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Dedicated to Professor Lech Górniewicz

PREFACE

The present volume contains a selection of papers submitted by the participants of the Third Polish Symposium on Nonlinear Analysis held in Łódź, January 29–31, 2001, and organized by the Faculty of Mathematics of the Łódź University and the Juliusz Schauder Center of Nonlinear Studies at the Nicholas Copernicus University in Toruń.

The main purpose of this Symposium was to integrate the large group of Polish researchers interested in different aspects of nonlinear problems, to present their recent results and to create a convenient platform for the exchange of scientific information and experience. Nonlinear Analysis is a major branch of mathematics encompassing various problems arising in mathematical, functional and convex analysis, topology, fixed point theory and their applications in the theory of ordinary and partial differential equations, inclusions and the dynamical systems, control and game theories. There is a number of strong Polish scientific centers where these topics are extensively studied.

During the Symposium a special session celebrating the 60-th anniversary of Professor Lech Górniewicz was organized. Professor Górniewicz, one of the leading specialists in the field of Nonlinear Analysis, is the head and a co-founder (together with Professor Andrzej Granas) of the Schauder Center, the Managing Editor of the journal "Topological Methods in Nonlinear Analysis" published by the Schauder Center and one of the persons promoting the development of nonlinear studies in Poland. The contributing authors and the editors are proud to dedicate this volume to Professor Górniewicz.

The papers, received by the editors in Fall 2001, were refereed and appear in alphabetical order.

The organizers of the Symposium and the editors express their gratitude to all the participants, the authors and all other persons who contributed to the program and activities of the Symposium, and to the publishers of the Lecture Notes of the Juliusz Schauder Center of Nonlinear Studies, and the Nicholas Copernicus University for the their help in preparing this volume for publication.

Wojciech Kryszewski
Andrzej Nowakowski

Łódź–Toruń, February 2002

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POINCARÉ'S TRANSLATION MULTIOPERATOR REVISITED

Jan Andres

Dedicated to Professor Lech Górniewicz on the occasion of his 60th birthday

Abstract. Poincaré's translation multioperator is revisited for the associated systems of ordinary, functional, random and discontinuous differential equations and inclusions (with or without constraints) in Euclidean as well as in Banach spaces. Applications are related to periodic solutions and, less traditionally, to other types of boundary value problems. Existence and multiplicity results are presented on the basis of our recent papers [5]–[7].

1. Introduction (historical remarks)

Poincaré's idea of the translation operator along the trajectories of differential systems comes back to the end of the nineteenth century ([33]). Since it was effectively applied for investigating periodic orbits by A. A. Andronow ([10]) in the late 20's and by N. Levinson ([28]) in 1944, its name is also sometimes related to them. This topic became popular due to the monographs [26], [32] of M. A. Krasnosel'skiĭ and V. A. Pliss, dealing with ODEs, and [22] of J. K. Hale, dealing with functional differential equations. On the other hand, comparing the results (see e.g. [34] and the references therein) with those obtained by functional-analytic methods, one should have additionally assumed the uniqueness for Cauchy problems. This obstruction might have been eliminated by applying the standard limiting argument (see e.g. [23], [26]), but such

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a procedure can be considered (especially w.r.t. differential inclusions) as rather technical.

It is difficult to recognize when the more promissible (in the above light) idea of multivalued translation operator appeared for the first time. Perhaps in Remarque 12 of [27] saying that "La multi-application $x_0 \to x(t) \mid x \in \mathcal{T}$, $x(0) = x_0$ est un exemple naturel de multi-application pseudo-acyclique", where $\mathcal{T}$ denotes the set of solutions of a Carathéodory system of inclusions. In Chapter III of [12], entitled *Existence without uniqueness*, where [27] is quoted, the multivalued translation operator has been proved to be pseudoacyclic (for the definitions, see below) and then applied for the existence of (harmonic) periodic solutions of differential equations in Banach spaces.

Nevertheless, the systematic study of admissible (= pseudoacyclic) maps and, in particular, translation multivalued operators, was allowed after Ph. D. Thesis of L. Górniewicz ([17], cf. also [19]). Since 1976, i.e. the year of publication of both [27] and [17], Poincaré's multioperator has been treated and applied on various levels of abstraction (see e.g. [1]–[3], [5]–[9], [11]–[15], [17]–[21], [24], [25], [29]–[31]).

Below, these levels will be considered in detail separately. Finally, some very recent nontrivial applications in [5]–[7] will be given. However, before doing it, we need to recall some facts (for more details, see e.g. [19]).

Let X_1 and X_2 be two metric spaces. We say that $\varphi: X_1 \to 2^{X_2} \setminus \{\emptyset\}$ is a *multivalued mapping* form X_1 to X_2, and we write $\varphi: X_1 \rightsquigarrow X_2$.

A multivalued mapping $\varphi: X_1 \rightsquigarrow X_2$ is *upper semi continuous* (u.s.c.) if, for any open subset $B \subset X_2$, $\{x \in X_1 : \varphi(x) \subset B\}$ is an open subset of X_1.

A metric space X_2 is called an *absolute retract* (AR) or an *absolute neighbourhood retract* (ANR) if, for every metrizable X_1 and any closed $A \subset X_1$, every continuous mapping $f: A \to X_2$ is extendable over X_1 or over an open neighbourhood U of A in X_1, respectively.

By an *R_δ-set*, we mean the one homeomorphic to the intersection of a decreasing sequence of compact AR-spaces and by an *R_δ-mapping* the u.s.c. one with R_δ-values.

An R_δ-set is well-known to be, in particular, nonempty, compact and *acyclic* (w.r.t. any continuous theory of cohomology), i.e. it is homologicaly the same as a one point space. By an *acyclic mapping*, we mean the u.s.c. one with nonempty, compact and acyclic values.

An u.s.c. mapping is called *admissible* (in the sense of [17]; cf. also [19]) if it possesses a multivalued selection which can be composed by acyclic maps.

Furthermore, let X_1 be a real Banach space and $\mathcal{B}(X_1)$ the family of all nonempty, bounded subsets of X_1. Then the function $\chi: \mathcal{B}(X_1) \to [0, \infty)$ defined by

$$\chi(X_1) := \inf\{r > 0 : \Omega \text{ can be covered by finitely many balls of radius } r\}$$

is called the *Hausdorff measure of noncompactness* (MNC). It is well-known that χ is monotone, i.e. $\Omega_1 \subset \Omega_2$ implies $\chi(\Omega_1) \leq \chi(\Omega_2)$, and algebraically semiadditive, i.e. $\chi(\Omega_1 + \Omega_2) \leq \chi(\Omega_1) + \chi(\Omega_2)$, for bounded $\Omega_1, \Omega_2 \subset X_1$.

An u.s.c. self-mapping $\varphi\colon X_1 \multimap X_1$, where X_1 is a real Banach space, is called *condensing* if, for every bounded $\Omega \subset X_1$, the set $\varphi(\Omega) \subset X_1$ is bounded and such that

$$\chi(\varphi(\Omega)) < \chi(\Omega), \quad \text{whenever } \chi(\Omega) \neq 0.$$

Finally, by a fixed-point of a multivalued mapping $\varphi\colon X_1 \multimap X_2$, we mean the point $\widehat{x} \in X_1$ with $\widehat{x} \in \varphi(\widehat{x})$.

2. Translation multioperator for ordinary systems

Consider the Carathéodory system

$$X' \in F(t, X), \quad X \in \mathbb{R}^n, \tag{1}$$

where

(i) the set of values of F is nonempty, compact and convex, for all $(t, X) \in [0, \tau] \times \mathbb{R}^n$,
(ii) $F(t, \cdot\,)$ is u.s.c. for a.a. $t \in [0, \tau]$,
(iii) $F(\,\cdot\,, X)$ is measurable, for every $X \in \mathbb{R}^n$, i.e. for any open $U \subset \mathbb{R}^n$ and every $X \in \mathbb{R}^n$, the set $\{t \in [0, \tau] : F(\,\cdot\,, X) \cap U \neq \emptyset\}$ is measurable,
(iv) $|F(t, X)| \leq \alpha + \beta|X|$, for every $X \in \mathbb{R}^n$ and a.a. $t \in [0, \tau]$, where α, β are suitable positive constants.

By a *solution* $X(t)$ of (1), we mean an absolutely continuous function $X(t) \in AC([0, \tau], \mathbb{R}^n)$ satisfying (1), for a.a. $t \in [0, \tau]$, i.e. the one in the sense of Carathéodory; such solutions of (1) exist on $[0, \tau]$.

Hence, if $X(t, X_0) := X(t, 0, X_0)$ is a solution of (1) with $X(0, X_0) = X_0 \in \mathbb{R}^n$, then the translation multioperator $T_\tau\colon \mathbb{R}^n \multimap \mathbb{R}^n$ at the time $\tau > 0$ along the trajectories of (1) is defined as follows:

$$T_\tau(X_0) := \{X(\tau, X_0) : X(\,\cdot\,, X_0) \text{ is a solution of (1) with } X(0, X_0) = X_0\}. \tag{2}$$

More precisely, T_τ can be considered as the composition of two maps, namely $T_\tau = \psi \circ \varphi$,

$$\mathbb{R}^n \overset{\varphi}{\multimap} AC([0, \tau], \mathbb{R}^n) \overset{\psi}{\to} \mathbb{R}^n,$$

where

$$\varphi(X_0)\colon X_0 \multimap \{X(t, X_0) : X(t, X_0) \text{ is a solution of (1) with } X(0, X_0) = X_0\}$$

is well-known (cf. e.g. [19] and the references therein) to be an R_δ-mapping and $\psi(y)\colon y(t) \to y(\tau)$ is obviously a continuous (single-valued) evaluation mapping.

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \overset{\mathscr{L}}{\rightsquigarrow} & AC([0,\tau],\mathbb{R}^n) \\ & \ddots & \downarrow \psi \\ & & \overset{T_\tau}{\rightsquigarrow} \mathbb{R}^n \end{array}$$

The following characterization of T_τ has been proved on various levels of abstraction in [12], [14], [20], [29], [27], etc.

Theorem 1. *T_τ defined by* (2) *is admissible and homotopic to identity. More precisely, T_τ is a composition of an R_δ-mapping and a continuous* (*single-valued*) *evaluation mapping.*

Remark 1. Since a composition of admissible maps is admissible as well (cf. [19]), T_τ can be still composed with further admissible maps ϕ such that $\phi\circ T_\tau$ becomes an (admissible) self-map on a compact ENR-space (i.e. homeomorphic to ANR in $\mathbb{R}^n$), for computation of the well-defined (cf. [19]) generalized Lefschetz number:

$$\Lambda(\phi \circ T_\tau) = \Lambda(\phi).$$

T_τ considered on ENRs can be even composed e.g. with suitable homomorphisms $\mathcal{H}$ (again considered on ENRs), namely $\mathcal{H}\circ T_\tau$, for computation of the well-defined (cf. e.g. [19]) fixed-point index:

$$\operatorname{ind}(\mathcal{H} \circ T_\tau) = \operatorname{ind}\mathcal{H},$$

provided the fixed-point set of $\mathcal{H} \circ T_{\lambda\tau}$ is compact, for $\lambda \in [0,1]$.

3. Translation multioperator for functional systems

Consider the functional system

$$X' \in F(t, X_t), \quad X \in \mathbb{R}^n, \tag{3}$$

where $X_t(\,\cdot\,) = X(t + \,\cdot\,)$, for $t \in [0,\tau]$, denotes as usual a function from $[-\delta, 0]$, $\delta \geq 0$, into $\mathbb{R}^n$ and $F\colon [0,\tau] \times \mathcal{C} \rightsquigarrow \mathbb{R}^n$, where $\mathcal{C} = AC([-\delta,0],\mathbb{R}^n)$, is a Carathéodory multifunction, i.e.

(i) the set of values of $F(t,Y)$ is nonempty, compact and convex, for all $(t,Y) \in [0,\tau] \times \mathcal{C}$,
(ii) $F(t,\,\cdot\,)$ is u.s.c., for a.a. $t \in [0,\tau]$,
(iii) $F(\,\cdot\,,Y)$ is measurable, for all $Y \in \mathcal{C}$, i.e. for any open $U \subset \mathbb{R}^n$ and every $Y \in \mathcal{C}$, the set $\{t \in [0,\tau] : F(\,\cdot\,,Y) \cap U \neq \emptyset\}$ is measurable,
(iv) $|F(t,Y)| \leq \alpha + \beta\|Y\|$, for every $Y \in \mathcal{C}$ and a.a. $t \in [0,\tau]$, where α, β are suitable positive constants.

By a *solution* $X(t)$ of (the initial problem to) (3), we mean again an absolutely continuous function $X(t) \in AC([-\delta,\tau],\mathbb{R}^n)$ (with $X(t) = X_*$, $t \in [-\delta, 0]$,) satisfying (3), for a.a. $t \in [-\delta,\tau]$; such solutions exist on $[-\delta,\tau]$, $\delta \geq 0$.

Hence, if $X(t,X_*) := X(t,[-\delta,0],X_*)$ is a solution of (3) with $X(0,X_*) = X_* \in E$, for $t \in [-\delta,0]$, where E consists of equicontinuous functions, then the translation multioperator $T_\tau\colon AC([-\delta,0],\mathbb{R}^n) \multimap AC([-\delta,0],\mathbb{R}^n)$ at the time $\tau > 0$ along the trajectories of (3) is defined as follows:

$$T_\tau(X_*) := \{X(\tau,X_*) : X(\,\cdot\,,X_*) \text{ is a solution of (3) with } X(t,X_*) = X_*, \text{ for } t \in [-\delta,0]\}. \tag{4}$$

More precisely, T_τ can be considered as the composition of two maps, namely $T_\tau = \psi \circ \varphi$,

$$AC([-\delta,0],\mathbb{R}^n) \overset{\varphi}{\multimap} AC([-\delta,\tau],\mathbb{R}^n) \overset{\psi}{\to} AC([-\delta,0],\mathbb{R}^n),$$

where $\varphi(X_*)\colon X_* \multimap \{X(t,X_*) : X(t,X_*)$ is a solution of (3) with $X(t,X_*) = X_*$, for $t \in [-\delta,0]\}$ is known (cf. e.g. [30]) to be an R_δ-mapping and $\psi(y)\colon y(t) \to y(\tau)$ is a continuous (single-valued) evaluation mapping.

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} AC([-\delta,0],\mathbb{R}^n) & \overset{\varphi}{\multimap} & AC([-\delta,\tau],\mathbb{R}^n) \\ & \ddots & \downarrow \psi \\ & \overset{T_\tau}{\multimap} & AC([-\delta,0],\mathbb{R}^n) \end{array}$$

The following characterization of T_τ has been proved on various levels of abstraction in [15], [21], [24], [30], etc.

Theorem 2. *T_τ defined by* (4) *is admissible and homotopic to identity. More precisely, T_τ is a composition of an R_δ-mapping and a continuous* (*single-valued*) *evaluation mapping.*

Remark 2. Theorem 2 reduces to Theorem 1, for $\delta = 0$, and Remark 1 can be appropriately modified here as well.

4. Translation multioperator for systems with constraints

In view of Remark 2, consider again system (3), where $F\colon [0,\tau] \times \mathcal{C} \multimap \mathbb{R}^n$ is the same as above. For a nonempty, compact and convex set $K \subset \mathbb{R}^n$, the constraint, denote $\mathcal{K} = \{\xi \in \mathcal{C} : \xi(t) \in K, \text{ for } t \in [-\delta,0]\}$ and assume that the Nagumo-type condition holds,

$$F(t,Y) \cap T_K(Y(0)) \neq \emptyset, \quad \text{for all } (t,Y) \in [0,\tau] \times \mathcal{K}, \tag{5}$$

where

$$T_K(Y(0)) = \left\{ y \in \mathbb{R}^n : \liminf_{h \to 0+} \frac{d(Y(0)+hy,K)}{h} = 0 \right\}$$

is the tangent cone (in the sense of Bouligand). Observe, that (iv) can be reduced to

(iv') $\sup_{(t,Y)\in[0,\tau]\times\mathcal{K}} |F(t,Y)| < \infty$.

Then, for every $X_* \in \mathcal{K}$, there exists at least one Caratéodory solution $X(t, X_*)$ of (3) (see e.g. [24]) such that $X(t, X_*) = X_* \in E$, for $t \in [-\delta, 0]$, and $X(t, X_*) \in K$, for $t \in [0, \tau]$. Hence, we can define, under (5), the associated translation multioperator $T_\tau: \mathcal{K} \rightsquigarrow \mathcal{K}$ at the time $\tau > 0$ along the trajectories of (3), which makes the set $\mathcal{K}$ invariant, as follows:

$$
(6) \quad T_\tau(X_*) := \{X(\tau, X_*) : X(\,\cdot\,, X_*) \text{ is a solution of (3) with } X(t, X_*) = X_* \\ \text{for } t \in [-\delta, 0] \text{ and } X(t, X_*) \in K, \text{ for } t \in [0, \tau]\}.
$$

More precisely, T_τ can be considered as the composition of two maps, namely $T_\tau = \psi \circ \varphi$,

$$
\mathcal{K} \overset{\varphi}{\rightsquigarrow} \{y \in AC([-\delta, \tau], \mathbb{R}^n) : y(t) \in K, \quad \text{for } t \in [-\delta, \tau]\} \overset{\psi}{\to} \mathcal{K},
$$

where

$$
\varphi(X_*): X_* \rightsquigarrow \{X(\tau, X_*) : X(\,\cdot\,, X_*) \text{ is a solution of (3)} \\ \text{with } X(t, X_*) = X_*, \text{ for } t \in [-\delta, 0], \text{ and } X(t, X_*) \in K, \text{ for } t \in [0, \tau]\}
$$

is known (see e.g. [24]) to be an R_δ-mapping and $\psi(y): y(t) \to y(\tau)$ is a continuous (single-valued) evaluation mapping.

In other words, we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{K} & \overset{\varphi}{\rightsquigarrow} & \{y \in AC([-\delta, \tau], \mathbb{R}^n) : y(t) \in K, \text{ for } t \in [-\delta, \tau]\} \\
& \ddots & \downarrow \psi \\
& & \overset{T_\tau}{\rightsquigarrow} \mathcal{K}
\end{array}
$$

The folowing characterization of T_τ has been proved on various levels of abstraction in [11], [21], [24], [31], etc.

Theorem 3. *T_τ defined by (6) is, under (5), admissible and homotopic to identity. More precisely, T_τ is a composition of an R_δ-mapping and a continuous (single-valued) evaluation mapping.*

Remark 3. Theorem 3 coincides with Theorem 2, for $K = \mathbb{R}^n$, in spite of the fact that K is assumed to be bounded and closed. Remark 1 can be appropriately modified as well.

5. Translation multioperator for systems in Banach spaces

Consider the functional system

$$
(7) \qquad X' + AX \in F(t, X_t), \quad X \in \mathcal{B},
$$

where $X_t(\,\cdot\,) = X(t+\,\cdot\,)$, for $t \in [0,\tau]$, denotes as above the mapping from $[-\delta, 0]$, $\delta \geq 0$, into a real separable Banach space $\mathcal{B}$. Let, furthermore, the following assumptions be satisfied:

(i) A is a closed, linear (not necessarily bounded) operator in $\mathcal{B}$, generating an analytic semigroup e^{At},

(ii) the set of $F(t,Y)\colon [0,\tau] \times \mathcal{C} \multimap \mathcal{B}$, where $\mathcal{C} = C([-\delta,0],\mathcal{B})$ and $\delta \geq 0$, is nonempty, compact and convex, for all $(t,Y) \in [0,\tau] \times \mathcal{C}$,

(iii) $F(t,\,\cdot\,)$ is u.s.c. for a.a. $t \in [0,\tau]$,

(iv) $F(\,\cdot\,,Y)$ is measurable, for all $Y \in \mathcal{C}$, i.e. for any open $U \subset \mathbb{R}^n$ and every $Y \in \mathcal{C}$, the set $\{t \in [0,\tau] : F(\,\cdot\,,Y) \cap U \neq \emptyset\}$ is measurable,

(v) (cf. [30]) for every nonempty, bounded, equicontinuous set $D \subset \mathcal{C}$, we have
$$\chi(F(t,D)) \leq g(t,\xi(D)) \quad \text{for a.a. } t \in [0,\tau]$$
where $\xi(D) \in C([-\delta,0] \times [0,\infty))$, $\xi(D)(\theta) = \chi(D(\theta))$ and $g\colon [0,\tau] \times C([-\delta,0] \times [0,\infty)) \to [0,\infty)$ is a Caratéodory-type function such that

  (a) $g(t,\,\cdot\,)$ is nondecreasing, for a.a. $t \in [0,\tau]$, in the sense that if $\varphi,\psi \in C([-\delta,0] \times [0,\infty))$ satisfy $\varphi(\theta) < \psi(\theta)$, for every $\theta \in [-\delta,0]$, then $g(t,\varphi) < g(t,\psi)$,

  (b) $|g(t,\varphi) - g(t,\psi)| < k(t)\|\varphi - \psi\|_1$, for a.a. $t \in [0,\tau]$ and for all $\varphi,\psi \in C([-\delta,0] \times [0,\infty))$, where k is a Lebesgue measurable function and $\|\cdot\|_1$ denotes the norm in the space $C([-\delta,0] \times [0,\infty))$,

  (c) $g(t,0) = 0$ for a.a. $t \in [0,\tau]$,

(vi) there exists a continuous bounded function $h\colon [0,\infty) \to [0,\infty)$ such that
$$\chi(e^{At}S) \leq h(t) \quad \text{for } t \in [0,\infty),$$
where S denotes the unit sphere in $\mathcal{B}$, and
$$\sup_{t\in[0,\infty)} \int_0^t h(t-s)k(s)\,ds < 1,$$

(vii) the solutions of the problem
$$\begin{cases} w(t) = v(t) & \text{for } t \in [-\delta,0], \\ w(t) = \dfrac{1}{h(0)}h(t)w(0) + \displaystyle\int_0^t h(t-s)g(s,w_s)\,ds & \text{for } t \in [0,\tau], \end{cases} \tag{8}$$
are uniformly asymptotically bounded in the sense that there exists a function $\sigma\colon [0,\infty) \to [0,\infty)$ such that
$$\limsup_{t\to\infty} \sigma(t) < \frac{1}{h(0)}$$
and, for every solution $w(t,v)$ of (8), we have
$$\|w_t\|_1 \leq \sigma(t)\|v\|_1 \quad \text{for } t \in [0,\infty),$$

(viii) $\|F(t,Y)\| \le \alpha + \beta\|Y\|_0$, for every $Y \in \mathcal{C}$ and a.a. $t \in [0,\tau]$, where α, β are suitable positive constants and $\|\cdot\|_0$ denotes the norm in $\mathcal{C}$.

By a *solution* $X(t)$ of (the initial problem to) (7) we mean this time a *mild solution,* namely $X(t) \in C([-\delta,\tau],\mathcal{B})$ such that

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-s)} f(s)\, ds \quad \text{for } t \in [0,\tau],$$

(with $X(t) = X_*$, for $t \in [-\delta,0]$), where f is an (existing) measurable selection of $F(s, X_s(t))$, $t \in [-\delta,0]$; such solutions exist on $[-\delta,\tau]$, $\delta \ge 0$.

Hence, if $X(t,X_*) = X_* \in E$ for $t \in [-\delta,0]$, where E consists of equicontinuous functions, then the translation multioperator $T_\tau\colon \mathcal{C} \to \mathcal{C}$ at the time $\tau > 0$ along the trajectories of (7) is defined as follows:

$$(9) \qquad T_\tau(X_*) := \{X(\tau,X_*) : X(\,\cdot\,,X_*) \text{ is a solution of (7) with } X(t,X_*) = X_* \text{ for } t \in [-\delta,0]\}.$$

More precisely, T_τ can be considered as the composition of two maps, namely $T_\tau = \psi \circ \varphi$,

$$\mathcal{C}([-\delta,0],\mathcal{B}) \overset{\varphi}{\rightsquigarrow} \mathcal{C}([-\delta,\tau],\mathcal{B}) \overset{\psi}{\to} \mathcal{C}([-\delta,0],\mathcal{B}),$$

where $\varphi(X_*)\colon X_* \rightsquigarrow \{X(t,X_*) : X(t,X_*)$ is a solution of (7) with $X(t,X_*) = X_*$ for $t \in [-\delta,0]\}$ is known (see [30]) to be an R_δ-mapping and $\psi(y)\colon y(t) \to y(\tau)$ is a continuous (single-valued) evaluation mapping.

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}([-\delta,0],\mathcal{B}) & \overset{\varphi}{\rightsquigarrow} & \mathcal{C}([-\delta,\tau],\mathcal{B}) \\ & \ddots & \downarrow \psi \\ & \overset{T_\tau}{\rightsquigarrow} & \mathcal{C}([-\delta,0],\mathcal{B}) \end{array}$$

The following characterization of T_τ has been proved on various levels of abstraction in [11], [12], [25], [30], etc.

Theorem 4. *T_τ defined by* (9) *is admissible and homotopic to identity. More precisely, T_τ is a composition of an R_δ-mapping and a continuous* (*single-valued*) *evaluation mapping, provided* (i)–(v) *hold. Under* (i)–(viii)*, T_τ is χ-condensing on equicontinuous sets, i.e. with the Hausdorff* MNC *in $\mathcal{C}$, provided $\tau > \inf\{t' : \sigma(t) < 1/h(0)$ for all $t \ge t'\}$.*

Remark 4. In the ordinary case ($\delta = 0$), the Banach space need not be necessarily separable (see e.g. [11]), condition (v) can be weaken and condition (vii) can be avoided (see e.g. [11], [25]).

Remark 5. It is a question whether Theorem 4 can be reformulated in an appropriate way for functional Caratéodory systems in Banach spaces with constraints, i.e. similarly as Theorem 3, but for $\mathbb{R}^n$ replaced by $\mathcal{B}$. So far, only particular cases were considered with this respect (see e.g. [11] and the references therein).

6. Translation multioperator for random systems

Consider the random system

$$X'(\kappa,t) \in F(\kappa,t,X(\kappa,t)), \quad \kappa \in \Omega,\ X \in \mathbb{R}^n, \tag{10}$$

where Ω is a complete probability space and

- (i) the set of values of $F(\kappa,t,X)$ is nonempty, compact and convex, for all $(\kappa,t,X) \in \Omega \times [0,\tau] \times \mathbb{R}^n$,
- (ii) $F(\kappa,t,\cdot\,)$ is u.s.c., for a.a. $(\kappa,t) \in \Omega \times [0,\tau]$,
- (iii) $F(\,\cdot\,,\,\cdot\,,X)$ is measurable, for every $X \in \mathbb{R}^n$, i.e. for any open $U \subset \mathbb{R}^n$ and every $X \in \mathbb{R}^n$, the set $\{(\kappa,t) \in \Omega \times [0,\tau] : F(\,\cdot\,,\,\cdot\,,X) \cap U \neq \emptyset\}$ is measurable,
- (iv) $|F(\kappa,t,X)| \le \mu(\kappa,1)(1+|X|)$, for a.a. $(\kappa,1) \in \Omega \times [0,\tau]$ and all $X \in \mathbb{R}^n$, where $\mu\colon \Omega \times [0,\tau] \to [0,\infty)$ is a map such that $\mu(\,\cdot\,,1)$ is measurable and $\mu(\kappa,\,\cdot\,)$ is Lebesque integrable.

The operator F satisfying conditions (i)–(iv) is called a *random Carathéodory operator*. Similarly, for metric spaces X_1 and X_2, we say that a multivalued mapping with nonempty closed values $\varphi\colon \Omega \times X_1 \rightsquigarrow X_2$ is a *random operator* if φ is product-measurable and $\varphi(\kappa,\,\cdot\,)$ is u.s.c. for every $\kappa \in \Omega$. By a *random homotopy* $\chi : \Omega \times X_1 \times [0,1] \rightsquigarrow X_2$, we understand a product-measurable mapping with nonempty closed values which is u.s.c. w.r.t. the last variable and that, for every $\lambda \in [0,1]$, $\chi(\,\cdot\,,\,\cdot\,,\lambda)$ is a random operator.

Furthemore, we say that a measurable map (a random variable) $\widehat{X}\colon \Omega \to X_1 \cap X_2$ is a *random fixed-point* of a random operator $\varphi\colon \Omega \times X_1 \rightsquigarrow X_2$ if $\widehat{X}(\kappa) \in \varphi(\kappa,\widehat{X}(\kappa))$, for a.a. $\kappa \in \Omega$.

The following proposition, proved in [19, Proposition 31.3], is crucial for further investigations.

Proposition 1. *Let $\varphi\colon \Omega \times A \rightsquigarrow Y$, where A is a closed subset of a metric space Y, be a random operator with compact values such that, for every $\kappa \in \Omega$, the set of fixed-points of $\varphi(\kappa,\,\cdot\,)$ is nonempty. Then φ has a random fixed-point.*

Because of Proposition 1, we can define the random translation multioperator T_τ in a "deterministic" way. We can namely employ, for every $\kappa \in \Omega$ and $X_0 \in \mathbb{R}^n$, Carathéodory solutions $X(t,X_0)$ of the deterministic Cauchy problems

$$\begin{cases} X' \in F_\kappa(t,X) = F(\kappa,t,X), \\ X(0,X_0) = X_0. \end{cases} \tag{11}$$

On the other hand, by a *solution* $X(\kappa,t)$ of (10), we mean a function such that $X(\,\cdot\,,t)$ is measurable, $X(\kappa,\,\cdot\,)$ is absolutely continuous and $X(\kappa,t)$ satisfies (11), for a.a. $(\kappa,t) \in \Omega \times [0,\tau]$; the derivative $X'(\kappa,t)$ is considered w.r.t. t.

Hence, the associated random translation multioperator $T_\tau \colon \Omega \times \mathbb{R}^n \multimap \mathbb{R}^n$ at the time $\tau > 0$ along the trajectories of the system $X' \in F_\kappa(t, X)$ is defined as follows:

$$T_\tau(\kappa, X_0) := \{X(\tau, X_0) : X(\,\cdot\,, X_0) \text{ is a solution of (11)}\}. \tag{12}$$

More precisily, T_τ can be considered as the composition of two maps, namely $T_\tau = \psi \circ \varphi$,

$$\Omega \times \mathbb{R}^n \overset{\varphi}{\multimap} AC([0,\tau], \mathbb{R}^n) \overset{\psi}{\to} \mathbb{R}^n,$$

where $\varphi(\kappa, X_0) \colon (\kappa, X_0) \multimap \{X(t, X_0) : X(t, X_0) \text{ is a solution of (11)}\}$ is, according to [19], an R_δ-mapping, for every $X \in \Omega$, and the (single-valued) evaluation mapping $\psi(Z) \colon Z(t) \to Z(\tau)$ is obviously continuous.

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \Omega \times \mathbb{R}^n & \overset{\varphi}{\multimap} & AC([0,\tau], \mathbb{R}^n) \\ & \ddots & \downarrow \psi \\ & & \overset{T_\tau}{\multimap}\ \mathbb{R}^n \end{array}$$

Applying Proposition 1, one can show the following characterization of T_τ (for more details, see [19]).

Theorem 5. *T_τ defined by* (12) *is a random operator with compact values composed by a random operator with R_δ-values and a continuous* (*single-valued*) *evaluation mapping. Moreover, it is randomly homotopic to identity.*

Thus, $\widehat{X}$ is a random fixed-point of T_ω if and only if the original system (10) has a solution $X(\kappa, t)$ such that $X(\kappa, 0) = X(\kappa, \omega) = \widehat{X}(\kappa)$, for a.a. $\kappa \in \Omega$.

Remark 6. In [19, pp. 156–157], the random degree theory is sketched, having quite anologous properties as in the deterministic case, and so it is available for proving the random fixed-points of the random translation operator T_ω.

Remark 7. Theorem 5 reduces to Theorem 1 in the deterministic case, i.e. in the absence of Ω. Remark 1 can be appropriately modified here as well.

7. Translation multioperator for directionally u.s.c. systems

Let $M \in \mathbb{R}$ and $\Gamma^M = \{(t, X) \in \mathbb{R} \times \mathbb{R}^n : |X| \leq Mt\} \subset \mathbb{R} \times \mathbb{R}^n$ be a closed, convex cone. Following [9], we say that a multivalued mapping with nonempty closed values $F = \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is Γ^M-*directionally u.s.c.* if, at each point $(t_0, X_0) \in \mathbb{R} \times \mathbb{R}^n$, and for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $(t, X) \in B((t_0, X_0), \delta)$ (i.e. (t, X) belonging to an open ball with the radius δ and centered at (t_0, X_0)) satisfying $|X - X_0| \leq M(t - t_0)$ holds $F(t, X) \subset F(t_0, X_0) + \varepsilon B$.

Consider the Γ^M-directionally u.s.c. system (1). We will show that the solution set of (1) can be characterized by means of the Filippov-like regularization of (1). Subsequently, the related translation multioperator to (1) can be associated to the regularized system.

Definition 1 (cf. [16])**.** Let $F(t,X)\colon [a,b]\times\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be closed, convex-valued, locally bounded and measurable. Then the mapping

$$\phi(t,X) = \bigcap_{\delta>0} \bigcap_{\substack{N\subset\mathbb{R}^{n+1}\\ \mu(N)=0}} \overline{\mathrm{conv}}\, F(B((t,X),\delta)\setminus N),$$

called the *regularization of* F, satisfies the following properties:

(i) $\phi(t,X)$ is u.s.c., for all $(t,X)\in[a,b]\times\mathbb{R}^n$,
(ii) $F(t,X)\subset\phi(t,X)$, for all $(t,X)\in[a,b]\times\mathbb{R}^n$,
(iii) ϕ is minimal in the following sense: if $\psi\colon[a,b]\times\mathbb{R}^n\rightsquigarrow\mathbb{R}^n$ satisfies (i) and (ii), then $\phi(t,X)\subset\psi(t,X)$, for all $(t,X)\subset[a,b]\times\mathbb{R}^n$,

where $\mu(N)$ stands for the Lebesgue measure of N and $\overline{\mathrm{conv}}$ denotes the closed-convex hull of a set.

The following statement has been proved in [9].

Proposition 2. *Let* $\Omega\subset\mathbb{R}\times\mathbb{R}^n$ *and* $F\colon\Omega\rightsquigarrow\mathbb{R}^n$ *be closed, convex-valued* Γ^M*-directionally u.s.c. and bounded in the following way* $F(\Omega)\subset B(0,L)$*, where* $0<L<M$. *Let* $\phi\colon\Omega\rightsquigarrow\mathbb{R}^n$ *be the regularization of* F *in the sense of Definition* 1. *Then every solution of the regularized inclusion*

$$X'(t)\in\phi(t,X(t)) \tag{13}$$

is the solution of the original inclusion (1) *and vice versa.*

Hence, if $X(t,X_0):=X(t,0,X_0)$ is a solution of (1) with $X(0,X_0)=X_0\in\mathbb{R}^n$, then the translation multioperator $T_\tau\colon\mathbb{R}^n\rightsquigarrow\mathbb{R}^n$, at the time $\tau>0$, along the trajectories of (1) can be defined as follows:

$$T_\tau(X_0) := \{X(\tau,X_0) : X(\,\cdot\,,X_0) \text{ is a solution of (1) with } X(0,X_0)=X_0\}. \tag{14}$$

In fact, according to Proposition 2, we also have

$$T_\tau(X_0) := \{X(\tau,X_0) : X(\,\cdot\,,X_0) \text{ is a solution of (13) with } X(0,X_0)=X_0\}. \tag{15}$$

T_τ in (15) can be considered as the composition of two maps, namely $T_\tau=\psi\circ\varphi$,

$$\mathbb{R}^n \overset{\varphi}{\rightsquigarrow} AC([0,\tau],\mathbb{R}^n) \overset{\psi}{\to} \mathbb{R}^n,$$

where

$$\varphi(X_0)\colon X_0 \rightsquigarrow \{X(t,X_0) : X(t,X_0) \text{ is a solution of (13) with } X(0,X_0)=X_0\}$$

is, according to Theorem 1, an R_δ-mapping and $\psi(Z): Z(t) \to Z(\tau)$ is obviously a continuous (single-valued) evaluation mapping. The same is true, according to Proposition 2, for (14).

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \overset{\varphi}{\rightsquigarrow} & AC([0,\tau],\mathbb{R}^n) \\ & \ddots & \downarrow \psi \\ & \overset{T_\tau}{\rightsquigarrow} & \mathbb{R}^n \end{array}$$

Hence, we can sumarize our investigations as follows.

Theorem 6. *T_τ defined by* (14) *is admissible and admissibly homotopic to identity. More precisely, T_τ is a composition of an R_δ-mapping and a continuous* (*single-valued*) *evaluation mapping.*

Remark 8. An appropriately modified version of Remark 1 is true here as well.

8. Some applications: existence results

A typical application of Poincaré's translation multioperator concern the existence results for periodic solutions. The standard conditions are either related to a weak semi-flow invariance of a suitable set $\mathcal{K}$, under (5), (as in Chapter 4); see e.g. [11], [21], [24], [31]; or to a uniform boundedness and a uniform ultimate boundedness of solutions $X(t, X_*)$ of a given system (e.g. (7)), namely there exist $D > 0$ and $t_D > 0$ such that

$$\|X_t(s, X_*)\|_0 \le D, \quad \text{for } t \ge t_D,$$

see e.g. [1], [13]–[15], [20], [25], [29], [30].

For example, the main result in [24] can be easily generalized in the following way.

Theorem 7. *Let the assumptions in Chapter* 4 *be satisfied, for $\tau = \omega > 0$. If $\phi: \mathcal{K} \to \mathcal{K}$ is a continuous* (*single-valued*) *self-mapping whose generalized Lefschetz number* (*for the definition, see e.g.* [19]) *is nontrivial, $\Lambda(\phi) \neq 0$, then the system* (3) *admits a solution $X(t)$ such that*

$$X(0) = \phi(X(\omega)).$$

If, in particular, $\phi = \mathrm{id}$ (*i.e. an identity*) *and $F(t, Y) \equiv F(t + \omega, Y)$, then the system* (3) *admits an ω-periodic solution.*

Remark 9. A similar generalization can be performed to the results (e.g. in [30]) for system (7) in Banach spaces, when applying conditions guaranteeing the strong flow-invariance.

On the other hand, rather rarely conditions like (10) are only related to some part of components of solutions, while the other components satisfy certain dual (expanding-like) conditions (see e.g. [2], [4], [8]). Let us note that, in the case of uniqueness (for differential equations), this type of conditions cannot be apparently regarded as a semi-flow invariance.

Applying the relative generalized Lefschetz number in [6], we can modify Theorem 6.9 in [6] as follows.

Theorem 8. *Let the assumptions in Chapter* 3 *be satisfied, for* $\tau = \omega > 0$. *Let, furthermore,* A *and* B, *where* $B \subset A \subset \mathcal{C}$, *be compact* ANR*-spaces which are* (*strongly*) *semi-flow invariant, under Poincaré's operator defined by* (4), *and such that* $\chi(A) \neq \chi(B)$, *where* $\chi(\,\cdot\,)$ *stands for the Euler–Poincaré characteristic. Then system* (3), *where* $F(t, Y) \equiv F(t + \omega, Y)$, *admits an* ω*-periodic solution* $X(t, X_*)$ *with* $X_* \in \overline{A \setminus B}$.

Remark 10. Conditions for a strong semi-invariance can be expressed in terms of bounding functions (cf. [6]). Observe that Theorem 8 gives us an additional information about the location of initial values of ω-periodic solutions and that the set $\overline{A \setminus B}$ need not be (semi-) flow invariant.

9. Some applications: multiplicity results

Application of Poincaré's translation multioperator for obtaining multiplicity results is much more delicate than those for the sole existence. In [3], [5]–[7], the generalized Nielsen fixed-point theory, developed in [5]–[7], has been applied to this aim.

For example, the following statement represents Theorem 12.8 in [7].

Theorem 9. *Assume that* $F(t, \dots, x_j + 1, \dots) \equiv F(t, \dots, x_j, \dots)$, *for* $j = 1, \dots, n$, *where* $X = (x_1, \dots, x_n)$, *and consider system* (1) *on the set* $[0, \infty) \times \mathbb{R}^n/\mathbb{Z}^n$. *Let the assumptions in Chapter* 2 *be satisfied, for a sufficiently big* $\tau > 0$. *Assume, futhermore, that* ϕ *is a continuous* (*single-valued*) *self-mapping on* $\mathbb{R}^n/\mathbb{Z}^n$ *such that* $\det(\mathrm{id} - A^k) \neq 0$, *for some* $k \in \mathbb{N}$, *where* A *is the associated* $(n \times n)$*-matrix with integer coeficients representing the induced homomorphism of the fundamental group, which corresponds to* ϕ *and which is called the linearization of* ϕ. *Then the number of geometrically distinct* k*-tuples of solutions of* (1), *satisfying*

$$\underbrace{\phi \circ X(\omega; \phi \circ X(\omega; \dots \phi \circ X(\omega;}_{k\text{-times}} X(0, X_0) \dots))) = X(0, X_0) \pmod 1,$$

with the minimal period k *is at least*

$$\frac{1}{k} \sum_{m|k} \mu(k/m) |\det(\mathrm{id} - A^m)|,$$

where $\mu(d)$, $d \in \mathbb{N}$ *is the Möbius function,*

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes}, \\ 0 & \text{if } d \text{ is not square-free}, \end{cases}$$

and $\omega > 0$ *is a given real number.*

Two particular cases of Theorem 9 below, representing Theorem 7.8 and Corollary 7.9 in [5], are more transparent.

Corollary 1. *Assume that* $F(t, \dots, x_j + 1, \dots) \equiv F(t, \dots, x_j, \dots)$, *for* $j = 1, \dots, n$, *where* $X = (x_1, \dots, x_n)$, *and consider system* (1) *on the set* $[0, \omega] \times \mathbb{R}^n/\mathbb{Z}^n$. *Let the assumptions in Chapter* 2 *be satisfied, for a sufficiently big* $\tau = \omega > 0$. *If* ϕ *is a continuous* (*single-valued*) *self-mapping on* $\mathbb{R}^n/\mathbb{Z}^n$, *then system* (1) *has at least* $|\det(\mathrm{id} - A)|$ *solutions* $X(t)$ *such that*

$$X(0) = \phi(X(\omega)) \pmod 1,$$

where A *has the same meaning as above.*

Corollary 2. *If, additionally to the assumptions of Corollary* 1, $F(t+\omega, -X) \equiv -F(t, X)$, *then system* (1) *possesses* (*for* $\phi = \mathrm{id}$) *at least* 2^n *anti-*ω*-periodic* (*or* 2ω*-periodic*) *solutions* $X(t)$ *such that* $X(t + \omega) \equiv -X(t) \pmod 1$.

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ENTROPY METHODS AND INTERMEDIATE ASYMPTOTICS FOR NONLINEAR DRIFT-DIFFUSION SYSTEMS

Piotr Biler

Abstract. We review some recent results obtained with the use of entropy methods for the asymptotics of solutions of drift-diffusion systems with Poisson coupling.

1. Introduction

This paper deals with some aspects of the development in describing long time behavior of solutions of nonlinear drift-diffusion systems with Poisson coupling obtained jointly with Jean Dolbeault, Maria J. Esteban and Peter A. Markowich, cf. for preliminary results [2], [3], [5].

In the model problem we consider the evolution of the densities $n \geq 0$ and $p \geq 0$ of the negatively and (resp.) positively charged particles in a bipolar plasma in $\mathbb{R}^d \times \mathbb{R}^+ \ni \langle y, \tau \rangle$, when the drift-diffusion equations for $\langle n, p \rangle$ are coupled to a mean-field Poisson equation

$$\begin{aligned} n_\tau &= \nabla \cdot (\nabla f(n) + n \nabla \psi), \\ p_\tau &= \nabla \cdot (\nabla f(p) - p \nabla \psi), \\ \Delta \psi &= p - n. \end{aligned} \tag{1}$$

Here, the *nonlinear* ($\equiv$ not necessarily linear) *diffusion* is described by a power function $f(s) = s^m$ with $m > 0$. The case $m = 1$ corresponds to the usual *linear*

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Brownian diffusion, $m > 1$ — to the *porous media diffusion*, and $0 < m < 1$ — to the case of *"fast" diffusion*.

The system (1) is supplemented by the initial data prescribed at $t = 0$

$$n(y,0) = n_0(y), \quad p(y,0) = p_0(y), \tag{2}$$

which we suppose integrable: $n_0, p_0 \in L^1(\mathbb{R}^d)$.

We will consider sufficiently regular solutions of the Cauchy problem (1)–(2), so that the conservation of the total charges

$$\begin{aligned} M_n &= \int_{\mathbb{R}^d} n(y,\tau)\,dy = \int_{\mathbb{R}^d} n_0(y)\,dy, \\ M_p &= \int_{\mathbb{R}^d} p(y,\tau)\,dy = \int_{\mathbb{R}^d} p_0(y)\,dy \end{aligned}$$

holds. Formally, these relations are obtained by integrating the first two equations in (1) over $\mathbb{R}^d$.

Our aim is to describe the long time asymptotics of solutions of (1)–(2). Taking into account the repulsive character of the self-interaction of particles of each of the species, one can expect that particles disperse, and the Poisson coupling through the third equation in (1) becomes asymptotically weaker and weaker. Thus, one anticipates that the evolution of $\langle n, p \rangle$ charges is decoupled in the limit $\tau = \infty$, so that the densities resemble solutions ϱ of the nonlinear diffusion equation in $\mathbb{R}^d$

$$\varrho_\tau = \nabla \cdot (\nabla f(\varrho)), \tag{3}$$

e.g. their $L^q(\mathbb{R}^d)$-norms, $1 < q \le \infty$, tend to 0 at an algebraic rate, or they behave like self-similar solutions of (3) (i.e. those enjoying some invariance properties).

This conjecture on the *intermediate asymptotics* of solutions has a strong physical background, but only recently appropriate mathematical tools have been developed to deal with such problems, esp. when one looks for the optimal decay rates. The idea is to combine a space-time rescaling, a priori estimates obtained from the relative entropies ([1]), and the generalized Sobolev inequalities ([7], [8]) extending the usual and logarithmic Sobolev inequalities.

Standard notation $|\cdot|_q$ is used for the Lebesgue norms of functions defined on $\mathbb{R}^d$. The integrals without integration limits are over the whole $\mathbb{R}^d$ space.

2. Results and methods

The first step in our analysis of the system (1) is the space-time rescaling

$$t = \log R(\tau), \quad x = \frac{y}{R(\tau)},$$

where $m > 1 - 2/d$, and the function R satisfies

$$\dot{R}R^{d(m-1)+1} = 1 \quad \text{and} \quad R(0) = 1.$$

This change of variables applied to (1) leads to the nonautonomous system

$$
\begin{aligned}
u_t &= \nabla \cdot (\nabla f(u) + ux + \beta(t)u\nabla\varphi),\\
v_t &= \nabla \cdot (\nabla f(v) + vx - \beta(t)v\nabla\varphi),\\
\Delta\varphi &= v - u,
\end{aligned} \tag{4}
$$

for the rescaled functions

$$
\begin{aligned}
u(x,t) &= R^d(\tau)n(y,\tau),\\
v(x,t) &= R^d(\tau)p(y,\tau),\\
\varphi(x,t) &= R^{d-2}(\tau)\psi(y,\tau),
\end{aligned}
$$

and $\beta(t) = R(\tau)^{2-d} = e^{(2-d)t}$. The system (4) for $\langle u, v\rangle$ (which can be viewed as a system of Fokker–Planck equations with nonlinear diffusions) has the same initial data (2) as (1) for $\langle n, p\rangle$ because $R(0) = 1$. The solution φ of the Poisson equation in (4) is taken as the Newtonian potential

$$
\varphi = E_d * (v - u), \tag{5}
$$

where E_d is the fundamental solution of the Laplacian in $\mathbb{R}^d$. Observe that for $d \geq 3$, the damping factor $\beta(t)$ makes the transport terms $u\nabla\varphi$, $v\nabla\varphi$ in (4) negligible in the limit $t \to \infty$.

Now consider the system (4) with $\beta = 0$ which corresponds to the *asymptotic* (uncoupled) problem. Both u and v then satisfy the equation of the type

$$
z_t = \nabla \cdot (\nabla f(z) + zx) \tag{6}
$$

with suitable initial condition $z(.,0) = z_0 \geq 0$. Define the entropy functional

$$
W[z] = \int \left(z\left(\frac{1}{2}x^2 + h(z)\right) - f(z)\right) dx \tag{7}
$$

with the enthalpy function

$$
h(z) = \int_1^z f'(s)s^{-1}\, ds. \tag{8}
$$

Simple computations valid for sufficiently regular solutions of (6) show the conservation of the $L^1(\mathbb{R}^d)$-norm

$$
\int z(x,t)\, dx = \int z_0(x)\, dx,
$$

and the production of entropy formula

$$
\frac{d}{dt}W[z(t)] + \int z\left|\nabla\left(\frac{1}{2}x^2 + h(z)\right)\right|^2 dx = 0. \tag{9}
$$

Therefore, all the steady states z_∞ of (6) satisfy the relation

$$z_\infty(x) = \widetilde{h^{-1}}\left(C_z - \frac{1}{2}x^2\right), \tag{10}$$

where C_z is a real constant with $C_z \le h(\infty)$, and $\widetilde{h^{-1}}$ is the extension of h^{-1} given by $\widetilde{h^{-1}}(s) = h^{-1}(s)$ if $h(0^+) < s < h(\infty)$, and $\widetilde{h^{-1}}(s) = 0$ if $s \le h(0^+)$.

Remark that in the fast diffusion case ($m < 1$) $h(0^+) = -\infty$, $h(\infty) = m/(1-m)$, while $h(0^+) = -m/(m-1)$, $h(\infty) = \infty$ in the porous media case ($m > 1$). In the case of linear diffusion steady states satisfy $z_\infty(x) = \exp(C_z - x^2/2)$.

If $0 \le M < \infty$ satisfy the inequality $M \le \int \widetilde{h^{-1}}(h(\infty) - x^2/2)\,dx$, then the steady state z_∞ is uniquely determined by the requirement $\int z_\infty(x)\,dx = M$ (e.g. if $f(s) = s^m$ with $m > d/2 - 1$). It can be shown ([1], [7]) that (9) implies the decay of the entropy functional $W[z(t)]$ to $W[z_\infty]$, where z_∞ is the unique steady state of the form (10) with $\int z_\infty\,dx = \int z_0\,dx$. Thus, the *relative entropy*

$$W[z|z_\infty] = W[z] - W[z_\infty] \tag{11}$$

can be defined so that $\lim_{t\to\infty} W[z(t)|z_\infty] = 0$.

Let us come back to the system (4) and define, using (11), for solutions $\langle u, v\rangle$ with $\int u_0\,dx = M_u \ge 0$, $\int v_0\,dx = M_v \ge 0$, the relative entropy

$$\mathcal{W}[\langle u,v\rangle|\langle u_\infty, v_\infty\rangle] = W[u|u_\infty] + W[v|v_\infty] + \frac{\beta}{2}|\nabla\varphi|_2^2. \tag{12}$$

Here u_∞, v_∞ are steady states of the equation (6) corresponding to M_u, M_v, resp.

Taking into account (4) we compute the time derivative of (12)

$$\begin{aligned}\frac{d}{dt}\mathcal{W}[\langle u(t),v(t)\rangle|\langle u_\infty, v_\infty\rangle] &= -J - \beta^2\int (u+v)|\nabla\varphi|^2\,dx \\ &- 2\beta\int (f(u)-f(v))(u-v)\,dx + 2\beta\int \Delta\varphi\nabla\varphi\cdot x\,dx + \frac{1}{2}\frac{d\beta}{dt}|\nabla\varphi|_2^2,\end{aligned} \tag{13}$$

where

$$J = \int u\left|\nabla h(u) + x\right|^2 dx + \int v\left|\nabla h(v) + x\right|^2 dx. \tag{14}$$

In the next step we show that $\mathcal{W}$ in (12) is a Lyapunov functional for (4). This is obtained in the result below. Its proof is rather involved, cf. [2], [5]. In particular, to show the inequality $d\mathcal{W}/dt + \lambda\mathcal{W} \le 0$ we use (13) and, in estimations of the quantity J in (14), the *generalized Sobolev inequalities* (from [1], [7], [8]) of the form

$$W[z|z_\infty] \le K\int z|\nabla h(z) + x|^2\,dx$$

for some $K > 0$. These are nonlinear generalizations of the classical Sobolev inequalities as well as the (Gross) logarithmic Sobolev inequality in [8].

Theorem. *Let $d \geq 3$ and $f(s) = s^m$ for $m \geq 1 - 1/d$. Consider a sufficiently regular, global in time solution of* (4) *such that $M_u + M_v > 0$. Then there exists a constant $\lambda > 0$ such that*

$$\mathcal{W}[\langle u(t), v(t)\rangle | \langle u_\infty, v_\infty\rangle] \leq e^{-\lambda t} \mathcal{W}[\langle u_0, v_0\rangle | \langle u_\infty, v_\infty\rangle].$$

Remark 1. If we assume *charge neutrality* $M_u = M_v$, and $|\nabla\varphi(0)|_2^2 < \infty$ (which is a consequence of $\mathcal{W}[\langle u_0, v_0\rangle | \langle u_\infty, v_\infty\rangle] < \infty$), then the result of Theorem remains true also in the one- and two-dimensional cases (since for φ defined in (5), under the electroneutrality condition, the relation $\lim_{|x|\to\infty} |\nabla\varphi(x,t)| = 0$ holds).

Remark 2. As it follows from [1], [7], [9] and [2], the exponential decay of the relative entropy implies the exponential convergence in $L^1(\mathbb{R}^d)$ of $\langle u, v\rangle$ to the steady state $\langle u_\infty, v_\infty\rangle$ of (4), and thus, by the space-time rescaling, the convergence to a self-similar solution of (1) at an algebraic decay rate in the $L^1(\mathbb{R}^d)$-norm. The latter result much improves upon the algebraic decay estimate for (3) in L^q-norms, $1 < q \leq \infty$. Therefore, the solutions of (1) asymptotically resemble self-similar solutions as $\tau \to \infty$. The relations between the entropy and the L^1-norm are a consequence of *Csiszár–Kullback inequalities* for relative entropies, cf. [9].

In particular, in the case of the Brownian diffusion $m = 1$, we recover the result from [2]

$$|n(t) - n_{\mathrm{as}}(t)|_1^2 + |p(t) - p_{\mathrm{as}}(t)|_1^2 + |\nabla\psi(t)|_2^2 \leq CH(t),$$

where the asymptotic state $\langle n_{\mathrm{as}}, p_{\mathrm{as}}\rangle$ is given by

$$n_{\mathrm{as}}(x,t) = M_n(2\pi(2t+1))^{-d/2} \exp\left(\frac{-|x|^2}{2(2t+1)}\right),$$

$$p_{\mathrm{as}}(x,t) = M_p(2\pi(2t+1))^{-d/2} \exp\left(\frac{-|x|^2}{2(2t+1)}\right),$$

and $H(t) = (2t+1)^{-1/2}$ if $d = 3$, $H(t) = (2t+1)^{-1}(\log(2t+1)+1)$ if $d = 4$, and $H(t) = (2t+1)^{-1}$ if $d \geq 4$. Moreover, in the electroneutrality case $M_n = M_p$, $H(t) = (2t+1)^{-1}$ for each $d \geq 3$. The asymptotic states n_{as}, p_{as} are self-similar solutions of the heat equation which is exactly the equation (3) associated with the problem (1) with the linear diffusion, i.e. when $m = 1$.

3. Extensions and generalizations

The result in Theorem can be extended in different directions.

First, one can consider, instead of the terms ux, vx in (4) generated by the potential $V(x) = x^2/2$, the terms $u\nabla V$, $v\nabla V$, where V is a *confining potential* which tends to ∞ fast enough as $|x| \to \infty$. A sufficient condition for that is $2D^2V(x) - \mathrm{Tr}(D^2V(x)) \geq cI$ with a constant $c > 0$.

Second, it suffices that the relaxation factor β satisfies $\beta_t \leq -2\omega\beta$ for some $\omega > 0$, instead of $\beta(t) = e^{(2-d)t}$. For these extensions see [5].

Third, the power nonlinearity $f(s) = s^m$ can be replaced by a more general function f, with the power-like behavior at $s = 0$, see [3] for the case of a single equation (3). Another possible choice of f in (1) when $d = 3$ is that from the Fermi–Dirac thermodynamical framework, i.e. $f(s) = sF^{-1}(s) - \int_0^s F^{-1}(\tau)\,d\tau$, where $F(\sigma) = \int_{\mathbb{R}^3} dv/(\varepsilon + \exp(|v|^2/2 - \sigma))$, $\varepsilon > 0$, considered in [5].

Remark 3. One can also study the problem (1) in bounded domains of $\mathbb{R}^d$ with suitable no-flux boundary conditions guaranteeing the conservation of charge (as was in [6]). The physical interpretations include models in semiconductors and electrolytes theory. In that case, the unique nontrivial steady states for (1) exist for each M_n, $M_p > 0$, and the solutions of the evolution problem approach them in L^q-norm, $1 \leq q \leq \infty$, at an *exponential* rate as $\tau \to \infty$. This is shown using similar tools (relative entropy methods and generalized Poincaré–Sobolev inequalities) in [2], improving on earlier results in [6].

Remark 4. Analogous methods can be applied to a more general system that takes also into account the evolution of the temperature (the, so called, Streater's energy-transport model of second kind). The solutions of that model obey the first and the second laws of thermodynamics, and it is shown in Section 3.2 of [4] that for radially symmetric solutions of the rescaled model a relative entropy decays. This implies, as before, an asymptotically self-similar behavior of solutions of the original (unscaled) system.

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ON A NEW FUNCTION OF MINIMAL DISPLACEMENT

Krzysztof Bolibok

Dedicated to Professor Lech Górniewicz

Abstract. We introduce a new function of minimal displacement and show its asymptotic behaviour. Next we discuss applications to the retraction problem.

1. Introduction

Let $(X, \|\cdot\|)$ be an infinitely dimensional Banach space with the closed unit ball B and the unit sphere S. The term of minimal displacement $d_T = \inf_{x\in B} \|x - Tx\|$ for lipschitzian mappings T was introduced by Goebel (see [8]). He also defined some functions describing this problem. Let recall one of them, the so-called minimal displacement characteristic of X $\psi_X(k) = \sup\{d_T : T \in L(k),\ k \geq 1\}$, where $L(k)$ denotes the class of lipschitzian mappings $T\colon B \to B$ with constant k. It is known that $\psi_X(k) \leq 1 - 1/k$ for any space X. There are some "square" spaces like c_0, $C[0,1]$ for which $\psi_X(k) = 1 - 1/k$. Moreover, it is known that $\lim_{k\to\infty} \psi_X(k) = 1$ for any space X. The aim of this paper is to show that the same is true for the function $\overline{\psi}_X$ defined as follows $\overline{\psi}_X(k) = \sup\{d_T : T \in L(k),\ T(S) = 0,\ k \geq 1\}$. This function can be very useful in evaluations of the retraction constant $k_0(X)$ being the infimum of the set of all numbers $k > 1$, for which there exists a retraction $R\colon B \to S$ belonging to $L(k)$. It is known that $k_0(X) \geq 3$ for any space X. The existence of such retraction was proved by Nowak ([15]) for some spaces and by Benyamini and Sternfeld

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([1]) for all infinite dimensional normed spaces. For a wider discussion of the topics mentioned above we refer the reader to [11]. The latest results concerning the minimal displacement and retraction problems can be found in [2]–[6], [9], [10], [13].

Before we proceed to the second part of our paper let us recall that the radial projection $P\colon X \to B$ is the mapping defined as

$$Px = \begin{cases} x & \text{if } x \in B, \\ \dfrac{x}{\|x\|} & \text{if } x \notin B. \end{cases}$$

It is known that its Lipschitz constant $1 \le P(X) \le 2$ for any space X (see [16]).

2. Results

Theorem 1. *In every infinite dimensional Banach space X*

$$\overline{\psi}_X(k) \ge 1 - \frac{2P(X)k_0(X)}{k} \quad \textit{for any } k > k_0(X).$$

Proof. Let $0 < \varepsilon < 1/4$ and $R_\varepsilon\colon B(\varepsilon) \to S(\varepsilon)$ be a lipschitzian retraction with constant k_{R_ε}, where $B(\varepsilon)$ and $S(\varepsilon)$ denote respectively the closed ball and the sphere of radius ε. Define a mapping $T\colon B \to B$ as follows

$$Tx = \begin{cases} P_{1-\varepsilon}R_\varepsilon(-x) & \text{for } \|x\| \le \varepsilon, \\ P_{1-\varepsilon}(-x) & \text{for } \varepsilon \le \|x\| \le 1-\varepsilon, \\ \left(\dfrac{1-\varepsilon}{\varepsilon}\|x\| + \dfrac{\varepsilon - 1}{\varepsilon}\right)x & \text{for } \|x\| \ge 1-\varepsilon, \end{cases}$$

where $P_{1-\varepsilon}$ is a radial projection onto the ball $B(1-\varepsilon)$.

Observe that $Tx = 0$ for any $x \in S$. To show that T is lipschitzian we consider three cases.

(i) If $\|x\| \le \varepsilon$ and $\|y\| \le \varepsilon$, then

$$\|Tx - Ty\| = \|P_{1-\varepsilon}R_\varepsilon(-x) - P_{1-\varepsilon}R_\varepsilon(-y)\| \le \frac{P(X)k_{R_\varepsilon}}{\varepsilon}\|x-y\|.$$

(ii) If $\varepsilon \le \|x\| \le 1-\varepsilon$ and $\varepsilon \le \|y\| \le 1-\varepsilon$, then

$$\|Tx - Ty\| = \|P_{1-\varepsilon}(-x) - P_{1-\varepsilon}(-y)\| \le \frac{P(X)}{\varepsilon}\|x-y\|.$$

(iii) If $\|x\| \ge 1-\varepsilon$ and $\|y\| \ge 1-\varepsilon$, then

$$\begin{aligned} \|Tx - Ty\| &= \left\| \left(\frac{1-\varepsilon}{\varepsilon}\|x\| + \frac{\varepsilon-1}{\varepsilon}\right)x - \left(\frac{1-\varepsilon}{\varepsilon}\|y\| + \frac{\varepsilon-1}{\varepsilon}\right)y \right\| \\ &\le \frac{1-\varepsilon}{\varepsilon}\|x\|\|x-y\| + \frac{1-\varepsilon}{\varepsilon}\|y\|\big|\|x\| - \|y\|\big| + \frac{1-\varepsilon}{\varepsilon}\|x-y\| \\ &\le \frac{3(1-\varepsilon)}{\varepsilon}\|x-y\|, \end{aligned}$$

which finally shows that T is lipschitzian with constant $P(X)k_{R_\varepsilon}/\varepsilon$.

Now let us estimate the minimal displacement of T. Let us also consider three cases.

(i) If $\|x\| \le \varepsilon$, then

$$\|x - Tx\| = \|x - P_{1-\varepsilon}R_\varepsilon(-x)\| \ge \|P_{1-\varepsilon}R_\varepsilon(-x)\| - \|x\| \ge 1 - 2\varepsilon.$$

(ii) If $\varepsilon \le \|x\| \le 1-\varepsilon$, then

$$\|x - Tx\| = \|x - P_{1-\varepsilon}(-x)\| = \left\| x + (1-\varepsilon)\frac{x}{\|x\|}\right\| = \|x\| + (1-\varepsilon) \ge 1.$$

(iii) If $\|x\| \ge 1-\varepsilon$, then

$$\|x - Tx\| = \left\| x - \left(\frac{1-\varepsilon}{\varepsilon}\|x\| + \frac{\varepsilon - 1}{\varepsilon}\right)x\right\| = \frac{1}{\varepsilon}\|x\| - \frac{1-\varepsilon}{\varepsilon}\|x\|^2 \ge 1,$$

which finally shows that $d_T \ge 1 - 2\varepsilon$.

Because ε can be arbitrary small and the Lipschitz constant of retraction R_ε can be arbitrary close to $k_0(X)$ we get our theorem. Moreover, observe that as corollary we get. □

Theorem 2. *In every infinite dimensional Banach space X*

$$\lim_{k\to\infty} \overline{\psi}_X(k) = 1.$$

In the proof of Theorem 1 we use the fact of the existence of the Lipschitz retraction. Without this assumption we can get worse result.

Example 1. Let $T: B \to B$, $T \in L(k)$ with $d_T > \psi_X(k) - \varepsilon$. Define a mapping $\overline{T}: B \to B$ as follows $\overline{T}x = (1 - \|x\|^n)Tx$, where $n \in \mathbb{N}$. Observe that $\overline{T}x = 0$ for any $x \in S$. The map $\overline{T}$ is lipschitzian. Indeed

$$\begin{aligned}\|\overline{T}x - \overline{T}y\| &\le (1 - \|x\|^n)\|Tx - Ty\| + |\|x\|^n - \|y\|^n|\|Ty\| \\ &\le k(1 - \|x\|^n)\|x - y\| + \left(\sum_{i=0}^{n-1}\|x\|^{n-1-i}\|y\|^i\right)\|x - y\| \\ &\le \left(k(1 - \|x\|^n) + \sum_{i=0}^{n-1}\|x\|^{n-1-i}\|y\|^i\right)\|x - y\| = L_1\|x - y\|,\end{aligned}$$

and analogously

$$\|\overline{T}x - \overline{T}y\| \le \left(k(1 - \|y\|^n) + \sum_{i=0}^{n-1}\|y\|^{n-1-i}\|x\|^i\right)\|x - y\| = L_2\|x - y\|.$$

Finally we get

$$\|\overline{T}x - \overline{T}y\| \le \min\{L_1, L_2\}\|x - y\|.$$

Without loss of generality suppose that $\|x\| \geq \|y\|$. Then

$$\begin{aligned}\|\overline{T}x - \overline{T}y\| &\leq \min\{L_1, L_2\}\|x-y\| \leq L_1\|x-y\| \\ &= (k(1-\|x\|^n) + \sum_{i=0}^{n-1} \|x\|^{n-1-i}\|y\|^i)\|x-y\| \\ &\leq (k(1-\|x\|^n) + n\|x\|^{n-1})\|x-y\| \leq L\|x-y\|,\end{aligned}$$

where (calculations)

$$L = \begin{cases} n & \text{for } n-1 > k, \\ k + \left(\dfrac{n-1}{k}\right)^{n-1} & \text{for } n-1 \leq k. \end{cases}$$

The minimal displacement of $\overline{T}$ can be evaluated as follows

$$\|x - \overline{T}x\| = \|x - (1-\|x\|^n)Tx\| \geq \|x\| - (1-\|x\|^n)\|Tx\| \geq \|x\|^n + \|x\| - 1.$$

On the other hand

$$\|x - \overline{T}x\| = \|x - (1-\|x\|^n)Tx\| \geq \|x - Tx\| - \|x\|^n\|Tx\| \geq d_T - \|x\|^n.$$

We get that

$$\|x - \overline{T}x\| \geq \max\{\|x\|^n + \|x\| - 1, d_T - \|x\|^n\}$$

and because d_T can be arbitrary close to $\psi_X(k)$ we obtain

$$\|x - \overline{T}x\| \geq \max\{\|x\|^n + \|x\| - 1, \psi_X(k) - \|x\|^n\}.$$

Putting $n-1 = k$ we have

$$\overline{\psi}_X(n) \geq \max\{\|x\|^n + \|x\| - 1, \psi_X(n-1) - \|x\|^n\}.$$

Observe that

(i) If $\|x\| \in [0, \sqrt[n]{1/2}]$, then $\overline{\psi}_X(n) \geq \psi_X(n-1) - \|x\|^n \geq \psi_X(n-1) - 1/2$.
(ii) If $\|x\| \in [\sqrt[n]{1/2}, 1]$, then $\overline{\psi}_X(n) \geq \|x\|^n + \|x\| - 1 \geq \sqrt[n]{1/2} - 1/2$.

In both cases we have only that $\lim_{n\to\infty} \overline{\psi}_X(n) \geq 1/2$.

Observe that the exact value of the function $\overline{\psi}_X$ would be very helpful in estimation from above of the retraction constant $k_0(X)$.

Example 2. Define a retraction $R: B \to S$ as

$$Rx = \frac{x - Tx}{\|x - Tx\|},$$

where $T \in L(k)$, $T(S) = 0$ and $d_T \geq \overline{\psi}_X(k) - \varepsilon$. Observe that $Rx = x$ for any $x \in S$ and

$$\|Rx - Ry\| \leq \frac{k(X)(k+1)}{\lim_{z\in B}\|z - Tz\|}\|x-y\| = \frac{P(X)(k+1)}{\overline{\psi}_X(k)}\|x-y\|.$$

Using evaluation of $\overline{\psi}_X(k)$ from the previous example one can get that $k_0(X) \leq 37.74$ in spaces where $\psi_X(k) = 1 - 1/k$. For details see [5]. The problem of exact evaluation of $k_0(X)$ for at least one space is still open.

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A NEW PROOF OF THE SHAFRIR LEMMA

Monika Budzyńska and Tadeusz Kuczumow

Dedicated to Professor Lech Górniewicz on the occasion of his 60th birthday

Abstract. In this paper we give a new proof of the Shafrir lemma. In his proof, I. Shafrir used the so-called "cosine rule". In our proof we will not apply this rule — we will only use the basic properties of the Kobayashi distance k_{B_H}.

1. Introduction

Let B_H denote the open unit ball of a complex Hilbert space $(H, (\,\cdot\,,\,\cdot\,))$ and let k_{B_H} denote the Kobayashi distance in B_H (see [5], [6]). For each $a \in B_H$, consider the Möbius transformation defined by

$$M_a(x) = \frac{1}{1 + (x, a)}(\sqrt{1 - \|a\|}Q_a + P_a)(x + a),$$

where $x \in B_H$, P_a is the orthogonal projection of H onto $\mathrm{lin}(a) = \{\lambda a : \lambda \in \mathbb{C}\}$ and $Q_a = I - P_a$. Each M_a is a k_{B_H}-isometry and has a norm continuous injective extension from $\overline{B_H}$ onto itself. We also note that $M_a(0) = a$, $M_a^{-1} = M_{-a}$, $M_0 = I$, and for every $a, b \in B_H$ the mapping $M_b \circ M_{-a}$ takes a to b. Applying these properties of Möbius transformations we get the following explicit formula for k_{B_H}, namely,

$$k_{B_H}(x, y) = \arg\tanh(1 - \sigma(x, y))^{1/2},$$

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where $x, y \in B_H$ and (see [2], [3])

$$\sigma(x,y) = \frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-(x,y)|^2}.$$

Let $k_{B_H^n}$ denotes the Kobayashi distance in the Cartesian product B_H^n of n open unit balls B_H. It is not difficult to check that

$$k_{B_H^n}(x,y) = \max_{1\le j\le n} k_{B_H}(x_j, y_j)$$

(see [2], [9]).

In this paper $N(B_H^n)$ will denote the class of all $k_{B_H^n}$-nonexpansive self-mapping on B_H^n. Note that each holomorphic selfmapping of B_H^n is $k_{B_H^n}$-nonexpansive. The class of those mappings in $N(B_H^n)$, which have a continuous (in norm) extension to $\overline{B_H^n}$, will be denoted by $CN(\overline{B_H^n})$. It will also be convenient to consider a slightly more general class of mappings $N(\overline{B_H^n})$ which consists of all norm continuous mappings $f\colon \overline{B_H^n} \to \overline{B_H^n}$ such that $tf_{|B_H^n} \in N(B_H^n)$ for all $0 < t < 1$ ([8]). This class contains all holomorphic mappings $f : B_H^n \to \overline{B_H^n}$ which have a continuous (in norm) extension to $\overline{B_H^n}$ ([8]).

In [12] I. Shafrir proved the following theorem.

Theorem 1 ([12]). *A commuting family of mappings $\{f_\alpha\}_{\alpha\in I}$ in $N(\overline{B_H^n})$ has a common fixed point in $\overline{B_H^n}$.*

This theorem is a generalization of the following existence theorem due to A. Stachura and T. Kuczumow ([8], [9], see also [1], [3], [4], [7], [10] and [11]).

Theorem 2 ([8], [9]). *If $f \in N(\overline{B_H^n})$, then* $\mathrm{Fix}(f) \neq \emptyset$.

An application of the given bellow lemma is a crucial point in the proof of the Shafrir theorem.

Lemma 3 ([12]). *Let $\{x_\alpha\}_{\alpha\in I}$ be a k_{B_H}-unbounded net in B_H satisfying*

$$\sup_{\alpha,\beta\in I,\ \alpha\le\beta} \{k_B(x_\alpha, x_\beta) - k_B(x, x_\beta)\} = R < \infty$$

for some $x \in B$ and $\alpha \in I$. Then there is a point $\xi \in \partial B$ such that $\xi = \lim_\alpha x_\alpha$.

A new proof of this lemma will be presented in the next section.

2. Proof of Lemma 3

In his proof of Lemma 3, I. Shafrir used the so-called "cosine rule". In our proof we will not apply this rule — we will only use basic properties of the Kobayashi distance k_{B_H}.

Proof. It is obvious that the point x can be replaced by 0. We first note that

$$\lim_\alpha k_B(0, x_\alpha) = \infty.$$

Otherwise, there would exist M and, for each index $i \in I$, an $\alpha_i \geq i$ such that $k_{B_H}(0, x_{\alpha_i}) \leq M$. But then, for all i, we would have

$$k_{B_H}(0, x_i) \leq k_{B_H}(0, x_{\alpha_i}) + k_{B_H}(x_{\alpha_i}, x_i) \leq 2k_B(0, x_{\alpha_i}) + R \leq 2M + R,$$

contradicting the k_{B_H}-unboundedness of $\{x_\alpha\}_{\alpha \in I}$. Now, without loss of generality we can assume that $x_\alpha \neq 0$ for all $\alpha \in I$. Then there exists $0 < r < 1$ such that for each

$$z_\alpha = -\frac{r}{\|x_\alpha\|} x_\alpha,$$

$\alpha \in I$, we have

$$k_{B_H}(z_\alpha, x_\alpha) = k_{B_H}(0, x_\alpha) + R.$$

This implies

$$\begin{aligned} k_{B_H}(M_{-z_\beta} x_\alpha, M_{-z_\beta} x_\beta) &= k_{B_H}(x_\alpha, x_\beta) \leq k_{B_H}(0, x_\beta) + R \\ &= k_{B_H}(z_\beta, x_\beta) = k_{B_H}(0, M_{-z_\beta} x_\beta) \end{aligned}$$

for $\alpha \leq \beta$, or equivalently,

$$|1 - (M_{-z_\beta} x_\alpha, M_{-z_\beta} x_\beta)| \leq 1 - \|M_{-z_\beta} x_\alpha\|^2$$

for $\alpha \leq \beta$. Next, we observe that, for $\|x_\alpha\| \geq r$,

$$\min_{\|z\|=r} k_{B_H}(M_{-z} x_\alpha, 0) = k_{B_H}(x_\alpha, 0) - R,$$

and therefore

$$\lim_\alpha (\min_{\|z\|=r} \|M_{-z} x_\alpha\|) = 1.$$

Now, let $\{x_{\alpha_j}\}_{j \in J}$ and $\{x_{\beta_{j'}}\}_{j' \in J'}$ be two subnets of $\{x_\alpha\}_{\alpha \in I}$ which are weakly convergent to ξ and ξ', respectively. Then we obtain

$$\begin{aligned} 1 &= \lim_j \lim_{j'} (M_{-z_{\beta_{j'}}} x_{\alpha_j}, M_{-z_{\beta_{j'}}} x_{\beta_{j'}}) \\ &= \lim_j \lim_{j'} \frac{(\|x_{\beta_{j'}}\| + r)[((x_{\alpha_j}, x_{\beta_{j'}}) + r\|x_{\beta_{j'}}\|)]}{(1 + r\|x_{\beta_{j'}}\|)[r(x_{\alpha_j}, x_{\beta_{j'}}) + \|x_{\beta_{j'}}\|]} = \frac{(\xi, \xi') + r}{r(\xi, \xi') + 1} \end{aligned}$$

and finally,

$$0 = 1 - \frac{(\xi, \xi') + r}{r(\xi, \xi') + 1} = (1 - r)\frac{1 - (\xi, \xi')}{r(\xi, \xi') + 1}$$

which implies

$$\xi = \xi' \in \partial B.$$

This concludes the proof of Lemma 3. □

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ON SOME APPLICATIONS OF REICHERT'S PRINCIPLE TO DIFFERENTIAL EQUATIONS IN LOCALLY CONVEX SPACES

Daria Bugajewska

Abstract. In this paper we investigate a topological structure of solution sets of the initial value problem for some equation of nth order in locally convex spaces. The method of our proofs is based on the connectness principle from [4]. Our results complement similar theorems from [8] and [7] for Banach spaces.

1. Introduction

Let E be a quasi-complete locally convex space and let $\mathcal{P}$ be a family of seminorms which generate the topology of E. Moreover, let $I = [0, a]$ be a compact interval in $\mathbb{R}$ and $B = \{x \in E : p_i(x) \leq b,\ i = 1, \ldots, k\}$, $b > 0$, $k \in \mathbb{N}$ and $p_1, \ldots, p_k \in \mathcal{P}$. Consider the problem

$$
\begin{aligned}
x^{(n)} &= f(t, x) \\
x^{(j)}(0) &= x_j, \qquad j = 0, \ldots, n-1,
\end{aligned}
\tag{1}
$$

where $x_j \in E$ for $j = 0, \ldots, n-1$, $x_0 = 0$ and $f\colon I \times B \to E$ is a bounded, continuous function.

Denote by $(\beta_p(\,\cdot\,))_{p\in\mathcal{P}}$ the Sadovski measure of noncompactness (see [5] for the definition and basic properties). In the sequel we shall need the following two lemmas.

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Lemma 1 ([6]). *Let D be a compact subset of $\mathbb{R}^n$. Denote by $C = C(D, E)$ the space of all continuous functions $D \to E$ with the topology of uniform convergence. Let $p \in \mathcal{P}$ and $H \subset C$. If H is p-equiuniformly continuous and $H(D)$ is p-bounded, then the function $t \to \beta_p(H(t))$, $t \in D$ is continuous and*

$$\beta_p(H(D)) = \sup\{\beta_p(H(t)) : t \in D\}.$$

Lemma 2 ([7]). *Let $\omega\colon [0, 2b] \to \mathbb{R}_+$ be a continuous, nondecreasing function and $g\colon [0, c) \to [0, 2b]$ be a C^n function satisfying inequalities:*

$$\begin{aligned} &g^{(j)}(t) \geq 0, && j = 0, 1, \dots, n, \\ &g^{(j)}(0) = 0, && j = 0, 1, \dots, n-1, \\ &g^{(n)}(t) \leq \omega(g(t)), && t \in [0, c). \end{aligned}$$

If $\omega(0) = 0$, $\omega(r) > 0$ for $r > 0$ and

$$\int_{0^+} \frac{dr}{\sqrt[n]{r^{n-1}\omega(r)}} = +\infty,$$

then $g \equiv 0$.

Finally, define

$$\varphi_p(t, X) = \lim_{r \to 0^+} \beta_p(f(I_{tr} \times X)) \quad \text{for } t \in (0, a) \text{ and } X \subset B,$$

where $I_{tr} = (t-r, t+r) \cap I$ (cf. [3]). Moreover, set $B_p(0, r) = \{x \in E : p(x) \leq r\}$.

2. Results and proofs

At the beginning of this section we prove the following

Theorem 1. *Assume that for every seminorm $p \in \mathcal{P}$ there exists a continuous function u_p, defined on I, such that $u_p(t) > 0$ for $t > 0$, $u_p(0) = \ldots = u_p^{(n-1)}(0) = 0$, $u_p^{(n)}(t)$ is positive and integrable in the Lebesgue sense and*

$$\varphi_p(t, X) \leq \frac{u_p^{(n)}(t)}{u_p(t)} \beta_p(X) \tag{2}$$

for $t \in (0, a)$ and for every bounded set $X \subset B$. Moreover, let

$$\lim_{\substack{t \to 0^+ \\ r \to 0^+}} \frac{\varphi_p(t, B_p(0, r))}{u_p^{(n)}(t)} = 0. \tag{3}$$

Then there exists an interval $J = [0, d] \subset I$ such that the set S of all solutions of (1), defined on J, is nonempty, compact and connected in $C(J, E)$.

Proof. Let $m_p = \sup\{p(f(t, x)) : t \in I,\ x \in B\}$, where $p \in \mathcal{P}$ nad let $M = \sup\{p_i(f(t, x)) : t \in I,\ x \in B,\ i = 1, \dots, k\}$. Choose a positive number d such that $d \leq a$ and $\sum_{j=1}^{n-1} p_i(x_j) d_j / j! + M d^n / n! \leq b$ for $i = 1, \dots, k$. Let

$C_0 := \{x \in C(J,E) : x(0) = 0\}$ and $U_0 := \{x \in C_0 : p_i(x(t)) < b$ for $t \in J$ and $i = 1, \dots, k\}$. Define $F\colon \overline{U}_0 \to \overline{U}_0$ in the following way

$$F(x)(t) = \sum_{j=1}^{n-1} x_j \frac{t^j}{j!} + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s, x(s))\, ds \quad \text{for } t \in J.$$

The above defined operator F is a Volterra type on $\overline{U}_0$, obviously. Further, put $K_0 := \{0\}$, $K_\alpha := \{t \in J : t \le \alpha d\}$ for $0 < \alpha < 1$. For $0 < \lambda < 1$ set $K_{\lambda,\nu} := K_{\lambda\nu}$ for $\nu = 1, 2, \dots, n(\lambda) - 1$ and $K_{\lambda, n(\lambda)} := K$, where

$$n(\lambda) := \begin{cases} 1/\lambda & \text{for } 1/\lambda \in \mathbb{N}, \\ [1/\lambda] + 1 & \text{for } 1/\lambda \notin \mathbb{N}. \end{cases}$$

Then $K_{\lambda,1} = [0, \lambda d]$. Define

$$x_\lambda(t) := \begin{cases} 0 & \text{for } t \in K_{\lambda,1}, \\ x(t - \lambda d) & \text{for } t \in K \setminus K_{\lambda,1}. \end{cases} \tag{4}$$

It can be easily shown that F is continuous, $F(\overline{U}_0)$ is a bounded and equiuniformly continuous set, and $F(\overline{U}_0) \subset W$, where

$$W := \Big\{ x \in C_0 : p(x(t)) \le \sum_{j=1}^{n-1} p(x_j) \frac{d^j}{j!} + m_p \frac{d^n}{n!} \text{ and } \\ p(x(t) - x(s)) \le \Big(\sum_{j=1}^{n-1} p(x_j) \frac{d^{j-1}}{(j-1)!} + m_p \frac{d^{n-1}}{(n-1)!} \Big) |t - s|, \\ t, s \in I \quad \text{and} \quad p \in \mathcal{P} \Big\}. \tag{5}$$

Let

$$\mathcal{M} := \{ M \subset C_0 : M \ne \emptyset, \\ M \text{ is bounded and } p\text{-equiuniformly continuous for } p \in \mathcal{P} \}.$$

It is clear that $\mathcal{M}$ is suitable in the sense of [4]. For $M \in \mathcal{M}$ define

$$\mu_p(M(t)) := \sup_{0 \le \tau \le t} \beta_p(M(\tau)) = \beta_p\Big(\bigcup_{0 \le \tau \le t} M(\tau) \Big) \tag{6}$$

for $t \in [0, d]$ and $p \in \mathcal{P}$, where $M(t) := \{x(t) : x \in M\}$ (note that the second equality above is a consequence of Lemma 1).

In view of the p-equiuniform continuity of M, the function $t \to \beta_p(M(t))$ is continuous on J and therefore the function $t \to \mu_p(M(t))$ is also continuous on this interval for every $p \in \mathcal{P}$.

Let $\mathcal{R}$ be a set of families $\{\mu_p(t)\}_{p\in\mathcal{P}}$, where μ_p are continuous, nonnegative functions, defined on J. The set $\mathcal{R}$ is partially ordered by the relation defined as follows

$$\{\mu_p(t)\}_{p\in\mathcal{P}} \leq \{\overline{\mu}_p(t)\}_{p\in\mathcal{P}} \Leftrightarrow \mu_p(t) \leq \overline{\mu}_p(t)$$

for all $t \in J$ and $p \in \mathcal{P}$. Define $\Psi\colon \mathcal{M} \to \mathcal{R}$ by the formula

$$\Psi(M) := \{\mu_p(M(t))\}_{p\in\mathcal{P}}.$$

In view of the properties of the measure $(\beta_p(\,\cdot\,))_{p\in\mathcal{P}}$ it is clear that the mapping Ψ is a measure of noncompactness in the sense of [4] (in particular, $\Psi(M) = \Psi(\bigcup_{0\leq\lambda\leq1} M_\lambda)$, where $M_\lambda := \{x_\lambda : x \in M\}$ for $0 \leq \lambda \leq 1$ and x_λ are defined in (4)).

Now, we verify that F is a condensing operator on $\overline{U}_0$. Let $M \in \mathcal{M}$ be such that $\Psi(M) \leq \Psi(F(M))$. By (3), for every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\varphi_p(t, B_p(0,\eta)) \leq \varepsilon u_p^{(n)}(t), \quad 0 \leq t \leq \eta.$$

On the other hand, in view of the equiuniform continuity of $F(M)$, there exists $\delta \in (0,\eta]$ such that

$$p(x(t)) \leq \eta \quad \text{for } t \in [0,\delta],\ x \in F(M).$$

Hence, arguing similarly as in [1, Theorem 1], one can show that

$$\text{(7)} \quad \beta_p(F(M)(t)) \leq \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\varepsilon u_p^{(n)}(s)\,ds = \varepsilon u_p(t) \quad \text{for } 0 \leq t \leq \delta.$$

Fix $t \in J$. Since the function $t \to \beta_p(F(M)(t))$ is continuous, by (6) there exists $t_m \leq t$ such that

$$\mu_p(F(M)(t)) = \beta_p(F(M)(t_m)).$$

Again, arguing similarly as in [1, Theorem 1] we infer that

$$\beta_p(F(M)(t_m)) \leq \frac{1}{(n-1)!}\int_0^{t_m} (t_m-s)^{n-1}\frac{u_p^{(n)}(s)}{u_p(s)}\beta_p(M(s))\,ds,$$

so

$$\begin{aligned}\beta_p(F(M)(t_m)) &\leq \frac{1}{(n-1)!}\int_0^{t_m} (t_m-s)^{n-1}\frac{u_p^{(n)}(s)}{u_p(s)}\beta_p\Big(\bigcup_{0\leq\sigma\leq s} M(\sigma)\Big)\,ds\\ &\leq \frac{1}{(n-1)!}\int_0^{t} (t-s)^{n-1}\frac{u_p^{(n)}(s)}{u_p(s)}\mu_p(M(s))\,ds,\end{aligned}$$

because $t_m \leq t$. Hence, we obtain

$$\mu_p(F(M)(t)) \leq \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\frac{u_p^{(n)}(s)}{u_p(s)}\mu_p(M(s))\,ds \quad \text{for } t \in J.$$

Because $\mu_p(M(t)) \le \mu_p(F(M)(t))$, so

$$\mu_p(M(t)) \le \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\frac{u_p^{(n)}(s)}{u_p(s)}\mu_p(M(s))\,ds \quad \text{for } t \in J. \tag{8}$$

Now, we apply an idea from [2]. Let $w_p(t) = \mu_p(M(t))/u_p(t)$ for $t \in (0,d]$ and $w_p(0) = 0$. By (7), the function w_p is continuous. Suppose that w_p is not identically equal to zero on J. Let τ be the smallest real number such that $w_p(\tau) = \max_{t\in J} w_p(t)$. Then $0 \le w_p(s) < w_p(\tau)$ for $s \in [0,\tau)$. In view of (8), we obtain

$$\begin{aligned} w_p(\tau) &= \frac{\mu_p(M(\tau))}{u_p(\tau)} \le \frac{1}{u_p(\tau)(n-1)!}\int_0^\tau (\tau-s)^{n-1}\frac{u_p^{(n)}(s)}{u_p(s)}\mu_p(M(s))\,ds \\ &= \frac{1}{u_p(\tau)(n-1)!}\int_0^\tau (\tau-s)^{n-1}u_p^{(n)}(s)w_p(s)\,ds \\ &< \frac{w_p(\tau)}{u_p(\tau)}\frac{1}{(n-1)!}\int_0^\tau (\tau-s)^{n-1}u_p^{(n)}(s)\,ds = \frac{w_p(\tau)}{u_p(\tau)}u_p(\tau) = w_p(\tau), \end{aligned}$$

what gives a contradiction. Thus $\mu_p(M(\,\cdot\,)) \equiv 0$ on J.

Hence F is condensing on $\overline{U}_0$. Therefore in view of [4, Theorem 1.9] the set of all fixed points is nonempty and compact. To obtain the conectedness of S by this theorem we must have a set $W \subset U_0$. Note that the set W defined in (5) is not contained in U_0. Therefore consider again the problem (1) in the space $C_0(\widetilde{J},E)$, where $\widetilde{J} = [0,\widetilde{d}] \subset J$ and $0 < \widetilde{d} < d$. Restricting F to $C_0(\widetilde{J},E)$ we obtain the required inclusion $W \subset U_0$. Then by [4, Theorem 1.9] we infer that the set of all fixed points of F restricted to $C_0(\widetilde{J},E)$ is connected in this space for every $\widetilde{d} \in (0,d)$. This implies that the set S is connected in $C_0(J,E)$, because S is compact in $C_0(J,E)$ and every fixed point of the operator F restricted to the space $C_0(\widetilde{J},E)$ can be extended to a fixed point of F. □

Now we prove the second result of this section, namely

Theorem 2. *Assume that for every seminorm $p \in \mathcal{P}$ there exists a continuous nondecreasing function $w_p\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $w_p(0) = 0$, $w_p(r) > 0$ for $r > 0$,*

$$\int_{0^+} \frac{dr}{\sqrt[n]{r^{n-1}w(r)}} = +\infty \tag{9}$$

and

$$\beta_p(f(T\times X)) \le w_p(\beta_p(X)) \tag{10}$$

for $T \subset I$ and any bounded set $X \subset B$. Then the set S of all solutions of the problem (1)*, defined on some interval $J \subset I$, is nonempty, compact and connected in the space $C(J,E)$.*

Proof. We will keep the same notation as in the proof of Theorem 3.

Fix $p \in \mathcal{P}$ and $t \in J$. The functions $s \to \beta_p(M(s))$ and $s \to \beta_p(F(M)(s))$ are continuous on J, so in particular, there exists $\tau_m \le t$ such that $\mu_p(F(M)(t)) = \beta_p(F(M)(\tau_m))$. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$|(\tau_m - \tau)^{n-1} w_p(\beta_p(M(\rho))) - (\tau_m - s)^{n-1} w_p(\beta_p(M(s)))| < \varepsilon$$

for $|\tau - s| < \delta$, $|\rho - s| < \delta$, $\rho, s, \tau \in [0, \tau_m]$. Divide the interval $[0, \tau_m]$ into k parts: $0 = t_0 < \ldots < t_k = \tau_m$ in such a way that $t_i - t_{i-1} < \rho$ for $i = 1, \ldots, k$.

In view of the properties of the measure $(\beta_p(\,\cdot\,))_{p \in \mathcal{P}}$, (10) and Lemma 1, we have

$$\begin{aligned}
&\beta_p(\{(\tau_m - s)^{n-1} f(s, x(s)) : s \in [t_{i-1}, t_i],\ x \in M\}) \\
&\qquad \le (\tau_m - t_{i-1})^{n-1} \beta_p(f([t_{i-1}, t_i] \times M([t_{i-1}, t_i]))) \\
&\qquad \le (\tau_m - t_{i-1})^{n-1} w_p(\beta_p(M([t_{i-1}, t_i]))) = (\tau_m - t_{i-1})^{n-1} w_p(\beta_p(M(s_i)))
\end{aligned}$$

for some $s_i \in [t_{i-1}, t_i]$. Thus we obtain

$$\begin{aligned}
\mu_p(F(M)(t)) &= \beta_p(F(M)(\tau_m)) \\
&= \beta_p\biggl(\biggl\{\frac{1}{(n-1)!} \int_0^{\tau_m} (\tau_m - s)^{n-1} f(s, x(s))\, ds : x \in M\biggr\}\biggr) \\
&\le \beta_p\biggl(\biggl\{\frac{1}{(n-1)!} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (\tau_m - s)^{n-1} f(s, x(s))\, ds : x \in M\biggr\}\biggr) \\
&\le \frac{1}{(n-1)!} \sum_{i=1}^{k} (t_i - t_{i-1}) \\
&\qquad \cdot \beta_p(\overline{\operatorname{conv}}\,\{(\tau_m - s)^{n-1} f(s, x(s)) : s \in [t_{i-1}, t_i],\ x \in M\}) \\
&= \frac{1}{(n-1)!} \sum_{i=1}^{k} (t_i - t_{i-1}) \\
&\qquad \cdot \beta_p(\{(\tau_m - s)^{n-1} f(s, x(s)) : s \in [t_{i-1}, t_i],\ x \in M\}) \\
&\le \frac{1}{(n-1)!} \sum_{i=1}^{k} (\tau_m - t_{i-1})^{n-1} w_p(\beta_p(M(s_i)))(t_i - t_{i-1}) \\
&\le \frac{1}{(n-1)!} \int_0^{\tau_m} (\tau_m - s)^{n-1} w_p(\beta_p(M(s)))\, ds + \frac{\varepsilon \tau_m}{(n-1)!} \\
&\le \frac{1}{(n-1)!} \int_0^{t} (t - s)^{n-1} w_p(\mu_p(M(s)))\, ds + \frac{\varepsilon t}{(n-1)!}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we infer that

$$\mu_p(F(M)(t)) \le \frac{1}{(n-1)!} \int_0^t (t - s)^{n-1} w_p(\mu_p(M(s)))\, ds.$$

On the other hand, we know that $\mu_p(M(t)) \le \mu_p(F(M)(t))$, so

$$\mu_p(M(t)) \le \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} w_p(\mu_p(M(s)))\,ds, \quad \text{for } t \in J.$$

Put

$$g(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} w_p(\mu_p(M(s)))\,ds$$

for $t \in J$. Then $g \in C^n$, $\mu_p(M(t)) \le g(t)$, $g^{(j)}(t) \ge 0$ for $j = 0, \dots, n$, $g^{(j)}(0) = 0$ for $j = 0, \dots, n-1$ and $g^{(n)}(t) = w_p(\mu_p(M(t))) \le w_p(g(t))$ for $t \in J$. In view of (9) and Lemma 2 we deduce that $g(t) \equiv 0$ for $t \in J$. Thus $\mu_p(M(t)) = 0$ for $t \in J$ and $p \in \mathcal{P}$. Further we argue similarly as in the proof of Theorem 3. □

Note that Theorem 4 extends Theorem 2(a) from [6].

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ON THE POINCARÉ–BENDIXSON THEOREM

Krzysztof Ciesielski

Abstract. The famous Poincaré–Bendixon Theorem, its generalizations and the development of the theory (up to the most recent results) are presented.

1. Introduction

The Poincaré–Bendixson theorem plays an important role in the study of the qualitative behaviour of autonomous differential equations and dynamical systems on $\mathbb{R}^2$. It describes very precisely the structure of limit sets in such systems. In 1901, precisely one hundred years ago, the famous Bendixson paper on this theorem and related subjects was published in "Acta Mathematica". In this paper, we present this theorem and its development during the last century – from the first results to the modern generalizations.

2. Preliminaries

By a *Jordan arc* (a *Jordan curve*) we mean a homeomorphic image of a compact segment $[-1,1]$ (a unit circle). A homeomorphism $\varphi\colon[-1,1]\to T$ will be called a *parametrization* of an arc T. A 2-manifold M is called *dichotomic* if any Jordan curve cuts M into two open connected domains.

Let X be a metric space. A *flow* (*dynamical system*) on X (which is called a *phase space*) is a triplet $(X,\mathbb{R},\pi)$ where $\pi\colon\mathbb{R}\times X\to X$ is a continuous function such that $\pi(0,x)=x$ and $\pi(t,\pi(u,x))=\pi(t+u,x)$ for any t,u,x.

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Consider an autonomous differential equation $x' = f(x)$. We say that it defines a flow if for any x the solution $x(t)$ through x exists for any $t \in (-\infty, \infty)$ and is unique. It is well known that the function given by the solutions of autonomous differential equation fulfilling the above conditions satisfies the conditions required in the definition of a flow.

The definitions in the sequel will be given for flows, but they are applicable in the obvious way for systems of differential equations.

By *an orbit* (*a trajectory*) through a point x we mean the set $\gamma(x) = \{\pi(t,x) : t \in \mathbb{R}\}$. By *a positive semiorbit* (*semitrajectory*) we mean the set $\gamma^+(x) = \{\pi(t,x) : t \geq 0\}$. By *a negative semiorbit* (*semitrajectory*) we mean the set $\gamma^-(x) = \{\pi(t,x) : t \leq 0\}$. We denote by $\gamma([s,t],x)$ the set $\{\pi(u,x) : s \leq u \leq t\}$ and by $\gamma((s,t),x)$ the set $\{\pi(u,x) : s < u < t\}$.

A point x is said to be

- *stationary* if $\pi(t,x) = x$ for every $t \geq 0$,
- *periodic* if there exists a $t > 0$ such that $\pi(t,x) = x$ and x is not stationary,
- *regular* if it is neither periodic nor stationary.

For a given point x we define *the positive limit set of* x (or ω*-limit set of* x) as $\omega(x) = \{y \in X \colon \pi(t_n,x) \to y \text{ for some } t_n \to \infty\}$ and *the negative limit set of* x (or α*-limit set of* x) as $\alpha(x) = \{y \in X \colon \pi(t_n,x) \to y \text{ for some } t_n \to -\infty\}$.

A point x is said to be *positively Poisson stable* if $x \in \omega(x)$. It is said to be *negatively Poisson stable* if $x \in \alpha(x)$.

A set A is *invariant* if $\pi(\mathbb{R} \times A) = A$. A set A is *minimal* if it is nonempty, closed, invariant and no proper subset of A has all these properties.

A set A is *a saddle set* if there exists a neighbourhood U of A such that every neighbourhood V of A contains a point x with $\gamma^+(x) \nsubseteq U$ and $\gamma^-(x) \nsubseteq U$.

By *a section through* x we mean a set S containing x such that for some $\lambda > 0$ the set $U = \pi((-\lambda,\lambda), S)$ is a neighbourhood (not necessarily open) of x and for every $y \in U$ there are a unique $z \in S$ and a unique $t \in (-\lambda,\lambda)$ with $\pi(t,z) = y$. In such a neigbourhood, the local parallelizability of the system is fulfilled.

For an autonomous system of differential equations which defines a flow, by *a transversal* we mean a Jordan arc which is not tangent to any orbit of the system in any of its points.

A *semiflow* (*semi-dynamical system*) on X is a triplet $(X, \mathbb{R}_+, \pi)$ where $\pi \colon \mathbb{R}_+ \times X \to X$ is a continuous function such that $\pi(0,x) = x$ and $\pi(t,\pi(u,x)) = \pi(t+u,x)$ for any t, u, x.

For a given semiflow, we define stationary, periodic and regular points and positive semiorbits in the same way as in the case of flows. We define *a negative solution through* x as a function $\sigma \colon (-\infty, 0] \to X$ such that $\sigma(0) = x$ and $\pi(t, \sigma(u)) = \sigma(t+u)$ for any t, u with $u \leq 0$, $t \geq 0$, $t+u \leq 0$. The image of a negative solution is called a *negative semiorbit* (*semitrajectory*) *through* x. Usually, the definition of a negative solution is stated in a more general way, but for the subject considered in this paper we may restrict ourselves to such definition. This will be explained in Chapter 8. We put

$$\begin{aligned} F(t,x) &= \{y \in X : \pi(t,y) = x\}, \\ F(\Delta, A) &= \bigcup \{F(u,y) : u \in \Delta,\ y \in A\} \end{aligned}$$

for $A \subset X$, $\Delta \subset \mathbb{R}_+$. A set A is *positively invariant* if $\pi(\mathbb{R}_+ \times A) = A$. A set A is *negatively invariant* if $F([0,\infty), A) = A$. We call the set A *weakly negatively invariant* if for any $x \in A$ there is a negative solution σ through x with $\sigma(-\infty, 0] \subset A$. A set A is *positively* (*weakly*) *minimal* if it is nonempty, closed, positively (weakly negatively) invariant and no proper subset of A has all these properties.

In the case of semiflows we define ω-limit sets in the same way as in the case of flows. However, it may happen that there are many negative solutions through a point x. For a given negative solution σ we define the α-limit set $\alpha_\sigma(x)$ as $\{y \in X\colon \sigma(t_n) \to y$ for some $t_n \to -\infty\}$. Note that for a given point x, different negative solutions may give different negative limit sets. It is known ([12]) that limit sets in semiflows on manifolds are positively and weakly negatively invariant.

3. The Poincaré–Bendixson theorem

First, we formulate the Poincaré–Bendixson Theorem in its classical version.

Theorem 3.1. *Consider a plane autonomous system $x' = f(x)$ where $x \in \mathbb{R}^2$ and assume that this system defines a flow. Assume that the positive semiorbit $\gamma^+(p)$ through a point $p \in \mathbb{R}^2$ is bounded and that the positive limit set $\omega(p)$ does not contain any stationary point. Then $\omega(p)$ is a periodic orbit. Moreover, either p is a periodic point or $\gamma^+(p)$ spirals towards a limit cycle of the system. The analogous result holds for the negative limit set $\alpha(p)$.*

There are several proofs of this theorem. Generally, they are based on two important facts: the Jordan Curve Theorem and the local parallelizability of a small neighbourhood of a non-stationary point (i.e. the properties of transversals). Here we describe the outline of the proof.

Step 1. Let T be a transversal. Then T is a section according to the definition of sections for flows (the most frequently presented proof uses the Implicit Function Theorem).

Step 2. Let T be a transversal and $\gamma([s,t],p)$ be a segment of the orbit through p. Then the intersection $T \cap \gamma([s,t],p)$ is finite (possibly empty).

Step 3. Let x_1, x_2, x_3 be common points of the transversal T and the orbit $\gamma(p)$ of a regular point p. Let $x_i = h(u_i)$ where h is a parametrization of the transversal and let $x_i = \gamma(t_i, x)$. Assume that $u_1 < u_2 < u_3$. Then either $t_1 < t_2 < t_3$ or $t_3 < t_2 < t_1$. In other words, for the common points of the transversal and the orbit, the order on the transversal given by its parametrization coincides with the order on the orbit given by the time variable.

Step 4. If T is a transversal and $\gamma(p)$ is a periodic orbit, then $T \cap \gamma(p)$ has at most one element.

Step 5. If L is an ω-limit set (an α-limit set) and T is a transversal, then $T \cap L$ has at most one element.

Then it is shown that if any bounded limit set L contains a periodic orbit then L is equal to that periodic orbit. The property that any bounded ω-limit set (an α-limit set) is compact and invariant is also used.

The detailed proof can be found, for instance, in [21], [24], [37], [57], [59], [74]. Usually, the proofs are done in different ways by different authors. For example, as the corollary of the Poincaré–Bendixson Theorem one can get the characterization of the minimal sets on the plane. Some authors base the proof of the Poincaré–Bendixson Theorem on this characterization, obtained earlier. Some other techniques are used also in the proofs for flows; this will be described in Chapter 7.

As an immediate consequence we have

Corollary 3.2. *Under the above assumptions, if a bounded closed region does not contain any stationary point and contains a semiorbit of some point, then it contains also a closed orbit.*

The generalized version of the theorem says:

Theorem 3.3. *Consider a plane autonomous system $x' = f(x)$ where $x \in \mathbb{R}^2$ and assume that this system defines a flow. Assume that the positive semiorbit $\gamma^+(p)$ through a point $p \in \mathbb{R}^2$ is bounded. Then either*

(a) *the positive limit set $\omega(p)$ is a periodic orbit, or*
(b) *for any $q \in \omega(p)$ the limit sets $\alpha(q)$ and $\omega(q)$ are nonempty and contain only stationary poins.*

The theorem can be immediately adopted for the systems on the 2-dimensional sphere S^2. Then, it leads to the characterization of the minimal sets and Poisson stable points. We have

Theorem 3.4. *Consider an autonomous system $x' = f(x)$ where $x \in S^2$ and assume that this system defines a flow. Then any minimal set is either a stationary point or a periodic trajectory.*

Theorem 3.5. *Consider an autonomous system $x' = f(x)$ where $x \in S^2$ and assume that this system defines a flow. If $p \in S^2$ is neither stationary nor periodic, then $p \notin \alpha(p)$ and $p \notin \omega(p)$.*

The analogous theorems hold for systems on $\mathbb{R}^2$ (in the case of Theorem 3.4 we consider compact minimal sets). We have also

Corollary 3.6. *Let an autonomous system $x' = f(x)$ where $x \in S^2$ or $x \in \mathbb{R}^2$ be given and assume that this system defines a flow. Then any minimal set of the system is a single trajectory.*

The Poincaré–Bendixson Theorem has several very important applications. Let us mention here only some of them. First of all, it guarantees (under some assumptions) the existence of periodic orbits (frequently it guarantees the existence of limit cycles). Moreover, it gives the existence of stationary points (as for the system defined on the plane each periodic orbit must surround a stationary point). There are several examples of differential equations where the existence of a periodic orbit can be proved just with the use of this theorem. Consequently, the theorem is applicable to the real second order equations. The theorem has also very much other deep consequences, not only in looking for the properties of solutions of differential equations. For more details about the fundamental applications, the reader is referred for instance to [3], [24], [74].

4. The early years

Henri Poincaré (1854–1912) can be regarded as the father of the qualitative theory of differential equations. In his four-part memorable paper [61], published in 1881–1886, he studied celestial mechanics and two–dimensional systems. He made the investigations of the phase portrait of the solutions. However, he considered only the systems $x' = f(x)$ given by an analytic function f. Theorem 3.1 in the analytic case is due to part III of the work [61]. Poincaré stated also Corollary 3.2 for systems given by analytic functions.

It was Poincaré who introduced the concept of an orbit (or, in other words, a trajectory), i.e. a curve in the (x, x') plane parametrized by the time variable t. Such a curve, which was called by Poincaré *a characteristic* (in French: *caractéristique*) can be obtained by eliminating the variable t from the given equations. In such a way Poincaré gave a geometric framework for studying qualitative behaviour of planar differential equations. Poincaré did not investigate the method of solving the particular equations. He analysed possible behaviours of second order differential equations. Investigating trajectories, Poincaré formulated and solved several problems in the theory of differential equations as topological problems.

Poincaré was the first who investigated a geometric picture of the trajectories of a system given by a differential equation without integrating this equation. The geometric picture of the phase portrait of the system would have lead to understanding physical phenomena of the system given by the equation.

Note also that it was just Poincaré who introduced the term “limit cycle”.

About fifteen years later, in the beginning of the twentieth century Ivar Bendixson (1861–1935) in his paper [11] (published in 1901) completed the analysis of stationary points by making a more detailed classification. He proved Theorem 3.1 with much weaker assumption on function f. He assumed that

(4.1) *$f = (f_1, f_2)$ is continuous and each of f_1, f_2 has continuous partial derivatives.*

Using this theorem, he proved also some important theorems characterizing the behaviour of orbits near an isolated stationary point and near a periodic orbit. He showed, in particular, the following theorems.

Theorem 4.1. *Consider an autonomous system $x' = f(x)$ and assume that* (4.1) *is fulfilled. Let p be an isolated stationary point. Then at least one of the following conditions holds:*

(4.1.1) *in any neighbourhood of p there exist infinitely many periodic orbits surrounding p*

(4.1.2) *there exists a point $x \neq p$ such that $\omega(x) = \{p\}$ or $\alpha(x) = \{p\}$.*

Theorem 4.2. *Consider an autonomous system $x' = f(x)$ and assume that* (4.1) *is fulfilled. Let $\gamma(p)$ be a periodic orbit. Then at least one of the following conditions holds:*

(4.2.1) *in any neighbourhood of $\gamma(p)$ there exist infinitely many periodic orbits,*

(4.2.2) *there exists a point $x \notin \gamma(p)$ such that $\omega(x) = \gamma(p)$ or $\alpha(x) = \gamma(p)$.*

From Theorem 4.1 Bendixson obtained the classification of isolated stationary points. A stationary point which fulfills the condition (4.1.1) is called a centre.

There are two types of centres which are now known as Poincaré centres and Bendixson centres. By a Poincaré centre we mean the isolated stationary point p such that there exists a neighbourhood U of p fulfilling the properties:

(4.3.1) *U is invariant*,

(4.3.2) *all points in U but p are periodic*,

(4.3.3) *any periodic orbit contained in U surrounds p.*

For example, p is a Poincaré centre in the system given by the equations:

$$r'(t) = 0, \quad \theta'(t) = 1 \quad \text{(in polar coordinates)}$$

Assume now that (4.1.1) holds and for any periodic orbit $\gamma(q)$ which surrounds p there exists a regular point v contained in the bounded component of $\mathbb{R}^2 \setminus \gamma(q)$ (then, of course, also $\gamma(v)$ is contained in this component). Such stationary point is called a Bendixson centre. For example, 0 is a Bendixson centre in the system given by the equations:

$$r'(t) = g(r), \quad \theta'(t) = 1 \quad \text{(in polar coordinates)}$$

where $g(r) = r^2 \sin(\pi/r)$ for $r \neq 0$ and $g(r) = 0$ for $r = 0$.

In this system, 0 is an isolated stationary point which is surrounded by infinitely many periodic orbits. Any circle of radius $r = 1/n$ ($n = 1, 2, \dots$) is a periodic orbit. The orbits between the circle of radius $r = 1/n$ and the circle of radius $r = 1/(n+1)$ are spirals which spiral from one circle to the second one. For each of them, the α-limit set is the bigger circle and the ω-limit set is the smaller circle.

The existence of Bendixson centres is impossible for planar systems given by analytic functions. Such situation was discovered by Bendixson in [11]. Poincaré centres were considered by Poincaré.

In his further investigations, Bendixson analysed the stationary points fulfilling the condition (4.1.2) and divided a neighbourhood of such stationary point into several subsets according to the behaviour of orbits in this subsets (called sectors). On the base of this division, he introduced the number which is now known as a Bendixson index. We will not follow here this idea and concentrate on the Poincaré–Bendixson Theorem. Let us only note that Bendixson's work set a new direction which a large number of mathematicians followed.

It must be pointed out that Bendixson considered only such orbits for which the ω-limit set contains finite number of stationary points.

Finally, note that Poincaré used rather analytic methods as Bendixson's reasoning was rather purely geometrical.

It should be also mentioned that the terminology: *α-limit points* and *ω-limit points* are due to George David Birkhoff (1884–1944) who introduced it in [15].

5. The systems with infinite number of stationary points

As was mentioned above, neither Poincaré nor Bendixson investigated limit sets with infinite number of stationary points. However, the Poincaré–Bendixson Theorem can be generalized to such case. This was done in 1945 by J. K. Solntzev in [70].

Solntzev split each compact limit set $\omega(p)$ onto two parts, $\omega(p) = \omega_S(p) \cup \omega_O(p)$. By $\omega_S(p)$ he defined the set of all stationary points contained in $\omega(p)$, by $\omega_O(p)$ the set of all nonstationary points contained in $\omega(p)$. Any component of $\omega_S(p)$ was called *a singular component*. He proved the following theorem:

Theorem 5.1. *Consider an autonomous system $x' = f(x)$ where $x \in \mathbb{R}^2$ and assume that this system defines a flow. Assume that the positive semiorbit $\gamma^+(p)$ through a point $p \in \mathbb{R}^2$ is bounded. Then either*

(a) *the positive limit set $\omega(p)$ is a periodic orbit, or*

(b) *the set of nonstationary orbits contained in $\omega(p)$ is at most countable. Then for any nonstationary point q contained in $\omega(p)$: the set $\alpha(q)$ is contained in some singular component of $\omega_S(p)$ and the set $\omega(q)$ is contained in some singular component of $\omega_S(p)$.*

As one can easily see, this is a more general version of Theorem 3.3.

According to the results of Solntzev ([70]) and Vinograd ([75]) we have also the further precise characterization of limit sets. We present another theorem of Solntzev (formulated in a different way than in the original paper; compare also [9]).

Consider the limit set $\omega(p)$ of p and define an equivalence class in $\omega(p)$. We say that $x \sim y$ if $x = y$ or x, y are stationary point belonging to the same singular component of $\omega_S(p)$. Let $\Omega(p) = \omega(p)/\sim$, $\Omega_S(p) = \omega_S(p)/\sim$, $\Omega_O(p) = \omega_O(p)/\sim$. Of course, we may identify $\Omega_O(p)$ with $\omega_O(p)$. Then we have

Theorem 5.2. *Consider an autonomous system $x' = f(x)$ where $x \in \mathbb{R}^2$ and assume that this system defines a flow. Assume that the positive semiorbit $\gamma^+(p)$ through a point $p \in \mathbb{R}^2$ is bounded. Then there exists a continuous surjective mapping h from a circle to $\Omega(p)$ such that $h\,|_{h^{-1}(\Omega_O(p))}$ is a homeomorphism from a subset of S^1 onto $\Omega_O(p)$.*

Roughly speaking, the theorem says that we can go along the whole limit set like along "cyclic paths" and meet any non-singular point precisely once.

The similar result concerning unbounded limit sets was obtained by Vinograd in [75].

The limit sets may be of different shape. For instance, a limit set may be in the shape of the circle and contain infinitely many regular trajectories (coming from one singular component to another one). On the other hand, it may be in the shape of a finite-leafed rose or infinite-leafed rose, with only one stationary point common for all the leaves.

According to above theorems we have also

Theorem 5.3. *Consider an autonomous system $x' = f(x)$, where $x \in \mathbb{R}^2$, and assume that this system defines a flow. Assume that the positive semiorbit $\gamma^+(p)$ through a point $p \in \mathbb{R}^2$ is bounded and the limit set $\omega(p)$ contains precisely one stationary point and infinitely many regular trajectories. Then the regular trajectories form a sequence of planar subsets with the diameters tending to* 0.

6. The systems on 2-manifolds

Poincaré and Bendixson considered only planar systems. The obtained theorems can be in an obvious way adopted for the systems on the sphere S^2. Then the natural question arises about the similar results for the systems on other 2-dimensional compact manifolds.

The famous example of the system on the torus, in which all the orbits are dense was already known to Poincaré. Poincaré called non-trivial trajectories such that $x \in \omega(x)$ "non-closed Poisson stable". This example shows that the Poincaré–Bendixson Theorem in its classical form cannot be generalized for all 2-dimensional manifolds. However, many questions connected with this problems arose. The systems on 2-manifolds were investigated by many mathematicians.

Poincaré himself posed the question if for a flow on a torus $\mathbb{T}^2$ given by the analytic function f, the only possible minimal sets are points, periodic trajectories and the whole torus $\mathbb{T}^2$. This was proved in 1932 by Arnaud Denjoy (1884–1974) in his celebrated paper [25]. Denjoy showed the theorem in stronger version. He proved several results about the systems on the torus $\mathbb{T}^2$. On the other hand, his results were important and gave a good description of the phenomena occurring in such systems, on the other hand they gave the base fur further investigations. Among others, he considered phenomena based on ergodicity and rotation numbers. We present here the theorem mostly connected with the Poincaré–Bendixson theory.

Theorem 6.1. *Assume that $x' = f(x)$ ($x = (x_1, x_2)$ with a suitable identification) is an autonomous system on the torus $\mathbb{T}^2$ where f is of class C^2. Assume that this system defines a flow. Let M be a minimal set for this system. Then either M is a stationary point or M is homeomorphic to the circle (i.e. is a periodic orbit) or $M = \mathbb{T}^2$.*

As was also shown by Denjoy, the assumption that f is of class C^2 is essential. He gave an example of the system given by the function of class C^1 for which the assertion of the above theorem did not hold.

The work of Denjoy was continued by many others. In [33] the similar theorem for orientable manifolds was stated. However, the proof was not correct, as was pointed out by Peixoto ([58]). The theorem of Denjoy was generalized in 1963 by Arthur J. Schwartz who proved ([64]):

Theorem 6.2. *Assume that $x' = f(x)$ is an autonomous system on a compact, connected 2-dimensional manifold X of class C^2 where f is of class C^2. Assume that this system defines a flow. Let M be a minimal set for this system. Then either M is a stationary point or M is homeomorphic to the circle (i.e. is a periodic orbit) or $M = X$ (i.e. is the whole manifold); in the last case the manifold X must be equal to the two-dimensional torus $\mathbb{T}^2$.*

and

6.3. Corollary. *Assume that $x' = f(x)$ is an autonomous system on a compact, connected 2-dimensional orientable manifold X of class C^2 where f is of class C^2. Assume that this system defines a flow and that the manifold X is not a minimal set. Then, if the ω-limit set $\omega(p)$ of a point p does not contain any fixed point, then $\omega(p)$ must be homeomorphic to the circle S^1.*

For a non-orientable manifold, a similar theorem was obtained in 1969 by N. Markley. Moreover, Markley did not assume the differentiability of the flow. This will be presented in the next chapter devoted to the results without the assumption of differentiability.

7. The Poincaré–Bendixson theorem for flows

In 1927 George David Birkhoff wrote his celebrate monograph [15] which was the crucial step for the development of the qualitative theory of differential equations and dynamical systems. Soon later, in the early thirties, the abstract definition of a dynamical system (a flow) was formulated independently in 1931 by Andrei Andreievich Markov (1903–1979) ([49]) and in 1932 by Hassler Whitney (1907–1989) ([77]). In 1933 Whitney ([78]) introduced the concept now known as parallelizability. Independently, Whitney in 1933 ([78]) and M. Bebutov in 1939 ([10]) defined sections for flows and proved the existence theorem which is now called the Whitney–Bebutov Theorem. In these papers, the authors not only presented different proofs of the theorem, but they even approached these problems from different sides. Their results were of great influence and are still a subject of further investigation. Even recently, for example, the important results of flowbox manifolds were obtained in 1991 by J. M. Aarts and L. G. Oversteegen ([1]).

Come back to the existence theorem. We have

Theorem 7.1. *Let X be a metric space and $(X, \mathbb{R}, \pi)$ be a flow. If p is not a stationary point then there exists a section through p.*

This theorem allows to give a very good local description of a non-stationary point p and helps with a qualitative analysis of the behaviour of orbits. An important phenomenon connected with the Poincaré–Bendixson Theorem, proved with the use of Theorem 3.1, was obtained in 1936 by H. Bohr and W. Fenchel ([16]). They showed the following

Theorem 7.2. *Let $(\mathbb{R}^2, \mathbb{R}, \pi)$ be a flow and $p \in \mathbb{R}^2$ be a regular point. Then $p \notin \omega(p)$.*

This result was several years later (in 1967) proved without the use of sections by P. Seibert and P. Tulley. They obtained the more general theorem:

Theorem 7.3. *Let $X \subset \mathbb{R}^2$ and $(X, \mathbb{R}, \pi)$ be a flow. Assume that $p \in \mathbb{R}^2$ is a regular point. Then $p \notin \omega(p)$.*

In fact, a theorem of the Poincaré–Bendixson type for flows was earlier obtained by H. Kneser. However, then the flows were not formally introduced yet. In 1924 Kneser proved in [43] the theorem which now may be formulated in the following way:

Theorem 7.4. *Denote the Klein bottle by $\mathbb{K}$ and let $(\mathbb{K}, \mathbb{R}, \pi)$ be a flow without stationary point. Then there exists an $x \in \mathbb{K}$ such that the orbit through x is periodic.*

One of the most important results which could help in the generalization of the Poincaré–Bendixson theorem for flows is a local parallelizability of flows, i.e. the existence of sections. This is guaranteed by the Whitney–Bebutov Theorem.

However, the difficulties are moved to another point. In the case of flows given by differential equations, fulfilling suitable continuity assumptions, we find without any difficulty transversal curves through any non-stationary point of the phase space. Now, it must be proved that the system behaves well in a neighbourhood of this transversal, i.e. it can be described by the local parallelizability. In fact, one need to show that a transversal curve is a section which is not quite immediate. When we do not assume differentiability, we have the required properties of sections guaranteed from the definition and the Whitney–Bebutov Theorem, but we do not know anything about the topological shape of sections.

This problem was solved by O. Hajek, who proved in 1965 ([35]) the following

Theorem 7.5. *Let X be a 2-dimensional manifold and let $(X, \mathbb{R}, \pi)$ be a flow. Then every section which is a locally connected continuum is either a Jordan arc or a Jordan curve.*

This helped with the generalization of the Poincaré–Bendixson Theorem for flows in the 2-dimensional case. It was obtained by O. Hajek ([34]). Hajek gave a very precise description of limit sets in planar flows. He considered mainly dichotomic 2–dimensional manifolds, not necessarily compact.

Theorem 7.6. *Let $(X, \mathbb{R}, \pi)$ be a flow on a dichotomic manifold X. Assume that $\omega(p) \neq \emptyset$ for some $p \in X$. Then $\omega(p)$ consists of stationary points and at most countable family $\{T_n : n \in A\}$ $(A \subset \mathbb{N})$ of non-stationary orbits. Moreover, each compact subset of X without stationary points has common points with at most finite number of T_n.*

Theorem 7.7. *Let $(X, \mathbb{R}, \pi)$ be a flow on a dichotomic manifold X and let the closure of $\omega(p)$ be compact for some non-stationary point $p \in X$. Then either p is a periodic point, or $\omega(p)$ is a periodic orbit and a limit cycle, or for every $x \in \omega(p)$ both $\alpha(x)$ and $\omega(x)$ are non-void compact connected sets containing only stationary points.*

Hajek also carried some results of Solntzev ([70]) and Vinograd ([75]) over to flows on dichotomic 2-manifolds.

Generally, the main points in Hajek's proof were similar to that used in the differentiable case. However, several parts had to be done in a different way because of the lack of the assumption of differentiation. Also, several other techniques were used. One of them was considering the inherent topology. For a given regular orbit, we may consider the euclidean topology induced from the plane. On the other hand, we may define a topology on a regular orbit taking as the base the images of open intervals through the solution (i.e. $\gamma((\beta_1, \beta_2), x)$. One of the main point of the proof is to show that these topologies are equal.

There are also other proofs of the Poincaré–Bendixson Theorem for flows (see for example [2], [66]). Also the proof in [20] which will be discussed in the next chapter, may be adopted (as another proof) to the case of flows.

The results showed that the Poincaré–Bendixson Theorem is purely topological and does not depend on differentiability assumptions. As it turned out about twenty years later, it was caused by much more general deep property describing not only limit sets. In 1986 C. Gutierrez published a paper on smoothing continuous flows ([32]). The result that any continuous flow on a 2-dimensional compact manifold X of class C^∞ is topologically equivalent to a C^1 flow on X was the corollary of the main theorem of this paper. Not defining topological equivalence precisely here, note only that it preserves topological properties of orbits, in particular the properties investigated in the Poincaré–Bendixson Theorem.

From the theorem of Gutierrez, we conclude immediately that topological properties of differential planar autonomous systems hold for flows, the Poincaré–Bendixson Theorem among others. Nevertheless, it should be pointed out that Gutierrez used in his proof many really advanced and complicated modern techniques and results. Thus, the other proofs of the Poincaré–Bendixson Theorem for flows, however not easy, were really much elementary.

Another theorem of the Poincaré–Bendixson type was obtained in 1969 by N. G. Markley ([48]).

Theorem 7.8. *Denote the Klein bottle by* $\mathbb{K}$ *and let* $(\mathbb{K}, \mathbb{R}, \pi)$ *be a flow. Assume that* $p \in \omega(p)$ *or* $p \in \alpha(p)$. *Then* p *is either stationary or periodic.*

From this theorem, one may get as a simple corollary the early result of Kneser mentioned above (Theorem 7.4).

Let us come back to the paper of C. Gutierrez ([32]). In this paper the following theorem, which generalized the theorem of Schwartz (Theorem 6.2) was proved.

Theorem 7.9. *Let* $(X, \mathbb{R}, \pi)$ *be a flow on a compact* 2*-dimensional manifold* X *of class* C^∞. *Then the following conditions are equivalent:*

(7.9.1) $(X, \mathbb{R}, \pi)$ *is topologically equivalent to a* C^2 *flow on* X,

(7.9.2) $(X, \mathbb{R}, \pi)$ *is topologically equivalent to a* C^∞ *flow on* X,

(7.9.3) *if* M *is a minimal set in the flow* $(X, \mathbb{R}, \pi)$ *then either* M *is a stationary point or* M *is homeomorphic to the circle* (*i.e. is a periodic orbit*) *or* $M = X$ (*i.e. the whole manifold*); *in the last case the manifold* X *must be equal to the two–dimensional torus* $\mathbb{T}^2$.

8. The Poincaré–Bendixson theorem for semiflows

In flows, we have the movement defined in both directions. However, one may consider only the movement defined in positive direction. This leads to the abstract definition of a semiflow, which was first formulated in 1965 by Hajek ([36]). The theory of semiflows was developed soon later in the book [12] published in 1969.

Semiflows have the movement defined only in positive direction, but a natural question about negative continuations arises. For a given point x, we may introduce negative semiorbits coming to x (it may happen that there are many such orbits, on the other hand it is possible that there is no one) and consider negative limit sets, depending not only on the point but also on a negative semiorbit.

Thus a natural question arises about the Poincaré–Bendixson properties for 2-dimensional semiflows, not only for ω-limit sets but also for α_σ-limit sets, where σ is a negative solution through x.

In 1977 R. C. McCann wrote an important paper about isomorphisms of semiflows ([51]). In particular, his results implied that during the investigation of the topological properties of semiflows on 2-dimensional manifolds one could assume that any negative solution is defined on the interval $(-\infty, 0]$ (in the way we stated in Preliminaries).

In the proof of the Poincaré–Bendixson Theorem for flows, transversals and sections played an important role. The local parallelizability of the flows was fundamental for the local characterization of the neighbourhood of the system. However, for semiflows it is impossible to give such a good description, as here an orbit can "glue" with other orbits.

In 1992, the definition of section for semiflows was stated in [19]. These sections give a good local description of a suitable neighbourhood of a non-stationary point in general semiflows. Also, the existence of sections in the general case was proved. We have

Definition 8.1. A closed set S containing x is called a *section* through x if there are a $\lambda > 0$ and a closed set B such that:

(a) $F(\lambda, B) = S$,
(b) $F([0, 2\lambda], B)$ is a neighbourhood (not necessarily open) of x,
(c) $F(\mu, B) \cap F(\nu, B) = \emptyset$ for $0 \leq \mu < \nu \leq 2\lambda$.

In the case of flows, this definition gives a Whitney–Bebutov section.

Theorem 8.2. *Let a semiflow $(X, \mathbb{R}_+, \pi)$ on a metric space X be given. Then for any non-stationary point x there exists a section through x. Moreover, if X is a manifold, then for any non-stationary point x there exists a compact section through x.*

According to this theorem and McCann's results we can contain in a suitable neighbourhood any non-stationary point x in a planar semiflow. This neighbourhood is a parallelizable "box" in which all the segments of trajectories go perfectly from one side to the opposite one in the time interval 2λ (all segments start in one side on the box and no other trajectory joins these segments).

This local characterization was an important step for the Poincaré–Bendixson Theorem for semiflows, which was proved in 1994 ([20]). We have the following results. In Theorems 8.3–8.6 by a limit set Λ we mean either an ω-limit set $\omega(p)$ or an α_σ-limit set $\alpha_\sigma(p)$ where σ is a negative solution through a non-stationary point p.

Theorem 8.3. *Let a semiflow $(X, \mathbb{R}_+, \pi)$ on a dichotomic 2-manifold X be given. If a limit set Λ is connected and does not contain stationary points, then Λ is a single trajectory.*

Theorem 8.4. *Let a semiflow $(\mathbb{R}^2, \mathbb{R}_+, \pi)$ be given and let a semi-orbit (positive or negative) be bounded. Then either the limit set Λ associated with this orbit is a periodic orbit or any semi-orbit, contained in Λ, may contain in its limit set only stationary points.*

Theorem 8.5. *Let a semiflow $(X, \mathbb{R}_+, \pi)$ on a dichotomic 2-manifold X be given. If $p \in \omega(p)$ (or $p \in \omega_\sigma(p)$) then p is either stationary or periodic.*

Theorem 8.6. *Let a semiflow $(X, \mathbb{R}_+, \pi)$ on a dichotomic 2-manifold X be given. If a compact set A is either positively minimal or weakly minimal, then it is either a stationary point or a periodic trajectory.*

In the proofs, except of the existence of sections, the following properties played an important role:

(8.6.1) Any compact section in a semiflow on a 2-manifold X is either a Jordan arc or a Jordan curve.

(8.6.2) For any non-stationary point y contained in a limit set Λ in a semiflow on a 2-manifold and for any $t > 0$ the set $\Lambda \cap F(t, y)$ has precisely one element.

(8.6.3) If a limit set Λ in a semiflow on a 2-manifold X does not contain any stationary point, then the semiflow induced from X on Λ is a system with negative unicity and (after an obvious introducing the values of $\pi(t, x)$ for negative t) gives a flow on Λ.

Also, the continuity properties in semiflows proved in [18] played an important role in the proof. Moreover, the properties analogous to that mentioned in Steps 2–5 in Chapter 3 had to be shown and the inherent topology (see Chapter 7) was also used. Generally, the proofs of these properties were not a simple analogy to the case of differential systems (or even to the case of flows) as semiflows admit complicated situations which are impossible for flows. Moreover, all the earlier proofs of the Poincaré–Bendixson Theorem depended on the uniqueness of the negative semi–solutions and the continuity of movement in both directions.

Note that the Gutierrez theorem about the topological equivalence says only about 2-dimensional flows, not semiflows. Because of the complicated structure of semiflows and the phenomena connected with singular points in the finite dimensional case one would not expect that the analogous theorem for semiflows would hold.

For the end of this section it should be pointed out that the Poincaré–Bendixson Theorem for semiflows shows that this theorem is not only purely topological, but in fact it depends only on the continuous movement defined for positive values of the time variable t. Roughly speaking, "the reason" of this theorem is a possibility of a continuous movement forward and we do not need bother about backward direction.

9. Some other generalizations

As was mentioned in Chapter 4, in his paper [11] Bendixson proved also several other results. The Poincaré–Bendixson Theorem and those results gave the beginning of other investigations and generalizations; all of them could now form a large collection which could be regarded as the developed Poincaré–Bendixson theory. All these results together would be a good subject for a mathematical monograph. Here, we only mention some of this generalizations and directions in which the theory developed. It should also be noted that many applications and continuations of the work on this subject are contained in papers not cited here.

One of the advantages of the Poincaré–Bendixson Theorem was a precise description of planar systems in a neighbourhood of a periodic trajectory or a stationary point (see Theorems 4.1 and 4.2). This suggests a possible generalization and a question about the behaviour of the system in the neighbourhood of a compact invariant set, not necessarily for 2-dimensional systems. This was obtained (in the general case of flows) in 1960 by T. Ura and I. Kimura in ([73]) and later developed by T. Saito in 1968 ([62]). The detailed description of the Ura–Kimura Theorem can be found in [14]. This theorem turned out to be of great importance for the theory of persistence, which rapidly developed in the late eighties and nineties of the twentieth century.

In the development of modern theory of dynamical systems and invariant sets, chain recurrence introduced in 1972 by C. Conley ([23]) (see also [22], [29], [30]) played an important role. In particular, chain recurrence is of importance for the Conley index theory. Also, there are remarkable connections of chain recurrence with the persistence theory (see for instance [31]). In 1996 K. Athanassopoulos ([5], compare also [6]) proved that the assertion of the Poincaré–Bendixson Theorem for flows on S^2 holds for really larger class than compact limit sets. He proved an analogous theorem for any 1-dimensional invariant chain recurrent continuum. Also, some other results connecting chain recurrence, sections and the Poincaré–Bendixson type properties, in a very interesting way were proved by M. W. Hirsh and C. C. Pugh in 1988 (see [39]). Other properties, also grown from recurrence and lead to some connections to the Poincaré–Bendixson Theorem were shown in 1970 by N. G. Markley ([47]).

Another generalization of the Poincaré–Bendixson Theorem was given in 1988 by K. Athanassopoulos and P. Strantzalos ([8]). This was a generalization following the results of Schwartz and the assertion of the Ura–Kimura Theorem simultaneously. They proved that for a flow on a 2-dimensional manifold a compact minimal stable set is as in the assertion of the theorem of Schwartz. Also, they proved the assertion of the Poincaré–Bendixson Theorem for any compact minimal saddle set in such flows. Another theorem giving the condition for the existence of periodic orbits in flows on closed orientable 2-dimensional manifolds was presented by D. Neumann in 1978 ([53]). The results of Athanassopoulos and Strantzalos lead to another interesting characterization of the Poincaré–Bendixson type gave by K. Athanassopoulos in 1996 ([4]). It should be also mentioned that the property of the Poincaré–Bendixson type was proved for the Poisson stable points in a special kind of flows (so called D-stable flows) on an orientable 2-manifold of finite genus. This was obtained by K. Athanassopoulos, T. Petrescou and P. Strantzalos in 1997 ([7]). We should also note here about the study of 2-dimensional flows on 2-manifolds presented by D. Neumann and T. O'Brien in 1976 ([54]).

In 1956, L. Markus ([50]) presented a very interesting theorem of the Poincaré–Bendixson type for some kind of autonomous differential equations in the plane. Later on, in 1960 another version of this theorem was obtained by Z. Opial ([56]). These theorems had many interesting applications (compare [72]). In 1992, H. R. Thieme ([72]) extended the Markus theorem for more general case. Another analogue of the differentiable version of the Poincaré–Bendixson Theorem was proved by V. V.Filippov ([28]) in 1993. Using these results, B. Klebanov in 1997 ([42]) gave a precise description of orbits for some kind of planar equations and extended the results of Markus in a similar way to Thieme's theorems (the results of Klebanov and Thieme did not cover each other).

As was noted in Chapter 4, Bendixson introduced the index of a stationary point of a planar differential system. This was also a subject of further development. In particular, recently some interesting results (connected also with the Conley index theory) was obtained in 1996 by M. Izydorek, S. Rybicki and Z. Szafraniec in ([40]). The reader is referred to [40] for details and more information about this aspects of the Poincaré–Bendixson theory.

The classical Poincaré–Bendixson Theorem is strictly 2-dimensional. However, there are some kind of generalizations of this theorem for higher dimensions. In 1979 H. M. Hastings ([38]) proved a theorem for semiflows defined on 2-dimensional submanifolds of $\mathbb{R}^n$. Assuming that the semiorbit of a given point p is contained in a compact set A, he obtained some conditions of the style of K. Borsuk's shape theory for A. In the paper, there also pointed out some differences between 2-dimensional cases and higher dimensional cases from the point of view of this theorem. Some ideas of the Poincaré–Bendixson Theorem were also transformed for some classes of n-dimensional equations by R. A. Smith in 1980 ([67]). Continuing the research on the stability aspects of the Poincaré–Bendixson Theorem, in 1987 R. Smith ([68]) proved several results on orbital stability, extending the Poincaré–Bendixson Theorem in some way.

The stability of periodic orbits and its connections with Poincaré–Bendixson Theorem was also considered from another point of view, generally for strictly two-dimensional case. Some stability problems were investigated by Athanassopoulos in some of his papers mentioned above. Also, in 1979 D. Erle ([26]) proved some properties, particularly interesting from the point of view of mathematical models and applications.

Also, some infinite dimensional generalizations of the Poincaré–Bendixson Theorem are known. The investigations in this direction began in 1975 with the work of J. L. Kaplan and J. A. Yorke ([41]) where the authors considered differential delay equations. This was followed by many others in the papers concerning scalar equations and slowly oscillating solutions. For some type of ordinary differential delay equations the Poincaré–Bendixson type theorems were proved by J. Mallet–Parret and H. L. Smith in 1990 ([46]). For some type of scalar partial differential equations the result of this kind were obtained in 1989 by B. Fiedler and J. Mallet–Parret ([27]). These results were followed in 1996 by the paper on more general equations written by J. Mallet–Parret and G. Sell ([45]). Some results for another type of infinite dimensional systems were obtained by R. A. Smith. In 1992, he proved that the assertion of the Poincaré–Bendixson Theorem holds (under several additional assumptions) for bounded positive semiorbits in some autonomous retarded differential equations. This can be found in [69].

As was noted in Chapter 3, the Poincaré–Bendixson Theorem has a lot of important applications, mainly for the problems connected with solving particular differential equations. In the end, let us mention that in 1966 N. P. Bhatia, A. C. Lazer and W. Leighton showed ([13]) that among other applications, it is possible to prove the Brouwer Fixed Point Theorem (in the 2-dimensional case) as the corollary from the Poincaré–Bendixson Theorem.

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ON MULTIVALUED MAPPINGS WITH SYMMETRIES

Zdzisław Dzedzej

Dedicated to Prof. Lech Górniewicz on his 60th birthday

Abstract. We define a homotopy invariant for a class of G-equivariant multivalued maps considered by Bader and Kryszewski. It detects the existence of invariant orbits, not necesarrily consisting of fixed points of a mapping.

1. Notations

Let G denote a compact Lie group. We shall use some standard notations of the compact transformation group theory (comp. [2]).

If $H \subset G$ is a closed subgroup of G, then (H) denotes the conjugacy class of H in G. By $\Psi(G)$ we denote the set of all conjugacy classes in G. There is a natural partial order in $\Psi(G)$: $(K) \leq (H)$ if and only if H is conjugate to a subgroup of K.

Let X be a topological space. An *action* of G on X is a continuous map $\rho: G \times X \to X$ such that

$$\begin{aligned} \rho(g, \rho(h, x)) &= \rho(gh, x) && \text{for } g, h \in G,\ x \in X, \\ \rho(e, x) &= x && \text{for } x \in X,\ e \in G \text{ unit.} \end{aligned}$$

A *G-space* X is a space with a given action of G. The element $\rho(g, x)$ is usually denoted by gx.

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Suppose X and Y are G-spaces. A continous map $f: X \to Y$ is G-*equivariant* (*a G-map*) if for all $x \in X$ and $g \in G$ the relation $f(gx) = gf(x)$ holds. A notion of a G-*homotopy* is analogous. Similarily to the nonequivariant case we obtain:

Definition 1. A G-space X is called a G-ANR (G-*absolute neighbourhood retract*), if X is metrizable and for every G-pair (Y, B) where B is closed in Y, and every G-map $f: B \to X$ there exists a G-invariant neighbourhood U of B in Y and a G-equivariant extension $\overline{f}: U \to X$ of f.

For a systematic presentation of the theory of G-ANR's we refer to [9].

Suppose that X is a G-space. For each $x \in X$ the set $Gx = \{gx \in X : g \in G\}$ is called *an orbit through* x. The set $G_x = \{g \in G : gx = x\}$ is a closed subgroup of G called *the isotropy group* of x.

We shall use the following subspaces of X:

$$\begin{aligned} X^H &:= \{x \in X : H \subset G_x\}, \\ X^{(H)} &:= \{gx \in X : g \in G \text{ and } x \in X^H\} = GX^H, \\ X^{[H]} &:= \bigcup \{X^K : H \subset K \text{ and } H \neq K\}, \\ X^{\{H\}} &:= GX^{[H]}, \quad X_{(H)} := X^{(H)} \setminus X^{\{H\}}. \end{aligned}$$

The following is true (see [9]).

Proposition 1. *Let X be a G-*ANR. *Then for every closed subgroup $H \subset G$ the sets X^H, $X^{(H)}$, $X^{\{H\}}$, $X_{(H)}$ are* ANR*'s. Moreover, $X^{(H)}$, $X^{\{H\}}$, $X_{(H)}$ are G-*ANR*'s, and the quotient spaces $X^{(H)}/G$, $X^{\{H\}}/G$ are* ANR*'s.*

For a given G-map $f: X \to Y$ we denote by f^H, $f^{(H)}$, $f^{[H]}$, $f^{\{H\}}$ its' restrictions to the corresponding subsets. By $f^*: X/G \to Y/G$ we denote the induced map on quotient spaces, which will be denoted by $X^* = X/G$, and analogously $X^{(H)*} = X^{(H)}/G$, $X^{\{H\}*} = X^{\{H\}}/G$.

2. Multivalued maps

Let X, Y be two spaces. We say that $\varphi: X \to Y$ is *a multivalued map* if for every point $x \in X$ a nonempty subset $\varphi(x)$ of Y is given.

We associate with φ *the graph* to be the set

$$\Gamma_\varphi := \{(x, y) \in X \times Y : y \in \varphi(x)\}.$$

The *image* of a subset $A \subset X$ is the set $\varphi(A) := \bigcup_{x \in A} \varphi(x)$.

For a subset $B \subset Y$ we can define two types of *a counterimage*:

$$\begin{aligned} \varphi^{-1}(B) &:= \{x \in X : \varphi(x) \subset B\}, \\ \varphi_+^{-1}(B) &:= \{x \in X : \varphi(x) \cap B \neq \emptyset\}. \end{aligned}$$

They both coincide if φ is a singlevalued map.

One defines a composition of $\varphi: X \to Y$ and $\psi: Y \to Z$ as a map $\gamma: X \to Z$ given by $\gamma(x) = \psi(\varphi(x))$.

A multivalued map $\varphi: X \to Y$ is *upper semicontinuous* (*u.s.c.*) provided

(i) for each $x \in X$ $\varphi(x) \subset Y$ is compact,
(ii) for every open subset $V \subset Y$ the set $\varphi^{-1}(V)$ is open in X.

Let us recall some basic properties of u.s.c. maps:

(1) The image of a compact set is a compact set.
(2) The graph Γ_φ is a closed subset of $X \times Y$.
(3) The composition of two u.s.c. maps is an u.s.c. map, too.

We would like to remind a class of admissible multivalued maps considered by Górniewicz ([6]).

We say that a space X is *acyclic* if $H^*(X) = H^*(point)$.

Definition 2. An u.s.c. map $\varphi: X \to Y$ is *acyclic* if all the values $\varphi(x)$ are acyclic sets.

A continuous map $p: X \to Y$ is *a Vietoris map* if:

(i) $p(X) = Y$,
(ii) p is proper (i.e. $p^{-1}(A)$ is compact whenever $A \subset Y$ is compact),
(iii) for every $y \in Y$ the set $p^{-1}(y)$ is acyclic.

An important feature of Vietoris maps is the famous Vietoris–Begle Mapping Theorem (see [10]) which says that if X, Y are paracompact spaces and $p: X \to Y$ is a Vietoris map, then it induces an isomorphism on cohomology.

Definition 3. An u.s.c. map $\varphi: X \to Y$ is *admissible* provided there exists a space Γ, and two continuous maps $p: \Gamma \to X$, $q: \Gamma \to Y$ such that

(i) p is a Vietoris map,
(ii) for every $x \in X$ $q(p^{-1}(x)) \subset \varphi(x)$.

We call every such a pair (p, q) of maps *a selected pair* for φ.

The class of admissible maps is very broad. It includes all u.s.c maps with acyclic values (see [6]), and in particular with convex values, if Y is a normed space. Moreover, a composition of two admissible maps is also admissible ([6]). Many results from topological fixed point theory of singlevalued maps carry onto this class of maps.

There are several classes of multivalued mappings, for which a fixed point index theory has been constructed. See [6] for various approaches. As an example we follow the one presented in [1].

Definition 4. (i) A compact subset K of a space X is *proximally ∞-connected* if , for each ε there is $0 < \delta \leq \varepsilon$ such that the inclusion $O_\delta(K) \to O_\varepsilon(K)$ induces the trivial homomorphism $\pi_n(O_\delta(K)) \to \pi_n(O_\varepsilon(K))$ for any $n \geq 0$.

(ii) An u.s.c. mapping $\varphi: X \to Y$ belongs to $J(X,Y)$ if, for any $x \in X$, $\varphi(x)$ is proximally ∞-connected.

In particular, these maps are acyclic in the sense of the Alexander–Spanier cohomology with integer coefficients. We denote by $J_c(X,Y)$ the class of compositions of maps from J. The main advantage of the considered maps is the existence of arbitrarily close graph approximations, i.e. single-valued maps $f: X \to Y$ such that $\Gamma_f \subset O_\varepsilon(\Gamma_\varphi)$. Namely, the following was proved in [7].

Theorem 1. *Let X, Y be compact* ANR*'s and $\varphi \in J(X,Y)$. Then, for any $\varepsilon > 0$, there is an ε-approximation of φ. Moreover, there is $0 < \delta < \varepsilon$ such that any two δ-approximations of φ are homotopic via ε-approximations.*

Let W be an open subset of a compact ANR X and let $\varphi: \operatorname{cl} W \to X$ be a mapping such that $x \notin \varphi(x)$ for $x \in \partial W$. Moreover let φ be given by a composition

$$D_\varphi: \operatorname{cl} W \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} X_n = X,$$

where $\varphi_i \in J(X_{i-1}, X_i)$ and $X_i \in$ ANR.

Definition 5. We define an *index* of the decomposition D_φ

$$\operatorname{Ind}(X, D_\varphi, W) := \operatorname{ind}(X, f_n \circ \ldots \circ f_1, W) \in Z$$

where ind stands for the ordinary fixed point index and f_i are sufficiently fine approximations of φ_i.

By using standard retraction arguments this index can be defined for compact mappings on open subsets of arbitrary metric ANR's. For detailed proofs see e.g. [6], [8].

Let X, Y be two G-spaces.

Definition 6. An u.s.c. map $\varphi: X \to Y$ is *G-equivariant* provided

(i) $\varphi(gx) = g\varphi(x)$ for all $x \in X$ and $g \in G$,
(ii) if $y, gy \in \varphi(x)$ then $y = gy$.

Note that in the case of a singlevalued map φ the condition (ii) is automatically fullfilled. However without (ii) the following natural fact would not be true.

Proposition 2. *If $\varphi: X \to Y$ is G-equivariant, then for each subgroup $H \subset G$*

(i) $\varphi(X^H) \subset Y^H$,
(ii) $\varphi(X^{(H)}) \subset Y^{(H)}$.

Proof. Let $x \in X^H$, $y \in \varphi(x)$ and $g \in H$. Then $y \in \varphi(x) = \varphi(g^{-1}x) = g^{-1}\varphi(x)$. Thus we have $y \in \varphi(x)$ and $gy \in \varphi(x)$, and thus $y = gy$. Since $g \in H$ was arbitrary, $y \in Y^H$. Assertion (ii) is a corollary of (i). □

3. Fixed orbit index

In [8] a definition of a fixed point index for compact G-mappings on metric G-ANR' has been given. In a very similar way in [5], [3] a fixed orbit index has been defined for singlevalued compact G-mappings. Using results of [1] we now extend this invariant to the case of multivalued G-mappings (decompositions in fact).

Let us denote by $U(G)$ a free abelian group generated by $\Psi(G)$. It can be equipped also with a multiplicative structure (see [2]), thus we call it the *tom Dieck ring*.

Let $\operatorname{cl} W = X_0, X_1, \dots, X_n = X$ be compact G-ANR's, $W \subset X$ an open G-subset of X.

Suppose that $\varphi_i \in J(X_{i-1}, X_i)$ are G-equivariant and φ is their composition. Let $gx \notin \varphi(x)$ for all $x \in \partial W$, $g \in G$. Because of Proposition 2 for each closed subgroup $H \subset G$ restrictions of our mappings $\varphi_i\colon X_{i-1}^{(H)} \to X_i^{(H)}$ are well defined and so the induced maps φ_i^* on quotient spaces are well defined. We use here the same notation as for singlevalued mappings in the end of Section 1. Since the images of points $\varphi_i^*(x)$, $\varphi_i(x)$ are homeomorphic to each other, we have that $\varphi_i^* \in J(X_{i-1}^{(H)*}, X_i^{(H)*})$. Thus their composition $\varphi^{(H)*}$ belongs to $J_c(\operatorname{cl} W^{(H)*}, X^{(H)*})$ and it has no fixed points on the boundary $\partial W^{(H)*}$. The same is true for $\varphi^{\{H\}}$.

Therefore the following integer numbers are well defined

$$i_{(H)}(D_\varphi) = \operatorname{Ind}(X^{(H)*}, D_{\varphi^*}, W^{(H)*}) - \operatorname{Ind}(X^{\{H\}*}, D_{\varphi^*}, W^{\{H\}*})$$

where Ind denotes the fixed point index from Definition 5.

Since G is compact and $\operatorname{cl} W$ is a compact G-space, there are only finitely many orbit types involved, and thus only finite number of $i_{(H)}(\varphi)$ are different from zero.

Definition 7. The *fixed orbit index* of the decomposition D_φ is defined by the formula:

$$I_G(W, X, D_\varphi) := \sum_{(H) \in \Psi(G)} i_{(H)}(D_\varphi) \cdot (H).$$

The basic properties of this index are easy consequences of the corresponding properties of the fixed point index proved in [1].

Proposition 3. *If $W_0 \subset W$ is an open G-subset $gx \notin \varphi(x)$ for all $x \in \operatorname{cl} W \setminus W_0$ and $g \in G$, then*

$$I_G(W_0, X, D_\varphi) = I_G(W, X, D_\varphi).$$

Proposition 4. *Suppose that $W = W_1 \cup W_2, W_1 \cap W_2 = \emptyset$ and $gx \notin \varphi(x)$ for all $x \in \partial W_1 \cup \partial W_2$. Then*

$$I_G(W, X, D_\varphi) = I_G(W_1, X, D_\varphi) + I_G(W_2, X, D_\varphi).$$

Proposition 5. *Suppose that $F\colon \operatorname{cl} W \times [0,1] \to X$ is a G-homotopy such that $Gx \cap F(x \times [0,1]) = \emptyset$ for $x \in \partial W$. Then for each decomposition D_F we have*

$$I_G(W, X, D_{F(\cdot,0)}) = I_G(W, X, D_{F(\cdot,1)}).$$

Denote by $\pi^H\colon U(G) \to Z$ the projection map onto the (H)-coordinate, and $\chi^H\colon U(G) \to Z$; $\chi^H(\alpha) = \sum_{(K)\le(H)} \pi^K(\alpha)$.

Proposition 6. *If $Gx \cap \varphi(x) = \emptyset$ for all $x \in \operatorname{cl} W^{(H)}$, then*

$$\chi^K(I_G(W, X, D_\varphi)) = 0 \quad \text{for all } (K) \le (H).$$

Proof. By definition we have

$$\pi^K(I, W, X, D_\varphi)) = \operatorname{Ind}(X^{(K)*}, D_{\varphi^*}, W^{(K)*}) - \operatorname{Ind}(X^{\{K\}}, D_{\varphi^*}, W^{\{K\}})$$

and by our assumption $\varphi^{(K)*}$ has no fixed points in $\operatorname{cl} W^{(K)} \subset \operatorname{cl} W(H)$. The same is true for all subgroups $L \in G$ such that $(L) < (K)$. Thus all the indices in the expression of $\chi^K(I_G(W, X, D_\varphi)))$ are zero. □

Remark. One easily checks that by retraction arguments this index can be defined on noncompact G-ANR's for compact mappings (see [5]).

Analogously to the singlevalued case we can define a Lefschetz type number and then we have also a normalization property of the index. We omit the details here.

One can also define an equivariant fixed point index as in [8] by using the summation formula.

Notice that at least for the purpose of the definition we do not need to have equivariant versions of approximation lemmas as in [7]. However, a careful repeating of their arguments is possible at least for finite groups. One can assume that the group acts by isometries, and then the proof is virtually the same. Having such a singlevalued approximation f the fixed orbit index is obviously equal to the fixed orbit index of f defined in [5], [3].

One can also consider homological approach for admissible mappings (see [6]), or chain approximations. It has been considered for finite groups in [4].

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ON INVEXITY AND UNIQUENESS IN THE CALCULUS OF VARIATIONS

Marek Galewski

Abstract. We provide an existence and uniqueness theorem for the Dirichlet problem

$$-\frac{d}{dt}L_{\dot{x}}(t,\dot{x}(t))+F_x(t,x(t))=0, \quad x(0)=x(T)=0.$$

We assume that F is invex and apply direct varaitional method. Applications of the above theorem are shown.

1. Introduction

We shall consider the existence of solutions for the following equation

$$-\frac{d}{dt}L_{\dot{x}}(t,\dot{x}(t))+F_x(t,x(t))=0, \quad x(0)=x(T)=0. \tag{1}$$

Existence is obtained with the aid of a direct variational method and presents a generalization of results given in [8] to the case of possibly nonlinear differential operator and to the class of functions taking values in abstract spaces. The new idea of the paper bases on imposing invexity ([2]), on F with respect to the second variable instead of convexity. This still implies uniqueness and since the class of differentiable invex functions is much wider than the class of convex functions applies to some other nonlinear problems.

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Let H be a separable, real Hilbert space. We shall assume that

(H1) $L\colon [0,T]\times H \to R$, L is Gateaux differentaible with repsect to the second variable, L, L_x satisfy Caratheodory condition, there exists constants $c_1, c_2 > 0$, functions $d_1 \in L^1[0,T;R]$, $d_2 \in L^2[0,T;R]$ such that for all $v \in H$ and for almost all $t \in [0,T]$ the following condition holds

$$d_2(t) + c_2\|v\|^2 \le L(t,v) \le c_1\|v\|^2 + d_2(t),$$

(H2) $F\colon [0,T]\times H \to R$, F is Gateaux differentaible with repsect to the second variable, F, F_x satisfies Caratheodory condition, there exist functions $b \in L^2[0,T,R_+]$, $c \in L^1[0,T,R_+]$ and a constant $a < (4\pi^2/T^2)c_2$ such that for all $x \in H$ and for almost all $t \in [0,T]$ the following condition holds

$$-a\|x\|^2 - b(t)\|x\| - c(t) \le F(t,x),$$

for every $r > 0$ there exists a function $g_r \in L^1[0,T;R]$ such that for all $x \in H$, $\|x\| < r$, and for almost all $t \in [0,T]$ the following conditions hold

$$F(t,x) \le g_r(t),$$
$$\|F_x(t,x)\|_H \le g_r(t).$$

(H3) either L is convex with respect to the second variable for a.e. t and F is strictly invex, [11], with respect to the second variable for a.e. t or L is strictly convex in second variable for a.e. t and F is invex, [2], with respect to the second variable for a.e. t.

Solutions of (1) are sought on the space H_0^1 which comprises such functions $x\colon [0,T] \to H$ that x is absolutely continuous and $\dot{x} \in L^2[0,T,H]$, $x(0) = x(T) = 0$.

By Poincare inequality ([10]),

$$\|x\|_{L^2[0,T,H]} \le \frac{T^2}{4\pi^2}\|\dot{x}\|_{L^2[0,T,H]},$$

the norm in H_0^1, is equivalent to

$$\|x\|_{H_0^1} = \sqrt{\int_0^T \|\dot{x}(t)\|^2\,dt}.$$

Let us recall that invexity of a functional $f\colon H \to R$ means that there exists an operator $\eta\colon H \times H \to H$ such that for all $x, y \in H$ the following inequality holds

$$f(x) - f(y) \ge \langle \eta(x,y), \nabla f(y)\rangle,$$

where $\langle \eta(x,y), \nabla f(y)\rangle$ denotes the scalar product in H. In many applications it is not the form of the functional η that is required but its existence which may be obtained, pointwise, by use of separation theorems.

It is well known that for an invex functional all stationary points are global minima. The functional is called strictly invex if the inequality in the above is a strict one. It is obvious that for a strictly invex functional a stationary point is a unique minimizer.

Application of invexity instead of convexity in (1) appears to apply to much a wider class of nonlinear problems, compare Section 3.

2. Existence and uniqueness

The proof of existence is based on Proposition 1.3 in [8]. We have to prove in subsequent lemmas that the action functional $J\colon H_0^1 \to R$ given by the formula

$$J(x) = \int_0^T (L(t, \dot{x}(t)) + F(t, x(t)))\, dt \tag{2}$$

is coercive on H_0^1 and lower semicontinuous.

It follows, as in [9] (compare [3]), that under assumptions (H1), (H2), J is Gateaux differentiable and (1) constitutes its Euler–Lagrange equation.

Lemma 1. *An action functional J is coercive on H_0^1.*

Proof. Indeed by assumptions (H1) and (H2), Poincare inequality, Hölder inequality we obtain

$$\begin{aligned} J(x) &\geq c_2 \int_0^T \|\dot{x}(t)\|^2\, dt - a\int_0^T \|x\|^2\, dt - \int_0^T b(t)\|x\|\, dt \\ &\quad - \int_0^T c(t)\, dt - \int_0^T d_2(t)\, dt \\ &\geq \left(c_2 - \frac{T^2}{4\pi^2}a\right)\int_0^T \|\dot{x}(t)\|^2\, dt - d - \sqrt{\int_0^T \|\dot{x}(t)\|^2\, dt}\sqrt{\int_0^T b^2(t)\, dt} \\ &\geq \left(c_2 - \frac{T^2}{4\pi^2}a\right)\|x\|_{H_0^1}^2 - b\|x\|_{H_0^1} - d, \end{aligned}$$

where $d = \int_0^T c(t)\, dt + \int_0^T d_2(t)\, dt$, $b = \sqrt{\int_0^T b^2(t)\, dt}$. Letting $\|x\|_{H_0^1} \to \infty$ we obtain the assertion of the lemma. □

Lemma 2. *An action functional J is weakly lower semicontinuous on H_0^1.*

Proof. We shall prove that functionals

$$J_1(x) = \int_0^T L(t, \dot{x}(t))\, dt, \quad J_2(x) = \int_0^T F(t, x(t))\, dt$$

are weakly lower semicontinuous on H_0^1. For J_1 weak lower continuity is a consequence of convexity.In order to prove that J_2 is weakly lower semicontinuous we will observe that if a we observe that sequence $x_n \rightharpoonup \overline{x}$ (weakly in H_0^1) it follows that $x_n \to \overline{x}$ in $L^2[0,T;H]$. Hence there exists a function $g \in L^1[0;T;R]$

and a subsequnece of x_n, which we denote still by x_n, converging to $\overline{x}$ a.e. on $[0,T]$ and such that for a.e. $t \in [0,T]$

$$\|x_n(t)\|_H \leq g(t).$$

Since x_n is absolutely continuous and $\dot{x}_n$ is weakly convergent it follows for a.e. $t \in [0,T]$ that

$$\|x_n(t)\| \leq \int_0^t \|\dot{x}_n(t)\|\,dt \leq \int_0^T \|\dot{x}_n(t)\|\,dt \leq \sqrt{T}\sqrt{\int_0^T \|\dot{x}_n(t)\|^2\,dt} \leq r$$

for a certain constant $r > 0$. Combinig the above with growth conditions imposed on F and Fatou lemma we obtain the weak lower semicontinuity of J_2. □

Lemma 3. *Let A, B denote Hilbert spaces. If a functional $J_1\colon A \to R$ is striclty convex* (*resp. convex*) *and $J_2\colon B \to R$ is invex* (*resp. striclty invex*) *with respect to a certain operator η than the functional $J_1 + J_2\colon A \times B \to R$ is striclty invex.*

Proof. The assertion of the lemma in the first case is a consequence of the following inequalities for all $x_a, y_a \in A$ and all $x_b, y_b \in B$

$$J_1(x_a) - J_1(y_a) > \langle \nabla J_1(y_a), x_a - y_a \rangle,$$
$$J_2(x_b) - J_2(y_b) \geq \langle \nabla J_2(y_b), \eta(x_b, y_b) \rangle.$$

It follows that

$$J_1(x_a) + J_2(x_b) - J_1(y_a) - J_2(y_b) > \langle (\nabla J_1(y_a), \nabla J_2(y_b)), (x_a - y_b, \eta(x_b, y_b)) \rangle$$

for all $(x_a, x_b), (y_a, y_b) \in A \times B$. The remaining case follows in the same manner. □

We may now state and prove the main results of the paper.

Theorem 1 (Existence)**.** *Assume* (H1)–(H2) *to hold. Than there exists a solution to the Dirichlet problem* (1).

Proof. Since the action functional J is coercive, lower semicontinuous on H_0^1 it follows by Proposition 1.2 in [8] that there exists a solution to the problem

$$\inf_{x \in H_0^1} J(x)$$

and is obtained in such a point $\overline{x}$ for which $J'(\overline{x}) = 0$. Since, by convexity of L and the growth conditions (H1), it follows that $L_{\dot{x}}$ also satisfies grotwth condtions, the point $\overline{x}$ satisfies (1). □

Theorem 2 (Uniqueness)**.** *Assume* (H1)–(H5)*. Than there exist the unique solution to the Dirichlet problem* (1).

Proof. The existence follows by the Theorem 1. Uniqueness is a consequence of Lemma 3. □

3. Example

Example. Consider now the following Dirichlet problem

$$\ddot{x} = 6x^5 + 8x^3 + 9x^2 + 2x + 3.$$

Here $L(t,v) = (1/2)|v|^2$ and $F(t,x) = (x^3 + x)^2 + 3(x^3 + x)$. F is again not convex with respect to x. It is actually invex and its invexity can be shown as in [6]. By Theorem 2 it follows that the above problem possess unique solution. Again it can be shown that Corollary 1.3 [8] does not apply.

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MAPS WITH BOUNDED SEQUENCE OF INDICES OF ITERATIONS AND FINITELY MANY PERIODIC POINTS

Grzegorz Graff

Abstract. A class of self-maps f of a compact ANR X with a finite set of periodic points and bounded sequence of local indices of iterations is considered. Under this assumptions we study relations between global topological structure of X expressed in terms of the Euler–Poincaré characteristic of f and its local properties determined by the behaviour of f at periodic points.

1. k-adic expansion and the Euler characteristic of a map

Let f be a self-map of a compact ANR X and $I(f,x)$ be a fixed point index of f at $x \in \text{Fix}\,(f)$. Then there are relations among $I(f^m,x)$ for different m provided the all indices are well-defined. Let us define for any $m \in \mathbb{N}$ the numbers:

$$i_m(f,x) = \sum_{s|m} \mu(s) I(f^{m/s},x),$$

where μ denotes the Möbius function. If $i_m(f) \neq 0$ then the following congruences (called Dold's relations) hold (cf. [6]):

Theorem 1.1. *For every $m \in \mathbb{N}$ $i_m(f) \equiv 0 \pmod{m}$.*

The following fact is an important consequence of Dold relations (cf. [1]):

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Theorem 1.2. *Any bounded sequence of indices of iterations is periodic.*

The useful language for expressing periodicity of such sequences is provided by so-called k-adic expansion.

Definition 1.3. Define a sequence $\mathrm{reg}_k(q)$:

$$\mathrm{reg}_k(q) = \begin{cases} k & \text{if } k|q, \\ 0 & \text{if } k \not| q. \end{cases}$$

We can represent $\{I(f^q, x)\}_{q=1}^{\infty}$ in the form of k-adic expansion (cf. [12]):

$$I(f^q, x) = \sum_{k \in O(x)} a_k(x)\mathrm{reg}_k(q), \tag{1.1}$$

where $O(x) \subset \mathbb{N}$, $a_k(x) = i_k(f, x)/k$.

Define the set $P_n(f)$ by: $P_n(f) = \mathrm{Fix}\,(f^n) \setminus \bigcup_{0<k<n} \mathrm{Fix}\,(f^k)$.

The class of maps under consideration in this paper consists of continuous maps $f: X \to X$ of a compact ANR which satisfy the assumptions:

(i) the set $P(f)$ of periodic points is finite,
(ii) for every $x \in P_n(f)$ the set of local fixed point indices $\{I((f^n)^k, x)\}_{k=1}^{\infty}$ is bounded.

Remark 1.4. In general a set $O(x)$ of the expansion (1.1) may be an arbitrary subset of natural numbers, but for maps satisfying the condition (ii) it is finite (see Theorem 1.2). Among such maps there are C^1 self-maps of $\mathbb{R}^n$ (cf. [5], [13]), simplicial maps of smooth type (cf. [9]) and planar homeomorphisms (cf. [3]).

Let $x \in P_n(f)$ and $I((f^n)^q, x) = \sum_{k \in O(x)} a_k(x)\mathrm{reg}_k(q)$. Then we may rewrite it as k-adic expansion of the form: $I(f^q, x) = \sum_{k \in O'(x)} a'_k(x)\mathrm{reg}_k(q)$, where $O'(x) = n\ O(x) = \{nk : k \in O(x)\}$, $a'_k(x) = a_{k/n}(x)/n$; with the convention that $I(f^s, x) = 0$ if $x \notin \mathrm{Fix}\,(f^s)$. Let $[x] = \{x_i = f^i(x)\}_{i=1}^n$ denote the n-periodic orbit of x. Then we may define the k-adic expansion of a n-orbit $[x]$ by summing all elements of an orbit: $I(f^q, [x]) = \sum_{i=1}^n I(f^q, x_i)$. We obtain:

$$I(f^q, [x]) = \sum_{k \in O'(x)} a'_k[x]\mathrm{reg}_k(q), \tag{1.2}$$

where $a'_k[x] = a_{k/n}(x)$.

Definition 1.5. For a complex number λ define an integer $c_{[x]}(\lambda)$ by:

$$c_{[x]}(\lambda) = \sum_{k \in O'(x);\lambda^k=1} a'_k[x].$$

Let us remark that for $x \in \mathrm{Fix}\,(f)$ the invariant $c_x(1) = \sum_{k \in O(x)} a_k(x)$ was first studied by Chow, Mallet–Paret and Yorke for C^1-maps (cf. [5]).

Definition 1.6 (cf. [2], [7])**.** Let X be a compact ANR, $f: X \to X$. By the Euler–Poincaré characteristic of a map f we denote the number:

$$\chi(f) = \sum_i (-1)^i \dim \left[H_i(X;\mathbb{Q}) / \bigcup_{m\geq 1} \ker f_{*i}^m \right],$$

or equivalently:

$$\chi(f) = \sum_i (-1)^i \eta_i,$$

where η_i is the number of non-zero (counted with multiplicities) eigenvalues of the endomorphism $f_{*i}: H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q})$ and H is the singular homology functor.

Definition 1.7. Let X be a compact ANR, $f: X \to X$. For a complex number λ define an integer $\eta(\lambda)$ by:

$$\eta(\lambda) = \begin{cases} \sum_i (-1)^i \eta_i(\lambda) & \text{if } \lambda \text{ is an eigenvalue of } f_{*i}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\eta_i(\lambda)$ is the multiplicity of λ in $f_{*i} : H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q})$.

2. Main result

For the class of maps under consideration (i.e. self-maps of compact ANR satisfying (i) and (ii)), the sequence of Lefschetz numbers of iterations is periodic (cf. [1]). The comparison of this periodicity with the periodicity given by k-adic expansion at periodic orbits allows to find relations among eigenvalues of f_{*i} (represented by $\eta(\lambda)$) and $c_{[x]}(\lambda)$. Our approach is based on C^1-case, considered by Matsuoka and Shiraki (cf. [11]), which we generalize onto the class of maps with bounded indices of iterations.

Lemma 2.1.

$$\mathrm{reg}_k(q) = \left\{ \sum_{\lambda^k=1} \lambda^q \right\}_{q=1}^{\infty}.$$

Proof. Let λ_0 be a given root of unity of degree k, then:

$$\sum_{\lambda^k=1} \lambda^q = \sum_{\lambda^k=1} \lambda_0^q \lambda^q = \lambda_0^q \sum_{\lambda^k=1} \lambda^q.$$

If $\lambda_0^q \neq 1$ then $\sum_{\lambda^k=1} \lambda^q = 0$, if $\lambda_0^q = 1$ then $\sum_{\lambda^k=1} \lambda^q = k$. This gives the assertion of the Lemma. □

Theorem 2.2. *Let Γ be the set of periodic orbit of a map f. Then, for any non-zero complex number λ, we have:*

$$\eta(\lambda) = \sum_{[x]\in\Gamma} c_{[x]}(\lambda).$$

Proof. The proof is the same as in C^1-case (cf. [11]), except for more general meaning of $c_{[x]}(\lambda)$. For a sequence $\{z_q\}_{q=1}^{\infty}$ define the operator Φ by:

$$\Phi(\{z_q\}_{q=1}^{\infty}) = \sum_{q=1}^{\infty} z_q/q!.$$

From Lemma 2.1 it follows that:

$$\Phi(\mathrm{reg}_k(q)) = \Phi\left(\left\{\sum_{\lambda^k=1} \lambda^q\right\}_{q=1}^{\infty}\right) = \sum_{\lambda^k=1}\sum_{q=1}^{\infty} \lambda^q/q! = \sum_{\lambda^k=1} (e^{\lambda} - 1),$$

thus:

$$(*) \qquad \Phi\left(\left\{\sum_{[x]\in\Gamma} I(f^q,[x])\right\}_{q=1}^{\infty}\right) = \sum_{[x]\in\Gamma}\sum_{\lambda\neq 0} c_{[x]}(\lambda)(e^{\lambda}-1).$$

On the other hand $L(f^q) = \sum_{\lambda\neq 0}\eta(\lambda)\lambda^q$, which implies:

$$(**) \qquad \Phi(\{L(f^q)\}_{q=1}^{\infty}) = \sum_{\lambda\neq 0}\eta(\lambda)(e^{\lambda}-1).$$

Consider

$$s(\lambda) = \begin{cases} \displaystyle\sum_{[x]\in\Gamma} c_{[x]}(\lambda) - \eta(\lambda) & \text{if } \lambda \neq 0, \\ \displaystyle -\sum_{\lambda\neq 0} s(\lambda) & \text{if } \lambda = 0. \end{cases}$$

Left hand sides of the equalities $(*)$ and $(**)$ are equal, so we obtain:

$$\sum_{\lambda\in G} s(\lambda)e^{\lambda} = 0,$$

where G is a finite subset of algebraic numbers. Then, by the theorem of Lindemann (cf. [11]), $\{e^{\lambda}\}_{\lambda\in G}$ is the set of linear independent values over the field of algebraic numbers, which implies that $s(\lambda) = 0$ for all λ. □

Summing up by eigenvalues of f_* in the formula of Theorem 2.2 we get the following relation:

Theorem 2.3.

$$\chi(f) = \sum_{\lambda\neq 0}\sum_{[x]\in\Gamma} c_{[x]}(\lambda).$$

As a consequence we obtain a corollary formulated by Fuller for homeomorphisms (cf. [8]).

Corollary 2.4. *If X admits a map f with no periodic points and f_* is an isomorphism, then its Euler characteristic vanishes: $\chi(X) = 0$.*

The next corollary is based on Theorem 2.3 and the definition of $c_{[x]}(\lambda)$. It may be used to estimate the number of periodic orbits.

Corollary 2.5. *Let $f\colon X \to X$ be such that for each $k \in \mathbb{N}$ and $[x] \in \Gamma$ we have $|a'_k[x]| \le V$. Let Γ_n be a set of n-periodic orbits. Assume that for each $[x] \in \Gamma_n$ $O'(x) \subset A_n$. Define $\dim X = \dim \oplus H_i(X;\mathbb{Q})$. Then:*

$$|\chi(f)| \le \sum_n \sum_{[x]\in\Gamma_n} \dim X |A_n| V.$$

If for every n $|A_n| \le D$ (or equivalently for every $x \in P(f)$ $|O'(x)| \le D$), then:

$$|\chi(f)| \le |\Gamma| \ \dim X \ D \ V.$$

3. Examples and applications

3.1. Transversal maps. Let M be a compact manifold and $f\colon M \to M$ be a C^∞-map. We call a map f *transversal* if for any $m \in \mathbb{N}$ and $x \in \mathrm{Fix}\,(f^m)$ we have $1 \notin \sigma(Df^m(x))$.

Let $\sigma_+(x)$ $(\sigma_+(x))$ denote the number of real eigenvalues of $D(f^m(x))$ greater than 1 (smaller than -1), counted with multiplicities.

Then for $x \in \mathrm{Fix}\,(f)$ (cf. [5]):

$$I(f^m, x) = \begin{cases} (-1)^{\sigma_+(x)} & \text{for } m \text{ odd},\\ (-1)^{\sigma_+(x)+\sigma_-(x)} & \text{for } m \text{ even}.\end{cases}$$

We can divide $P_n(f)$ into the following subsets:

$$\begin{aligned} P_n^{\mathrm{EE}}(f) &= \{x \in P_n(f) : \sigma_+(x),\ \sigma_-(x) \text{ are even}\},\\ P_n^{\mathrm{EO}}(f) &= \{x \in P_n(f) : \sigma_+(x) \text{ is even},\ \sigma_-(x) \text{ is odd}\},\\ P_n^{\mathrm{OE}}(f) &= \{x \in P_n(f) : \sigma_+(x) \text{ is odd},\ \sigma_-(x) \text{ is even}\},\\ P_n^{\mathrm{OO}}(f) &= \{x \in P_n(f) : \sigma_+(x),\ \sigma_-(x) \text{ are odd}\}.\end{aligned}$$

As a result we have four types of k-adic expansion for n-periodic orbits (cf. also [10]):

$$\begin{aligned} I(f^m,[x]) &= \mathrm{reg}_n(m), && \text{for } x \in P_n^{\mathrm{EE}}(f),\\ I(f^m,[x]) &= -\mathrm{reg}_n(m), && \text{for } x \in P_n^{\mathrm{OE}}(f),\\ I(f^m,[x]) &= \mathrm{reg}_n(m) - \mathrm{reg}_{2n}(m), && \text{for } x \in P_n^{\mathrm{EO}}(f),\\ I(f^m,[x]) &= -\mathrm{reg}_n(m) + \mathrm{reg}_{2n}(m), && \text{for } x \in P_n^{\mathrm{OO}}(f).\end{aligned}$$

Let now f be a transversal self-map of a closed manifold M, which satisfies the conditions (i) and (ii). Then we get in the formula (1.2) for k-adic expansion of a periodic orbit $[x]$. For $x \in P_n^{\mathrm{EE}}(f)$: $a'_n[x] = 1$, $a'_k[x] = 0$ if $k \ne n$, $O'(x) = \{n\}$. For $x \in P_n^{\mathrm{OE}}(f)$: $a'_n[x] = -1$, $a'_k[x] = 0$ if $k \ne n$, $O'(x) = \{n\}$. For $x \in P_n^{\mathrm{EO}}(f)$: $a'_n[x] = 1$, $a'_{2n}[x] = -1$, $a'_k[x] = 0$ if $k \notin \{n, 2n\}$, $O'(x) = \{n, 2n\}$. For $x \in P_n^{\mathrm{OO}}(f)$: $a'_n[x] = -1$, $a'_{2n}[x] = 1$, $a'_k[x] = 0$ if $k \notin \{n, 2n\}$, $O'(x) = \{n, 2n\}$.

If λ is an eigenvalue of f_*, $x \in P_n(f)$ we obtain:

$$c_{[x]}(\lambda) = \begin{cases} 1 & \text{if } x \in P_n^{\mathrm{EE}}(f) \text{ and } \lambda^n = 1, \\ -1 & \text{if } x \in P_n^{\mathrm{OE}}(f) \text{ and } \lambda^n = 1, \\ 1 & \text{if } x \in P_n^{\mathrm{OO}}(f) \text{ and } \lambda^{2n} = 1,\ \lambda^n \neq 1, \\ -1 & \text{if } x \in P_n^{\mathrm{EO}}(f) \text{ and } \lambda^{2n} = 1,\ \lambda^n \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let Γ_n^{ij}, where $i, j \in \{E, O\}$, be the set of n-periodic orbits which corresponds to the set $P_n^{ij}(f)$. Let $\mathcal{R}(n) = \{\lambda : \lambda \text{ is an eigenvalue of } f_* \text{ and } \lambda^n = 1\}$. By Theorem 2.3 we have:

$$\begin{aligned} \chi(f) = \sum_n \biggl(& \sum_{[x]\in\Gamma_n^{\mathrm{EE}}} \sum_{\lambda\in\mathcal{R}(n)} 1 + \sum_{[x]\in\Gamma_n^{\mathrm{OE}}} \sum_{\lambda\in\mathcal{R}(n)} (-1) \\ & + \sum_{[x]\in\Gamma_n^{\mathrm{OO}}} \sum_{\lambda\in\mathcal{R}(2n)\setminus\mathcal{R}(n)} 1 + \sum_{[x]\in\Gamma_n^{\mathrm{EO}}} \sum_{\lambda\in\mathcal{R}(2n)\setminus\mathcal{R}(n)} (-1) \biggr) \\ = \sum_n & [|\mathcal{R}(n)|(|\Gamma_n^{\mathrm{EE}}| - |\Gamma_n^{\mathrm{OE}}|) + |\mathcal{R}(2n) \setminus \mathcal{R}(n)|(|\Gamma_n^{\mathrm{OO}}| - |\Gamma_n^{\mathrm{EO}}|)]. \end{aligned}$$

3.2. Homeomorphisms of surfaces. Let $f\colon M \to M$ be a homeomorphism of a surface M without boundary which preserves the orientation and fulfills the required assumptions (conditions (i) and (ii)). Let us assume additionally that the following condition of "hyperbolicity" is satisfied for each $x \in P_n(f)$:

(1) There is no V — a neighbourhood of x such, that

$$f^n(V) \subset V \quad \text{or} \quad V \subset f^n(V).$$

(2) There is W — a neighbourhood of x such, that

$$\bigcap_{k\in\mathbb{Z}} (f^n)^k(W) = \{x\}.$$

In this case the shape of k-adic expansion is known (cf. [4]): $I(f^m, [x]) = \mathrm{reg}_n(m) - r_x \mathrm{reg}_{nq(x)}(m)$, so $a'_n[x] = 1$, $a'_{nq(x)}[x] = -r_x$, $a'_k[x] = 0$ for $k \notin \{n, nq(x)\}$, $O'(x) = \{n, nq(x)\}$. What is more $r_x > 0$.

For λ an eigenvalue of f_*, $[x] \in \Gamma_n$ we have:

$$c_{[x]}(\lambda) = \begin{cases} -r_x & \text{if } \lambda^n \neq 1 \text{ and } \lambda^{nq(x)} = 1, \\ 1 - r_x & \text{if } \lambda^n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, from Theorem 2.3, we obtain:

$$\chi(f) = \chi(M) = \sum_n \sum_{[x]\in\Gamma_n} \biggl[\sum_{\lambda\in\mathcal{R}(n)} (1 - r_x) + \sum_{\lambda\in\mathcal{R}(nq(x))\setminus\mathcal{R}(n)} -r_x \biggr].$$

We get:

$$\chi(M)=\sum_{n}\sum_{[x]\in\Gamma_n}[|\mathcal{R}(n)|-r_x(|\mathcal{R}(n)|+|\mathcal{R}(nq(x))\setminus\mathcal{R}(n)|)].$$

Consequently, $\chi(M)\leq 0$ because $r_x>0$. This fact shows that there is no orientation preserving homeomorphism of S^2 with only hyperbolical periodic points (cf. [4]).

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EXISTENCE OF HETEROCLINIC ORBITS FOR SYSTEMS SATISFYING MONOTINICITY CONDITIONS

Bogdan Kaźmierczak

Dedicated to Professor Lech Górniewicz

Abstract. We use the implicit function theorem to prove the existence of heteroclinic orbits for systems of second order ordinary differential equations satisfying global monotonicity conditions.

1. Introduction

Travelling waves are a special kind of solutions, which can describe many physical phenomena, e.g. phase transitions, ionization processes in plasma physics or different types of species interaction in mathematical ecology. Many of these phenomena can be described by systems of PDEs of reaction-diffusion type. While looking for travelling wave solutions of such systems we arrive at a system of second order ODEs.

In this paper we are interested in heteroclinic solutions to the following system of ODEs:

$$a_i(u_i, u_i')u_i'' - qc_i(u_i, u_i')u_i' + M_i(u, u_i')u_i' + f_i(u) = 0, \tag{1}$$

where $i \in \{1, \ldots, n\}$, $u = (u_1, \ldots, u_n)$ and $'$ denotes differentiation with respect to $\xi \in \mathbb{R}^1$.

We prove the existence of heteroclinic solutions $u(\xi)$ to system (1) joining its stable equilibrium states $\mathbf{0}$ and $\mathbf{1}$ i.e. such that $\lim_{\xi \to -\infty} u(\xi) = \mathbf{0}$ and $\lim_{\xi \to \infty} u(\xi) = \mathbf{1}$.

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The basic tool of the proof is the implicit function theorem. We consider an appropriate family of systems depending on a parameter $\lambda \in [0,1]$ in such a way that for $\lambda = 1$ it coincides with system (1) and for $\lambda = 0$ it becomes a system, which can be easily analyzed. First, we prove that for all the possible strictly monotone heteroclinic solutions both $|u'|_{C^1}$ and $|q|$ are bounded from above by constants independent of $\lambda \in [0,1]$. Starting from the unique strictly monotone heteroclinic solution for $\lambda = 0$, by means of the implicit function theorem, we can show that the unique heteroclinic solution exists also for all $\lambda > 0$ sufficiently small. Having shown that heteroclinic solutions u_λ are strictly monotone for $\lambda \in [0,\lambda_1]$, λ_1 sufficiently close to 0, we can repeat the procedure. The monotonicity property enables us to take advantage of 'a priori' estimates and allows us to demonstrate that the linearization of the mapping generated by the left hand sides of the equations is boundedly invertible. It allows us to extend the interval of existence of heteroclinic pairs. It is necessary to underline that in this procedure the monotonicity conditions imposed on the source terms f_i are crucial.

The system analyzed here is not the most general one. The most important limitation consists in the fact that the nonlinear source vector function can be continuously transformed to a standard function, which is symmetric with respect to the components of u without changing the number and the character of its zeros (see Assumption 5). From this point of view much more general systems, e.g. in [4] and [8] and also in [13], were considered. (However, contrary to [4] and [13] the coefficients a_i and $_i$ depend on u_i and u_i'.) It seems that a proof by the use of the implicit function theorem is interesting. It is elementary an does not use advanced topological methods like the Leray Schauder degree theory ([4], [13]) or the Conley index theory ([11], [1]). Moreover, it gives the uniqueness of solutions at every stage of continuation.

In Section 5 we show that this method can be applied to prove the existence of travelling wave solutions in a multitemperature model of laser sustained plasma. These waves connect the two states of gas: the cold unionized and a hot ionized one. This time the travelling wave solutions describe the motion of the boundaries of the plasma region.

Most of the lemmas are stated without proofs. They will be inserted in the subsequent paper.

2. Main assumptions and preliminary lemmas

Assumption 1. *Assume that all the considered functions are sufficiently smooth.*

Assumption 2. *Assume that:*

(a) $f_{i,j} > 0$ *for all* $i,j \in \{1,\dots,n\}$, $j \neq i$ *(monotonicity conditions).*

(b) **0** *and* **1** *are solutions to the system*

$$f_1(u) = 0, \dots , f_n(u) = 0. \tag{2}$$

Both constant states **0** *and* **1** *are stable, i.e. all the eigenvalues of the matrices*

$$f_{i,j}(\mathbf{0}), \quad f_{i,j}(\mathbf{1}) \tag{3}$$

have negative real parts.

(c) *There is only one solution* $E_1 = (e_{11}, \dots , e_{1n}) \in (0, 1)^n$ *to system* (2) *different from* **0** *and* **1**. *This solution is unstable i.e.* $f_{i,j}(E_1)$ *has at least one eigenvalue with positive real part.*

For any natural $m \geq 1$ we put $\mathbb{R}^m_+ = \{y : y \in \mathbb{R}^m,\ y \geq \mathbf{0}\}$.

For $x, y \in \mathbb{R}^m$ we write $x \geq y$ ($x > y$) *if and only if* $x_i \geq y_i$ ($x_i > y_i$) for all $i = 1, \dots , m$.

For $Y \in \mathbb{R}^n$ we put

$$|Y| = \sup_i |Y_i|.$$

Assumption 3. $a_i(u_i, z_i) > 1$ *for all* $i \in \{1, \dots , n\}$, *all* $u_i \in [0, 1]$ *and all* $z_i \in \mathbb{R}^1_+$. *There exists* $c_0 > 0$ *such that* $c_i(u_i, z_i) > c_0$ *for all* $u_i \in [0, 1]$, $z_i \in \mathbb{R}^1_+$. *There exists* $b > 0$ *such that for all* $u, v \in [0, 1]$, $p, r \in \mathbb{R}^1_+$ *with* $p < r$ *we have* $c_i(u, p)(c_i(v, r))^{-1} \leq b$. $M_{i,u_j}(u, z_i) \geq 0$, $j \neq i$, *for all* u *from some open neighbourhood of the set* $[0, 1]^n$ *and all* $z_i \in \mathbb{R}^1$.

Remark 1. The condition $a_i(u_i, z_i) > 1$ in Assumption 3 can be obviously achieved if only $a_i(u_i, z_i) > C_{ai} > 0$ for all $u_i \in [0, 1]$ and all $z_i \in \mathbb{R}^1_+$.

For all $i \in \{1, \dots , m\}$, let $\chi_i: \mathbb{R}^1_+ \to \mathbb{R}^1_+$ denote a continuous and increasing function such that

$$\int_0^{z_i} a_{*i}(u_i, z_i) z_i \, dz_i \geq \chi_i(z_i),$$

where $a_{*i}(u_i, z_i) := \inf_{u_i \in [0,1]} a_i(u_i, z_i)$.

The next assumption guarantees the possibility of finding a priori estimates of the first derivatives of the solutions.

Assumption 4. *For each* $i \in \{1, \dots , n\}$ *one of the below conditions holds:*

(a) *For all* $u_i \in [0, 1]$ *and all* $z_i \in \mathbb{R}^1_+$ *either* $a_{i,u_i}(u_i, z_i) \leq 0$ *or* $a_{i,u_i}(u_i, z_i) \geq 0$. *The function* $M_i(u, z_i)$ *satisfies the estimation:*

$$|M_i(u, z_i)| \leq k(|u|)(1 + \beta_i(|z_i|)), \tag{4}$$

with $k: \mathbb{R}^1_+ \to \mathbb{R}^1_+$ *continuous, and* $\beta_i: \mathbb{R}^1_+ \to \mathbb{R}^1_+$ *continuous and such that* $\beta_i(y) y (\chi_i(y))^{-1} \to 0$ *as* $y \to \infty$.

(b) $c_i \equiv 1$ *and for all* $p, r \in \mathbb{R}^1_+$, $p \leq r$,

$$M_i(u, p) \geq M_i(v, r) - \widehat{M}_i(u, p, v, r)$$

for all $\mathbf{0} \le u \le v \le \mathbf{1}$, $\widehat{M}_i(u,p,v,r) \le k(u,v)(1+\beta_i(|r|))$, *with* $k\colon \mathbb{R}^{2n}_+ \to \mathbb{R}^1_+$ *continuous,* $\beta_i\colon \mathbb{R}^1_+ \to \mathbb{R}^1_+$ *continuous and such that* $\beta_i(y)y(\chi_i(y))^{-1} \to 0$ *as* $y \to \infty$.

(c) $c_i \equiv 1$ *and for all* $p, r \in \mathbb{R}^n_+$, $p \le r$,

$$M_i(u,p) \le M_i(v,r) + \widehat{M}_i(u,p,v,r)$$

for all $\mathbf{0} \le v \le u \le \mathbf{1}$, $\widehat{M}_i(u,p,v,r) \le k(u,v)(1+\beta_i(|r|))$, *with* $k\colon \mathbb{R}^{2n}_+ \to \mathbb{R}^1_+$ *continuous,* $\beta_i\colon \mathbb{R}^1_+ \to \mathbb{R}^1_+$ *continuous and such that* $\beta_i(y)y(\chi_i(y))^{-1} \to 0$ *as* $y \to \infty$.

(d) $a_i(u_i,z_i) = a_i(u_i)$ *and* $M_i(u,z_i)$ *satisfies condition* (4) *with* $\beta_i(y)y^{-1} \to 0$ *as* $y \to \infty$, *or the sum* $a_i(u_i)u_i'' + M_i(u,u_i')u_i'$ *can be written in the form* $(a_i(u_i)u_i')' + \mu_i(u,u_i')u_i'$, *where* $\mu_{i,u_j}(u,z_i) > 0$ *for all* $j \ne i$ *and* u *from some open neighbourhood of the set* $[0,1]^n$, *and* $\mu_i(u,z_i)$ *satisfies* (4) *with* $\beta_i(y)y^{-1} \to 0$ *as* $y \to \infty$.

Remark 2. Points (b) and (c) of Assumption 4 are taken from the paper [4]. Let us note that in this case we do not assume any growth condition on the term M_i.

Definition 1. Let $g(u) = (g_1(u),\dots,g_n(u))$ denote a $C^1(\mathbb{R}^n)$ function satisfying for all $i \in \{2,\dots,n\}$ the following conditions:

(a) $g_i(u_1,\dots,u_{i-1},u_i,u_{i+1},\dots,u_n) = g_1(u_i,\dots,u_{i-1},u_1,u_{i+1},\dots,u_n)$,
(b) $g_{1,1}(u) \le -k$, $g_{1,i}(u) \ge k$ for all $u \in \mathbb{R}^n$, $i \in \{2,\dots,n\}$, $k > 0$,
(c) $\sum_{i=1}^n g_{1,i}(\mathbf{0}) < 0$, $\sum_{i=1}^n g_{1,i}(\mathbf{1}) < 0$,
(d) the only solutions to the equation $g(u_1,u_1,\dots,u_1) = \mathbf{0}$ are $\mathbf{0}$, $\mathbf{1}$ and $E_0 = (e_{01},\dots,e_{0n})$.

Lemma 1. *The solutions to the equation* $g(u) = \mathbf{0}$ *must lie on the diagonal of* $\mathbb{R}^n$.

The crucial assumption of the paper consists in the demand that the function $f(u)$ can be deformed continuously to the function $g(u)$ in such a way that the intermediate functions retain the quantitative properties of $f(u)$.

Assumption 5. *There exists a function*

$$G_\lambda(u)\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}^n \tag{5}$$

such that

$$G_1(u) \equiv f(u), \quad G_0(u) \equiv g(u). \tag{6}$$

and for all $\lambda \in [0,1]$ *the function* $G_\lambda(u)$ *satisfies Assumption* 2. *In particular, for all* $\lambda \in [0,1]$, *there is only one solution* $E_\lambda = (e_{\lambda 1},\dots,e_{\lambda n}) \in (0,1)^n$ *to the system* $G_\lambda(u) = \mathbf{0}$ *different from* $\mathbf{0}$ *and* $\mathbf{1}$ *inside* $[0,1]^n$. *This solution is unstable i.e.* $G_{\lambda i,j}(E_\lambda)$ *has at least one eigenvalue with positive real part.*

3. Properties of the linearized operator

Let us consider a family of systems depending on the parameter $\lambda \in [0,1]$:

$$\mathcal{M}_i(\lambda, q, u) = 0, \tag{7}$$

$i = 1, \dots, n$, where

$$\begin{aligned} \mathcal{M}_i(\lambda, q, u) = {} & a_{\lambda i}(u_i, u_i')u_i'' - (1-\lambda)qu_i' \\ & + \lambda[-qc_i(u_i, u_i')u_i' + M_i(u, u_i')u_i'] + G_{\lambda i}(u), \end{aligned} \tag{8}$$

and

$$a_{\lambda i}(u_i, u_i') = \lambda a_i(u_i, u_i') + (1-\lambda). \tag{9}$$

Definition 2. A pair $(q_\lambda, u_\lambda) \in \mathbb{R}^1 \times C^2(\mathbb{R}^1, \mathbb{R}^n)$ is called a heteroclinic pair for system (7), if $u_\lambda(\xi)$ satisfies system (7) for $q = q_\lambda$, $u_\lambda(\xi) \to \mathbf{0}$ as $\xi \to -\infty$, $u_\lambda(\xi) \to \mathbf{1}$ as $\xi \to \infty$ and $u_\lambda'(\xi) \to \mathbf{0}$ as $\xi \to \pm\infty$. It is called strictly monotone, if $u_\lambda'(\xi) > \mathbf{0}$ for all $\xi \in \mathbb{R}^1$.

For all the possible strictly monotone heteroclinic pairs of system (7) independently of $\lambda \in [0,1]$ a priori estimates of the C^1-norm and the absolute value of the parameter q hold.

Lemma 2. *If $(q_\lambda, u_\lambda(\xi))$, $\lambda \in [0,1]$, is a strictly monotone heteroclinic pair for system (7) then there exists a finite constant m such that $|u_\lambda'|_{C^0(\mathbb{R}^1)} < m$. This constant is independent of λ, q_λ and u_λ.*

Proof. The proof of the more general lemma may be found in [8]. □

The next lemma states the boundedness of the parameter q.

Lemma 3. *If $\lambda \in [0,1]$ and (q_λ, u_λ) is a strictly monotone heteroclinic pair satisfying system (7), then $|q_\lambda| < Q$, where Q independent of λ and u_λ.*

Proof. The proof (modulo slight modifications) is contained in [4, Lemma 3.4] or in [8]. □

The boundedness of q_λ allows us to estimate the exponential behaviour of monotone solutions near the singular points $(u_\lambda, u_\lambda') = (\mathbf{0}, \mathbf{0})$ and $(\mathbf{1}, \mathbf{0})$.

Lemma 4 (see [13, Lemma 2.9])**.** *There exist a number $\widetilde{\varepsilon} > 0$, such that for all strictly monotone heteroclinic solutions u_λ of the problem (7) with $\lambda \in [0,1]$ and $q \in [-Q, Q]$ we have the following estimates*

$$|u_\lambda(\xi)| \le K_0 \widetilde{\varepsilon} \exp[\gamma(\xi - \xi_0)], \quad |u_\lambda'(\xi)| \le K_0 \widetilde{\varepsilon} \exp[\gamma(\xi - \xi_0)],$$

for all $\xi \le \xi_0$ and ξ_0 such that $|u_\lambda(\xi_0)| \le \widetilde{\varepsilon}$, and

$$|u_\lambda(\xi) - 1| \le K_1 \widetilde{\varepsilon} \exp[-\vartheta(\xi - \xi_0)], \quad |u_\lambda'(\xi)| \le K_1 \widetilde{\varepsilon} \exp[-\vartheta(\xi - \xi_0)],$$

for all $\xi \geq \xi_0$ *and* ξ_0 *such that* $|u_\lambda(\xi_0) - 1| \leq \widetilde{\varepsilon}$. *Moreover, the constants* $K_0, K_1, \gamma > 0$ *and* $\vartheta > 0$ *are independent of the solution* u_λ.

Proof. The system (7) can be written as a first order system. For $q \in [-Q, Q]$ all the eigenvalues of the linearization matrix for such a system at the points $(u, u') = (\mathbf{0}, \mathbf{0})$ and $(\mathbf{1}, \mathbf{0})$ have nonzero real parts (see [3, Theorem 3.3]). Now, the proof of Lemma 4 follows from the Hartman–Grobman theorem. □

Remark 3. Obviously the same estimations hold for the second derivatives of u_λ, i.e. for some K_2 and all $\lambda \in [0, 1]$,

$$|u''_\lambda(\xi)| \leq K_2 \widetilde{\varepsilon} \exp[\gamma(\xi - \xi_0)],$$

for all $\xi \leq \xi_0$ and ξ_0 such that $|u_\lambda(\xi_0)| \leq \widetilde{\varepsilon}$, and

$$|u''_\lambda(\xi)| \leq K_2 \widetilde{\varepsilon} \exp[-\vartheta(\xi - \xi_0)]$$

for all $\xi \geq \xi_0$ and ξ_0 such that $|u''_\lambda(\xi_0) - 1| \leq \widetilde{\varepsilon}$.

Definition 3. Let B_2 denote the Banach space of functions $u: \mathbb{R}^1 \to \mathbb{R}^n$ of $C^2(\mathbb{R}^1)$ class equipped with the norm

$$\|u\|_2 = \max_i \sup_\xi \left(\sum_{k=0}^{2} |u_i^{(k)}(\xi)| \right),$$

with u satisfying the following conditions:

(a) the limits $\lim_{\xi \to \infty} u(\xi)$ and $\lim_{\xi \to -\infty} u(\xi)$ exist,
(b) $u'(\xi), u''(\xi) \to \mathbf{0}$ as $|\xi| \to \infty$.

Let B_{20} denote the subspace of B_2 consisting of functions u such that

$$u_1(0) = \frac{1}{2} e_{1*}(u_1(-\infty) + u_1(\infty)),$$

where $e_{1*} = \min_{\lambda \in [0,1]} e_{\lambda 1}$.

Let B_0 denote the Banach space of functions $u: \mathbb{R}^1 \to \mathbb{R}^n$ of $C^0(\mathbb{R}^1)$ class such that the limits $\lim_{\xi \to \infty} u(\xi)$ and $\lim_{\xi \to -\infty} u(\xi)$ exist, equipped with the norm

$$\|u\|_0 = \max_i \sup_\xi (|u_i(\xi)|).$$

Let

$$\mathcal{M}(\lambda, q, u) = (\mathcal{M}_1, \dots, \mathcal{M}_n), \tag{10}$$

where $\mathcal{M}_i$ is defined in (7). $\mathcal{M}$ acts from the space $\mathbb{R}^1 \times \mathbb{R}^1 \times B_{20}$ to the spaceB_0. $\mathcal{M}$ is Frechet differentiable. (In particular its Frechet derivative with respect to (q, u) is continuous with respect to (λ, q, u).) It is easy to check that the Frechet derivative with respect to (q, u) at the point (λ, q, u) is the operator

$$D\mathcal{M}(\lambda, q, u)\,[\delta q, \delta u] = D_u \mathcal{M}(\lambda, q, u)\delta u + \mathcal{M}_{,q}(\lambda, q, u)\delta q,$$

with

$$\begin{aligned} D_u\mathcal{M}(\lambda,q,u)\delta u = &\mathcal{A}(\lambda,u,u')(\xi)(\delta u)'' \\ &+ \mathcal{C}(\lambda,q,u,u',u'')(\xi)(\delta u)' + \mathcal{B}(\lambda,q,u,u',u'')(\xi)\delta u, \end{aligned}$$

where

$$\begin{aligned} \mathcal{A} &= \operatorname{diag}(a_{\lambda 1}(u_1,u_1'),\dots,a_{\lambda n}(u_n,u_n'))(\xi), \\ \mathcal{C}_{ij} &= \{a_{\lambda i}(u_i,u_i')u_i'' - (1-\lambda)qu_i' + \lambda[-qc_i(u_i,u_i')u_i' + M_i(u,u_i')u_i']\}_{,u_j'}(\xi), \\ \mathcal{B}_{ij} &= \{a_{\lambda i}(u_i,u_i')u_i'' - \lambda qc_i(u_i,u_i')u_i' + \lambda M_i(u,u_i')u_i' + G_{\lambda i}(u)\}_{,u_j}(\xi), \end{aligned}$$

and

$$\delta u = (\delta u_1,\dots,\delta u_n)^T.$$

Consider a linear operator:

$$\text{Lu} = \mathbf{A}(\xi)u'' + \mathbf{C}(\xi)u' + \mathbf{B}(\xi)u, \tag{11}$$

where $\mathbf{A}(\xi)$, $\mathbf{B}(\xi)$, $\mathbf{C}(\xi)$ are matrices of C^1-class, $\mathbf{A}(\xi)$ and $\mathbf{C}(\xi)$ are diagonal matrices. $\mathbf{A}(\xi)$ has positive diagonal elements and $\mathbf{B}(\xi)$ has positive off-diagonal elements. Assume that the matrices $\mathbf{A}(\xi)$, $\mathbf{B}(\xi)$, $\mathbf{C}(\xi)$ have limits as $\xi \to \pm\infty$ and that the matrices $\mathbf{B}_\pm = \lim_{\xi\to\pm\infty}\mathbf{B}(\xi)$ have negative principal eigenvalues.

The following theorem, which can be found in [13, (p. 155)], will be of basic importance below.

Lemma 5. *Let us assume that a positive solution $w(\xi)$ exists for the equation*

$$\text{Lu} = \mathbf{0}, \tag{12}$$

such that $\lim_{\xi\pm\infty} w(\xi) = 0$. Then the following is true:

(a) *the equation*

$$\text{Lu} = \lambda u, \quad u(\pm\infty) = \mathbf{0} \tag{13}$$

has no solutions different from $\mathbf{0}$ for $\operatorname{Re}\lambda \geq 0$, $\lambda \neq 0$,

(b) *every solution of (13) has for $\lambda = 0$ the form $u(\xi) = kw(\xi)$, $k \in \mathbb{R}^1$,*

(c) *the adjoint equation*

$$L^*v = \mathbf{0}, \quad v(\pm\infty) = \mathbf{0} \tag{14}$$

has a positive solution. This solution is unique to within a constant factor.

Our starting point will be system (7) for $\lambda = 0$. Then the system has exactly one heteroclinic solution pair $(q_0, u_0(\xi))$ with $u_0 \in B_{20}$ and $u_0'(\xi) > 0$ for all $\xi \in \mathbb{R}^1$ joining the points $\mathbf{0}$ and $\mathbf{1}$ (see [13, Lemma 3.2, p. 173]). According to Lemma 5 there is a unique (up to a multipliactive constant) solution to the linearized system $D_u\mathcal{M}(0,q_0,u_0)\delta u = 0$, namely $\delta u = u_0'(\xi)$. Thus, for $\lambda = 0$, the problem is reduced to a scalar one. By standard arguments we infer that the linearized operator $D\mathcal{M}(0,q_0,u_0)$ is boundedly invertible, i.e. the equation

$$D\mathcal{M}(0,q_0,u_0)\,[\delta q,\delta u]=h,$$

has a unique solution in the space $B_{20}\times\mathbb{R}^1$.

According to the implicit function theorem (see e.g. [2]) there exists $\lambda_*>0$ such that for all $\lambda\in[0,\lambda_*]$ there exists a heteroclinic pair for system (7). If $\lambda_*<1$ but $D\mathcal{M}(\lambda_*,u_{\lambda_*},q_{\lambda_*})\,[\delta q,\delta u]$ is boundedly invertible, then we can prolong the interval of existence of heteroclinics to $[0,\lambda_{*1}]$, $\lambda_{*1}>\lambda_*$. If this procedure can be repeated, then after a finite number of steps we are able to extend the existence interval to the whole of $[0,1]$.

Lema 6. *Suppose that for $\lambda\in[0,\lambda_b]$, $\lambda_b\in(0,1]$, there exists a heteroclinic pair (q_λ,u_λ) satisfying the system (7), such that u_λ is strictly monotonic in all of its components. Then the linearized system*

$$D\mathcal{M}(\lambda,q_\lambda,u_\lambda)[\delta q,\delta u]=h, \tag{15}$$

has for all $h\in B_0$ a unique (up to a multiplication constant) solution in the space $B_{20}\times\mathbb{R}^1$. The norm of $[D\mathcal{M}(\lambda,q,u_\lambda)]^{-1}$ is bounded uniformly by a constant independent of $\lambda\in[0,\lambda_b]$.

4. Strict monotonicity of u_λ and the existence proof

In this section we demonstrate that the interval of λ values, for which strictly monotone heteroclinic solutions exist can be extended to the whole of $[0,1]$. Roughly speaking the proof consists in showing that this interval is both relatively closed and open in $[0,1]$, so it must coincide with $[0,1]$.

In the previous section we showed that the operator $\mathcal{M}$ linearized around a heteroclinic pair (q_λ,u_λ), $\lambda\in[0,1]$, is boundedly invertible provided the function u_λ is strictly monotone. For $\lambda=0$ the heteroclinic solution u_0 is strictly monotone. The question arises, whether the solution may become non monotone for larger values of λ. First, we will show that if u_λ is strictly monotone for $\lambda\in[0,\lambda_0)$ then it exists and is monotonic also for $\lambda=\lambda_0$.

Lemma 7. *Assume that (q_λ,u_λ), $\lambda\in[0,\lambda_*)$, $\lambda_*>0$, is a continuous family of heteroclinic pairs (obtained by means of the implicit function theorem) and that $u_\lambda(\xi)$ is strictly monotonic for all all $\lambda\in[0,\lambda_0)$, $\lambda_0\in[0,\lambda_*]$. Then for $\lambda=\lambda_0$ the heteroclinic pair $(q_{\lambda_0},u_{\lambda_0}(\xi))$ also exists and $u_{\lambda_0}(\xi)$ is a strictly monotone function of ξ.*

Lemma 8. *Assume that (q_λ,u_λ), $\lambda\in[0,\lambda_*)$, $\lambda_*>0$, be a continuous family of heteroclinic pairs (obtained by means of the implicit function theorem) and that $u_\lambda(\xi)\in B_{20}$ is strictly monotonic for all all $\lambda\in[0,\lambda_0]$, $\lambda_0\in[0,\lambda_*)$. Then $u_\lambda(\xi)$ is also a strictly monotonic for all $\lambda\geq\lambda_0$ sufficiently close to it.*

Lemma 9. *The family* (q_λ, u_λ) *of strictly monotone heteroclinic pairs can be continued at least up till* $\lambda = 1$.

We are thus in a position to formulate the main theorem of our paper.

Theorem 1. *There exists a unique family of heteroclinic pairs* $(q_\lambda, u_\lambda) \in \mathbb{R}^1 \times C^2(\mathbb{R}^1)$, $\lambda \in [0,1]$, *such that each* (q_λ, u_λ) *satisfies system* (7), $u_\lambda(-\infty) = \mathbf{0}$, $u_\lambda(\infty) = \mathbf{1}$ *and* $u'_\lambda(\xi) > \mathbf{0}$ *for* $x \in \mathbb{R}^1$. *This family is continuous, i.e. for all* $\lambda_1, \lambda_2 \in [0,1]$:

$$|q_{\lambda_1} - q_{\lambda_2}| + \|u_{\lambda_1} - u_{\lambda_2}\|_{B_{20}} \to 0 \quad \text{as } \lambda_2 \to \lambda_1.$$

In particular $(q,u) = (q_1, u_1)$ *is a heteroclinic pair for system* (1) *joining the points* $\mathbf{0}$ *and* $\mathbf{1}$.

Proof. Existence of (q_λ, u_λ) follows from Lemma 9. Let us prove the uniqueness of the pair. First, the pair (q_0, u_0) is unique. Suppose to the contrary that for some $\eta \in [0,1]$ we have at least two heteroclinic pairs $(q_{\eta i}, u_{\eta i})$, $i = 1,2$. These solutions can be continued back to the value $\lambda = 0$, so there must exist η_0 such that for $\lambda = \eta_0$ these two solutions merge for the first time, i.e. $(q_{\lambda 1}, u_{\lambda 1}) \neq (q_{\lambda 2}, u_{\lambda 2})$ for all $\lambda \in (\eta_0, \eta]$. But, then due to the implicit function theorem we would have also $(q_{\lambda 1}, u_{\lambda 1}) = (q_{\lambda 2}, u_{\lambda 2})$ for all λ in some vicinity of η_0. This is a contradiction, from which the uniqueness follows. □

5. Travelling waves in laser sustained plasma

As an example, let us investigate travelling waves in a system of equations describing multicomponent plasma sustained by a laser beam of a given intensity I.

Under a *constant pressure* p the temperatures T_1 of the light (electron) component and the temperatures of T_i, $i \in \{2, \dots, n\}$ of heavy particles (atoms and ions) of i-th kind are described by the following equations (see [5], [6], [9], [10], [12]):

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \overrightarrow{v}_1 \cdot \nabla\right)\left\{\frac{3}{2}k_B N_1 T_1 + \widetilde{E}(T_1)\right\} &= \nabla(k_1 \nabla T_1) + f_1(T),\\ \left(\frac{\partial}{\partial t} + \overrightarrow{v}_i \cdot \nabla\right)\left\{\frac{3}{2}k_B N_i T_i\right\} &= \nabla(k_i \nabla T_i) + f_i(T),\end{aligned} \tag{16}$$

where $i \in \{2, \dots, n\}$, $T = (T_1, \dots, T_n)$, $k_j = k_j(T_j)$, $j \in \{1, \dots, n\}$, is the heat conductivity coefficient, $N_1(T_1)$ is the number density of electrons, $N_i(T_i)$, $i \in \{2, \dots, n\}$ is the number density of the heavy component of i-th kind and $\overrightarrow{v}_j(T)$, $j \in \{1, \dots, n\}$, denotes the convectional velocity of the j-th component. k_B is the Boltzmann constant. $\widetilde{E}(T_1)$ is the average ionization energy for the given temperature T_1. (The energy necessary to the first ionization of an atom depends on the kind of the atom. If we have to do with a one component plasma, then $\widetilde{E}$ would be equal simply to $N_1(T_1)E$, where E is the first ionization energy for the given kind of atoms.) The functions f_i have the following form:

$$
\begin{aligned}
f_1 &= F_1(T_1) + \sum_{j\in\{2,\dots,n\}} c_{1j}(T)(T_j - T_1),\\
f_i &= \sum_{j\in\{1,\dots,n\},j\neq i} c_{ij}(T)(T_j - T_i) + K_i(T),
\end{aligned}
\tag{17}
$$

for $i = 2,\dots,n$. The term $F_1 = \kappa(T_1)I - \mathcal{E}_{\mathrm{rad}}(T_1)$ is responsible for the absorption of energy from the laser beam (κI) and its losses by through radiation ($\mathcal{E}_{\mathrm{rad}}$). The terms $K_i(T)$ describe the losses of energy in the process of heat conduction and convection. The terms $c_{ij}(T)(T_j - T_i)$ describe the transfer of energy from the i-th to the j-th component of the plasma.

Let us look for solutions in the form of travelling waves:

$$
T_i(x,t) = u_i(\overrightarrow{x}\cdot\overrightarrow{n} + qt), \quad i = 1,\dots,n,
\tag{18}
$$

where $\overrightarrow{n} \in \mathbb{R}^3$ is a chosen unit vector (a direction of propagation) and $\chi \in \mathbb{R}^1$ is the speed of the wave. If we denote $\xi := \overrightarrow{x}\cdot\overrightarrow{n} + \chi t$, then we arrive at a system of ordinary differential equations:

$$
(k_i u_i')' - qC_i(u_i)u_i' - \overrightarrow{v}_i\cdot\overrightarrow{n}C_i(u_i)u_i' + f_i(u) = 0,
\tag{19}
$$

$i = 1,\dots,n$, where $u := (u_1,\dots,u_n)$ and

$$
C_i(u_i) = \frac{\partial}{\partial u_i}\left\{\frac{3}{2}k_B N_i(u_i)u_i + \delta_{i1}\widetilde{E}(u_i)\right\},
$$

with δ_{i1} being the Kronecker's delta.

Assumption 6. *Assume that the function $F_1(u_1)$ has exactly three zeros: $0, 1$ and $U_0 \in (0,1)$ such that $F_1'(0) < 0$, $F_1'(U_0) > 0$ and $F_1'(1) < 0$.*

Assumption 7. $\sup_{i\in\{2,\dots,n\}}\sup_{u\in[-1,2]^n}(|K_i(u)| + |DK_i(u)|) < \tau$ *with τ sufficiently small. $K_i(\mathbf{0}) = 0$ for all $i \in \{2,\dots,n\}$.*

This assumption is reasonable, as both the absorption of energy (in the process of so called Inverse Brems–Strahlung) and the energetic losses are almost entirely carried out in the electron component.

Assumption 8. *$c_{ij}(u) > 0$, $c_{ij}(u) = c_{ji}(u)$ for all $i,j \in \{1,\dots,n\}$, $u \in \mathbb{R}^n$. For all $i,k \in \{1,\dots,n\}$, $k \neq i$, and all $u \in [0,1]^n$, we have $\sum_{j\neq i} c_{ij,k}(u)(u_j - u_i) + c_{ik}(u) > 0$.*

This assumption may be justified by the fact that the derivatives $c_{ij,k}(u)$ are relatively large only for small values of u thus they are, in a way, damped by the factors $(u_i - u_j)$.

Assumption 9. *$C_i(u) > C_{0i} > 0$ for all $u \in [0,1]^n$, $i \in \{1,\dots,n\}$.*

Assumption 10. *For all* $u \in [-1,2]^n$ *and* $i \in \{1,\dots,n\}$, $\{\overrightarrow{v}(u)\cdot\overrightarrow{n}\}_{i,u_j} \leq 0$ *for all* $j \neq i$.

This is a simplifying technical condition. It can be fulfilled e.g. if we assume that $\overrightarrow{v}_i(u) = \overrightarrow{v}_i(u_i)$. As C_i depend only on u_i, then in view of Assumption 9 system (19) satisfies Asumption 3. (The condition $a_i(u_i) > 1$ for all $u_i \in [0,1]$ can be achieved by dividing the i-th equation by $\min_{u_i\in[0,1]} a_i(u_i)$.) It also satisfies point 4 of Asumption 4.

Now, we will show that Assumptions 6–8 imply Assumption 2. We have, for $i \neq 1$, $k \neq i$,

$$f_{i,k}(u) = \sum_{j\neq i} c_{ij,k}(u_j - u_i) + c_{ik}(u) + K_{i,k}(u),$$

whereas, for $i = 1$, $k \neq 1$,

$$f_{1,k}(u) = \sum_{j\neq 1} c_{1j,k}(u_j - u_1) + c_{1k}(u).$$

From Assumption 8 it follows that for $\tau > 0$ sufficiently small $f_{i,k}(u) > 0$. Thus the monotonicity condition (see Assumption 2 point 1) is satisfied. Also the other points of Assumption 2 are satisfied. To prove it we must examine the roots of the system (2) and the structure of eigenvalues of Df at these roots. First, using the fact that the terms $K_i(u)$ were assumed sufficiently small, we will analyze the solutions to the simplified system of the form:

$$\begin{aligned} F_1(u_1) + \sum_{j\neq 1} c_{1j}(u)(u_j - u_1) &= 0, \\ \sum_{j\neq i} c_{ij}(u)(u_j - u_i) &= 0, \end{aligned} \tag{20}$$

where $i = 2,\dots,n$.

Lemma 10. *System* (20) *has only three solutions:* $\mathbf{0}$, $\mathbf{1}$ *and* $(U_0,\dots,U_0)$.

Proof. Adding the equations and using the symmetry $c_{ij} = c_{ji}$, we obtain:

$$F_1(u_1) = 0. \tag{21}$$

Hence the first component of the solution to system (20) is equal to one of the solutions to equation (21). The set of $n-1$ equations for $i = 2,\dots,n$ can be written in the form:

$$\mathcal{N}_{n-1}(u_2,\dots,u_n)^T = -u_1(c_{21}(u),\dots,c_{n1}(u))^T, \tag{22}$$

where

$$\mathcal{N}_{n-1} = \begin{pmatrix} -\sum_{j\neq 2} c_{2j}(u) & c_{23}(u) & \dots & c_{2n}(u) \\ c_{32}(u) & -\sum_{j\neq 3} c_{3j}(u) & \dots & c_{3n}(u) \\ \dots & \dots & \dots & \dots \\ c_{n2}(u) & c_{n3}(u) & \dots & -\sum_{j\neq n} c_{nj}(u) \end{pmatrix}.$$

Consider an auxiliary matrix arising from $\mathcal{N}_{n-1}$ by rejecting from the diagonal sums the terms c_{i1}, i.e.

$$\begin{pmatrix} -\sum_{j\neq 1,2} c_{2j}(u) & c_{23}(u) & \dots & c_{2n}(u) \\ c_{32}(u) & -\sum_{j\neq 1,3} c_{3j}(u) & \dots & c_{3n}(u) \\ \dots & \dots & \dots & \dots \\ c_{n2}(u) & c_{n3}(u) & \dots & -\sum_{j\neq 1,n} c_{nj}(u) \end{pmatrix}.$$

The Perron–Frobenius eigenvalue of this matrix (see e.g. [7], [3]) is equal to 0, whereas the eigenvector corresponding to this eigenvalue is equal to $(1,\dots,1)$. By using Lemma 3 in [8] we infer that all the eigenvalues of $\mathcal{N}_{n-1}$ will be negative, hence $\det \mathcal{N}_{n-1} \neq 0$. Thus system (22), for a given u_1 has exactly one solution. It is equal to $(u_1,\dots,u_1)$, where u_1 satisfies the equation $F_1(y) = 0$. The lemma is proved. □

Now, let us find the structure of eigenvalues of $Df(\widetilde{u})$ for $\tau = 0$ and $\widetilde{u}$ equal to $(0,\dots,0)$, $(1,\dots,1)$ and $(U_0,\dots,U_0)$. For $\tau = 0$, $Df(\widetilde{u})$ has the form:

$$Df(\widetilde{u}) = \begin{pmatrix} F_1'(\widetilde{u}_1) - \sum_{j\neq 1} c_{1j}(\widetilde{u}) & c_{12}(\widetilde{u}) & \dots & c_{1n}(\widetilde{u}) \\ c_{21}(\widetilde{u}) & -\sum_{j\neq 2} c_{2j}(\widetilde{u}) & \dots & c_{2n}(\widetilde{u}) \\ \dots & \dots & \dots & \dots\dots \\ c_{n1}(\widetilde{u}) & c_{n2}(\widetilde{u}) & \dots & -\sum_{j\neq 1,n} c_{nj}(\widetilde{u}) \end{pmatrix}.$$

(Note that the terms proportional to $c_{i,k}(u)(\widetilde{u}_i - \widetilde{u}_j)$ vanish.) Let us consider the matrix:

$$\begin{pmatrix} -\sum_{j\neq 1} c_{1j}(\widetilde{u}) & c_{12}(\widetilde{u}) & \dots & c_{1n}(\widetilde{u}) \\ c_{21}(\widetilde{u}) & -\sum_{j\neq 2} c_{2j}(\widetilde{u}) & \dots & c_{2n}(\widetilde{u}) \\ \dots & \dots & \dots & \dots\dots \\ c_{n1}(\widetilde{u}) & c_{n2}(\widetilde{u}) & \dots & -\sum_{j\neq 1,n} c_{nj}(\widetilde{u}) \end{pmatrix}.$$

As before one notes that the Perron–Frobenius eigenvalue of this matrix is equal to 0, whereas the eigenvector corresponding to this eigenvalue is equal to $(1,\dots,1)$. Thus by means of Lemma 3 in [8] we have proved the following lemma.

Lemma 11. *For $\tau = 0$, all the eigenvalues of $Df(\widetilde{u})$ have their real parts smaller than zero, if $F_1'(\widetilde{u}_1) < 0$ and larger than zero, if $F_1'(\widetilde{u}_1) > 0$.*

Lemma 10 and the implicit function theorem imply the following lemma.

Lemma 12. *Assume that the function $F_1(u_1)$ has exactly three zeros:* **0**, **1** *and $u_0 \in (0,1)$. Then the only solutions to systems* (2) *with f given by* (17) *are* $(0,\dots,0)$, $(\overline{u}_1,\dots,\overline{u}_n) = (1,\dots,1) + O(\tau)$ *and* $(\widehat{u}_1,\dots,\widehat{u}_1) = (u_0,\dots,u_0) + O(\tau)$.

By means of this lemma and the fact that the eigenvalues of a matrix depend continuously on parameters we may prove the lemma corresponding to Lemma 11.

Lemma 13. *For τ sufficiently small, all the eigenvalues of $Df(\widetilde{u})$, for $\widetilde{u}$ equal to one of the solutions to system (2), have their real parts smaller than zero, if $F_1'(\widetilde{u}_1) < 0$ and larger than zero, if $F_1'(\widetilde{u}_1) > 0$.*

By the linear change of variables $u_i \to (\overline{u}_i)^{-1} u_i$ the largest root of system (17) becomes equal to $(1, \dots, 1)$ and the intermediate one changes to $(u_{01}, \dots, u_{0n})$.

Now, we will construct a homotopy satisfying Assumption 5. We divide this homotopy into three stages.

(1) $\lambda \in [2/3, 1]$. Let $\widetilde{c}_{ij} = \min_{u \in [-1,2]} c_{ij}(u)$, $i, j \in \{1, \dots, n\}$ and let

$$c_{ij}(u) = \widetilde{c}_{ij} + c_{ij}^*(u).$$

Let

$$G_{\lambda i} = F_i(u) + 3\left(\lambda - \frac{2}{3}\right) K_i(u) + \sum_{j \neq i} h_{\lambda ij}(u)(u_j - u_i), \tag{23}$$

where $F_i(u) \equiv 0$ for $i \in \{2, \dots, n\}$ and

$$h_{\lambda ij}(u) = \widetilde{c}_{ij} + 3\left(\lambda - \frac{2}{3}\right) c_{ij}^*(u).$$

(2) $\lambda \in [1/3, 2/3]$.

$$G_{\lambda i} = \sum_{j \neq i} \left[3\left(\frac{2}{3} - \lambda\right) H + 3\left(\lambda - \frac{1}{3}\right) \widetilde{c}_{ij} \right] (u_j - u_i) \tag{24}$$

where $H > 0$ is *sufficiently large.*

(3) $\lambda \in [0, 1/3]$.

$$G_{\lambda i} = \sum_{j \neq i} H(u_j - u_i) + 3\left[\left(\frac{1}{3} - \lambda\right) + \lambda \delta_{i1} \right] F_1(u_i). \tag{25}$$

It is obvious that for $\lambda \in [1/3, 1]$ Assumption 2 is satisfied. Also points (a) and (b) of Assumption 2 are satisfied for $\lambda \in [0, 1/3]$. We will show that point (c) is satisfied too. As before we can replace system (2) by the system:

$$F_1(u_1) + 3\left(\frac{1}{3} - \lambda\right)(F_1(u_2) + \ldots + F_1(u_n)) = 0, \tag{26}$$

$$\begin{aligned} \mathcal{N}_{n-1}^*(u_2, \dots, u_n)^T = &-u_1(1, \dots, 1)^T \\ &- H^{-1} 3\left(\frac{1}{3} - \lambda\right)(F_1(u_2), \dots, F_1(u_n))^T, \end{aligned} \tag{27}$$

where $\mathcal{N}_{n-1}^*$ is an $(n-1) \times (n-1)$ matrix:

$$\mathcal{N}_{n-1}^* = \begin{pmatrix} -(n-1) & 1 & \dots & 1 \\ 1 & -(n-1) & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & -(n-1) \end{pmatrix}.$$

Let us note that $|\det \mathcal{N}^*_{n-1}| = n^{n-2}$. Hence due to the implicit function theorem for $H > 0$ sufficiently large and given the right hand sides there exists a unique solution $(u_2, \ldots, u_n)$ of system (27). This solution is equal to $(u_1, \ldots, u_1) + 3(1/3 - \lambda)O(H^{-1})$. Putting this relation into (26) we obtain $F_1(u_1) + (1/3 - \lambda)\sum_{i\neq 1} F_1(u_i) = F_1(u_1)(1 + (n-1)(1/3-\lambda)) + (n-1)(1/3-\lambda)O(H^{-1}) = (n-1)(1/3-\lambda)O(H^{-1})$. By the use of the implicit function theorem we conclude that for every solution $(u_1, \ldots, u_n)$ to (26) u_1 is equal to one of the states 0, u_0, 1, plus $O(H^{-1})$ terms. Hence in system (27) $(F_1(u_2), \ldots, F_1(u_n))^T = (0, \ldots, 0)^T + O(H^{-1})$. This implies that $u_i = u_1 + O(H^{-2})$, $i \in \{2, \ldots, n\}$. Now, we may succesively repeat the procedure, to conclude that $u_i = u_1 + O(H^{-k})$ for any natural k. This implies that $u_i = u_1$, $i \in \{2, \ldots, n\}$ for all $\lambda \in [0, 1/3]$. Thus Assumption 2 is satisfied for all $\lambda \in [0, 1]$.

Consequently using Theorem 1 we can state the following result.

Theorem 2. *Suppose that all the functions in system* (19) *are sufficiently smooth and that Assumption* 6–9 *are fulfilled. Then there exists $q^* \in \mathbb{R}^1$ such that for $q = q^*$ system* (19) *has a strictly monotone heteroclinic solution joining the states* $\mathbf{0}$ *and* $\mathbf{1}$.

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ALGORITHM FOR DERIVING HOMOTOPY MINIMAL PERIODS OF NILMANIFOLD AND SOLVMANIFOLD MAP

Rafał Komendarczyk and Wacław Marzantowicz

Abstract. A natural number m is called the homotopy minimal period of a selfmap $f\colon X \to X$ if it is a minimal period for every map g homotopic to f. In particular this invariant is stable for small perturbations of f. We present a survey of recent theory describing the set $\mathrm{HPer}\,(f)$ of homotopy minimal periods of a map of compact nilmanifold or exponential solvmanifold of dimension d. The first step of this construction is so called linearization of f, well-know for tori, in which a $d \times d$ integral matrix A_f is assigned to f. In this paper we present the background that is necessary to set up a computational procedure with the matrix A_f in the input and the set $\mathrm{HPer}\,(f)$ as the output. A computer implementation of this algorithm is written as a "Mathematica" notebook.

1. Introduction

The famous Šarkowski theorem characterizes the dynamics (minimal periods) of a map of interval [25]. The set of minimal periods of self-maps of the circle has been completely described by Block at al. [3] (see also [6]) in the terms of degree of map. This led to a natural problem of study the homotopy minimal periods of self-map $f\colon X \to X$ i.e. these periods which are also minimal periods for every map g homotopic to f. Since the homotopy minimal period of a manifold map f preserves under a small perturbation, one can say that homotopy minimal

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periods give an information about a rigid dynamics of f. A natural question: to give a complete determination of the set $\mathrm{HPer}\,(f)$ of all homotopy minimal periods in terms of the homological information on f was investigated by several authors.

After the case of maps of the circle ([3]) in the second instance maps of two-dimensional torus ($X = T^2$) has been investigated in a series of papers ([2]), where the Nielsen theory was first time used to study this problem. Next in the work of Jiang and Llibre [14] the problem was successfully studied for maps of d-dimensional torus, $d \geq 1$. Recently and Jezierski and Marzantowicz [9] shown that the analogous complete description of the set $\mathrm{HPer}\,(f)$ is possible for a map of a compact nilmanifold.

It is done by usage of the Nielsen theory, which for the torus maps has very nice algebraic description ([2], [5]) and prepossessing geometric properties ([12], [5], [8], [15], [22], [23]). Making use of these geometric properties Jiang and Llibre [14] could apply the Anosov and summation formulas

$$N(f) = |L(f)|, \quad N(f^m) = k \mid m \sum NP_k(f), \tag{1}$$

where $N(f)$ is the Nielsen number, $NP_k(f)$ is the k-periodic Nielsen number, and $L(f)$ is the Lefschetz number of f (see also [15]), to describe the set $\mathrm{HPer}\,(f)$. The proof employs very subtle and difficult combinatorial, and number theory, arguments, which carry over the case of a nilmanifold map ([9]).

It is worth of pointing out that the Anosov and summation formulas do not hold for every solvmanifold (cf. [15]). However Keppelmann and McCord observed that these formulas remain true for a map of exponential solvmanifolds i.e. quotient homogenous spaces G/Γ with the covering simply-connected Lie group G for which the exponent map is onto ([15]). This fact let us to show [9] that for a map of a compact solvmanifold (being the quotient G/Γ of a simple-connected completely solvable Lie group G by a lattice Γ) still holds an analogous theorem to the main theorems of [14] and [10]. This class of solvmanifolds is included into the exponential solvmanifolds.

In more detail, let $f\colon X \to X$ be a map of a compact exponential solvmanifold X, (thus of a compact nilmanifold a particularly of a torus) and $A = A_f$ be the integral matrix which corresponds to a given map f by the Fadell–Husseini and Mostow fibration (see [15] or [10] for a definition of A_f in the nilponent case and [15] in the case of exponential solvmanifold.

The most important property of A says that for every map $f\colon X \to X$ we have (cf. [15], [10])

$$L(f) = \det(\mathrm{I} - A), \quad \text{thus } L(f^m) = \det(\mathrm{I} - A^m) \tag{2}$$

for every $m \in \mathbb{N}$.

Let $T_A := \{m \in \mathbb{N} : \det(\mathrm{Id} - A^m) \neq 0\}$. We have (see [14, Theorem A], [10, Theorem A], or [9]):

Theorem 1.1.

(i) *Either* $L(f) = N(f) = 0$ (*then* $\mathrm{HPer}\,(f) = \emptyset)\}$,

(ii) or $N(f) \neq 0$ and the sequence $\{N(f^m)\}$ is bounded, then $\mathrm{HPer}\,(f)$ is finite and its cardinality depends on the size of A (= dimension of $X)\}$ only,

(iii) or $\{N(f^m)\}$ is unbounded then T_A is infinite, $\mathrm{HPer}\,(f) \subset T_A$, the set $T_A \setminus \mathrm{HPer}\,(f)$ is finite and its cardinality depends on the size of A (= dimension of $X)\}$ only.

The basic combinatorial argument of the Boju Jiang and Llibre proof of this theorem was the following observation ([14, Theorem 2])

Let $f\colon T^d \to T^d$ be a map of the torus. Suppose that $N(f) \neq 0$. Then a natural number $m \in T_A$ *does not belong* to $\mathrm{HPer}\,(f)$ if and only if there exist

$$p \mid m, \quad p \text{ prime} : N(f^{m/p}) = N(f^m). \tag{3}$$

This statement extends onto the case of a map of nilmanifold or exponential solvmanifold X (cf. [10], [9]).

The algebraic number theory argument of the Boju Jiang and Llibre proof of the main theorem says that the equality (3) could happen only for finitely many $m \in [1, m_0] \subset \mathbb{N}$ and the constant m_0 depends on d (the size of A = the dimension of X) only. In other words there exists a constant $m_0(d)$ such that for every map $f\colon X \to X$ and every $m \geq m_0$ we have $N(f^m) > N(f^{m/p})$ for every prime divisor p of m.

The above let us to construct an algorithm which checks whether a given natural number is a homotopy minimal period of a given map of a manifold from the discussed class. Consequently it gives the set $\mathrm{HPer}\,(f) \subset T_A \subset \mathbb{N}$. The starting point of the procedure is the integral $d \times d$ matrix A_f of the linearization of f.

The paper can be outlined as follows. In Section 1 we remind the information about the definition of the linearization A_f of a map f of a compact nilmanifold or exponential solvmanifold. We must emphasize that it is a topological part of consideration and there is not any algorithm for it. We present different definitions of this notion and give some examples.

Section 2 is devoted to a discussion of the effectiveness of computation of the constant $m_0(d)$ which appears in the mentioned Boju Jiang and Llibre theorem. We quote some known results of number theory. Finally we note that to describe the set $\mathrm{HPer}\,(f)$, for a given f it is enough to find $\breve{m}(f)$ such that for every $m \geq m(f)$ we have $N(f^m) > N(f^{m/p})$ for every prime divisor p of m. The constant $\breve{m}(f) = \breve{m}(A_f)$ is not universal with respect to f but unquestionably smaller then $m_0(d)$ of [14] and effectively estimated in the terms of spectral radius of A_f. Finally we discuss the class of integral matrices which spectrum intersect the unit circle at the nontrivial roots of unity only (Assumption 5).

Under this we are able to show that there exists $\widetilde{m}(A) \ll \breve{m}(A)$ also effectively given in terms of spectral invariants of A (Theorem 3.5, Corollary 3.6) with the same property as $\breve{m}$.

In Section 4 we describe an implementation of the computational procedure which for a given matrix $A \in M_{d\times d}(\mathbb{Z})$ derive the set $\mathrm{HPer}\,(f) \subset T_A$ by excluding these $m \in T_A \subset \mathbb{N}$ for which equality 3 happens for some $p \mid m$. The estimate of $\breve{m}(A)$, or $\widetilde{m}(A)$ if Assumption 5 is satisfied, let us stop the checking process at this point. The implementation is written as a "Mathematica" notebook.

In the last Section 5 we verify our program comparing its output with tabled lists of all minimal periods different then $\mathbb{N}$ of the three-dimensional torus T^3 of the paper [14]. The same we do for non-abelian three-dimensional nilmanifold comparing with the results of [11] and endowing with patterns of matrices corresponding given cases.

2. The linearization of a map

Reminding a standard terminology, let $f\colon X \to X$ be a self-map of a compact connected polyhedron X, and let n be a natural number. Let $\mathrm{Fix}\,(f)$ be the fixed point set of f, $P^n(f) := \mathrm{Fix}\,(f^n)$, and

$$P_n(f) := P^n(f) \setminus \bigcup_{k<n} P^k(f),$$

the set of periodic points with least period m.

Recall that $\mathrm{Per}\,(f)$ denotes the set of all minimal periods of f i.e. $\mathrm{Per}\,(f) := \{k \in \mathbb{N};\ P_k(f) \neq \emptyset\}$. When a map $g : X \to X$ is homotopic to f, we will write $g \simeq f$. Define the *set of homotopy minimal periods* as the set

$$\mathrm{HPer}\,(f) := \bigcap_{g\simeq f} \mathrm{Per}\,(g).$$

Boju Jiang and Llibre use the name "the minimal set of periods" but we hope that the one we use here more emphasizes that $n \in \mathrm{HPer}\,(f)$ if and only if n is a minimal period for every g homotopic to f.

Homogeneous spaces of nilpotent Lie groups are called nilmanifolds. A compact manifold X is a nilmanifold if and only if it is of the form G/Γ where G is a simply-connected nilpotent Lie group of dimension d and Γ is a lattice of rank d of G i.e. a discrete, torsion free, subgroup of G of rank d [18]. Since G is homeomorphic to the Euclidean space $\mathbb{R}^d$, $X = G/\Gamma$ is $K(\Gamma, 1)$ space and moreover the nilmanifold $X = G/\Gamma$ is determined by Γ, up to diffeomorphism.

We would like to remind that the simplest non-trivial examples of compact nilmanifolds are *Iwasawa manifolds* $\mathcal{N}_n(\mathbb{R})/\mathcal{N}_n(\mathbb{Z})$ and $\mathcal{N}_n(\mathbb{C})/\mathcal{N}_n(\mathbb{Z}[\imath])$, where $\mathbb{Z}[\imath]$ is the ring of Gaussian integers and for any ring $\mathcal{R}$ with unity, $\mathcal{N}_n(\mathcal{R})$ denotes the group of all unipotent upper triangular matrices whose entries are elements of the ring $\mathcal{R}$.

For the next we need a definition of nilpotent class in the sense of Fadell–Husseini ([5]). Let $\mathcal{N}$ denote a class of compact connected manifolds satisfying the following conditions:

(1) $\mathcal{N}$ contains all tori (products of circles)
(2) For any map $g\colon X \to X$, where $X \in \mathcal{N}$ is not a torus, there is a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{f_0} & T \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\overline{f}]{} & B \end{array}$$

where p is a principal T-fibration, T a torus, $B \in \mathcal{N}$ and $f \simeq g$.

We call a class of manifolds $\mathcal{N}$ satisfying N.1 and N.2 a nilpotent class.

Fadell and Husseini showed that the class of compact nilmanifolds is a nilpotent class ([7, Theorem 3.3]). This allows to prove the following theorem (cf. [17]).

Theorem 2.1. *Let $f\colon X \to X$ be a map of a compact nilmanifold X of dimension d. Then there exists a $d \times d$ matrix A with integral coefficients such that, for every $n \in \mathbb{N}$, $L(f^n) = \det(1 - A^n)$.*

Proof (see also [8], [15]). By [4] the proposition holds for tori. We may take as A the $n \times n$ matrix representing the induced endomorphism of the fundamental group. Now we prove the general case by induction. Suppose that our thesis holds for all nilmanifolds of dimension $< d$.

If a nilmanifold M^d is a torus then the statement holds by the above. Otherwise by [5] there exists a principal torus bundle $T^k \hookrightarrow X^d \xrightarrow{p} B$ where B is a nilmanifold of dimension $d-k < d$. Moreover any self-map $f\colon X^d \to X^d$ is homotopic to a fibre map $g\colon X^d \to X^d$ i.e. to a map such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\overline{g}]{} & B \end{array}$$

The map g could be obtained as follows.

(1) Since X is the $K(\Gamma, 1)$ space, homotopy classes of of selfmaps are in 1–1 correspondence with endomorphisms of Γ we take $f_*\colon \pi_1(X) = \Gamma \to \Gamma$ the induced homomorphism, which we denote by ϕ.
(2) Since G is a nilpotent group and $\Gamma \subset G$ a lattice, every homomorphism $\phi\colon \Gamma \to \Gamma$ extends (uniquely) to the homomorphism $\Phi\colon G \to G$.

(3) The homomorphism $\Phi: G \to G$ preserves the subgroup Γ, also the commutator $G_1 := [G,G]$, thus every term $G_i := [G, G_{i-1}]$, $G_n = e$, of the nilpotent central tower, and consequently every subgroup $\Gamma_i := G_i \cap \Gamma$.
(4) From (3) it follows that $\Phi: G \to G$ induces maps $[\Phi]: X = G/\Gamma \to G/\Gamma = X$, and also G_i/Γ_i, e.g. $[\Phi] : Z(G)/Z(\Gamma) = G_{n-1}/\Gamma_{n-1}$.
(5) By the construction $[\Phi]: X \to X$ induces the homomorphism f_* on the fundamental group, consequently we can take $[\Phi]$ as g.

Using the product property of Lefschetz number of a fibre map of the principal bundle we may assume that $\overline{g}$ has a fixed point b_0. By the induction assumption there exists a matrix $\overline{A} \in \mathcal{M}_{(d-k)\times(d-k)}(\mathbb{Z})$ such that $L(\overline{g}^n) = \det(\mathrm{I} - \overline{A}^n)$. Similarly let $A' \in \mathcal{M}_{k\times k}(\mathbb{Z})$ be a matrix satisfying $L(f_{b_0}^n) = \det(\mathrm{I} - (A')^n)$ where $f_{b_0}: T^k \to T^k$ denotes the restriction of f to the fibre $T^k = p^{-1}(b_0)$.

Since $L(f_{b_0})$ is independent on b_0, once more using the product formula we have

$$L(f^n) = L(\overline{f}^n)L((f')^n) = \det(\mathrm{I} - \overline{A}^n)\det(\mathrm{I} - (A')^m) = \det(\mathrm{I} - A^m),$$

where

$$A := \begin{bmatrix} \overline{A} & 0 \\ 0 & A' \end{bmatrix} \in \mathcal{M}_{r\times r}(\mathbb{Z}) \qquad \square$$

For given $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$ we set $T_A := \{\, n \in \mathbb{N} \mid \det(\mathrm{I} - A^n) \neq 0\}$. If $A = A_f$ is a matrix associated to a selfmap $f: X \to X$ of a compact nilmanifold X then we call T_A *the set of algebraic periods* of f.

Due the Nomizu and Hattori ([17], [7]) theorems the linearization matrix A_f can be constructed also in a differential way. Let $V = T_e$ be the tangent space to G at e, V^* its dual i.e. the cotangent space, $\Lambda\mathcal{G} := (\sum_0^d \wedge^i V^*, d)$ the complex of left invariant differential forms on G, called Eilenberg–Chevalley complex, with the differential d defined by the Lie bracket on $\mathcal{G}$. Now we would like to give a differential construction of the matrix A_f for completely solvable solvmanifolds.

We start with the following result of Hattori ([7]) generalizing the earlier result of Nomizu ([17]) for nilmanifolds.

Theorem 2.2. *Let $(\Lambda^*\mathcal{G}^*, \delta)$ denote the Eilenberg–Chevalley complex associated to the Lie algebra $\mathcal{G}$ of a simply connected completely solvable Lie group G. If $\Gamma \subset G$ is a lattice, then $H^*(G/\Gamma;\mathbb{R}) \cong H^*(\Lambda^*\mathcal{G}^*, \delta)$.*

On the other hand we have the following property of the compact completely solvable solvmanifolds.

Proposition 2.3. *Every continuous map $f: G_1/\Gamma_1 \to G_2/\Gamma_2$ between completely solvable solvmanifolds is homotopic to the map induced by a homomorphism $F: G_1 \to G_2$.*

Let G/Γ be a completely solvable compact solvmanifold $f: G/\Gamma \to G/\Gamma$ a map and $F: G \to G$ the homomorphism given by Proposition 2.3.

Let next A denotes a matrix representing the map $F^*\colon \mathcal{G}^* \to \mathcal{G}^*$ induced by F on the Lie algebra $\mathcal{G}$.

Theorem 2.4. *Let $f\colon G/\Gamma \to G/\Gamma$ be a self map of a compact completely solvable solvmanifold of dimension d. Then $d\times d$-matrix A defined above we have*

$$L(f^m) = \det(\mathrm{I} - A^m).$$

Proof. See [9]. □

At the end of this section we formulate a theorem which is rather a reformulation of what was stated before however it elucidates the matter more apparently. As we said every integral matrix $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$ induces a map $[A]\colon T^d \to T^d$, thus every $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$ is a linearization of a selfmap of the torus. On the other hand for a selfmap $f\colon X \to X$ of a compact nilmanifold, or solvmanifold X its linearization A_f has a special form namely it is a direct sum of $d_j \times d_j$, $1 \le j \le r$, blocks A_j, where r is the length of nil, respectively solv-tower and d_j is the dimension the consecutive abelian factor. Observe that to find the set $\mathrm{HPer}\,(f)$ we have to make only algebraic operation with the matrix A_f. This leads to the following theorem which says that the case of torus is the richest one from the point of view of all possible sets of homotopy minimal periods.

Theorem 2.5. *Let $f\colon X \to X$ be a map of a compact nilmanifold, or compact exponential solvmanifold of dimension d, $A_f \in \mathcal{M}_{d\times d}(\mathbb{Z})$ its linearization, and $[A_f]\colon T^d \to T^d$ the map induced by A_f. Then $\mathrm{HPer}\,(f) = \mathrm{HPer}\,([A_f])$. Consequently f belongs to the empty, finite, or generic case if and only if $[A_f]$ belongs to the corresponding case. Subsequently, if the pair of sets $(\mathrm{HPer}\,(f), T_{A_f})$ occurs for a map of X then it occurs for a selfmap of the torus T^d.*

But the above condition $A = \bigoplus_{j=1}^{r} A_j$ is not the only restriction. In [11] it is shown that for a map of three dimensional non-abelian nilmanifold there is additional relation for the matrix A_f. This shows that the variety dynamics of maps, measured by the sets of homotopy minimal periods, is essentially less complicated in this case.

3. Number theory

We begin with a short analysis of the proof of number theoretical part of Theorem 1.1. In [14] it is based on a deep theorem from algebraic number theory proved by the authors. For convenience of the reader we present the statement of this theorem. It is worth to emphasize that due to its character it is not connected with the geometry of the space thus holds as well for the torus as compact nilmanifold as compact exponential solvmanifold.

Let α be an algebraic number. Suppose its minimal polynomial is $a_0x^d + a_1x^{d-1} + \ldots + a_d$ with roots $\alpha_1, \ldots, \alpha_d$. Then d is called the degree of α. The measure of α is defined as

$$M(\alpha) := a_0 \prod_{i=1}^{d} \max\{1, |a_i|\}.$$

The crucial step of the proof of Theorem 1.1 (cf. [14, Theorem 4.2]) is the characterization of an algebraic number.

Theorem 3.1. *For every algebraic number α of degree d and every natural number m such that $\alpha^m \neq 1$, we have the inequality $|1-\alpha^m| > e^{-9\alpha H^2}/2$, where*

$$\begin{aligned} a &= \max\left\{20,\ 12.85|\log\alpha| + \frac{1}{2}\log M(\alpha)\right\}, \\ H &= \max\left\{17,\ \frac{d}{2}\log m + 0.66d + 3.25\right\}. \end{aligned}$$

As a consequence of this estimate Jiang and Llibre derived an estimate for $N(f^m)/N(f^n)$, providing $N(f^m) \neq 0$. Note that then $N(f^n) \neq 0$ as follows from [14, Theorem 1.6(ii)]. More precisely, the final inequality of the proof of Theorem 4.2 of [14] looks as follows. Let ρ be the maximal module of eigenvalues of A i.e. the spectral radius of A. Then

$$\frac{N(f^m)}{N(f^n)} > \frac{\rho^{m/2}-1}{e^{9d(41.4+(\log\rho)/2)(d\log m)^2}}. \tag{4}$$

In [14] it is shown that the right hand side is greater than 1 provided m is larger than $m_0(d)$ depending on d only. Moreover the value of m_0 can be derived in an effective, but complicated way [!].

We turn now to our task of showing how the inequality $N(f^m)/N(f^n) > 1$ follows from the mentioned result of Schinzel. To do this we need new notions and definitions. Let α, β be non-zero integers of an algebraic number field $\mathcal{K}$ of degree d. A prime ideal $\mathfrak{B}$ of $\mathcal{K}$ is called a primitive divisor of $\alpha^m - \beta^m$ if $\mathfrak{B} \mid \alpha^m - \beta^m$, but $\mathfrak{B}$ does not divide $\alpha^n - \beta^n$ if $n < m$. In 1974 A. Schinzel proved the following theorem (cf. [19, Theorem I]).

Theorem 3.2. *If $(\alpha,\beta) = 1$ and α/β is not a root of unity then $\alpha^m - \beta^m$ has a primitive divisor for all $m > \widetilde{m}_0(d)$, where d is the degree of α/β and $\widetilde{m}_0(d)$ is effectively computable.*

Let $N(f^m) = |\det(A^m - \mathrm{I})|$, $A \in M_{d\times d}(\mathbb{Z})$. For every eigenvalue $\lambda_j \in \sigma(A)$, $1 \le j \le d$, take $\alpha_j := \lambda_j$ and $\beta_j := 1$. By the definitions α_j, β_j are integers of the algebraic field given by the characteristic polynomial of A. If $m \in T_A$ i.e. $N(f^m) \neq 0$ then the hypothesis of the Schinzel theorem is satisfied. Note that if $n \mid m$, $k = m/n$, then

$$(\lambda^m - 1) = (\lambda^n - 1)[1 + \lambda^n + \lambda^{2n} + \ldots + \lambda^{(k-1)n}].$$

Consequently, for any $1 \leq j \leq d$ such that $|\lambda_j| > 1$ and $m \in T_A$, $\widetilde{m} > m_0(d_j)$ there exists a primitive ideal $\mathfrak{B}_j \subset \mathcal{K}$ such that $\mathfrak{B}_j \mid [1+\lambda_j^n+\lambda_j^{2n}+\ldots+\lambda_j^{(k-1)n}]$ as follows from the Schinzel theorem. Observe also that $d(\alpha_j) = d(\lambda_j)$ is a divisor of $d := \operatorname{degree}\mathcal{K}$ and $\widetilde{m}_0(d_j) \leq m_0(d)$, by an argument of proof of Theorem 3.4 contained in [19]. From this it follows that

$$\mathfrak{B}_j \mid N(f^m)/N(f^n) = \prod_1^j [1 + \lambda_j^n + \lambda_j^{2n} + \ldots + \lambda_j^{(k-1)n}], \quad \text{for every } 1 \leq j \leq d.$$

The above implies that then there exists a prime $q \in \mathcal{P} \subset \mathbb{N}$ such that q divides $N(f^m)/N(f^n)$ provided $m \in T_A$, $m > \widetilde{m}_0(d)$. Indeed it is enough to take $q \in \mathcal{P}$, where q^{f} is the norm $|\mathfrak{B}_j|$ of the ideal $\mathfrak{B}_j$, since $\mathfrak{B}_j \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. This shows that $N(f^m)/N(f^n) > 1$ and consequently proves Theorem 1.10.

It is worth of pointing out that either the proof of referred Schinzel theorem 3.2 or the Jiang Llibre theorem 3.1 are based on the Baker inequality (see [19] for more details).

We would like to point out that the problem of an effective estimate of $\widetilde{m}_0(d)$ is connected with so called Lucas-Lehmer numbers (see [20] for details).

On the other hand to exclude these numbers $m \in T_A$ which are not the homotopy minimal periods of f with $A_f = A$ we do not need to work with the mentioned constant $m_0(d)$, or $\widetilde{m}_0(d)$ but use the inequality 4.

Corollary 3.3. *Let $f\colon X \to X$ be a map of a compact nilmanifold or compact exponential solvmanifold X of dimension d and $A = A_f \in \mathcal{M}_{d\times d}$ its linearization. Let next $\rho \in \mathbb{R}$ be its spectral radius. Suppose that $\breve{m}_0(A)$ is the smallest number for which*

$$\frac{\rho^{m/2} - 1}{e^{9d(41.4+(\log\rho)/2)(d\log m)^2}} > 1.$$

Then $T_A \setminus \mathrm{HPer}\,(f) \subset [1, \breve{m}_0(A)]$ and $m \in T_A \setminus \mathrm{HPer}\,(f)$ if and only if there exists prime $p \mid m$ such that

$$N(f^m) = |\det(\mathrm{I} - A^m)| = |\det(\mathrm{I} - A^{m/p})| = N(f^{m/p}).$$

Remark 3.4. In other words for a matrix $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$ to determine the set $\mathrm{HPer}\,(f) \subset \mathbb{N}$ we check all ratios $N(f^m)/N(f^{m/p})$ for $m \in T_A$ $m \leq \breve{m}(\rho, d)$, $\rho = \mathrm{sp}\,(A)$ the spectral radius, p-prime, $p \mid m$.

Note that the number $\breve{m}$ is also large in general, which could lead to computational problems. If the spectrum of A intersected with the unit circle consists of the roots of unity only, in particular if it is separated from the unit circle, then the estimate is simpler. This let us to derive a constant $\widetilde{m}(f) = \widetilde{m}(A_f) < \breve{m}$ essentially smaller than $\breve{m}$ such that $T_A \setminus \mathrm{HPer}\,(f) \subset [1, \widetilde{m}]$. To do it we need new notions.

Let A be $d \times d$ integral matrix. Assume that $\mathrm{sp}\,(A) := \rho > 1$ and that

$$\sigma(A) \cap \{|z| = 1\} \subset \mathcal{C}_{q_1} \cup \ldots \cup \mathcal{C}_{q_r} \subset \mathcal{C}, \tag{5}$$

where $\mathcal{C}_q \subset S^1$ denote the set of roots from unity of the order q.

First we put an order in $\sigma(A)$ such that

$$\begin{aligned} \rho &= |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_r| > 1 \\ &= |\lambda_{r+1}| = \ldots = |\lambda_{r+s}| > |\lambda_{r+s+1}| \geq \ldots \geq |\lambda_{r+s+t}|. \end{aligned}$$

where $r+s+t = d$. We denote the set $\{\lambda_1, \ldots, \lambda_r\}$ by $\sigma_{>1}(A)$, the set $\{\lambda_{r+1}, \ldots, \lambda_{r+s}\}$ by $\sigma_1(A)$, and $\{\lambda_{r+s+1}, \ldots, \lambda_{r+s+t}\}$ by $\sigma_{<1}(A)$ respectively.

Since $|\lambda_j| > 1$, for $1 \leq j \leq r$, there exists $n(j) \geq 1$, such that $|\lambda_j|^{-n(j)} \geq 2$. Put $\widetilde{n} := \max_{1 \leq j \leq r} n(j)$. Note that $\widetilde{n} = n(r)$.

Set $\widetilde{\rho} := \prod_{1 \leq j \leq r} |\lambda_j|$. Next put $\overline{\rho} := |\lambda_r|$ the minimal absolute value of eigenvalues of A which are greater than 1.

Let $\delta_q > 0$ be the length of the side of the regular q-polygon in the unit circle i.e. $\delta_q = 2\sin(\pi/q)$. Let next $q_0 := \max_{r+1 \leq j \leq r+s} q_j$ and $\delta = \delta(A) := \delta_{q_0}$.

Finally, since $|\lambda_j| < 1$, for $r+s+1 \leq j \leq r+s+t$, there exists $k(j) \geq 1$, such that $|\lambda|^{k(j)} \leq 1/2$. Put $\widetilde{k} := \max_{r+s+1 \leq j \leq r+s+t} k(j)$. Note that $\widetilde{k} = k(r+s+1)$.

Theorem 3.5. *Let $f\colon X \to X$ be a map of a compact nilmanifold or exponential solvmanifold X of dimension d and $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$ be its linearization. Suppose that the spectral radius $\rho = \mathrm{sp}\,(A) > 1$ and*

$$\sigma(A) \cap \{|z| = 1\} \subset \mathcal{C}_{q_1} \cup \ldots \cup \mathcal{C}_{q_r} \subset \mathcal{C}.$$

Let $\widetilde{\rho}$, $\widetilde{n}$, $\delta(A)$, and $\widetilde{k}$ be the constants defined above. If $\widetilde{m}$ is the smallest natural number $m \geq \max\{\widetilde{n}, \widetilde{k}\}$ such that $\widetilde{\rho}^{\,m/2} > 2^d 2^r 2^t/\delta^s$ then $T_A \setminus \mathrm{HPer}\,(f) \subset [1, \widetilde{m}]$. The same holds if we take $\widetilde{m}$ is the smallest natural number $\widetilde{m} \geq \max\{\widetilde{n}, \widetilde{k}\}$ such that $\widetilde{\rho}^{\,\widetilde{m}/2} > 2^{2d}/\delta^s$.

Proof. We estimate the value

$$N(f^m) = |L(f^m)| = |\det(\mathrm{I} - A^m)| = \Big| \prod_{\lambda_j \in \sigma(A)} (\lambda_j^m - 1) \Big| \tag{6}$$

from above and below.

First note that the spectral radius of $\wedge^* A$, the map induced by A on the real cohomology of torus, is equal to $\widetilde{\rho} = |\lambda_1| \ldots |\lambda_r|$. Consequently,

$$|L(f^m)| \leq \widetilde{\rho}^{\,m} \operatorname{rank} H^*(T^d) = \widetilde{\rho}^{\,m} 2^d. \tag{7}$$

To give an estimate from below we split our multiply 6 into three factors

$$\begin{aligned} |L(f^m)| &= \Big| \prod_{\lambda_j \in \sigma_{>1}(A)} (\lambda_j^m - 1) \Big| \Big| \prod_{\lambda_j \in \sigma_1(A)} (\lambda_j^m - 1) \Big| \Big| \prod_{\lambda_j \in \sigma_{<1}(A)} (\lambda_j^m - 1) \Big| \\ &:= \Pi_1 \Pi_2 \Pi_3. \end{aligned} \tag{8}$$

We have

$$\tag{9} \begin{aligned} \Pi_1 &= \Big|\prod_{\lambda_j\in\sigma_{>1}(A)} (\lambda_j^m - 1)\Big| \geq \prod_{\lambda_j\in\sigma_{>1}(A)} (|\lambda_j|^m - 1) \\ &= \prod_{\lambda_j\in\sigma_{>1}(A)} |\lambda_j|^m \bigg(1 - \frac{1}{|\lambda_j|^m}\bigg) \geq \frac{\widetilde{\rho}^{\,m}}{2^r} \end{aligned}$$

if $m \geq \widetilde{n}$. Next note that for every $m \in T_A$ we have

$$\tag{10} \Pi_2 = \Big|\prod_{\lambda_j\in\sigma_1(A)} (\lambda_j^m - 1)\Big| = \prod_{\lambda_j\in\sigma_1(A)} |\lambda_j^m - 1| \geq \delta^s,$$

because here every λ_j is a root of unity and $\lambda_j^m \neq 1$. Finally

$$\tag{11} \Pi_3 = \Big|\prod_{\lambda_j\in\sigma_{<1}(A)} (\lambda_j^m - 1)\Big| \geq \prod_{\lambda_j\in\sigma_{<1}(A)} (1 - |\lambda_j|^m) \ \geq \frac{1}{2^t}$$

if $m \geq \widetilde{k}$. Combining (9), (10) and (11) we get the estimate from below

$$\tag{12} |L(f^m)| = |\det(\mathrm{I} - A^m)| \geq \widetilde{\rho}^{\,m} \frac{\delta^s}{2^r 2^t},$$

if $m \in T_A$ and $m \geq \max\{\widetilde{n}, \widetilde{k}\}$.

Suppose that $m \in T_A$ and $m \geq \max\{\widetilde{n}, \widetilde{k}\}$, and $n \mid m$. From (7) and (12) it follows that $|L(f^m)|/|L(f^n)| > 1$ if

$$\tag{13} \frac{\widetilde{\rho}^{\,m}}{\rho^n} \geq \frac{\widetilde{\rho}^{\,m}}{\widetilde{\rho}^{\,m/2}} > \frac{2^d 2^r 2^t}{\delta^s},$$

because $\widetilde{\rho}^m/\rho^n \geq \widetilde{\rho}^m/\rho^{m/2}$ as $n \leq m/2$. This proves the statement with respect to Theorem 1.1.

The second inequality gives stronger requirement on $\widetilde{m}$ than the first, because $r + s + t = d$, and consequently $r + s \geq d$. This completes the proof. □

Corollary 3.6. *Suppose that A satisfies Assumption 5. Then $T_A \backslash \mathrm{HPer}\,(f) \subset [1, \widetilde{m}]$ if $\widetilde{m} \geq \max\{\widetilde{n}, \widetilde{k}\}$ and*

$$\widetilde{m} > \frac{2(d + r + t)\log 2 - 2s\log\delta}{\log\widetilde{\rho}} \quad or \quad \widetilde{m} > \frac{2(d + r + t)\log 2 - 2s\log\delta}{r\log\overline{\rho}}.$$

Proof. Note that $\widetilde{\rho} \geq \overline{\rho}^{\,r}$, by the definition. The statement follows by taking the logarithms of the both sides of inequality of Theorem 3.5. □

Example 3.7. To illustrate the difference between the order of constants $\breve{m}(A)$ of Corollary 3.3 and $\widetilde{m}(A)$ of Corollary 3.6 we derive their approximate values for a 3×3 matrix. Let A be an integral 3×3 matrix i.e. $d = 3$ such that the spectral radius $\rho = 5$, $s = 2$, thus $t = 1$, $r = 1$ and $\overline{\rho} = \rho = 5$.

It is easy to check that $\widetilde{k} = 1$, $\widetilde{n} = 1$, and $q \leq 6$, which implies that $\delta \geq 1$, and consequently $\widetilde{m} = 2$ by the second inequality of Corollary 3.6.

On the other hand one can check that for this data $\breve{m} \geq 10000000$ by the inequality of Corollary 3.3.

4. Explication of the algorithm construction

Based on Theorem 1.1 and auxiliary Corollaries 3.3, 3.6 one can easily construct a computer algorithm which effectively calculates homotopy minimal periods for selfmaps of nilmanifolds. For the purpose of this work, the algorithm has been implemented as a procedure under *Wolfram's Mathematica* (available at `http://www.math.gatech.edu/~rako` in `Min_P.nb` file).

An input to the routine is an integral $d \times d$ matrix $A = A_f$ associated to the self-map $f\colon X \to X$, an output is a listing of sets T_A, $\mathrm{HPer}\,(f)$ and $T_A \setminus \mathrm{HPer}\,(f)$. In order to establish a content of T_A, the algorithm finds a characteristic polynomial χ_A of A and determines multiplicities of roots of unity (it is accomplished by reduction of χ_A through cyclotomic polynomials). Since $\mathrm{HPer}\,(f)$ is a subset of T_A, the next natural step is to establish which $m \in T_A$ belong to $\mathrm{HPer}\,(f)$, this is verified by checking the condition $N(f^m) = N(f^{m/p})$ of the main theorem. Although, it has to be done only for finitely many $m \in [1, m_0] \cap T_A$, the universal bound m_0 given by Corollary 3.3 comes out "too big" in general (i.e. in the generic case provided by the main theorem), which might lead to long execution time. A partial remedy for this situation is given by Corollary 3.6, which provides much smaller bound and is valid in most of the cases e.g. selfmaps of 3-nilmanifolds (see Theorem 14). Therefore a crucial step in the algorithm involves establishing a proper value for m_0, then verifying which $m \in [1, m_0] \cap T_A$ belongs to $\mathrm{HPer}\,(f)$. The last step can be done easily by applying the formula $N(f^m) = |\det(I - A^m)|$ in the condition $N(f^m) = N(f^{m/p})$.

Below, we give the list of steps which constitute the algorithm. Some parts of the code are also included, which allows a potential user to relate specific parts of the source code to a description of the steps. Parts of the code included are rather self-explanatory thus to avoid lengthy detours only short *comments*, are provided.

4.1. Determining the set of algebraic periods T_A. According to the definition Lefschetz of T_A it is clear that $T_A = \mathbb{N} \setminus M$, where M is an ideal in $\mathbb{N}$ generated by a set of multiplicities of unit roots of the characteristic polynomial χ_A. Equivalently an M can be viewed as a set of degrees of these cyclotomic polynomials which divide χ_A. Since the reduced polynomial $\widetilde{\chi}_A$ comes handy later in the process, it is useful to determine M during the reduction of χ_A via cyclotomic polynomials.

… *setting initial variables,* $\chi_A \equiv$`chp1`, $\widetilde{\chi}_A \equiv$`chp2`, … etc.

```
chp1 = Det[t Id - mat]; i = 1; (* characteristic polynomial *)
chp2 = chp1; cyc = Cyclotomic[i, t];
cyc1 = {}; cyc3 = {}; cyc0 = 1;
```

… *this loop divides* χ_A *by consecutive cyclotomic polynomials up to degree* $\deg(\chi_A)^2$

```
(*reduce by cyclotomic polynomials*)
While[deg[cyc, t] < (deg[chp2, t] + 1)^2,
chp3 = Simplify[chp2/cyc];
If[PolynomialQ[chp3, t], chp2 = chp3; cyc0 *= cyc;
cyc1 = Join[cyc1, cycroots[i]]; cyc3 = Append[cyc3, i]; i = 0];
cyc = Cyclotomic[++i, t];];
```

… *set of generators for* M *is defined in* `cyc3`,
`Union` *removes multiple entries* …

```
cyc = Union[cyc3, cyc3]; (* reuse cyc *)
...
```

4.2. Determining the case of A. By the main theorem, a self-map f can be classified into three cases (i)–(iii), called respectively empty (E), finite (F) and generic (G). In order to establish a case for the given A, it is required to verify a set of conditions posed in the theorem.

Case (E). $L(f) = N(f) = 0$.

Occurs if 1 is an eigenvalue of A, which in sequel implies $T_A = \emptyset$ (by Lefschetz). Since no calculation is necessary the procedure ends at this point.

```
(*CASE : EMPTY*)
If[Intersection[cyc, 1] == 1, Print["case: EMPTY"];
Print["T_A is empty."]; Return[]];
```

Case (F). $N(f) \neq 0$ and the sequence $\{N(f^m)\}$ is bounded.

Occurs if eigenvalues of A are zero or roots of unity. From Lefschetz it can be concluded that $\{N(f^m)\}_m$ is h-periodic. The constant h depends only on d, and can be defined as the least common multiple of the set $\{k \in \mathbb{N}\colon \phi(k) \leq d\}$, where $\phi(k)$ stands for the Euler function (i.e. $\phi(k)$ is a number of these $m \in \mathbb{N}$ which are relatively prime to k, see e.g. [9]). Consequently, in order to determine $\mathrm{HPer}(f)$, it suffices to check the condition $N(f^m) = N(f^{m/p})$ only for $m \in [1, h] \cap T_A$.

```
(*CASE : FINITE*)
Print["case: FINITE"];
(*calculation of h(n)*)
```

… *computing a constant* $h \equiv$ `hn`, …

... (invphi *returns an inverse image* ϕ^{-1} *of the Euler function* ϕ) ...

```
hn = LCM @@ Union @@ Table[invphi[i], {i, 1, d}];
Ta = Divisors[hn];
(*ind is a complement of the ideal*)
ind = Table[i, {i, 1, Max @@ Ta}];
ind = Complement[ind,Union @@ Map[Table[i - 1, {i, 1, Max @@ Ta,
#}] &, cyc]];
```

... T_A *is obtained as a set of divisors of* h*, except multip. of roots of unity* ...

```
Ta = Intersection[ind, Ta];
divs = Join[{{1}}, Map[Map[#[[1]] &, FactorInteger[#]] &,
Drop[Ta, 1]]];
...
```

Case (G). $\{N(f^m)\}$ is unbounded.

Occurs if a spectral radius of A is greater than 1 (i.e. there exists at least one eigenvalue of module > 1). In this case, to determine HPer (f), it suffices to check the condition $N(f^m) = N(f^{m/p})$ only for $m \in [1, m_0] \cap T_A$. As already mentioned, the crucial step here is to find an appropriate bound m_0, which limits a range of search. For this task the algorithm first checks the assumptions of Theorem 3.5 i.e. the "spectral condition", and if they hold the estimate from Corollary 3.6 is applied.

```
(*CASE: GENERIC*)
Print["case: GENERIC"];
```

... *checking the spectral condition* ...

```
If[(mrt > 1) && (mit > 0),(* spectral condition is satisfied *)
Print["Spectral condition satisfied ... "];
eigenv1 = Map[Abs[#] &, eigenv1]; ...
```

... *calculating constants* r, s, t *necessary to estimate* m_0 (*see Theorem* 3.5)
dots

```
tt = Length[eigsm]; (* # of < 1 *)
rt = Length[eigla]; (* # of > 1 *)
st = d - tt - rt; (* # of == 1 *)
nt = Round[(1/Log[2, Min[eigla]] + 1)]; (* n - tilde *)
If[st != 0, kt = (1/Log[2, 1/Max[eigsm]] + 1), kt = 0];
(* k - tilde *)
```

... *calculating* m_0 *from the formula in Corollary* 3.6 ...

```
mt = Round[(2*(d + rt + tt) - If[cyc=={}, 0, 2*st*Log[2,
2*Sin[Pi/Max[cyc]]]])/Log[2,Times @@ eigla]] + 1;
cr = mt;Print["m0=", cr];
```

If the "spectral condition" is not satisfied then m_0 has to be estimated from the formula obtained in Corollary 3.3

```
Print["Searching for an upper bound of m0 ..."];
... ratio calculates the right hand side of the inequality given in
Corollary 3.3 ...
ratio[m_, rho_, d_]:=(rho^(m/2) - 1)/(Exp[9 d (41.4 + .5 Log[rho])
(d Log[m])^2]);
... searching for the first m which is a candidate for m0 ...
cr = 1; While[ratio[2^(cr), mrt, dg2] <= 1, cr++;];
...  refining the previous m0 with the bisection method ...
Print["Searching for m0 with the bisection method ..."];
cr1 = 2^(cr - 1); cr2 = 2^cr;
While[Abs[cr1 - cr2] > 1, cr = Floor[(cr1 + cr2)/2];
m1 = ratio[cr, mrt, dg2];
If[m1 > 1, cr2 = cr, cr1 = cr];];
cr = cr2; (*< -final m0 for GENERIC case*)
Print["m_0=", cr];];
...
```

4.3. Final step — checking the condition $N(f^m) = N(f^{m/p})$ for $m \in [1, m_0] \cap T_A$ to determine $\mathrm{HPer}\,(f)$**.** Since $N(f^m) = |\det(I - A^m)|$, verifying the condition $N(f^m) = N(f^{m/p})$, may be accomplished by raising A to appropriate powers and calculating determinants. In order to speed up the procedure all the determinants are calculated in advance and buffered. It is worth pointing out that this calculation may be based on eigenvalues of A (i.e. one might exploit the formula (6)). This approach, however, may lead to the numerical precision problems, contrary to the above method, where the calculation is based purely on integer numbers.

```
... buffering determinants of matrices I - A^m ...
Print["Calculating determinants ..."];
matsd = Table[Abs[Det[Id - #]] &, mats];
Print["Calculating minimal periods ..."];
minp = {}; For[j = 1, j < Length[Ta], j++, i = Ta[[j]];
pom = matsd[[i + 1]];
... marking m's which satisfy the condition N(f^m) = N(f^(m/p)) ...
pom1 = Map[ If[(Greater[pom, N[matsd[[i/# + 1]]]] || ((i == 1) &&
(pom != 0))), 1, 0] &, divs[[j]]];
pom1 = Times @@ pom1;
If[pom1 == If[dg2 > 0, 0, 1], minp = Append[minp, i]] ];
(* end For *)
... printing out the results ...
If[cyc != {}, Print["T_A=ℕ", cyc, "ℕ"], Print["T_A=ℕ"]];
If[dg2 == 0,(*for finite case display HPer*)
Print["Result HPer=", minp];,
```

```
(*for generic case display TA/HPer*)
Print["Result T_A \ HPer =", minp];]
```

The last paragraph of this section is devoted to the example.

Example. MinP$[\{\{-5,-4,3\},\{7,4,2\},\{2,1,1\}\},100]$.

The first parameter in MinP routine is the matrix $A = A_f$ of a selfmap. The second parameter determines the bound m_0, which allows restricting the search in the case when the "spectral condition" is not satisfied and m_0 is $\gg 1$. However if the second parameter is set to zero then m_0 is calculated automatically (see the step 2 of the algorithm).

Results.

```
Characteristic polynomial: 1 - t + t^3
Reduced characteristic polynomial: 1 - t + t^3
Roots of unity: {}
Modules of roots of char. polynomial: 1.3247, 0.8688, 0.8688
Spectral radius: 1.3247
case: GENERIC
Spectral condition satisfied...
m_0=44
Assuming constant m_0=44
Calculating prime divisors...
Calculating matrices...
Calculating determinants...
Calculating minimal periods...
T_A = ℕ
Result T_A \ HPer = {2, 6, 8, 9, 10}
```

5. Examples of matrices

In this section we list all possible sets of homotopy minimal periods of maps of compact three nilmanifolds. For the selfmaps of the torus T^3 it was derived by Jiang and Llibre in [14] (cf. [14, Theorem 3]) and we rewrite it. They gave this classification in the term of the coefficients a, b, c of the characteristic polynomial of A. To enrich these classification we endow every case with the matrix A which has the given characteristic polynomial and the set T_A. Next we present the same list for maps of compact three dimensional nilmanifold X not diffeomorphic to the torus. This classification does not depend on the topological isomorphism class (homotopy type) of X and was given in [11]. And also here we present matrices that induce a given map.

The torus. For a given integral matrix $A \in \mathcal{M}_{3\times 3}(\mathbb{Z})$ let

$$\det(t\mathrm{I} - A) = t^3 - at^2 + bt - c$$

be its characteristic polynomial. Remind that $a = \operatorname{tr} A$, $b = \operatorname{tr} \wedge^2 A$, and $c = \operatorname{tr} \wedge^3 A = \det A$.

For selfmaps of T^3 we have the following 10 cases of type (F) — finite:

1. $(a,b,c) = (0,0,0)$, $T_A = \mathbb{N}$, $\mathrm{HPer}\,(A) = \{1\}$,
$$\text{matrix pattern:} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
2. $(a,b,c) = (-1,0,0)$, $T_A = \mathbb{N} \setminus 2\mathbb{N}$, $\mathrm{HPer}\,(A) = \{1\}$,
$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$
3. $(a,b,c) = (-2,-1,0)$, $T_A = \mathbb{N} \setminus 2\mathbb{N}$, $\mathrm{HPer}\,(A) = \{1\}$,
$$\text{matrix pattern:} \quad \begin{bmatrix} -2 & -2 & -2 \\ -2 & -1 & -2 \\ 1 & 2 & 1 \end{bmatrix}.$$
4. $(a,b,c) = (-1,-1,0)$, $T_A = \mathbb{N} \setminus 3\mathbb{N}$, $\mathrm{HPer}\,(A) = \{1\}$,
$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$
5. $(a,b,c) = (0,1,0)$, $T_A = \mathbb{N} \setminus 4\mathbb{N}$, $\mathrm{HPer}\,(A) = \{1,2\}$,
$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$
6. $(a,b,c) = (1,1,0)$, $T_A = \mathbb{N} \setminus 6\mathbb{N}$, $\mathrm{HPer}\,(A) = \{1,2,3\}$,
$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$
7. $(a,b,c) = (-3,3,0)$, $T_A = \mathbb{N} \setminus 2\mathbb{N}$, $\mathrm{HPer}\,(A) = \{1\}$,
$$\text{matrix pattern:} \quad \begin{bmatrix} -2 & -2 & -2 \\ 1 & -2 & -2 \\ -1 & 1 & 1 \end{bmatrix}.$$
8. $(a,b,c) = (-2,2,1)$, $T_A = \mathbb{N} \setminus (2\mathbb{N} \cup 3\mathbb{N})$, $\mathrm{HPer}\,(A) = \{1\}$,
$$\text{matrix pattern:} \quad \begin{bmatrix} -2 & -2 & 1 \\ 0 & -1 & 1 \\ -1 & -2 & 1 \end{bmatrix}.$$

9. $(a, b, c) = (-1, 1, -1)$, $T_A = \mathbb{N} \setminus 2\mathbb{N}$, $\mathrm{HPer}\,(A) = \{1\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

10. $(a, b, c) = (0, 0, -1)$, $T_A = \mathbb{N} \setminus 2\mathbb{N}$, $\mathrm{HPer}\,(A) = \{1,\ 3\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Remind that we say an integral matrix A with $\rho = \mathrm{sp}\,(A) > 1$ represents the special case if $T_A \neq \mathrm{HPer}\,(A)$. Additionally we say that a special case is exceptional if it corresponds to a a single point (a, b, c) in the parameter space $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ (not a line or two dimensional hyper-subspace).

For selfmaps of T^3 we have infinite series of special cases listed in Theorem C of [14] with $T_A \setminus \mathrm{HPer}\,(A) \subset \{2, 3, 3, 4\}$ and the following 9 exceptional cases of type (G) – generic (also listed there but without the matrix which gave them):

1. $(a, b, c) = (0, -1, 1)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2, 3, 5, 8\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}.$$

2. $(a, b, c) = (-1, 0, 1)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2, 3, 5, 8\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

3. $(a, b, c) = (0, 1, -1)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2, 4, 5, 6\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

4. $(a, b, c) = (-1, 0, 1)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2, 4, 5, 6\}$,

$$\text{matri pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

5. $(a, b, c) = (1, 0, -1)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2, 6, 8, 9, 10\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

6. $(a, b, c) = (0, -1, -1)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2, 6, 8, 9, 10\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

7. $(a, b, c) = (0, -2, -2)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2, 5\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

8. $(a, b, c) = (-1, 0, -2)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{5\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

9. $(a, b, c) = (-1, 0, -2)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2, 6\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

It is worth of pointing out that for finding an integral matrix with given traces $a = \mathrm{tr}\, A$, $b = \mathrm{tr}\, \wedge^2 A$, and $c = \mathrm{tr}\, \wedge^3 A = \det A$ we used a simple program which searches the suitable overlooking all 3×3 matrices with coefficients contained in an interval $[k, l]$. The program is written in "Delphi".

To check and confirm the list of finite and exceptional generic cases presented in [14] we used the program – notebook written in "Mathematica". The authors derived it by long theoretic consideration, but of course such an argument shows also that these are the only special generic cases.

Nonabelian three nilmanifolds. For selfmaps of a compact nilmanifold we have a specification of the linearizations that occur in this case. To present it we need an information about the three-dimensional compact nilmanifolds.

Examples of three dimensional compact nilmanifolds are the quotient spaces $\mathcal{N}_3(\mathbb{R})/\Gamma_{p,q,r}$, where $\mathcal{N}_n(\mathbb{R})$ denotes the group of all unipotent upper triangular matrices with real coefficients and $\Gamma_{p,q,r}$, with fixed $p, q, r \in \mathbb{N}$, the subgroup consisted of all matrices of the form

$$\begin{bmatrix} 1 & k/p & m/p \cdot q \cdot r \\ 0 & 1 & l/q \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where} \quad k, l, m \in \mathbb{Z}. \tag{14}$$

The nilmanifolds of the form $\mathcal{N}_3(\mathbb{R})/\Gamma_{p,q,r}$ are called Heisenberg manifolds since the group $\mathcal{N}_3(\mathbb{R})$ is also the Heisenberg group.

It is known that (cf. [16], also [11] for references):

Theorem 5.1. *Let X be a compact nilmanifold of dimension 3. Then X is diffeomorphic to T^3 or to $\mathcal{N}_3(\mathbb{R})/\Gamma_{1,1,r}$ with some $r \in \mathbb{N}$. Moreover every non-abelian $X = \mathcal{N}_3(\mathbb{R})/\Gamma_{1,1,r}$ form an S^1-bundle over T^2 with the Euler number equal to r.*

The following theorem ([11]) states that in this case for every fiber map $f\colon X \to X$ the degree of the map along the fiber is equal to the degree of the map along the base. Together with previously stated fact that every map f of X is homotopic to a fiber map this shows that the linearization $A_f = A$ of f is the direct sum of a 1×1 matrix A_1 (linearization along fiber) and 2×2 matrix A_2, and $\det(A_1) = \det(A_2)$.

Theorem 5.2. *Let $f\colon X \to X$ be a map of three-dimensional compact nilmanifold X not diffeomorphic to T^3. Let $A = A_1 \oplus \overline{A} \in \mathcal{M}_{3\times 3}(\mathbb{Z})$ be the matrix induced by the fibre map $f = (f_1, \overline{f})$ (Theorem 2.1) and $\chi_A(t) = \chi_{A_1}(t) \cdot \chi_{\overline{A}}(t) = (t-d)(t^2 - at + b)$ be its characteristic polynomial. Then $d = b$ and there are three types for the minimal homotopy periods of f: "empty", "finite", and "generic".*

Of course, $\mathrm{HPer}\,(f) = \emptyset$ if and only if or $d = 1$ or $-a + d + 1 = 0$, because then $1 \in \sigma(A)$.

The case of finite homotopy minimal periods is the following.

(F) $\mathrm{HPer}\,(f)$ is nonempty and finite only for 2 cases corresponding to $d = 0$ combined with one of the two pairs (a, b): $(0, 0)$ and $(-1, 0)$.

We have $\mathrm{HPer}\,(f) = \{1\}$ then. Moreover, the sets T_A, $\mathrm{HPer}\,(f)$, and the matrices patterns are the following:

1. $(d, a, b) = (0, 0, 0)$, $T_A = \mathbb{N}$, $\mathrm{HPer}\,(A) = \{1\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

2. $(d, a, b) = (0, -1, 0)$, $T_A = \mathbb{N} \setminus 2\mathbb{N}$, $\mathrm{HPer}\,(A) = \{1\}$,

$$\text{matrix pattern:} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(G) $\mathrm{HPer}\,(f)$ is infinite for the remaining $(d, a, b = d)$. Furthermore, $\mathrm{HPer}\,(f)$ is equal to $\mathbb{N}$ for all triples $(d, a, b = d) \in \mathbb{Z}^3$ with except the following special cases listed below.

An infinite series of special cases: (d, a, d) where $a + d + 1 = 0$, with $a \neq 0$, and $d \notin \{-2, -1, 0, 1\}$, $T_A = \mathbb{N} \setminus 2\mathbb{N}$, $\mathrm{HPer}\,(A) = \mathbb{N} \setminus 2\mathbb{N}$.

Also there is 6 exceptional cases:

1. $(d, a, d) = (0, -2, 0)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2\}$,
$$\text{matrix pattern: } \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
2. $(d, a, d) = (-1, 1, -1)$, $T_A = \mathbb{N} \setminus 2\mathbb{N}$ $T_A \setminus \mathrm{HPer}\,(A) = \emptyset$,
$$\text{matrix pattern: } \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}.$$
3. $(d, a, d) = (-1, -1, -1)$, $T_A = \mathbb{N} \setminus 2\mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \emptyset$,
$$\text{matrix pattern: } \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$
4. $(d, a, d) = (-2, -1, -2)$, $T_A = \mathbb{N} \setminus 2\mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \emptyset$,
$$\text{matrix pattern: } \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
5. $(d, a, d) = (-2, 0, -2)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2\}$,
$$\text{matrix pattern: } \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -1 & -2 \end{bmatrix}.$$
6. $(d, a, d) = (-2, 2, -2)$, $T_A = \mathbb{N}$, $T_A \setminus \mathrm{HPer}\,(A) = \{2\}$,
$$\text{matrix pattern: } \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 0 \end{bmatrix}.$$

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REPRESENTATION THEOREM FOR LOCALLY DEFINED OPERATORS IN THE SPACE OF WHITNEY DIFFERENTIABLE FUNCTIONS

Janusz Matkowski and Małgorzata Wróbel

Abstract. A representation formula for locally defined operators mapping the space of m-times continuously differentiable functions in the Whitney sense into the space of continuous functions is given.

1. Introduction

For a real interval $I \subset \mathbb{R}$ and a nonnegative integer m, by $C^m(I)$ denote the set of all m-times continuously differentiable functions $\varphi: I \to \mathbb{R}$. An operator $K: C^m(I) \to C^0(I)$ is said to be locally defined if for every two functions $\varphi, \psi \in C^m(I)$ and for every open subinterval $J \subset I$ the relation $\varphi|_J = \psi|_J$ implies that $K(\varphi)|_J = K(\psi)|_J$. Answering a question posed by F. Neuman, the authors of [2] proved that: *every locally defined operator $K: C^m(I) \to C^0(I)$ must be of the form*

$$K(\varphi)(x) = h(x, \varphi(x), \varphi'(x), \dots, \varphi^{(m)}(x)), \quad \varphi \in C^m(I),\ x \in I,$$

for a certain function $h: I \times \mathbb{R}^{m+1} \to \mathbb{R}$. Moreover, the assumption "K is locally defined" can be replaced here by a weaker one that "K is *left defined* and *right defined*".

In the present paper we generalize this result showing that analogous representation theorem holds true for locally defined operators $K: C^m(A) \to C^0(A)$

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where $A \subset \mathbb{R}^n$ is an arbitrary closed set and $C^m(A)$ is the space of m-times continuously differentiable functions in the sense of Whitney.

2. Preliminaries

In this paper the symbols $\mathbb{N}$, $\mathbb{R}$ denote, respectively, the set of positive integers, the set of real numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let $n \in \mathbb{N}$ be fixed. For $k \in \mathbb{N}_0^n$, $k = (k_1, \dots, k_n)$ we put

$$|k| := k_1 + \ldots + k_n \quad \text{and} \quad k! = (k_1!) \ldots (k_n!).$$

Moreover, for $i = (i_1, \dots, i_n) \in \mathbb{N}_0^n$ and $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, we put

$$x^i := x^{i_1} \cdot \ldots \cdot x^{i_n}, \quad \text{and} \quad \|x\| := \left(\sum_{i=1}^{n} (x^i)^2 \right)^{1/2}.$$

Definition 1 ([4], cf. also [3])**.** Let $A \subset \mathbb{R}^n$ be a nonempty set and let $m \in \mathbb{N}_0$. A function $f\colon A \to \mathbb{R}$ is said to be of the class C^m in the Whitney sense on A, if there exists a family of functions

$$\{f^k \mid f^k\colon A \to \mathbb{R},\ k \in \mathbb{N}_0^n,\ |k| \le m\}$$

with $f^{(0,\dots,0)} = f$ such that for all $k \in \mathbb{N}_0^n$, $|k| \le m$, $x_0 \in A$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x, y \in A$, the inequalities

$$\|x - x_0\| < \delta \quad \text{and} \quad \|y - x_0\| < \delta$$

imply that

$$\left| f^k(x) - \sum_{|i| \le m - |k|} \frac{f^{k+i}(y)}{i!} (x - y)^i \right| \le \varepsilon \|x - y\|^{m-|k|}.$$

Notation. Let $m \in \mathbb{N}_0$, $A \subset \mathbb{R}^n$ and suppose that $f\colon A \to \mathbb{R}^n$. The symbol $f \in C^m(A)$ stands for a family of functions

$$\{f^k \mid f^k\colon A \to \mathbb{R},\ k \in \mathbb{N}_0^n,\ |k| \le m\} \quad \text{with } f^{(0,\dots,0)} = f, \tag{1}$$

satisfying the conditions of the above definition.

Thus $f, g \in C^m(A)$ and $f = g$ imply that

$$\begin{aligned} f &= \{f^k \mid f^k\colon A \to \mathbb{R},\ k \in \mathbb{N}_0^n,\ |k| \le m\}, \\ g &= \{g^k \mid g^k\colon A \to \mathbb{R},\ k \in \mathbb{N}_0^n,\ |k| \le m\} \end{aligned}$$

and

$$f^k = g^k \quad \text{for all } k \in \mathbb{N}_0^n,\ |k| \le m.$$

Remark 1. A function $f\colon A \to \mathbb{R}$ is continuous if and only if $f \in C^0(A)$.

Remark 2. Let $A \subset \mathbb{R}^n$ be an open set or the closure of an open set. Then $f \in C^m(A)$ if and only if f is of the class C^m on A in the usual sense, that is, f is m-times continuously differentiable in A and, moreover,

$$f^k = \frac{\partial^{|k|} f}{\partial x^{i_1} \dots \partial x^{i_n}}, \quad k \in \mathbb{N}_0^n,\ |k| \le m.$$

We shall need the following Whitney Extension Theorem [4] (cf. also [3]).

Theorem 1 (Whitney). *Let $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and a closed set $A \subset \mathbb{R}^n$ be fixed. If a function $f\colon A \to \mathbb{R}$, with the family of functions*

$$\{f^k \mid f^k\colon A \to \mathbb{R},\ k \in \mathbb{N}_0^n,\ |k| \le m\},$$

is of the class C^m in the Whitney sense on the set A, then there exists a function $g\colon \mathbb{R}^n \to \mathbb{R}$ of the class C^m on $\mathbb{R}^n$ such that

$$\frac{\partial^{|k|} g}{\partial x^{i_1} \dots \partial x^{i_n}}(x) = f^k(x), \quad x \in A,\ k \in \mathbb{N}_0^n,\ |k| \le m. \tag{2}$$

Remark 3. Let $n \in \mathbb{N}$, $m \in \mathbb{N}_0 \cup \{\infty\}$ and $A \subset \mathbb{R}^n$ be a nonempty and compact. Then a function $f\colon A \to \mathbb{R}$, with the family of functions (1), is of the class C^m in the Whitney sense on the set A if and only if

$$f^k(x) - \sum_{|i| \le m-|k|} \frac{f^{k+i}(y)}{i!}(x-y)^i = o(\|x-y\|^{m-|k|}) \quad \text{as } \|x-y\| \to 0,$$

for all k, $|k| \le m$, and $x, y \in A$.

The following lemma is a consequence of Theorem 1.

Lemma 1. *Let $A \subset \mathbb{R}^n$ be a compact set with only one cluster point $z \in \mathbb{R}^n$. Suppose that $m \in \mathbb{N}_0 \cup \{\infty\}$ and $\{f^k \mid f^k\colon A \to \mathbb{R},\ k \in \mathbb{N}_0^n,\ |k| \le m\}$, is a family of functions satisfying the condition*

$$f^k(x) - \sum_{|i| \le m-|k|} \frac{f^{k+i}(z)}{i!}(x-z)^i = o(\|x-z\|^{m-|k|}) \quad \text{as } x \to z, \tag{3}$$

for all k, $|k| \le m$. If for some $\alpha \in (0,1)$,

$$x \ne y \Rightarrow \|x-y\| \ge \alpha \min(\|x-z\|, \|y-z\|), \quad x, y \in A, \tag{4}$$

then there exists a function $g \in C^m(\mathbb{R}^n)$ satisfying conditions (2).

Proof. Since z is the only cluster point of the set A, by Whitney's Theorem and Remark 3, it is enough to show that for all $k \in \mathbb{N}_0^n$, $|k| \le m$,

$$\lim_{x \to z, y \to z} \left(f^k(x) - \sum_{|i| \le m-|k|} \frac{f^{k+i}(y)}{i!}(x-y)^i \right) \frac{1}{\|x-y\|^{m-|k|}} = 0.$$

Define a polynomial $P\colon \mathbb{R}^n \to \mathbb{R}$ by

$$P(x) = \sum_{j\in\mathbb{N}_0^n,\ |j|\le m} \frac{f^j(z)}{j!}(x-z)^j.$$

By Taylor's formula, for every $k \in \mathbb{N}_0^n$, $|k| \le m$,

$$\frac{\partial^{|k|}P}{\partial x^{i_1}\dots\partial x^{i_n}}(x) = \sum_{|i|\le m-|k|} \frac{1}{i!}\frac{\partial^{|k|+|i|}P}{\partial x^{i_1}\dots\partial x^{i_n}}(y)(x-y)^i, \quad x,y\in\mathbb{R}^n. \tag{5}$$

Moreover, from the definition of the polynomial P, we have

$$\frac{\partial^{|k|}P}{\partial x^{i_1}\dots\partial x^{i_n}}(x) = \sum_{|j|\le m-|k|} \frac{f^{k+j}(z)}{i!}(x-z)^j, \quad k\in\mathbb{N}_0^n,\ |k|\le m,\ x\in\mathbb{R}^n. \tag{6}$$

Take $x, y \in A$, $x \ne y$. Making use of (3) for $k \in \mathbb{N}_0^n$, $|k| \le m$, we obtain

$$f^k(x) - \sum_{|i|\le m-|k|} \frac{f^{k+i}(y)}{i!}(x-y)^i = \sum_{|j|\le m-|k|} \frac{f^{k+j}(z)}{i!}(x-z)^j + o(\|x-z\|^{m-|k|})$$
$$- \sum_{|i|\le m-|k|} \frac{1}{i!}\Bigg(\sum_{|j|\le m-|k|-|i|} \frac{f^{k+i+j}(z)}{j!}(y-z)^j + o(\|y-z\|^{m-|k|-|i|})\Bigg)(x-y)^i,$$

when $\|x-z\| \to 0$ and $\|y-z\| \to 0$. Taking into account (5) and (6), we get

$$f^k(x) - \sum_{|i|\le m-|k|} \frac{f^{k+i}(y)}{i!}(x-y)^i$$
$$= \frac{\partial^{|k|}P}{\partial x^{i_1}\dots\partial x^{i_n}}(x) + o(\|x-z\|^{m-|k|}) - \sum_{|i|\le m-|k|} \frac{1}{i!}\frac{\partial^{|k|+|i|}P}{\partial x^{i_1}\dots\partial x^{i_n}}(y)(x-y)^i$$
$$- \sum_{|i|\le m-|k|} \frac{1}{i!} o(\|y-z\|^{m-|k|-|i|})(x-y)^i$$
$$= o(\|x-z\|^{m-|k|}) - \sum_{|i|\le m-|k|} \frac{1}{i!} o(\|y-z\|^{m-|k|-|i|})(x-y)^i$$

and, consequently,

$$\Bigg(f^k(x) - \sum_{|i|\le m-|k|} \frac{f^{k+i}(y)}{i!}(x-y)^i\Bigg)\frac{1}{\|x-y\|^{m-|k|}}$$
$$= \frac{o(\|x-y\|^{m-|k|})}{\|x-y\|^{m-|k|}} - \sum_{|i|\le m-|k|} \frac{1}{i!}\frac{o(\|y-z\|^{m-|k|-|i|})(x-y)^i}{\|x-y\|^{m-|k|-|i|}\|y-z\|^{|i|}}$$

when $\|x-z\| \to 0$ and $\|y-z\| \to 0$. Making use of (4) we get

$$\|x-y\|^{m-|k|} \ge \alpha^{m-|k|}\|x-z\|^{m-|k|}$$

and

$$\|x-y\|^{m-|k|-|i|} \geq \alpha^{m-|k|-|i|}\|y-z\|^{m-|k|-|i|}$$

for all $x \neq y$, $|k| \leq m$, $|i| \leq m-|k|$, $k, i \in \mathbb{N}_0^n$. Hence and from the inequalities

$$\begin{aligned}|(x-y)^i| &= |(x^1-y^1)^{i_1} \cdot \ldots \cdot (x^n-y^n)^{i_n}| \\ &\leq \|x-y\|^{i_1} \cdot \ldots \cdot \|x-y\|^{i_n} = \|x-y\|^i\end{aligned}$$

we have

$$\lim_{x\to z, y\to z} \frac{o(\|x-z\|^{m-|k|})}{\|x-y\|^{m-|k|}} = \lim_{x\to z, y\to z} \frac{o(\|x-z\|^{m-|k|})}{\|x-z\|^{m-|k|}} \frac{\|x-z\|^{m-|k|}}{\|x-y\|^{m-|k|}} = 0$$

and

$$\begin{aligned}&\lim_{x\to z, y\to z} \frac{o(\|y-z\|^{m-|k|-|i|})(x-y)^i}{\|x-y\|^{m-|k|}} \\ &\qquad = \lim_{x\to z, y\to z} \frac{o(\|y-z\|^{m-|k|-|i|})}{\|y-z\|^{m-|k|-|i|}} \frac{\|y-z\|^{m-|k|-|i|}}{\|x-y\|^{m-|k|-|i|}} \frac{(x-y)^i}{\|x-y\|^{\ |i|}} = 0\end{aligned}$$

for all $|k| \leq m$, $|i| \leq m-|k|$, $k, i \in \mathbb{N}_0^n$, which completes the proof. □

3. Locally defined and one-sided defined operators

We begin this section with definitions of *locally defined* and *one-sided defined* operators of the type $K: C^m(A) \to C^p(A)$.

Let $J_i \subset \mathbb{R}$, $i = 1, \ldots, n$, be open intervals. A set $J \subset \mathbb{R}^n$, $J = \mathbb{P}_{i=1}^n J_i$, the Cartesian product of the intervals J_i, will be called an open interval in $\mathbb{R}^n$.

Definition 2. Let $m, p \in \mathbb{N}_0$ and a nonempty and closed set $A \subset \mathbb{R}^n$ be fixed. An operator $K: C^m(A) \to C^p(A)$ is said to be locally defined if for every two functions $\varphi, \psi \in C^m(A)$ and for every open interval $J \subset \mathbb{R}^n$,

$$\varphi|_{A\cap J} = \psi|_{A\cap J} \Rightarrow K(\varphi)|_{A\cap J} = K(\psi)|_{A\cap J}.$$

Definition 3. Let $m, p \in \mathbb{N}_0$ and a nonempty and closed set $A \subset \mathbb{R}^n$ be fixed. An operator $K: C^m(A) \to C^p(A)$ is said to be left defined, if for every point $(x^1, \ldots, x^n) \in A$ and for all $\varphi, \psi \in C^m(A)$,

$$\begin{aligned}&\varphi|_{A\cap\mathbb{P}_{i=1}^n(-\infty,x^i)} = \psi|_{A\cap\mathbb{P}_{i=1}^n(-\infty,x^i)} \\ &\qquad\qquad \Rightarrow K(\varphi)|_{A\cap\mathbb{P}_{i=1}^n(-\infty,x^i)} = K(\psi)|_{A\cap\mathbb{P}_{i=1}^n(-\infty,x^i)}.\end{aligned}$$

An operator $K: C^m(A) \to C^p(A)$ is said to be right defined, if for every point $(x^1, \ldots, x^n) \in A$ and for all $\varphi, \psi \in C^m(A)$,

$$\varphi|_{A\cap\mathbb{P}_{i=1}^n(x^i,\infty)} = \psi|_{A\cap\mathbb{P}_{i=1}^n(x^i,\infty)} \Rightarrow K(\varphi)|_{A\cap\mathbb{P}_{i=1}^n(x^i,\infty)} = K(\psi)|_{A\cap\mathbb{P}_{i=1}^n(x^i,\infty)}.$$

Example 1. Let $A := [a^1, b^1] \times \ldots \times [a^n, b^n]$ for some $a^i, b^i \in \mathbb{R}$, $a^i < b^i$, $(i = 1, \ldots, n)$. Suppose that a continuous function $H : A \times \mathbb{R} \to \mathbb{R}$ is continuous. Then the operator $K\colon C^0(A) \to C^1(A)$ given by

$$K(\varphi)(x^1, \ldots, x^n) = \int_{a^1}^{x^1} \cdots \int_{a^n}^{x^n} H(y^1, \ldots, y^n, \varphi(y^1, \ldots, y^n))\, dy^1 \ldots dy^n$$

is left defined; and $K\colon C^0(A) \to C^1(A)$ given by

$$K(\varphi)(x^1, \ldots, x^n) = \int_{x^1}^{b^1} \cdots \int_{x^n}^{b^n} \int_{a^n}^{x^n} H(y^1, \ldots, y^n, \varphi(y^1, \ldots, y^n))\, dy^1 \ldots dy^n$$

is right defined.

Example 2. Let $A := [a^1, b^1] \times \ldots \times [a^n, b^n]$ for some $a^i, b^i \in \mathbb{R}$, $a^i < b^i$, $(i = 1, \ldots, n)$. Suppose that $H : A \times \mathbb{R} \to \mathbb{R}$ and $f_i\colon A \to [a^i, b^i]$, $i = 1, \ldots, n$, are continuous. Then the operator $K\colon C^0(A) \to C^0(A)$ given by

$$K(\varphi)(x) = H(x, \varphi(f_1(x), \ldots, f_n(x)))$$

is left defined if, for all $(x^1, \ldots, x^n) \in A$,

$$f_i(x^1, \ldots, x^n) \le x^i, \quad i = 1, \ldots, n;$$

and right defined if, for all $(x^1, \ldots, x^n) \in A$,

$$f_i(x^1, \ldots, x^n) \ge x^i, \quad i = 1, \ldots, n.$$

Theorem 2. *Let $m \in \mathbb{N}_0$ and a nonempty and closed set $A \subset \mathbb{R}^n$ be fixed. An operator $K\colon C^m(A) \to C^0(A)$ is locally defined if, and only if, it is left defined and right defined.*

Proof. If $K\colon C^m(A) \to C^0(A)$ is locally defined then, obviously, it is left defined and right defined.

Let K be left defined and right defined. Take arbitrary $\varphi, \psi \in C^m(A)$ and suppose that there exists an open interval $J \subset \mathbb{R}^n$ such that $\varphi|_{A\cap J} = \psi|_{A\cap J}$. There are $a^i, b^i \in \mathbb{R}$, $a^i < b^i$, for $i = 1, \ldots, n$, such that $J = \mathbb{P}_{i=1}^n (a^i, b^i)$.

Let us define a function $\gamma_0\colon A \cap [\mathbb{P}_{i=1}^n (-\infty, b^i] \cup \mathbb{P}_{i=1}^n [a^i, \infty)] \to \mathbb{R}$ by the formula

$$\gamma_0(x) := \begin{cases} \varphi(x) & \text{for } x \in A \cap \mathbb{P}_{i=1}^n (-\infty, b^i], \\ \psi(x) & \text{for } x \in A \cap \mathbb{P}_{i=1}^n [a^i, \infty). \end{cases}$$

Since γ_0 satisfies the assumptions of Theorem 1, there exists its extension function $\gamma \in C^m(\mathbb{R}^n)$. Consequently, we have

$$\gamma|_{A\cap\mathbb{P}_{i=1}^n(-\infty,b^i)} = \varphi|_{A\cap\mathbb{P}_{i=1}^n(-\infty,b^i)} \quad \text{and} \quad \gamma|_{A\cap\mathbb{P}_{i=1}^n(a^i,\infty)} = \psi|_{A\cap\mathbb{P}_{i=1}^n(a^i,\infty)}.$$

By the assumption and Definition 3,

$$\begin{aligned} K(\gamma)|_{A\cap\mathbb{P}_{i=1}^n(-\infty,b^i)} &= K(\varphi)|_{A\cap\mathbb{P}_{i=1}^n(-\infty,b^i)}, \\ K(\gamma)|_{A\cap\mathbb{P}_{i=1}^n(a^i,\infty)} &= K(\psi)|_{A\cap\mathbb{P}_{i=1}^n(a^i,\infty)}. \end{aligned}$$

It follows that$K(\varphi)|_{A\cap J} = K(\gamma)|_{A\cap J} = K(\psi)|_{A\cap J}$, which proves that K is locally defined. □

4. Representation theorem

For a number $m \in \mathbb{N}_0$ put

$$S(m) := \sum_{s=0}^{m} \binom{n+s-1}{s}.$$

Remark 4. If $g \in C^m(\mathbb{R}^n)$ then the set of all partial derivatives

$$\left\{ \frac{\partial^{|k|} g}{\partial x^{i_1} \dots \partial x^{i_n}} : |k| \le m \right\}$$

consists of $S(m)$ elements.

The main result reads as follows

Theorem 3. *Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, and a nonempty and closed set $A \subset \mathbb{R}^n$ be fixed. If an operator $K: C^m(A) \to C^0(A)$ is locally defined then there exists a unique function $h: A \times \mathbb{R}^{S(m)} \to \mathbb{R}$ such that*

$$K(\varphi)(x) = h(x, \varphi^{(0,\dots,0)}(x), \varphi^{(1,\dots,0)}(x), \dots, \\ \varphi^{(0,\dots,1)}(x), \dots, \varphi^{(m,\dots,0)}(x), \dots, \varphi^{(0,\dots,m)}(x))$$

for all $\varphi \in C^m(A)$, and $x \in A$.

Proof. Take two arbitrary functions $\varphi, \psi \in C^m(A)$. Thus there are two families of functions

$$\{\varphi^k \mid \varphi^k: A \to \mathbb{R},\ k \in \mathbb{N}_0^n,\ |k| \le m\} \quad \text{and} \quad \{\psi^k \mid \psi^k: A \to \mathbb{R},\ k \in \mathbb{N}_0^n,\ |k| \le m\},$$

satisfying the suitable conditions of Definition 1; in particular $\varphi = \varphi^{(0,\dots,0)}$, $\psi = \psi^{(0,\dots,0)}$.

We shall prove that, *for every $x_0 \in A$, if $\varphi^k(x_0) = \psi^k(x_0)$ for all $k \in \mathbb{N}_0^n$, $|k| \le m$, then $K(\varphi)(x_0) = K(\psi)(x_0)$.*

In the case when x_0 is an isolated point of the set A, this is an immediate consequence of Definition 2. In the opposite case we can always find a sequence $x_s \in A$, $s \in \mathbb{N}$, such that $\lim_{s\to\infty} x_s = x_0$ and, for all $s, t \in \mathbb{N}$,

$$s > t \Rightarrow \|x_s - x_t\| \ge \frac{1}{2}\|x_s - x_0\|.$$

Applying Lemma 1 for the family of functions $\{f^k \mid k \in \mathbb{N}_0^n,\ |k| \le m\}$ defined on the compact set $\{x_1, x_2, \dots\} \cup \{x_0\}$ by

$$f^k(x_s) := \begin{cases} \varphi^k(x_s) & \text{for even } s, \\ \psi^k(x_s) & \text{for odd } s, \\ \varphi^k(x_0) & \text{for } s = 0, \end{cases}$$

we obtain a function $g \in C^m$ such that

$$\frac{\partial^{|k|} g}{\partial x^{i_1} \dots \partial x^{i_n}}(x_{2s}) = \varphi^k(x_{2s}), \quad \frac{\partial^{|k|} g}{\partial x^{i_1} \dots \partial x^{i_n}}(x_{2s-1}) = \psi^k(x_{2s-1}),$$

for all $s \in \mathbb{N}$, $k \in \mathbb{N}_0^n$, $|k| \le m$. Hence, according to the previous case, we have

$$K(\varphi)(x_{2s}) = K(g)(x_{2s}), \quad K(\psi)(x_{2s-1}) = K(g)(x_{2s-1}), \quad s \in \mathbb{N}.$$

Letting here $s \to \infty$ and making use of the continuity of the functions $K(g)$, $K(\varphi)$ and $K(\psi)$, we obtain $K(\varphi)(x_0) = K(g)(x_0) = K(\psi)(x_0)$ which proves the desired claim.

To define the function $h\colon A \times \mathbb{R}^{S(m)} \to \mathbb{R}$, let us fix arbitrarily

$$x = (x^1, \dots, x^n) \in A,$$
$$y_{(j_1,\dots,j_n)} \in \mathbb{R}, \quad j_1, \dots, j_n \in \{0, \dots, m\}, \quad j_1 + \dots + j_n \le m,$$

then take the polynomial

$$P_{x^1,\dots,x^n, y_{(0,\dots,0)}, y_{(1,0\dots,0)},\dots, y_{(0,\dots,0,1)}}(z^1, \dots, z^n) := \sum_{j_1,\dots,j_n \in \{0,1,\dots,m\}, \quad j_1+\dots+j_n \le m} \frac{y_{(j_1,\dots,j_n)}}{j_1! \dots, j_{n!}} (z^1 - x^1)^{j_1} \cdot \ldots \cdot (z^n - x^n)^{j_n}$$

for all $z^1, \dots, z^n \in \mathbb{R}$, and put

$$h(x^1, \dots, x^n, y_{(0,\dots,0)}, y_{(1,0\dots,0)}, \dots, y_{(0,\dots,0,1)}) := K(P_{x^1,\dots,x^n, y_{(0,\dots,0)}, y_{(1,0\dots,0)},\dots, y_{(0,\dots,0,1)}})(x^1, \dots, x^n).$$

Now, for a $\varphi \in C^m(A)$, $\varphi = \{\varphi^k \mid k \in \mathbb{N}_0^n, \ |k| \le m\}$, we have

$$\varphi^k(x) = P^k{}_{x^1,\dots,x^n, \varphi^{(0,\dots,0)}(x), \varphi^{(1,\dots,0)}(x),\dots,\varphi^{(0,\dots,1)}(x),\dots,\varphi^{(m,\dots,0)}(x),\dots,\varphi^{(0,\dots,m)}(x)}(x)$$

for all $k \in \mathbb{N}_0^n$, $|k| \le m$. It follows that

$$K(\varphi)(x) = K(P_{x^1,\dots,x^n, \varphi^{(0,\dots,0)}(x), \varphi^{(1,\dots,0)}(x),\dots,\varphi^{(0,\dots,1)}(x),\dots,\varphi^{(m,\dots,0)}(x),\dots,\varphi^{(0,\dots,m)}(x)})(x),$$

which, by the definition of h, means that

$$K(\varphi)(x) = h(x, \varphi^{(0,\dots,0)}(x), \varphi^{(1,\dots,0)}(x), \dots, \varphi^{(0,\dots,1)}(x), \dots, \varphi^{(m,\dots,0)}(x), \dots, \varphi^{(0,\dots,m)}(x)),$$

which proves the representation formula for K.

Since the uniqueness of h is obvious, the proof is completed. □

Recall (cf. J. Appell and P. P. Zabreĭko [1, Theorem 6.3, p. 167]) the following:

Theorem 4. *Let X be a compact metric space and let $g: X \times \mathbb{R} \to R$ be an arbitrary function. Denote by $F(X)$ the set of all real function on X and by $C(X)$ the set of all real continuous functions on X. Then the superposition operator $G: C(X) \to F(X)$ defined by*

$$G(\varphi)(x) := g(x, \varphi(x)), \quad x \in X,$$

maps $C(X)$ into $C(X)$ if and only if the function g is continuous on the set $X' \times \mathbb{R}$, where X' denotes the set of all accumulation points of X.

From Theorem 3 and Theorem 4 we obtain

Corollary 1. *Let $n \in \mathbb{N}$ and a nonempty and closed set $A \subset \mathbb{R}^n$ be fixed. An operator $K: C^0(A) \to C^0(A)$ is locally defined if and only if it is a continuous superposition (or Nemytskiĭ) operator, i.e. there exists a unique continuous function $h: A \times \mathbb{R} \to \mathbb{R}$ such that*

$$K(\varphi)(x) = h(x, \varphi(x))$$

for all $\varphi \in C^0(A)$ and $x \in A$.

Remark 5. Applying Theorem 2 we infer that Theorem 3 generalizes the main result of [2] concerning the left and right defined operators in the case $n = 1$.

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WEAK SOLUTIONS TO STOCHASTIC DIFFERENTIAL INCLUSIONS A MARTINGALE APPROACH

Mariusz Michta

Abstract. A martingale problem approach is used to analyze the nonempties and compactness property of the set of weak solutions to stochastic differential inclusions of Ito type with convex integrands. Next a weak viability (or the viability under distribution constraints) problem is considered for such solutions.

1. Introduction

The theory of stochastic differential inclusions starts its history in the beginning of 90's of the last century. The fundamental studies can be found in the papers done by Hiai ([4]), and Kisielewicz ([8]). The major contributions in this field were connected with strong solutions. In the same time there have appeared papers connected with the viability problems for strong solutions to stochastic equations or stochastic differential inclusions due to Aubin and Da Prato ([1]), Gautier and Thibault ([3]), and others. On the other hand, Mazliak in [11] has studied the same problem for controlled diffusion equation in the sense of weak solution. Mazliak's approach was essentially based on so-called martingale problem which solutions are closely connected with weak solutions to stochastic equations. In the paper we use a similar approach in a multivalued case. We formulate the basic connection between weak solutions to the stochastic differential inclusion and solutions to the martingale problem for multivalued mappings.

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Next we study compactness property of solutions set for this problem. Similar results were recently and independently obtained by Kisielewicz ([9]). The final section presents a multivalued version of Mazliak's viability result.

2. The results

2.1. Weak solutions to stochastic differential inclusions. Let us consider the stochastic differential inclusion:

$$(1)\qquad \begin{gathered} d\xi_t \in F(t,\xi_t)\,dt + G(t,\xi_t)\,dW_t, \quad t\in[0,T], \\ P^{\xi_0} = \mu, \end{gathered}$$

where $F\colon[0,T]\times\mathbb{R}^d\to \mathrm{Conv}(\mathbb{R}^d)$, $G\colon[0,T]\times\mathbb{R}^d\to\mathrm{Conv}(\mathbb{R}^{d\times m})$ are measurable, compact and convex valued multifunctions, W is a m-dimensional Wiener process on the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge 0},P)$, and μ is a given probability measure on the space $(\mathbb{R}^d,\beta(\mathbb{R}^d))$. Here $\mathrm{Conv}(\,\cdot\,)$ denotes the space of nonempty, convex and compact subsets of the underlying space. By $\mathbb{R}^{d\times m}$ we denote the space of all $d\times m$ matrices $(g_{ij})_{d\times k}$ with real elements, equipped with the norm:

$$\|(g_{ij})_{d\times m}\| = \max_{1\le i\le d,\ 1\le j\le m} |g_{ij}|.$$

For $\mathbb{R}^d$-valued stochastic process X, let

$$X_t^* = \sup_{0\le s\le t}\|X_s\|.$$

By $\mathcal{F}_t^X$ we denote a σ-field generated by the process X to the time t, i.e. $\mathcal{F}_t^X = \sigma\{X_s : s\le t\}$. The basic notion in the paper is a weak solution.

Definition 1. By a weak solution to stochastic inclusion (1) we mean d-dimensional, continuous stochastic process ξ defined on some probability space $(\Omega,\mathcal{F},P)$, $(W_t,\mathcal{F}_t^\xi)$-Wiener process, and $(\mathcal{F}_t^\xi)$-adapted processes $f_t\in F(t,\xi_t)$, $g_t\in G(t,\xi_t) : dt\times dP - a.e.$ such that:

$$(2)\qquad \begin{gathered} \xi_t = \xi_0 + \int_0^t f_s\,ds + \int_0^t g_s\,dW_s, \quad t\in[0,T], \\ P^{\xi_0} = \mu. \end{gathered}$$

The major contribution on weak solutions to stochastic inclusions has been quoted recently in [9]. The main ideas were based on selection properties for multivalued mappings, Skorohod Representation Theorem and convergence in distribution of stochastic processes. In the case of stochastic differential equations an equivalent approach to weak solutions is to consider martingale problems on a path space (canonical space) (see e.g. [7], [12], [13]). It is possible to employ this approach in the case of weak solutions to stochastic inclusions. Let $C := C([0,T],\mathbb{R}^d)$ be the space of vector valued continuous functions. By $\beta(C)$ we denote a Borel σ-field in C. We shall use projections $\pi_t\colon C\to\mathbb{R}^d$, $\pi_t(x) = x(t)$, a natural filtration (β_t), $\beta_t = \sigma\{\pi_s : s \le t\}$, $t \in [0,T]$, and its right continuous version (Γ_t), $\Gamma_t = \beta_{t+}$. Let $a\colon [0,T]\times C \to \mathbb{R}^d$, $b\colon [0,T]\times C \to \mathbb{R}^{d\times m}$ be measurable functions. Let $u \in C_b^2(\mathbb{R}^d)$ i.e. a function $u\colon \mathbb{R}^d \to R$ is bounded and twice differentiable. Suppose $y \in C$. We use the following differential operator:

$$(\mathcal{A}_t u)(y) := \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d}\gamma_{ik}(t,y)\frac{\partial^2 u(y(t))}{\partial x_i \partial x_k} + \sum_{i=1}^{d} a_i(t,y)\frac{\partial u(y(t))}{\partial x_i},$$

where $\gamma_{ik}(t,y) = \sum_{j=1}^{m} b_{ij}(t,y)b_{kj}(t,y)$, $1 \le i,k \le d$.

Let $\mathcal{M}(C)$ denote the set of all probability measures on $(C, \beta(C))$. For given multifuctions F, G, and a probability measure μ on $(\mathbb{R}^d, \beta(\mathbb{R}^d))$ we introduce:

Definition 2. A probability measure $Q \in \mathcal{M}(C)$ is said to be a *solution to martingale* (*local martingale*) *problem* for (F,G,μ) if:

(i) $Q^{\pi_0} = \mu$,

(ii) there exist measurable mappings $a\colon [0,T]\times C \to \mathbb{R}^d$, and $b\colon [0,T]\times C \to \mathbb{R}^{d\times m}$, such that $a(t,y) \in F(t,y(t))$, $b(t,y) \in G(t,y(t))$ $dt \times dQ$-a.e. and, for every $f \in C_b^2(\mathbb{R}^d)$, the process (M_t^f) (on $(C, \beta(C), Q)$)

$$M_t^f := f\circ\pi_t - f\circ\pi_0 - \int_0^t (\mathcal{A}_s f)\, ds, \quad t \ge 0,$$

is a (Γ_t, Q)-local martingale.

Let $\mathcal{R}^{\mathrm{loc}}(F,G,\mu)$ denote the set of those measures $Q \in \mathcal{M}(C)$, which are solutions of the local martingale problem for (F,G,μ).

There is the following connection between the existence of weak solution to stochastic inclusion and solution set of the local martingale problem

Proposition 1. *Let* $F, G\colon [0,T]\times\mathbb{R}^d \to 2^{\mathbb{R}^d}, 2^{\mathbb{R}^{d\times m}}$ *be* $\beta([0,T]\times\mathbb{R}^d)$*-measurable multifunctions. Let* μ *be a probability measure on* $(\mathbb{R}^d, \beta(\mathbb{R}^d))$. *Then, there exists a weak solution to stochastic inclusion* (1) *if and only if* $\mathcal{R}^{\mathrm{loc}}(F,G,\mu) \ne \Phi$.

Proof. Let us suppose that $(\Omega, \mathcal{F}, P, W_t, \mathcal{F}_t^\xi, \xi, f, g)$ is a weak solution to stochastic inclusion (1). Since $f_t \in F(t,\xi_t)$, $g_t \in G(t,\xi_t)$: $dt \times dP$-a.e. and f_t, g_t are $\mathcal{F}_t^\xi$-measurable for $t \in [0,T]$, then by Lemma 4.9 in [10], there exist measurable functionals $a\colon [0,T]\times C \to \mathbb{R}^d$, and $b\colon [0,T]\times C \to \mathbb{R}^{d\times m}$, such that $a(t,\cdot)$ and $b(t,\cdot)$ are Γ_t-measurable, for $t \in [0,T]$, hence progressively measurable. Moreover, $a(t,\xi) = f_t$ and $b(t,\xi) = g_t$. Thus $a(t,y) \in F(t,y(t))$ and $b(t,y) \in G(t,y(t))$ $dt \times dP^\xi$-a.e. Next, we get $(\Omega, \mathcal{F}, P, W_t, \mathcal{F}_t^\xi, \xi)$ as a weak solution to stochastic differential equation

$$\begin{aligned} d\xi_t &= a(t,\xi)dt + b(t,\xi)dW_t, \\ P^{\xi_0} &= \mu. \end{aligned} \tag{3}$$

The rest of the proof follows from Proposition 4.11 in [7, Chapter 5], which indicates the equivalence between existence of weak solution to the stochastic differential equation (3) and existence of solution to local martingale problem for (a, b, μ). The measures P and Q are related by: $Q = P^{\xi}$. □

For lower semicontinuous multivalued mappings (see e.g. [6]), the nonemptiness of the set $\mathcal{R}^{\mathrm{loc}}(F, G, \mu)$, can be seen by applying selections results and reducing the problem to the single valued case. Then the theory of stochastic equations applies. By Proposition 1 it can be made the following observation.

Proposition 2. *Let*

$$F\colon [0,T] \times \mathbb{R}^d \to \mathrm{Conv}(\mathbb{R}^d) \quad \textit{and} \quad G\colon [0,T] \times \mathbb{R}^d \to \mathrm{Conv}(\mathbb{R}^{d\times m})$$

be $B([0,T]) \times B(\mathbb{R}^d)$-measurable multifunctions, such that $F(t, \cdot)$ and $G(t, \cdot)$ are lower semicontinuous for $t \in [0,T]$, and suppose that

$$\max\{\|F(t,x)\|^2, \|G(t,x)\|^2\} \le K(1 + \|x\|^2).$$

Then $\mathcal{R}^{\mathrm{loc}}(F, G, \mu) \neq \Phi$.

Proof. From Theorem 7.23 in [6], there exist mappings $f, g\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d\times m}$, being Caratheodory selections for F and G respectively. Let $a\colon [0,T] \times C \to \mathbb{R}^d$, and $b\colon [0,T] \times C \to \mathbb{R}^{d\times m}$ be funtionals defined by $a(t,y) = f(t, y(t))$ and $b(t,y) = g(t, y(t))$. Hence they are bounded, jointly measurable and continuous with respect to the second variable. Now applying Theorem 2 in [4, Chapter 5, Section 2] there exists a weak solution to stochastic equation:

$$d\xi_t = a(t,\xi)\,dt + b(t,\xi)\,dW_t, \quad P^{\xi_0} = \mu,$$

what completes the proof. □

2.2. Weak compactness property of the set $\mathcal{R}^{\mathrm{loc}}(F, G, \mu)$. In [9] the topological properties of the set of weak solutions to stochastic inclusion (1) has been studied.

Let $\mathrm{SI}(F, G, \mu)$ denote the set of all weak solutions to stochastic inclusion (1) and let $\mathrm{SI}(\Omega, F, G, \mu)$ be a set of weak solutions defined on a common probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$. It was proved the following result.

Theorem 1 ([9, Theorem 13])**.** *If $F\colon [0,T] \times \mathbb{R}^d \to \mathrm{Conv}(\mathbb{R}^d)$, and $G\colon [0,T] \times \mathbb{R}^d \to \mathrm{Conv}(\mathbb{R}^{d\times m})$ are measurable and bounded set-valued mappings such that $F(t, \cdot)$ and $G(t, \cdot)$ are continuous for each fixed $t \in [0,T]$, then for every filtered probability space $(\Omega, F, (F_t)_{t\in[0,T]}, P)$ and every probability measure μ on $(\mathbb{R}^d, \beta(\mathbb{R}^d))$ the set $\mathrm{SI}(\Omega, F, G, \mu)$ is nonempty and relatively compact with respect to convergence in distribution.*

It is worth seeing that a similar result can be proved for the whole set of solutions, without restriction to the common space. Moreover, the result holds true with a weaker-lower semicontinuity assumptions imposed on F and G.

Proposition 3. *Let* $F:[0,T]\times\mathbb{R}^d\to\mathrm{Conv}(\mathbb{R}^d)$, *and* $G:[0,T]\times\mathbb{R}^d\to\mathrm{Conv}(\mathbb{R}^{d\times m})$ *be measurable and bounded set-valued mappings such that* $F(t,\cdot)$ *and* $G(t,\cdot)$ *are lower continuous for each fixed* $t\in[0,T]$, *and let a sequence of probability measures* (μ^k) *be tight. Then* $\bigcup_{k\geq 1}\mathcal{R}^{\mathrm{loc}}(F,G,\mu^k)$ *is nonempty and relatively compact subset of* $\mathcal{M}(C)$.

Proof. The nonemptiness of the set $\bigcup_{k\geq 1}\mathcal{R}^{\mathrm{loc}}(F,G,\mu^k)$ follows from Proposition 1. By Prokhorov's Theorem ([2]), it is enough to show that this set is tight. Let us note first that

$$\lim_{a\to\infty}\sup_{Q\in\bigcup_{k\geq 1}\mathcal{R}^{\mathrm{loc}}(F,G,\mu^k)} Q\{y\in\mathcal{C}:\|y(0)\|>a\}$$
$$\leq \lim_{a\to\infty}\sup_{k\geq 1}\mu^k\{x\in\mathbb{R}^d:\|x\|>a\}=0,$$

because the sequence (μ^k) is tight. Hence, by [2, Theorem 8.2], it is sufficient to show that, for every $\varepsilon>0$,

$$\lim_{n\to\infty}\sup_{Q\in\bigcup_{k\geq 1}\mathcal{R}^{\mathrm{loc}}(F,G,\mu^k)} Q\{y\in C:\Delta(1/n,y)>\varepsilon\}=0, \tag{4}$$

where $\Delta(\delta,y)=\sup\{\|y(t)-y(s)\|:s,t\in[0,T],\ |s-t|<\delta\}$, for $y\in\mathcal{C}$. Let Q be arbitrary chosen from the set $\bigcup_{k\geq 1}\mathcal{R}^{\mathrm{loc}}(F,G,\mu^k)$. Then by the definition there exist $k\geq 1$, and measurable and bounded (say by a constant $L>0$) functionals $a^k,b^k:[0,T]\times\mathcal{C}\to\mathbb{R}^d,\mathbb{R}^{d\times m}$ such that $a^k(t,y)\in F(t,y(t))$, $b^k(t,y)\in G(t,y(t))$ $dt\times dQ$-a.e. and $Q\in\mathcal{R}^{\mathrm{loc}}(a^k,b^k,\mu^k)$. Hence, taking functions $f:\mathbb{R}^d\to R$: $f(x)=x_i$, $i=1,\ldots,d$, we obtain continuous Q-local martingales (on $\mathcal{C}$)

$$M_t^{k,i}=\pi_t^i-\int_0^t a_i^k(s,\cdot)\,ds,$$

with quadratic covariations

$$[M^{k,i},M^{k,j}]_t=\int_0^t (b^k(b^k)^T)_{ij}(s,\cdot)\,ds,$$

$i,j=1,\ldots,d$. Let $M^k=(M^{k,1},\ldots,M^{k,d})$. For $0\leq t_0<t_1<T$, let us introduce the stopping time

$$\tau(y)=\inf\{u>0:\|\pi_{t_0+u}(y)-\pi_{t_0}(y)\|>\varepsilon/3\}\wedge(t_1-t_0),$$

where $y\in C$. Then the process $M^k_{t_0+t\wedge\tau}-M^k_{t_0}$ is a continuous (Γ_{t_0+t},Q)-martingale. We let $t_0=0$ for simplicity. Then one can show that

$$\|\pi_{t\wedge\tau}-\pi_0\|^2\leq 2\|M^k_{t\wedge\tau}-M^k_0\|^2+2\|\int_0^{t\wedge\tau}a^k(s,\cdot)\,ds\|^2$$

(Q-a.e.), and consequently

$$(\pi-\pi_0)^{*2}_{t\wedge\tau}\leq 2(M^k-M^k_0)^{*2}_{t\wedge\tau}+2L^2\tau^2. \tag{5}$$

Thus, by virtue of (5), we get, for any $p \geq 1$,

$$E_Q(\pi - \pi_0)^{*2p}_{\tau} \leq 2^p E_Q(M^k - M^k_0)^{*2p}_{\tau} + 2^p L^{2p} E_Q(\tau^{2p}). \tag{6}$$

For a continuous Q-local martingale $M^k - M^k_0$ we apply Burholder's inequality (see e.g. [12]) (with the same p) getting

$$E_Q(M^k - M^k_0)^{*2p}_{\tau} \leq C_{2p} E_Q \bigg\{ \int_0^{\tau} \sum_{i=1}^{d} (b^k (b^k)^T)_{ii}(s, \cdot\,)\, ds \bigg\}^p,$$

where $C_{2p} = \{((2p-1)/2p)^{2p} p(2p-1)\}^p$. Thus, by virtue of the boundness of multifunctions F and G, the inequality (6) has the form: $E_Q(\pi - \pi_0)^{*2p}_{\tau} \leq 2^p A L^p C_{2p} E_Q(\tau^p) + 2^p L^{2p} E_Q(\tau^{2p})$, where A is some constant depending on L, p and d. Thus restoring t_0 and setting $\alpha = t_1 - t_0$, $p = 2$, we obtain:

$$E_Q(\pi - \pi_{t_0})^{*4}_{\alpha} \leq 4AL^2 C_4 \alpha^2 + 4L^4 \alpha^4.$$

Hence, by Tchebyshev inequality, we get for each $\varepsilon > 0$

$$Q\{\sup_{s \leq \alpha} \|\pi_{t_0+s} - \pi_{t_0}\| > \varepsilon\} \leq \frac{4AL^2 C_4 \alpha^2 + 4L^4 \alpha^4}{\varepsilon^4} \tag{7}$$

For arbitrary $n \in \mathbb{N}$, let us divide the interval $[0, T]$ by points $\{i/n\}$, $i = 0, \ldots, Tn$. Then

$$Q\{y : \Delta(1/n, y) > \varepsilon\} = Q\bigg\{ \bigcup_{i=0}^{Tn-1} \{ \sup_{0 \leq s \leq 1/n} \|\pi_{i/n+s} - \pi_{i/n}\| > \varepsilon/3\} \bigg\}.$$

Hence, using (7), with $\alpha = 1/n$, we get

$$Q\{y : \Delta(1/n, y) > \varepsilon\} \leq 3^4 T \bigg(\frac{4AL^2 C_4}{n\varepsilon^4} + \frac{4L^4}{n^3 \varepsilon^4} \bigg),$$

what proves (4) and completes the proof. □

The closedness of $\mathcal{R}^{\mathrm{loc}}(F, G, \mu)$ is related to the same property of the set of weak solutions to the stochastic inclusion (1). Recall (see [9]) that a set $U \subset \mathbb{R}^{n \times d}$ is said to be diagonally convex if a set $D(U) = \{uu^T : u \in U\}$ is a convex subset of $\mathbb{R}^{n \times n}$. Consequently, a multivalued mapping G is said to be diagonally convex valued if the set $G(t, x)$ is diagonally convex for all $t \in [0, T], x \in \mathbb{R}^d$. In [9] it was proved that if multivalued mappings $F(t, \cdot\,)$, and $G(t, \cdot\,)$ are continuous for all $t \in [0, T]$, and G is diagonally convex valued, then the set $\mathrm{SI}(F, G, \mu)$ is closed in the topology of convergence in distribution.

Corollary. *Let $F\colon [0, T] \times \mathbb{R}^d \to \mathrm{Conv}(\mathbb{R}^d)$, $G\colon [0, T] \times \mathbb{R}^d \to \mathrm{Conv}(\mathbb{R}^{d \times m})$ be measurable and bounded set-valued mappings such that $F(t, \cdot\,)$ and $G(t, \cdot\,)$ are continuous for each fixed $t \in [0, T]$. Assume also that G is diagonally convex valued. Then, for every probability measure μ on $(\mathbb{R}^d, \beta(\mathbb{R}^d))$, the set $\mathcal{R}^{\mathrm{loc}}(F, G, \mu)$ is nonempty and compact in $\mathcal{M}(C)$.*

2.3. The weak viability problem. Let us consider the set $\Pi(K,\varepsilon) = \{\mu \in \mathcal{M}(\mathbb{R}^d) : \mu\{K\} \geq 1-\varepsilon\}$, for a fixed, nonempty and closed set $K \subset \mathbb{R}^d$ and $\varepsilon \in [0,1)$. We say that the stochastic inclusion (1) has *a weakly viable* (or *ε-viable*) *solution in* K, if for every $\mu \in \Pi(K,\varepsilon)$, there exists its weak solution $(\Omega, \mathcal{F}, P, W_t, (\mathcal{F}_t^\xi)_{t\in[0,T]}, \xi)$ such that $P^{\xi_t} \in \Pi(K,\varepsilon)$, for every $t \in [0,T]$. For strong solutions the viability problem for $\varepsilon = 0$ was studied first by Aubin and Da Prato in [1]. The case of weakly viable solutions to controlled diffusion equation was considered in [11]. By Proposition 1, it follows that there exists a weakly viable solution to stochastic differential inclusion (1) if and only if, there exists $Q \in \mathcal{R}^{\text{loc}}(F,G,\mu)$ such that $Q^{\pi_t}\{K\} \geq 1-\varepsilon$, for $t \in [0,T]$, and $\mu \in \Pi_\varepsilon(K)$. To ensure a weak viability property, we introduce a multivalued version of *the weak tangential condition* used in [11].

Definition 3. We say that *the weak tangential condition* holds for F, G, μ and K if, for every $Q \in \mathcal{R}^{\text{loc}}(F,G,\mu)$ and for every Γ_t-measurable random vector α on $(C, \beta(C))$, with $Q^\alpha\{K\} \geq 1-\varepsilon$, there exist: $t' \in (t,T)$ and sequences of processes $\{D^{(n)}\}$, $\{a^{(n)}\}$, $\{b^{(n)}\}$ on $(C,\beta(C))$ such that:

(a) $D_t^n = \alpha$ and $(D_s^{(n)})_{t\leq s\leq t'}$ is a diffusion process with generator

$$\mathcal{A}^{(n)} = \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d}(b^{(n)}b^{(n)^T})_{ik}\frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{d} a_i^{(n)}\frac{\partial}{\partial x_i},$$

(b) $Q^{D_s^{(n)}}\{K\} \geq 1-\varepsilon$ for all $s \in (t,t']$,

(c) $\sup_{t\leq s\leq t'}\{\text{dist}\{a_s^{(n)}, F(s,\pi_s(\,\cdot\,))\} + \text{dist}\{b_s^{(n)}, G(s,\pi_s(\,\cdot\,).)\}\} \to^Q 0$, as $n \to \infty$, where the symbol "$\to^Q 0$" means the convergence in probability Q.

Let $\mathcal{R}_{K,\varepsilon}(F,G,\mu) = \{Q \in \mathcal{R}^{\text{loc}}(F,G,\mu),\ \text{for all } s \in [0,T]; Q^{\pi_s}\{K\} \geq 1-\varepsilon\}$.

Theorem 3. *Let $\mu \in \Pi_\varepsilon(K)$ and suppose that F and G satisfy assumptions of the Corollary. If the weak tangential condition holds for F, G, μ and K, then $\mathcal{R}_{K,\varepsilon}(F,G,\mu)$ is nonempty and compact subset of the space $\mathcal{M}(C)$.*

Proof. Let us define the set $A = \{t \in [0,T] : \text{exists } Q \in \mathcal{R}^{\text{loc}}(F,G,\mu),\ \text{for all } s \in [0,t]; Q^{\pi_s}\{K\} \geq 1-\varepsilon\}$. For nonempties of the set $\mathcal{R}_{K,\varepsilon}(F,G,\mu)$, it is enough to show that $A = [0,T]$. Since $\mathcal{R}^{\text{loc}}(F,G,\mu) \neq \Phi$ we get $0 \in A$.

Next we can claim that A is closed in the same way as in [11]. We show that $\sup A = T$. Let $c = \sup A$, and suppose $c < T$. Since $c \in A$, thus there exists $Q \in \mathcal{R}^{\text{loc}}(F,G,\mu)$ such that $Q^{\pi_t}\{K\} \geq 1-\varepsilon$, $t \in [0,c]$.

Let us take now $\alpha = \pi_c$. Then we can find $c' > c$, and processes $D^{(n)}$, $a^{(n)}$, $b^{(n)}$ satisfying conditions in Definition 3. For every $n \geq 1$, we can extend the process $D^{(n)}$ by: $D_s^{(n)} = \pi_s$, for $s \in [0,c]$. Hence by (b) (in Definition 3) $Q^{D_s^{(n)}}\{K\} \geq 1-\varepsilon$, for all $s \in [0,c']$. From (c) and assumptions imposed on multifunctions F, and G we can choose Caratheodory's selections $a'^{(n)}(s,y) \in F(s,y(t)), b'^{(n)}(s,y) \in G(s,y(t))$ such that $\sup_{c\leq s\leq c'} \|a_s^{(n)} - a'^{(n)}(s,\,\cdot\,)\| \to^Q 0$, and $\sup_{c\leq s\leq c'} \|b_s^{(n)} - b'^{(n)}(s,\,\cdot\,)\| \to^Q 0$.

Let $D'^{(n)}$ be a diffusion process with generator $\mathcal{A}'^{(n)}$ built on funtionals $a'^{(n)}$, and $b'^{(n)}$. Let us extend this process for $s \in [0,c]$ in the same way as $D^{(n)}$. Due to the convergence of coefficients $a^{(n)}, a'^{(n)}$ and $b^{(n)}, b'^{(n)}$, we have convergence of distributions in Levy–Prokhorov's metric: $d_{L-P}(Q^{\{D_s^{(n)}\}_{0\leq s\leq c'}}, Q^{\{D'^{(n)}_s\}_{0\leq s\leq c'}}) \to 0$, as $n \to \infty$. On the other hand the choice of selections $a'^{(n)}$ and $b'^{(n)}$ implies that $Q^{\{D'^{(n)}_s\}_{0\leq s\leq c'}} \in \mathcal{R}^{\rm loc}(F,G,\mu)$, for every $n \geq 1$. Thus (passing to the subsequence if needed) we get $Q^{\{D'^{(n)}_s\}_{0\leq s\leq c'}} \Rightarrow Q^{\tilde{}}$, for some $Q^{\tilde{}} \in \mathcal{R}^{\rm loc}(F,G,\mu)$, what finally implies $Q^{\{D_s^{(n)}\}_{0\leq s\leq c'}} \Rightarrow Q^{\tilde{}}$. Now using Continuous Mapping Theorem and Theorem 2.1 in [2, Section 4, Chapter 1], one has $Q^{\tilde{}\pi_s}\{K\} \geq 1-\varepsilon$, for all $s \in [0,c']$. Thus $c' \in A$, what contradicts with the choice of c. The compactness of $\mathcal{R}_{K,\varepsilon}(F,G,\mu)$ follows from its closedness. □

Remark. Let us note that if $\mu\{K\} > 1-\varepsilon$, for some $\varepsilon \in (0,1)$, and $\mu\{\partial K\} = 0$, where ∂K denotes a boundary of the set K, then under the same assumptions as in Corollary, it can be proved that there exists $Q \in \mathcal{R}^{\rm loc}(F,G,\mu)$, and $T_0 \in (0,T]$ such that $Q^{\pi_t}\{K\} > 1-\varepsilon$, for $t \in [0,T_0]$.

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PICARD ITERATIONS SCHEME FOR NONLOCAL ELLIPTIC PROBLEMS WITH DECREASING NONLINEARITY

Tadeusz Nadzieja and Ewa Sylwestrzak

Abstract. We prove that for some class of nonlocal elliptic problems appearing in mathematical physics the Picard iteration sequence is convergent.

1. Introduction

In this paper we study the following nonlocal elliptic equation

$$-\Delta\varphi = M\frac{f(\varphi)}{(\int_\Omega f(\varphi))^p}, \tag{1}$$

with the homogeneous boundary Dirichlet condition

$$\varphi_{|\partial\Omega} = 0. \tag{2}$$

Here $\varphi\colon\Omega\to\mathbb{R}$ is an unknown function from a bounded subdomain Ω of $\mathbb{R}^n$ into $\mathbb{R}$, $n\geq 2$, $f\colon\mathbb{R}\to\mathbb{R}^+$ is a given continuous function and $M>0$, $p>0$ are real parameters.

The roots for the study of the problems of the type (1), (2) lie in statistical mechanics ([2], [7], [9]), the theory of electrolytes ([6], [17]), and the theory of thermistors ([10], [15]).

If the parameter p equals 1 and the nonlinearity $f(\varphi)$ has the exponential form $f(\varphi)=e^{-\varphi}$ then (1) is the well-known Poisson–Boltzmann equation. The

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physical interpretation of φ is the electric potential of a system of charged Brownian particles in thermodynamical equilibrium, and the parameter M is the total charge of the particles.

The Poisson–Boltzmann equation arises also in investigations of phenomena associated with the occurrence of shear bands in metals being deformed under high strain rates [4], and in the modelling of turbulent behaviour of flows [9].

The general problem (1), (2) with given $f(\varphi)$ and $p = 2$ appears in modelling the stationary temperature φ, which results when an electric current flows through a material with temperature-dependent electrical resistivity $f(\varphi)$, subject to a fixed potential difference $\sqrt{M}$ [10], [15].

The integral form of (1), (2) is

$$\varphi(x) = \frac{M}{(\int_\Omega f(\varphi))^p} \int_\Omega K(x,y) f(\varphi(y))\, dy, \tag{3}$$

where K is the Green function corresponding to $-\Delta$ and zero boundary data. We will consider the equation (3) with general symmetric kernel K satisfying the integrability condition

$$\sup_{x\in\Omega} \int_\Omega |K^r(x,y)|\, dy < \infty \tag{4}$$

for some $r > 1$. This condition implies that the operator

$$T(\varphi)(x) = \frac{M}{(\int_\Omega f(\varphi))^p} \int_\Omega K(x,y) f(\varphi(y))\, dy, \tag{5}$$

defined on the space of continuous functions on $\overline{\Omega}$, $C^0(\overline{\Omega})$, with the uniform norm $|\varphi|_\infty = \sup_{x\in\Omega} |\varphi(x)|$, is compact and continuous [13].

The assumption that $\partial\Omega$ is of class $C^{1+\epsilon}$ guarantees that the Green function satisfies the estimate (4) with any $r < n/(n-2)$ ([12]).

The problem (3) is also motivated by the study of the stationary solutions of parabolic-elliptic systems describing the temporal evolution of a cloud of self-interacting Brownian particles [8], [18]. The kernel K in (3) depends on the kind of interaction between the particles under consideration.

It is known [1] that certain assumptions on the function f guarantee that the Picard iteration sequence for the local elliptic problem $-\Delta\varphi = f(\varphi)$, $\varphi_{|\partial\Omega} = 0$, is convergent in an appropriate space of functions. Results of this kind are not known for the nonlocal problems. Our main goal here is to prove that for decreasing f and $0 < p < 2$, the Picard iteration sequence for the problem (1), (2) is convergent (Theorem 7).

We prove also some new results about the existence of solutions of (3) (Theorems 4 and 5).

2. Existence of solutions

The existence and the uniqueness of the solution of the Poisson–Boltzmann equation with zero Dirichlet data was proved in [9]. The key point in the proof is that the Poisson–Boltzmann equation is the Euler–Lagrange equation for the functional $M \log \int_\Omega e^{-\varphi} + (1/2) \int_\Omega |\nabla\varphi|^2$. It seems that the variational methods work only when the nonlinearity f is of the form $f(\varphi) = e^{-\varphi}$ or $f(\varphi) = e^{\varphi}$.

It is worth noting that, the following result can be proved, as a consequence of the well-known Pohozaev identity [3].

Theorem 1 ([3]). *Assume that Ω is a star-shaped domain, and $n > 1$. If $f(\varphi) = e^{\varphi}$, $0 \le p \le 1$ or $f(\varphi) = e^{-\varphi}$, $p \ge 2$, then the problem* (1), (2) *has no solution for sufficiently large M.*

In the proof of Theorem 1 the particular form of f is essential.

The construction of specific subsolutions for the problem $\Delta\varphi + \lambda f(\varphi) = 0$ allows the authors of [3] to extend the proof of nonexistence of solutions, for sufficiently large M, for any integrable and decreasing function f, even when Ω is not necessarily star-shaped domain.

Theorem 2 ([3]). *Suppose that $\int_0^\infty f < \infty$ and f is decreasing. If $p \ge 2$ then* (1), (2) *has no solutions for sufficiently large M.*

The technique of sub- and supersolutions applied to our problem gives the following results

Theorem 3 ([3]). *Assume that $\int_0^\infty f = A < \infty$. If $p < 2$, then* (1), (2) *has a solution for all M. For $p = 2$ the solution exists for $M < 2A|\partial\Omega|^2$.*

An alternative approach to the problem of existence of solutions of (1), (2) or (3) is based on the topological Leray–Schauder principle [6], [14], [16].

We will use the following form of the Leray–Schauder theorem.

Theorem (Leray–Schauder). *Assume that T is a continuous and compact operator on a Banach space and there exists a constant B such that $\|x_\lambda\| \le B$ for all possible solutions x_λ of the equation*

$$x = \lambda T x, \quad \lambda \in [0, 1]. \tag{6}$$

Then there exists a continuous curve $x(\lambda)$, starting from 0, of solutions of the family of equations (6).

We may assume that the origin $0 \in \Omega$ and the volume $|\Omega| = 1$. In fact, the function $\psi(x) = \varphi(x/\alpha)$ with $\alpha = |\Omega|^{-1/n}$ is defined on a set of measure 1 and satisfies (3) with M replaced by $M\alpha^{np-2}$.

Various inessential constants depending on f and Ω only will be denoted by C, even if they may vary from line to line.

As we mentioned above, the operator (5) is compact and continuous. Hence for the proof of existence of a fixed point of T, it is enough to find a prior estimate for all solutions of (3).

Theorem 4. *Assume that the kernel K satisfies (4), $f \geq 0$ is continuous and bounded, $0 \leq f \leq F$, then (3) has a solution for all $M \geq 0$ if one of the following conditions is satisfied:*

(i) $p > 0$, $f \geq a > 0$,
(ii) $0 \leq p \leq 1$ *and f is convex,*
(iii) $p > 1$, *f is decreasing, convex and* $\lim_{z\to\infty} z^{1/(p-1)} f(z) = \infty$.

Proof. From (3) we get

$$|\varphi|_\infty \leq MC\left(\int_\Omega f(\varphi)\right)^{-p}. \tag{8}$$

Now what we need is to estimate the integral $\int_\Omega f(\varphi)$ from below. If (i) is satisfied then $|\varphi|_\infty \leq MCa^{-p}$.

For convex f from the Jensen inequality we have $f(\int_\Omega \varphi) \leq \int_\Omega f(\varphi)$ (we assumed, with no loss of generality, $|\Omega| = 1$).

To have the desired estimate we must to find a prior estimate for $\int_\Omega \varphi$. Integrating (3) over Ω we obtain

$$\left|\int_\Omega \varphi\right| \leq \frac{M}{(\int_\Omega f(\varphi))^p}\int_\Omega\int_\Omega |K(x,y)| f(\varphi(y))\,dy\,dx \leq MC\left(\int_\Omega f(\varphi)\right)^{1-p}. \tag{9}$$

If (ii) holds, $p \leq 1$, then the right hand side of (9) is less than MCF^{1-p}.

For f convex and decreasing we have

$$f\left(MC\left(\int_\Omega f(\varphi)\right)^{1-p}\right) \leq f\left(\int_\Omega \varphi\right) \leq \int_\Omega f(\varphi). \tag{10}$$

Denoting $z = MC(\int_\Omega f(\varphi))^{1-p}$, (10) can be written in the form

$$z^{1/(p-1)} f(z) \leq (MC)^{1/(p-1)}.$$

The last inequality and (iii) give the prior estimate for z, and thus for $|\varphi|_\infty$. □

Theorem 4 does not guarantee the existence of a solution of (3) for an unbounded function f, for example as in the Poisson–Boltzmann equation.

It was proved in [7], [16], that if Ω is star-shaped and K is the fundamental solution of the Laplacian, $f(\varphi) = e^{-\varphi}$ and $p = 1$ then the problem (3) has no solution for sufficiently large M.

This result as well as the Theorems 1 and 2 suggest that we should assume some additional properties of K to obtain the existence of solutions of (3) for an unbounded f.

Theorem 5. *Assume that K satisfies* (4), $K \geq A$, $f \geq 0$ *is continuous, convex, bounded on* $[0, \infty)$ *and* $p = 1$. *Then the problem* (3) *has a solution for all* $M \geq 0$.

Proof. Suppose that φ is a solution of (3). Obviously $\varphi(x) \geq AM$, and so $f(\varphi(x))$ is bounded. Next, proceeding as in the proof of the Theorem 4(ii), we obtain the desired estimate for $|\varphi|_\infty$. □

Remark. Note that the fundamental solution of $-\Delta$ and the Green funtion of $-\Delta$ are bounded from below, so they satisfy the assumption imposed on K in Theorem 5.

3. Picard iterations

We consider the following elliptic problem:

$$-\Delta\varphi = \lambda f(\varphi), \quad \varphi|_{\partial\Omega} = 0, \tag{11}$$

where $\lambda > 0$ is a given constant. The following result is standard and can be found in books on PDE; we give its simple proof for the completeness of the exposition.

Theorem 6. *If f is a positive decreasing function, then the problem* (11) *has a unique solution.*

Proof. We transform (11) to the integral form

$$\varphi(x) = \lambda \int_\Omega G(x,y) f(\varphi(y))\, dy, \tag{12}$$

where G is the Green function of $-\Delta$ in Ω.

The right hand side of (12) defines a continuous and compact operator on $C^0(\overline{\Omega})$. To obtain the prior estimate for solutions we note that $\varphi(x) \leq \lambda f(0) \sup_{x\in\Omega} \int_\Omega G(x,y)\, dy$.

To prove the uniqueness, we take the difference of equations (11) written for φ_1 and φ_2, multiply it by $\varphi_1 - \varphi_2$ and integrate over Ω, which gives

$$\int_\Omega |\nabla(\varphi_1 - \varphi_2)|^2 = \lambda \int_\Omega (f(\varphi_1) - f(\varphi_2))(\varphi_1 - \varphi_2).$$

The right hand side of the last equation is nonpositive, hence $\nabla(\varphi_1 - \varphi_2) = 0$, which implies $\varphi_1 = \varphi_2$ in Ω due to $\varphi_1 = \varphi_2$ on $\partial\Omega$. □

For $\varphi \in C^0(\overline{\Omega})$ we define $S(\varphi)$ as the unique solution of (11) with $\lambda = M(\int_\Omega f(\varphi))^{-p}$. The sequence of iterations $S^n(\varphi)$ of φ we call the *Picard iterations sequence* for the problem (1), (2).

Theorem 7. *If f is a positive decreasing function and $0 < p < 2$, then for every $\varphi \in C^0(\overline{\Omega})$ the Picard iteration sequence is convergent in the supremum norm to a solution of* (1), (2).

Proof. We denote by φ_λ the solution of (11). Note that if $s > t$ then $\varphi_s > \varphi_t$ in Ω. If not, there exists x_0 such that $\varphi_s(x_0) \le \varphi_t(x_0)$ and $\Delta(\varphi_t - \varphi_s)(x_0) \le 0$, whereas the difference $Msf(\varphi_s(x_0)) - Mtf(\varphi_t(x_0))$ of the right hand side terms of (11) at x_0 is positive, a contradiction. Thus the mapping $F(\lambda) = M(\int_\Omega f(\varphi_\lambda))^{-p}$ is increasing and due to the continuous dependence of φ_λ on the parameter λ; obviously F is continuous in λ.

Hence the iterations $F^n(\lambda) = \lambda_n$ of any λ tend to a fixed point of F or diverge to ∞. However, it was proved in [3] that $\left(\int_\Omega f(\varphi_\lambda)\right)^{-1}$ grows like $C\sqrt{\lambda}$ as λ tends to ∞. This implies that $F(\lambda) \sim C\lambda^{p/2} < \lambda$ for sufficiently large λ. Thus for each λ, $F^n(\lambda)$ converges to a fixed point $\overline{\lambda}$ of F. Hence for any $\varphi_0 \in C^0(\overline{\Omega})$ the sequence $\varphi_{\lambda_n} = S^{n+1}(\varphi_0)$, $\lambda_n = F^n(\lambda_0)$, $\lambda_0 = M(\int_\Omega f(\varphi_0))^{-p}$, $n = 0, 1, \ldots$, tends to a solution of the nonlocal problem (1), (2). □

Problem 1. Theorem 3 ([14]) implies that for $0 \le p \le 1$ the operator S has a unique fixed point and due to Theorem 7 we know that S has no periodic orbits. It follows from theorem of Bessaga ([5]) that there exists a metric on $C^0(\overline{\Omega})$) such that S is a contraction. It would be interesting to find such a metric.

Problem 2. For which conditions imposed on K and f the sequence $T^n\varphi$ converges for each $\varphi \in C^0(\overline{\Omega})$?

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SHADOWING AND LIKES AS C^0 GENERIC PROPERTIES

Jerzy Ombach and Marcin Mazur

Abstract. We review some known results on shadowing, inverse shadowing, tolerance stability and related concepts. We also announce here a few yet unpublished results, among them the solution of the main part of Zeeman's Tolerance Stability Conjecture, Statement 2 in Theorem 8.

1. Introduction

While simulating behavior of a dynamical system we often encounter the following problems.

(1) Does the orbit displayed on the computer screen really correspond to some true orbit?
(2) Can every true orbit be recovered, at least with a given accuracy?

The first problem is in fact a question about the shadowing property of the system while the second one corresponds to the property known as inverse shadowing. The shadowing or the pseudo-orbits tracing property (abbr. POTP), was established by Anosov and Bowen ([17], [3]), for theoretical purposes about 30 years ago. It says that any δ-pseudo-orbit can be uniformly approximated by a "true" orbit with a given accuracy if $\delta > 0$ is sufficiently small. Inverse shadowing was established by Corless and Pilyugin ([4]), and also as a part of the concept of bishadowing by Diamond at al ([8]–[11]). Kloeden and Ombach ([19]) redefined this property using the concept of a δ-method. Generally speaking, a dynamical

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system is inverse shadowing with respect to a class of methods if any "true" orbit can be uniformly approximated with given accuracy by a δ-pseudo-orbit generated by a method from the chosen class if $\delta > 0$ is small enough. Another concepts closely related to the above ones are (strong) tolerance stability and weak shadowing.

In Sections 2, 3 and 4 we present basic definitions and theorems on shadowing, inverse shadowing and tolerance stability. In Section 5 we establish several results on C^0 genericity of the mentioned properties.

The main result is that strong tolerance stability is C^0 generic in the space of all homeomorphisms of a compact and smooth manifold without boundary. It is stated in Statement 2 of Theorem 8. In fact, it gives the positive answer to the main part of the Zeeman's Tolerance Stability Conjecture in the topological formulation suggested by Takens ([42]).

Let M be a compact topological manifold (in this paper we consider manifolds without boundary). The following figure summarizes present knowledge on the concept of shadowing and the likes. The bottom part of each box, beginning with the letter "G", indicates a situation when the related property is C^0 generic.

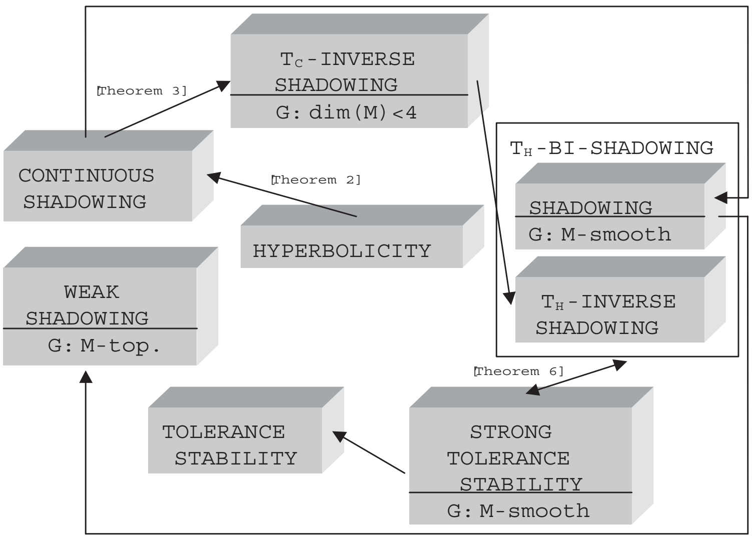

Figure 1. Shadowing and likes

2. Shadowing

Firstly we introduce some notation which remains obligatory throughout this paper.

Let (X, d) be a metric space, $f: X \to X$ a continuous map, $J \subset \mathbb{Z}$ an interval of the form $J = \{0, \dots, N\}$, $J = \{0, 1, \dots\}$ or $J = \mathbb{Z}$. By $O_f(x)$ we denote the orbit of the point $x \in X$.

A sequence $\{y_n\}_{n\in J} \subset X$ is called a δ-pseudo-orbit if

$$d(fy_n, y_{n+1}) \leq \delta \quad \text{for all } n \in J.$$

Definition. The map f is shadowing (f has POTP) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo-orbit $\{y_n\}_{n\in J}$ is ε-traced by the orbit of some point $x \in X$, i.e.

$$d(y_n, f^n x) \leq \varepsilon \quad \text{for all } n \in J.$$

A classical result, known as the Shadowing Lemma, was first proved by Anosov and Bowen.

Shadowing Lemma (Anosov, Bowen)**.** *Hyperbolicity condition implies pseudo-orbits traicing property* (POTP)*.*

See [18], [30], [32], [37] or another book on dynamical systems for the precise statement and a proof. In the simplest, though non-trivial case, when f is a linear map a proof is elementary and instructive ([29]). On the other hand, a proof in a rather general general situation of semi-hyperbolic sets (not necessarily invariant) can be found in [20].

Another classical result closely related to the above one is due to Robinson ([36]).

Theorem 1 (Robinson)**.** *A structurally stable diffeomorphism on a compact smooth manifold has* POTP*.*

Hyperbolicity is not a necessary condition for shadowing. Namely, we have the following

Remark. Let $J = \{0, 1, ...\}$. By a result of [5] we know that:

1. The tent map $fx = 1 - |2x - 1|$ has POTP on the unit interval.

And then:

2. The logistics map $fx = 4x(1 - x)$ has POTP on the unit interval.

Otherwise, there are many simple dynamical systems that are not shadowing.

Example. It is easily seen that rotation of the unit circle S^1 does not have POTP.

One can also consider more general property than shadowing, see for example [4].

Definition. If f is a homeomorfism then f is weakly shadowing if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo-orbit $\mathrm{y} = \{y_n\}_{n\in Z}$ there exists a point $x \in M$ such that $\mathrm{y} \subset V_\varepsilon(O_f(x))$, where $V_\varepsilon(A)$ denotes the ε-neighbourhood of a set $A \subset X$.

More restrictive condition than shadowing is continuous shadowing.

Definition (Kloeden, Ombach, Pokrovskiĭ ([20])). The map f is continuously shadowing if for every $\varepsilon > 0$ there exist $\delta > 0$ and a continuous map $W_\delta: \mathcal{T}_\delta \to X$ such that for any δ-pseudo-orbit y $= \{y_n\}_{n\in J}$

$$d(f^n W_\delta(\mathbf{y}), y_n) \le \varepsilon,$$

for all $n \in J$, where $\mathcal{T}_\delta$ is the set of all δ-pseudo-orbits with coordinate-wise topology.

At the end of this section we recall the notion of expansiveness.

Definition. If f is a homeomorphism then f is expansive if there exists $e > 0$ such that for all points $x, y \in X$, $x \ne y$ there is $n \in \mathbb{Z}$ such that $d(f^n x, f^n y) > e$.

Theorem 2. *If f is a hyperbolic* (*i.e. expansive and shadowing*) *homeomorphism then f is continuously shadowing. In this case, the map W_δ is uniquely determined.*

The proof of the above theorem is similar to the proof of Theorem 1 in [19].

3. Inverse shadowing

We begin this section with a definition of a δ-method. Let X^J denote the family of all sequences of elements of X indexed by J.

Definition (Kloeden, Ombach ([19])). A map $\varphi: X \to X^J$ is called a δ-method if the following conditions hold:

1. $\varphi(y)_0 = y$, for all $y \in X$,
2. $\varphi(y)$ is a δ-pseudo-orbit.

Example. Let $g: X \to X$ satisfy $D_\infty(f, g) < \delta$, where

$$D_\infty(f, g) \stackrel{\text{df}}{=} \sup_{x\in X} d(fx, gx).$$

We define the map $\varphi(y) = O_g(y)$. Then φ is a δ-method.

Example. Let $I = \{0, 1, \dots\}$ and let $g_i: X \to X$ satisfy $D_\infty(g_i, f) < \delta$, $i \in I$. We define the map

$$\varphi(y) = \{y,\ g_{\alpha_1} y,\ g_{\alpha_2} g_{\alpha_1} y,\ g_{\alpha_3} g_{\alpha_2} g_{\alpha_1} y, \dots\},$$

with $\alpha_1, \alpha_2, \alpha_3, \dots \in I$. Then φ is a δ-method.

In [4] Corless and Pilyugin introduced and examined the following concept of inverse shadowing. Actually, they did not use a notion of δ-method there.

Definition (Corless, Pilyugin)**.** If f is a homeomorphism then f is inverse shadowing if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any orbit $\{x_n\}_{n\in\mathbb{Z}}$ and any δ-method $\varphi: X \to X^{\mathbb{Z}}$ there is $y \in X$ such that

$$d(x_n, \varphi(y)_n) \leq \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

In the same paper the authors showed that this definition was, in fact, of a limited interest.

Denote by $\mathcal{T}$ a collection of δ-methods satisfying condition: for any positive δ there is a δ-method $\varphi \in \mathcal{T}$.

Definition (Kloeden, Ombach ([19]))**.** The map f is $\mathcal{T}$-inverse shadowing if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any orbit $\{x_n\}_{n\in J}$ and any δ-method $\varphi \in \mathcal{T}$ there is $y \in X$ such that

$$d(x_n, \varphi(y)_n) < \varepsilon \quad \text{for all } n \in J.$$

Remark. Let f be a homeomorphism and let $J = \mathbb{Z}$. Consider the class of methods:

$$\mathcal{T}_h \stackrel{\text{df}}{=} \{\varphi : \text{ there exists a homeomorfism } g: X \to X \text{ such that } \varphi(y) = O_g(y), \text{ for all } y \in X\}.$$

Then $\mathcal{T}_h$-inverse shadowing is equivalent to persistency (see [22] for the definition).

Hence, by the result of [22] we then have

Corollary. *A topologically stable diffeomorphism of a compact smooth manifold is $\mathcal{T}_h$-inverse shadowing.*

Consider now a broader class of methods:

$$\mathcal{T}_c \stackrel{\text{df}}{=} \{\varphi : \varphi \text{ is a continuous } \delta\text{-method, for some } \delta > 0\}.$$

Theorem 3. *Assume that $f: M \to M$ is a continuously shadowing map of a compact topological manifold M. Then f is $\mathcal{T}_c$-inverse shadowing.*

Proof. Choose δ and W_δ to ε by continuous shadowing. Let φ be a continuous δ-method, $y \in X$.

$$d(f^n W_\delta(\varphi(y)), \varphi(y)_n) \leq \varepsilon \quad \text{for all } n \in \mathbb{Z}. \tag{1}$$

In particular, putting $n = 0$ we have: $d(W_\delta(\varphi(y)), y) \leq \varepsilon$ for all $y \in X$. Since $W_\delta \circ \varphi$ is continuous, it is *onto* for small ε. For any point $x \in X$ there exists $y \in X$ such that $(W_\delta \circ \varphi)(y) = x$. From inequality (1) we then have: $d(f^n x, \varphi(y)_n) \leq \varepsilon$, for all $n \in \mathbb{Z}$. $\square$

Corollary (Kloeden, Ombach ([19])). *Hyperbolic homeomorphism is $\mathcal{T}_c$-inverse shadowing.*

The above result can be extended to some noninvertible maps.

Definition (Reddy ([35])). The map f is called expanding if f is continuous, onto, open and there exist a compatible metric ϱ, constants $\varepsilon > 0$ and $\lambda > 1$ such that

$$0 < \varrho(x, y) < \varepsilon \Rightarrow \varrho(fx, fy) > \lambda\varrho(x, y).$$

Theorem 4 (Kloeden, Ombach ([19])). *An expanding map of a compact manifold is $\mathcal{T}_c$-inverse shadowing (here $J = \{0, 1, \dots\}$).*

Theorem 5 (Pilyugin ([34])). *C^0-structurally stable homeomorphism is $\mathcal{T}_c$-inverse shadowing.*

Example. Tent map f is not $\mathcal{T}_c$-inverse shadowing with $J = \{0, 1, \dots\}$. It is enough to consider the orbit $O_f(1) = \{1, 0, 0, \dots\}$ and δ-methods

$$\varphi_\delta(y) = O_g(y), \quad \text{where} \quad gy = \begin{cases} 1 - |2y - 1| & \text{for } 0 \le y \le 1 - \delta, \\ 2\delta & \text{for } 1 - \delta \le y \le 1. \end{cases}$$

4. Tolerance stability

The notion of tolerance stability was introduced by Zeeman and considered by Takens in [42]. The ideas behind it and the concept of shadowing were close to each other, still it seems that they were established independently.

In the sequel we assume that $f\colon X \to X$ is a homeomorphism. Let $H(X)$ denotes the space of all homeomorphisms of X with the C^0 topology.

Definition. A sequence $\mathrm{y} = \{y_n\}_{n\in\mathbb{Z}}$ is ε-set-traced by the orbit of the point $x \in X$ if

$$\varrho_H(\mathrm{Cl}\,(O_f(x)), \mathrm{Cl}\,(\mathrm{y})) \le \varepsilon.$$

Here ϱ_H denotes the Hausdorff metric induced by d.

Definition. The homeomorphism f is tolerance stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $g \in H(X)$, $d(f, g) < \delta$, each f-orbit is ε-set-traced by some g-orbit and each g-orbit is ε-set-traced by some f-orbit.

Definition. The homeomorphism f is strongly tolerance stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $g \in H(X)$, $d(f, g) < \delta$, each f-orbit is ε-traced by some g-orbit and each g-orbit is ε-traced by some f-orbit.

One can prove that strong tolerance stability implies shadowing in the case of homeomorpisms of a compact topological manifold ([28][1]). So we have

[1] Actually, in this paper Odani assumed a differential structure on a manifold. However, the proof was a simple modification of the proof of the classical Walters' result saying that

Theorem 6. *Let M be a compact topological manifold and let $f\colon M \to M$ be a homeomorphism. Then f is strongly tolerance stable if and only if f is shadowing and T_h-inverse shadowing.*

5. C^0 genericity

One can ask how many dynamical systems share the shadowing and/or the related properties discussed above. More particular question is whether a specific family of dynamical systems is generic. The answer to this question depends on a space of dynamical systems we are interested in. We consider here the space of all homeomorphisms $H(X)$ with the C^0 topology. Takens ([42]) established topological version of Zeeman's Tolerance Stability Conjecture which says that for a subspace $\mathcal{D} \subset H(X)$, equipped with the topology not coarser than that of $H(X)$, the set of all $f \in \mathcal{D}$ having tolerance stability property is residual in $\mathcal{D}$, i.e. it includes a countable intersection of open and dense subsets of $\mathcal{D}$. This condition means in fact that tolerance stability is a generic property in the space $\mathcal{D}$. Then, White ([45]) presented the counterexample showing that the set $\mathcal{D}$ cannot be chosen arbitrarily. There were also proved several results in the direction of Zeeman's Tolerance Stability Conjecture ([7], [16], [31], [43]). Odani ([28]) showed that the set of all homeomorphisms satisfying the strong tolerance stability condition is residual in $H(M)$, where M is a compact differentiable manifold of the dimension at most 3. Corless and Pilyugin ([4]) proved that weak shadowing is generic in $H(M)$ under additional assumption that M has also a smooth differential structure.

Let M be a compact topological manifold.

One of the most important recent results is the following

Theorem 7 (Pilyugin, Plamenevskaya ([32], [33]))**.** *If M is a smooth manifold then shadowing property is generic in $H(M)$.*

Now, we also have

Theorem 8 (Mazur ([25]–[27]))**.** *The following properties are generic in $H(M)$:*

(1) *T_C-inverse shadowing, if M is a smooth manifold and* $\dim(M) \leq 3$*,*
(2) *strong tolerance stability, if M is a smooth manifold,*
(3) *the chain recurrent set CR is a Cantor set, if M is a smooth manifold,*
(4) *weak shadowing.*

It seems that Statement 2 is the most remarkable one. It extends the corresponding result ([28]) established for the case when $\dim(M) \leq 3$. Actually, it

topological stability implies POTP ([44]), which can be also proved in the case of topological manifold ([2]).

resolves the main part of Zeeman's and Taken's Tolerance Stability Conjecture. Statement 4 extends Corless' and Pilyugin's result ([4]) mentioned above.

Statement 3 is of a different meaning than the others. The analogous theorem, under additional assumption that $\dim(M) \leq 3$, was proved by Hurley ([15]). In fact, it says that asymptotic behaviour of a C^0 generic homeomorphism concentrates on a Cantor set. For the concept of the chain recurrent set CR we refer, for example, to [36]. We just mention here that C^0 generically CR is the closure of the set of all periodic orbits ([6], [41]). So, in the other words, Statement 3 means that C^0 generically dynamics of a homeomorphism is in a specific way chaotic.

Proof of the above theorem is established in [25]–[27] and will be published elsewhere.

The proof of Statement 1 uses Shub's Density Theorem ([40]). It is very similar to the proof of Odani's result ([28]), mentioned above.

The proofs of Statements 2 and 3 employ the ideas and techniques of handle decomposition established by Pilyugin and Plamenevskaya ([33]) to the proof of Theorem 7.

In the proof of Statement 4 we use a topological method of characterization of residual sets by continuity points of semi-continuous multivalued maps, which was established by Takens ([42]) as a reformulation of a Kuratowski's theorem ([21]). This technique was previously applied by Takens to the proof of a similar but weaker result ([42]).

For a characterization of diffeomorphisms having POTP we refer to Sakai's papers [38], [39].

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ON THE STRUCTURE OF FIXED POINT SETS OF NONEXPANSIVE MAPPINGS

Andrzej Wiśnicki

Dedicated to Professor Lech Górniewicz

Abstract. We use nonstandard analysis methods and a recently introduced notion of neocompact sets to study the structure of fixed point sets of nonexpansive mappings in nonstandard hulls of Banach spaces.

1. Introduction

Throughout the paper E will always denote a Banach space, C a nonempty bounded closed and convex subset of E and $T\colon C \to C$ a nonexpansive mapping. We say that C has the generic fixed point property (GFPP) if every nonexpansive mapping $T\colon C \to C$ has a fixed point in every nonempty closed convex subset C_0 of C for which $T(C_0) \subset C_0$. We will denote by $\operatorname{Fix} T$ the set of fixed points of T in C.

One of very deep results in metric fixed point theory is a theorem of R. E. Bruck ([4]) which asserts that if C is a weakly compact set which has the GFPP, then $\operatorname{Fix} T$ is a nonexpansive retract of C. The applications of Bruck's theorem are immense (see for instance [5], [11], [12]).

It turns out (see [4, Theorem 3]) that if $\operatorname{Fix} T$ is a nonexpansive retract of C, then $\operatorname{Fix} T$ is metrically convex, that is, for every $x, y \in \operatorname{Fix} T$ there exists $z \in \operatorname{Fix} T$ such that

$$\|x - y\| = \|x - z\| + \|z - y\|.$$

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On the other hand, it is not difficult to find examples of bounded closed and convex subsets of E and nonexpansive self-mappings whose fixed-point sets are not metrically convex.

The situation is different if we consider Banach ultrapowers. Let us recall that the ultrapower $\widetilde{E}$ (or $(E)_U$) of a Banach space E over a free ultrafilter $U \subset 2^{\mathbb{N}}$ is the quotient space of

$$l_\infty(E) = \{(x_n) : x_n \in E \text{ for all } n \in \mathbb{N} \text{ and } \|(x_n)\| = \sup_n \|x_n\| < \infty\}$$

by $\ker \mathcal{N} = \{(x_n) \in l_\infty(E) : \lim_U \|x_n\| = 0\}$. Here $\lim_U$ denotes the ultralimit over U. One can prove that the quotient norm on $(E)_U$ is given by $\|(x_n)_U\| = \lim_U \|x_n\|$, where $(x_n)_U$ is the equivalence class of (x_n). For a given set $C \subset E$ and nonexpansive $T: C \to C$ set

$$\widetilde{C} = \{(x_n)_U \in \widetilde{E} : x_n \in C \quad \text{for all } n \in \mathbb{N}\}$$

and $\widetilde{T}((x_n)_U) = (Tx_n)_U$. It is not difficult to see that $\widetilde{T}: \widetilde{C} \to \widetilde{C}$ is a well defined nonexpansive mapping and that $\operatorname{Fix} \widetilde{T} \neq \emptyset$ is characterized as those points from $\widetilde{C}$ represented by sequences (x_n) in C for which $\lim_U \|Tx_n - x_n\| = 0$.

It was proved by B. Maurey in [13] that, under our standard assumptions, $\operatorname{Fix} \widetilde{T}$ is metrically convex (see also [15] for a short proof of even more general result). This is now one of the basic facts for the so-called nonstandard methods in fixed point theory (see [1], [14] and the references given there). A positive answer to the following open question might be regarded as a generalization of Maurey's result:

Question. If C is a bounded closed and convex subset of a Banach space and if $T: C \to C$ is a nonexpansive mapping, is then $\operatorname{Fix} \widetilde{T}$ a nonexpansive retract of $\widetilde{C}$?

It is shown in [16] that a positive answer to the above question yields a positive solution to the long-standing problem concerning the existence of common approximate fixed points for commuting nonexpansive mappings (see [10, p. 11]).

Our paper is a first step to answer the question in the affirmative. It is shown that $\operatorname{Fix} \widetilde{T}$ is an 'almost' nonexpansive retract of $\widetilde{C}$ in the sense given below. As corollary, we obtain a nonexpansive mapping of $\widetilde{C}$ into $\operatorname{Fix} \widetilde{T}$ which leaves a countable set $D \subset \operatorname{Fix} \widetilde{T}$ fixed. We use nonstandard analysis approach instead of the ultraproduct language and a recently introduced notion of neocompactness due to S. Fajardo and H. J. Keisler [6], [7].

2. Results

Let us first recall the terminology. We assume the reader is familiar with basic notions of nonstandard analysis, including the transfer principle, the overspill principle and the notion of internal sets (see [2], [9]). We work in an $\aleph_1$-saturated nonstandard universe $(V(\Xi), V({}^*\Xi), {}^*)$. Recall that for a given metric space (M, ρ) in the original superstructure $V(\Xi)$, the standard part of an element $X \in {}^*M$ is the equivalence class ${}^\circ X = \{Y \in {}^*M : {}^*\rho(X, Y) \approx 0\}$. For each $U \in {}^*M$, the nonstandard hull $\mathcal{H}({}^*M, U)$ is the set $\{{}^\circ X : {}^*\rho(X, U) \text{ is finite}\}$ with the metric $\rho({}^\circ X, {}^\circ Y) = \operatorname{st}({}^*\rho(X, Y))$, where $\operatorname{st} x$ denotes the standard part of $x \in {}^*\mathbb{R}$. It is well known that each nonstandard hull is a complete metric space. The monad of a subset $A \subset \mathcal{H}({}^*M, U)$ is the set

$$\operatorname{monad}(A) = \{X \in {}^*M : {}^\circ X \in A\}.$$

Conversely, if $B \subset \operatorname{monad}(\mathcal{H}({}^*M, U))$, then ${}^\circ B = \{{}^\circ X : X \in B\}$.

Note that the concept of the nonstandard hull of a Banach space is closely related to the Banach ultrapower construction. If E is a Banach space (in $V(\Xi)$) and $A \subset E$ is bounded, we shall write $\widetilde{E}$ for the nonstandard hull $\mathcal{H}({}^*E, 0)$ and $\widetilde{A}$ for the set ${}^\circ({}^*A)$. Let $T\colon C \to C$ be a nonexpansive mapping. Then, by transfer, we obtain a *nonexpansive mapping ${}^*T\colon {}^*C \to {}^*C$ and we may define a nonexpansive mapping $\widetilde{T}\colon \widetilde{C} \to \widetilde{C}$ putting $\widetilde{T}({}^\circ X) = {}^\circ({}^*TX)$ for $X \in {}^*C$. We shall identify each $x \in E$ with ${}^\circ({}^*x)$. Hence E may be regarded as a subspace of $\widetilde{E}$ and $\widetilde{T}$ as an extension of T.

Let $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ be the huge neometric family for the nonstandard universe $(V(\Xi), V({}^*\Xi), {}^*)$, see [6]. Recall that D is neocompact in $\widetilde{E}$ if there exists a countable collection $\{B_n\}$ of internal subsets of $\operatorname{monad}(\widetilde{E})$ such that $D = {}^\circ \bigcap_{n=1}^{\infty} B_n$. In particular, $\widetilde{A}$ is neocompact in $\widetilde{E}$ for every bounded $A \subset E$. Moreover, it follows from [6, Theorem 4.16] that $\widetilde{T}$ is a neocontinuous mapping from $\widetilde{C}$ to $\widetilde{C}$ (see [6] for the precise definition of neocontinuity).

In further considerations we shall need the following result which is a direct consequence of the existence of approximate fixed points for nonexpansive self-mappings and the Approximation Theorem [7, Theorem 6.1].

Theorem 1 (see also [3, Theorem 2.3])**.** *Let E be a Banach space* (*from the original superstructure*)*. Suppose D is a nonempty convex and neocompact* (*hence closed and bounded*) *subset of a nonstandard hull $\widetilde{E}$ and let $S\colon D \to D$ be a nonexpansive neocontinuous mapping. Then* $\operatorname{Fix} S \neq \emptyset$.

Let $\omega \in {}^*\mathbb{N}$ and set

$$\operatorname{Fix}_\omega \widetilde{T} = \{x \in \widetilde{C} : \text{ there exists } X \in {}^*C \text{ such that } {}^\circ X = x \text{ and } \|{}^*TX - X\|_* \leq 1/\omega\}.$$

It is not difficult to see that $\operatorname{Fix}_\omega \widetilde{T} \subset \operatorname{Fix} \widetilde{T}$ for every hyperinteger $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$ and that $\operatorname{Fix} \widetilde{T} = \bigcup_{\omega \in {}^*\mathbb{N} \setminus \mathbb{N}} \operatorname{Fix}_\omega \widetilde{T}$.

Theorem 2. *Let C be a nonempty bounded closed and convex subset of a Banach space E and let $T\colon C \to C$ be a nonexpansive mapping. Then, for* *every* $\omega \in {}^*\mathbb{N}\setminus\mathbb{N}$*, there exists a nonexpansive mapping* $r_\omega\colon \widetilde{C} \to \operatorname{Fix}\widetilde{T}$ *such that* $r_\omega x = x$ *for every* $x \in \operatorname{Fix}_\omega \widetilde{T}$.

Proof. Denote by $B(C,E)$ the Banach space of all bounded mappings $\varphi\colon C \to E$ with the standard uniform norm

$$\|\varphi\| = \sup\{\|\varphi x\| : x \in C\}$$

and let $\widetilde{B(C,E)}$ be its nonstandard hull $\mathcal{H}({}^*B(C,E),\mathbf{0})$. Set

$$D = \{S \in B(C,E) : S \text{ is nonexpansive from } C \text{ to } C\}.$$

Fix $\omega \in {}^*\mathbb{N}\setminus\mathbb{N}$, $k \in \mathbb{N}$ and let

$$\overline{N}_{\omega,k} = \{F \in {}^*D : (\text{for all } Y \in {}^*C)\ (\|{}^*TY - Y\|_* \le 1/\omega \Rightarrow \|FY - Y\|_* \le 1/k)\}.$$

By Keisler's Internal Definition Principle (see for instance [9]), $\overline{N}_{\omega,k}$ is internal for each $k \in \mathbb{N}$ and consequently

$$N(\operatorname{Fix}_\omega \widetilde{T}) = {}^\circ \bigcap_{k=1}^{\infty} \overline{N}_{\omega,k} = \Big\{ {}^\circ F : F \in \bigcap_{k=1}^{\infty} \overline{N}_{\omega,k} \Big\}$$

is neocompact in $\widetilde{D}$ (here ${}^\circ F$ denotes the standard part of F with respect to the *sup norm, that is, ${}^\circ F = \{G \in {}^*D : \|F - G\|_* \approx 0\}$). Let us notice that a nonexpansive mapping $T\colon C \to C$ induces a nonexpansive mapping $\mathcal{T}\colon D \to D$ in the following way:

$$(\mathcal{T} f)x = (T \circ f)x,\ \ f \in D,\ \ x \in C.$$

Define $\widetilde{\mathcal{T}} : \widetilde{D} \to \widetilde{D}$ putting

$$\widetilde{\mathcal{T}}({}^\circ F) = {}^\circ({}^*\mathcal{T}F),\ \ F \in {}^*D.$$

It is not difficult to see that $\widetilde{\mathcal{T}}$ is a well defined neocontinuous nonexpansive mapping from $\widetilde{D}$ to $\widetilde{D}$ (see [6, Theorem 4.16]). Moreover $N(\operatorname{Fix}_\omega \widetilde{T})$ is convex and invariant under $\widetilde{\mathcal{T}}$. Indeed, if $f, g \in N(\operatorname{Fix}_\omega \widetilde{T})$ and $\alpha \in (0,1)$, then there exist $F, G \in \bigcap_{k=1}^{\infty} \overline{N}_{\omega,k}$ such that ${}^\circ F = f$ and ${}^\circ G = g$. Hence

$$\alpha f + (1-\alpha) g = \alpha\, {}^\circ F + (1-\alpha)\, {}^\circ G = {}^\circ(\alpha F + (1-\alpha) G) \in N(\operatorname{Fix}_\omega \widetilde{T})$$

and

$$\widetilde{\mathcal{T}} f = \widetilde{\mathcal{T}}(\,{}^\circ F) = {}^\circ({}^*\mathcal{T}F) = {}^\circ({}^*TF) \in N(\operatorname{Fix}_\omega \widetilde{T})$$

since

$$\begin{aligned}\|{}^*TFY - Y\|_* &\le \|{}^*TFY - {}^*TY\|_* + \|{}^*TY - Y\|_* \\ &\le \|FY - Y\|_* + \|{}^*TY - Y\|_* \le \frac{1}{k+1} + \frac{1}{\omega} < \frac{1}{k},\end{aligned}$$

whenever $Y \in {}^*C$, $\|{}^*TY - Y\|_* \le 1/\omega$ and $k \in \mathbb{N}$. Therefore, we may use Theorem 1 to obtain a fixed point of $\widetilde{\mathcal{T}}$ in $N(\operatorname{Fix}_\omega \widetilde{T})$. Hence, there exists $R_\omega \in$

$\bigcap_{k=1}^{\infty} \overline{N}_{\omega,k}$ such that $\|{}^*TR_\omega - R_\omega\|_* \approx 0$ and we may define a nonexpansive mapping $r_\omega : \widetilde{C} \to \widetilde{C}$ putting $r_\omega({}^\circ Y) = {}^\circ(R_\omega Y)$ for $Y \in {}^*C$. It is easy to see that $\widetilde{T} r_\omega x = r_\omega x$ for every $x \in \widetilde{C}$ and thus $r_\omega(\widetilde{C}) \subset \operatorname{Fix} \widetilde{T}$. If $x \in \operatorname{Fix}_\omega \widetilde{T}$, then $\|{}^*TX - X\|_* \le 1/\omega$ for some $X \in {}^*C$ with ${}^\circ X = x$. Hence $\|R_\omega X - X\|_* \le 1/k$ for all $k \in \mathbb{N}$, that is, $\|R_\omega X - X\|_* \approx 0$. It follows that $r_\omega x = x$ and the proof is complete. □

Corollary. *For any countable set $A \subset \operatorname{Fix} \widetilde{T}$ there exists a nonexpansive mapping $r : \widetilde{C} \to \operatorname{Fix} \widetilde{T}$ such that $rx = x$ for $x \in A$.*

Proof. For each $x_i \in A$, $i = 1, 2, \dots$, there exists $X_i \in {}^*C$ such that ${}^\circ X_i = x_i$ and $\|{}^*TX_i - X_i\|_* \le 1/k$ for all $k \in \mathbb{N}$. It follows from the overspill principle that there exist $\omega_1, \omega_2, \dots \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $\|{}^*TX_i - X_i\|_* \le 1/\omega_i$, $i = 1, 2, \dots$. By $\aleph_1$-saturation, $\|{}^*TX_i - X_i\|_* \le 1/\omega$ for some $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$ and it is enough to apply Theorem 2. □

In some special cases one can use Theorem 2 to obtain a nonexpansive retraction of $\widetilde{C}$ onto $\operatorname{Fix} \widetilde{T}$. Assume that $\widetilde{C}$ is weakly compact, fix $\omega_0 \in {}^*\mathbb{N} \setminus \mathbb{N}$ and notice that the set $J = \{\omega \in {}^*\mathbb{N} \setminus \mathbb{N} : \omega \le \omega_0\}$ is directed by the relation $\le$ (or $\le_*$ to be precise), and can be considered as a net itself. Let $(j_\beta)_{\beta \in J'}$ be an ultranet in J. It follows from Theorem 2 that there exist nonexpansive mappings $r_{j_\beta} : \widetilde{C} \to \operatorname{Fix} \widetilde{T}$ such that $r_{j_\beta} x = x$ for $x \in \operatorname{Fix}_{j_\beta} \widetilde{T}$ and $\beta \in J'$. By the weak compactness of $\widetilde{C}$, the ultranet $(r_{j_\beta} x)_{\beta \in J'}$ tends weakly to $r(x) \in \widetilde{C}$ for each $x \in \widetilde{C}$. One can check that since the norm is weak lower semicontinuous a mapping $r : \widetilde{C} \to \widetilde{C}$ is nonexpansive. Moreover, if $x \in \operatorname{Fix} \widetilde{T}$, then (by overspill) $x \in \operatorname{Fix}_{\omega_1} \widetilde{T}$ for some $\omega_1 \in J$ and consequently there exists $\beta_1 \in J'$ such that $r_{j_\beta} x = x$ for $\beta \ge \beta_1$. Hence $rx = x$ for $x \in \operatorname{Fix} \widetilde{T}$. What is left is to show that $r(\widetilde{C}) \subset \operatorname{Fix} \widetilde{T}$. This is rather easy to prove in the case $\widetilde{E}$ is uniformly convex or has the Opial property for nets but we cannot do that in general.

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COMPLICATED DYNAMICS IN PERIODICALLY FORCED ODE'S

Klaudiusz Wójcik

Dedicated to Professor Lech Górniewicz on his 60th birthday

Abstract. We present two methods for detecting complicated dynamics in nonautonomous ODE's. First is based on the Waewski Retract Theorem and the Lefschetz Fixed Point Theorem. Second is based on the continuation of the considered system to the model one.

1. Introduction

In recent years there has been a growing interest in the differential equations that generate chaotic dynamics. This interest was inspired by common access to fast computers, which give numerical evidence of the existence of chaos in many equations. Examples of complicated dynamics are ubiquitous, extending well beyond the mathematical literature into the realm of theoretical sciences and engineering. However, there seem to be few methods that permit proving the existence of chaos in a rigorous mathematical way. The set of examples for which chaos has been rigorously demonstrated is quite small. The methods that had been most frequently applied is based on the existence of homoclinic trajectory with transversal intersection of the stable and unstable manifolds, which is sufficient to the existence of Smale's horseshoe (based mainly on ideas of Melnikov and Shilnikov, see [5], [9]).

A different topological methods in the field of chaotic dynamics are present for example in [5], [7], [16], [17]. In [8] a new method for the detection of chaos

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in dynamical systems generated by time periodic non-autonomous differential equations was introduced. The method is based on the concept of a periodic isolating segment introduced by Srzednicki (see [6]). The notion of a periodic isolating segment is a modification of an isolating block from the Conley index theory (see [2]) adapted to the setting of nonautonomous ODE's. In all practical applications the isolating segments are manifolds with corners contained in the extended phase space, such that in any point on the boundary of the segment the vector field is directed either outward or inward with respect of the segment. It was first observed by Srzednicki in [6], that the fixed point index of the Poincaré map inside the segment is equal to the Lefschetz number of the monodromy homeomorphism given by the segment. This result was used in [8] (see also [10], [11]) to prove a sufficient condition for chaos in the sense that there exists a compact invariant set I for the Poincaré map P such that P is semiconjugated to the shift on two symbols and the counterimage by the semiconjugacy of any periodic point in the shift contains a periodic point of the Poincaré map. As an application the authors in [8] considered the following planar periodic equation of the variable $z \in \mathbb{C}$

$$(*) \qquad \dot{z} = \overline{z}(1 + |z|^2 e^{i\kappa t}) \quad \text{for some } \kappa \in \mathbb{R}.$$

In this paper we describe a new method introduced in [13] (see also [14], [15]) based on the continuation theorem and a construction of the topological model map. This continuation result enables us to dig deeper into the structure of the set of periodic solutions than does the Lefschetz Fixed Point Theorem used originally in [8]. As a result we are able to show that there exists a symbolic dynamics on three symbols for the system $(*)$ for $0 < \kappa \leq 0.495$. In [8] it was proved that the above equation has symbolic dynamics on two symbols for the parameter range $0 < \kappa \leq 1/288$. Compared to [8] two new ingredients are added: the continuation theorem and a topological model for equation $(*)$. Combining the topological data with local hyperbolic behavior we are also able to prove the existence of infinitely many homoclinic solutions.

Our continuation theorem allows us to prove that the dynamics of the topological model continues to that of equation $(*)$. This is a rare phenomenon in the theory of dynamical systems. Usually one cannot claim rigorously that the dynamics of the model reflects that of the system under consideration.

The continuation theorem gives conditions for determining when the chaotic system can be homotoped to another system. The main idea is well known: find a simple system for which a fixed point can be found, determine that the fixed point index is non-zero, and homotopy to the system of interest. The important point is that index is insensitive to bifurcations: as long as fixed points do not encroach on the boundary of the domain of interest, the index remains constant. While the idea is clear, the details are difficult to handle. The continuation theorem presents the algebraic invariants that remain constant under the appropriate homotopies and simultanously provide a minimal description of the complexity of the dynamics of the considered system.

2. Semiprocesses and periodic isolating segments

Some notation: $\mathbb{R}_+ = [0,\infty)$, ρ euclidean distance function, $B(Z,\delta)$ a ball of size δ arround the set Z, $\operatorname{ind}(F,D)$-fixed point index of map F relatively to the set D (see [4]).

We start with introducing the notion of a local semiprocess which formalizes the notion of a continuous family of local forward trajectories in an extended phase-space.

Definition 1. Assume that X is a topological space and $\varphi\colon D \to X$ is a continuous mapping, $D \subset \mathbb{R} \times \mathbb{R}_+ \times X$ is an open set. We will denote by $\varphi_{(\sigma,t)}$ the function $\varphi(\sigma,t,\,\cdot\,)$. φ is called a *T-periodic local semiprocess* if the following conditions are satisfied

- (S1) $\{t \in \mathbb{R}_+ : (\sigma,t,x) \in D\}$ is an open interval for all $\sigma \in \mathbb{R}$, $x \in X$,
- (S2) $\varphi_{(\sigma,0)} = \mathrm{id}_X$ for all $\sigma \in \mathbb{R}$,
- (S3) $\varphi_{(\sigma,s+t)} = \varphi_{(\sigma+s,t)} \circ \varphi_{(\sigma,s)}$ for all $\sigma \in \mathbb{R}$, and for all $s,t \in \mathbb{R}_+$,
- (S4) $\varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)}$ for all $\sigma, t \in \mathbb{R}_+$.

A local semiprocess φ on X determines a local semiflow Φ on $\mathbb{R} \times X$ by the formula

$$\Phi_t(\sigma,x) = (\sigma+t, \varphi_{(\sigma,t)}(x)).$$

Let φ be a T-periodic local semiprocess and let Φ be a local flow associated to φ. It follows by (S1) and (S2) that for every $z = (\sigma,x)$ there is an $0 < \omega_z \le \infty$ such that $(\sigma,t,x) \in D$ if and only if $\le t < \omega_z$. Let $x \in X$, $\sigma \in \mathbb{R}$, then a left solution trough $z = (\sigma,x)$ is a continuous map $v\colon (a,0] \to \mathbb{R} \times X$ for some $a \in [-\infty,0)$ such that $v(0) = z$ and for all $t \in (a,0]$ and $s > 0$ with $s + t \le 0$ it follows that $s < \omega_{v(t)}$ and $\Phi_s(v(t)) = v(t+s)$. If $a = -\infty$ then we call v a full left solution. We can extend a left solution through z onto (a,ω_z) by setting $v(t) = \Phi_t((\sigma,x))$ for $0 \le t < \omega_z$, to obtain a solution through z.

Remark. The differential equation

$$\dot{x} = f(t,x)$$

such that f is regular enough to guarantee the uniqueness for the solutions of the Cauchy problems associated to f generates a local process as follows: for $x(t_0,x_0;\,\cdot\,)$ the solution of the Cauchy problem $x(t_0,x_0;t_0) = x_0$ we put

$$\varphi_{(t_0,\tau)}(x_0) = x(t_0,x_0;t_0+\tau).$$

If f is T-periodic with respect to t then φ is a T-periodic local process. By P we will be denote a Poincaré map $\varphi_{(0,T)}$.

We use the following notation: by $\pi_1\colon \mathbb{R}\times X\to\mathbb{R}$ and $\pi_2\colon \mathbb{R}\times X\to X$ we denote the projections and for a subset $Z\subset\mathbb{R}\times X$ and $t\in\mathbb{R}$ we put

$$Z_t=\{x\in X:(t,x)\in Z\}.$$

Definition 2. We will say that a set $Z\subset\mathbb{R}\times X$ is *T-periodic*, if and only if $Z_{nT+t}=Z_t$ for every $n\in\mathbb{N}$ and $t\in\mathbb{R}$.

Definition 3. Let $(W,W^-)\subset\mathbb{R}\times X$ be a pair of subsets. We call W a *T-periodic isolating segment* for the T-periodic semiprocess φ if:

(a) W, W^- are T-periodic,
(b) $(W,W^-)\cap([0,T]\times X)$ is a pair of compact sets,
(c) for every $\sigma\in\mathbb{R}$, $x\in\partial W_\sigma$ there exists $\delta>0$ such that for all $t\in(0,\delta)$ $\varphi_{(\sigma,t)}(x)\notin W_{\sigma+t}$ or $\varphi_{(\sigma,t)}(x)\in \operatorname{int}W_{\sigma+t}$,
(d) $W^-=\{(\sigma,x)\in W:$ exists $\delta>0$ for all $t\in(0,\delta)$ such that $\varphi_{(\sigma,t)}(x)\notin W_{\sigma+t}\}$, $W^+:=\operatorname{cl}(\partial W\setminus W^-)$,
(e) for all $z\in W^+$ and all $v\colon(a,0]\to\mathbb{R}\times X$ a left solution through z there is $a\le b<0$ such that for all $t\in(b,0)$ $v(t)\notin W$,
(f) there exists $\eta>0$ such that for al $x\in W^-$ there exists $t>0$ such that for all $\tau\in(0,t]$ $\Phi_\tau(x)\notin W$ and $\rho(\Phi_t(x),W)>\eta$.

Roughly speaking, W^- and W^+ are sections for the semiflows, through which trajectories leave and enter W, respectively. One can check that a T-periodic isolating segment is a Waewski set (see [2]).

3. Chaos via Lefschetz Fixed Point Theorem

Recall that a topological space is an ENR if and only if it is homeomorphic with a retract of an open set in some Euclidean space.

Definition 4. Let $(W,W^-)\subset\mathbb{R}\times X$ be a pair of subsets. We call W a *T-periodic regular isolating segment* for the T-periodic semiprocess φ if W is T-periodic isolating segment for φ and the following conditions hold

(b') $(W,W^-)\cap([0,T]\times X)$ is a pair of compact ENR's
(g) there exists a T-periodic homeomorphism of pairs

$$h\colon\mathbb{R}\times(W_0,W_0^-)\to(W,W^-)$$

such that $\pi_1=\pi_1\circ h$.

Let W be a T-periodic regular isolating segment for a given T-periodic process φ. Following [8] we define a homeomorphism

$$\tilde h\colon(W_0,W_0^-)\to(W_T,W_T^-)=(W_0,W_0^-)$$

by $\tilde{h}(x) = \pi_2(h(T, \pi_2 h^{-1}(0, x)))$ for $x \in W_0$. A different choice of the homeomorphism h in (vii) leads to a map which is homotopic to $\tilde{h}$ (see [6]), hence the automorphism

$$\mu_W = \tilde{h}_*: H(W_0, W_0^-) \to H(W_0, W_0^-)$$

induced by $\tilde{h}$ in singular homology, is an invariant of the segment W. Recall that its Lefschetz number is defined as

$$\mathrm{Lef}(\mu_w) = \sum_{n=0}^{\infty} (-1)^n \mathrm{tr} \tilde{h}_{*n}.$$

In particular, if $\mu_W = \mathrm{id}_{H(W_0, W_0^-)}$ then it is equal to the Euler characterictic $\chi(W_0, W_0^-)$.

Let $\Sigma_k = \{0, \ldots, k-1\}^{\mathbb{Z}}$ and $\sigma: \Sigma_k \to \Sigma_k$ be a shift map. For $\alpha \in \Sigma_k$ by $p(\alpha)$ we will denote its principal period. For α non-periodic we set $p(\alpha) = \infty$.

Definition 5. We say that a T-periodic semiprocess φ is Σ_k-*chaotic* if there exists a compact set I invariant with respect to the Poincaré map $\varphi_{(0,T)}$ and a continuous surjective map $g: I \to \Sigma_k$ such that

(j) $\sigma \circ g = g \circ \varphi_{(0,T)}$,
(jj) for every periodic sequence $\alpha \in \Sigma_k$ its counterimage $g^{-1}(\alpha)$ contains at least one $p(\alpha)$ periodic point of $\varphi_{(0,T)}$.

Since I is compact and the set of periodic points is dense in Σ_k, condition (jj) implies that g must be a surjection. It follows in particular that a Σ_k-chaotic semiprocess, for every $l \in \mathbb{N}$, has a periodic orbit with the principal period lT and the topological entropy of the Poincaré map is positive.

The following theorem was proved in [8]

Theorem 1. *Let U, W be two T-periodic regular isolating segments for the T-periodic process φ. Assume that*

(a) $U \subset W$, $U_0 = W_0$, $U_0^- = W_0^-$,
(b) $\mu_U = \mathrm{id}_{H(W_0, W_0^-)} = \mu_W^2$,
(c) $\mathrm{Lef}(\mu_W) \neq \chi(U_0, U_0^-) \neq 0$.

Then φ is Σ_2-chaotic.

In [8] the authors applied this result to prove that the equation $(*)$ is Σ_2-chaotic provided $0 < \kappa \leq 1/288$. The result follows by the construction of two $T = 2\pi/\kappa$-periodic regular isolating segments U and W satisfying assumptions of the above theorem. The symbolic dynamics is obtained as follows. Let

$$I_W := \{x \in W_0 : \varphi(0, t, x) \in W_t, \text{ for } t \in \mathbb{R}\}$$

The semiconjugacy map $g: I_W \to \Sigma_2$ is given by the rule: $g(x)_l = 1$, when a trajectory of $x \in I_W$ leaves a small segment U in a moment between lT and $(l+1)T$, and $g(x)_l = 0$, otherwise.

4. Main result

Let $T = 2\pi/\kappa$ be a period of equation $(*)$ and φ be a T-periodic local process generated by $(*)$. Let $P = \varphi_{(0,T)}$ be a Poincaré map.

Theorem 2. *There exists a compact set I, such that $P(I) = I$ and a continuous map $g: I \to \Sigma_3$, such that*

(a) $\sigma \circ g = g \circ P$,
(b) $g(I) = \Sigma_3$,
(c) *if $\alpha \in \Sigma_3$ is a periodic sequence, then $g^{-1}(\alpha)$ contains a point with a period equal to $p(\alpha)$,*
(d) *for any $\alpha \in \Sigma_3$ such that $\alpha_i = 0$ for $i > i_0$, $g^{-1}(\alpha)$ contains a point x such that $\lim_{t\to\infty} \varphi(0,t,x) = (0,0)$,*
(e) *for any $\alpha \in \Sigma_3$ such that $\alpha_i = 0$ for $i < i_0$, $g^{-1}(\alpha)$ contains a point x such that $\lim_{t\to-\infty} \varphi(0,t,x) = (0,0)$,*
(f) *for any $\alpha \in \Sigma_3$ such that $\alpha_i = 0$ for $|i| > i_0$ $g^{-1}(\alpha)$ contains a point x such that $\lim_{t\to\pm\infty} \varphi(0,t,x) = (0,0)$.*

The main idea of the proof will be described in the next sections. For more details we refer the reader to [13] and [14]. Assertion (a)–(c) are proved in [13]. The new element in [14] are the statements about existence of complicated orbits (α in (d)–(f) can be an arbitrary finite sequence of $-1, 0, 1$) which are asymptotic to the hyperbolic periodic orbits passing through the origin. Assertion (vi) is a statement about an existence of plenty of orbits homoclinic to the origin, in literature (see [1], [3] and references cited there) such orbits are often called multibump solutions. The proof consists in the construction of a topological model for a process induced by $(*)$, for which assertions (a)–(f) hold. Then we show by a continuation argument, that these assertion also hold for φ. Note that to obtain a symbolic dynamics (assertion (a)–(c) we have built in [13] (see also [8]) "large" isolating segments. To deal with solutions which are asymptotic to the origin we shrink the isolating segments almost to a line. Obviously the actual proof is much more complicated (see [14]).

5. Continuation result

Let $H: [0,1] \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous family of T-periodic semiprocesses and Φ^λ be a T-periodic semiflow generated by H_λ.

Let U, W be two T-periodic isolating segments for the semiprocess H_λ for every $\lambda \in [0,1]$, such that

$$U \subset W, \quad U_0 = W_0, \quad U_0^- = W_0^-.$$

Let

$$U^- = \bigcup_{l=1}^{K} U_l^-$$

be a decomposition of the exit set U^- into a disjoint union of closed T-periodic sets U_l^-.

Definition 6. Let $\alpha = (\alpha_0, \dots, \alpha_{n-1}) \in \{0, \dots, K\}^n$ and $\lambda \in [0,1]$. We define $(\operatorname{int}(W_0)_\alpha(H_\lambda)$ as a set of points fulfilling the following conditions

(a) $H_\lambda(0, lT, x) \in \operatorname{int} W_0$, $l = 0, \dots, n$,
(b) $\Phi_t^\lambda(0,x) = (t, H_\lambda(0,t,x)) \in \operatorname{int} W$, $t \in (0, nT)$,
(c) if $\alpha_l = 0$, then $\Phi_{lT+t}(0,x) \in \operatorname{int} U$, for $t \in (0,T)$,
(d) if $\alpha_l \neq 0$, then $\Phi_{lT}(0,x)$ leaves U in time less then T through $U_{\alpha_l}^-$.

Under the above assumptions and notation we have (see [13])

Theorem 3. *Assume that there exists $\eta > 0$ such that for every $\lambda \in [0,1]$ and for every $x \in W^-$ ($x \in U^-$) there exists $t > 0$ such that for $0 < \tau \le t$ holds $\Phi_\tau^\lambda(x) \notin W$ and $\rho(\Phi_t^\lambda(x), W) > \eta$ (resp. $\Phi_\tau^\lambda \notin U$ and $\rho(\Phi_t^\lambda(x), U) > \eta$). Then for every $\alpha = (\alpha_0, \dots, \alpha_{n-1}) \in \{0, \dots, K\}^n$ the fixed point indices*

$$\operatorname{ind}(H_\lambda(0,nT), (\operatorname{int} W_0)_\alpha(H_\lambda))$$

are well defined and equal (i.e. do not depend on λ).

It should be stressed that we have a good reason to state our definitions and the above continuation theorem for semiprocesses: a Poincaré map of our model T-periodic semiprocess constructed in next section is 1-dimensional, hence is not invertible. This fact (one-dimensionality of this map) enables us to calculate the various fixed point indices of interest.

6. Model semiprocess

Let $U \subset W$ be a periodic isolating segments for $(*)$ constructed in [13] for $0 < \kappa \le 0.495$. We start with the description of U and W. W (the big segment) is a twisted prism with a square base centered at origin. Its cross-sections W_t will be obtained by rotating a base $W_0 = U_0 = [-R,R]^2$ with the angular velocity $\kappa/2$ over the t-interval $[0, 2\pi/\kappa]$. The segment U is a regular square-based prism with broadening ends. Its cross-sections U_t corresponding to t near the center of the interval have the side of length $2r < 2R$ and they are broadened to the length $2R$ when t approaches to 0 or $2\pi/\kappa$. The exit sets $W_0^- = U_0^-$ consist of two components. We refer the reader to [8] for the picture.

Let U^{+1}, U^{-1} be two connected components of U^-, the right one and the left one, respectively.

The following theorem was proved in [13].

Theorem 4. *There exists a semiprocess φ^M such that there are disjoint closed segments $J_{-1} = [-b,-a]$, $J_0 = [-c,c]$, $J_1 = [a,b]$, $J_l \subset (-R,R)$ for $l = -1, 0, 1$ and a continuous function $f\colon J_{-1} \cup J_0 \cup J_1 \to [-R,R]$ such that*

(a) $Z := \{p \in W_0 : \varphi^M(0,t,p) \in W,\ t \in [0,T]\} = \{J_{-1} \cup J_0 \cup J_1\} \times [-R,R]$,

(b) $Z_0 := \{p \in W_0 : \varphi_M(0,t,p) \in U,\, t \in [0,T]\} = J_0 \times [-R,R]$,

(c) $Z_l = \{p \in Z : p$ leaves U through U^l in time $\leq T\} = J_l \times [-R,R]$, $l = -1, 1$,

(d) $\varphi^M(0,T,(x,y)) = (f(x),0)$ *where* $(x,y) \in Z$,

(e) $f(-x) = -f(x)$, $f(c) = R$, $f(a) = R$, $f(b) = -R$,

(f) f *is strictly increasing on* J_0 *and is strictly decreasing on* J_{-1} *and* J_1,

(g) *there exists a continuous family of* T*-periodic semiprocesses* H_λ *such that* $H_0 = \varphi$, $H_1 = \varphi^M$,

(h) *for every* $\lambda \in [0,1]$ *the pairs* (U,U^-) *and* (W,W^-) *are periodic isolating segments for* H_λ *and the assumptions in continuation theorem hold,*

(i) *for every* $\alpha = (\alpha_0, \dots, \alpha_{n-1}) \in \{-1,0,1\}^n$ *the fixed point index*

$$\mathrm{ind}(\varphi^M_{0,nT}, (\mathrm{int}\, W_0)_\alpha)$$

is nontrivial.

The assertion (a)–(c) in the main theorem (Theorem 2) are obtained as follows. Let

$$I = I_W = \{x \in W_0 : \varphi(0,t,x) \in W, \text{ for } t \in \mathbb{R}\}.$$

The semiconjugacy map $g\colon I \to \Sigma_3$ is given by $g(x)_l = 0$ if $\varphi(0,(lt,(l+1)T),x) \subset U$, $g(x)_l = -1$ if $\varphi(0,lT,x)$ leaves U in time less than T through U^{-1} and $g(x)_l = +1$ if $\varphi(0,lT,x)$ leaves U in time less than T through U^{+1}. The above theorem imply that all fixed point indices for periodic points with prescribed periodic sequence of symbols are nontrivial. Hence $g(I)$ contains all periodic sequences from Σ_3. But the set of all periodic sequences is dense in Σ_3, so $g(I) = \sigma_3$. The proof of assertion (d)–(f) of Theorem 2 is technically more complicated and we refer the reader to [14] for details.

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EXISTENCE OF SOLUTIONS OF INITIAL-BOUNDARY VALUE PROBLEMS FOR THE HEAT-EQUATION IN SOBOLEV SPACES WITH THE WEIGHT AS A POWER OF THE DISTANCE TO SOME AXIS

Wojciech M. Zajączkowski

Abstract. We examine initial-boundary value problems for the heat-equation in a domain in $\mathbb{R}^3$ which contains an axis. Assuming that data functions belong to Sobolev spaces with weights equal to a power of the distance from the axis we prove existence of solutions in the same kind of weighted Sobolev spaces.

1. Introduction

In this paper we prove the existence and show some regularity properties of solutions to the Dirichlet

$$\begin{aligned} &u_t - \Delta u = f && \text{in } \Omega \times (0,T) \equiv \Omega^T, \\ &u = b && \text{on } S \times (0,T) \equiv S^T, \\ &u|_{t=0} = u_0 && \text{in } \Omega, \end{aligned} \tag{1.1}$$

and the Neumann

$$\begin{aligned} &u_t - \Delta u = f && \text{in } \Omega^T, \\ &\frac{\partial u}{\partial n} = b && \text{on } S^T, \\ &u|_{t=0} = u_0 && \text{in } \Omega, \end{aligned} \tag{1.2}$$

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problems in a bounded domain $\Omega \subset \mathbb{R}^3$ with the boundary S.

We assume that through the domain Ω passes a given axis L. We consider problems (1.1) and (1.2) in Sobolev spaces $H^{k+2}_\mu(\Omega^T)$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mu \in \mathbb{R}_+$, with the weight equal to the power function of the distance from the axis L (for the definition see Section 2).

To prove the existence of solutions to problems (1.1) and (1.2) we use the technique of regularizer. Therefore we have to consider problems (1.1) and (1.2) locally. The most difficult problems appear when we localize them to any neighbourhood of points from the distinguished axis. Therefore we shall restrict our considerations to such neighbourhoods only. Moreover, we consider the internal points only. To examine a behaviour of solutions we apply the Kondratiev theory (see [1]) so following [6], [7] we consider the following artificial problem

$$\begin{aligned} &\Delta' u = f \quad \text{in } \mathbb{R}^2, \\ &u|_{\gamma_0} = |_{\gamma_{2\pi}}, \quad u_\varphi|_{\gamma_0} = u_\varphi|_{\gamma_{2\pi}}, \end{aligned} \tag{1.3}$$

where $\Delta' = \partial^2_{x_1} + \partial^2_{x_2}$, $\gamma_0 = \gamma_{2\pi} = \{x \in \mathbb{R}^3 : x_2 = 0\}$ and r, φ are the polar coordinates.

Moreover, we introduce such system of coordinates that the axis x_3 is the distinguished axis L.

2. Notation and auxiliary results

Let (x_1, x_2, x_3) be a system of local coordinates such that the axis x_3 is the distinguished axis L passing through Ω. Let $x' = (x_1, x_2)$ and $|x'| = \sqrt{x_1^2 + x_2^2}$. Then we introduce space $H^k_\mu(\mathbb{R}^2)$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mu \in \mathbb{R}_+$, with the norm

$$\|u\|_{H^k_\mu(\mathbb{R}^2)} = \Big(\sum_{|\alpha'| \le k} \int_{\mathbb{R}^2} |D^{\alpha'}_{x'} u|^2 |x'|^{2(\mu - (k - |\alpha'|))}\, dx' \Big)^{1/2},$$

where $\alpha' = (\alpha_1, \alpha_2)$, $|\alpha'| = \alpha_1 + \alpha_2$ is multiindex and $D^{\alpha'}_{x'} = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2}$. Moreover, $H^0_\mu(\mathbb{R}^2) = L_{2,\mu}(\mathbb{R}^2)$ and $\|u\|_{L^k_{2,\mu}(\mathbb{R}^2)} = \sum_{|\alpha'|=k} \|D^{\alpha'}_{x'} u\|_{L_{2,\mu}(\mathbb{R}^2)}$.

Let $k \in \mathbb{N}_0$, $\mu \ge 0$. By $\overset{\circ}{L}{}^{k,k/2}_{2,\mu}(\mathbb{R}^3 \times (0,T))$ we mean a closure of a set of smooth functions with compact support vanishing for $t \le 0$ in the norm

$$\begin{aligned} \|u\|_{\overset{\circ}{L}{}^{k,k/2}_{2,\mu}(\mathbb{R}^3 \times (0,T))} = \Big(&\sum_{|\alpha| + 2a = k} \int_0^T dt \int_{\mathbb{R}^3} |D^\alpha_x \partial^a_t u(x,t)|^2 |x'|^{2\mu}\, dx \\ &+ \sum_{2a + |\alpha| = k-1} \int_{\mathbb{R}^3} |x'|^{2\mu} \int_{-\infty}^T dt \\ &\cdot \int_{-\infty}^T dt' |D^\alpha_x \partial^a_t u(x,t) - D^\alpha_x \partial^a_{t'} u(x,t')|^2 \frac{1}{|t-t'|^2} \Big)^{1/2}, \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $D_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$. Boundedness of the norm means that the following conditions are satisfied

$$\frac{\partial^i u}{\partial t^i}|_{t=0} = 0 \quad \text{for} \begin{cases} i & \le [k/2] \quad \text{for } k \text{ odd}, \\ i & < [k/2] \quad \text{for } k \text{ even}. \end{cases}$$

Taking the Fourier transform

$$\widetilde{u}(x', \xi, \xi_0) = (2\pi)^{-1} \int_{\mathbb{R}} dx_3 \int_{-\infty}^{\infty} u(x,t) e^{-i(x_3\cdot\xi + t\xi_0)}\, dt$$

we write the norm of $\overset{\circ}{L}{}_{2,\mu}^{k,k/2}(\mathbb{R}^3 \times (0,\infty))$ in the form

$$\|u\|_{\overset{\circ}{L}{}_{2,\mu}^{k,k/2}(\mathbb{R}^3\times(0,\infty))} = \bigg(\int_{\mathbb{R}} d\xi \int_{-\infty}^{\infty} d\xi_0 \sum_{j=0}^{k} \|\widetilde{u}\|_{L_{2,\mu}^{k-j}(\mathbb{R}^2)} (|\xi|^2 + |\xi_0|)^j \bigg)^{1/2}.$$

Next we introduce anisotropic spaces $H_\mu^{k,k/2}(\mathbb{R}^3 \times (0,T))$ with the norm

$$\|u\|_{H_\mu^{k,k/2}(\mathbb{R}^3\times(0,T))} = \bigg(\sum_{|\alpha|+2a\le k} \int_0^T dt \int_{\mathbb{R}^3} |D_x^\alpha \partial_t^a u|^2 |x'|^{2(\mu-(k-|\alpha|-2a))}\, dx \bigg)^{1/2}.$$

Let $\zeta(t) \in C_0^\infty(\mathbb{R}_+)$ be a monotone function such that $\zeta(t) = 1$ for $t \le 1/2$ and $\zeta(t) = 0$ for $t \ge 1$.

We introduce also the spaces

$$\|u\|_{W_{2,\mu}^{k,k/2}(\Omega^T)} = \bigg(\sum_{|\alpha|+2a\le k} \int_{\Omega^T} |D_x^\alpha \partial_t^a u|^2 |x'|^{2\mu}\, dx\, dt \bigg)^{1/2}, \quad k \in \mathbb{N}_0,$$

and

$$\|u\|_{L_{p,q}(\Omega^T)} = \bigg(\int_0^T \int_\Omega |u(x,t)|^p\, dx \bigg)^{q/p} dt)^{1/q}, \quad p, q \in [1,\infty].$$

From [6] and [7] we know that (1.3) has eigenvalues equal to integer numbers. Moreover from [6] and [7] we have

Theorem 2.1. *Assume that $f \in H_\mu^k(\mathbb{R}^2)$, $\mu \in (0,1)$, $k \in \mathbb{N}_0$, $h = 1 + k - \mu \neq 0$. Then there exists a unique solution $u \in H_\mu^{k+2}(\mathbb{R}^2)$ of (1.3) such that*

$$\|u\|_{H_\mu^{k+2}(\mathbb{R}^2)} \le c\|f\|_{H_\mu^k(\mathbb{R}^2)}. \tag{2.1}$$

Moreover,

Theorem 2.2. *Assume that $f \in H_\mu^k(\mathbb{R}^2) \cap H_\mu^{k'}(\mathbb{R}^2)$, $\mu, \mu' \in (0,1)$, $k, k' \in \mathbb{N}_0$ and $h' = 1 + k' - \mu' > 1 + k - \mu = h$, where $h, h' \notin \mathbb{Z}$. Assume that $l_1, l_2, \ldots, l_\mu \in \mathbb{Z}$ and $l_1, l_2, \ldots, l_\mu \in (h, h')$. Then there exist two solutions of problem (1.3), $u \in H_\mu^{k+2}(\mathbb{R}^2)$ and $u' \in H_{\mu'}^{k'+2}(\mathbb{R}^2)$ such that*

$$\|u\|_{H_\mu^{k+2}(\mathbb{R}^2)} \le c\|f\|_{H_\mu^k(\mathbb{R}^2)}, \qquad \|u'\|_{H_{\mu'}^{k'+2}(\mathbb{R}^2)} \le c\|f\|_{H_{\mu'}^{k'}(\mathbb{R}^2)}, \tag{2.2}$$

$$u = \sum_{\sigma=l_1}^{l_\mu} (a_\sigma r^\sigma \sin\sigma\varphi + b_\sigma r^\sigma \cos\sigma\varphi) + u'. \tag{2.3}$$

3. Existence of solutions for (1.1) in a neighbourhood of L

To show existence of solutions to problem (1.1) in some weighted Sobolev spaces we apply the methods from [3], [5], [7]. Localizing problem (1.1) in a neighbourhood of the axis L we consider the following problem

$$\begin{gathered} u_t - \Delta u = f, \\ u|_{\Gamma_0} = u|_{\Gamma_{2\pi}}, \quad \frac{\partial u}{\partial x_2}\bigg|_{\Gamma_0} = \frac{\partial u}{\partial x_2}\bigg|_{\Gamma_{2\pi}}, \quad u|_{t=0} = u_0, \end{gathered} \tag{3.1}$$

where $\Gamma_0 = \Gamma_{2\pi} = \{x \in \mathbb{R}^3 : x_2 = 0\}$ and f has a compact support. In the cylindrical coordinates r, φ, z the axis L is determined by $r = 0$.

First we have to consider the problem with vanishing initial data. Let $\widetilde{u}_0$ be an extension of u_0 such that

$$\widetilde{u}|_{t=0} = u_0. \tag{3.2}$$

Then introducing the new quantity

$$v = u - \widetilde{u}_0, \tag{3.3}$$

we write (3.1) in the form

$$\begin{gathered} v_t - \Delta v = f - (\widetilde{u}_{0t} - \Delta\widetilde{u}_0) \equiv g, \\ v|_{\Gamma_0} = v|_{\Gamma_{2\pi}}, \quad v_{x_2}|_{\Gamma_0} = v_{x_2}|_{\Gamma_{2\pi}}, \quad v|_{t=0} = 0. \end{gathered} \tag{3.4}$$

where we used the compatibility conditions for u_0.

By a weak solution to (3.4) we mean a function v which satisfies the following integral identity

$$\int_0^T dt \int_{\mathbb{R}^3} (-v\eta_t + \nabla v \cdot \nabla \eta)\, dx = \int_0^T dt \int_{\mathbb{R}^3} dx g\eta, \tag{3.5}$$

which holds for any η such that $\eta_t, \nabla\eta \in L_2(\mathbb{R}^3 \times (0,T))$ and $\eta(T) = 0$.

For the weak solutions (3.5) we have the estimate

$$\begin{aligned} &\int_{\mathbb{R}^3} |v|^2\, dx + \int_0^T dt \int_{\mathbb{R}^3} |\nabla v|^2\, dx + \int_0^T dt \int_{\mathbb{R}^3} \frac{1}{t} |v(x,t)|^2\, dx \\ &\quad + \int_{\mathbb{R}^3} dx \int_0^T dt \int_0^T dt' \frac{|v(x,t) - v(x,t')|^2}{|t-t'|^2} \le c\|g\|_{L_{r',q'}(\mathbb{R}^3 \times (0,T))}, \end{aligned} \tag{3.6}$$

where $1/r + 1/r' = 1$, $1/q + 1/q' = 1$, $1/r + 3/2q = 3/4$.

In view of (3.6) we have existence of weak solutions in such classes that (3.6) holds.

Now we show higher regularity of the weak solution.

Lemma 3.1. *Assume that* $g \in L_{2,\mu}(\mathbb{R}^3 \times (0,T))$. *Then the weak solution of* (3.4) *is such that* $v \in \overset{\circ}{L}{}^{2,1}_{2,\mu}(\mathbb{R}^3 \times (0,T))$ *and the estimate holds*

$$\|v\|_{\overset{\circ}{L}{}^{2,1}_{2,\mu}(\mathbb{R}^3\times(0,T))} \le c\|g\|_{L_{2,\mu}(\mathbb{R}^3\times(0,T))}. \tag{3.7}$$

Moreover, let $v|_{r=0} = v_*$. *Then* $v - v_* \in H^{2,1}_\mu(\mathbb{R}^3 \times (0,T))$ *and*

$$\|v - v_*\|_{H^{2,1}_\mu(\mathbb{R}^3\times(0,T))} \le c\|g\|_{L_{2,\mu}(\mathbb{R}^3\times(0,T))}. \tag{3.8}$$

Proof. The first part of the proof follows easily from the proof of Theorem 2.1 from [3]. Extending function g with respect to t on $(0,\infty)$ we apply the Fourier transform with respect to x_3 and t to the problem (3.4). Then we obtain

$$\begin{aligned} -\Delta'\widetilde{v} + q\widetilde{v} &= \widetilde{g} \quad \text{in } \mathbb{R}^2, \\ \widetilde{v}|_{\gamma_0} = \widetilde{v}|_{\gamma_{2\pi}}, &\quad \widetilde{v}_{x_2}|_{\gamma_0} = \widetilde{v}_{x_2}|_{\gamma_{2\pi}}, \end{aligned} \tag{3.9}$$

where $q = \xi^2 + i\xi_0$.

The problem (3.9) has a generalized solution $\widetilde{v} \in H^1(\mathbb{R}^2)$ satisfying the integral identity

$$\int_{\mathbb{R}^2} (\nabla'\widetilde{v}\cdot\nabla'\varphi + q\widetilde{v}\varphi)\,dx' = \int_{\mathbb{R}^2} \widetilde{g}\varphi\,dx', \tag{3.10}$$

which holds for any $\varphi \in H^1(\mathbb{R}^2)$. Inserting $\varphi = \widetilde{v}(1 - i\,\mathrm{sgn}\,\xi_0)|q|^{1-\mu}$ we obtain

$$|q|^{1-\mu}\int_{\mathbb{R}^2} (|\nabla'\widetilde{v}|^2 + |q||\widetilde{v}|^2)\,dx' \le c\int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{2\mu}\,dx'. \tag{3.11}$$

Next inserting $\varphi = \widetilde{v}V_s(x',q)(1 - i\,\mathrm{sgn}\,\xi_0)$, where

$$V_s = \min(s|q|^{-\mu}, \max(|x'|^{2\mu}, |q|^{-\mu}))|q|,$$

into (3.11) and passing with s to ∞ we obtain

$$|q|\int_{\mathbb{R}^2} (|\nabla'\widetilde{v}|^2 + |q||\widetilde{v}|^2)|x'|^{2\mu}\,dx' \le c\int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{2\mu}\,dx'. \tag{3.12}$$

Now using Theorem 2.1 and (3.12) we obtain

$$\int_{\mathbb{R}^2}\bigg(\sum_{|\alpha'|=2} |D^{\alpha'}_{x'}\widetilde{v}|^2 + |q||\nabla'\widetilde{v}|^2 + |q|^2|\widetilde{v}|^2\bigg)|x'|^{2\mu}\,dx' \le c\int_{\mathbb{R}^2} |\widetilde{g}|^2|x'|^{2\mu}\,dx'. \tag{3.13}$$

Taking the inverse Fourier transform we get (3.7).

To show (3.8) we repeat the considerations from the proof of Lemma 3.1 from [7]. Introducing the function

$$\widetilde{v}_R = \widetilde{v}\zeta\bigg(\frac{|q|^{1/2}|x'|}{R}\bigg), \tag{3.14}$$

where R will be chosen sufficiently large, we write problem (3.9) in the form

$$\begin{aligned} &-\Delta'\widetilde{v}_R + q\widetilde{v}_R = \widetilde{g}\zeta - 2\nabla'\widetilde{v}\nabla'\zeta - \widetilde{v}\Delta'\zeta \equiv h_R,\\ &\widetilde{v}_R|_{\gamma_0} = \widetilde{v}_R|_{\gamma_{2\pi}}, \quad \widetilde{v}_{R,\varphi}|_{\gamma_0} = \widetilde{v}_{R,\varphi}|_{\gamma_{2\pi}}. \end{aligned} \tag{3.15}$$

where $\widetilde{v}$ in the *r.h.s.* is the weak solution.

Now, exactly as in [7], we show existence of two solutions $\widetilde{v}_R^1 \in H_\mu^2(\mathbb{R}^2)$ and $\widetilde{v}_R^2 \in H_{1+\mu}^2(\mathbb{R}^2)$ and the relation $\widetilde{v}_R^2 = \widetilde{v}_R^1 + c_0$, where $c_0 = \widetilde{v}|_{r=0}$.

Therefore we obtain (3.8). This concludes the proof. □

Considering problem (3.1) with nonvanishing initial data we assume that $u_0 \in H_\mu^1(\mathbb{R}^3)$ so there exists an extension for $t > 0$ denoted by $\widetilde{u}_0 \in H_\mu^{2,1}(\mathbb{R}^3 \times (0,T))$ and

$$\|\widetilde{u}_0\|_{H_\mu^{2,1}(\mathbb{R}^3\times(0,T))} \le c\|u_0\|_{H_\mu^1(\mathbb{R}^3)}. \tag{3.16}$$

Then we have that

$$\begin{aligned} \|g\|_{L_{2,\mu}(\mathbb{R}^3\times(0,T))} &\le c(\|f\|_{L_{2,\mu}(\mathbb{R}^3\times(0,T))} + \|\widetilde{u}_0\|_{H_\mu^{2,1}(\mathbb{R}^3\times(0,T))})\\ &\le c(\|f\|_{L_{2,\mu}(\mathbb{R}^3\times(0,T))} + \|u_0\|_{H_\mu^1(\mathbb{R}^3)}). \end{aligned} \tag{3.17}$$

In view of (3.16) and (3.17) we have

Lemma 3.2. *Assume that $f \in L_{2,\mu}(\mathbb{R}^3 \times (0,T))$, $\mu \in (0,1)$, $u_0 \in H_\mu^1(\mathbb{R}^3)$. Then the weak solution of* (3.1) *is such that $u \in W_{2,\mu}^{2,1}(\mathbb{R}^3 \times (0,T))$ and*

$$\|u\|_{W_{2,\mu}^{2,1}(\mathbb{R}^3\times(0,T))} \le c(\|f\|_{L_{2,\mu}(\mathbb{R}^3\times(0,T))} + \|u_0\|_{H_\mu^1(\mathbb{R}^3)}). \tag{3.18}$$

Moreover, we have

$$\|u - u(0)\|_{H_{2,\mu}^{2,1}(\mathbb{R}^3\times(0,T))} \le c(\|f\|_{L_{2,\mu}(\mathbb{R}^3\times(0,T))} + \|u_0\|_{H_\mu^1(\mathbb{R}^3)}). \tag{3.19}$$

where $u(0) = u|_L$.

Repeating the proof of Theorem 3.4 and 3.5 from [7] we have

Lemma 3.3. *Assume that $f \in W_{2,\mu}^{k,k/2}(\mathbb{R}^3 \times (0,T))$, $u_0 \in W_{2,\mu}^{k+1}(\mathbb{R}^3)$, $\mu \in (0,1)$. Then there exists a solution to problem* (3.1) *such that*

$$\begin{aligned} &u \in W_{2,\mu}^{k+2,k/2+1}(\mathbb{R}^3 \times (0,T)),\\ &\|u\|_{W_{2,\mu}^{k+2,k/2+1}(\mathbb{R}^3\times(0,T))} \le c(\|f\|_{W_{2,\mu}^{k,k/2}(\mathbb{R}^3\times(0,T))} + \|u_0\|_{W_{2,\mu}^{k+1}(\mathbb{R}^3)}). \end{aligned} \tag{3.20}$$

To show Lemma 3.3 some considerations from [4] must be used.

4. Existence of solutions to problem (1.1)

First we consider problem (1.1) with $b=0$.

Definition 4.1. By a weak to problem (1.1) with $b=0$ we mean a function satisfying the following integral indentity

$$\int_0^T dt\int_\Omega[-u\eta_t+\nabla u\cdot\nabla\eta]\,dx=\int_\Omega u_0\eta(x,0)\,dx+\int_0^T dt\int_\Omega f\eta\,dx, \tag{4.1}$$

which holds for any $\eta\in H^1(\Omega^T)$ such that $\eta(T)=0$ and $\eta|_S=0$.

From (4.1) we prove the existence of the weak solution and the corresponding estimate

Lemma 4.2. *Assume that $u_0\in L_2(\Omega)$, $f\in L_{r',q'}(\Omega\times(0,T))$, $b=0$, $1/r+1/r'=1$, $1/q+1/q'=1$, $1/r+3/2q=3/4$. Then there exists a weak solution to problem* (1.1) *such that $u\in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;H^1(\Omega))$ and*

$$\int_\Omega|u|^2\,dx+\int_0^T dt\int_\Omega|\nabla u|^2\,dx\le c\|f\|^2_{L_{r',q'}(\Omega^T)}+\|u_0\|^2_{L_2(\Omega)}. \tag{4.2}$$

To show higher regularity we apply the idea of regularizer (see [2]). For this purpose we consider problem (1.1) locally. We distinguish four kinds of subdomains:

(1) neighbourhoods of internal points of L,
(2) neighbourhoods of the points where L meets S,
(3) neighbourhoods of internal points which are in a positive distance from L,
(4) neighbourhoods of points of S which are in a positive distance from L.

Assuming that L is perpendicular to S and that the boundary conditions vanish we can extend the problem (1.1) by reflection with respect to a plane perpendicular to L. In this case we can restrict our considerations to neighbourhoods of kind (1) only because regularity problems in neighbourhood (3) and (4) are well known.

Therefore we have

Theorem 4.3. *Assume that $u_0\in W^{k+1}_{2,\mu}(\Omega)$, $b\in W^{k+3/2,k/2+3/4}_{2,\mu}(S\times(0,T))$, $f\in L_{r,q}(\Omega\times(0,T))\cap W^{k,k/2}_{2,\mu}(\Omega\times(0,T))$, $1/r+3/2q=7/4$, $k\in\mathbb{N}_0$, $\mu\in(0,1)$. Then $u\in W^{k+2,k/2+1}_{2,\mu}(\Omega\times(0,T))$ and*

$$\begin{aligned}\|u\|_{W^{k+2,k/2+1}_{2,\mu}(\Omega\times(0,T))}\le c(&\|u_0\|_{W^{k+1}_{2,\mu}(\Omega)}+\|b\|_{W^{k+3/2,k/2+3/4}_{2,\mu}(S\times(0,T))}\\&+\|f\|_{W^{k,k/2}_{2,\mu}(\Omega\times(0,T))}+\|f\|_{L_{r,q}(\Omega\times(0,T))}).\end{aligned} \tag{4.3}$$

5. Existence of solutions to (1.2)

Similarly as Theorem 4.3 we can prove

Theorem 5.1. *Assume that* $u_0 \in W^{k+1}_{2,\mu}(\Omega)$, $b \in W^{k+1/2,k/2+1/4}_{2,\mu}(S\times(0,T))$, $f \in L_{r,q}(\Omega\times(0,T)) \cap W^{k,k/2}_{2,\mu}(\Omega\times(0,T))$, $1/r + 3/2q = 7/4$, $\mu \in (0,1)$, $k \in \mathbb{N}_0$. *Then* $u \in W^{k+2,k/2+1}_{2,\mu}(\Omega\times(0,T))$ *and*

$$\begin{aligned}(5.1)\quad \|u\|_{W^{k+2,k/2+1}_{2,\mu}(\Omega\times(0,T))} \le c(\|u_0\|_{W^{k+1}_{2,\mu}(\Omega)} &+ \|b\|_{W^{k+1/2,k/2+1/4}_{2,\mu}(S\times(0,T))} \\ &+ \|f\|_{W^{k,k/2}_{2,\mu}(\Omega\times(0,T))} + \|f\|_{L_{r,q}(\Omega\times(0,T))}).\end{aligned}$$

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