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THE ROLE OF VARIOUS KINDS OF CONTINUITY IN THE FIXED POINT THEORY OF SET-VALUED MAPPINGS

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CONTENTS

INTRODUCTION

Let X be a topological space and f — a mapping assigning a nonempty set $f(x) \subset X$ to every point $x \in X$. The point x is called the fixed point of the mapping f if $x \in f(x)$. This work was intended as an attempt to answer the following question: What assumptions are sufficient for the mapping f to have at least one fixed point? There are three kinds of the natural assumptions:

- on the space X,
- on the values $f(x)$,
- on the kind of the continuity of the mapping $f\colon X \to 2^X$.

The space X, generally in this dissertation, is the compact disc B^n in the n-dimensional Euclidean space. One of our fixed point results is formulated for closed topological manifolds. It is interesting that manifolds appear as well as an instrument of the analysis of the set-valued mappings on B^n. The classical fixed point theory deals with the more extensive classes of spaces, e.g. the absolute neighbourhood retracts (ANRs). We work with the smaller class, since our interest is focused on the other two types of assumptions.

The assumptions of the second type mean that some topological properties are satisfied by every value considered as the subspace of X. One of these properties is the compactness, which is assumed throughout the whole thesis. We will impose some conditions not only on every set $f(x)$, but also on the family of all values of the mapping f. Sets which are equally locally contractible (e.l.c.), [2], or equally locally connected in the dimension k (eLC^k), [47], provide with examples of such conditions.

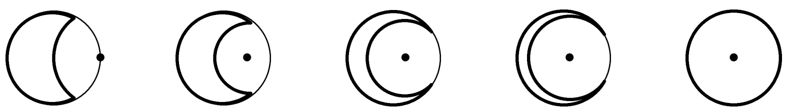
The continuity of set-valued mappings will be considered with respect to one of three metrics on some subsets of 2^X: the Hausdorff metric ρ_s, the Borsuk metric of continuity ρ_c, the Borsuk metric of homotopy ρ_h, [2]. Occasionally there will appear the upper semicontinuity (u.s.c.) or the lower semicontinuity (l.s.c.). We apply the Dyer–Hamstrom and Ferry's results on the completely regular mappings [13] and strongly regular mappings [20] to translate the Borsuk continuities into the language of the fibrations.

We now recall these results of the classical fixed point theory which are especially important for our thesis to create its mathematical context.

- In 1912 Brouwer proved that every continuous map $f\colon B^n \to B^n$ has a fixed point, [4]. The Brouwer Fixed Point Theorem was generalized by Schauder for compact maps of infinite-dimensional normed spaces, [60]. The Schauder Theorem is basic for many applications to differential equations, [12]. Another excellent generalization is the Lefschetz Fixed Point Theorem, [41], [42], [5], [11], [27].
- In 1946 Eilenberg and Montgomery proved the Lefschetz Fixed Point Theorem for u.s.c. set-valued mappings with Q-acyclic values, [16]. The set is Q-acyclic, if its Čech cohomology groups with rational coefficients are isomorphic to these of the one-point space. The star-shaped sets known from the Poincaré Lemma are Q-acyclic. Another example is the real projective space RP^{2n}. The Gćrniewicz generalization of the Eilenberg–Montgomery theorem, [22], opened the doors to many applications of set-valued mappings to differential equations and inclusions, [21].
- In 1947 O'Neill gave an example of the fixed point free mapping $f\colon B^2 \to 2^{B^2}$ which is ρ_s-continuous and takes values homeomorphic to S^1, [57]. Set $\eta(x) = 1 - \|x\| + \|x\|^2$ for $x \in B^2$. Then

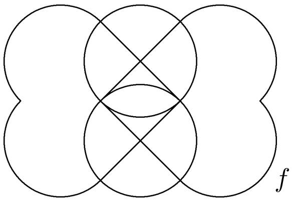
  $$f(x) = \{y \in S^1 : \|y - x\| \geq \eta(x)\} \cup \{y \in B^2 : \|y - x\| = \eta(x)\}.$$

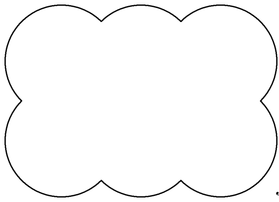
  The following pictures show "moon-shaped" values $f(p) \cong S^1$ for $p = (x, 0)$ with $x = 1, 1/2, 1/4, 1/8, 0$. Each picture shows S^1, p, $f(p)$.

  Of course, if $q = e^{it} \cdot p$ then $f(q) = e^{it} \cdot f(p)$, so arguments and values rotate by the same angle.
- In 1987 Jezierski proved that there is a fixed point free mapping $B^2 \to 2^{B^2}$ which is ρ_s-continuous and takes the finite values who have 1, 2 or 3 elements, [37].
- In 1990 Schirmer defined the fixed point index for bimaps, [61]. The bimap is the ρ_s-continuous mapping which takes values having one or two elements.
- In 1977 Gćrniewicz defined the spheric mappings, [24]. This notion was developed by Gćrniewicz's student Dawidowicz in [9] and [10]. The most general definition comes from [23]. Denote by $Bf(x)$ the set which is

the sum of all bounded components of the complement of $f(x)$ in $\mathbb{R}^n$. Set $\widetilde{f}(x) = f(x) \cup Bf(x)$.





The above figure shows a 1-dimensional continuum $f(x)$ in $\mathbb{R}^2$ shaped as two hearts joined by two wedding rings and a 2-dimensional continuum $\widetilde{f}(x)$ which has a form of the gingerbread "katarzynka" baked in the town Toru as a souvenir, and is connected with a beautiful ancient legend.

The map $f\colon B^n \to 2^{B^n}$ is called the *spheric mapping* if it satisfies the following conditions:

- f is u.s.c.;
- the graph $\Gamma(Bf)$ of the mapping Bf is an open subset of $B^n \times \mathbb{R}^n$ (or equivalently of $B^n \times B^n$, since $Bf(x) \subset IntB^n$);
- the mapping $\widetilde{f}$ has a fixed point.

Let us recall two properties of spheric mappings:

- every spheric mapping has a fixed point, [23];
- every ρ_c-continuous mapping $f\colon B^2 \to 2^{B^2}$ with compact connected values is a spheric mapping, [23].

The following problem (called in [49] the Gćrniewicz Conjecture) was the main source of inspiration for the author of this dissertation.

Problem 0.1 ([23])**.** *Is the Brouwer Fixed Point Theorem true for ρ_c-continuous mappings with compact connected values?*

We now formulate the main results of our thesis:

(1) There is a fixed point free mapping $f\colon B^4 \to 2^{B^4}$, which is ρ_c-continuous and has compact connected values, [51, Theorem 1].
(2) Every ρ_h-continuous mapping $f\colon B^n \to 2^{B^n}$ has a continuous selector and a fixed point, [51, Theorem 2A].
(3) Every ρ_s-continuous mapping $f\colon B^n \to 2^{B^n}$ with eLC^{n-2} values, such that the mapping $\widetilde{f} = f \cup Bf$ has eLC^{n-1} values, is a spheric mapping and has a fixed point, [51, Theorem 3].
(4) Every ρ_c-continuous mapping $f\colon B^n \to 2^{B^n}$ such that $n \neq 6$, and for every $x \in B^n$ $f(x)$ is homeomorphic to either a point or the $n-2$-sphere S^{n-2}, has a fixed point, [49, Theorem 2] (for $n = 3$); [50, Theorem 4] (for $n = 3, 4, 5$); [52] (for $n \neq 6$).

In this thesis the above results are Theorems 3.1, 3.2, 4.4, and 7.5. The Reader can ask: why does the author write this dissertation if the main results are published in the separate papers?

The first reason is that the author's way to Theorem 7.5 is contained in four papers. The truth was revealed little by little in the consecutive steps. On this way we find some conjectures which are only partially confirmed in the following papers, (e.g. see Conjectures 1 and 2 in [48]). The status of these conjectures is clarified in this thesis by means of several examples. On the other hand there were some problems with the formulation of Theorem 7.4, which is the basic tool for proving Theorem 7.5, (see [50, Theorem 2] and the footnote 4 given there, p. 71). We overcome these difficulties with the help of the Stiefel–Whitney classes and more piecewise-linear (p.l.) topology (see Lemmas 5.4, 5.5 and proof of Theorem 7.4, Case 2).

The second reason is that some theorems given in this dissertation are not formulated in the author's earlier papers. These are Theorem 2.1 and Theorem 2.2, where we claim that graphs of some set-valued mappings, which are continuous with respect to the Borsuk metrics, are spaces of some fibrations and bundles. Chapter 1 contains some preliminary notions and theorems which make proofs of Theorem 2.1 and Theorem 2.2 possible (these proofs are given in Chapter 2). The contents of Chapter 1: The continuities; Completely regular mappings; Approximating homotopy equivalences by homeomorphisms; Strongly regular mappings; show that our starting point assumes rather difficult results of Borsuk, Dyer, Hamstrom, Edwards, Kirby, Ferry, Chapman and Jakobsche. In the earlier papers author cited these results, but he did not write Theorems 2.1 and 2.2 explicitly and did not prove it.

We will not describe here the contents of all chapters, but there are two matters which are worth pointing out.

The first matter are the special cases. The eLC^{n-2}-result in Chapter 4 (Theorem 4.4) is followed by its two-dimensional version (Theorem 4.9) in Section 4.4. This low-dimensional version has weaker assumptions. Author does not know if the assumptions of Theorem 4.4 can be weakened in the similar way. This problem is posed in Chapter 8 where some other open problems are collected. Similarly, the Section 7.2 on the $1 - S^{n-2}$-mappings of B^n is followed by the Section 7.3 on the $1 - S^1$-mappings of B^3. This special case was inserted by another reason: author heard someone's opinion that everyone who does not know K-theory is not a mathematician. In these circumstances giving up the proofs of Lemma 7.8 and Theorem 7.7 was impossible.

The second matter is the question, whether all lemmas in this dissertation which have no references to the literature are new results. Author suspects that some of these lemmas could be known earlier (e.g. Lemma 4.6, Lemma 7.6), but he does not know the corresponding references. Though it is difficult to claim that one can say something new about the Stiefel–Whitney classes the author hopes that the description of classes $w_j^{\triangle}$ in Chapter 5 is new. The proof of the fact that $w_j^{\triangle}$ are some characteristic classes is given in the Appendix. The question if these are the Stiefel–Whitney classes is posed in Chapter 8. Theorem 5.8 answers this question for *mod* 2 Euler classes only. We stress the fact that without the notion of the classes $w_j^{\triangle}$ the proof of Theorem 7.5 was known for low dimensions ($n = 3, 4, 5$) only and it used results of Kneser [39], Smale [63] and Hatcher [32] on the equivalence $\mathrm{TOP}(S^{n-2}) \simeq O(n-1)$ for these dimensions, (see [50]). The fixed point theorem for the Brouwer mappings (Theorem 6.3) is a special case of the Saveliev result on the coincidences, [59, Corollary 5.1], but it was obtained independently and in another way in [48].

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CHAPTER 1

PRELIMINARIES

1.1. The continuities

Let X and Y be compact metric spaces and $f: X \to 2^Y$ — the function with nonempty compact values. Let us recall that f is u.s.c. (l.s.c.) if the inverse image $\{x \in X : f(x) \subset U\}$ (respectively, $\{x \in X : f(x) \cap U \neq \emptyset\}$) of any open set $U \subset Y$ is an open subset of X. Under our assumptions, f is u.s.c. and l.s.c. if and only if f is continuous with respect to the *Hausdorff metric* ρ_s, [21], which is defined by the equivalence of the following two conditions:

(a) $\rho_s(A, B) < \varepsilon$,

(b) there are functions $g: A \to B$, $h: B \to A$, not necessarily continuous, which locate each value at a point that is closer than ε to the argument,

for all compact subsets A, B of Y. In other words,

$$\rho_s(A, B) = \inf\{\varepsilon : \text{the condition (b)}\}, \tag{1.1}$$

or equivalently, $\rho_s(A, B) = \inf\{\varepsilon : A \subset O_\varepsilon B \text{ and } B \subset O_\varepsilon A\}$ with

$$O_\varepsilon C \overset{\text{def}}{=} \{y \in Y : d(y, C) \overset{\text{def}}{=} \inf\{d(y, c) : c \in C\} < \varepsilon\} \quad \text{for } C \subset Y.$$

The *Borsuk metric of continuity* ρ_c is defined similarly, by the formula (1.1) with the continuous functions g and h in (b), [2].

If A, B are homeomorphic, then we repeat the same schema and define $\rho_t(A, B)$ by (1.1) with the homeomorphisms g, h in (b). The metric ρ_t is a notion of an auxiliary character.

Let us note that

- $\rho_c \geq \rho_s$,
- $\rho_c = \rho_s$ on finite sets.

The *Borsuk metric of homotopy* ρ_h, [2], is defined on the set of all compact ANRs in Y. We would like to present a definition of ρ_h which is slightly modified when compared with [2], (the difference concerns the description of the function λ_A).

Let us fix $t \geq 0$ and a locally contractible set A which is a compact subset of Y. We define $\phi_A(t)$ to be the lower bound of the set which is composed of 1 and all $s \geq t$ such that every set $T \subset A$ with the diameter $\operatorname{diam}(T) \leq t$ is contractible in a set $S \subset A$ with $\operatorname{diam}(S) \leq s$.

We say that sets from the class $\Theta \subset 2^Y$ are *equally locally contractible* (e.l.c.) if

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{A\in\Theta} \quad \phi_A(\delta) < \varepsilon.$$

Let $\Gamma_{sub}(\phi_A) = \{(t,u) : -\infty < u \leq \phi_A(t) \text{ and } t \geq 0\}$ and

$$\lambda_A(t) = \sup\{s \in R : (t,s) \in \operatorname{conv}(\Gamma_{\mathrm{sub}}(\phi_A))\} \quad \text{for } t \geq 0.$$

Then

$$\rho_h(A,B) = \rho_c(A,B) + \sup\{|\lambda_A(t) - \lambda_B(t)| : t \geq 0\}. \tag{1.2}$$

The explicit formula (1.2) can be replaced by the observation that for the finite-dimensional space Y the mapping $f: X \to 2^Y$ is ρ_h-continuous if and only if f is ρ_s-continuous and has e.l.c. values. This follows from two facts (see [2]):

(i) $\lim_{n\to\infty} \rho_h(A_n, A) = 0$ if and only if $\lim_{n\to\infty} \rho_s(A_n, A) = 0$ and sets $A_1, \ldots, A_n, \ldots,$ are e.l.c.

(ii) the sets forming any ρ_h-compact subset of 2^Y are e.l.c.

In particular, if we replace ρ_c in (1.2) with ρ_s, we get a metric which is equivalent to ρ_h, [2].

The following lemma is of great importance:

Lemma 1.1 (Borsuk, [2, p. 187])**.** *Suppose that all sets from the class $\Theta \subset 2^{\mathbb{R}^n}$ are compact* e.l.c. *Then there is a continuous increasing function $\alpha: (0,1] \to (0,1]$ with $\alpha(\varepsilon) \leq \varepsilon$ (for all ε) such that for every set $\theta \in \Theta$ there exists a retraction*

$$r_\theta: O_{\alpha(1)}(\theta) \to \theta,$$

with $\|r_\theta(x) - x\| < \varepsilon$ for all $x \in O_{\alpha(\varepsilon)}(\theta)$ and $\varepsilon \in (0,1]$.

We prove this lemma (in a slightly modified version) in the Appendix.

1.2. Completely regular mappings

Let B, E be compact metric spaces. The metrics in both spaces are denoted by d.

Definition 1.2 ([13])**.** The map $p: E \to B$ is called the *completely regular mapping* if p is onto B, and for every $b \in B$ and $\varepsilon > 0$ there is $\delta > 0$ such that for any $c \in B$ with $d(b,c) < \delta$ there exists a homeomorphism $h: p^{-1}(b) \to p^{-1}(c)$ such that $d(h(x), x) < \varepsilon$ for all x.

Theorem 1.3 (Dyer–Hamstrom, [13], [7, Proposition 7.1]). *Let $p: E \to B$ be a completely regular mapping,* $\dim(B) < \infty$. *If the group of all homeomorphisms of $p^{-1}(b)$ is a locally contractible space for every b, then p is a locally trivial bundle.*

Theorem 1.4 (Edwards–Kirby, [15]). *The group of all homeomorphisms of any compact topological manifold is a locally contractible space.*

1.3. Approximating homotopy equivalences by homeomorphisms

Let X, Y, Z be compact metric spaces, $\varepsilon > 0$.

Definition 1.5 ([7]). The maps $f_0, f_1: X \to Y$ are called *ε-homotopic* ($f_0 \overset{\varepsilon}{\simeq} f_1$) if there exists a homotopy $f_t: f_0 \simeq f_1$ such that $\operatorname{diam}\{f_t(x) : t \in I\} < \varepsilon$ for every $x \in X$.

Definition 1.6 ([7]). Let us fix $g: Y \to Z$. The mappings $f_0, f_1: X \to Y$ are called *$g^{-1}(\varepsilon)$-homotopic* ($f_0 \overset{g^{-1}(\varepsilon)}{\simeq} f_1$) if there exists a homotopy $f_t: f_0 \simeq f_1$ such that $\operatorname{diam}\{g(f_t(x)) : t \in I\} < \varepsilon$ for every $x \in X$.

Definition 1.7 ([19]). The mapping $f: X \to Y$ is called an ε-domination if there exists a mapping $g: Y \to X$ such that

$$f \circ g \overset{\varepsilon}{\simeq} \mathrm{id}.$$

Definition 1.8 ([19]). The mapping $f: X \to Y$ is called an *ε-equivalence* if there exists a mapping $g: Y \to X$ such that

$$f \circ g \overset{\varepsilon}{\simeq} \mathrm{id} \quad \text{and} \quad g \circ f \overset{f^{-1}(\varepsilon)}{\simeq} \mathrm{id}.$$

Theorem 1.9 (Ferry, [19, Theorem 3]). *Let N be a closed connected topological n-manifold. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that for every n-manifold M (such as N) each δ-domination $f: M \to N$ is an ε-equivalence.*

Theorem 1.10 (Chapman–Ferry–Jakobsche, [7], [35], [36]). *Let N be a closed connected topological n-manifold and $n \neq 4$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that for every closed connected topological n-manifold M and δ-equivalence $f: M \to N$, there exists a homeomorphism $h: M \to N$ which is ε-close to the mapping f, (i.e. $d(f(x), h(x)) < \varepsilon$ for all x). In case $n = 3$ we additionally assume that the manifold M does not contain any "fake 3-cell".*

The "fake 3-cell" is any set homeomorphic to $S_H \setminus \operatorname{Int}(\Delta)$ where S_H is a non-standard p.l. homotopy 3-sphere and Δ — one of its 3-simplices. According to some press reports, we hope that the Poincaré Conjecture is true and, consequently, there is no "fake 3-cell" in topology.[1] The fact, which is essential in this thesis, is that S^3 does not contain any "fake 3-cell", [29].

1.4. Strongly regular mappings

Let B, E be compact metric spaces.

Definition 1.11 ([20])**.** The map $p\colon E \to B$ is called the *strongly regular mapping*, if for every $b \in B$ and $\varepsilon > 0$ there is $\delta > 0$ such that for every $c \in B$ with $d(b,c) < \delta$ there are mappings

$$g\colon p^{-1}(b) \to p^{-1}(c), \qquad h\colon p^{-1}(c) \to p^{-1}(b)$$

and the homotopies

$$j\colon h \circ g \simeq \mathrm{id}, \qquad k\colon g \circ h \simeq \mathrm{id}$$

satisfying the inequalities

$$d(g(x),x) < \varepsilon, \quad d(j_t(x),x) < \varepsilon, \quad d(h(y),y) < \varepsilon, \quad d(k_t(y),y) < \varepsilon$$

for every x, y, t.

Theorem 1.12 (Ferry, [20])**.** *Let $p\colon E \to B$ be the strongly regular mapping onto B, $\dim(B) < \infty$. Suppose that the set $p^{-1}(b)$ is an* ANR *for every $b \in B$. Then p is a Hurewicz fibration.*

[1] For a fuller treatment we refer the reader to the Milnor's article "Towards the Poincaré conjecture and the classification of 3-manifolds", http://www.ams.org/notices/200310/fea–milnor.pdf.

CHAPTER 2

CONTINUITIES AND FIBRATIONS

This is known very well that some conditions on the mapping $f\colon X \to 2^Y$ reflect in the properties of the graph $\Gamma(f)$ or the projection $p\colon \Gamma(f) \to X$. (We will also use $\Gamma(f|X)$ or Γ_X^f to denote the graph $\Gamma(f)$ of f.) Though the previous sentence sounds trivially, because each function is identified with its graph in the set theory, yet, there exist connections which are unexpected. We recall two classical examples, the first — not difficult, the second — very deep and not easy to prove. We assume, that X and Y are compact metric spaces.

- The map f is u.s.c. if and only if the graph $\Gamma(f)$ is a closed subset of $X \times Y$, [21].
- If the map f is u.s.c. with acyclic values, then the projection $p\colon \Gamma(f) \to X$ induces an isomorphism of the Čech cohomology groups, (the Vietoris Theorem). For a fuller treatment we refer the reader to [40].

We now formulate two results, which connect the continuity types of f to some fibre properties of p.

Theorem 2.1. *Let X, Y be compact finite-dimensional metric spaces and $f\colon X \to 2^Y$ — a ρ_h-continuous mapping* (*with values* ANR*s*)*. Then the projection $p\colon \Gamma(f) \to X$ is a Hurewicz fibration.*

Theorem 2.2. *Let X, Y be compact metric spaces, $n \neq 4$ — a fixed natural number,* $\dim(X) < \infty$*, $f\colon X \to 2^Y$ — a ρ_c-continuous mapping with values which are compact connected topological n-manifolds without boundary. Suppose that for $n = 3$ the values contain no "fake* 3*-cell". Then the projection $p\colon \Gamma(f) \to X$ is a locally trivial bundle.*

The proofs of these theorems are straightforward, but only modulo the deep theorems from the Preliminaries. We give these proofs below.

Proof of Theorem 2.2. If f is ρ_t-continuous, then the mapping $p^{-1}\colon X \to 2^{\Gamma(f)}$ with $p^{-1}(x) = \{x\} \times f(x)$ is ρ_t-continuous, which means that p is the completely regular mapping. By the Dyer–Hamstrom Theorem, p is a locally trivial bundle.

What is left is to show that the ρ_c-continuity of f implies (under our assumptions) that f is ρ_t-continuous.

Let us fix a point $b \in X$. We have the sequence of implications:

$$\begin{aligned}
&d(b,c) < \varepsilon_1\\
&\qquad \Rightarrow \rho_c(f(b), f(c)) < \varepsilon_2\\
&\qquad \Rightarrow \text{there are } g\colon f(b) \to f(c) \text{ and } h\colon f(c) \to f(b)\\
&\qquad\quad\ \text{such that } d(g(x),x) < \varepsilon_2 \text{ and } d(h(y),y) < \varepsilon_2 \text{ (for all } x,y)\\
&\qquad \Rightarrow d(h(g(x)),x) \le d(h(g(x)),g(x)) + d(g(x),x) < 2\varepsilon_2\\
&\qquad \Rightarrow h \circ g \overset{\varepsilon_3}{\simeq} \mathrm{id}\\
&\qquad\quad\ \text{(since } f(b)\text{, as the compact ANR, is uniformly locally contractible)}\\
&\qquad \Rightarrow h\colon f(c) \to f(b) \text{ is an } \varepsilon_3\text{-domination}\\
&\qquad \Rightarrow h \text{ is an } \varepsilon_4\text{-equivalence}\\
&\qquad \Rightarrow \text{there is a homeomorphism } k\colon f(c) \to f(b) \text{ with } d(h,k) < \varepsilon_5\\
&\qquad \Rightarrow d(k(y),y) < \varepsilon_5 + \varepsilon_2 \text{ (for all } y)\\
&\qquad \Rightarrow \rho_t(f(b), f(c)) < \varepsilon_5 + \varepsilon_2 < \varepsilon_6.
\end{aligned}$$

Having ε_i we can choose ε_{i-1} for $i = 6, \dots, 2$, such that all the abovementioned inequalities are true. This proves that f is ρ_t-continuous. □

Proof of Theorem 2.1. By assumption, the mapping $p^{-1}\colon X \to 2^{\Gamma(f)}$, $p^{-1}(x) = \{x\} \times f(x)$ is ρ_h-continuous. By the Ferry Theorem, it suffices to prove that p is a strongly regular mapping. Since $\dim(X \times Y) \le \dim(X) + \dim(Y) < \infty$, we can assume that $X \times Y \subset \mathbb{R}^m$ for some m, by the Menger–Nöbeling Theorem.

Let us consider the function $\alpha\colon (0,1] \to (0,1]$ from the Borsuk Lemma for the class

$$\Theta = \{p^{-1}(x) : x \in X\} \subset 2^{\mathbb{R}^m}.$$

The sets from Θ are e.l.c., because p^{-1} is ρ_h-continuous and X is a compact space. We fix a point $b \in X$ and $\varepsilon \in (0,1]$. Choose $\delta > 0$ such that

$$d(b,c) < \delta \ \Rightarrow\ \rho_h(p^{-1}(b), p^{-1}(c)) < \alpha \circ \alpha(\varepsilon).$$

Since $\rho_s \le \rho_c \le \rho_h$, we have

$$p^{-1}(b) \subset O_{\alpha^2(\varepsilon)}(p^{-1}(c)), \qquad p^{-1}(c) \subset O_{\alpha^2(\varepsilon)}(p^{-1}(b)).$$

The retraction

$$r_{p^{-1}(x)}\colon O_{\alpha(1)}(p^{-1}(x)) \to p^{-1}(x)$$

from the Borsuk Lemma will be denoted by r_x.

We define the mappings

$$g: p^{-1}(b) \to p^{-1}(c), \qquad h: p^{-1}(c) \to p^{-1}(b),$$

and homotopies $j_t: h \circ g \simeq \mathrm{id}$, $k_t: g \circ h \simeq \mathrm{id}$, by the formulae:

$$\begin{aligned} g(y) &= r_c(y), & h(z) &= r_b(z), \\ j_t(y) &= r_b((1-t)r_c(y) + ty), & k_t(z) &= r_c((1-t)r_b(z) + tz). \end{aligned}$$

Since $y \in p^{-1}(b) \subset O_{\alpha^2(\varepsilon)}(p^{-1}(c))$,

$$d(g(y), y) = \|r_c(y) - y\| < \alpha(\varepsilon).$$

The homotopy j_t is well-defined, because

$$\begin{aligned} d((1-t)r_c(y) + ty, p^{-1}(b)) &\le \|(1-t)r_c(y) + ty - y\| \\ &= (1-t)\|r_c(y) - y\| < \alpha(\varepsilon) \le \alpha(1). \end{aligned}$$

Set $q = (1-t)r_c(y) + ty$. Then

$$d(j_t(y), y) \le \|r_b(q) - q\| + \|q - y\| < \varepsilon + \alpha(\varepsilon) \le 2\varepsilon.$$

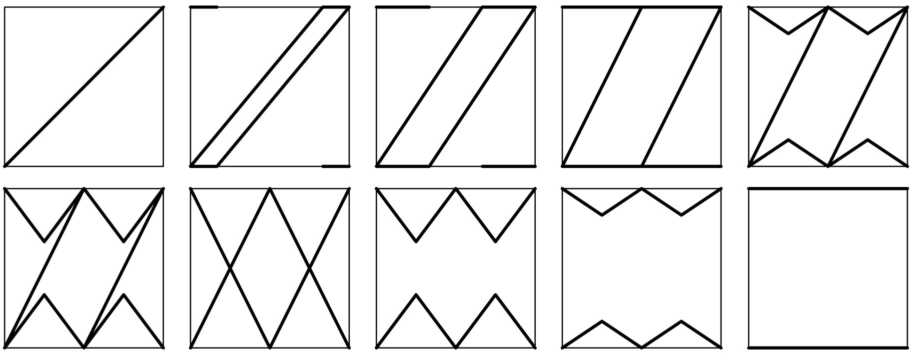
The estimations for h and k_t can be obtained in the similar manner. We conclude that p is a strongly regular mapping. □

CHAPTER 3

THE BORSUK CONTINUITIES AND FIXED POINTS

3.1. A fixed point free ρ_c-continuous mapping

We give here the negative answer to the Problem 0.1, applying the Jezierski example of a fixed point free ϱ_s-continuous mapping of B^2 with values being finite sets, [37]. For the sake of completeness of our considerations we recall below Jezierski's ρ_s-continuous homotopy $\chi^J\colon S^1 \times I \to 2^{S^1}$ between the single-valued "identity" and single-valued constant mapping.



This picture shows the graphs of χ_t^J for $t = 0, 1/9, \ldots, 8/9, 1$ ("as time goes by") under the obvious identification of $S^1 \times S^1$ and $I \times I$.

Theorem 3.1 ([51, Theorem 2.1])**.** *There exists a fixed point free ρ_c-continuous mapping of B^4 with compact connected values.*

Proof. Let us recall that

$$\begin{aligned} \rho_c(X,Y) &= \max\{d_c(X,Y), d_c(Y,X)\}, \\ d_c(X,Y) &= \inf\{\max\{\|\alpha(x) - x\| : x \in X\}\}, \end{aligned}$$

where the infimum is taken over all continuous functions $\alpha\colon X \to Y$; $(X, Y \subset B^4)$. The disc B^4 will be identified with $B^2 \times B^2$. Set $H(\,\cdot\,,t) = \chi^J(\,\cdot\,, 1-t)$. This formula yields a ρ_s-continuous homotopy $H: S^1 \times I \to 2^{S^1}$ joining $H(z,0) = \{z_0\}$ and $H(z,1) = \{z\}$ such that $H(z,t)$ is a finite subset of S^1, which has at most 3 elements for every $(z,t) \in S^1 \times I$. The multivalued retraction $r: B^2 \to 2^{S^1}$ is the standard one:

$$r(x) = \begin{cases} \{z_0\} & \text{for } \|x\| \le 1/2, \\ H(x/\|x\|, 2\|x\| - 1) & \text{for } \|x\| \in [1/2, 1]. \end{cases}$$

We define $J: B^2 \to 2^{S^1}$ by

$$J(x) = \begin{cases} -r(3x) & \text{for } \|x\| \le 1/3, \\ \{-x/\|x\|\} & \text{for } \|x\| \in [1/3, 1]. \end{cases}$$

Of course, J is ρ_s-continuous and has finite values. Since $\rho_s = \rho_c$ on finite sets, J is ρ_c-continuous. The mapping J is fixed point free, moreover

$$x \notin [1/2, 1]J(x) \quad \text{for every } x \in B^2.$$

This is easy to check that the join of sets $A \subset S^1 \times \{0\}$ and $B \subset \{0\} \times S^1$ in $B^2 \times B^2$ is well defined by

$$A * B = \{(1-t)a + tb : t \in [0,1],\ a \in A,\ b \in B\},$$

(i.e. each point $(1-t)a + tb$ with $0 < t < 1$, $a \in A$, $b \in B$, is uniquely written in this form). Let $\phi_1, \phi_2, f: B^2 \times B^2 \to 2^{B^2 \times B^2}$ be given by

$$\phi_1(x,y) = J(x) \times \{0\}, \qquad \phi_2(x,y) = \{0\} \times J(y), \qquad f(p) = \phi_1(p) * \phi_2(p).$$

We check now that f is ρ_c-continuous. Take an $\varepsilon > 0$. Since ϕ_i is ρ_c-continuous, there is a positive δ such that

$$\|p - q\| < \delta \ \Rightarrow\ \varrho_c(\phi_i(p), \phi_i(q)) < \varepsilon$$

for $i = 1,2$. Fix $p, q \in B^2 \times B^2$ with $\|p - q\| < \delta$. By the definition of ρ_c, there is a continuous map $\alpha_i: \phi_i(p) \to \phi_i(q)$ such that $\|\alpha_i(v) - v\| < \varepsilon$ for every $v \in \phi_i(p)$. Let $\alpha_1 * \alpha_2: f(p) \to f(q)$ be the join of maps α_1 and α_2. Take $x = (1-t)u_1 + tu_2 \in f(p)$ with $u_i \in \phi_i(p)$. Thus $\|\alpha_1 * \alpha_2(x) - x\| = \|(1-t)\alpha_1(u_1) + t\alpha_2(u_2) - (1-t)u_1 - tu_2\| \le (1-t)\|\alpha_1(u_1) - u_1\| + t\|\alpha_2(u_2) - u_2\| < \varepsilon$. Hence $d_c(f(p), f(q)) < \varepsilon$. Likewise, $d_c(f(q), f(p)) < \varepsilon$, $\rho_c(f(p), f(q)) < \varepsilon$.

The mapping f is fixed point free. Otherwise, there is $(x,y) \in B^2 \times B^2$ such that

$$(x,y) \in f(x,y) = \{((1-t)a, tb) : t \in [0,1],\ a \in J(x),\ b \in J(y)\}.$$

As we have already noticed $x \notin [1/2, 1]J(x)$. Now $x = (1-t)a$, $a \in J(x)$ implies $t \in (1/2, 1]$. But then $y = t \cdot b$, $b \in J(y)$ gives $y \in [1/2, 1]J(y)$ contradicting to the above.

The values of f being joins of some finite sets are compact and connected (these are graphs of 4 homotopy types: $\bullet$, $\bigcirc$, $\ominus$, $\oplus$). $\square$

3.2. The ρ_h-continuity and fixed points

Theorem 3.2 ([51, Theorem 2.2]). *Every ρ_h-continuous mapping $f\colon B^n \to 2^{B^n}$ (with values* ANR*s) has a continuous selector and a fixed point.*

Proof. Let $f\colon B^n \to 2^{B^n}$ be a ρ_h-continuous mapping. By Theorem 2.1, the projection $p\colon \Gamma(f) \to B^n$ is a Hurewicz fibration. Since B^n is contractible, p has a section. The second coordinate of this section is a continuous selector of f, which proves the theorem. □

We now formulate our main result for the set-valued mappings on the manifold. Let K be a field and $L(F;K)$ denote the *Lefschetz number* of the *J-mapping F* in K. For the definition and properties of *J-mappings* we refer the reader to [26] or [21]. We recall that $L(F;K)$ is defined as the Lefschetz number of the sufficiently close single-valued graph approximation of F. In other words, there is an $\varepsilon > 0$ such that $L(F;K) = L(s;K)$ for any mapping $s\colon M \to M$ satisfying $\Gamma(s) \subset O_\varepsilon(\Gamma(F))$. This definition is correct, since any two ε-graph approximations of F are homotopic, for a small ε, [26], [21].

Theorem 3.3 ([51, Theorem 2.9]). *Let M be a closed connected K-oriented topological n-manifold. Let $f\colon M \to 2^M$ be a ρ_h-continuous mapping (with values connected* ANR*s). Assume that there exists a* u.s.c. *mapping $W\colon M \to 2^M$ with values homeomorphic to the n-disc such that $L(W;K) \neq 0$ and*

$$f(x) \subset W(x) \quad \textit{for every } x \in M.$$

If the Hurewicz fibration $p\colon \Gamma(f) \to M$ (see Theorem 2.1) is K-oriented and

$$H_{n-1}(M \times f(x);K) = H_{n-1}(M;K),$$

then f has a fixed point.

Proof. We fix a sufficiently close graph approximation s of W. Let us consider the composition

$$M \xrightarrow{D} M^2 \xrightarrow{1\times s} M^2 \xrightarrow{j} (M^2, M^2 \setminus \Delta)$$

with $M^2 = M \times M$, $D(x) = (x,x)$, j — an inclusion and Δ — the diagonal in M^2.

Let $U \in H^n(M^2, M^2 \setminus \Delta; K)$ denote a K — orientation class of the manifold M and $\lambda(s) = D^\star \circ (1 \times s)^\star \circ j^\star(U)$ — the Lefschetz class of the function s. The following diagram

$$\begin{array}{ccc} \Gamma(f) & \xrightarrow{i} & M^2 \\ {\scriptstyle p}\downarrow & \simeq & \uparrow{\scriptstyle 1\times s} \\ M & \xrightarrow[D]{} & M^2 \end{array}$$

is homotopy commutative. This is because the mappings $i(x,y) = (x,y)$ (with $y \in f(x)$) and $(1 \times s) \circ D \circ p(x,y) = (x, s(x))$ are two sufficiently close graph approximations of the J-mapping $\Psi(x,y) = \{x\} \times W(x)$, [26], [21], between compact ANR's. The set $\Gamma(f)$ is an ANR, since it is a space of the Hurewicz fibration with the ANR fibres over an ANR, [17], [20].

To obtain a contradiction suppose that f is fixed point free. From this the following diagram

$$\begin{array}{ccccc} \Gamma(f) & \xrightarrow{\ i\ } & M^2 & \xrightarrow{\ j\ } & (M^2, M^2 \setminus \Delta) \\ {\scriptstyle h}\big\downarrow & & \big\uparrow{\scriptstyle k} & & \\ M^2 \setminus \Delta & \xrightarrow[\text{id}]{} & M^2 \setminus \Delta & & \end{array}$$

commutes, (h, k — inclusions).

Since $L(s;K) \neq 0$, we see that $\lambda(s) \neq 0$. By our diagrams,

$$p^\star(\lambda(s)) = p^\star D^\star (1 \times s)^\star j^\star(U) = i^\star j^\star(U) = h^\star k^\star j^\star(U) = 0.$$

(The last equality follows from the long exact sequence of the pair $(M^2, M^2 \setminus \Delta)$.) Hence $p^\star \colon H^n(M;K) \to H^n(\Gamma(f);K)$ is not a monomorphism. Equivalently, $p_\star \colon H_n(\Gamma(f);K) \to H_n(M;K)$ is not an epimorphism.

On the other hand, $p_\star$ can be described in terms of the Leray–Serre spectral sequence as the composition

$$H_n(\Gamma(f);K) \overset{\text{onto}}{\longrightarrow} E^\infty_{n,0} \overset{\mu}{\longrightarrow} E^2_{n,0} \cong H_n(M;K),$$

(see [67]). The monomorphism μ is the composition of inclusions

$$E^{r+1}_{n,0} = \ker(E^r_{n,0} \to E^r_{n-r,r-1}) \subset E^r_{n,0}$$

for $r = n, \ldots, 2$. Clearly, $E^{n+1}_{n,0} = E^\infty_{n,0}$. By the assumption,

$$H_{n-1}(M;K) = H_{n-1}(M \times f(x);K) = \bigoplus_{r=1}^{n} H_{n-r}(M;K) \otimes_K H_{r-1}(f(x);K),$$

hence

$$0 = H_{n-r}(M;K) \otimes_K H_{r-1}(f(x);K) = E^2_{n-r,r-1} \quad \text{for} \quad r = n, \ldots, 2,$$

which suffices to conclude that $E^r_{n-r,r-1} = 0$ and μ is an isomorphism. Thus $p_\star$ is an epimorphism, a contradiction. $\square$

Example 3.4. Let $M = S^2$, $r < 1$, $f(x) = M \cap S^2(x;r)$, $W(x) = M \cap B^3(x;r)$ for $x \in M$. Then $L(W;Q) = L(id;Q) = 2$. We have $f(x) \cong S^1$, $H_1(M \times f(x);Q) = Q \neq H_1(M;Q) = 0$. In fact the equality $H_1(M \times f(x);Q) = H_1(M;Q)$ can not hold, since otherwise Theorem 3.3 would imply a fixed point. But f is obviously fixed point free. Note that there are Q-orientations of the manifold M and of the fibration $p \colon \Gamma(f) \to M$, because M is simply-connected.

CHAPTER 4

MAPPINGS WITH eLC^k-VALUES

4.1. The eLC^{n-2}-result: the formulation

Recall that the sets from a class $\{X_\lambda : \lambda \in \Lambda\}$ are eLC^k (equally locally connected in the dimension k) if and only if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for all λ, $x \in X_\lambda$, $r = 0, \dots, k$, every map $\omega\colon S^r \to K(x, \delta(\varepsilon)) \cap X_\lambda$ has a continuous extension $\overline{\omega}\colon B^{r+1} \to K(x, \varepsilon) \cap X_\lambda$.

Here $K(x, \varepsilon)$ denotes the open ball with the center x and radius ε.

Lemma 4.1. *Sets from a class $\Theta \subset 2^{\mathbb{R}^n}$ are* e.l.c. *if and only if these sets are eLC^{n-1}.*

Proof. Clearly, the nontrivial implication is $eLC^{n-1} \Rightarrow$ e.l.c. The basic observation is that the condition e.l.c. in the Borsuk Lemma 1.1 can be replaced by eLC^{n-1}, (see Appendix). Roughly speaking, this is because the proof of the Borsuk Lemma follows by induction on the dimension of the skeleton of a simplicial decomposition of $\mathbb{R}^n \setminus \theta$ (for $\theta \in \Theta$) and n-simplices are maximal. Having the Borsuk Lemma we see that

$$\phi_\theta(\alpha(\varepsilon)/2) \le 4\varepsilon.$$

Really, take $T \subset \theta$ with $\operatorname{diam}(T) \le \alpha(\varepsilon)/2$ and choose $x \in T$. For any $y \in T$, $y_t \overset{\text{def}}{=} (1-t)y + tx \in O_{\alpha(\varepsilon)}(x)$, $\|r_\theta(y_t) - x\| \le \|r_\theta(y_t) - y_t\| + \|y_t - x\| < \varepsilon + \alpha(\varepsilon) \le 2\varepsilon$. Thus $r_\theta(\,\cdot\,_t)$ is a homotopy from id_T to the constant map x in the set $\theta \cap O_{2\varepsilon}(x)$ with the diameter $\le 4\varepsilon$. □

By the above lemma, the remark that every ρ_s-continuous multivalued map with e.l.c. values is ρ_h-continuous and Theorem 3.2 we have:

Corollary 4.2. *Every ρ_s-continuous mapping $f\colon B^n \to 2^{B^n}$ with eLC^{n-1}-values has a continuous selector and a fixed point.*

This result has in its origin the Michael selection theorem, [47], which is applied in the proof of the Ferry Theorem 1.12, which is used in the proof of

Theorem 2.1 — basic for proving Theorem 3.2. We can compare this corollary with that which is obtained by applying directly the Michael theorem:

Corollary 4.3 ([47]). *Every* l.s.c. *mapping* $f\colon B^n \to 2^{B^n}$ *with* eLC^{n-1} *and* C^{n-1} *values* ($n-1$*-connected*) *has a continuous selector and a fixed point.*

The proof of the next result is based on the concept of the spheric mapping, (see Introduction). This is the main theorem in this chapter.

Theorem 4.4 (eLC^{n-2}-result). *If*

(a) $f\colon B^n \to 2^{B^n}$ *is* ρ_s*-continuous;*
(b) *the sets in* $\{f(x) : x \in B^n\}$ *are* eLC^{n-2}*;*
(c) *the sets in* $\{\widetilde{f}(x) : x \in B^n\}$ *are* eLC^{n-1}*,*

then f *is a spheric mapping and has a fixed point.*

Example 4.5. There is a fixed point free ρ_s-continuous mapping $f\colon B^n \to 2^{B^n}$ such that sets in $\{f(x) : x \in B^n\}$ are eLC^{n-3}. Set

$$f(x) = \{y \in S^{n-1} : \langle y, x\rangle \leq (1 - \|x\|)\|x\|\}.$$

4.2. The eLC^{n-2}-result: two lemmas

Let X be a compact subset of $\mathbb{R}^n$. We will denote by $B(X)$ the sum of all bounded components of the complement of the set X in $\mathbb{R}^n$. The unbounded component of $\mathbb{R}^n \setminus X$ will be denoted by $D(X)$. Set $\widetilde{X} = X \cup B(X)$.

Lemma 4.6. *Let* $X \subset \mathbb{R}^n$ *be a compact* ANR *and* $x \in \mathbb{R}^n \setminus X$*. Then* $x \in B(X)$ *if and only if the homomorphism*

$$j_\star\colon H_{n-1}(X;Q) \to H_{n-1}(\mathbb{R}^n \setminus \{x\};Q)$$

(*induced by inclusion*) *is an epimorphism.*

Proof. Choose $r_1, r_2 > 0$ such that $\mathring{D}_1 \overset{\text{def}}{=} \{y \in \mathbb{R}^n : \|y - x\| < r_1\} \subset \mathbb{R}^n \setminus X$ and $D_2 \overset{\text{def}}{=} \{y \in \mathbb{R}^n : \|y - x\| \leq r_2\} \supset X$.

The part "if" does not require that X is an ANR. By assumption, the homomorphism $j_\star\colon H_{n-1}(X;Q) \to H_{n-1}(\mathbb{R}^n \setminus \{x\};Q)$ is nontrivial. Suppose contrary to our claim that $x \in D(X)$. Fix in $\mathbb{R}^n$ a point $y \notin D_2$. Since $D(X)$ is a domain in $\mathbb{R}^n$, there are points $z_0 = x, z_1, \ldots, z_q = y$ such that each interval $z_i z_{i+1}$ lies in $D(X)$. The diagram

$$\begin{array}{ccc} X & \xrightarrow{\;j_{i\star}\;} & \mathbb{R}^n \setminus \{z_i\} \\ {\scriptstyle \mathrm{id}}\downarrow & & \cong \downarrow {\scriptstyle T_{i\star}} \\ X & \xrightarrow[\;j_{i+1\star}\;]{} & \mathbb{R}^n \setminus \{z_{i+1}\} \end{array}$$

with $T_i(z) = z + z_{i+1} - z_i$ is homotopy commutative for $i = 0, \ldots, q-1$. Indeed, $H(z,t) = (1-t)z + tT_i(z)$ is a homotopy $H\colon j_{i+1} \simeq T_i \circ j_i$. It follows that $j_{q\star}\colon H_{n-1}(X;Q) \to H_{n-1}(\mathbb{R}^n \setminus \{y\};Q)$ is a nontrivial homomorphism. But $X \subset D_2 \subset \mathbb{R}^n \setminus \{y\}$ and $H_{n-1}(D_2;Q) = 0$, a contradiction.

We now prove "only if". Let $A = D_2 \setminus \overset{\circ}{D}_1$, $S^n = \mathbb{R}^n \cup \{\infty\}$, $x \in B(X)$.

Let $\check{H}^\star$ denote the Čech cohomology, $\widetilde{H}_\star$ - the reduced singular homology. Consider the following commutative diagram

$$\begin{array}{ccccl}
\check{H}^{n-1}(A;Q) & \xrightarrow{\check{\beta}^\star} & \check{H}^{n-1}(X;Q) & & \\
\downarrow\cong & & \downarrow\cong & & \text{the Alexander duality} \\
H_1(S^n, S^n \setminus A;Q) & \longrightarrow & H_1(S^n, S^n \setminus X;Q) & & \\
\downarrow\cong & & \downarrow\cong & & \text{excision} \\
H_1(\mathbb{R}^n, \mathbb{R}^n \setminus A;Q) & \longrightarrow & H_1(\mathbb{R}^n, \mathbb{R}^n \setminus X;Q) & & \\
\downarrow\cong & & \downarrow\cong & & \widetilde{\partial} \\
\widetilde{H}_0(\mathbb{R}^n \setminus A;Q) & \xrightarrow[\widetilde{\alpha}_\star]{} & \widetilde{H}_0(\mathbb{R}^n \setminus X;Q) & &
\end{array}$$

with inclusions $\beta\colon X \to A$ and $\alpha\colon \mathbb{R}^n \setminus A \to \mathbb{R}^n \setminus X$. We have

$$\alpha\colon (\mathbb{R}^n \setminus D_2) \cup \overset{\circ}{D}_1 \to D(X) \cup B(X),$$

$\mathbb{R}^n \setminus D_2 \subset D(X)$, $\overset{\circ}{D}_1 \subset B_\mu \subset B(X) = \bigcup_\lambda B_\lambda$, where $\{B_\lambda\}$ is a family of all bounded components of $\mathbb{R}^n \setminus X$. Thus $\alpha_\star\colon H_0(\mathbb{R}^n \setminus A;Q) \to H_0(\mathbb{R}^n \setminus X;Q)$ is the homomorphism

$$(s,t) \in Q \oplus Q \to Q \oplus \bigoplus_\lambda Q \ni (s, i_\mu(t)),$$

where $i_\mu\colon Q \to \bigoplus_\lambda Q$ denotes the μ-th canonical inclusion. Choose $y \in \mathbb{R}^n \setminus D_2$. Since $\widetilde{H}_0(\mathbb{R}^n \setminus A;Q) = \operatorname{coker}(H_0(\{y\};Q) \to H_0(\mathbb{R}^n \setminus A;Q))$ and the same is true for X in place of A, $\widetilde{\alpha}_\star = i_\mu \neq 0$. Consequently $\check{\beta}^\star \neq 0$. Since A and X are compact ANR's and the (co)homology coefficients are in Q, it follows that $\beta_\star\colon H_{n-1}(X;Q) \to H_{n-1}(A;Q)$ is nontrivial. Clearly, the same holds for the composition $j_\star\colon H_{n-1}(X;Q) \to H_{n-1}(A;Q) \to H_{n-1}(\mathbb{R}^n \setminus \{x\};Q)$ which proves the lemma. □

Author does not know, whether the above result remains true with integer coefficients.

Lemma 4.7. *Let $X_1, X_2, \ldots$ be compact eLC^{n-2} subsets of $\mathbb{R}^n$ such that*

$$\lim_{k\to\infty} \rho_s(X_k, X) = 0$$

for a compact $X \subset \mathbb{R}^n$. *Then*

$$\forall_{x\in B(X)}\ \exists_{k_0}\ \forall_{k>k_0}\quad x \in B(X_k),$$

or equivalently

$$B(X) \subset \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} B(X_k).$$

Proof. Let η denote a positive number. Fix $x \in B(X)$ and a compact polyhedron P such that

$$X \subset P \subset O_\eta(X).$$

Clearly, $\rho_s(X,P) < \eta$. Since $D(P) \subset D(X)$, $x \in \widetilde{P}$. Assuming that $\eta < \operatorname{dist}(x,X)$ gives $x \in B(P)$. By Lemma 4.6, there is a singular $(n-1)$-cycle $Z_{n-1} = \sum_\sigma c_\sigma \sigma$ in P with $c_\sigma \in Q$, which does not bound in $\mathbb{R}^n \setminus \{x\}$. There is no loss of generality in assuming that

$$\forall_\sigma\ (c_\sigma \neq 0\ \Rightarrow\ \operatorname{diam}(\sigma(\Delta_{n-1})) < \eta),$$

(we apply to Z_{n-1} the multiple barycentric subdivision, if necessary).

Choose $k_0 \in N$ with $\rho_s(X_k,X) < \eta$ for every $k > k_0$ and take such a k. Hence $\rho_s(X_k,P) < 2\eta$. The main point of this proof is the construction of the $(n-1)$-cycle $\phi(Z_{n-1})$ in X_k which is homologous to Z_{n-1} in $\mathbb{R}^n \setminus \{x\}$. Finding such a $\phi(Z_{n-1})$ will complete the proof, by Lemma 4.6.

Let $\delta\colon R_+ \to R_+$ be a function, which appears in the definition of the eLC^{n-2} property for sets in the class $\{X_k : k \in N\}$. Set $\mu(\varepsilon) = \delta(\varepsilon/4)$. Let $\mu^{(r)}$ denote the r-th iteration of the function μ. Fix the positive numbers

$$\varepsilon < \operatorname{dist}(x,X) \quad\text{and}\quad \eta < \min\left\{\frac{1}{5}\mu^{(n-1)}(\varepsilon), \frac{1}{8}\varepsilon\right\}.$$

Let Ξ be the set of all simplices of the cycle Z_{n-1} and all their faces. Thus

- $\Xi = \bigcup_{i=0}^{n-1} \Xi_i$,
- $\Xi_{n-1} = \{\sigma : c_\sigma \neq 0\}$,
- $\Xi_{i-1} = \{\tau \circ F_p^i : \tau \in \Xi_i,\ 0 \le p \le i\}$ for $1 \le i \le n-1$.

The function $F_p^i\colon \Delta_{i-1} \to \Delta_i$ is the p-th face mapping.

Our strategy is to make a copy $\phi(\tau)$ in X_k of every simplex $\tau \in \Xi$. Take $\tau \in \Xi_0$. By an obvious convention, $\tau \in P$. We choose $\phi(\tau)$ to be any point of X_k such that $\|\phi(\tau) - \tau\| < 2\eta$.

Take $\tau \in \Xi_1$. We have

$$\begin{aligned}
&\|\phi(\tau\circ F_1^1)\ - \phi(\tau \circ F_0^1)\| \\
&\qquad \le \|\phi(\tau\circ F_1^1) - \tau\circ F_1^1\| + \|\tau\circ F_1^1 - \tau\circ F_0^1\| + \|\tau\circ F_0^1 - \phi(\tau\circ F_0^1)\| \\
&\qquad < 2\eta + \operatorname{diam}(\tau(\Delta_1)) + 2\eta < 5\eta < \mu^{(n-1)}(\varepsilon) = \delta(\mu^{(n-2)}(\varepsilon)/4).
\end{aligned}$$

We choose $\phi(\tau)\colon \Delta_1 \to X_k$ to be any path joining $\phi(\tau\circ F_1^1)$ and $\phi(\tau\circ F_0^1)$ in $K(\phi(\tau\circ F_0^1), \mu^{(n-2)}(\varepsilon)/4) \cap X_k$.

Suppose that ϕ is defined on Ξ_{i-1} for an $i \leq n-1$ in this way that $\phi(\tau)(\Delta_{i-1})$ lies in an open ball of the radius $\mu^{(n-i)}(\varepsilon)/4$ in X_k, for every $\tau \in \Xi_{i-1}$.

Take $\tau \in \Xi_i$. We have

$$\operatorname{diam}(\phi(\tau \circ F_p^i)(\Delta_{i-1})) < \mu^{(n-i)}(\varepsilon)/2$$

for $p = 0, \dots, i$. Define $\omega : \partial\Delta_i \to X_k$ by

$$\omega(F_p^i(x)) = \phi(\tau \circ F_p^i)(x).$$

Clearly,

$$\operatorname{diam}(\omega(\partial\Delta_i)) < \mu^{(n-i)}(\varepsilon) = \delta(\mu^{(n-i-1)}(\varepsilon)/4).$$

Take any point $q \in \omega(\partial\Delta_i)$. We choose $\phi(\tau)$ to be a continuous extension $\widetilde{\omega} : \Delta_i \to X_k$ of ω such that $\widetilde{\omega}(\Delta_i) \subset K(q, \mu^{(n-i-1)}(\varepsilon)/4)$. In particular,

$$\phi(\tau) \circ F_p^i = \phi(\tau \circ F_p^i) \quad \text{for } \tau \in \Xi_i.$$

This condition on Ξ_{i-1} makes ω well defined. Since $F_p^i \circ F_q^{i-1} = F_q^i \circ F_{p-1}^{i-1}$ for $q < p$ [28],

$$\begin{aligned}
\omega(F_p^i \circ F_q^{i-1}(y)) &= \phi(\tau \circ F_p^i)(F_q^{i-1}(y)) \\
&= \phi(\tau \circ F_p^i \circ F_q^{i-1})(y) = \phi(\tau \circ F_q^i \circ F_{p-1}^{i-1})(y) \\
&= \phi(\tau \circ F_q^i)(F_{p-1}^{i-1}(y)) = \omega(F_q^i \circ F_{p-1}^{i-1}(y)).
\end{aligned}$$

The induction completes the construction of ϕ on Ξ. In particular,

$$\operatorname{diam}(\phi(\sigma)(\Delta_{n-1})) < \mu^{(0)}(\varepsilon)/2 = \varepsilon/2.$$

We now define the $(n-1)$-chain $\phi(Z_{n-1})$ in X_k to be $\sum_\sigma c_\sigma \phi(\sigma)$. Since

$$\partial Z_{n-1} = \sum_\sigma \sum_{p=0}^{n-1} (-1)^p c_\sigma \cdot \sigma \circ F_p^{n-1} = 0,$$

we see that

$$\begin{aligned}
\partial\phi(Z_{n-1}) &= \sum_\sigma \sum_{p=0}^{n-1} (-1)^p c_\sigma \cdot \phi(\sigma) \circ F_p^{n-1} \\
&= \sum_\sigma \sum_{p=0}^{n-1} (-1)^p c_\sigma \cdot \phi(\sigma \circ F_p^{n-1}) = 0.
\end{aligned}$$

What is left is to show that the cycle $\phi(Z_{n-1})$ is homologous to Z_{n-1} in $\mathbb{R}^n \setminus \{x\}$.

We follow the notation of [28]: $E_0, \dots, E_q$ — the vertices of Δ_q; $\delta_q = \mathrm{id}_{\Delta_q}$; $S_q(Y)$ — the group of the singular q — chains in Y (with rational coefficients); $P_q\colon S_q(Y) \to S_{q+1}(Y \times I)$ — the homomorphism defined by

$$\begin{aligned} P_q(\sigma) &= S_{q+1}(\sigma \times \mathrm{id}) \circ P_q(\delta_q) \quad \text{for } \sigma\colon \Delta_q \to Y, \\ P_q(\delta_q) &= \sum_{i=0}^{q} (-1)^i \cdot ((E_0,0)\dots(E_i,0)(E_i,1)\dots(E_q,1)). \end{aligned}$$

Let $\lambda_t\colon Y \to Y \times I$ be given by $\lambda_t(y) = (y,t)$. By [28],

$$\partial \circ P_q + P_{q-1} \circ \partial = S_q(\lambda_1) - S_q(\lambda_0).$$

We now define $G_q(\sigma,\tau)\colon \Delta_q \times I \to \mathbb{R}^n \setminus \{x\}$ by

$$G_q(\sigma,\tau)(E,t) = (1-t)\sigma(E) + t\tau(E)$$

for all q-simplices σ, τ in $\mathbb{R}^n \setminus \{x\}$ such that the above expression takes values apart from $\{x\}$. Note that

$$\begin{aligned} &\mathrm{dist}((1-t)\sigma(E) + t\phi(\sigma)(E), X) \\ &\qquad \le t\|\phi(\sigma)(E) - \phi(\sigma)(E_0)\| + t\|\phi(\sigma)(E_0) - \sigma(E_0)\| \\ &\qquad\quad + t\|\sigma(E_0) - \sigma(E)\| + \mathrm{dist}(\sigma(E), X) \\ &\qquad \le \varepsilon/2 + 2\eta + \eta + \rho_s(P,X) < \varepsilon/2 + 4\eta < \varepsilon < \mathrm{dist}(x,X), \end{aligned}$$

for every $\sigma \in \Xi_q$ and $E \in \Delta_q$.

It follows that $G_q(\sigma,\phi(\sigma))$ is well defined for every $\sigma \in \Xi_q$. Clearly, $\sigma = G_q(\sigma,\phi(\sigma)) \circ \lambda_0$ and $\phi(\sigma) = G_q(\sigma,\phi(\sigma)) \circ \lambda_1$. Moreover,

$$G_q(\sigma,\tau) \circ (F \times \mathrm{id}) = G_{q-1}(\sigma \circ F, \tau \circ F)$$

for any $F\colon \Delta_{q-1} \to \Delta_q$. Thus

$$\begin{aligned} \phi(\sigma) - \sigma &= S_q(G_q(\sigma,\phi(\sigma))) \circ (S_q(\lambda_1) - S_q(\lambda_0))(\delta_q) \\ &= S_q(G_q(\sigma,\phi(\sigma))) \circ (\partial P_q + P_{q-1}\partial)(\delta_q) \\ &= \partial S_{q+1}(G_q(\sigma,\phi(\sigma)))P_q(\delta_q) + S_q(G_q(\sigma,\phi(\sigma)))P_{q-1}\partial(\delta_q). \end{aligned}$$

The second summand is equal to

$$\begin{aligned} &\sum_{j=0}^{q} (-1)^j S_q(G_q(\sigma,\phi(\sigma))) \circ P_{q-1}(F_j^q) \\ &\qquad = \sum_{j=0}^{q} (-1)^j S_q(G_q(\sigma,\phi(\sigma))) \circ S_q(F_j^q \times id) \circ P_{q-1}(\delta_{q-1}) \\ &\qquad = \sum_{j=0}^{q} (-1)^j S_q(G_{q-1}(\sigma \circ F_j^q, \phi(\sigma) \circ F_j^q)) \circ P_{q-1}(\delta_{q-1}). \end{aligned}$$

Take $q = n-1$. The cycle $\phi(Z_{n-1}) - Z_{n-1}$ is homologous in $\mathbb{R}^n \setminus \{x\}$ to

$$\sum_\sigma c_\sigma \cdot \sum_{j=0}^{q} (-1)^j S_q(G_{q-1}(\sigma \circ F_j^q, \phi(\sigma \circ F_j^q))) \circ P_{q-1}(\delta_{q-1}),$$

which equals to zero, because $\sum_\sigma c_\sigma \cdot \sum_{j=0}^{q} (-1)^j \sigma \circ F_j^q = \partial Z_{n-1} = 0$. □

4.3. Proof of the eLC^{n-2}-result

We will prove that $f\colon B^n \to 2^{B^n}$ satisfying assumptions of Theorem 4.4 is spheric in the following sense:

(a) f is u.s.c. with compact values;
(b) the graph $\Gamma(Bf)$ is open in $B^n \times \mathbb{R}^n$;
(c) $\widetilde{f}$ has a fixed point.

The only point which needs our attention is (b). Indeed, f is ρ_s-continuous $\Rightarrow$ f is u.s.c. and l.s.c.; f is u.s.c. $\Rightarrow$ $\widetilde{f}$ is u.s.c., [23]; (f is l.s.c. and (b)) $\Rightarrow$ $\widetilde{f}$ is l.s.c.; $\widetilde{f}$ is u.s.c. and l.s.c. $\Rightarrow$ $\widetilde{f}$ is ρ_s-continuous. Corollary 4.2 and assumption (c) of Theorem 4.4 now imply (c).

Suppose, (b) is false. Then

$$\exists_{(x,y)\in\Gamma(Bf)}\ \exists_{\{(x_k,y_k)\}} \quad \lim_{k\to\infty} (x_k, y_k) = (x, y) \quad \text{and} \quad \forall_k\ (x_k, y_k) \notin \Gamma(Bf).$$

Thus $y \in Bf(x)$, $y_k \in f(x_k) \cup Df(x_k)$. Since $\lim_{k\to\infty} \rho_s(f(x_k), f(x)) = 0$, Lemma 4.7 shows that $y \in Bf(x_k)$ for $k > k_0$. By connectedness of the interval yy_k, there is $c_k \in yy_k$ such that $c_k \in f(x_k)$ for $k > k_0$. This gives $y \in f(x)$, a contradiction. □

4.4. A two-dimensional eLC^{n-2}-result

The following theorem is the origin of the idea that for $n = 2$ the third assumption in Theorem 4.4 is superfluous.

Theorem 4.8 (Borsuk, [3, p. 212]). *The nonempty closed subset A of $\mathbb{R}^2$ is the retract of $\mathbb{R}^2$ if and only if the set A is connected and locally connected and all the components of $\mathbb{R}^2 \setminus A$ are unbounded sets.*

In this way LC^0 for $f(x) \subset \mathbb{R}^2$ implies LC^1 for $\widetilde{f}(x)$. Omitting the question, if eLC^0 for $\{f(x) : x \in B^2\}$ implies eLC^1 for $\{\widetilde{f}(x) : x \in B^2\}$ (see Chapter 8), we prove the following

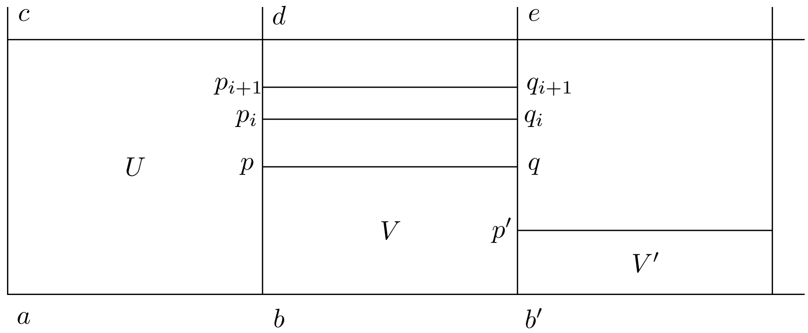
Theorem 4.9. *If $f\colon B^2 \to 2^{B^2}$ is ρ_s-continuous and sets in $\{f(x) : x \in B^2\}$ are eLC^0 then f has a fixed point.*

Example 4.10. Let us recall, that every ρ_c-continuous mapping $f\colon B^2 \to 2^{B^2}$ with compact connected values has a fixed point, [23, Theorem 4.4]. The map f defined by $f(x) = \{y \in D^2 : \|y\| \geq \|x\|\}$ is not ρ_c-continuous but satisfies the assumptions of Theorem 4.9.

Proof of Theorem 4.9. We give two proofs.

Proof I. Let us identify B^2 with I^2 and consider a ρ_s-continuous mapping $f\colon I^2 \to I^2$ with eLC^0 values. Fix $\eta > 0$. Choose $\varepsilon > 0$ small enough that for all $x \in I^2$ and $y, y' \in f(x)$ with $\|y - y'\| < 4\varepsilon$ there is a path $\sigma\colon I \to I^2$ from y to y' in $f(x)$ satisfying $\operatorname{diam}(\sigma(I)) < \eta$. Take $\delta > 0$ such that $\rho_s(f(x), f(x')) < \varepsilon$ for all x, x' with $\|x - x'\| < \delta$. We assume that $\delta < \varepsilon < \eta$.

Let us divide I^2 into squares, each with the edge of the same length less than δ. Our purpose is to find a single-valued continuous map $s\colon I^2 \to I^2$ which approximates f.



We follow the notation of the figure. Fix $A \in f(a)$. Then choose $B \in f(b)$, $C \in f(c)$ and $D, P \in f(d)$ such that $\|B - A\|$, $\|C - A\|$, $\|D - C\|$, $\|P - B\|$ are all less than ε. It follows that $\|D - P\| < 4\varepsilon$.

Set $p = (b + d)/2$ and $s(a) = A$, $s(b) = B$, $s(c) = C$, $s(d) = D$, $s(p) = P$. Find a path $\sigma\colon I \to I^2$ from P to D in $f(d)$ with $\operatorname{diam}(\sigma(I)) < \eta$. Choose $r_0 = 0, r_1, \dots, r_k = 1$ in I such that $\operatorname{diam}(\sigma([r_i, r_{i+1}])) < \varepsilon$ for $i < k$. Set $p_i = p + i(d - p)/k$ and $s((1 - t)p_i + tp_{i+1}) = \sigma((1 - t)r_i + tr_{i+1})$ for $t \in I$. Define $P_i = s(p_i)$ and note, that $\|P_{i+1} - P_i\| < \varepsilon$. Extend s to be linear on the segments ab, ac, cd, bp; e.g. $s((1 - t)b + tp) = (1 - t)B + tP$.

Let U be the square $abdc$. Clearly, $\operatorname{diam}(s(\partial U)) < 3\varepsilon + \eta$. Obviously map s into a convex set admits an extension $\widetilde{s}\colon U \to \operatorname{conv}(s(\partial U))$. To simplify notation, we continue to write s for $\widetilde{s}$. Clearly, $\operatorname{diam}(s(U)) < 3\varepsilon + \eta$. Let $q = (b' + e)/2$ and V be the rectangle $bb'qp$ in I^2. Choose $Q \in f(e)$ such that $\|Q - P\| < \varepsilon$ and repeat the construction of s on V after that on U. We stress that $s(q) = Q \in f(e)$, moreover s maps the segment $p'q$ into $f(e)$, where $p' = (b' + q)/2$.

Since $P_i \in f(d)$, there is $Q_i \in f(e)$ with $\|Q_i - P_i\| < \varepsilon$ for $i = 0, \dots, k$ and $Q_0 = Q$. Set $E = Q_k$, $q_i = q + i \cdot (e - q)/k$, $s(q_i) = Q_i$. Thus $\|Q_{i+1} - Q_i\| < 3\varepsilon$. Find a path $\alpha_i\colon I \to I^2$ from Q_i to Q_{i+1} in $f(e)$ with $\operatorname{diam}(\alpha_i(I)) < \eta$.

Let V_i be the rectangle $p_i q_i q_{i+1} p_{i+1}$. We extend s to be linear on segment $p_i q_i$ and by $s((1-t)q_i + tq_{i+1}) = \alpha_i(t)$ on $q_i q_{i+1}$. Thus $\operatorname{diam}(s(\partial V_i)) < 2\varepsilon + \eta$. Clearly, there is an extension $s\colon V_i \to \operatorname{conv}(s(\partial V_i))$ with $\operatorname{diam}(s(V_i)) < 2\varepsilon + \eta$ for $i = 0, \dots, k-1$.

The map s is now defined on U and on the square $bb'ed = V \cup \bigcup_{i=0}^{k-1} V_i$. In the same manner we extend s, square by square, on the first row of our subdivision of I^2. It is worth pointing out that passing to the third square we forget points q_i and define $p'_j = p' + j \cdot (e - p')/k'$ with k' such that $\operatorname{diam}(s(p'_j p'_{j+1})) < \varepsilon$, for $j = 0, \dots, k'-1$. The definition of s on the other rows is straightforward.

It remains to prove that $s\colon I^2 \to I^2$ approximates f. For every $x \in I^2$ there are $R = r^0 r^1 r^2 r^3$ and $T = t^0 t^1 t^2 t^3$ such that:

- R is a rectangle, T is a square and $x \in R \subset T$,
- $\operatorname{diam}(T) < \sqrt{2} \cdot \delta$ and $\operatorname{diam}(s(R)) < 3\varepsilon + \eta$,
- $s(r^2) \in f(t^2)$.

Thus $s(x) \in s(R) \subset O_{3\varepsilon+\eta}\{s(r^2)\} \subset O_{4\eta} f(t^2) \subset O_{4\eta} f(O_{2\delta}\{x\}) \subset O_{6\eta} f(x)$. □

Proof II. Another way of proving Theorem 4.9 is the analysis similar to that in the proof of Theorem 4.4. The only difference is the argument which shows that $\widetilde{f}$ has a fixed point. We will see that the values of $\widetilde{f}$ have a fixed finite number of acyclic components. Therefore $\widetilde{f}$ is in a class of mappings which is equipped with the fixed point index, [14].

Let $\operatorname{nc}(X)$ denote the number of the components of the space X. Since $f(x)$ is compact and LC^0, $\operatorname{nc}(f(x)) < \infty$. By the Alexander duality, $\check{H}^i(\widetilde{f}(x)) = \widetilde{H}_{1-i}(Df(x)) = 0$ for $i \geq 1$.

It suffices to show that $\operatorname{nc}(\widetilde{f}(x))$ is finite and does not depend on x. Since every component of $\widetilde{f}(x)$ contains a point of the set $f(x)$, we have $\operatorname{nc}(\widetilde{f}(x)) \leq \operatorname{nc}(f(x)) < \infty$. Because sets in $\{f(x) : x \in B^2\}$ are eLC^0, there is an $\varepsilon > 0$ such that the distance of any two components of $f(x)$ is not less than ε, for every $x \in B^2$. The same is true for the components of $\widetilde{f}(x)$. Indeed, if C, C' are two components of $\widetilde{f}(x)$, then

$$\partial C \subset f(x), \quad \partial C' \subset f(x), \quad \operatorname{dist}(C, C') = \operatorname{dist}(\partial C, \partial C') \geq \varepsilon.$$

Since $\widetilde{f}$ is ρ_s-continuous, there is $\delta > 0$ such that $\operatorname{nc}(\widetilde{f}(x')) \geq \operatorname{nc}(\widetilde{f}(x))$, whenever $\|x - x'\| < \delta$. Thus $\operatorname{nc}(\widetilde{f}(x')) = \operatorname{nc}(\widetilde{f}(x))$ for every $x' \in O_\delta\{x\}$. The connectedness of B^2 finishes the proof. □

CHAPTER 5

PRELIMINARIES TO THE STIEFEL–WHITNEY CLASSES

The notion of the Stiefel–Whitney classes originated from [66] and [69]. Each class can be considered as the mod(2) reduction of the obstruction to extending a section of an associted bundle [64], an element of the cohomology algebra of the classifying space [43] or a homology class ([31]). The approach to this subject is sometimes diversified according to the kind of fibration we wish to describe. In Section 5.1 we recall the Thom–Cartan definition of Stiefel–Whitney classes [68], (see also [45]). This definition is suitable for all spherical fibrations, i.e. the Hurewicz fibrations having the fibers homotopy equivalent to the sphere. We also recall these properties of Stiefel–Whitney classes which will be useful for our purposes. Section 5.2 contains two easy consequences of the definition. The sphere bundles are these spherical fibrations which are locally trivial and which are equipped with a structural group. Some properties of the characteristic classes of the sphere bundle are conditioned by the possibility of passing to the projective space bundle and, moreover, by the assumption that all maps in its structural group are the linear isomorphisms. In Section 5.3 we show how to generalize one of these properties (coming from the Vector Bundles Theory) for all locally trivial fibrations with fibre S^k.

5.1. Definition and basic properties

The good starting point of the journey to the world of characteristic classes is the following

Theorem 5.1 (Leray–Hirsch, [67, 15.47])**.** *Let $p\colon (E_2, E_1) \to B$ be a pair of the Hurewicz fibrations over a pathwise connected base B with fibre (F_2, F_1). Assume that there are elements $e_1, \dots, e_r \in H^\star(E_2, E_1)$ such that their restrictions to the fibre form a base of the free $H^\star(pt)$-module $H^\star(F_2, F_1)$. Then the set*

$\{e_1, \dots, e_r\}$ is a base of the free $H^\star(B)$-module $H^\star(E_2, E_1)$ with

$$be = p^\star(b) \cup e$$

defining the multiplication.

Let $p\colon \Gamma \to B$ be a spherical fibration with fibre $S \simeq S^k$. Unless otherwise stated, we assume that B is a pathwise connected topological space. Set $Z = (I \times \Gamma \cup B)/\sim$, where $(1, x) \sim p(x)$ for every x. The map $\overline{p}\colon Z \to B$, $\overline{p}[t, x] = p(x)$ is easily checked to be the Hurewicz fibration with fibre $D = I \times S / 1 \times S \simeq B^{k+1}$. The identification $x = [0, x]$ makes Γ a subset of Z and $S \subset D$. Let $\tau\colon (D, S) \to (Z, \Gamma)$ be the inclusion. There exists $t \in H^{k+1}(Z, \Gamma; Z_2)$, called the Thom class of the bundle p, such that $\tau^\star t$ generates $H^{k+1}(D, S; Z_2)$. The Leray–Hirsch theorem provides with Thom's isomorphism $\Phi\colon H^\star(B; Z_2) \to H^{\star+k+1}(Z, \Gamma; Z_2)$, $\Phi(x) = \overline{p}^\star x \cup t$.

Definition 5.2 ([68])**.** The element

$$w_j = \Phi^{-1} Sq^j \Phi(1) \in H^j(B; Z_2)$$

is called the j-th Stiefel–Whitney class of the fibration p.

Since the Steenrod square $Sq^j\colon H^q(\,\cdot\,, \cdot\,; Z_2) \to H^{q+j}(\,\cdot\,, \cdot\,; Z_2)$ equals zero for $j > q$,

$$w_j = 0 \quad \text{for } j > k+1.$$

Thom proved that the Stiefel–Whitney classes are invariants of the fibre homotopy equivalences, [68]. Moreover, for any spherical fibrations $p_i\colon \Gamma_i \to B_i$, $i = 1, 2$ and every $f\colon B_1 \to B_2$ such that there is an $\overline{f}\colon \Gamma_1 \to \Gamma_2$ which satisfies $p_2 \circ \overline{f} = f \circ p_1$ and induces a homotopy equivalence of the corresponding fibres, we have $f^\star w_{j,2} = w_{j,1}$, (see Appendix). This property is called the naturalness of characteristic classes. It implies that classes $w_0, w_1, w_2, \dots$ are natural in the sense of the category theory:

$$f^\star w_j([p_2]) = w_j([f^\star p_2]).$$

Here $[\,\cdot\,]$ denotes the fibre homotopy equivalence class, and $f^\star p_2\colon f^\star \Gamma_2 \to B_1$ (called the induced fibration) with

$$f^\star \Gamma_2 = \{(b, e) \in B_1 \times \Gamma_2 : f(b) = p_2(e)\}$$

is given by $f^\star p_2(b, e) = b$.

One can list some properties of the Stiefel-Whitney classes which considered as the axioms, uniquely determine these classes for all sphere bundles with the structural group $O(k+1)$, (such bundles are in some sense the same as the vector bundles), [54], [33]:

- $w_j(p) \in H^j(B; Z_2)$ for $j = 0, 1, 2, \dots$ and $w_0(p) = 1$;
- The naturalness of $w_j(\,\cdot\,)$;

- $w_j(p \oplus q) = \sum_{i=0}^{j} w_i(p) \cup w_{j-i}(q)$, ($\oplus$ denotes the Whitney sum of two spheric bundles, which has the fibre being the join of the fibre of p and that of q);
- The class w_1 of the covering $S^1 \to RP^1$ is nontrivial.

This axiomatic approach fails for the locally trivial bundles with the fibre S^k. The reason is that the algebra of Z_2-cohomology of the classifying space of these bundles is much more complicated, [6]. Such algebra for the spherical fibrations is described in [53], (see Appendix).

Assume for the moment that p is a sphere bundle with fibre S^k and the structural group $O(k+1)$. In this case a free fibre preserving action of Z_2 on Γ is defined with the help of the local trivializations $\phi = \phi_U \colon p^{-1}U \to U \times S^k$ by

$$1_{Z_2} x = \phi^{-1} \circ \alpha \circ \phi(x), \quad \text{where } \alpha(u,s) = (u,-s). \tag{5.1}$$

The right-hand side of (5.1) does not depend on the local trivialization, if only all elements of the structural group are odd mappings (which is true for $O(k+1)$). Having the above action we proceed as follows. Let c be the first (= the last) Stiefel–Whitney class of the S^0-bundle $\Gamma \to \Gamma/Z_2$. The map $q \colon \Gamma/Z_2 \to B$ induced by p is a locally trivial bundle with fibre RP^k. We call q the projective space bundle. By the Leray–Hirsch theorem,

$$H^\star(\Gamma/Z_2; Z_2) \text{ is a free } H^\star(B; Z_2)\text{-module with a base } 1, c, \dots, c^k, \tag{5.2}$$

and the multiplication $\beta\gamma = q^\star(\beta) \cup \gamma$. More exhaustive arguments: the algebra $H^\star(RP^k; Z_2)$ is the truncated polynomial algebra with one generator x of degree 1 such that $x^{k+1} = 0$, [11]; x is the first Stiefel–Whitney class of the S^0-bundle $S^k \to RP^k$, (for $k = 1$ — an axiom); at last $c|RP^k = x$, by the naturalness of characteristic classes; the Leray–Hirsch theorem now applies. Consequently,

$$c^{k+1} = \sum_{j=1}^{k+1} q^\star(\widetilde{w}_j) \cup c^{k+1-j} \quad \text{for some } \widetilde{w}_1, \dots, \widetilde{w}_{k+1}. \tag{5.3}$$

Surprisingly,

$$\widetilde{w}_j = w_j \quad \text{for } j = 1, \dots, k+1. \tag{5.4}$$

Proofs of this equality, (see for instance [33, III.5]), use the Splitting Principle [30, p. 106] which is a property of the sphere bundles having the structural group $O(k+1)$.

The next property generalizes the well-known Borsuk–Ulam theorem (see [12], for further generalizations in this direction see [56]).

Theorem 5.3. *Let E be a topological space, $T\colon E \to E$ — a fixed point free involution, $c \in H^1(E/T; Z_2)$ — the first Stiefel–Whitney class of the 0-sphere bundle $\pi\colon E \to E/T$ and $g\colon E \to \mathbb{R}^n$ — a continuous function. Suppose that $c^n \neq 0$. Then there is $y \in E$ such that $g(y) = g(Ty)$.*

The proof of this theorem is an easy application of the naturalness of the Stiefel–Whitney classes to the map $f\colon E/T \to RP^{n-1}$ induced by $\overline{f}\colon E \to S^{n-1}$,

$$\overline{f}(y) = (g(y) - g(Ty))/\|g(y) - g(Ty)\|.$$

Suppose that $\overline{f}$ is well-defined. Then $c^n = f^\star x^n = 0$ for $x =$ generator of $H^1(RP^{n-1}; Z_2)$, which contradicts our assumption. □

5.2. Preliminary results

The following lemma is an immediate consequence of the naturalness of characteristic classes.

Lemma 5.4. *Let $p\colon \Gamma \to B$ be a spherical fibration. If B_1 is a deformation retract of B_2, $B_1 \subset B_2 \subset B$ and $w_{j,i}$ denotes the j-th Stiefel–Whitney class of $p|\Gamma_i\colon p^{-1}(B_i) \to B_i$, $(i = 1, 2)$, then $w_{j,1} = 0$ if and only if $w_{j,2} = 0$.*

The second result generalizes [49, Lemma 1].

Lemma 5.5. *Let $p\colon \Gamma \to B$ be a Hurewicz fibration with compact* ANR *fibres which are homotopy equivalent to S^k, B — a compact* ANR, *Γ — a compact space. Then the following conditions are equivalent:*

$$\dim H_k(\Gamma; Z_2) > \dim H_k(B; Z_2), \tag{5.5}$$

$$w_{k+1} = 0, \tag{5.6}$$

$$0 \to H_k(S^k; Z_2) \to H_k(\Gamma; Z_2) \tag{5.7}$$

is an exact sequence (an inclusion of the fibre in Γ induces a monomorphism on $H_k(\,\cdot\,, Z_2)$).

Example 5.6. The Hopf fibration $p\colon S^3 \to S^2$ with fibre S^1 has $w_2 \neq 0$, since $\dim H_1(S^3; Z_2) = \dim H_1(S^2; Z_2) = 0$.

Proof of Lemma 5.5. The equivalence of (5.5) and (5.7) follows from [46, Example 5.D, p. 145]. We now prove that (5.5) and (5.6) are equivalent. By [20, p. 374], Γ is an ANR. Since Γ, B are compact ANRs, the inequality (5.5) is equivalent to $\dim H^k(\Gamma; Z_2) > \dim H^k(B; Z_2)$. The Gysin exact sequence for p

$$0 = H^{-1}(B; Z_2) \longrightarrow H^k(B; Z_2) \xrightarrow{p^\star} H^k(\Gamma; Z_2) \longrightarrow \cdots$$

shows that $p^\star$ is a monomorphism. Thus (5.5) does not hold if and only if $p^\star$ is an epimorphism. The commutative diagram

$$\begin{array}{ccccccc} H^kZ & \xrightarrow{j^\star} & H^k\Gamma & \xrightarrow{\delta} & H^{k+1}(Z,\Gamma) & \xrightarrow{i^\star} & H^{k+1}Z \\ {\scriptstyle \overline{p}^\star}\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle p^\star} & & & & \\ H^kB & = & H^kB & & & & \end{array}$$

with the first row exact (and Z_2-cohomology coefficients) yields that $p^\star$ — epimorphism $\Leftrightarrow j^\star$ — epimorphism $\Leftrightarrow \delta = 0 \Leftrightarrow i^\star$ — monomorphism.

If $t \in H^{k+1}(Z,\Gamma; Z_2)$ is the Thom class of p, then $t \to t|Z \to w_{k+1}$ under the homomorphism

$$H^{k+1}(Z,\Gamma;Z_2) \xrightarrow{i^\star} H^{k+1}(Z;Z_2) \xrightarrow{(\overline{p}^\star)^{-1}} H^{k+1}(B;Z_2).$$

Moreover, $H^{k+1}(Z,\Gamma;Z_2) = Z_2 = \{0,t\}$. These well-known relationships show that $i^\star$ — monomorphism $\Leftrightarrow t|Z \neq 0 \Leftrightarrow w_{k+1} \neq 0$, which completes the proof.□

The following example shows that the compactness of Γ does not follow from the condition that $p\colon \Gamma \to B$ is the Hurewicz fibration with compact fibres.

Example 5.7. Set $\Gamma = (0;1] \times [0;1] \cup O \subset \mathbb{R}^2$, $O = (0,0)$, $B = [0;1]$, $p(x,y) = x$. The map $p\colon \Gamma \to B$ is a Hurewicz fibration. It is easy to check that for every path σ in B and $e \in p^{-1}(\sigma(0))$ there is a path $\overline{\sigma}_e$ from e which covers σ and which is continuous with respect to the pair (σ, e).

5.3. Classes $w_j^\triangle$

It is of interest to know whether the polynomial formula (5.3) with (5.4) holds for the fibrations which are not equipped with the structural group $O(k+1)$. We give here a partial answer to this question. Let $p\colon \Gamma \to M$ be a locally trivial fibration with fibre $S = S^k$. Set $S^\triangle = S \times S \setminus \Delta$ with Δ — the diagonal. It is easy to check that maps $\phi, \psi\colon S^\triangle \to S$, which are defined by the formulae

$$\phi(x,y) = \|x-y\|^{-1} \cdot (x-y), \quad \psi(x,y) = x,$$

are the homotopy equivalences with the homotopy inverse $\omega(x) = (x, -x)$. Let

$$\Gamma^\triangle = \{(x,y) \in \Gamma \times \Gamma \setminus \Delta : p(x) = p(y)\}$$

and $p^\triangle(x,y) = p(x)$. The group Z_2 acts on $S^\triangle$ and $\Gamma^\triangle$ by the transposition. The map $p^\triangle\colon \Gamma^\triangle \to M$ is a locally trivial bundle with the fibre $S^\triangle$. The orbit spaces of the fibres of $p^\triangle$ are homeomorphic to $S^\triangle/Z_2 \simeq RP^k$, the homotopy equivalence being induced by ϕ, (the idea of such an equivalence is due to Cohen [8, Proposition III]). The map

$$\Gamma^\triangle \ni (x,y) \xrightarrow{\chi} x \in \Gamma,$$

covers the identity on M (with respect to $p^{\triangle}$ and p). Moreover, $\chi|S^{\triangle} = \psi$ is a homotopy equivalence of the fibres, (we skip here the obvious homeomorphisms which identify these fixed fibres with the standard ones). By the naturalness property, both bundles p, $p^{\triangle}$ have the same Stiefel–Whitney classes.

It is a simple matter to obtain the polynomial formula (5.3)

$$(c^{\triangle})^{k+1} = \sum_{j=1}^{k+1} (q^{\triangle})^{\star}(w_j^{\triangle}) \cup (c^{\triangle})^{k+1-j} \tag{5.8}$$

in the new situation, with c, q, $\widetilde{w}_j$ replaced by $c^{\triangle} \in H^1(\Gamma^{\triangle}/Z_2; Z_2)$, $q^{\triangle}\colon \Gamma^{\triangle}/Z_2 \to M$ and $w_j^{\triangle} \in H^j(M; Z_2)$.

Theorem 5.8. *Let M be a pathwise connected topological space. With the above notation, we have $w_{k+1}^{\triangle} = w_{k+1}$.*

Proof. Our proof will resemble a classical inductive reasoning in theory of characteristic classes which makes use of the Gysin sequence, (see [54, 14.5]).

To simplify notation, we continue to write

- $E = \Gamma^{\triangle}$, $E_2 = \Gamma^{\triangle}/Z_2$;
- $w = w_{k+1}$, $v_j = w_j^{\triangle}$, $e = c^{\triangle}$;
- $r = p^{\triangle}$, $g = q^{\triangle}$, $\rho\colon E \to E_2$ (the projection).

($*$) By the Leray–Hirsch theorem, $H^{\star}(E_2; Z_2)$ is an $H^{\star}(M; Z_2)$-module freely generated by $1, e, \dots, e^k$ with the multiplication

$$H^{\star}(M; Z_2) \times H^{\star}(E_2; Z_2) \ni (\mu, \eta) \to \mu \cdot \eta = g^{\star}(\mu) \cup \eta.$$

The Gysin exact sequences of bundles r and ρ form the following commutative diagram with Z_2-coefficients:

$$\begin{array}{ccccccccccc}
\cdots H^{-1}M & \longrightarrow & H^kM & \xrightarrow{r^{\star}} & H^kE & \xrightarrow{\alpha} & H^0M & \xrightarrow{\cup w} & H^{k+1}M & \xrightarrow{r^{\star}} & H^{k+1}E \cdots \\
 & & \Big\downarrow{\scriptstyle g^{\star}} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & & & & & \\
\cdots H^{k-1}E_2 & \xrightarrow[\cup e]{} & H^kE_2 & \xrightarrow[\rho^{\star}]{} & H^kE & \xrightarrow[\beta]{} & H^kE_2 & \xrightarrow[\cup e]{} & H^{k+1}E_2 & \xrightarrow[\rho^{\star}]{} & H^{k+1}E \cdots
\end{array}$$

Since $H^{-1}M = 0$, $r^{\star k}\colon H^kM \to H^kE$ is a monomorphism. Similarly, $r^{\star j}$ is an isomorphism for $j < k$. By our assumption, $H^0M = Z_2$.

Case 1. Assume that $\alpha = 0$. Since $r^{\star j}$ is now an isomorphism for $j \le k$ and $r^{\star} = \rho^{\star} \circ g^{\star}$, we have

$$H^jE_2 = \ker(\rho^{\star}) \oplus \mathrm{im}(g^{\star}) = \mathrm{im}(\cup e) \oplus \mathrm{im}(g^{\star}) \tag{5.9}$$

for $j \le k$. Since $w = 1 \cup w \in \ker(r^{\star}) = \ker(\rho^{\star} \circ g^{\star})$, $g^{\star}w \in \ker(\rho^{\star}) = \mathrm{im}(\cup e)$. There is $x_1 \in H^kE_2$ with $g^{\star}w = x_1 \cup e$. By (5.9), there are $x_2 \in H^{k-1}E_2$ and $v_k \in H^kM$ with $x_1 = x_2 \cup e + g^{\star}v_k$. By induction on $j \le k$,

$$x_j = x_{j+1} \cup e + g^{\star}v_{k-j+1}$$

for some $x_{j+1} \in H^{k-j}E_2$, $v_{k-j+1} \in H^{k-j+1}M$. Thus $g^\star w = x_{k+1} \cup e^{k+1} + \sum_{j=1}^{k} g^\star v_{k-j+1} \cup e^j$ for an $x_{k+1} \in H^0 E_2 = \{0,1\}$. If $x_{k+1} = 0$ then $(*)$ shows that $w = 0$. In this way α is an epimorphism, $\alpha \neq 0$, which contradicts our assumption. We conclude that $x_{k+1} = 1$ and

$$e^{k+1} = g^\star w + \sum_{j=1}^{k} g^\star v_{k-j+1} \cup e^j,$$

which gives $w = v_{k+1}$, as required.

Case 2. Assume that $\alpha \neq 0$. Thus α is an epimorphism, $\ker(\cup w) = H^0 M$, $w = 1 \cup w = 0$.

Case 2.1. Suppose that $\beta = 0$. Hence $\rho^{\star k}$ is an epimorphism. By $(*)$,

$$H^k E_2 = g^\star H^k M \oplus \bigoplus_{j=1}^{k} (g^\star H^{k-j} M) \cup e^j.$$

Since $\rho^\star \circ (\cup e) = 0$, we have $\rho^\star H^k E_2 = \rho^\star g^\star H^k M$. Thus $H^k E = r^\star H^k M = \ker(\alpha)$, $\alpha = 0$, a contradiction.

Case 2.2. Suppose that $\beta \neq 0$. There is an $x_1 \in \operatorname{im}(\beta)$, $x_1 \neq 0$. Clearly, $0 = x_1 \cup e$.

Case 2.2.1. Assume that $\alpha \rho^\star x_1 = 1$. Fix $x \in H^k E$. If $\alpha(x) = 0$, then $x \in \operatorname{im}(r^\star) \subset \operatorname{im}(\rho^\star)$. If $\alpha(x) = 1$, then $x = (x - \rho^\star x_1) + \rho^\star x_1 \in \operatorname{im}(\rho^\star)$, because $\alpha(x - \rho^\star x_1) = 0$. Hence $\operatorname{im}(\rho^\star) = H^k E$ and $\beta = 0$, a contradiction.

Case 2.2.2. Assume that $\alpha \rho^\star x_1 = 0$. Thus $\rho^\star x_1 \in \ker(\alpha) = \operatorname{im}(r^\star) = \operatorname{im}(\rho^\star \circ g^\star)$. There is an $v_k \in H^k M$ with $\rho^\star x_1 = \rho^\star g^\star v_k$. Thus

$$x_1 - g^\star v_k \in \ker(\rho^\star) = \operatorname{im}(\cup e).$$

There is $x_2 \in H^{k-1} E_2$ with $x_1 = x_2 \cup e + q^\star v_k$. We will now proceed by induction. Just as in Case 1, $x_j = x_{j+1} \cup e + g^\star v_{k-j+1}$,

$$g^\star w = 0 = x_1 \cup e = x_{k+1} \cup e^{k+1} + \sum_{j=1}^{k} g^\star v_{k-j+1} \cup e^j.$$

Next we claim that $x_j \neq 0$ for every $j = 1, \ldots, k+1$. Conversely, suppose that $j + 1 = \min\{m : x_m = 0\}$. Thus $0 = x_1 \cup e = \sum_{i=1}^{j} g^\star v_{k-i+1} \cup e^i$. By $(*)$, $v_{k-i+1} = 0$ for $i = 1, \ldots, j$. Thus $x_j = 0$, contrary to the choice of j. We have proved that $x_{k+1} \neq 0$, and so $x_{k+1} = 1$. Thus $e^{k+1} = g^\star w + \sum_{j=1}^{k} g^\star v_{k-j+1} \cup e^j$, and $w = v_{k+1}$, as required. The proof of Theorem 5.8 is complete. □

The Stiefel–Whitney classes of topological manifolds were considered by Fadell in [18]. His results differ from ours.

CHAPTER 6

THE BROUWER MAPPINGS

The homology theory will provide us here with the basic tools of proving that some set-valued mappings have fixed points. In other words, we show that a homology property of graphs forces that the corresponding mappings have fixed points. We call these mappings the Brouwer mappings.

Let $\check{H}_\star$ denote the Čech homology functor, F be a field, B — the closed unit ball in $\mathbb{R}^n$, $S = \partial B$, $\Gamma(\varphi|A)$ — the graph of the restriction of the mapping $\varphi: B \to 2^B$ to any $A \subset B$, $p: \Gamma(\varphi|A) \to A$ — the projection.

Definition 6.1. The upper-semicontinuous compact-valued map $\varphi: B \to 2^B$ is called an *F-Brouwer mapping* if and only if

$$\check{H}_n(\Gamma(\varphi|B), \Gamma(\varphi|S); F) \xrightarrow{i_\star} \check{H}_n(B \times B, S \times B; F),$$

induced by inclusion, is a non-zero homomorphism.

Let us note that single-valued mappings, mappings with a continuous selector and acyclic mappings are Brouwer mappings, (see Lemmas 6.2, 6.4, 6.6). On the other hand, there are Brouwer mappings, which are neither acyclic nor continuously selectionable, (see Theorem 7.11(b), (c)). From now on we consider upper-semicontinuous compact-valued mappings only.

Lemma 6.2. *The following conditions are equivalent:*

(a) *φ is an F-Brouwer mapping;*
(b) $i_\star: \check{H}_n(\Gamma(\varphi|B), \Gamma(\varphi|S); F) \to \check{H}_n(B \times B, S \times B; F)$ *is an epimorphism;*
(c) $p_\star: \check{H}_n(\Gamma(\varphi|B), \Gamma(\varphi|S); F) \to \check{H}_n(B, S; F)$ *is non-zero;*
(d) *$p_\star$ is an epimorphism.*

Proof. Since $\check{H}_n(B, S; F) = F$ and F is a field, (c) and (d) are equivalent. The homomorphism $j_\star: \check{H}_n(B, S; F) \to \check{H}_n(B \times B, S \times B; F)$ which is induced by the homotopy equivalence $j(x) = (x, 0)$, is an isomorphism. Moreover, $j \circ p \simeq i$, which proves the lemma. □

The next theorem follows from the Saveliev result on the coincidences in [59, Corollary 5.1], but it was obtained independently and in another way (which we recall here) in [48].

Theorem 6.3. *Every F-Brouwer mapping has a fixed point.*

Proof. On the contrary, suppose that an F-Brouwer mapping φ has no fixed point. Let $\Delta = \{(x,x) : x \in B\}$. The following diagram

$$\begin{array}{ccc} \check{H}_n(\Gamma(\varphi|B,\Gamma(\varphi|S);F) & \xrightarrow{i_\star} & H_n(B\times B, S\times B; F) \\ \| & & \uparrow \\ \check{H}_n(\Gamma(\varphi|B),\Gamma(\varphi|S);F) & \longrightarrow & H_n(B\times B\setminus\Delta, S\times B\setminus\Delta; F) \end{array}$$

with all arrows induced by inclusions, is commutative. (We omit $(\check{\cdot})$ for some Čech homology groups which are isomorphic to the singular ones). Since $S\times B\setminus\Delta$ is the deformation retract of $B\times B\setminus\Delta$, $H_n(B\times B\setminus\Delta, S\times B\setminus\Delta) = 0$ and $i_\star = 0$, a contradiction.

It remains to define a suitable deformation retraction. For $x \neq y \in B$ we denote by $s(x,y)$ the unique point $s \in S$ such that $s = y + \lambda\cdot(x-y)$ for a positive number λ. Define $r\colon (B\times B\setminus\Delta)\times I \to B\times B\setminus\Delta$ by the formula $r((x,y),t) = ((1-t)x + ts(x,y), y)$. It follows that $r\colon \mathrm{id} \simeq r_1$ and r_1 is a strong deformation retraction from $B\times B\setminus\Delta$ onto $S\times B\setminus\Delta$. □

The mapping $\psi\colon B\to 2^B$ is called a (*multivalued*) *selector* of φ if $\psi(x)\subset\varphi(x)$ for every $x\in B$. The inclusion $(\Gamma(\psi|B),\Gamma(\psi|S))\subset(\Gamma(\varphi|B),\Gamma(\varphi|S))$ implies the following

Lemma 6.4. *Every map having an F-Brouwer selector is an F-Brouwer mapping.*

Any compact neighbourhood U of $\Gamma(\varphi|B)$ in $B\times B$ determines a set-valued map $\varphi_U\colon B\to 2^B$ such that $\varphi_U(x) = \{y\in B : (x,y)\in U\}$. We have

$$(\Gamma(\varphi_U|B),\Gamma(\varphi_U|S)) = (U, U\cap(S\times B)).$$

Recall that on the category of compact pairs functors $\check{H}_\star$ and $\mathrm{Hom}_F\circ\check{H}^\star$ are naturally isomorphic, [22, Theorem 1.1]. The above fact, the continuity of the Čech cohomology functor $\check{H}^\star$ [11] and the formula $\mathrm{Hom}(\,\cdot\,;F)\circ\text{dir lim} = \text{inv lim}\circ\mathrm{Hom}(\,\cdot\,;F)$ give

$$\check{H}_n(\Gamma(\varphi|B),\Gamma(\varphi|S);F) = \text{inv lim}\left\{\check{H}_n(U, U\cap(S\times B);F)\right\},$$

where U runs over the set of all neighbourhoods of $\Gamma(\varphi|B)$.

We say that the set-valued map $\varphi\colon B\to 2^B$ is approximable by F-Brouwer mappings if for every compact neighbourhood U of $\Gamma(\varphi|B)$ in $B\times B$ the map φ_U has an F-Brouwer selector. We have the following generalization of Lemma 6.4.

Lemma 6.5. *Every map approximable by F-Brouwer mappings is an F-Brouwer mapping too.*

Proof. Our assertion follows from three facts:

- $\check{H}_n((\Gamma(\varphi|B), \Gamma(\varphi|S)) \to (B \times B, S \times B); F) = \operatorname{inv} \lim\{\check{H}_n(i_U; F)\}$ with the inclusion $i_U \colon (U, U \cap (S \times B)) \to (B \times B, S \times B)$, where U runs over the set of all neighbourhoods of $\Gamma(\varphi|B)$;
- The family of polyhedral neighbourhoods of $\Gamma(\varphi|B)$ in $B \times B$ is cofinal in the family of all compact neighbourhoods of $\Gamma(\varphi|B)$ in $B \times B$;
- The functor inv lim of the inverse limit is exact on the category of the inverse systems of finite dimensional vector spaces. □

The map having F-acylic values is called an F-acyclic map.

Lemma 6.6. *The composition $B \xrightarrow{\varphi} B \xrightarrow{\psi} B$ of an F-acyclic map ψ and an F-Brouwer mapping φ is an F-Brouwer mapping.*

Proof. We will follow the ideas of Gérniewicz [25]. Let C be B or S. Consider the following commutative diagram

$$\begin{array}{ccccc} \Gamma(\varphi|C) & \xrightarrow{\;q\;} & B & \xleftarrow{\;p\;} & \Gamma(\psi|B) \\ & \nwarrow^{\overline{p}} & & \nearrow_{\overline{q}} & \\ & & \Gamma(\varphi|C) * \Gamma(\psi|B) & & \end{array}$$

where $q(x, y) = y$, $p(y_1, z) = y_1$,

$$\Gamma(\varphi|C) * \Gamma(\psi|B) = \{(x, y, y, z) : (x, y) \in \Gamma(\varphi|C), (y, z) \in \Gamma(\psi|B)\},$$

and $\overline{p}(x, y, y, z) = (x, y)$, $\overline{q}(x, y, y, z) = (y, z)$. The assumption that ψ is an F-acyclic map implies that p, $\overline{p}$ are Vietoris maps and $\overline{p}_\star \colon \check{H}_\star(\Gamma(\varphi|C) * \Gamma(\psi|B)) \to \check{H}_\star(\Gamma(\varphi|C))$ is an isomorphism [25]. By Five-Lemma, another $\overline{p}_\star$ is an isomorphism in the following commutative diagram

$$\begin{array}{ccccc} \check{H}_n(\Gamma^\varphi_B * \Gamma^\psi_B, \Gamma^\varphi_S * \Gamma^\psi_B) & \xrightarrow{\pi_\star} & \check{H}_n(\Gamma^{\psi\circ\varphi}_B, \Gamma^{\psi\circ\varphi}_S) & \xrightarrow{i_\star} & \check{H}_n(B \times B, S \times B) \\ \downarrow{\scriptstyle \overline{p}_\star} & & & & \\ \check{H}_n(\Gamma^\varphi_B, \Gamma^\varphi_S) & & & & \uparrow{\scriptstyle j_\star} \\ \downarrow{\scriptstyle p_\star} & & & & \\ \check{H}_n(B, S) & = & = & = & \check{H}_n(B, S) \end{array}$$

where $\pi(x, y, y, z) = (x, z)$, $j(x) = (x, 0)$ and $\Gamma^\chi_C = \Gamma(\chi|C)$. Since $j_\star p_\star \overline{p}_\star$ is an epimorphism, $i_\star$ is an epimorphism too. □

CHAPTER 7

THE $1 - S^k$-MAPPINGS

7.1. Definition and the basic theorem

Let us begin with the following definition for $B = B^n$.

Definition 7.1. We call $f\colon B \to 2^B$ an $1 - S^k$-*mapping* if f is ρ_c-continuous and for every x in B, $f(x)$ is homeomorphic to either a point or the k-sphere.

The motivation for this definition comes from [61] and [23]. Results of these papers show that $1 - S^0$-mappings and $1 - S^{n-1}$-mappings of B^n have fixed points, though for different reasons. The $1 - S^0$-mappings (called bimaps) are equipped with the fixed point index, [61]. These mappings can be considered as the single-valued maps of B^n into its second symmetric product. The fixed point theory for this case was developed in [44]. The $1 - S^{n-1}$-mappings are simplest spheric mappings which have been studied in [23]. The main idea was there to "fill" each value $f(x) \cong S^{n-1}$ with the bounded component $Bf(x)$ of $\mathbb{R}^n \setminus f(x)$ and consider the mapping $\widetilde{f}$ with acyclic values $\widetilde{f}(x) = f(x) \cup Bf(x)$; (note, that $\widetilde{f}(x)$ does not have to be a disc). Of course, both methods mentioned above do not apply to $1 - S^k$-mappings with $0 < k < n - 1$.

We now describe the method of the approximation of $1 - S^k$-mappings by the mappings from the same class, but having the more regular set of all these points, where the corresponding values are spheres. The general reference for the notions and results of the p.l. topology is [58].

Definition 7.2. Let U be an open subset of $B = B^n$ and $\varepsilon > 0$. We say that an $n-1$-dimensional p.l. manifold M ε-*approximates* $\operatorname{Fr} U = \operatorname{Fr}_{\mathbb{R}^n} U$ in U, if there exists a compact n-dimensional p.l. manifold K such that $\partial K = M$ and $U \supset K \supset U \setminus O_\varepsilon(\operatorname{Fr}_B U)$.

Let us observe that for every U and ε there is a p.l. manifold K such that ∂K ε-approximates $\operatorname{Fr} U$ in U: it suffices to take a simplicial decomposition of B

with mesh $\leq \varepsilon/2$ and define K to be a small regular neighbourhood of the union of all simplices intersecting $U \setminus O_\varepsilon(\mathrm{Fr}_B U)$.

Let $f, \varphi\colon B \to 2^B$ be mappings. We say that φ ε-approximates f if $\varphi(x) \subset O_\varepsilon f(x)$ for every $x \in B$.

Lemma 7.3. *Let $f\colon B \to 2^B$ be an $1 - S^k$ mapping and $U_f = \{x \in B : f(x) \cong S^k\}$. Then for every $\varepsilon > 0$ there is an $r > 0$ such that for any compact p.l. manifold K with ∂K r-approximating $\mathrm{Fr}\, U_f$ in U_f there is an $1 - S^k$-mapping φ with $U_\varphi = Int_B K$, which ε-approximates f.*

Proof. Fix an $\varepsilon > 0$. Set $U = U_f$. Take $r > 0$ such that $\operatorname{diam} f(x) < \varepsilon$ for all $x \in O_{2r}(\mathrm{Fr}_B U)$. Let K be a p.l. manifold with ∂K r-approximating $\mathrm{Fr}\, U$ in U. Take $\zeta < r$. Then

$$U \supset K \supset K \setminus O_\zeta(\mathrm{Fr}_B K) \supset U \setminus O_{2r}(\mathrm{Fr}_B U).$$

For every compact convex subset C of $\mathbb{R}^n$ we will denote by $s(C)$ the Steiner point of C ([1], [55]). We have $s(C) \in C$ and $\|s(C_1) - s(C_2)\| \leq n \cdot \rho_s(C_1, C_2)$. Let

$$\begin{aligned} \lambda(x) &= \zeta^{-1} \cdot \min(\zeta, \mathrm{d}(x, B \setminus K)), \quad b(x) = s(\mathrm{cl}(\mathrm{conv}(f(x)))), \\ \varphi(x) &= b(x) + \lambda(x) \cdot (f(x) - b(x)) \quad \text{for } x \in B. \end{aligned} \tag{7.1}$$

Since $\varphi(x) \subset \mathrm{cl}(\mathrm{conv}(f(x)))$ for every x and $\{x : \varphi(x) \neq f(x)\} \subset O_{2r}(\mathrm{Fr}_B U)$, φ is an ε-approximation of f. One can check that φ is a ρ_c-continuous mapping which takes values homeomorphic to S^k on $\mathrm{Int}_B K$ and which is single-valued elsewhere. $\square$

Theorem 7.4. *Let $f\colon B \to 2^B$ be an $1 - S^k$-mapping with $0 < k \neq 4$ and $U = \{x \in B : f(x) \cong S^k\}$. Assume that for every $\varepsilon > 0$ there exists a p.l. manifold M, which ε-approximates $\mathrm{Fr}\, U$ in U and satisfies the inequality*

$$\dim H_k(\Gamma(f|M_i); Z_2) > \dim H_k(M_i; Z_2) \tag{7.2}$$

for all components M_i of M. Then f is a Z_2-Brouwer mapping.

Proof. The basic observation is that $p\colon \Gamma(f|U) \to U$ is a locally trivial bundle with fibre S^k by Theorem 2.2.

Case 1. The case where $U \subset \mathrm{Int}\, B$.

Fix $\varepsilon > 0$. Take $r > 0$ from the Lemma 7.3. Choose K with $M \stackrel{\text{def}}{=} \partial K$ r-approximating $\mathrm{Fr}\, U$ in U and satisfying (7.2). Define φ to be the ε-approximation of f, which is the one we have described in the proof of Lemma 7.3.

By Lemma 6.5, it suffices to prove that φ is a Z_2-Brouwer mapping. Consider the following diagram with Z_2-coefficients

$$\begin{array}{ccccc} H_n(\Gamma(\varphi|B),\Gamma(\varphi|S)) & \longrightarrow & H_{n-1}(\Gamma(\varphi|S)) & \xrightarrow{i_\star} & H_{n-1}(\Gamma(\varphi|B)) \\ \downarrow{\scriptstyle p_\star} & & \downarrow{\scriptstyle \cong} & & \\ H_n(B,S) & \xrightarrow[\cong]{} & H_{n-1}(S) & = & Z_2 \end{array}$$

and the first row exact; $n = \dim B$. The right vertical arrow represents an isomorphism because $\varphi|S$ is single-valued. Note that the condition on φ to be a Z_2-Brouwer mapping ($p_\star \neq 0$) is equivalent to $i_\star = 0$. We shall define a Z_2-cycle which generates $H_{n-1}\Gamma(\varphi|S)$, and which is zero in $H_{n-1}\Gamma(\varphi|B)$.

There exists a simplicial decomposition $\mathcal{T}$ of B and a subcomplex $\mathcal{K}$ of $\mathcal{T}$ such that $K = |\mathcal{K}|$. Let us denote by K_i — components of K, by M_{ij} — components of ∂K_i and by $\mathcal{K}_i$, $\mathcal{M}_{ij}$ — corresponding subcomplexes of the simplicial decomposition $\mathcal{T}$ of B. Let $\mathcal{S} \subset \mathcal{T}$ be such that $S = |\mathcal{S}|$. Fix a linear order in the set of all vertices of $\mathcal{T}$. Ordered and singular simplices determined by $\sigma \in \mathcal{T}$ will be denoted by the same letter σ. If $\varphi|\sigma$ is single-valued then $\widetilde{\sigma}$ denotes the singular simplex $\widetilde{\sigma}(x) = (\sigma(x), \varphi(\sigma(x)))$. We use the same notation for chains. All considered chain complexes have Z_2-coefficients. For every $\mathcal{T}' \subset \mathcal{T}$ let $\sum \mathcal{T}'(p)$ denote the chain equal to the sum of all p-simplices of $\mathcal{T}'$. If $1_S = \sum \mathcal{S}(n-1)$, $1_{ij} = \sum \mathcal{M}_{ij}(n-1)$, $c = \sum(\mathcal{T} \setminus \mathcal{K})(n)$, then $1_S = \partial c + \sum_{i,j} 1_{ij}$, $\widetilde{1_S} = \partial\widetilde{c} + \sum_{i,j} \widetilde{1_{ij}}$ and $\widetilde{1_S}$ is a generator of $H_{n-1}\Gamma(\varphi|S)$. It suffices to prove that $\sum_j \widetilde{1_{ij}} = 0$ in $H_{n-1}\Gamma(\varphi|K_i)$.

Without loss of generality we can assume that K is connected and we omit the index i. There exists a neighbourhood N_1 of ∂K in K (the collar of ∂K in K) and a homeomorphism $h_1\colon N_1 \to \partial K \times [0,2]$ such that $h_1(x) = (x,0)$ for $x \in \partial K$. For simplicity of notation we write $N_1 = \partial K \times [0,2]$. Let $N = \partial K \times [0,1] \subset N_1$ and $L = \mathrm{cl}(K \setminus N)$. We define a homeomorphism $h\colon L \to K$ by the formula: $h(y) = y$ for $y \in L \setminus \partial K \times [1,2]$, $h(x,t) = (x, 2t-2)$ for $(x,t) \in \partial K \times [1,2]$. In particular, $h(x,1) = (x,0)$, i.e. $h(\partial L) = \partial K$.

Let $M'_j = h^{-1}(M_j)$ and $1'_j = h^{-1}1_j$. Of course, M'_j is a component of $M' = \partial L$ and the cycle $1'_j$ is a generator of $H_{n-1}(M'_j)$. We assume that ζ (chosen in the proof of Lemma 7.3) is small enough, i.e. that $O_\zeta(\partial K) \subset N$ and, consequently, $\varphi = f$ on M'.

Consider the following commutative diagram

$$\begin{array}{ccccc} & & H_{n-1}\Gamma(\varphi|\partial K) & = & H_{n-1}\Gamma(\varphi|\partial K) \\ & & \downarrow{\scriptstyle (0,u)} & & \downarrow{\scriptstyle v} \\ H_{n-1}\Gamma(\varphi|\partial L) & \xrightarrow[(-\beta,\alpha)]{} & H_{n-1}\Gamma(\varphi|L) \oplus H_{n-1}\Gamma(\varphi|N) & \longrightarrow & H_{n-1}\Gamma(\varphi|K) \end{array}$$

where α, β, u, v are induced by inclusions and the second row is a segment of the Mayer–Vietoris exact sequence. We are reduced to proving that $v(\sum_j \widetilde{1_j}) = 0$, which is equivalent to $(0, u(\sum_j \widetilde{1_j})) \in \operatorname{im}(-\beta, \alpha)$. Rows of the next diagram are segments of Gysin exact sequences:

$$\begin{array}{ccccc}
H_{n-k-1}\partial L & \xrightarrow{\gamma} & H_{n-1}\Gamma(\varphi|\partial L) & \xrightarrow{p_\star} & H_{n-1}\partial L \\
\delta\downarrow & & \downarrow\beta & & \downarrow\eta \\
H_{n-k-1}L & \xrightarrow[\varepsilon]{} & H_{n-1}\Gamma(\varphi|L) & \xrightarrow[\pi]{} & H_{n-1}L
\end{array}$$

We first prove that $(0, \sum_j 1'_j) \in \operatorname{im}(\beta, p_\star)$. The first row of the above diagram is the direct sum of the following exact sequences:

$$H_{n-k-1}M'_j \xrightarrow{\gamma_j} H_{n-1}\Gamma(\varphi|M'_j) \xrightarrow{p_{j\star}} H_{n-1}M'_j.$$

By the Poincaré duality,

$$\begin{aligned}
\dim H_{n-1}M'_j &= 1,\\
\dim H_{n-1}\Gamma(\varphi|M'_j) &= \dim H_k\Gamma(\varphi|M'_j),\\
\dim H_{n-k-1}M'_j &= \dim H_k M'_j.
\end{aligned}$$

By (7.2) and Lemmas 5.4, 5.5, $\dim H_k\Gamma(\varphi|M'_j) > \dim H_k M'_j$. Hence γ_j is not an epimorphism, $p_{j\star} \neq 0$, $p_{j\star}$ is onto, $p_\star$ is an epimorphism. Another epimorphism is δ. This follows from the Mayer–Vietoris exact sequence:

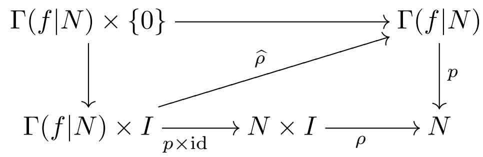
$$H_{n-k-1}\partial L \to H_{n-k-1}L \oplus H_{n-k-1}\operatorname{cl}(\mathbb{R}^3 \setminus L) \to H_{n-k-1}\mathbb{R}^3.$$

Since $p_\star$ is onto, there exists $z \in H_{n-1}\Gamma(\varphi|\partial L)$ such that $p_\star z = \sum_j 1'_j$. Of course, $\eta \sum_j 1'_j = 0$. Hence $0 = \eta p_\star z = \pi\beta z$, $\beta z \in \operatorname{im}\varepsilon$. Let $y \in H_{n-k-1}L$ and $a \in H_{n-k-1}\partial L$ be such that $\varepsilon y = \beta z$ and $\delta a = y$. Thus $\beta z = \varepsilon\delta a = \beta\gamma a$, $\beta(z - \gamma a) = 0$, $(\beta, p_\star)(z - \gamma a) = (0, p_\star z) = (0, \sum_j 1'_j)$. It remains to prove that $p_\star x = \sum_j 1'_j$ implies that $\alpha x = u(\sum_j \widetilde{1_j})$ for $x \in H_{n-1}\Gamma(\varphi|\partial L)$, (here $x = z - \gamma a$). This is a corollary from the following diagram:

$$\begin{array}{ccccc}
H_{n-1}\Gamma(\varphi|\partial L) & \xrightarrow{p_\star} & H_{n-1}\partial L & \xrightarrow{h_\star} & H_{n-1}\partial K \\
\alpha\downarrow & & & & \downarrow(\mathrm{id},\varphi)_\star \\
H_{n-1}\Gamma(\varphi|N) & & \xleftarrow[u]{\qquad\qquad} & & H_{n-1}\Gamma(\varphi|\partial K)
\end{array}$$

If $p_\star x = \sum_j 1'_j$ then $h_\star \sum_j 1'_j = \sum_j 1_j$, $(\mathrm{id}, \varphi)_\star \sum_j 1_j = \sum_j \widetilde{1_j}$ and finally $\alpha x = u(\sum_j \widetilde{1_j})$. The only remaining point concerns the commutativity of the above diagram. Let $r\colon N \to \partial K$ be the retraction $r(x, s) = x$ and $\overline{r}\colon \Gamma(\varphi|N) \to \Gamma(\varphi|\partial K)$ be given by $\overline{r}(x, y) = (r(x), \varphi(r(x)))$. If $\overline{r}$ is a strong deformation retraction, then $u = (\overline{r}_\star)^{-1}$ and reversing the lower arrow makes the corresponding diagram of mappings commutative. We now prove that this is the case.

Define $\rho\colon N \times I \to N$ by the formula $\rho((x,s),t) = (x,(1-t)s)$ for $(x,s) \in \partial K \times [0,1] = N$. Of course, $\rho\colon \mathrm{id}_N \simeq r$. By the Homotopy Lifting Property, there exists $\widehat{\rho}\colon \Gamma(f|N) \times I \to \Gamma(f|N)$ which makes the following diagram



commutative. Recall that $\varphi(x) = b(x) + \lambda(x)(f(x) - b(x))$, (see (7.1)). Let $\hbar\colon \Gamma(f|N \setminus \partial K) \to \Gamma(\varphi|N \setminus \partial K)$ be a homeomorphism defined by the formula

$$\hbar(x,y) = (x, b(x) + \lambda(x)(y - b(x))).$$

One can check that $\overline{\rho}\colon \Gamma(\varphi|N) \times I \to \Gamma(\varphi|N)$ defined by

$$\overline{\rho}((x,y),t) = \begin{cases} \hbar(\widehat{\rho}(\hbar^{-1}(x,y),t)) & \text{for } x \in N \setminus \partial K \text{ and } t \neq 1, \\ (\rho(x,1), \varphi(\rho(x,1))) & \text{for } x \in \partial K \text{ or } t = 1, \end{cases}$$

is continuous and $\overline{\rho}\colon \mathrm{id} \simeq \overline{r}$, which proves the theorem in the Case 1. The continuity of $\overline{\rho}$ follows from the fact that

$$\hbar(\widehat{\rho}(\hbar^{-1}(x,y),t)) \in \{\rho(x,t)\} \times \varphi(\rho(x,t)).$$

Case 2. The case where $U \cap \partial B \neq \emptyset$.

We replace f by $\psi\colon 2B \to 2B$,

$$\psi(x) = \begin{cases} f(x) & \text{if } \|x\| \in [0,1], \\ (2 - \|x\|) f(x/\|x\|) & \text{if } \|x\| \in [1,2], \end{cases}$$

which is singlevalued on $2S$ and satisfies all assumptions which were made on f. Let us check the condition (7.2) for ψ. Let $V = U \cap \partial B$. The new U is the set

$$U_1 = U \cup [1,2) \cdot V \cong U \times \{0\} \cup V \times [0,1).$$

Take an $\varepsilon > 0$. We find the $n-1$-manifold $L \subset V$ such that ∂L ε-approximates $\mathrm{Fr}_{\partial B} V$ in V. Then we find the n-manifold $K \subset U$ such that ∂K δ-approximates $\mathrm{Fr}\, U$ in U and satisfies (7.2). For simplicity of notation we may assume that ∂K is connected. If $\delta > 0$ is sufficiently small, then L is contained in ∂K (even with a collar). Then the set

$$K_1 = K \cup [1, 2-\delta] \cdot L \cong K \times \{0\} \cup L \times I$$

is a p.l. manifold and ∂K_1 well approximates $\mathrm{Fr}\, U_1$ in U_1. We have

$$\partial K_1 = (\partial K \setminus L) \times \{0\} \cup (\partial L) \times I \cup L \times \{1\}.$$

Since $(L, \partial L)$ is a Borsuk pair, the set $(\partial L) \times I \cup L \times \{1\}$ is a strong deformation retract of $L \times I$ and consequently, ∂K_1 is a strong deformation retract of the set

$C = (\partial K) \times \{0\} \cup L \times I$. Of course, also ∂K is a strong deformation retract of C. By Lemmas 5.4, 5.5, the conditions (7.2) for ∂K and for ∂K_1 are equivalent.

The Case 1 now implies that ψ is a Z_2-Brouwer mapping.

It suffices to prove that if ψ is a Z_2-Brouwer mapping, so is f. Let $P = 2B \setminus \mathrm{Int}(B)$. Consider the following diagram

$$\begin{array}{ccccc} \check{H}_n(\Gamma^f_B, \Gamma^f_S) & \xrightarrow{\cong} & \check{H}_n(\Gamma^\psi_{2B}, \Gamma^\psi_P) & \xleftarrow{\cong} & \check{H}_n(\Gamma^\psi_{2B}, \Gamma^\psi_{2S}) \\ \downarrow{\scriptstyle p_\star} & & \downarrow & & \downarrow{\scriptstyle p^\psi_\star} \\ \check{H}_n(B, S) & \xrightarrow[\cong]{} & \check{H}_n(2B, P) & \xleftarrow[\cong]{} & \check{H}_n(2B, 2S) \end{array}$$

All horizontal arrows represent isomorphisms: left arrows are excisions, on the right-hand side $2S$ and Γ^ψ_{2S} are strong deformation retracts of P and Γ^ψ_P. Thus $p^\psi_\star \neq 0$ implies that $p_\star \neq 0$. □

7.2. The $1 - S^{n-2}$-mappings of B^n

The next theorem is in the author's opinion the main result of this dissertation.

Theorem 7.5. *Every $1 - S^{n-2}$-mapping of B^n is a Z_2-Brouwer mapping and has a fixed point.*

We will need the following lemma, which states that the raising to the n-th power in the Z_2-cohomology algebra of any closed n-manifold in $\mathbb{R}^{n+1}$ is a trivial operation.

Lemma 7.6. *Let $M \subset \mathbb{R}^{n+1}$ be an n-dimensional compact connected topological manifold without boundary, $n \geq 2$. Then $x^n = 0$ for every $x \in H^1(M; Z_2)$.*

The situation described in the assumptions of this lemma is known very well in the literature. Let us gather some facts before the proof. First, $M \subset \mathbb{R}^{n+1} \cup \{\infty\} \cong S^{n+1}$; $S^{n+1} \setminus M = U \cup V$, ($U$, V — connected). The closures $A = \overline{U}$, $B = \overline{V}$ are ANRs, [11, VIII.4.8]. By the Alexander duality, $H^n(A; Z_2) = H^n(B; Z_2) = 0$. Let $i: M \to A$, $j: M \to B$ be inclusions. The Mayer–Vietoris exact sequence shows that $\varphi: H^s(A; Z_2) \oplus H^s(B; Z_2) \to H^s(M; Z_2)$, $\varphi(\alpha, \beta) = i^\star\alpha + j^\star\beta$ is an isomorphism for $1 \leq s < n$. Moreover,

(1) $Sq^{n-r}y = 0$ for every $y \in H^r(M; Z_2)$, $1 \leq r < n$, [65, III.2.3];
(2) $i^\star Sq^1 a \cup j^\star b = i^\star a \cup j^\star Sq^1 b$ for all $a \in H^r(A; Z_2)$, $b \in H^{n-1-r}(B; Z_2)$, [65, III.2.4] (see also [65, II.4, III.1.4]).
(3) $Sq^i u^k = \binom{k}{i} u^{k+i}$ if $\dim(u) = 1$, [65, I.2.4].

Proof of Lemma 7.6. *Case 1.* Let $n \neq 2^m - 1$ for every natural m. Since $0 = Sq^{n-r}x^r = \binom{r}{n-r}x^n$, by (1), (3), it suffices to find r such that $\binom{r}{n-r}$ is odd and $1 \leq n - r \leq r < n$. If $n = 2t$ then $r = t$ satisfies the above conditions. If $n = 2t - 1$ then $t \neq 2^{m-1}$ for every m. Thus $t = 2^{i-1} + j$ for some $i \geq 2$ and

$1 \le j \le 2^{i-1} - 1$. It is easy to check that $\binom{2^i-1}{k}$ is odd for every $k = 0, \dots, 2^i - 1$, (by induction on i, $(x+y)^q \bmod 2 = x^q + y^q$ for $q = 2^i$, so $\binom{2^i}{k}$ is even for $k = 1, \dots, 2^i - 1$), and $r \overset{\text{def}}{=} 2^i - 1$ satisfies $1 < n - r < r < n$.

Case 2. Let $n = 2^m - 1$. Then

$$
\begin{aligned}
x^n &= (i^\star\alpha + j^\star\beta)^n = \sum_{k=0}^{n} \binom{n}{k} i^\star\alpha^k \cup j^\star\beta^{n-k} = \sum_{k=1}^{n-1} i^\star\alpha^k \cup j^\star\beta^{n-k} \\
&= \sum_{p=1}^{(n-1)/2} \left(i^\star\alpha^{2p-1} \cup j^\star\beta^{n-2p+1} + i^\star\alpha^{2p} \cup j^\star\beta^{n-2p} \right) \\
&= \sum_{p=1}^{(n-1)/2} \left(i^\star\alpha^{2p-1} \cup j^\star \binom{n-2p}{1} \beta^{n-2p+1} + i^\star \binom{2p-1}{1} \alpha^{2p} \cup j^\star\beta^{n-2p} \right) \\
&= \sum_{p=1}^{(n-1)/2} \left(i^\star\alpha^{2p-1} \cup j^\star Sq^1\beta^{n-2p} + i^\star Sq^1\alpha^{2p-1} \cup j^\star\beta^{n-2p} \right) = 0,
\end{aligned}
$$

by (2), which proves the lemma. □

Proof of Theorem 7.5. By Theorem 7.4, it suffices to check the inequality (7.2). Let $f \colon B^n \to 2^{B^n}$ be a $1 - S^{n-2}$-mapping, M — a closed $n-1$-manifold in B^n; $p \colon \Gamma(f|M) \to M$ — a projection. By Theorem 2.2, p is a locally trivial bundle with the fibre $S = S^{n-2}$. Let us denote by w_j the j-th Stiefel–Whitney class of p. Set $\Gamma = \Gamma(f|M)$. Consider the bundle $p^\triangle \colon \Gamma^\triangle \to M$ and the map $g \colon \Gamma^\triangle \to \mathbb{R}^n$; $g((x,y),(x,z)) = y$. Recall that $c^\triangle$ denotes the first Stiefel–Whitney class of the S^0-bundle $\Gamma^\triangle \to \Gamma^\triangle/Z_2$ with the Z_2 action given by the transposition $T((x,y),(x,z)) = ((x,z),(x,y))$. We must have $(c^\triangle)^n = 0$, for otherwise, by Theorem 5.3, $y = z$ for a $((x,y),(x,z)) \in \Gamma^\triangle$, a contradiction. Therefore, applying twice the formula (5.8), we see that

$$
\begin{aligned}
0 &= (c^\triangle)^{n-1} \cup c^\triangle = \sum_{j=1}^{n-1} (q^\triangle)^\star(w_j^\triangle) \cup (c^\triangle)^{n-1-j} \cup c^\triangle \\
&= (q^\triangle)^\star(w_1^\triangle) \cup (c^\triangle)^{n-1} + \sum_{j=2}^{n-1} (q^\triangle)^\star(w_j^\triangle) \cup (c^\triangle)^{n-j} \\
&= \sum_{j=1}^{n-1} (q^\triangle)^\star(w_1^\triangle \cup w_j^\triangle) \cup (c^\triangle)^{n-1-j} + \sum_{j=1}^{n-2} (q^\triangle)^\star(w_{j+1}^\triangle) \cup (c^\triangle)^{n-j-1} \\
&= \sum_{j=1}^{n-2} (q^\triangle)^\star(w_1^\triangle \cup w_j^\triangle + w_{j+1}^\triangle) \cup (c^\triangle)^{n-j-1}.
\end{aligned}
$$

This gives $w^{\triangle}_{j+1} = w^{\triangle}_1 \cup w^{\triangle}_j$ for $j = 1, \ldots, n-2$, and $w_{n-1} = w^{\triangle}_{n-1} = (w^{\triangle}_1)^{n-1} = 0$, by Theorem 5.8 and Lemma 7.6. According to Lemma 5.5, this completes the proof. □

7.3. The $1 - S^1$-mappings of B^3

For $n = 3$ Theorem 7.5 sounds especially visually:

Theorem 7.7. *Every ρ_c-continuous mapping of the closed 3-dimensional disc, taking values which are points or knots, has a fixed point.*

Note that the values in two points of the same component of the set U_f (see Lemma 7.3) can be knots of different types.

We give an alternative proof of this special case of Theorem 7.5, which is based on another lemma.

Lemma 7.8. *If M_g is a closed orientable surface of genus g then*

$$\widetilde{K}(M_g) = (Z_2)^{2g+1}.$$

Proof. (All results of K-theory which will be needed here, can be found in [33] and [38].)

We begin by recalling that $\widetilde{K}(S^1) = Z_2$ and $\widetilde{K}(S^2) = Z_2$. Now suppose that $g \geq 1$. Let SX denote the reduced suspension of the space X and $\widetilde{K}^{-1}(X) = \widetilde{K}(SX)$. Let Y be a closed subset of X. Consider the following exact sequence of abelian groups (see [33, 9.2.8], [38, II.3.29]):

$$\widetilde{K}^{-1}(X) \xrightarrow{\gamma} \widetilde{K}^{-1}(Y) \xrightarrow{\delta} \widetilde{K}(X/Y) \xrightarrow{\alpha} \widetilde{K}(X) \xrightarrow{\beta} \widetilde{K}(Y).$$

Take $X = M_g$ and $Y = \bigvee_{i=1}^{2g} Y_i$, $Y_i \cong S^1$ for $i = 1, \ldots, 2g$. If the surface M_g is represented as a polygon (with $4g$ angles and standard identifications) then Y is its boundary. Of course, $X/Y \cong S^2$. Homomorphisms γ and β have their right inverses. Indeed, let $r_i \colon X \to Y_i$ be a retraction such that $r_i(Y_j) = *$ for $j \neq i$. Then

$$\widetilde{K}(Y) \cong \bigoplus_{i=1}^{2g} \widetilde{K}(Y_i) \xrightarrow{(r_i^!)} \widetilde{K}(X)$$

is a right inverse of β, (fortunately, $\widetilde{K}(*) = 0$). Replacing $\widetilde{K}$ by $\widetilde{K}^{-1}$ we obtain a right inverse of γ. Consequently, γ and β are epimorphisms. We obtain an exact sequence

$$0 \longrightarrow \widetilde{K}(S^2) \xrightarrow{\alpha} \widetilde{K}(M_g) \xrightarrow{\beta} \bigoplus_{i=1}^{2g} \widetilde{K}(S^1) \longrightarrow 0,$$

which splits. Thus

$$\widetilde{K}(M_g) \cong \widetilde{K}(S^2) \oplus \bigoplus_{i=1}^{2g} \widetilde{K}(S^1) = (Z_2)^{2g+1}.$$ □

Proof of Theorem 7.7. The structural group of the locally trivial bundle with fibre S^1 reduces to $O(2)$, (see [67, 11.45], [48, Fact 2]). For this reason we can rewrite the proof of Theorem 7.5 omitting the triangles in all the symbols $(\,\cdot\,)^{\triangle}$. We now change our last argument in that proof.

Let $\overrightarrow{p}$ be the vector bundle with the fibre $\mathbb{R}^2$, which corresponds to p. By Lemma 7.8, $\overrightarrow{p} \oplus \overrightarrow{p}$ represents zero in $\widetilde{K}(M)$. This gives $\overrightarrow{p} \oplus \overrightarrow{p} \oplus \overrightarrow{\theta} = \overrightarrow{\Theta}$ for some trivial vector bundles $\overrightarrow{\theta}$, $\overrightarrow{\Theta}$. It follows that $1 = w(p \oplus p) = w(p) \cup w(p) = (1 + w_1 + w_2)^2 = 1 + [w_1]^2 + [w_2]^2 = 1 + [w_1]^2$. Therefore $[w_1]^2 = 0$, which finishes the proof. □

Proof of Lemma 7.8 was based on the classification of the closed 2-manifolds. We now describe the construction of any S^1-bundle E over the closed oriented surface $M = M_g$ of genus g. The question, which of these bundles are weakly equivalent, is answered in [62]. We get a necessary condition for such bundles to be graphs of the restrictions of some $1 - S^1$-mappings.

Let $g > 0$. Remove an open disc D from M. Then $M \setminus D$ is a "thickened" wedge W of $2g$ circles $S_1, \dots, S_{2g}$. There are two S^1-bundles over S^1: the torus and the Klein bottle. The S^1-bundle $E|(M \setminus D)$ is determined by choosing one of these bundles over each S_i. The bundle $E|D$ is trivial (equivalent to $D \times S^1$). The bundles $E|(M \setminus D)$ and $E|\overline{D}$ admit the sections a, b, (the first one — because there exists a section over

$$W = \bigvee_{i=1}^{2g} S_i$$

and W is a strong deformation retract of $M \setminus D$). Let α, β be the restrictions of sections a, b to ∂D and γ be any loop representing the fiber. The set $T = p^{-1}(\partial D)$ is a torus and there is an integer j such that

$$\alpha = \beta + j \cdot \gamma \quad \text{in } H_1(T; Z).$$

Lemma 7.9.

$$\dim H_1(E; Z_2) = \begin{cases} 2g + 1 & \textit{if } j = 2s, \\ 2g & \textit{if } j = 2s + 1. \end{cases}$$

Proof. The reduced Mayer–Vietoris exact sequence for $E|(M \setminus D)$ and $E|\overline{D}$ with Z_2-coefficients

$$H_1(E|\partial D) \xrightarrow{\xi} H_1(E|(M \setminus D)) \oplus H_1(E|\overline{D}) \xrightarrow{\zeta} H_1(E) \longrightarrow 0$$

yields $\dim H_1 E = [(2g + 1) + 1] - \dim im(\xi)$. Let us denote by a_i the restriction of the section a to the circle S_i. In $H_1(M \setminus D)$ we have:

$$\alpha = (a_1 + a_2 - a_1 - a_2) + \dots + (a_{2g-1} + a_{2g} - a_{2g-1} - a_{2g}) = 0.$$

The cycles $a_1, \dots, a_{2g}, \gamma$, freely generate $H_1(E|(M \setminus D))$. The basis of $H_1(E|\partial D)$ is $\{\alpha, \gamma\}$. Moreover, $H_1(E|\overline{D})$ is generated by γ. Let $[\xi]$ be the matrix of ξ with respect to the systems of free generators which are given above. Then

$$[\xi] = \begin{bmatrix} \xi_{1,1} & \xi_{1,2} \\ \vdots & \vdots \\ \xi_{2g,1} & \xi_{2g,2} \\ \xi_{2g+1,1} & \xi_{2g+1,2} \\ \xi_{2g+2,1} & \xi_{2g+2,2} \end{bmatrix} = \begin{bmatrix} (1-1) & 0 \\ \vdots & \vdots \\ (1-1) & 0 \\ 0 & 1 \\ j & 1 \end{bmatrix}.$$

Thus

$$\dim \operatorname{im}(\xi) = \operatorname{rank}_{Z_2}[\xi] = \begin{cases} 1 & \text{for } j \equiv 0 \bmod 2, \\ 2 & \text{for } j \equiv 1 \bmod 2, \end{cases}$$

which proves the lemma. □

Since $\dim H_1(M_g; Z_2) = 2g$, $w_2(E) = 0$ if and only if j is even, by Lemma 5.5. For this reason, for the surfaces with $g > 0$, we have

Corollary 7.10. *The only bundles, which can be graphs of the $1 - S^1$-mappings of B^3 over M_g, are these with even j.*

Proof. It suffices to consider the case $g = 0$. According to the classification given in [34, pp. 143–144], every S^1-bundle over S^2 is equivalent to $p: L_j \to S^2$, where $L_j = S^3/Z_j$ is a lens space and p is induced by the Hopf fibration $S^3 \to S^2$. By [11, V.6.16],

$$H_1(L_j; Z_2) = \begin{cases} 0 & \text{if } j = 2s+1, \\ Z_2 & \text{if } j = 2s. \end{cases}$$

□

7.4. Examples

We give here three examples, which are summarized in the following

Theorem 7.11. *Let* $\operatorname{char}(F)$ *denote the characteristic of the field F.*

(a) *There is an $1 - S^1$-mapping of B^4, which for every field F is not an F-Brouwer mapping.*
(b) *There is an $1 - S^1$-mapping of B^3, which is an F-Brouwer mapping if and only if* $\operatorname{char}(F) = 2$.
(c) *There is an $1 - S^0$-mapping of B^2, which is an F-Brouwer mapping if and only if* $\operatorname{char}(F) \neq 2$.

Proof. Let us observe that f is an F-Brouwer mapping for a field F with $\operatorname{char}(F) = 2$ if and only if f is a Z_2-Brouwer mapping, which follows from the Universal Coefficients Theorem.

(a) (It is worth pointing out that our example does have an obvious fixed point.)

Part 1. Write $x = \sum_{i=1}^4 x_i e_i \in \mathbb{R}^4$, $(e_i)_{i=1}^4$ — the standard basis in $\mathbb{R}^4$, $\mathbb{R}^i = \operatorname{Span}\{e_1, \dots, e_i\}$ for $i \leq 4$, S^3 — the unit sphere in $\mathbb{R}^4$, $S^{i-1} = S^3 \cap \mathbb{R}^i$, $E_x = \operatorname{Span}\{x, e_3, e_4\}$ for $x \in S^1$, $S_x = S^3 \cap E_x$. Define $\varphi_x : S^2 \to S_x$, $\varphi_x(y) = y_1 x + y_2 e_3 + y_3 e_4$ for $x \in S^1$ and $\varphi : S^1 \times S^2 \to \mathbb{R}^4$, $\varphi(x, y) = x/2 + \varphi_x(y)/4$. The map φ is an embedding of $S^1 \times S^2$ in $\mathbb{R}^4$. Set $K = \{x/2 + r \cdot \varphi_x(y)/4 : (x, y) \in S^1 \times S^2,\ 0 \leq r \leq 1\}$. Clearly, $K \cong S^1 \times B^3$. Let $q : S^3 \to S^2$ be the Hopf fibration. We define $f : B^4 \to 2^{B^4}$ by the formula:

$$\begin{cases} f\left(\dfrac{1}{2}x + r \cdot \dfrac{1}{4}\varphi_x(y)\right) = r(1-r) \cdot q^{-1}(y) & \text{on } K, \\ f(z) = 0 & \text{on } B^4 \setminus K. \end{cases}$$

Part 2. Suppose contrary to our claim that there is a field F making f an F-Brouwer map. Set $B = B^4$, $S = \partial B$ and $H_\star(\cdot) = H_\star(\cdot\,; F)$. Note that $f|_S = 0$. The commutative diagram

$$\begin{array}{ccc} H_4(\Gamma_B, \Gamma_S) & \xrightarrow{\neq 0} & H_4(B \times B, S \times B) \\ {\scriptstyle p_\star}\downarrow & & \uparrow{\scriptstyle j_\star}\ \cong \\ H_4(B, S) & = & H_4(B, S) \end{array}$$

with $j(x) = (x, 0)$ yields $p_\star \neq 0$. The diagram

$$\begin{array}{ccccc} H_4(\Gamma_B, \Gamma_S) & \longrightarrow & H_3\Gamma_S & \xrightarrow{i_\star} & H_3\Gamma_B \\ {\scriptstyle p_\star}\downarrow & & \cong\ \downarrow{\scriptstyle p_\star} & & \\ H_4(B, S) & \xrightarrow[\cong]{} & H_3 S & = & F \end{array}$$

with the first row exact shows that $i_\star = 0$. Let $C = B \setminus \operatorname{Int}(K)$. Consider the segment of the Mayer–Vietoris exact sequence:

$$H_4 B \to H_3(\partial K) \to H_3 C \oplus H_3 K \to H_3 B.$$

Since $H_3 K = H_3(S^1 \times B^3) = 0$ and $H_3(\partial K) = H_3(S^1 \times S^2) = F$, we have $H_3 C = F$. Take $v \in \operatorname{Int}(K)$. Since S is a strong deformation retract of $[B \setminus \{v\}]$, the composition

$$F = H_3 S \xrightarrow{\eta} H_3 C \longrightarrow H_3(B \setminus \{v\})$$

of homomorphisms induced by inclusions is an isomorphism. Therefore η is a monomorphism. Now, the equality $H_3 C = F$ shows that η is an isomorphism. Since $\Gamma_S = S \times 0$ and $\Gamma_C = C \times 0$, $\overline{\eta} : H_3\Gamma_S \to H_3\Gamma_C$ is an isomorphism too. It follows that $j_1 : H_3\Gamma_C \to H_3\Gamma_B$ is zero, because $0 = i_\star = j_1 \circ \overline{\eta}$. Summarizing, we have: $j_1 = 0$, $H_3\Gamma_C = H_3 C = F$, $H_3\Gamma_{\partial K} = H_3(\partial K) = F$.

Part 3. Our next goal is to determine the group $H_3\Gamma_K$. Note, that $K = L \cup N$, where $L = \{x/2 + r \cdot \varphi_x(y)/4 : (x, y) \in S^1 \times S^2,\ 0 \leq r \leq 1/2\}$, $N = \{x/2 + r \cdot \varphi_x(y)/4 : (x, y) \in S^1 \times S^2,\ 1/2 \leq r \leq 1\}$ and $L \cap N = \partial(L)$. For abbreviation, we let Ω stand for $(x/2 + r\cdot\varphi_x(y)/4, r(1-r)z)$, $z \in q^{-1}(y)$. The homotopy

$$G_t(\Omega) = \left(\frac{1}{2}x + [t + (1-t)r]\cdot\frac{1}{4}\varphi_x(y), [t+(1-t)r][1-t-(1-t)r]z\right)$$

shows that $\Gamma_{\partial K}$ is a strong deformation retract of Γ_N. Another homotopy

$$H_t(\Omega) = \left(\frac{1}{2}x + tr\cdot\frac{1}{4}\varphi_x(y), tr(1-tr)z\right)$$

with $H_0(\Omega) = (x/2, 0)$ gives $\Gamma_L \simeq S^1$. We also have a homeomorphism $h\colon \Gamma_{\partial L} \to S^1 \times S^3$, which sends $(x/2 + \varphi_x(y)/8, z/4)$ to (x,z) for $z \in q^{-1}(y)$. Consider the segment of the Mayer–Vietoris exact sequence:

$$H_3\Gamma_{\partial L} \xrightarrow{\lambda} H_3\Gamma_L \oplus H_3\Gamma_N \xrightarrow{\psi} H_3\Gamma_K \longrightarrow H_2\Gamma_{\partial L}.$$

Since $H_2\Gamma_{\partial L} = H_2(S^1\times S^3) = 0$, ψ is an epimorphism. Clearly, $H_3\Gamma_L = H_3S^1 = 0$. If $\lambda = 0$ then ψ is an isomorphism and $H_3\Gamma_K \cong H_3\Gamma_N \cong H_3\Gamma_{\partial K} \cong H_3(\partial K) = F$. What is left is to show that $\lambda = 0$ or equivalently, that the inclusion $\omega\colon \Gamma_{\partial L} \to \Gamma_N$ induces the zero homomorphism on H_3-groups. This is equivalent to $0 = \xi_\star\colon H_3(S^1\times S^3) \to H_3(S^1\times S^2)$ for $\xi = \varphi^{-1}\circ G_1\circ\omega\circ h^{-1}$ where $G_1(\Gamma_N) = \Gamma_{\partial K} = \partial(K)\times 0$ is identified with $\partial(K) = \varphi(S^1\times S^2)$. It is easy to check that $\xi(x,z) = (x, q(z))$. Thus $\xi = \mathrm{id}\times q$. By the Künneth theorem, the diagram

$$\begin{array}{ccc} H_3(S^1\times S^3) & \xleftarrow{\cong} & \bigoplus_{i=0}^{3} H_iS^1\otimes H_{3-i}S^3 \\ \downarrow{\scriptstyle \xi_\star} & & \downarrow \\ H_3(S^1\times S^2) & \xleftarrow[\cong]{} & \bigoplus_{i=0}^{3} H_iS^1\otimes H_{3-i}S^2 \end{array}$$

commutes. The i-th component of the direct sum is non-zero only for $i = 0$ in the first row, and only for $i = 1$ in the second row of the above diagram. Hence $\xi_\star = 0$.

Part 4. Consider the segment of the Mayer–Vietoris exact sequence:

$$\begin{array}{ccccc} H_3\Gamma_{\partial K} & \xrightarrow{\alpha} & H_3\Gamma_C \oplus H_3\Gamma_K & \xrightarrow{\beta} & H_3\Gamma_B \\ \| & & \| & & \\ F & & F^2 & & \end{array}$$

where $\alpha(x) = (i_1x, i_2x)$ and $\beta(x,y) = j_2y - j_1x = j_2y$ (see Part 2). Since i_2 is a composition

$$H_3\Gamma_{\partial K} \xrightarrow{\cong} H_3\Gamma_N \xrightarrow{\psi} H_3\Gamma_K,$$

i_2 is an isomorphism (see Part 3). Now, $\dim \mathrm{im}\alpha = 1 = \dim\ker\beta$. Thus $\dim \mathrm{im} j_2 = \dim im\beta = 2 - \dim\ker\beta = 1$. But $0 = \beta \circ \alpha = j_2 \circ i_2$. Therefore $j_2 = 0$, a contradiction.

(b) Fix $r \in (-1, 1)$. Let $A(x) = \{rx + y : y \in \mathbb{R}^3,\ \langle y, x\rangle = 0\}$ and $F(x) = S^2 \cap A(x)$ for $x \in S^2$. Define $\varphi\colon B^3 \to B^3$ by the formula $\varphi(sx) = s \cdot F(x)$ for $s \in [0, 1]$, $x \in S^2$. By Theorem 7.5, φ is a Z_2-Brouwer map.

Of course Γ^φ_B is contractible and $\Gamma^\varphi_S \cong RP^3$, [34, p. 144]. By the exact sequence

$$H_3(\Gamma^\varphi_B; F) \to H_3(\Gamma^\varphi_B, \Gamma^\varphi_S; F) \to H_2(\Gamma^\varphi_S; F) \to H_2(\Gamma^\varphi_B; F),$$

we have

$$H_3(\Gamma^\varphi_B, \Gamma^\varphi_S; F) = H_2(\Gamma^\varphi_S; F).$$

By the Poincaré duality,

$$\dim H_2(\Gamma^\varphi_S; F) = \dim H_1(\Gamma^\varphi_S; F),$$

but

$$H_1(\Gamma^\varphi_S; F) = H_1(RP^3; F) = Z_2 \otimes F = F/_{2F} = 0.$$

The last equality holds if and only if $\mathrm{char}(F) \neq 2$. In this case φ is not an F-Brouwer map.

(c) Take $r\colon S^1 \to S^1$, $r(z) = \sqrt{z}$ for $z \in S^1 = S \subset C$. Let $f\colon B^2 \to B^2$ be the cone of r. Then the set Γ^f_B is contractible and the projection $p\colon \Gamma^f_S \to S$ is a double covering. Since the following diagram

$$\begin{array}{ccc} H_2(\Gamma^f_B, \Gamma^f_S; F) & \xrightarrow{\cong} & H_1(\Gamma^f_S; F) \\ {\scriptstyle p_\star}\downarrow & & \downarrow{\scriptstyle 2\times} \\ H_2(B, S; F) & \xrightarrow[\cong]{} & H_1(S; F) \end{array}$$

commutes, our assertion follows. □

Theorem 7.11(c) and the next result throw light on the difference beetwen $1 - S^0$-mappings and $1 - S^{n-1}$-mappings of B^n. Recall that the earlier methods of proving the fixed point theorems for these classes of mappings have been of the quite different natures, [61], [23].

Theorem 7.12. *All $1 - S^{n-1}$-mappings of B^n are Z_2-Brouwer mappings.*

Proof. We repeat, line by line, the proof of Theorem 7.5, (let me omit the triangles). The conclusion is now simpler:

$$0 = c^n = \sum_{j=1}^{n} w_j c^{n-j} \ \Rightarrow\ w_j = 0 \quad \text{for all } j,$$

in particular $w_n = 0$, which proves the theorem. □

Using the procedure which is described in the proof of Theorem 7.5, we can obtain some necessary conditions on the sphere bundle to be the graph of an $1 - S^k$-mapping over a closed $n-1$-manifold M in B^n.

Example 7.13. Let $p\colon \Gamma \to M$ be a S^1-bundle over M. Set $x = w_1(p)$, $y = w_2(p)$. We have

$$c^2 = cx + y.$$

Set $u_1 = 1$, $v_1 = 0$, $c^j = cu_j + v_j$ for $j = 1, \dots, n$. Then

$$u_{j+1} = xu_j + v_j, \qquad v_{j+1} = yu_j.$$

Since $\dim(M) = n-1$, $v_n = 0$. The condition $c^n = 0$, which is necessary for Γ to be a graph, is now equivalent to $u_n(x, y) = 0$.

Of course, the possibility of getting an application of this observation relies on our knowledge of the Z_2-cohomology algebra of M. Let us compute some initial polynomials u_n, v_n:

$$\begin{aligned} u_3 &= x^2 + y, & v_3 &= xy; \\ u_4 &= x^3, & v_4 &= x^2y + y^2; \\ u_5 &= x^4 + x^2y + y^2, & & \dots\dots\dots\dots\dots \end{aligned}$$

If $n = 3$, then $u_3 = y$, (and $y = 0$ yields the fixed points in Theorem 7.7).

If $n = 4$, then $u_4 = 0$, (there is no restriction). This fact combined with the property of the Hopf fibration of having $w_2 \neq 0$ (see Example 5.6) originated the first example from Theorem 7.11.

If $n = 5$, then $u_5 = x^2y$, (see Lemma 7.6 and its proof). The next lemma shows that this polynomial can take a non-zero value (a priori, i.e. forgetting the question, whether x_0 and y_0 satisfying $x_0^2 \cup y_0 \neq 0$ are the Stiefel–Whitney classes of any bundle).

Lemma 7.14. *There is a closed* 4*-manifold* M *in* $\mathbb{R}^5$ *such that there are* $x \in H^1(M; Z_2)$ *and* $y \in H^2(M; Z_2)$ *with* $x^2 \cup y \neq 0$.

Proof. We choose M to be the boundary of a regular neighbourhood N of the projective space RP^2 in $\mathbb{R}^5$. Consider the commutative diagram of the Poincaré dualities D

$$\begin{array}{ccccc} H^k(N, M) & \longrightarrow & H^k(N) & \xrightarrow{i^*} & H^k(M) \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} \\ H_{5-k}(N) & \longrightarrow & H_{5-k}(N, M) & \longrightarrow & H_{4-k}(M) \end{array}$$

with Z_2-coefficients and $k = 2$. Since $N \simeq RP^2$, $H_3(N) = 0$. Thus down-right, up-right arrows in our diagram represent monomorphisms. Of course,

$H^1(N) = Z_2 = H^2(N)$. Moreover, taking η which generates $H^1(N)$, we see that η^2 generates $H^2(N)$. Consequently, $i^\star(\eta^2) \neq 0$ in $H^2(M)$. Since

$$\cup\colon H^2(M) \times H^2(M) \longrightarrow H^4(M) \xrightarrow{\langle\,\cdot\,, o(M)\rangle} Z_2$$

is a non-degenerate pairing, there is a $y \in H^2(M)$ such that $[i^\star(\eta)]^2 \cup y \neq 0$, which is our claim. $\square$

CHAPTER 8

OPEN PROBLEMS

(1) Has every $1 - S^k$-mapping of B^n a fixed point for $1 \leq k \leq n-3$?

(2) Has every ρ_s-continuous mapping of B^n with eLC^{n-2}-values:
   (a) a fixed point?
   (b) a single-valued selector?
   (c) a single-valued approximation?

(3) Is it true that for every class $\{A_\lambda : \lambda \in \Lambda\}$ of eLC^{n-2} compact sets in $\mathbb{R}^n$, the sets $\widetilde{A}_\lambda = A_\lambda \cup B(A_\lambda)$ are eLC^{n-1} for $\lambda \in \Lambda$? Is this true for the one-point set Λ?

(4) Is there a free fibre-preserving Z_2-action on the space of any locally trivial sphere bundle?

(5) Is the group of all odd homeomorphisms of S^k the strong deformation retract of the space $\mathrm{TOP}(S^k)$ of all homeomorphisms of S^k?

(6) Are the classes $w_j^\triangle$ the Stiefel–Whitney classes for every j?

(7) Is there for every spherical fibration such a fibration, which would be an equivalent of the bundle $\Gamma^\triangle$?

(8) Is the equality $w_{n-1} = 0$ the sufficient condition for the equivalence class of the S^{n-2}-bundles over an $n-1$-manifold M in $\mathbb{R}^n$ to be represented by a graph of a $1 - S^{n-2}$-mapping on M?

(9) Is every $1 - S^{n-1}$-mapping of B^n an F-Brouwer mapping for every field F?

(10) How to generalize Theorems 7.4, 7.5
   (a) for ρ_c-continuous $1 - M$-mappings with $M \neq S^k$?
   (b) for ρ_s-continuous $1 - S(x)$-mappings with $S(x) \simeq S^k$, which are ρ_h-continuous on the set $U_f = \{x \in B^n : f(x) = S(x)\}$?

(11) Are there some fixed point theorems for compositions of mappings from this dissertation? See also [23, p. 177].

CHAPTER 9

APPENDIX

9.1. The Borsuk Lemma

We prove here the modified version of the Borsuk Lemma (see [2, p. 187]).

Lemma 9.1 (Borsuk, eLC^{n-1}-version)**.** *Suppose that all the sets from the class* $\Theta \subset 2^{\mathbb{R}^n}$ *are* eLC^{n-1} *and compact. Then there is a continuous increasing function* $\alpha\colon (0,1] \to (0,1]$ *with* $\alpha(\varepsilon) \leq \varepsilon$ *(for all* ε*) such that for every set* $\theta \in \Theta$ *there exists a retraction*

$$r_\theta\colon O_{\alpha(1)}(\theta) \to \theta,$$

with $\|r_\theta(x) - x\| < \varepsilon$ *for all* $x \in O_{\alpha(\varepsilon)}(\theta)$ *and* $\varepsilon \in (0,1]$.

Proof. The definition of the eLC^{n-1}-condition for Θ can be written in the following form: There are $d > 0$ and $\lambda\colon (0,d] \to (0,1]$ such that for every $\theta \in \Theta$, $x \in \theta$, $p \leq n-1$, every continuous map

$$\partial(\Delta^{p+1}) \to K_\theta(x,\delta)$$

for $\delta \leq d$ has a continuous extension

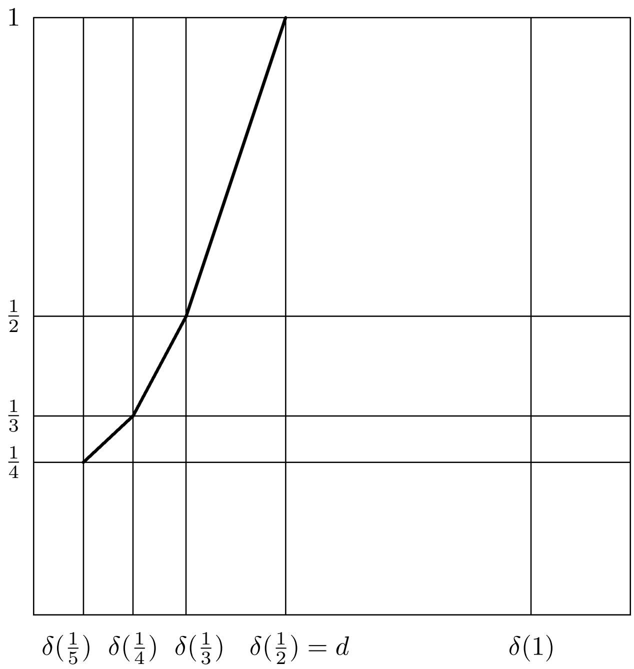
$$\Delta^{p+1} \to K_\theta(x,\lambda(\delta)).$$

Here $K_\theta(x,\delta)$ denotes an open ball in θ with the center x and radius δ. In addition we can assume that

- the function λ is increasing and continuous;
- $\lim_{\delta\to 0} \lambda(\delta) = 0$;
- $\delta < \lambda(\delta)$ for all $\delta \leq d$.

Assuming that $\delta(\varepsilon) < \varepsilon$ and $\delta(1/(n+1)) < \delta(1/n)$ in the eLC^{n-1}-condition (see the beginning of Chapter 4), the figure on the page 64 shows the definition of the function $\lambda = \lambda(\delta)$.

The formula $\eta(\delta) = \lambda(4\delta)$ defines the function $\eta\colon (0,d/4] \to (0,1]$. There is $b < d/4$ such that the function $\mu = \eta^{(n)} = \eta \circ \ldots \circ \eta$ is well defined on $(0,b]$.



We choose an increasing continuous function $\alpha\colon (0,1] \to (0,b]$ such that

- $0 < t \le 4\alpha(\varepsilon) \ \Rightarrow\ \mu(t) < \varepsilon/2$;
- $\alpha(\varepsilon) < \varepsilon/4$.

Fix the set $\theta \in \Theta$. Let Λ be an infinite simplicial decomposition of $\mathbb{R}^n \setminus \theta$ such that for every $\varepsilon \in (0,1]$ and $\Delta \in \Lambda$

$$\operatorname{dist}(\Delta,\theta) < \alpha(\varepsilon) \ \Rightarrow\ \operatorname{diam}(\Delta) < \frac{\alpha(\varepsilon)}{2}.$$

To every vertex p of Λ we assign a point $r(p) \in \theta$ such that

$$d(p, r(p)) = \operatorname{dist}(p,\theta).$$

Let $\Delta \in \Lambda$ be such that $\operatorname{dist}(\Delta,\theta) < \alpha(\varepsilon)$ for an $\varepsilon \in (0,1]$. Then

$$\operatorname{dist}(p,\theta) < \frac{3}{2}\alpha(\varepsilon)$$

for every vertex $p \in \Delta$. We have

$$d(r(p_0), r(p_1)) < \frac{\alpha(\varepsilon)}{2} + 2\cdot\frac{3}{2}\alpha(\varepsilon) < 4\alpha(\varepsilon)$$

for every vertices $p_0, p_1 \in \Delta$. By the eLC^{n-1}-condition, we extend r on all edges of the decomposition Λ. We have

$$r(\Delta^1(p_0,p_1)) \subset K(r(p_0), \lambda(4\alpha(\varepsilon))) = K(r(p_0), \eta(\alpha(\varepsilon))),$$
$$\operatorname{diam}(r(\Delta^1(p_0,p_1))) < 2\eta(\alpha(\varepsilon))),$$
$$r(\partial\Delta^2(p_0,p_1,p_2)) \subset K(r(p_0), 4\eta(\alpha(\varepsilon))).$$

We now extend r on all triangles of Λ. We have

$$r(\Delta^2(p_0,p_1,p_2)) \subset K(r(p_0), \lambda(4\eta(\alpha(\varepsilon)))) = K(r(p_0), \eta^{(2)}(\alpha(\varepsilon))).$$

Assuming that

$$r(\Delta^i(p_0,\dots,p_i)) \subset K(r(p_0), \eta^{(i)}(\alpha(\varepsilon))),$$

we have

$$\operatorname{diam}(r(\Delta^i(p_0,\dots,p_i))) < 2\eta^{(i)}(\alpha(\varepsilon)),$$
$$r(\partial\Delta^{i+1}(p_0,\dots,p_{i+1})) \subset K(r(p_0), 4\eta^{(i)}(\alpha(\varepsilon))).$$

We extend r on all $(i+1)$-faces of the decomposition Λ. We have

$$r(\Delta^{i+1}(p_0,\dots,p_{i+1})) \subset K(r(p_0), \lambda(4\eta^{(i)}(\alpha(\varepsilon)))) = K(r(p_0), \eta^{(i+1)}(\alpha(\varepsilon))).$$

By induction,

$$r(\Delta) = r(\Delta^n) \subset K(r(p_0), \eta^{(n)}(\alpha(\varepsilon))) = K(r(p_0), \mu(\alpha(\varepsilon))).$$

We define the retraction $r_\theta\colon O_{\alpha(1)}(\theta) \to \theta$ by

$$r_\theta(x) = \begin{cases} r(x) & \text{for } x \in \Delta \in \Lambda, \\ x & \text{for } x \in \theta. \end{cases}$$

Take $x \in O_{\alpha(\varepsilon)}(\theta) \setminus \theta$. Choose $\Delta \in \Lambda$ such that $x \in \Delta$. Of course, $\operatorname{dist}(\Delta, \theta) < \alpha(\varepsilon)$. Fix a vertex $p_0 \in \Delta$. Then $\|r_\theta(x) - x\| = \|r(x) - x\| \le \|r(x) - r(p_0)\| + \|r(p_0) - p_0\| + \|p_0 - x\| < \mu(\alpha(\varepsilon)) + 3\alpha(\varepsilon)/2 + \alpha(\varepsilon)/2 < \varepsilon/2 + 2\cdot\varepsilon/4 = \varepsilon$, which proves the lemma. □

9.2. More about characteristic classes

For completeness of this thesis we give Thom's proof of the naturalness of the Stiefel–Whitney classes. Then we prove that elements $w_j^{\triangle}$ are natural. Finally, we cite the Milgram Theorem on the algebra $H^\star(BG_{k+1}, Z_2)$ of the classifying space BG_{k+1}.

Lemma 9.2 (Thom)**.** *The Stiefel–Whitney classes are natural.*

Proof. Let $p_i\colon \Gamma_i \to B_i$ be a Hurewicz fibration with fibre $\simeq S^k$ for $i = 0, 1$. Let $f\colon B_0 \to B_1$ be covered by $F\colon \Gamma_0 \to \Gamma_1$, i.e.

$$p_1 \circ F = f \circ p_0.$$

Moreover, we assume that the restriction $F|\colon p_0^{-1}(b_0) \to p_1^{-1}(f(b_0))$ is a homotopy equivalence. Set $b_1 = f(b_0)$. Let

$$Z_i = (I \times \Gamma_i \cup B_i)/\sim \quad \text{with } (1, x) \sim p_i(x).$$

We have the mappings

- $\overline{p}_i\colon Z_i \to B_i$, $\overline{p}_i[t, \gamma_i] = p_i(\gamma_i)$;
- $\overline{F}\colon Z_0 \to Z_1$, $\overline{F}[t, \gamma_0] = [t, F(\gamma_0)]$.

Of course,

$$\overline{p}_1 \circ \overline{F} = f \circ \overline{p}_0.$$

Let $t_1 \in H^{k+1}(Z_1, \Gamma_1; Z_2)$ be the Thom class of the fibration p_1. Then $\tau_1^\star(t_1)$ generates $H^{k+1}(D_1, S_1; Z_2)$. Here τ_i denotes the inclusion

$$(D_i, S_i) = (\overline{p}_i^{-1}(b_i), p_i^{-1}(b_i)) \subset (Z_i, \Gamma_i).$$

The following diagram

$$\begin{array}{ccc} H^{k+1}(Z_1, \Gamma_1) & \xrightarrow{\overline{F}^\star} & H^{k+1}(Z_0, \Gamma_0) \\ \downarrow{\scriptstyle \tau_1^\star} & & \downarrow{\scriptstyle \tau_0^\star} \\ H^{k+1}(D_1, S_1) & \xrightarrow[(\overline{F}|)^\star]{} & H^{k+1}(D_0, S_0) \end{array}$$

with Z_2-coefficients commutes. Since $(\overline{F}|)^\star$ is an isomorphism, the element

$$(\overline{F}|)^\star \circ \tau_1^\star(t_1) = \tau_0^\star(\overline{F}^\star(t_1))$$

generates $H^{k+1}(D_0, S_0)$. Thus $t_0 = \overline{F}^\star(t_1)$ is the Thom class of the fibration p_0. Let

$$\Phi_i\colon H^\star(B_i; Z_2) \to H^{\star+k+1}(Z_i, \Gamma_i; Z_2),$$
$$\Phi_i(x) = \overline{p}_i^\star x \cup t_i$$

be the Thom isomorphism for p_i. We have

$$\overline{F}^\star \circ \Phi_1 = \Phi_0 \circ f^\star.$$

Indeed,

$$\overline{F}^\star \circ \Phi_1(x) = (\overline{p}_1 \circ \overline{F})^\star(x) \cup \overline{F}^\star(t_1) = (f \circ \overline{p}_0)^\star(x) \cup t_0 = \overline{p}_0^\star(f^\star x) \cup t_0 = \Phi_0(f^\star x).$$

We now recall Thom's definition of the Stiefel–Whitney classes:

$$w_{j,i} = \Phi_i^{-1} Sq^j \Phi_i(1) \in H^j(B_i; Z_2).$$

Thus

$$\begin{aligned} f^\star w_{j,1} &= f^\star \circ \Phi_1^{-1} Sq^j \Phi_1(1) = \Phi_0^{-1} \circ \overline{F}^\star Sq^j \Phi_1(1) \\ &= \Phi_0^{-1} Sq^j \overline{F}^\star \circ \Phi_1(1) = \Phi_0^{-1} Sq^j \Phi_0(f^\star(1)) = \Phi_0^{-1} Sq^j \Phi_0(1) = w_{j,0}, \end{aligned}$$

which proves the naturalness of the j-th Stiefel–Whitney class. □

Theorem 9.3. *Elements $w_j^\triangle$ are some characteristic classes for the locally trivial bundles with the fibre S^k.*

We stress the fact that we did not define $w_j^\triangle$ for spherical (= Hurewicz) fibrations other than locally trivial bundles. The classifying space for the locally trivial bundles with fibre S^k exists and it is denoted by $\mathrm{BTOP}(S^k)$. In this way, the elements $w_j^\triangle \in H^j(B; Z_2)$ are called the characteristic classes of the bundle $p\colon \Gamma \to B$, if only these are defined for all such bundles and are natural.

This naturalness property differs from the one described for the spherical fibrations: If $p_i\colon \Gamma_i \to B_i$ are the locally trivial bundles fith fibre S^k for $i = 0, 1$ and $f\colon B_0 \to B_1$ is covered by the map $F\colon \Gamma_0 \to \Gamma_1$ such that

$$F|\colon p_0^{-1}(b) \to p_1^{-1}(f(b))$$

is a homeomorphism for every $b \in B_0$, then

$$f^\star(w_{j,1}^\triangle) = w_{j,0}^\triangle$$

Proof of Theorem 9.3. Take p_0, p_1, f, F, as above. Note, that the mapping

$$F^\triangle\colon \Gamma_0^\triangle \to \Gamma_1^\triangle, \qquad F^\triangle = (F \times F)|\Gamma_0^\triangle,$$

is well-defined. The following diagram

$$\begin{array}{ccc} H^j\Gamma_1^\triangle & \xrightarrow{(F^\triangle)^\star} & H^j\Gamma_0^\triangle \\ \uparrow & & \uparrow \\ H^j\Gamma_1^\triangle/Z_2 & \xrightarrow{G^\star} & H^j\Gamma_0^\triangle/Z_2 \\ {\scriptstyle q^\star}\uparrow & & \uparrow{\scriptstyle q^\star} \\ H^jB_1 & \xrightarrow[f^\star]{} & H^jB_0 \end{array}$$

with G induced by $F^\triangle$ commutes. By the naturalness of the Stiefel–Whitney classes,

$$c_0^\triangle = G^\star c_1^\triangle.$$

Let us recall that

$$(c_i^{\triangle})^{k+1} = \sum_{j=1}^{k+1} q^\star w_{j,i}^{\triangle} \cup (c_i^{\triangle})^{k+1-j}.$$

Then

$$\begin{aligned}(c_0^{\triangle})^{k+1} = G^\star(c_1^{\triangle})^{k+1} &= \sum_{j=1}^{k+1} G^\star q^\star w_{j,1}^{\triangle} \cup (G^\star c_1^{\triangle})^{k+1-j} \\ &= \sum_{j=1}^{k+1} q^\star f^\star w_{j,1}^{\triangle} \cup (c_0^{\triangle})^{k+1-j}.\end{aligned}$$

We conclude that $f^\star w_{j,1}^{\triangle} = w_{j,0}^{\triangle}$, which proves the theorem. □

Let G_{k+1} denote the set of all homotopy equivalences $S^k \to S^k$.

Theorem 9.4 (Milgram, [53]).

(a) $H^\star(BG_{k+1}, Z_2) = P(w_1, \dots, w_{k+1}) \otimes E(\dots, e_I, \dots)$, *where* $P(\dots)$ *is a polynomial algebra,* $E(\dots)$ *— an exterior algebra,* w_j *— the* j*-th Stiefel–Whitney class;*

(b) $I = (i_1, \dots, i_m)$ *runs over all sequences of integers* $0 \le i_1 \le \dots \le i_m \le k-1$ *with* $m \ge 2$*. Additionally,* $i_1 = 0$ *implies* $m = 2$ *and* $i_2 > 0$*;*

(c) $\dim(w_j) = 1$*;*

(d) $\dim(e_I) = 1 + i_1 + 2i_2 + 4i_3 + \dots + 2^{m-1} i_m$.

Milgram's Theorem is loosely-linked with the subject of this thesis but it is interesting because of some open problems in Chapter 8. Author does not know the corresponding result for $\mathrm{BTOP}(S^k)$, (the reference is [6], but the explicit formulae for $H^\star(\mathrm{BTOP}(S^k), Z_2)$ are not written there).

9.3. Addendum to Theorems 7.4 and 7.5

Let $f: B^n \to 2^{B^n}$ be an $1 - S^{n-2}$-mapping and $U = \{x : f(x) \cong S^{n-2}\}$. In the proof of Theorem 7.4 we deal with the set K such that $M = \partial K$ well approximates $\mathrm{Fr}\, U$ in U (see also Lemma 7.3). Next, especially in the proof of Theorem 7.5, we study the bundle $p: \Gamma(f|M_i) \to M_i$ with M_i — a component of M. This is the point, where the following questions come into being:

(a) What about the bundle $\pi: \Gamma(f|K) \to K$?

(b) Does π bear an information which could make the proof of the equality $w_{n-1}(p) = 0$ easier?

In general the answer to (b) is negative. It suffices to consider the example of $K = M_0 \times [0, 1]$. In this case the bundles π and p are in some sense "the same" and are expected to make the same difficulties.

The (very special) assumption that K and M are connected makes the situation quite different. With these assumptions we give a short proof of the equality $w_{n-1}(p) = 0$.

Let us consider the following commutative diagram

$$
\begin{array}{ccc}
H^{n-1}(K; Z_2) & \longrightarrow & H_1(K, M; Z_2) \\
{\scriptstyle i^\star}\big\downarrow & & \big\downarrow{\scriptstyle \partial_\star} \\
H^{n-1}(M; Z_2) & \longrightarrow & H_0(M; Z_2) \\
& & \big\downarrow{\scriptstyle i_\star} \\
& & H_0(K; Z_2)
\end{array}
$$

with i — the inclusion and the horizontal arrows representing the Poincaré dualities. Since $i_\star$ is an isomorphism, we have $\partial_\star = 0$, $i^\star = 0$ and

$$w_{n-1}(p) = i^\star(w_{n-1}(\pi)) = 0.$$

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