $\mathrm{id}_{\mathbb{U}(X;M)}$ consisting of the identity maps $i: U \to U$.

Composition of mutations $\mathbf{f} = \{f: U \to V\}$, $\mathbf{g} = \{g: V \to W\}$ from X to Y and from Y to Z, respectively, is defined in an straigthforward way. Two mutations $\mathbf{f} = \{f: U \to V\}$ and $\mathbf{f}' = \{f': U' \to V'\}$ (both from X to Y) are said to be *homotopic* if for every pair of maps $f: U \to V$ and $f': U' \to V$ belonging to $\mathbf{f}$ and $\mathbf{f}'$, respectively, there exists a neighbourhod $U_0 \in \mathbb{U}(X;M)$, $U_0 \subset U \cap U'$ such that $f|U_0 \simeq f'|U_0$. It is easy to see that homotopy of mutations is an equivalence relation.

Two metric spaces X and Y have the same *shape* if they can be embedded as closed sets in ANRs M and N in such a way that there exist mutations $\mathbf{f} = \{f: U \to V\}$ and $\mathbf{g} = \{g: V \to U\}$ such that the compositions $\mathbf{gf}$ and $\mathbf{fg}$ are homotopic to the identity mutations $\mathrm{id}_{\mathbb{U}(X;M)}$ and $\mathrm{id}_{\mathbb{U}(Y;N)}$, respectively.

- The notion of shape of sets depends neither on the ANRs they are embedded in nor on the particular embeddings.
- Spaces belonging to the same homotopy type have the same shape.
- ANRs have the same shape if and only if they have the same homotopy type.

Concerning dynamical systems, our setting will be that of a continuous flow φ defined on a metric space M. If K is a compact invariant set of the flow then its *region or basin of attraction* is defined by $\mathcal{A}(K) = \{x \in M \mid d(xt, K) \to 0$ when $t \to \infty\}$. We say that K is an *attractor* if $\mathcal{A}(K)$ is a neighbourhood of K in M. An invariant compactum K is said to be *stable* if given an arbitrary neighbourhod V of K in M there exists another neighbourhood $U \subset V$ such that $xt \in V$ whenever $x \in U$ and $t \geq 0$. Stable attractors are called *asymptotically stable* sets. Most of the flows that we shall consider are defined in locally compact metric spaces M and the following definitions are understood in such a context. Given any $x \in M$, the set

$$J^+(x) = \{y \in M \mid y = \lim x_n t_n \text{ for some } x_n \to x,\ t_n \to \infty\}$$

is called the *positive prolongational limit set* of x and it is easy to check that K is stable if, and only if, $J^+(x) \subset K$ for every $x \in K$. The sets $J^+(x)$ are always closed and invariant and, when compact, also connected (at least in locally compact phase spaces). If an attractor K is stable then $J^+(x) \subset K$ for every $x \in K$, but in fact much more is true since $J^+(x) \subset K$ for all $x \in \mathcal{A}(K)$. If we agree to call $x \in \mathcal{A}(K)$ an *explosion point* if $J^+(x) \nsubseteq K$ then an attractor K is unstable if and only if there exists some explosion point in K. When we consider unstable attractors, we shall be primarily interested in those which have only internal explosions, that is, such that every explosion point is in K.

Following Conley we shall deal often with *isolated invariant sets*. These are compact invariant sets K which possess a so-called isolating neighbourhood, that is, a compact neighbourhood N such that K is the maximal invariant set in N, or setting

$$N^+ = \{x \in N \mid x[0, +\infty) \subset N\}, \quad N^- = \{x \in N \mid x(-\infty, 0] \subset N\}$$

such that $K = N^+ \cap N^-$. We shall make use of a special type of isolating neighbourhoods, the so-called *isolating blocks*, which have good topological properties. More precisely, an isolating block N is an isolating neighbourhood such that there are compact sets $N^i; N^o \subset \partial N$, called the *entrance* and *exit* sets, satisfying:

(1) $\partial N = N^i \cup N^o$,
(2) for every $x \in N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset M - N$ and for every $x \in N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset M - N$,
(3) for every $x \in \partial N - N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset \operatorname{int} N$ and for every $x \in \partial N - N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset \operatorname{int} N$.

These blocks form a neighbourhood basis of K in M (see [13] and [14]).

A general reference for dynamical systems, which we shall follow closely, is [9]; the reference [70] will be also useful for some aspects of the theory. Conley's index theory can be found in his monograph [14], see also [15], [75] and [53]. The Conley shape index was defined in [68]. On the topological side, [39] gives complete information about ANR's, and shape theory is thoroughly exposed in [11], [16], [19], [51] (see also [63], [64] and [87]). Finally, should a complement on algebraic topology be needed, [37] and [98] cover everything used in this article. In the paper singular and Čech homology and cohomology are used throughout.

2. Asymptotically stable attractors. Topology and embeddings

We see in this section that the global topological properties of asymptotically stable attractors are largely determined by those of their basin of attraction. This is a consequence of the first result in the paper, which is a slightly generalized version of a theorem of Kapitanski and Rodnianski [44].

Theorem 2.1. *Let $\varphi\colon M \times \mathbb{R} \to M$ be a flow on a metric space M and let K be an asymptotically stable attractor of φ. Then the inclusion $i\colon K \to \mathcal{A}(K)$ is a shape equivalence.*

Proof. By the Kuratowski–Wojdislawski Theorem [39] it is possible to embed $\mathcal{A}(K)$ as a closed set into an ANR-space N. Let $L\colon \mathcal{A}(K) \to \mathbb{R}_+$ be a Lyapunov function for K such that $d(x,K) \leq L(x)$ for every $x \in \mathcal{A}(K)$. We recall that a Lyapunov function is a continuous mapping such that $L(xt) < L(x)$ for every $x \in \mathcal{A}(K) - K$ and every $t > 0$ and $L(x) = 0$ for $x \in K$ (the existence of L can be proved following the pattern of [9, Theorem 2.2], see also [22, Lemma 2.2]).

Consider a basis of open neighbourhoods V_n of K in N such that $V_{n+1} \subset V_n$ for every n and a sequence c_n of positive numbers such that $c_{n+1} < c_n \to 0$ and $L^{-1}([0,c_n]) \subset V_n$ for every n. If $L(x) \geq c_n$ then there exists an unique $t_n(x) \geq 0$ such that $L(\varphi(x,t_n(x))) = c_n$. We define

$$r_n\colon \mathcal{A}(M) \to V_n, \qquad r_n(x) = \begin{cases} \varphi(x,t_n(x)) & \text{if } L(x) \geq c_n, \\ x & \text{otherwise.} \end{cases}$$

We see that r_n is continuous.

Suppose on the contrary that $L(x) \geq c_n$ and that there exists a sequence $x_k \to x$ such that $\varphi(x_k,t_n(x_k))$ is at distance $\geq \varepsilon$ of $\varphi(x,t_n(x))$ for $k = 1,2$. It is easy to see that the sequence $t_n(x_k)$ is bounded and, thus, there is no loss of generality in assuming that $t_n(x_k)$ converges to a certain t_0, which obviously implies that $\varphi(x_k,t_n(x_k) \to \varphi(x,t_0)$ and hence $c_n = L(\varphi(x_k,t_n(x_k))) = L(\varphi(x,t_0))$. It follows that $t_n(x) = t_0$ and $r_n(x_k) \to r_n(x)$, which is in contradiction with the choice of the sequence x_k. This proves that r_n is continuous.

Now, if j denotes the inclusion $V_{n+1} \to V_n$, we have that $r_n \simeq jr_{n+1}$ in V_n rel. K (through the homotopy $\Phi(x,s) = \varphi(r_n(x), st_{n+1}(r_n(x)))$ if $L(x) > c_{n+1}$ and $\Phi(x,s) = x$ otherwise).

Extend now r_n to a map $\widehat{r}_n\colon \mathcal{A}(M) \subset U_n \to V_n$, where U_n is a neighbourhood of $\mathcal{A}(M)$ in N. We define a mutation $\mathbf{r}$ from $\mathcal{A}(K)$ to K (both of them lying in the same ANR, N) consisting of all the maps of the form

$$U \longrightarrow U_n \xrightarrow{\widehat{r}_n} V_n \longrightarrow V$$

where the unlabelled arrows are inclusions. It can be readily seen that the inclusion mutation $\mathbf{i}\colon K \to \mathcal{A}(K)$ is a homotopy inverse of $\mathbf{r}$. □

More general results than Theorem 2.1 can be found in [26] and [28]. In particular, a version of this theorem has been proved by A. Giraldo, M. A. Morón, F. R. Ruiz del Portal and J. M. R. Sanjurjo in [28] for Hausdorff topological spaces.

An important result that predated Theorem 2.1 can be obtained, however, as a consequence of this theorem. It has been proved (at different levels of generality) by Bogatyi-Gutsu [10], Günther-Segal [33] and Sanjurjo [88] and [89]. This result is specially useful from an operational perspective since it guarantees that all the homological and cohomological invariants of asymptotically stable attractors are of finite type.

Theorem 2.2. *Let $\varphi\colon M \times \mathbb{R} \to M$ be a flow on a (not necessarily locally compact) ANR, M and let K be an asymptotically stable attractor of φ. Then K is shape dominated by a finite polyhedron, the Čech homologies and cohomologies of K are finitely generated and $\check{H}_q$ and $\check{H}^q$ are trivial for all sufficiently large q. Moreover, if M is locally compact then K has the shape of a finite polyhedron.*

Proof. Since K has the shape of $\mathcal{A}(K)$ and $\mathcal{A}(K)$ (being an open subset of the ANR, M) is an ANR, we have that K is an FANR. Hence K is shape dominated by a finite polyhedron and, as a consequence, its homological and cohomological invariants are of finite type. If M is locally compact then it can be readily seen from the proof of the previous theorem that K is a shape deformation retract of any of the sets $L^{-1}[0,r]$. But for r sufficiently small $L^{-1}[0,r]$ (which is a retract of $\mathcal{A}(K)$) is a compact ANR and hence it has the homotopy type of a finite polyhedron [106]. $\square$

From Theorem 2.1 it can also be obtained the following result which was previously established by B. M. Garay [22].

Theorem 2.3. *Suppose K is an asymptotically stable global attractor of a flow in an infinite-dimensional Banach space M. Then K has the shape of a point. In particular K is strongly cellular and all its Čech homology and cohomology groups are trivial (except in dimension 0, that are isomorphic to $\mathbb{Z}$).*

Theorem is also true (and easier) when the Banach space M is finite-dimensional. However, in this case, strong cellularity is not an automatic consequence of shape triviality.

Under more stringent conditions we can get even stronger results. The following one, established by J. A. Langa and J. C. Robinson in [47], is interesting in the context of dissipative evolution equations in Hilbert spaces. See [47] for definitions of the notions involved in the statement and for a proof.

Theorem 2.4. *If an invariant exponential global attractor K of a flow in a Hilbert space H, is flow-normally hyperbolic then K is a retract of H. As a consequence K is homotopically trivial.*

Some results by W. M. Oliva give sufficient conditions (in terms of the existence of certain Lipschitz constants) for attractors of retarded functional differential equations on compact manifolds to be compact manifolds themselves (see [59] and [34]). Another interesting result concerning the topological properties that the phase space induces on an attractor of a discrete dynamical system has been obtained by M. A. Morón and F. R. Ruiz del Portal in [56].

Our aim now is to prove a celebrated result by B. Günther an J. Segal which provides sufficient conditions for a compactum to be embedded in the Euclidean space as an attractor of a flow. We start with an auxiliary result.

Lemma 2.5. *Suppose P is a compact polyhedron in $\mathbb{R}^n$. Then there is a flow φ on $\mathbb{R}^n$ such that P is an asymptotically stable attractor of φ.*

Proof. Consider a a regular neighbourhood N of P in $\mathbb{R}^n$ endowed with a triangulation such that P is a full subcomplex, N is the simplicial neighbourhood of P and $\dot{N}$ (the simplicial boundary of N) is a triangulation of ∂N (see C. P. Rourke and B. J. Sanderson [72] for definitions of these notions). If we denote by v_i the vertices of the simplexes in N then every point $x \in N$ can be uniquely expressed in the form

$$x = \sum_{v_i \notin P} x_i v_i + \sum_{v_j \in P} y_j v_j.$$

Consider the simplicial map $\pi\colon N \to [0,1]$ defined by $\pi(v_i) = 0$ if $v_i \in P$ and $\pi(v_i) = 1$ if $v_i \notin P$. Since N is a regular neighbourhood of P we have that $P = \pi^{-1}(0)$ and $\partial N = \pi^{-1}(1)$. Consider a flow ψ on $[0,1]$ such that $\{0,1\}$ is an attractor-repeller pair of ψ. Define $\varphi_0\colon N\times \mathbb{R} \to N$ by

$$(x,t) \to \begin{cases} \psi(\pi(x),t) \displaystyle\sum_{v_i \notin P} \frac{x_i}{\pi(x)} v_i & \\ \qquad\qquad + (1-\psi(\pi(x),t)) \displaystyle\sum_{v_j \in P} \frac{y_j}{1-\pi(x)} v_j & \text{if } \pi(x) \neq 0, \\ x & \text{otherwise.} \end{cases}$$

The mapping φ_0 can be extended to a mapping $\varphi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by letting stationary all points outside N. It can be easily seen that φ is a flow and since $\{0\}$ is an asymptotically stable attractor of ψ it follows that $P = \pi^{-1}(0)$ is an asimptotically stable attractor of φ. □

We shall prove now B. Günther and J. Segal's Theorem [33] on embeddings of finite-dimensional compacta as attractors of flows. Our proof is different from the original one. We adapt some ideas of J. J. Sánchez-Gabites in [81], where he shows how to use a theorem by H. Whitney ([107] and [108]) about recognition of flows in a similar context. The proof also makes use of T. A. Chapman's famous finite-dimensional complement theorem [12].

Theorem 2.6. *Suppose K is a finite-dimensional compactum with the shape of a finite polyhedron. Then K can be embedded in $\mathbb{R}^n$ for suitable n in such a way that there is a flow on $\mathbb{R}^n$ having K as an asymptotically stable attractor.*

Proof. Suppose K has the shape of the finite polyhedron P. Then by [12] we can assume that K and P lie in *standard position* in $\mathbb{R}^n$ with $n \geq 2 + 2\max(\dim X, \dim Y)$. We refer the reader to [12] for the definition of standard position and observe that any subpolyhedron of $\mathbb{R}^n$ is in standard position. By T. A. Chapman's finite-dimensional complement theorem [12] (see also the reformulation of this theorem given in [33] by B. Günther and J. Segal) there exists a homeomorphism $h\colon \mathbb{R}^n - P \to \mathbb{R}^n - K$ that can be extended to a homeomorphism $\widehat{h}\colon \mathbb{R}^n/P \approx \mathbb{R}^n/K$. By Lemma 2.5 there exists a flow φ in $\mathbb{R}^n$ having P as an attractor. Let $\mathcal{A}(P)$ be the basin of attraction of P and denote by U its image under h. Clearly $V = U \cup K$ is an open neighbourhood of K in $\mathbb{R}^n$.

Consider the partition of $\mathcal{A}(P) - P$ defined by the oriented trajectories of φ; namely $\mathcal{C}_1 = \{\varphi(p \times \mathbb{R}) \mid p \in \mathcal{A}(P) - P\}$. This partition is transformed by h in a partition $\mathcal{C}_2 = h(\mathcal{C}_1)$ of U. We enlarge $\mathcal{C}_2$ to a partition $\mathcal{D}_2$ of $\mathbb{R}^n$ by adding the singletons $\{\{q\}, q \in \mathbb{R}^n - U\}$. Since the trajectories of $\mathcal{C}_1$ lie in the basin of attraction of P all of them are homeomorphic images of $\mathbb{R}$ and the same is true of the curves in $\mathcal{C}_2$.

Now it is straigthforward to see that $\mathcal{C}_2$ is regular. Let us recall that a family of oriented curves $\mathcal{C}$ is regular if given an oriented arc $pq \subset \gamma \in \mathcal{C}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $p' \in \gamma' \in \mathcal{C}$ and $d(p, p') < \delta$ then there is a point $q' \in \gamma'$ such that the oriented arcs pq and $p'q'$ have a parameter distance less than ε (that is, there exist parametrizations $f\colon [0,1] \to pq$ and $f'\colon [0,1] \to p'q'$ such that $d(f(t), f'(t)) < \varepsilon$ for every $t \in [0,1]$). Now by Whitney's Theorem 27A in [107] there exists a flow ϕ in $\mathbb{R}^n$ whose oriented trajectories correspond to the elements of the partition $\mathcal{D}_2$. From this it is immediate to see that K is an attractor of ϕ such that $\mathcal{A}(K) = V$. □

A related result, according to which every strongly cellular subset of a Banach space M is a global attractor of a flow defined in M had been previously proved by B. M. Garay [22]. On the other hand, B. Günther [32] improved Theorem 2.6 by proving that the flow on $\mathbb{R}^n$ can be chosen to be of class $\mathcal{C}^r$ for every finite r. Recently, J. J. Sánchez-Gabites [81] has given necessary and sufficient conditions for a compact subset K of a 3-manifold M to be an attractor of a flow in M. Three-dimensional flows may exhibit very complicated behaviour, in particular they may have any number of coexisting strange attractors with strong properties of persistence. Moreover, infinitely many of them exist simultaneously (see the book [66] by A. Pumariño and J. A. Rodríguez).

We don't want to end this section without stating Hasting's theorem ([35] and [36]). He was the first to successfully apply the techniques of shape theory to dynamical systems. We leave the proof of this result to the reader. It can be obtained without much difficulty by using the ideas previously discussed.

Theorem 2.7. *Let $\varphi: \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ be a continuous semiflow such that there exists a compact n-manifold $P \subseteq \mathbb{R}^n$ with the property that every orbit through ∂P enters P (for increasing time) then there exists a compact invariant set $K \subseteq \operatorname{int}(P)$ which is positively asymptotically stable and such that the inclusion $K \hookrightarrow P$ is a shape equivalence.*

3. Non-saddle sets and unstable attractors

Asymptotically stable attractors are only a particular case of a large family of invariant compacta whose global topological structure is regular (by which is essentially meant that they have polyhedral shape). We devote this section to introducing this class of compacta, the non-saddle sets (which have been largely studied by N. P. Bhatia [8]). We also relate these compacta with some forms of unstable attractors which still retain many of the properties of the asymptotically stable ones.

Definition 3.1. Let $\varphi: M \times \mathbb{R} \to M$ be a flow. A compact set $K \subset M$ is said to be a saddle set if there is a neighbourhood U of K in M such that every neighbourhood V of K contains a point $x \in V$ such that $\gamma^+(x) \not\subseteq U$ and $\gamma^-(x) \not\subseteq U$. We say that K is non-saddle if it is not a saddle set, i.e. if for every neighbourhood U of K there exists a neighbourhood V of K such that for every $x \in V$, $\gamma^+(x) \subset U$ or $\gamma^-(x) \subset U$

In the rest of the paper we will assume, without further mention, that all non-saddle sets are invariant. Attractors and repellers are examples of non-saddle sets. The following result, proved by A. Giraldo, M. Morón, J. J. Ruiz del Portal and J. M. R. Sanjurjo [27] extends Theorem 2.2 to a larger context.

Theorem 3.2. *Let K be an isolated non-saddle set of a flow $\varphi: M \times \mathbb{R} \to M$, where M is a locally compact ANR. Then K has the shape of a finite polyhedron.*

Proof. Consider an isolating neighbourhood U for K in M. Since K is non-saddle there exists another neighbourhood $V \subset U$ of K with the property that for every $x \in V$ at least one of the semi-orbits $\gamma^+(x)$ or $\gamma^-(x)$ is contained in U. We define

$$N = \{x \in U \mid \gamma^+(x) \subset U \text{ or } \gamma^-(x) \subset U\}.$$

Since $V \subset N \subset U$, N is an isolating neighbourhood for K (the compactness is a consequence of the compactness of U). Moreover, N can be decomposed as $N = N^+ \cup N^-$ where

$$N^+ = \{x \in N \mid \gamma^+(x) \subset N\} \quad \text{and} \quad N^- = \{x \in N \mid \gamma^-(x) \subset N\}.$$

Observe that $N^+ \cap N^- = K$.

By [68] there exists a map $f: N \to \mathbb{R}$ such that $f(x) = 0$ if $x \in K$ and $f(xt) < f(x)$ if $x[0,t] \subset N - K$ and $t > 0$. Now $\omega(x) \subset K$ for every $x \in N^+$, hence $f(x) > 0$ if $x \in N^+ - K$. Similarly, $f(x) < 0$ if $x \in N^- - K$. If W is an arbitrary open neighbourhood with its closure contained in the interior of N then there is a $t_0 > 0$ such that $f^{-1}([-t_0, t_0]) \subset W$. Otherwise there would exist a sequence of points $(x_n) \subset N - W$ and a null sequence of positive numbers (t_n) such that $|f(x_n)| < t_n$. From this we could deduce the existence of a point $x_0 \in N - K$ with $f(x_0) = 0$, in contradiction with the previous remark. This implies in particular that $f^{-1}(-t_0, t_0)$ is open in M and that for any null sequence $(t_n)_{n \geq 0}$ starting in t_0, the sets $f^{-1}([-t_n, t_n])$ form a neighbourhood basis of K in M.

Consider any such null sequence $(t_n)_{n \geq 0}$ with $t_{n+1} < t_n$ for every n. We first construct a retraction

$$r: f^{-1}([-t_0, t_0]) \to f^{-1}([-t_1, t_1])$$

in the following way: if $x \in f^{-1}([-t_1, t_1])$ we define $r(x) = x$. On the other hand, if $x \in f^{-1}([-t_0, t_0]) - f^{-1}([-t_1, t_1])$ and $f(x) > t_1$ then $x \in N^+$ and there exists a unique $t_x > 0$ such that $f(xt_x) = t_1$. We define $r(x) = xt_x$. In a similar way, if $f(x) < -t_1$ we define $r(x) = xt_x$ where $t_x < 0$ is the unique negative number satisfying $f(xt_x) = -t_1$. A strong deformation retraction from $f^{-1}([-t_0, t_0])$ to $f^{-1}([-t_1, t_1])$ is given by the homotopy

$$\theta: f^{-1}([-t_0, t_0]) \times \mathbb{R} \to f^{-1}([-t_0, t_0])$$

defined as $\theta(x, s) = x(t_x s)$ if $f(x) \notin [-t_1, t_1]$ and $\theta(x, s) = x$ otherwise.

In an analogous way we may construct a strong deformation retraction from $f^{-1}([-t_1, t_1])$ to $f^{-1}([-t_2, t_2])$ and, in general, from $f^{-1}([-t_n, t_n]$ to $f^{-1}([-t_{n+1}, t_{n+1}]$ for every $n \in \mathbb{N}$.

Since the sets $f^{-1}([-t_n, t_n]$ form a neighbourhood basis of K in M we can define a strong shape deformation retraction from $f^{-1}([-t_1, t_1])$ to K. Therefore K has the shape of $f^{-1}([-t_1, t_1])$. But, since $f^{-1}(-t_0, t_0)$ is an open set of M, then it is an ANR, and since $f^{-1}([-t_1, t_1])$ is a retract of it then it is also an ANR. Therefore $f^{-1}([-t_1, t_1])$ has the homotopy type, and hence the shape, of a finite polyhedron [106]. $\square$

As a consequence of Theorems 2.6 and 3.2 we have the following

Corollary 3.3. *A finite dimensional compactum can be an isolated non-saddle set of a continuous flow on a manifold if and only if it has the shape of a finite polyhedron.*

Our following result (see [27]) shows that the topological condition of shape triviality has an important dynamical implication for isolated non-saddle sets.

Theorem 3.4. *Let K be an isolated non-saddle set of the flow $\varphi: M \times \mathbb{R} \to M$, where M is an $n-$manifold with $n > 1$. If K has trivial shape then K is an attractor or a repeller.*

Proof. Consider an isolating neighbourhood N for K in M such that $N = N^+ \cup N^-$, as in the proof of the previous theorem. Let U be a connected open neighbourhood of K in N. Then by a result in [17, p. 121] $U - K$ is still connected. Since $N^+ \cap (U - K)$ and $N^- \cap (U - K)$ are disjoint closed subsets of $U - K$ whose union is $U - K$ we deduce that either $N^+ \cap (U - K) = \emptyset$ or $N^- \cap (U - K) = \emptyset$. In the first case $N = N^-$ and K is a repeller while in the second case $N = N^+$ and K is an attractor. □

In some occasions saddle-sets can be detected by a cohomology criterion as the next result shows.

Theorem 3.5. *Let K be a connected isolated compactum. If there is a connected isolating block N such that $\check{H}^*(K) \neq \check{H}^*(N)$ then K is a saddle-set.*

Proof. Suppose on the contrary that K is non-saddle. We shall show that $N = N^+ \cup N^-$, for which it is sufficient to prove that $N^+ \cup N^-$ is open and closed in N. Otherwise there would exist a sequence of points $x_n \in N - (N^+ \cup N^-)$ converging to $x \in (N^+ \cup N^-) - K$ (x is not in K since we are assuming K is non-saddle). Suppose that $x \in N^+ - K$, the other case being similar. We obtain sequences of times $s_n \leq 0 \leq t_n$ with $x_n s_n \in N^i$, $x_n t_n \in N^o$ and $x_n[s_n, t_n] \subset N$. We can assume without loss of generality that $x_n s_n \to y \in N^i$, $x_n t_n \to z \in N^o$. It is easily seen that t_n is bounded (otherwise $z \in N^-$ and the trajectories of the points x_n would be arbitrarily close to K, which would be saddle). Then we can assume, without loss of generality, that $t_n \to t_0$ and $x_n t_n \to x t_0 = z$, but then $x t_0 \in N^o$, which is a contradiction with the fact that $x \in N^+$. Hence $N = N^+ \cup N^-$ and from the construction in the proof of Theorem 3.2 it can be easily seen that the inclusion $i: K \to N^+ \cup N^-$ is a shape equivalence, which implies that $\check{H}^*(K) = \check{H}^*(N)$, a contradiction. Hence K is saddle. □

We see now that when the exit set of an index pair for an isolated non-saddle set K has trivial shape, its shape index is completely determined by the shape of K.

Theorem 3.6. *Let K be a connected isolated non-saddle set ($K \neq \emptyset$) of a flow φ defined on locally compact metric ANR, M. Suppose that K admits an index pair (N, L) such that N is a connected isolating neighbourhood of K and L (the exit set) has trivial shape. Then $\mathrm{Sh}(K)$ agrees with the unpointed (Conley) shape index of K.*

Proof. We already know by the proof of the previous result that the inclusion $i: K \to N$ induces a shape equivalence. Since $\mathrm{Sh}(L)$ is trivial we have that the shape index $\mathrm{Sh}(N/L) = \mathrm{Sh}(N)$ (see [51]). □

Remark 3.7. Theorem is not true if K is saddle. Consider the example of the Hawaiian earring below.

In our search for topological regularity we could think that all isolated invariant sets have polyedral shape. However this is not the case, as the following example shows.

Example 3.8. Consider a dynamical system defined in the cylinder $D \times [0,1]$, where D stands for the unit disk: The points in the Hawaiian earring $H = \bigcup_{n=1}^{\infty} S((1/2n, 0, 1/2), 1/2n)$ are stationary points. All the points in $D \times \{0,1\}$ are also stationary. The orbits of the rest of the points are vertical straight lines joining two stationary points.

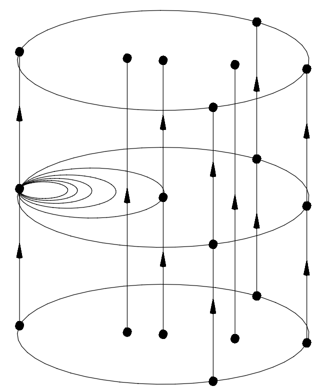

Figure 1

Then the set H is an isolated invariant set which does not have the shape of a finite polyhedron. Similar systems can be defined in $\mathbb{R}^3$, the sphere S^3 or in a solid torus.

In fact, the previous example is only a particular case of a very general situation (see [27]).

Theorem 3.9. *Any finite-dimensional compactum K can be embedded in $\mathbb{R}^n$, for suitable n, in such a way that there is a flow in $\mathbb{R}^n$ having K as an isolated invariant set.*

Proof. First, embed the compact set K as a subset of the diagonal of some $\mathbb{R}^{2n}$. Let φ be a translation flow on $\mathbb{R}^{2n}$, given by $\varphi((x,y),t) = (x, y+at)$ for some non-zero a. Then φ has no fixed points and all of the flow lines hit the diagonal in at most one point. By a theorem of Beck [7], φ can be modified to a new flow ϕ in such a way that all the orbits of φ not containing a point of K are preserved in ϕ while the orbits containing a point of K are decomposed in two orbits together with that point of K. Then K is an isolated invariant set for the flow ϕ. $\square$

An important example of non-saddle sets are the unstable attractors having only internal explosions. In fact, this condition is characteristic for this type of attractors.

Theorem 3.10. *Let K be an attractor of a flow on a locally compact metric space M. Then K has external explosions if and only if it is a saddle set.*

Proof. Suppose K has an external explosion in a point $x \in \mathcal{A}(K) - K$. Suppose $y \in J^+(x) - K$ and let U be a compact neighbourhood of K not containing either x or y. Consider an arbitrary neighbourhood V of K and take $t_0 > 0$ such that $xt_0 \in V$. Since there exist sequences $x_n \to x$, $t_n \to \infty$ such that $x_n t_n \to y$ we easily deduce that there is an $x_n \notin U$ such that $x_n t_0 \in V$ and $x_n t_n \notin U$. Hence K is saddle.

On the other hand if K is a saddle set consider a compact neighbourhod U of K such that there exists a sequence $x_n \to K$ and sequences $s_n \leq 0 \leq t_n$ with $x_n s_n$ and $x_n t_n$ belonging to ∂U and $x_n[s_n, t_n] \subset U$. We can assume that $x_n s_n \to y \in \partial U$, $x_n t_n \to z \in \partial U$ and $t_n - s_n \to \infty$ (otherwise z would belong to K by a simple argument). Then $z \in J^+(y)$ and the flow has an external explosion in y. □

Corollary 3.11. *Let K be an isolated unstable attractor having only internal explosions of a flow on a locally compact ANR, M. Then K has finite polyhedral shape.*

Unstable attractors having only internal explosions appear only in certain kind of manifolds (in surfaces like the torus for instance). We see that in the case of the plane, connected isolated global attractors are always stable and, on the other hand, in $\mathbb{R}^n$, isolated unstable attractors always have external explosions. The following result has been proved by M. A. Morón, J. J. Sánchez-Gabites and J. M. R. Sanjurjo in [57].

Theorem 3.12. *Every connected isolated global attractor K in $\mathbb{R}^2$ is stable.*

Proof. If K were unstable there would exist a point $x_0 \in \mathbb{R}^2 - K$ such that $\emptyset \neq \omega(x_0), \alpha(x_0) \subset K$ (see [9, Theorem 1.1, p. 114 and Corollary 1.2, p. 116]) and we can assume that x_0 lies in the unbounded component U of $\mathbb{R}^2 - K$ (if not, the argument is only sligthly different). Collapse K to a single point p and consider the flow $\widehat{\varphi}$ induced in the quotient space $\mathbb{R}^2/K$. Then $\{p\}$ is an isolated global attractor of $\widehat{\varphi}$ and $\overline{U} = U \cup \{p\}$ is homeomorphic to $\mathbb{R}^2$ (where the closure of U is taken in $\mathbb{R}^2/K$). This last assertion can be proved as follows: the set $K^* = \mathbb{R}^2 - U \supset K$ (equal to K plus the bounded components of $\mathbb{R}^2 - K$) does not disconnect the plane. Then $\mathcal{D} = \{K^*\} \cup \{\{x\} : x \notin K^*\}$ is un upper semicontinuous decomposition of $\mathbb{R}^2$ none of whose elements separates the plane, hence the quotient space $\mathbb{R}^2/K^* \cong \mathbb{R}^2/\mathcal{D}$ is homeomorphic to $\mathbb{R}^2$ by [54, Theorem 22]. But the closure of U in $\mathbb{R}^2/K$ is homemorphic to $\mathbb{R}^2/K^*$ so the asertion follows and we have reduced the proof to the case when K is a single point p.

In $\mathbb{R}^2/K$ the condition about the limit sets of x_0 says $\omega(x_0) = \alpha(x_0) = \{p\}$. This implies that $\gamma(x_0)$ is disjoint from its limits sets and homemomorphic to $\mathbb{R}$. But then $\overline{\gamma(x_0)} = \gamma(x_0) \cup \{p\}$ is a one-point compactification of $\gamma(x_0) \cong \mathbb{R}$, hence it must be homeomorphic to S^1. It follows that $\overline{\gamma(x_0)}$ separates $\overline{U}$ into two connected components, exactly one of which is bounded (say U_{x_0}), and with common boundary $\overline{\gamma(x_0)} = \gamma(x_0) \cup \{p\}$. Observe that U_{x_0} and $\overline{U}_{x_0}$ are invariant and homeomorphic to an open disk and a closed disk, respectively.

Now $x_0 \in J^+(p)$, so $x_0 \in \widehat{\{p\}}$ (the stabilization of the attractor $\{p\}$), and since $\widehat{\{p\}}$ is compact and invariant, $\overline{\gamma(x_0)} \subset \widehat{\{p\}}$. But now $\widehat{\{p\}}$ is a global attractor in $\overline{U} \cong \mathbb{R}^2$, hence its shape must be trivial, so it does not disconnect $\overline{U}$ and it follows $U_{x_0} \subset \widehat{\{p\}}$. By [4] Prop. 4.4 p. 211 this implies that $\alpha(x) = \{p\}$ for every $x \in U_{x_0}$ so the argument and notations introduced above for x_0 extend to all $x \in \overline{U}_{x_0}$. That is, if $x \in \overline{U}_{x_0}$ then $\overline{\gamma(x)} = \gamma(x) \cup \{p\}$ separates $\mathbb{R}^2$ into two connected components. If we denote by U_x the bounded one, it is an invariant set with boundary $\overline{\gamma(x)}$. Observe that if $y \in U_x$ then $\gamma(y) \subset U_x$, and since U_x is homeomorphic to an open disk, $U_y \subset U_x$.

Let N be an isolating neighbourhood for p. It is clear that for every $p \neq x \in U_{x_0}$ the inclusion $\gamma(x) \subset N$ cannot hold since otherwise p would not be isolated by N, hence $\gamma(x) \cap \partial N \neq \emptyset$ and $\overline{U}_{x_0} \cap \partial N \neq \emptyset$. If $x, y \in U_{x_0}$ are not in the same trajectory, then $x \in U_y$ or $y \in U_x$ so $\overline{U}_x \subset \overline{U}_y$ or $\overline{U}_y \subset \overline{U}_x$. In any case, the intersection $\overline{U}_x \cap \overline{U}_y \cap \partial N$ coincides with either $\overline{U}_x \cap \partial N$ or $\overline{U}_y \cap \partial N$ and therefore the family $\{\overline{U}_x \cap \partial N\}_{p \neq x \in U_{x_0}}$ has the finite intersection property. By the compactnes of ∂N there exists $y \in \bigcap_{p \neq x \in U_{x_0}} \overline{U}_x \cap \partial N$ and in particular $y \neq p$. However, $y \in \overline{U}_{x_0}$, hence U_y is an open disk whose boundary contains p. Consequently there must exist some $x \in U_y \cap \operatorname{int} N$, which implies $U_x \subset U_y$ and $y \in \overline{U}_x = U_x \cup \gamma(x) \cup \{p\} \subset U_y \cup \{p\}$; but this is a contradiction since $y \notin U_y \cup \{p\}$. □

Let us remark here that the conclusion of Theorem 3.12 is false if the attractor K is not global, as Mendelson's famous example of an isolated unstable attractor in the plane shows [52]. However, *every isolated invariant continuum $K \subset \mathbb{R}^2$ has polyhedral shape*. To prove this note that by Alexander's duality $\check{H}^1(K) = \widetilde{H}_0(\mathbb{R}^2 - K)$. Since K is isolated it follows easily from this that $\check{H}^1(K)$ is free and finitely generated. Now it follows from a theorem of K. Borsuk on plane continua ([11, Theorem 9.1, p. 52]) that K has the shape of a polyhedron (in fact, a finite bouquet of circles).

The following result, that we present without proof, was established by M. A. Morón, J. J. Sánchez-Gabites and J. M. R. Sanjurjo in [57].

Theorem 3.13. *Let $K \subset \mathbb{R}^n$ be a connected isolated attractor. If K is unstable, then it must have external explosions.*

Hence if we are interested in examples of flows with attractors having internal explosions we must look to places other than the Euclidean space. Sánchez Gabites has given in [80] a sufficient condition for a manifold to have global attractors of that kind.

Theorem 3.14. *Let M be a closed oriented smooth manifold. If $H^1(M,\mathbb{Z}) \neq 0$ then M contains a connected isolated unstable global attractor having no external explosions.*

For examples of flows having only internal explosions see [57] and [80].

4. An example: the Lorenz attractor

We shall see how the previous results can be applied to an example related to the Lorenz attractor. This attractor has been studied for a long time by many authors since E. Lorenz introduced his famous equations [48], but only recently has its existence been rigourously proved by W. Tucker ([101] and [102]). We recommend the book by C. Sparrow [99] and the expository paper [104] by M. Viana for information about this subject. The results of C. Morales, M. J. Pacífico and E. Pujals [55] provide a unified framework for robust strange attractors in dimension 3 of which the Lorenz attractor is a particular case. See also the paper [18] by L. Díaz, E. Pujals and R. Ures for related results about discrete-time systems and [62] for other recent related results. The topological classification of the Lorenz attractors (for different parameter values) can be found in the paper [67] by D. Rand. More general results about the classification of Lorenz maps are due to J. H. Hubbard and C. Sparrow [40].

The Lorenz equations provide an example of a Hopf bifurcation which takes place at parameter values very close to those which correspond to the creation of the Lorenz attractor. The equations are

$$dx/dt = \sigma(y-x), \quad dy/dt = rx - y - xz, \quad dz/dt = xy - bz,$$

where σ, r and b are three real positive parameters. As we vary the parameters, we change the behaviour of the flow determined by the equations in $\mathbb{R}^3$. The values $\sigma = 10$ and $b = 8/3$ have deserved special attention in the literature. We shall fix them from now on, and we shall consider the family of flows obtained when we vary the remaining parameter r. In the sequel we follow C. Sparrow [99] for the presentation of all the aspects concerning the basic properties of the Lorenz equations. Sparrow's book was written long before Tucker's work was available and some of the global statements made in it are only tentative. However, except for a few details, they have proved to agree with Tucker's results.

The origin is a stationary point for all the parameter values. If $0 < r < 1$, it is a global attractor. At $r = 1$ there is a bifurcation of a simple kind, and for

$r > 1$ the origin is non-stable and there are two other stationary points,

$$\begin{aligned} C_1 &= (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, (r-1)), \\ C_2 &= (+\sqrt{b(r-1)}, +\sqrt{b(r-1)}, (r-1)), \end{aligned}$$

both of them attractors in the parameter range $1 < r < 470/19 \approx 24.74$. When r is slightly larger than one, the unstable manifold of the origin is a one-dimensional manifold composed of the origin and two trajectories α_1 and α_2 spiralling towards C_1 and C_2, respectively. For a larger value of r, approximately equal to $13.926\ldots$, the behaviour of the flow changes in an important way: the trajectories started on the unstable manifold of the origin will also lie in the stable manifold of the origin producing two homoclinic orbits. For values of r larger than the critical value $r_0 = 13.926\ldots$ the trajectories are again attracted by the stationary points but α_1 is now spiralling towards C_2 and α_2 is spiralling towards C_1. We say that a *homoclinic explosion* has taken place at this critical value of the parameter. As a consequence, a "strange invariant set" has been created. This set consists of a countable infinity of periodic orbits, an uncountable infinity of aperiodic orbits, and an uncountable infinity of trajectories which terminate in the origin. For values of r close to r_0 the strange invariant set is non-stable: trajectories of many points close to it escape, spiralling towards C_1 or C_2. However, at the critical r-value $r_A \approx 24.06$ this set becomes attracting. The resulting attractor, called the Lorenz attractor, coexists with the two attracting points C_1 and C_2 until the r-value $r_H \approx 24.74$, when a Hopf bifurcation takes place and C_1 and C_2 lose their stability. This bifurcation is subcritical, i.e. C_1 and C_2 lose their stability by absorbing a non-stable periodic orbit.

The numerically computed solutions to the Lorenz equations projected onto the xz plane give a visual image of the attractor with its characteristic butterfly aspect. In fact, the stable manifold of the origin divides the phase space into points that first go to one wing of the butterfly and those that first go to the other wing when approaching the attractor. See [61] for very suggestive computer images.

4.1. The global attractor $\mathbf{E}_\infty$ and the Lorenz attractor. E. N. Lorenz in [48] proved that for every value of r there is an ellipsoid E in $\mathbb{R}^3$ which all trajectories eventually enter. C. Sparrow [99] describes the situation in this way:

"At times $1, 2, \ldots$ the surface of the ellipsoid E is taken by the flow into surfaces $S_1, S_2, \ldots$ which enclose regions $E_1, E_2, \ldots$ such that the volumes of the E_i decrease exponentially to zero as i increases. Because all trajectories cross the boundary inwards we know that $E \supset E_1 \supset E_2 \supset \ldots \supset E_i \supset \ldots$ and hence every trajectory is ultimately trapped in a region, E_∞, of zero volume given by

$$E_\infty = \bigcap_{i \in Z^+} E_i."$$

E_∞ is therefore a global attractor.

By using elementary notions of shape theory we infer from this that the flow $\varphi: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ induces, in a natural way, maps $r_k: E \to E_k$ and $x \to \varphi(x,k)$ that define an *approximative sequence*

$$\mathbf{r} = \{r_k,\, E \to E_\infty\}$$

in the sense of K. Borsuk [11]. This means that for every neighbourhood V of E_∞ in $\mathbb{R}^3$ there is a k_0 such that $r_k \simeq r_{k+1}$ in V for $k \geq k_0$. The approximative sequence $\mathbf{r}$ induces a shape isomorphism whose inverse is induced by the inclusion $i: E_\infty \to E$. This proves that E_∞ *has trivial shape.* This is also a consequence of the fact that the shape of a global attractor agrees with that of the phase space ([10], [33], [44], [88]). We remark that *at least for some values of the parameter* r*,* E_∞ *is not homotopically trivial* since there are trajectories spiralling into C_1 (as well as trajectories spiralling into C_2) which lie in a path-component of E_∞ not containing C_1 (resp. C_2).

For r-values $r < r_H$ close to the Hopf bifurcation, the non-wandering set of the flow, Ω, is the union of the Lorenz attractor L, the stationary points C_1 and C_2 and two periodic orbits γ_1 and γ_2 which are responsible for the Hopf bifurcation at the critical value r_H. The non-wandering set defines in a natural way a Morse decomposition of the global attractor E_∞. If we want to study the global topological structure of the Lorenz attractor (in particular its shape) we need to kow the evolution of the flow inside the ellipsoid E. At $r = r_H$ the periodic orbits γ_1 and γ_2 are absorbed by the stationary points C_1 and C_2 and for $r \geq r_H$ the points C_1 and C_2 lose their stability. The non-wandering set becomes simpler. In fact, Ω is

$$\Omega = L \cup \{C_1\} \cup \{C_2\}.$$

For r-values $r \geq r_H$ near the Hopf bifurcation the flow defines a semi-dynamical system in the ellipsoid E whose trajectories are all attracted by L except those which compose the stable manifolds of C_1 and C_2. These are one-dimensional manifolds whose intersection with E consists of closed arcs, l_1 and l_2 respectively, with their ends in the boundary of E and such that $l_1 \cap l_2 = \emptyset$. In other words, the Lorenz attractor L is an attractor of a semi-dynamical system in E whose basin of attraction is $E - (l_1 \cup l_2)$. Now Theorem 2.1 in this paper is also valid for semi-dynamical systems. If we apply this result to the semidynamical system induced by the Lorenz flow in $M = E - (l_1 \cup l_2)$ we deduce that the inclusion $i: L \to E - (l_1 \cup l_2)$ is a shape equivalence and, therefore, the shape of the Lorenz attractor is that of a disc with two holes or, equivalently, that of a wedge of two circles. We have only considered r-values $r \geq r_H$, hence our conclusion is limited, for the moment, to those r-values. We now apply the following result that we have proved in [89].

Theorem 4.1. *Let $\varphi_\lambda : X \times \mathbb{R} \to X$, $\lambda \in I$, be a parametrized family of flows defined on a locally compact ANR, X. If K is an attractor of φ_0 then for every neighbourhood V of K contained in the basin of attraction of K there exists a λ_0, with $0 < \lambda_0 \leq 1$, such that for every $\lambda \leq \lambda_0$ there exists an attractor $K_\lambda \subset V$ of the flow φ_λ with $\mathrm{Sh}(K_\lambda) = \mathrm{Sh}(K)$. Moreover V, is contained in the basin of attraction of K_λ.*

It follows from Theorem 4.1 that the shape of attractors is preserved by local continuation and, hence, the shape of the Lorenz attractor for r-values $r < r_H$ is the same as the one at r_H. Moreover, the cohomology Conley index of an attractor is also determined by its shape. In conclusion we have the following result (see [93]).

Theorem 4.2. *The Lorenz attractor L, has the shape of $S^1 \vee S^1$ (a wedge of two circles) for r-values close to r_H (the critical value of the Hopf bifurcation). As a consequence, the cohomology Conley indexes of L are $\mathrm{CH}^0(L) \cong \mathbb{Z}$, $\mathrm{CH}^1(L) \cong \mathbb{Z} \oplus Z$ and $\mathrm{CH}^q(L) \cong 0$ for $q > 1$.*

5. Bifurcations

We have seen that an important feature of the Lorenz equations is the existence of a Hopf bifurcation at a certain value of the parameter. Our next result describes a general bifurcation phenomenon consisting of a transition from asymptotic stability to instability. In such circumstances an attractor expels a family of new attractors that are created in the process. This happens, for instance, in some Hopf bifurcations (see [38] and [95]). All the attractors considered in this result are asymptotically stable. This and related results can be found in [93].

Theorem 5.1. *Let W be an orientable n-dimensional manifold. Let φ_λ: $W \times \mathbb{R} \to W$ be a parametrized family of flows with $\lambda \in I$ such that the compact connected set $A \subset W$ is an attractor of φ_0 and a repeller of φ_λ for $\lambda > 0$. Suppose that for a fixed $k \leq n$ the reduced homology groups $\widetilde{H}_k(\mathcal{B})$ and $\widetilde{H}_{k-1}(\mathcal{B})$ are trivial, where $\mathcal{B}$ is the basin of attraction of A for φ_0. Then for every compact neighbourhood V of A in $\mathcal{B}$ there is a $\lambda_0 > 0$ such that for every λ with $0 < \lambda \leq \lambda_0$ there is an attractor K_λ of φ_λ contained in $V - A$, attracting $V - A$ and such that $\check{H}^{n-k}(K_\lambda) = H^{n-k}(\mathcal{B})$ if (the previously fixed) $k \neq 1$ and $\check{H}^{n-1}(K_\lambda) = \mathbb{Z} \oplus H^{n-1}(\mathcal{B})$ if (the previously fixed) $k = 1$. In particular, if $\mathcal{B}$ is contractible then $\check{H}^{n-1}(K_\lambda) = \check{H}^0(K_\lambda) = \mathbb{Z}$ and $\check{H}^{n-k}(K_\lambda) = \{0\}$ when (the previously fixed) $k \neq 1, n$. The attractors K_λ are in arbitrarily small neighbourhoods of A for values of λ close to 0.*

Proof. The theorem is based on two facts of a dynamical-topological nature whose proof we omit. They describe a general phenomenon which takes place when an attractor becomes a repeller. We refer the reader to [93] for a detailed proof.

(1) There is a $\lambda_0 > 0$ such that for every $\lambda \leq \lambda_0$ there exists an attractor $A_\lambda \subset V$ attracting V and such that $\mathrm{Sh}(A_\lambda) = \mathrm{Sh}(A)$. Moreover, since the inclusion $A \to \mathcal{B}$ is a shape equivalence, we also have that $\mathrm{Sh}(A_\lambda) = \mathrm{Sh}(\mathcal{B})$.

(2) If $\mathcal{R}_\lambda$ is the basin of repulsion of A for φ_λ then $\mathcal{R}_\lambda \subset A_\lambda$ and

$$K_\lambda = A_\lambda - \mathcal{R}_\lambda$$

is an attractor such that $V - A$ is contained in its basin of attraction.

Now, by the Alexander duality theorem applied to the orientable manifold $\mathcal{B}$, we have that

$$\check{H}^{n-k}(K_\lambda) = H_k(\mathcal{B}, \mathcal{B}-K_\lambda).$$

If we consider the long homology sequence of the pair $(\mathcal{B}, \mathcal{B}-K_\lambda)$

$$\ldots \to \widetilde{H}_k(\mathcal{B}) \to H_k(\mathcal{B}, \mathcal{B}-K_\lambda) \to \widetilde{H}_{k-1}(\mathcal{B}-K_\lambda) \to \widetilde{H}_{k-1}(\mathcal{B}) \to \ldots,$$

since $\widetilde{H}_k(\mathcal{B}) = \widetilde{H}_{k-1}(\mathcal{B}) = \{0\}$ we have that

$$H_k(\mathcal{B}, \mathcal{B}-K_\lambda) \cong \widetilde{H}_{k-1}(\mathcal{B}-K_\lambda).$$

Since $\mathcal{B}-K_\lambda = \mathcal{R}_\lambda(A) \cup (\mathcal{B}-A_\lambda)$ we obtain

$$\widetilde{H}_{k-1}(\mathcal{B}-K_\lambda) = \widetilde{H}_{k-1}(\mathcal{R}_\lambda(A)) \oplus \widetilde{H}_{k-1}((\mathcal{B}-A_\lambda)) \quad \text{if } k \neq 1.$$

On the other hand, from the long exact sequence for the pair $(\mathcal{B}, \mathcal{B}-A_\lambda)$

$$\ldots \to \widetilde{H}_k(\mathcal{B}) \to H_k(\mathcal{B}, \mathcal{B}-A_\lambda) \to \widetilde{H}_{k-1}(\mathcal{B}-A_\lambda) \to \widetilde{H}_{k-1}(\mathcal{B}) \to \ldots$$

we get

$$\widetilde{H}_{k-1}(\mathcal{B}-A_\lambda) \cong H_k(\mathcal{B}, \mathcal{B}-A_\lambda).$$

But, using again the Alexander duality theorem for the compactum A_λ in the manifold $\mathcal{B}$ and recalling that Čech homology and cohomology are shape invariants, we obtain

$$H_k(\mathcal{B}, \mathcal{B}-A_\lambda) \cong \check{H}^{n-k}(A_\lambda) \cong \check{H}^{n-k}(A) \cong H^{n-k}(\mathcal{B}).$$

Hence, from the fact that the inclusion $A \to \mathcal{R}_\lambda(A)$ is a shape equivalence we deduce that, if $k \neq 1$,

$$\begin{aligned} \check{H}^{n-k}(K_\lambda) &= H_{k-1}(\mathcal{R}_\lambda(A)) \oplus H^{n-k}(\mathcal{B}) \\ &= \check{H}_{k-1}(A) \oplus H^{n-k}(\mathcal{B}) = H_{k-1}(\mathcal{B}) \oplus \check{H}^{n-k}(\mathcal{B}) = H^{n-k}(\mathcal{B}). \end{aligned}$$

If $k = 1$, the only difference from the previous argument is that

$$\widetilde{H}_0(\mathcal{B}-K_\lambda) = H_0(\mathcal{R}_\lambda(A)) \oplus \widetilde{H}_0((\mathcal{B}-A_\lambda))$$

and , since $\mathcal{R}_\lambda(A)$ is a connected open subset of W, we have that $H_0(\mathcal{R}_\lambda(A)) = \mathbb{Z}$. The rest of the argument is exactly the same and we conclude that

$$\check{H}^{n-1}(K_\lambda) = \mathbb{Z} \oplus H^{n-1}(\mathcal{B})$$

if $k = 1$. This ends the proof of the theorem. □

If we have have more information about the attractor A then more can be said about the attractors K_λ created in the bifurcation. For instance, the following result is proved in [93].

Theorem 5.2 (Bifurcations from equilibrium points)**.** *Let W be an n-dimensional manifold. Let $\varphi_\lambda : W \times \mathbb{R} \to W$ be a parametrized family of flows with $\lambda \in I$ (the unit interval) and such that the point $p \in W$ is an attractor of the flow φ_0. Then the two following statements hold:*

(1) *If p is a repeller of φ_λ for every $\lambda > 0$, then for every compact neighbourhood V of p contained in the basin of attraction of p for the flow φ_0, there exists a λ_0 such that for every λ, with $0 < \lambda \leq \lambda_0$, there exists an attractor K_λ of φ_λ with the shape (and hence with the Čech homology and cohomology) of S^{n-1}. The attractor K_λ is contained in $V - \{p\}$ and attracts all points in $V - \{p\}$. Moreover, if p is monotone, then the multivalued function $\Theta : [0, \lambda_0] \to W$ defined by $\Theta(0) = \{p\}$ and $\Theta(\lambda) = K_\lambda$ (when $\lambda \neq 0$) is upper semi-continuous.*

(2) *If the following conditions hold for $\lambda > 0$:*

- (a) *there exists a k-dimensional submanifold W_0 of W such that W_0 is invariant by φ_λ,*
- (b) *there exists a neighbourhood U of p (the same for all λ) such that the maximal invariant set of φ_λ inside U is contained in W_0,*
- (c) *p is a repeller of the restriction flow $\varphi_\lambda | W_0 : W_0 \times \mathbb{R} \to W_0$,*

then there is a λ_0 such that for every λ, with $0 < \lambda \leq \lambda_0$, there is an attractor $K_\lambda \subset U$ of the unrestricted flow $\varphi_\lambda : W \times \mathbb{R} \to W$ with the shape (and hence with the Čech homology and cohomology) of S^{k-1}. In particular, K_λ has the shape of S^1 when W_0 is of dimension 2. The attractors K_λ are contained in arbitrarily small neighbourhoods of p for values of λ close to 0.

6. Some open problems

The following is a list of problems which, up to the author's knowledge, are open at the moment of writing this paper.

Problem 6.1 (B. Günther and J. Segal)**.** Is it possible to characterize those compacta K that can be attractors of flows in manifolds such that K contains a dense orbit?

Problem 6.2 (B. Günther and J. Segal)**.** Is it possible to characterize attractors of discrete dynamical systems?

Problem 6.3 (J. C. Robinson)**.** Suppose $\varphi\colon K\times\mathbb{R}\to K$ is a flow on a finitely dimensional compactum K. Is is true that K can be embedded in $\mathbb{R}^n$ for suitable n in such a way that there is a flow $\widehat{\varphi}$ on $\mathbb{R}^n$ having K as an attractor and such that the restriction of $\widehat{\varphi}$ to K agrees with φ?

Problem 6.4 (M. A. Morón, J. J. Sánchez-Gabites and J. M. R. Sanjurjo)**.** To what extent does the shape of the Freudenthal compactification of the unstable manifold of an isolated invariant set K endowed with its intrinsic topology carry more information than the Conley shape index of K? Is there a satisfactory theory relating the dynamical properties of K to the topological properties of such a compactification?

Problem 6.5 (J. M. R. Sanjurjo)**.** Is there a dynamical condition C such that a finite dimensional metric compactum K is movable if and only if K can be embedded as an invariant subset of a flow on a manifold in such a way that K satisfies condition C?

Problem 6.6 (A. Giraldo, M. A. Morón, F. Ruiz del Portal and J. M. R. Sanjurjo)**.** Given a set K in a Hausdorff space X such that the inclusion $i\colon K\to X$ induces an H-shape equivalence, when is K the global attractor for some semi-dynamical system in X?

Problem 6.7 (K. Kuperberg)**.** Characterize those invariant compacta A of flows on a manifold such that every neighbourhood of A contains a movable compact invariant set containing A.

Problems 6.1 and 6.2 are from [33]. Problem 6.3 is suggested in [69]. A partial answer is contained in that paper. Problems 6.4 and 6.5 are from [94]. Problem 6.6 appears formulated in [28]. A partial answer has been given by J. J. Sánchez-Gabites in [81]. Problem 6.7 was posed, according to P. Šindelářová, by K. Kuperberg (with a different formulation), see [97]. That paper contains some partial answers.

7. Some recent developments

We complete this article with a quick review of some additional results. We briefly discuss some of our own contributions and select material from other authors, guided mainly by criteria of personal preference. Most of the results are quite recent and give an idea of several directions of current research in the area.

7.1. Intrinsic topology of the unstable manifold. J. W. Robbin and D. Salamon introduced in [68] the *intrinsic topology* of the unstable manifold W^u of an isolated invariant compactum. A similar definition applies to the stable manifold. This topology has some interesting features. For instance it can be used to characterize the shape index of an isolated compactum; namely, the shape index of K is the shape of the Alexandroff compactification of the unstable manifold of K endowed with the intrinsic topology [68]. This topology has been studied by Sanjurjo in [91], where he proved that the compactum K is a global repeller of the flow restricted to W^u endowed with the intrinsic topology. Similarly K is a global asymtoptically stable attractor of the flow restricted to the stable manifold with the intrinsic topology. In that paper it is also proved that this property turns out to be a necessary and sufficient condition for the intrinsic and the extrinsic topologies to agree. J. J. Sánchez-Gabites has carried out a thorough study of the intrinsic topology in [82]. Among other things he proves that this topology can be defined without using isolating blocks of K, solving a problem in [91]. This topology has also been used by K. Athanassopoulos [5] to study the complexity of the flow in the region of attraction of an isolated invariant set. In [6] he proved that if the intrinsic topology of the region of attraction of an isolated 1-dimensional compact minimal set K of a continuous flow on a locally compact metric space is locally connected at every point of K, then K is a periodic orbit.

7.2. The Lusternik–Schnirelmann category of isolated invariant compacta. M. Pozniak [65] has used a modification of the classical Lusternik–Schnirelman category to study properties of isolated invariant compacta K of flows. For instance, he has given an estimation from below of the rank of $\check{H}^*(K)$ using what he calls the *cohomological category* of N/L where N is an isolating block of K and L is the exit set. Sanjurjo has given in [90] a relation between the Lusternik–Schnirelmann category of the unstable manifold of an isolated invariant compactum satisfying some aditional conditions and the sum of the Lusternik–Schnirelmann categories of the members of any Morse decomposition of K. This result is used in [90] to detect connecting orbits in attractor repeller decompositions, saddle sets and fixed points of flows in the plane. In [92] it is proved that if the intrinsic topology is used on W^u then the mentioned result follows in full generality (without any restrictions on K).

7.3. Dynamical systems and hyperspaces. The theory of multivalued maps and hyperspaces plays a natural role in Dynamical Systems. Under suitable hypotheses the notion of first prolongational set J^+ gives rise to a multivalued map $\psi\colon X \to 2^X$ which is continuous when Michael's upper semifinite topology is considered in the hyperspace of X. It is possible to take advantage of this theory to obtain neat characterizations of such important notions as stability and attracting sets. In fact, stable sets and attractors have been characterized by J. J. Sánchez-Gabites and J. M. R. Sanjurjo in [86] as fixed points of certain maps in such hyperspaces. This kind of hyperspaces have been extensively studied by M. Alonso-Morón, E. Cuchillo-Ibáñez, A. González-Gómez and A. Luzón in [2] and [3] and their ideas will probably lead to further advances in applications of the theory of hyperspaces to dynamical systems. In [73], F. R. Ruiz del Portal and J. M. Salazar study the fixed point index in hyperspaces and use it to obtain a characterization of isolating neighbourhoods of compact invariant sets with non-empty attracting part and also a characterization of those isolated minimal sets that are attractors. In [76] and [77], J. M. Salazar considers a locally compact metric ANR, X, a semidynamical system $f\colon U \subset X \to X$ and a compact isolated invariant set $K \subset U$ with respect to f and he constructs the fixed point index of the map that f induces in the spaces $F_n(X)$ of the non-empty finite subsets of X with at most n elements, endowed with the Hausdorff metric (these spaces were defined in 1931 by Borsuk and Ulam). This fixed point index detects the existence of periodic orbits of f in K of period less than or equal to n.

7.4. Conley index and Ważewski theories. The Ważewski and Conley index theories are the source of a large number of applications. We briefly summarize some of them here (all quite recent); others are discussed in [53] and [100].

An application to the existence of asymptotic solutions is contained in Ortega's paper [60]. These are non-trivial solutions tending to the origin as time increases to infinity and they appear in systems of differential equations having the trivial solution. The classical method for proving their existence consists on the reduction of the problem to an integral equation. Once this equation has been found one uses the method of successive approximations or the contraction principle. This analytical method leads to the Principle of Linearization and to the Stable Manifold Theorem for autonomous equations. Ważewski applied the theory of retracts and developed an alternative method for constructing asymptotic solutions in his paper [105]. Ortega illustrates Ważewski's ideas in a concrete situation, and later he discusses the connections with the analytical approach. In the process he finds that other tools such as topological degree and global continuation are also applicable to this problems.

In [24] K. Gęba, M. Izydorek and A. Pruszko developed a new infinite-dimensional extension of the classical Conley index and subsequently M. Izydorek defined and studied in [42] a cohomological version of this notion. The theory proved to be fruitful in applications to strongly indefinite problems and he obtained new results concerning the existence of periodic solutions of certain Hamiltonian systems. In [43] he developed an equivariant version of this index and gave applications to asymptotically linear problems with and without resonance and to certain local bifurcation problems.

Another interesting application of the Conley index has been given by K. Wójcik [109] in Permanence Theory (a theory which plays an important role in mathematical ecology). The so-called criterion of permanence for biological systems is a condition ensuring the long-term survival of the species. Wójcik considers flows in $\mathbb{R}^n \times [0,\infty)$ and proves in [109] that if $S \subset \mathbb{R}^n \times \{0\}$ is an isolated invariant set with nonzero homological Conley index, then there exists an $x \in \mathbb{R}^n \times (0,\infty)$ such that $\omega(x) \subset S$. This may be understood as a strong violation of permanence.

In [74] Ruiz del Portal and Salazar develop some techniques based on Conley index ideas to give a short and simple proof of a theorem of Le Calvez and Yoccoz about the non-existence of minimal homeomorphisms of $\mathbb{R}^2 - K$ for any finite set K. They also obtain in this paper a general theorem that allows one to compute the fixed-point index of every iteration of any local homeomorphism of $\mathbb{R}^2$ at any non-repeller fixed point which is a locally maximal invariant set.

In [20] Z. Dzedzej and W. Kryszewski develop a theory of a particular cohomological Conley index which allows them to detect invariant sets of multivalued dynamical systems generated by semilinear differential inclusions in infinite dimensional Hilbert spaces. They give applications to the existence of periodic orbits of asymptotically linear Hamiltonian inclusions.

In [49], S. Maier-Paape, U. Miller, K. Mischaikow and T. Wanner show how to use rigorous computational techniques to establish computer-assisted existence proofs for equilibria of the Cahn–Hilliard equation on the unit square. Their method combines rigorous computations with Conley index techniques. They establish branches of equilibria and, under more restrictive conditions, even the local uniqueness of specific equilibrium solutions.

7.5. Regularity of isolating blocks. In [25] A. Gierzkiewicz and K. Wójcik consider the following question: given an isolated invariant set S, under what conditions does it possess an isolating block N such that the inclusion $j\colon S \to N$ induces isomorphisms in Čech cohomology? (for the sake of brevity we shall term those isolating blocks “regular”, following [79] and [83]. This is a natural problem to consider, since one might expect to obtain more information about S (which is not an “observable” object) from N (which is an “observable” object). The authors of [25] acknowledge R. Easton [21] as the first to consider this question.

In [25] they consider flows on locally compact metric spaces and their main theorem gives sufficient conditions for regular isolating blocks to exist when certain assumption about how the set n^- sits in $\mathrm{fr}(N)$ is satisfied. Then they specialize (following R. Easton [21] and fixing an essential gap in his arguments) their result to the case of continuous flows in 3-manifolds: using a classical lemma about neighbourhoods of compacta in surfaces, they prove that the condition on n^- is satisfied in this case, so for isolated invariant sets (of finite type) in 3-manifolds regular isolating blocks always exist.

In [79] and [83] J. J. Sánchez-Gabites studies the same problem, although in this case it is only isolated invariant sets in 3-manifolds which are considered. The author arrives at the same characterization of regular isolating blocks as the one given in [25] but, since now it is flows in 3-manifolds which are considered, the stronger result is obtained that when N is a regular isolating block for S the inclusion $j\colon S \to N$ is a shape equivalence.

An existence theorem is then proved in [79] and [83] but it is also complemented by a uniqueness statement: if N_1 and N_2 are regular isolating blocks of S, there exists a topological equivalence of flows $h: N_1 \to N_2$. The existence theorem is used to draw consequences about the shape-theoretic properties of isolated invariant sets, and also to show that if $\mathrm{Sh}(S) = *$ then S has isolating blocks which are balls. Finally, it is proved that any isolating block N for S is obtained from a regular one by adding handles onto it. This can be used to yield lower bounds on the complexity of S in terms of observable features of the flow on $\mathrm{fr}(N)$, as shown by example in [79] and [83].

7.6. Continuations and robustness. A. Giraldo and J. M. R. Sanjurjo study in [31] preservation of dynamical and shape theoretical properties under continuation for parametrized families of flows. They show that, although attractors continue, the same does not hold for non-saddle sets. However, when they continue (i.e. when they are dinamically robust), their shape is preserved in quite general settings (i.e. they are topologically robust). More concretely, they prove [31] the following result:

Let $\varphi_\lambda: M \times \ R \to M$ be a parametrized differentiable family of flows (parametrized by $\lambda \in I$, the unit interval) in an n-manifold, M, and let K_0 be a connected isolated non-saddle set for φ_0. If K_0 is dynamically robust, then K_0 is topologically robust.

If we have a continuous parametrized family of flows $\varphi_\lambda: X \times \mathbb{R} \to X$ with $\lambda \in [0, 1]$, and $\{K_\lambda \mid \lambda \in I\}$ is a continuation relating two attractors, K_0 and K_1, then on some occasions it is possible to replace at the parameter value $\lambda = 1$ the attractor K_1 by another one $\widehat{K}_1$ in such a way that the same continuation (with $\lambda < 1$) also relates K_0 and the new attractor $\widehat{K}_1$. In some particular cases we may even have a nested sequence of attractors of φ_1, $\dots K_1^{n+1} \subset K_1^n \dots$, all of which are related to K_0 through the same continuation. In this situation the "natural" continuation of K_0 through $\{K_\lambda\}$ seems to be $K_1' = \bigcap K_1^n$ in spite of the fact that K_1' may be non-isolated and hence, possibly, a non-attractor. K_1' is called a *singularity* of the continuation $\{K_\lambda\}$. K_1' is, in fact, a quasi-attractor, often with a complicated topological structure.

This situation is far from being exceptional in dynamical systems. As was remarked by Kennedy and Yorke in [46] "bizarre topology is natural in dynamical systems". Moreover, J. A. Kennedy, dealing with discrete dynamical systems (generated by a homeomorphism) in a large class of compact metric spaces, including manifolds of dimension at least two, proved in [45] that the property of admitting an infinite collection of attractors, each of which has nonempty interior and cannot be reduced to a "smallest" attractor, is generic. Hurley generalized this result in [41] by showing that this property holds for *all* attractors of a generic homeomorphism (see [1] for many related matters).

A. Giraldo and J. M. R. Sanjurjo study in [30] the quasi-attractors obtained in situations similar to the one described before. They focus, in particular, on properties of the singularities of continuations. They introduce the *continuation* *skeleton* of an attractor K_0, which gathers information from all the continuations of K_0, and the related *spectrum* of K_0, which is the quasi-attractor of the terminal flow, φ_1, which "survives" all possible continuations of K_0. In spite of their weird local topological structure, singularities of continuations and spectra of attractors have rather regular global topological properties, which agree with those of K_0.

7.7. Boundary of the region of attraction and boundary of attractors. Given an asymptotically invariant compactum K and a positively invariant compact neighbourhood P of K contained in $\mathcal{A}(K)$ it is natural to ask whether any relation can be ascertained between the shapes of their boundaries ∂K and ∂P. The paper by J. C. Robinson and O. M. Tearne [71] contains a proof of the fact that ∂K agrees with the ω-limit of ∂P, which shows a dynamical connection between them. The paper [84] is devoted to a general study of the relations between ∂P and ∂K. It is proved, among other things, that if K is an attractor in a locally compact metric space M and $P \subset \mathcal{A}(K)$ is a positively invariant compact neighbourhood of K such that $\mathrm{int}(K)$ contains a homotopical spine L of P, then $\mathrm{Sh}(\partial K) \geq \mathrm{Sh}(\partial P)$. A consequence of this result is that if ∂K can be bicollared (for example, this happens if the phase space is a differentiable manifold and ∂K is an orientable hypersurface), then $\mathrm{Sh}(\partial K) \geq \mathrm{Sh}(\partial P)$.

Similarly, some properties of the boundary of $\mathcal{A}(K)$ are studied in [85]. Among other things the following results are proved:

(1) Let $K \subset M$ be an attractor in the locally compact ANR M and suppose that the boundary D of the basin of attraction $\mathcal{A}(K)$ of K is an isolated compactum whose 1-dimensional cohomological Conley index satisfies $CH^1(D) = 0$. Then D has the shape of a finite polyhedron.

(2) Let $K, K' \subseteq S^n$ be, respectively, an attractor and repeller whose basins $\mathcal{A}(K)$ and $\mathcal{R}(K')$ have a common boundary D and such that $S^n = \mathcal{A}(K) \cup D \cup \mathcal{R}(K')$. If K has trivial shape, then $\check{H}_k(K') \cong CH^{n-k-1}(D)$.

(3) Let K be an attractor in a compact phase space M. Assume that $D = \partial\mathcal{A}(K)$ is an isolated set whose 1-dimensional cohomological Conley index is zero. Then there exists a repeller K' in M such that the boundary of $\mathcal{R}(K')$ coincides with D and $M = \mathcal{A}(K) \cup D \cup \mathcal{R}(K')$, where the union is disjoint.

(4) Let K be an attractor for a differentiable flow in an orientable compact n-manifold M and suppose that $\mathrm{CH}^1(D) = 0$. Then the number of connected components of the dual repeller K' of $\overline{\mathcal{A}(K)}$ is bounded above by $\mathrm{rk}\ \mathrm{CH}^{n-1}(D) +$ $\#$(components of D).

The complexity of the basin topology and specially of its boundary has been observed in many occasions. See, for instance, the papers [58] by Nusse and Yorke and [96] by Seoane, Aguirre, Sanjuán and Lai for the description of interesting features.

7.8. Dynamical systems and exterior spaces. A new topological approach to the study of flows is adopted by J. M. García-Calcines, L. J. Hernández-Paricio and M. T. Rivas in [23] where they present some applications of the theory of exterior spaces to dynamics. The structure of exterior space is given by the quasi-filter of all open absorbing sets of the flow. In this way they can translate notions like limit space or end space to the realm of flows. When one considers all the end points of a dynamical system one has an induced decomposition of the system as a disjoint union of stable (at infinity) subflows. Their theory makes possible the construction of some natural compactifications of the flows.

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FROM WAŻEWSKI SETS TO CHAOTIC DYNAMICS

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Abstract. The aim of this note is to present a survey of results concerning chaotic dynamics based on the Waewski retract method and the fixed point index theory.

1. Introduction

Topological approaches are frequently used in the study of dynamics generated by differential equations and proving results in problems of the existence of solutions satisfying some boundary conditions ([4], [11], [12], [14], [15], [26], [28], [29]). The very common strategy, restricted to equations describing some evolution in time, applies the fixed point theory ([9]) or degree theory to translation operators along solutions ([11], [12]). In the case of ordinary differential equations those operators are finite dimensional. For dissipative systems, the existence of a required solution is a consequence of the fact that the translation operator preserves some subset of the phase space of the equation with the fixed point property. Frequently, those invariant subsets are compact and convex, so the Brouwer fixed point theorem applies. However, for non-dissipative equations usually there are no reasonable compact subsets which are preserved by the translation operator. The aim of this note is to describe a geometric method which some times can be applied in that context.

In recent years there has been growing interest in the study complicated dynamics by means of topological tools ([5], [1]–[3], [6], [13], [18]–[23], [25], [30],

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[34], [35]). Chaotic dynamics are difficult to study in general and there are few rigorous results concerning chaotic dynamics in concrete dynamical systems. Among the first topological criteria for chaos were two criteria presented in [16]. The first one was based on the Conley index, the other on the fixed point index and continuation methods. The Conley index criterion was applied in [17] to prove chaos in the famous Lorenz equations. The fixed point index criterion developed by Zgliczyński [35] was applied to the Hénon map and the Rössler equations.

Another topological criterion is based on the work of Srzednicki ([26], [27], [31]), who developed the machinery of periodic segments to compute the fixed point index of the Poincaré map of a flow directly from the features of the vector field in the phase space. The method applies the fixed point index, the Lefschetz fixed point theorem and Ważewski retract method to provide results on the existence of periodic solutions and chaotic dynamics generated by periodic in time ODEs.

2. Local flows, Ważewski sets and isolating blocks

In 1947, Tadeusz Ważewski ([32]) described a new topological method for detecting of solutions remaining in a given set for positive values of time. Later, in [5] Charles Conley presented a version of the Ważewski theorem with a more convenient assaumptions. He introduced the notion of the exit set and with its use he defined the concept of Ważewski sets. The method is based on the observation which roughly assert that there is a solution contained in a Ważewski set for all positive values of time if the exit set is not a retract of the whole space. For a compact Ważewski sets which do not contain any full solutions intersecting their boundaries, Conley discovered a homotopical invariant, called the Conley index, which provides a quantitative information on their invariant part ([5], [27]).

Let M be a topological manifold and let D be an open subset of $\mathbb{R} \times M$ such that $\{0\} \times M \subset D$. A continuous map $\Phi\colon D \to M$ is called a *local flow* on M if for every $x \in M$ the set $\{t \in \mathbb{R} : (t,x) \in D\}$ is equal to an open interval $(\alpha_x, \omega_x) \subset \mathbb{R}$,

$$\Phi(0,x) = x,$$

and if $(t,x) \in D$, $(s, \Phi(t,x)) \in D$ then $(t+s, x) \in D$ and

$$\Phi(s+t, x) = \Phi(s, \Phi(t,x)).$$

We write $\Phi_t(x)$ instead of $\Phi(t,x)$, hence $\Phi_0 = \mathrm{id}$ and $\Phi_{s+t} = \Phi_t \circ \Phi_s$.

The map $t \to \Phi_t(x)$ is called an *orbit* of x. If it is a constant map then x is called a *stationary point*. If the orbit of x is a periodic map then x is called a *periodic point*. We say that a periodic point x is non-trivial if x is not a stationary point. The *trajectory* of x is defined as $O(x) = \Phi((\alpha_x, \omega_x) \times \{x\})$.

A set $I \subset M$ is called *invariant* if for all $x \in I$ we have $(\alpha_x, \omega_x) = \mathbb{R}$ and $O(x) \subset I$.

The most natural examples of local flows come from the theory of ordinary differential equations ([8]): if $v: M \to TM$ is a smooth vector-field on a manifold M and $u_{x_0}: (\alpha_{x_0}, \omega_{x_0}) \to M$ is the unique solution of the initial value problem

$$\dot{x} = v(x), \quad x(0) = x_0,$$

then ϕ defined by $\Phi(t,x) := u_{x_0}(t)$ is the local flow generated by v. Let us observe that in that case a point x_0 is stationary if and only if $v(x_0) = 0$.

Let $W \subset M$. Define the *exit set* of W as

$$W^- := \{x \in W : \Phi([0,t] \times \{x\}) \not\subset W \text{ for all } t \in (0, \omega_x)\},$$

and the *entrance set* as

$$W^+ = \{x \in W : \Phi([-t,0] \times \{x\}) \not\subset W \text{ for all } t \in (\alpha_x, 0)\}.$$

We call W a *Ważewski set* for Φ if W and its exit set W^- are closed. The *asymptotic part* of W is defined by

$$W^* = \{x \in W : \text{ there exists } t \in (0, \omega_x) : \Phi_t(x) \notin W\}.$$

The most important property of the notion of Ważewski set is given in the following lemma.

Lemma 2.1 ([5], [27]). *If W is a Ważewski set then the mapping*

$$\sigma: W^* \ni x \to \sup\{t \in [0, \omega_x) : \Phi([0,t] \times \{x\}) \subset W\} \in [0, \infty)$$

is continuous.

As a consequence we get the following Ważewski retract theorem.

Theorem 2.2 ([5], [27]). *If W^- is not a strong deformation retract of a Ważewski set W then there exists an $x_0 \in W$ such that $O^+(x_0) := \Phi([0, \omega_{x_0}) \times \{x_0\}) \subset W$. Moreover, if W is compact then $\omega_{x_0} = \infty$ and the ω-limit set of x_0 defined by $\omega(x_0) = \bigcap_{t \geq 0} \overline{\Phi([t, \infty))}$ is a non-empty compact invariant set.*

Indeed, otherwise $W = W^*$ and the mapping $r: W \ni x \to \Phi_{\sigma(x)}(x) \in W^-$ is a strong deformation retraction.

Example 2.3. Let Φ be a local flow on the plane generated by a smooth vector field $v: \mathbb{C} \to \mathbb{C}$.

Assume that for the annulus $A = \{z \in \mathbb{C} : 0 < r \leq |z| \leq R\}$ we have

$$\begin{aligned} z \cdot v(z) &> 0, \quad |z| = R, \\ z \cdot v(z) &< 0, \quad |z| = r, \end{aligned}$$

where $\cdot$ denotes a scalar product. Then A is a Ważewski set with the exit set $A^- = \{z \in A : |z| = r\} \cup \{z \in A : |z| = R\}$, so by the Ważewski retract method there exists $x \in A$ such that $O^+(x) \subset W$. Since A is compact, hence the invariant part of A is non-empty. Moreover, if $v(z) \neq 0$ for $z \in A$ then it follows by the Poincaré–Bendixson theorem ([8]) that there is a non-trivial periodic point in A.

We define $\sigma^{\pm}\colon W \to [0,\infty]$ by

$$\begin{aligned}\sigma^+(x) &= \sup\{t \in (0,\omega_x) : \Phi([0,t]\times\{x\}) \subset W\},\\ \sigma^-(x) &= \inf\{t \in (\alpha_x,0) : \Phi([t,0]\times\{x\}) \subset W\}.\end{aligned}$$

Theorem 2.4 ([5], [27])**.** *If W is a compact Ważewski set, and $W^{\pm}$ are compact, then the functions $\sigma^{\pm}\colon W \to [0,\infty]$ are continuous.*

A set $W \subset M$ is called an *isolating block* if W, $W^{\pm}$ are compact, $W = \overline{\operatorname{int}(W)}$ and for every $x \in \partial W \setminus (W^- \cup W^+)$

$$\sigma^{\pm}(x) < \infty, \quad \Phi([-\sigma^-(x),\sigma^+(x)]\times\{x\}) \subset \partial W.$$

The notion of an isolating block plays a crucial role in the Conley index theory (see [5], [27]).

3. Detection of stationary and periodic solutions

3.1. Lefschetz number and the fixed point index. Let $E = \{E_n\}_{n\geq 0}$ be a graded vector space (over $\mathbb{Q}$) and let $h = \{h_n\}_{n\geq 0}$ be an endomorphism of degree zero, i.e. $h_n\colon E_n \to E_n$ is a linear map of degree zero for all $n \geq 0$. Assume that E is of a finite type, so $E_n = 0$ for almost all $n \in \mathbb{N}$ and $\dim E_n < \infty$ for all $n \in \mathbb{N}$. Then the *Lefschetz number* of h is well defined by

$$L(h) := \sum_{n\geq 0} (-1)^n \operatorname{trace}(h_n).$$

In particular, if $h = I = \{\mathrm{id}_{E_n}\}_{n\geq 0}$, then the Lefschetz number is the *Euler–Poincaré characteristic* of E,

$$L(I) = \sum_{n\geq 0} (-1)^n \dim E_n = \chi(E).$$

We recall, that a metrizable space X is called an *Euclidean neighborhood retract* (shortly ENR) if there is an $n \in \mathbb{N}$, an open set U in $\mathbb{R}^n$, and a map $h\colon X \to \mathbb{R}^n$ which is a homeomorphism onto its image $h(X)$ such that $h(X)$ is a retract of U. Examples of ENRs include all compact polyhedra and manifolds with boundary. It follows that if X is a compact ENR and H is a singular homology functor with coefficients in $\mathbb{Q}$, then $H(X)$ is of finite type. In particular, for a continuous map $f\colon X \to X$ its Lefschetz number $L(f) := L(H(f))$ is well defined.

Assume that X is an ENR. For a continuous map $f\colon D \to X$, where $D \subset X$ is open, define its set of fixed points as

$$\mathrm{Fix}(f) := \{x \in D : f(x) = x\}.$$

We say that $f\colon D \to X$ is *admissible* if $\mathrm{Fix}(f)$ is compact.

To such an admissible map $f\colon D \to X$ one can associate an integer number $\mathrm{ind}(f)$, called the *fixed point index* with the properties (compare [9]):

(1) (Solvability) If $\mathrm{ind}(f) \neq 0$, then there exists $x \in D$ such that $f(x) = x$.
(2) (Lefschetz fixed point theorem) If X is a compact ENR and $f\colon X \to X$ then $L(f) = \mathrm{ind}(f)$.

Example 3.1. Suppose that Φ is a flow on X and $D \subset X$ is a compact ENR, $\chi(D) \neq 0$ and such that $\Phi_t(D) \subset D$ for $t \geq 0$. Observe that $\Phi_t|_D \simeq \mathrm{id}_D$ and the homotopy is given by

$$h(s,x) = \Phi_{st}(x), \quad s \in [0,1],\ x \in D.$$

Consequently, $L(\Phi_t|_D) = L(\mathrm{id}_D) = \chi(D)$ so by the Lefschetz fixed point theorem Φ_t has a fixed point in D. By the compactness of D there exists a stationary point for Φ in D. In the next section we present a generalization of this observation to the sets D that are not positively invariant with respect to Φ.

3.2. Periodic segments. Let $f\colon \mathbb{R}\times M \to TM$ be a smooth time dependent vector field on a manifold M. Then the vector field $v(t,x) = (1, f(t,x))$ generates a local flow Φ on the extended phase space $\mathbb{R} \times M$. It is given by

$$\Phi_s(t_0, x_0) = (t_0 + s, \phi_{(t_0,s)}(x_0))$$

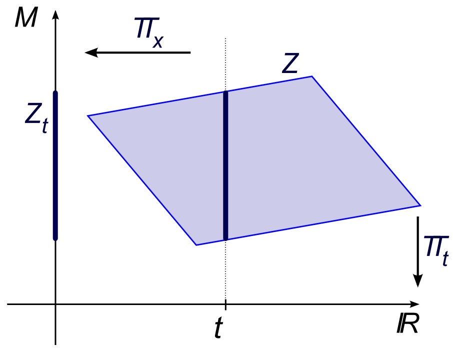
where $\phi_{(t_0,s)}(x_0)$ is the value of the unique solution of the Cauchy problem

$$\dot{t} = 1, \quad \dot{x} = f(t,x), \quad x(t) = x_0$$

at time $t_0 + s$. The map ϕ is called a *local process* on M. Them map ϕ is called a *local process*.

Assume that $T > 0$ and f is T-periodic with respect to t. The map $\phi_{(0,T)}$ is called the *Poincaré map*. Observe that in this case fixed points of the Poincaré map correspond to initial points of T-periodic solutions of the equation $\dot{x} = f(t,x)$.

In order to establish results on fixed points of the Poincaré map we introduce a special class of Ważewski sets, called periodic segments in the extended phase space. At first we introduce the following notation: we denote by π_t and π_x the projections of $\mathbb{R} \times M$ onto $\mathbb{R}$ and, respectively, M, and if $Z \subset \mathbb{R} \times M$ and $t \in \mathbb{R}$ then we put $Z_t = \{x \in M : (t,x) \in Z\}$.



FIGIRE 1. Projections in the phase space

A set $W \subset [0,T]\times M$ is called a *periodic segment* over $[0,T]$ if it is a compact Ważewski set (i.e. W and W^- are compact) with respect to the local flow Φ if the following conditions hold:

- there exists a compact subset W^{--} of W^- (called the essential exit set) such that
$$W^- = W^{--} \cup (\{T\}\times W_T), \quad W^- \cap ([0,T)\times M) \subset W^{--},$$
- there exists a homeomorphism $h\colon [0,T]\times W_0 \to W$ such that $\pi_t \circ h = \pi_t$ and
$$h([0,T]\times W_0^{--}) = W^{--},$$
- $W_0 = W_T$, $W_0^{--} = W_T^{--}$,
- W and W^- are ENRs.

We define the corresponding homeomorphism

$$m\colon (W_0, W_0^{--}) \ni x \to \pi_x h(T, \pi_x h^{-1}(0,x)) \in (W_0, W_0^{--}),$$

called a *monodromy map*, and the isomorphism induced in homologies

$$\mu_W = H(m)\colon H(W_0, W_0^{--}) \to H(W_0, W_0^{--}).$$

Theorem 3.2 ([26], [27], [31]). *Let W be a periodic segment over $[0,T]$. Then the set*

$$U_W := \{x \in W_0 : \phi_{(0,t)}(x) \in W_t \setminus W_t^{--} \textit{ for all } t \in [0,T]\}$$

is open in W_0 and the set of fixed points of the restriction $\phi_{(0,T)}|_{U_W}\colon U_W \to W_0$ is compact. Moreover,

$$L(\mu_W) = \operatorname{ind}(\phi_{(0,T)}|_{U_W}).$$

In particular, if $L(\mu_W) \neq 0$, then the equation $\dot{x} = f(t,x)$ has a T-periodic solution passing through W_0 at time 0.

Example 3.3. Consider a planar non-autonomous equation

$$\dot{z} = \overline{z} + f(t,z), \quad z \in \mathbb{C} \tag{3.1}$$

where $f\colon \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is a smooth function T-periodic with respect to t for some $T > 0$. Assume that

$$\frac{f(t,z)}{|z|} \to 0, \text{ as } |z| \to \infty \text{ uniformly in } t.$$

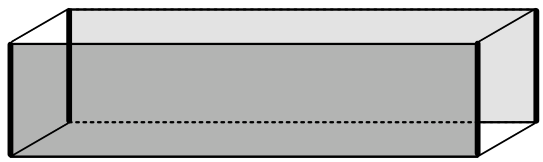


Figure 2. An isolating segment over $[0,T]$ for the equation (3.1)

For $r > 0$ we put

$$\begin{aligned} V_1\colon \mathbb{R} \times \mathbb{R}^2 \ni (t,x,y) &\to \frac{x^2}{r^2} - 1 \in \mathbb{R}, \\ V_2\colon \mathbb{R} \times \mathbb{R}^2 \ni (t,x,y) &\to \frac{y^2}{r^2} - 1 \in \mathbb{R}. \end{aligned}$$

If $r > 0$ is sufficiently large then

$$\nabla V_1(t,x,0) \cdot v(t,x,0) > 0, \quad |x| = r, \ t \in \mathbb{R},$$

where $v(t,x,y) = (1, f_1(t,x,y), f_2(t,x,y))$. Indeed,

$$\begin{aligned} \nabla V_1(t,x,0) \cdot v(t,x,0) &= \left(0 \frac{2x}{r^2}, 0\right) \cdot (1, x + f_1(t,x,0), f_2(t,x,0)) \\ &= \frac{2x^2}{r^2} + \frac{2x f_1(t,x,0)}{r^2} \geq \frac{2x^2}{r^2}(1 - \frac{f_1(t,x,0)}{|x|}) > 0, \end{aligned}$$

if $|x| = r$ is sufficiently large. In a similar way, we get that

$$\nabla V_2(t,0,y) \cdot v(t,0,y) < 0, \quad |y| = r, \ t \in \mathbb{R},$$

for sufficiently large r. It follows, that $W = [0,T] \times [-r,r] \times [-r,r]$ is a periodic segment for the equation (3.1) with the essential exit set $W^{--} = [0,T] \times \{-r,r\} \times [-r,r]$ and such that

$$L(\mu_W) = L(\mathrm{id}_{H(W_0, W_0^{--})}) = \chi(W_0) - \chi(W_0^{--}) = -1,$$

hence there exists a T-periodic solution of the equation (3.1).

Example 3.3. Let us modify the previous example to the equation

$$\dot{z} = e^{it}\overline{z} + f(t,z), \quad z \in \mathbb{C} \tag{3.2}$$

where $f\colon \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is a smooth function 2π-periodic with respect to t. Assume that

$$\frac{f(t,z)}{|z|} \to 0, \quad \text{as } |z| \to \infty \text{ uniformly in } t.$$

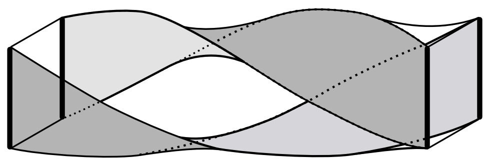


Figure 3. An isolating segment over $[0,2\pi]$ for the equation (3.2)

A direct calculation shows that there exists a periodic segment W over $[0,2\pi]$ depicted in Figure 3. It follows that $L(\mu_W) = 1$, hence there exists a 2π-periodic solution of the equation (3.2).

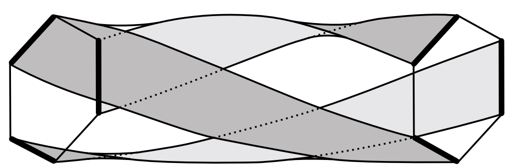
Corollary 3.5. *Assume that $U \subset W$ are two periodic segments over $[0,T]$ for the T-periodic equation $\dot{x} = v(t,x)$. If $L(\mu_U) \neq L(\mu_W)$ then there exists a T-periodic solution u such that*

$$u(t) \in W_t, \quad t \in [0,T]$$

and there is $t_0 \in [0,T]$ such that $u(t_0) \notin U_{t_0}$.

Example 3.6. Let us consider a 2π-periodic equation

$$\dot{z} = e^{it}\overline{z}^2 + \overline{z}, \quad z \in \mathbb{C}. \tag{3.3}$$



Figire 4. Isolating segment W for equation (3.3)

Observe that the zero solution is 2π-periodic, so one should look for a nontrivial one. One can check there is a large segment W for the equation being a twisted prism with hexagonal base centered at the origin and its time sections W_t are obtained by rotating the base with the angular velocity $1/3$ over time interval $[0,2\pi]$. The essential exit set W^{--} consists of 3 disjoint ribbons winding around the prism. One can choose the rotation by the angle $2\pi/3$ as a monodromy map of the segment W, hence $L(\mu_W) = 1$. It can be proved that there is another segment $U \subset W$ for that equation. It is a prism having a sufficiently small square centered at origin as a base and such that $L(\mu_U) = -1$. Since the zero solution is contained in the segment U, so by the above corollary we get a non-trivial solution.

Remark 3.7. Assume that Φ is a local flow generated by smooth vector field $v\colon M \to TM$ on a manifold M. Then a corresponding local process is given by $\phi_{(a,t)} := \Phi_t$ for each $a \in \mathbb{R}$. If B is a compact Ważewski set for Φ such that B, B^- are ENRs, then $[0,T] \times B$ is periodic segment over $[0,T]$ and its proper exit set is equal to $W = [0,T] \times B^-$. Since the monodromy map is the identity, one has that $L(\mu_W) = \chi(B) - \chi(B^-)$. It follows that the set

$$U = \{x \in B : \Phi_t(x) \in B \setminus B^- \text{ for all } t \in [0,T]\}$$

is an open subset of B and the set of fixed points of the restriction $\Phi_T|_U\colon U \to W$ is compact. Moreover,

$$\operatorname{ind}(\Phi_T|_U) = \chi(B) - \chi(B^-).$$

In particular, if $\chi(B) \neq \chi(B^-)$ then Φ_T has a fixed point in B. One can check that, by the compactness of B, there exists a stationary point in B.

We say that a flow $\Phi\colon \mathbb{R} \times M \to M$ is bounded if there exist a compact set $K \in M$ such that

$$K \cap \Phi([0,\infty) \times \{x\}) \neq \emptyset$$

for all $x \in M$. As an application of the above discussion one can prove the following theorem.

Theorem 3.8 ([29])**.** *If $\Phi\colon \mathbb{R} \times M \to M$ is bounded and $\chi(M) \neq 0$, then Φ has a stationary point.*

Remark 3.9. Let Φ be a local flow generated by the smooth vector field $v\colon \mathbb{R}^n \to \mathbb{R}^n$. In that case (see [19], [27]), if B is an isolating block for Φ such that B, B^- are ENRs then

$$\deg(0, v, \operatorname{int} B) = (-1)^n(\chi(B) - \chi(B^-)),$$

where $\deg(0, v, \operatorname{int} B)$ is a Brouwer degree.

We recall that a *Lusternik–Schnirelmann category* $\operatorname{cat}(X)$ of topological space X is an element of $\mathbb{N} \cup \{\infty\}$ such that $\operatorname{cat}(X) \leq n$, if there exist closed sets $A_i \subset X$ such that $X = \bigcup_{i=1}^n$ and A_i is contractible in X for $i = 1, \dots, n$. We say that

$A \subset X$ is *contractible in* X if there exists a continuous map $h: A \times [0,1] \to X$ such that

$$\begin{aligned} h(x,0) &= x, && x \in A, \\ h(x,1) &= h(y,1), && x,y \in A. \end{aligned}$$

The classical result of the critical point theory says that if $f: M \to \mathbb{R}$ is a smooth map on compact, connected Riemannian manifold M, then f has at least $\operatorname{cat}(M)$ critical points, i.e. such that derivative of f is zero. Observe that critical points of f corresponds to the stationary points of the gradient flow Φ generated by the equation

$$\dot{x} = -\nabla f(x),$$

where $\nabla f: M \to TM$ is a gradient of f.

We say that a (local) flow Φ is a *gradient like* if there is a continuous function $g : M \to \mathbb{R}$ such that if x is not a stationary point, then

$$g(\Phi(t,x)) < g(\Phi(s,x)), \quad s < t.$$

Theorem 3.10 ([27])**.** *If Φ is a gradient like, $B \subset M$ is an isolating block such that B, B^- are ENRs, then*

$$\operatorname{card}\{x \in B : x \textit{ stationary}\} \geq \operatorname{cat}(B/B^-) - 1,$$

where B/B^- is a quotient space.

By definition, $X/\emptyset$ is equal to $X \cup \{[\emptyset]\}$ and its topology is equal to the sum topology of X and the one-point space $\{[\emptyset]\}$.

Remark 3.11. The method of guiding functions for detecting periodic solutions of the equation $x' = f(t,x)$, where $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth and T-periodic with respect to t was developed in [11], [12]. We recall that $V: \mathbb{R}^n \to \mathbb{R}$ is a *guiding function* for the vector field f if there is $R > 0$ such that

$$\nabla V(x) \cdot f(t,x) > 0$$

for $\|x\| \geq R$, $t \in \mathbb{R}$. We define *the index of* V by

$$\operatorname{Ind} V = \deg(0, \nabla V, D_R),$$

where $D_R = \{x \in \mathbb{R}^n : \|x\| \leq R\}$ and deg is a Brouwer degree.

Let $V_1, \ldots, V_k: \mathbb{R}^n \to \mathbb{R}$ be a guiding functions for the vector field f. We say that $V_1, \ldots, V_k$ is a *complete set of guiding functions* if

$$\lim_{\|x\| \to \infty} |V_1(x)| + \ldots + |V_k(x)| = \infty.$$

Theorem 3.12 ([12])**.** *If $V_1, \dots, V_k$ is a complete set of guiding functions for f and $\operatorname{Ind} V_1 \neq 0$ then the equation $x' = f(t,x)$ has at least one T-periodic solutions.*

It was proved in [10], that if $V_1, \dots, V_k$ is a complete set of guiding functions for f, then there exists a periodic segment W over $[0,T]$ such that

$$L(\mu_W) = (-1)^n \operatorname{Ind} V_1,$$

hence Theorem 3.12 is a special case of Theorem 3.2.

4. Detection of chaotic dynamics

In order to formulate results on chaotic dynamics we use the notion of *shift on* 2 *symbols*. It is a pair (Σ_2, σ), where $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ is the set of bi-infinite sequences of 2 symbols (called the shift space), and the shift map $\sigma\colon \Sigma_2 \to \Sigma_2$ given by

$$\sigma(\dots s_{-1}.s_0 s_1 \dots) = (\dots s_0.s_1 s_2 \dots),$$

hence σ moves the sequence by one position to the left. The shift space Σ_2 is a compact metric space with the product topology and the shift map $\sigma\colon \Sigma_2 \to \Sigma_2$ is a homeomorphism.

By $c_k^n \in \Sigma_2$ we will denote a n-periodic sequence of symbols 0 and 1 such that 1 appears $k \in \{0, \dots, n\}$ times in $(c_0, \dots, c_{n-1})$. Let $\Sigma_2(n,k) \subset \Sigma_2$ be the set of all such sequences.

For a periodic isolating segment W we consider a Lefschetz sequence given by

$$L_k = L(\mu_W^k), \quad k \geq 0.$$

It is well-defined, since W_0, W_0^- are compact ENRs, so $H(W_0, W_0^-)$ is of finite type. We also define the *dual sequence* $\{L_k^*\}_{k\geq 0}$ of $\{L_k\}_{k\geq 0}$ by

$$L_k^* = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} L_l, \quad k \geq 0.$$

In particular, $L_0^* = L_0 = L(I) = \chi(E)$.

We assume that $U \subset W$ are two periodic isolating segments over $[0,T]$ for the equation $\dot{x} = f(t,x)$ (with the vector field f being T-periodic in time) such that

$$U_0 = W_0, \quad U_0^{--} = W_0^{--}, \quad \mu_U = \operatorname{id}_{H(W_0, W_0^-)}.$$

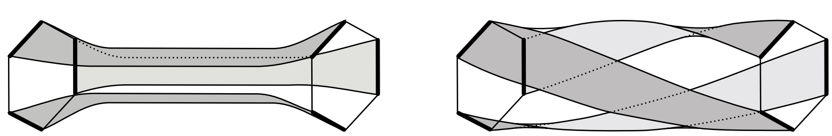


FIGURE 5. Two isolating segments with equal 0-sections

The following was essentially proved in [33] (see also [30], [31]).

Theorem 4.1. *There are a compact set $I \subset W_0$ invariant for the Poincaré map P and a continuous map $g\colon I \to \Sigma_2$ such that*

$$\sigma \circ g = g \circ P,$$

and with the property that $L_k^* \neq 0$ implies that for all sequence c_k^n, there exists $x \in g^{-1}(c_k^n)$ such that $P^n(x) = x$.

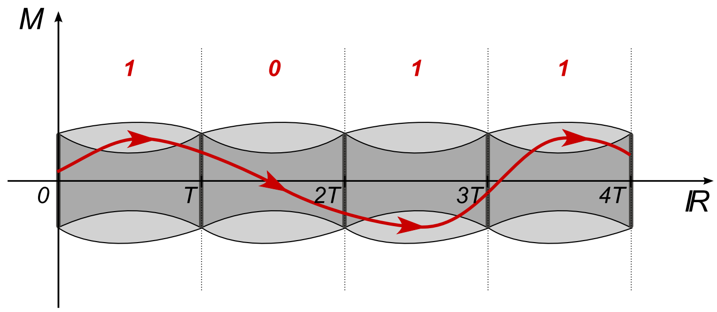


FIGURE 6. The map g. If the trajectory of the point $x \in I$ leaves the smaller segment U on the time interval $[kT, (k+1)T]$, then $g(x)_k = 1$. Otherwise its equal to 0.

The set I is the set of all points in W_0 whose full trajectories are contained in the segment W, i.e.

$$I = \bigcap_{n=-\infty}^{\infty} \{x \in W_0 : \phi_{(0,t+nT)}(x) \in W_t \text{ for all } t \in [0,T]\}.$$

For $x \in I$ we define $g(x) \in \Sigma_2$ by the following rule:

- if on the time interval $[iT, (i+1)T]$ the trajectory of x is contained in the smaller segment U, then $g(x)_i = 0$,
- if $\phi^i_{(0,T)}(x)$ leaves U in time less then T, then $g(x)_i = 1$.

It follows that $g\colon I \to \Sigma_2$ is continuous and $\sigma \circ g = g \circ P$. The last part of the above theorem is a consequence of the next lemma ([33], [31]).

For the sequence c_k^n we define a periodic segment $W(c_k^n)$ over $[0, nT]$ obtained by gluing translated copies of segments U and W with the following rule: $W(c_k^n)|_{[iT,(i+1)T]} = W$ if $c_i = 1$ and $W(c_k^n)|_{[iT,(i+1)T]} = U$ if $c_i = 0$ $(i = 0, \dots, n-1)$.

Lemma 4.2. *The set*

$$V = \{x \in W_0 : \phi_{(0,t)}(x) \in W(c_k^n)_t \setminus W(c_k^n)_t^{--} \textit{ for all } t \in [0, nT]\} \cap g^{-1}(c_k^n)$$

is open in W_0*. Moreover, the set of fixed points of the restriction* $P^n|_V : V \to W_0$ *is compact and*

$$\operatorname{ind}(P^n|_V) = L_k^*.$$

We define the sets

$$\mathbb{P}^n = \{x \in I : P^n(x) = x\}, \quad n \geq 1, \qquad \mathbb{P}_n = \mathbb{P}^n \setminus \bigcup_{1 \leq k < n} \mathbb{P}^k, \qquad \mathbb{P} = \bigcup_{n \geq 1} \mathbb{P}^n.$$

It follows that the condition $L_k^* \neq 0$ implies that $g^{-1}(c_k^n) \cap \mathbb{P}^n \neq \emptyset$, for all $n \geq k$ and all sequences $c_k^n \in \Sigma_2(n, k)$. Let us observe that by the compactness of $g(I)$ we have

$$\overline{\{c_k^n : L_k^* \neq 0,\ n \geq k\}} \subset g(I) \subset \Sigma_2,$$

so if $\{c_k^n : L_k^* \neq 0,\ n \geq k\}$ is dense in Σ_2, then g is surjective. We define a set

$$\nu(L) = \{k \in \mathbb{N} : L_k^* \neq 0\}.$$

Corollary 4.3. *If* $\nu(L)$ *is infinite, then* g *is surjective. In particular, the topological entropy of* f *is positive.*

We show that in the context described above the map g is usually surjective and the set $\mathbb{P}$ of periodic points of P is infinite.

We recall that the Möbius function $\mu : \mathbb{N} \to \mathbb{Z}$ is given by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \cdot \ldots \cdot p_k,\ p_i \text{ different primes}, \\ 0 & \text{otherwise}. \end{cases}$$

We say that a sequence a_n of integers satisfies the Dold's relations (see [9]) if, for all $n \in \mathbb{N}$,

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) a_d \cong 0 \mod n.$$

In particular, if p is prime then $a_1 \cong a_p \bmod p$.

Proposition 4.4 ([13])**.** *For the authomorphism* μ_W *the following conditions hold*

(a) $L_k^* = L((\mu_W - I)^k)$ *for* $k > 0$,
(b) *sequences* $\{L_k\}$ *and* $\{L_k^*\}$ *satisfy Dold's relations,*
(c) *If* L_k *is* m*-periodic* ($m > 1$)*, then there exists* k_0 *such that* $L_{2mk}^* \neq 0$ *for* $k > k_0$*. Moreover, there exists* $\rho > 1$ *such that*

$$\lim_{k \to \infty} \frac{L_{2mk}^*}{\rho^{2mk}} = a \neq 0.$$

In particular, the sequence $\{L_k^*\}$ *is unbounded.*

Example 4.5. Let $\{L_k\}$ be a Lefschetz sequence such that $L_1 \neq L_0$. We show that $\{p \in \mathbb{N} : |L_1 - L_0| < p,\ p \text{ prime}\} \subset \nu(L)$. Indeed, suppose that $L_p^* = 0$ for some prime $p > |L_1 - L_0|$. Since $L_1^* = L_1 - L_0$, and the dual sequence L_k^* satisfies the Dold's relations, so $0 = L_p^* \cong L_1^* \bmod p$, hence $p | L_1 - L_0 \neq 0$, and we get a contradiction. In particular, g is surjective.

The following result, concerning the periodic Lefschetz sequences of the type $(L_0, L_1, L_1, \dots, L_1, L_0, L_1, \dots)$ was essentially proved in [24].

Proposition 4.6. *Let $\{L_k\}$ be a Lefschetz sequence such that $L_1 \neq L_0$. If $\{L_k\}$ is m-periodic and $L_1 = \dots = L_{m-1} \neq L_m = L_0$, then*

(a) $\nu(L) = \mathbb{N}$, *for an even m,*
(b) $\nu(L) = \mathbb{N} \setminus \{nm : n\text{– odd}\}$, *for an odd m.*

Example 4.7. If $\{L_k\}$ is p-periodic with p odd prime, then $\nu(L) = \mathbb{N} \setminus \{np : n - \text{odd}\}$. It is sufficient to show that $L_1 = \dots = L_{p-1}$. Let $(r, p) = 1$. We prove that $L_r = L_1$. Since $(r, p) = 1$, then by Dirichlet's theorem there exists $n_k \to \infty$ such that $p_k = r + n_k p$ is a prime number for each $k \in \mathbb{N}$. In particular, there is a k such that $p_k > |L_r - L_1|$. Then, from the Dold's relation follows that $L_r = L_{p_k} \cong L_1 \bmod p_k$, so we get that $p_k | L_r - L_1$, hence $L_r = L_1$.

Theorem 4.8 ([7])**.** *Assume that $\{L_k\}_{k \geq 0}$ is p-periodic with p prime and $L_1 \neq L_0$. Then $g \colon I \to \Sigma_2$ is surjective. Moreover,*

(a) *if $p = 2$, then $g^{-1}(c_k^n) \cap \mathbb{P}^n$ is non-empty for each n-periodic sequence c_k^n and for q prime we have*

$$\operatorname{card} \mathbb{P}_q \geq 2^q - 2.$$

(b) *if p is odd prime, then the set $g^{-1}(c_k^n) \cap \mathbb{P}^n$ is non-empty for each n-periodic sequence c_k^n such that k is not an odd multiplicity of p and for q prime we have*

$$\operatorname{card} \mathbb{P}_q \geq \sum_{k \in C(q)} \binom{q}{k},$$

where $C(q) = \{1 \leq k < q : p \nmid k \text{ or } k \text{ is even}\}$.

Example 4.9. The method developed in this paper was motivated by the study of the dynamics generated by the equation

$$z' = (1 + e^{i\eta t}|z|^2)\overline{z}, \quad z \in \mathbb{C}. \tag{4.1}$$

It was first studied in [30], [34], where the symbolic dynamics was proved for the parameter value $\eta \in (0, 0.495]$. The authors showed that the equation (4.1) satisfies the assumptions of Theorem 4.8 with $p = 2$. Later it was improved in [21] to the parameter value $\eta \in (0, 0.5044]$.

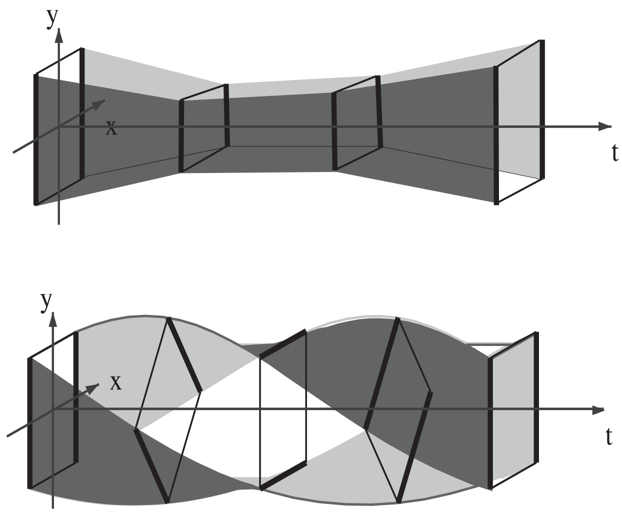


Figure 7. Two isolating segments with equal 0-sections for the equation (4.1)

The dynamics generated by this equation is very difficult to computer simulating. The equation exhibits extremely strong expansion and most of its trajectories escape to infinity in a very short time. The method introduced in [18] shows how to go around this problem using partial Poincaré sections and the main result in [18] prove that the equation (4.1) is chaotic for $\eta = 1$.

Example 4.10. The equation

$$\dot{z} = \overline{z}^2(1 + |z|^2 e^{i\eta t}), \quad z \in \mathbb{C} \tag{4.2}$$

satisfies the assumptions of the above theorem (with $p = 3$) for sufficiently small parameters values $\eta > 0$. The Lefschetz sequence for the bigger segment W is given by $(-2, 1, 1, -2, 1, 1, \ldots)$.

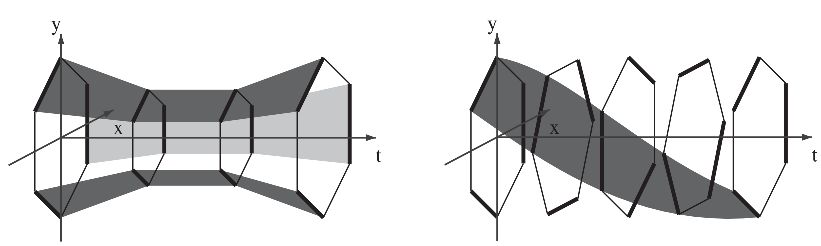


Figure 8. Two isolating segments with equal 0-sections for the equation (4.2)

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