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Theory of Hyperconvex Metric Spaces.
A Beginner's Guide

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1 Introduction

The topic of hyperconvex metric spaces is a very interesting one, and a lucky one (in a sense), too.

One of the reasons which make it interesting is that hyperconvexity seems to be useful for more than one thing. The very first research on hyperconvex spaces was Aronszajn's, who sought to prove an analogue of the Hahn–Banach extension theorem in metric setting. Although the theory was first developed in his doctoral dissertation in the 1930s, it was not until 1956 when his results were actually published in a joint paper with Panitchpakdi. Like Athena of the head of Zeus, the theory popped out of their paper already well-developed: with a handful of general theorems, a few notions connected to hyperconvexity, and a set of open problems for others to work on. Surprisingly, it seems that it was not until the sixties that other mathematicians started catching up.

Although Aronszajn and Panitchpakdi explained the etymology behind the term "hyperconvex" (and indeed, hyperconvex sets bear some kind of similarity to convex sets), they seemingly did not try to investigate the analogy beyond the connection with retracts. In 1964, Isbell published a paper about *injective* metric spaces (which is basically another term for hyperconvexity, as Aronszajn and Panitchpakdi proved; hyperconvexity is defined in terms of ball intersections and injectivity in terms of extending mappings – but the two properties are equivalent), which contained (among others) a brilliant proof of existence of the so-called "injective envelope" (which is called a "hyperconvex hull" in this book).

Fast forward to 1988, and Baillon publishes yet another paper on hyperconvex metric spaces, this time concerning a very interesting and useful intersection property and applying it to a remarkable fixed point theorem: any nonexpansive mapping acting on a bounded hyperconvex space turns out to have a fixed point. Not only that, but also its fixed point set must be hyperconvex. This marks the beginning of popularity of hyperconvex metric spaces in metric fixed point theory – numerous papers exploring fixed points for mappings of hyperconvex spaces have been appearing since then. It is especially interesting (and, frankly speaking, quite mysterious to the author) that many fixed point theorems for convex sets have hyperconvex analogues, which are often just identical aside from the change of the word "convex" to "hyperconvex" – only that the techniques used to prove them are completely different!

These milestones for development of the theory of hyperconvex spaces are, in fact, more than enough to justify the research on them – but it is not the whole story. In 1984, Dress rediscovered the notion of "injective envelope" or "hyperconvex hull", this time calling it a "tight span" of a metric space, and applied it to phylogenetic analysis. One of the problems he examined was: given a few points with some distances between them, find a tree spanning all these points. (One of the applications is the reconstruction of phylogenetic trees given a finite number of species with well-defined "distances" between them; another one was examining the structure of networks.) And as if applications to fixed point theory (inside pure mathematics) and biology (outside) were not enough, there appeared connections with graph theory, combinatorics, computational geometry, algebra, game theory... Also, some other notions, like $\mathbb{R}$-trees (briefly mentioned in chapter on hyperconvex geometry) or $\mathrm{CAT}(0)$ spaces (which are outside the scope of this book) seem to be relatives of hyperconvex spaces.

This book does not try to be ambitious enough to cover all these fascinating areas. Instead, we focus first on the origin of the theory of hyperconvex spaces, *i.e.*, the work of Aronszajn and Panitchpakdi on extending mappings (Chapter 3). Then, in Chapter 4, we investigate some geometric properties of hyperconvex spaces – Baillon's intersection theorem and Isbell's hyperconvex hull being the main protagonists, soon joined by Kirk's theorem on equivalence between hyperconvex spaces with unique metric segments and complete real trees. Chapter 5 is devoted to the author's favorite part of the theory: fixed points. All the theorems here look similar to their "convex" counterparts, but – as we will see – for the most part the proofs do not. Finally, we finish the book with a short chapter on multivalued functions on hyperconvex spaces (Chapter 6); the theory is much more developed and this chapter is in fact only an appetizer.

On a bit more personal note: I am ignorant in a lot of areas, but in the few in which I am less ignorant, a substantial portion of what I know I taught myself. Remembering the frustrations of a self-taught student, I tried not to omit parts of proofs often deemed "trivial", "obvious" *etc.* I implore more advanced readers to forgive me the verbosity, and hope to at least partially repay the debt I incurred – by studying from patient teachers – to the beginners in the ways of mathematics. Except for the case of a few theorems, especially the ones which are not directly related to hyperconvexity, all proofs are presented in full detail.

The text of the book is typeset in *TEXGyrePagella*, a font developed by Bogusław Jackowski and Janusz M. Nowacki and based on Herman Zapf's *Palatino*. The mathematical formulae are typeset in the charming *AMS Euler* font, also designed of Zapf with the assistance of Donald E. Knuth.

I would like to express my profound gratitude to the One who created the world in which beautiful mathematical theories worth discovering (or is it inventing?) exist. I also thank my thesis advisor, Dr. Dariusz Bugajewski, for introducing me to the fascinating and mysterious land of hyperconvex spaces (in fact, substantial parts of this very book are based on my Master's and Doctoral theses, written with his invaluable assistance). A special thank-you note must go to Bethany Soule and Daniel Reeves, without whose productivity tool this book would never be finished on time. Maria J. Szelatyńska M.S., who was responsible for proof-reading and typesetting the book, found

numerous typographic errors, and her effort made the book look much better. Last but not least, I wholeheartedly thank my wife and daughter who patiently bore a husband and a father disappearing almost every day to the realm of his papers, laptop, headphones and hyperconvexity for the last few months, sometimes returning to them on the verge of insanity.

I apologize in advance for all errors, mistakes and omissions obscuring the striking beauty of the theory of hyperconvex metric spaces; needless to say, I take full responsibility for them. I would be delighted to inspire the reader to investigate this theory further (and also to read the original articles!); there is a lot to learn and do there. The theory had a great luck (so to speak) to have been developed by many great mathematicians in many wonderful papers. In such a short and humble book only a small selection of what is known can be presented – but I sincerely hope that the reader will stand in awe studying even the most fundamental results in this beautiful area of mathematics.

Poznań, 2015. AMDG.

Preliminaries 2

In this (very short) chapter we gather some conventions and notation we use in the sequel. Definitions will be introduced at the point when they are needed.

The symbol “:=” means “is equal to by definition”. It will be used not only in definitions themselves, but also in reasonings, to emphasize that the given equality follows directly from a definition of the symbol on its left. On a very rare occasion, we will employ its variant, “=:”, with the meanings of the left- and right-hand-sides swapped. The symbol “$\to$” (used in contexts other than “$f\colon X \to Y$”) means “converges to” (usually, “as n tends to ∞”).

The word *order* (without any adjective) will mean a partial order. The word “space” without any adjective will mean a metric space; the word “subspace” by itself will mean a metric, linear or topological subspace, depending on the nature of the space whose subset is being considered.

The elements of a set will be written in curly braces, as usual. If, however, the set is (at least partially) ordered, and this order is important (for instance, when talking about ordered pairs or sequences), we will use angle brackets. Even though it is formally incorrect, we will use notation like “$\langle x_n \rangle_{n=1}^{\infty} \subseteq X$” to mean that $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence of elements of the set X. Another formally incorrect custom we will sometimes use will be identifying singletons with their unique elements. The letters Λ and I, if not explicitly defined otherwise, will denote some index sets (possibly uncountable). For the sake of formality, we will always assume that index sets denoted by different symbols are disjoint.

If X is a subset of Y, we will sometimes express this by saying that Y is a superset of X.

If X is a metric (or topological) space and A its nonempty subset, the expressions "A is a (sub)space with property P" and "A is a (sub)set with property P" will be synonymous; for instance, a "complete subspace", "complete subset" and "complete set" are all the same thing.

Formally, by a *metric space* we will understand a pair $\langle X, d_X \rangle$ consisting of a nonempty set X and a metric d_X on X. Oftentimes we will write just X for simplicity, and omit the index in the symbol for the metric (using a single letter d), if this is unambiguous. Similar conventions will hold for other well-known notions like topological spaces etc., and also some lesser-known ones, like hyperconvex hull.

The symbol $\|\cdot\|$ will mean any norm; we will sometimes need to consider some special kinds of norm on some function spaces, and we will denote them (as usually) by $\|\cdot\|_1$ (the *Manhattan norm*) or $\|\cdot\|_\infty$ (the (*essential*) *supremum norm*).

All balls (unless explicitly stated otherwise) in all metric spaces will be closed; we will denote a ball with center x and radius $r \geqslant 0$ in a metric space $\langle X, d \rangle$ by $\bar{B}_X(x, r)$ or $\bar{B}_d(x, r)$, or $\bar{B}_{\langle X, d \rangle}(x, r)$; most often (unless it is ambiguous) we will write just $\bar{B}(x, r)$.

The *distance* of a point b to some nonempty subset A of a metric space X, i.e. the number $\inf\{d(b, a) \mid a \in A\}$, will be denoted by $\operatorname{dist}(b, A)$.

A mapping between two metric spaces X and Y will be called an *isometric embedding*, if $d(f(x_1), f(x_2)) = d(x_1, x_2)$ for any $x_1, x_2 \in X$. A surjective isometric embedding will be called an *isometry*. If $X \subseteq Y$, then the isometric embedding $X \ni x \mapsto x \in Y$ will be called the *identity embedding*.

If $F\colon X \to Y_F$ and $G\colon X \to Y_G$ are some mappings, we will write $F = G$, if $F(x) = G(x)$ for all $x \in X$ (regardless whether $Y_F = Y_G$ or not). We will also sometimes write $F \leqslant G$ if $F(x) \leqslant G(x)$ for all x's in the (common) domain of F and G, and even $F \leqslant a$ (where a

is a constant) when $F(x) \leqslant a$ for all x's in the domain of F. For any mapping $F\colon X \to X$, we will define $F^1 := F$ and $F^n := F \circ F^{n-1}$ for $n = 2, 3, \ldots$. We will call a subset $A \subseteq X$ F-invariant, if $F(A) \subseteq A$. Finally, the set of all fixed points of a mapping $F\colon X \to Y$, where $X \subseteq Y$, will be denoted by $\operatorname{Fix} F$, that is, $\operatorname{Fix} F := \{x \in X \mid x = F(x)\}$ (note that this convention *will* be violated in Chapter 6, but for a good reason – see Definitions 6.1.2).

The boundary and closure of a set A will be denoted by ∂A and $\overline{A}$ respectively. Any family of sets, when considered as a partially ordered set, will be ordered by inclusion; in particular, a *chain of sets* will mean a family of sets totally ordered by inclusion. (An alert reader has probably already guessed in what direction this goes: we will apply the Kuratowski–Zorn Lemma numerous times. When doing so, to avoid using too many symbols, we will usually reuse the same symbol for the bound of a chain and the extremal element in the ordered set considered; this should never lead to a confusion.)

All linear spaces considered will be real. If E is a linear space, $A, B \subseteq E$ and $\alpha \in \mathbb{R}$, we will define $\alpha A := \{\alpha x \mid x \in A\}$ and $A + B := \{x + y \mid x \in A, y \in B\}$. Further, if x, y are points of a linear space E, then an *affine segment* joining x with y is defined as the set $[x, y] := \{\alpha x + \beta y \mid \alpha, \beta \geqslant 0, \alpha + \beta = 1\}$. The convex hull of a set A will be denoted as usual by $\operatorname{conv} A$.

Of course, in case of $a, b \in \mathbb{R}$, the symbol $[a, b]$ retains its usual meaning of a closed interval. Also, when we speak of any interval of the form $[a, b]$, (a, b), $[a, b)$ or $(a, b]$, we will implicitly assume that $a, b \in \mathbb{R}$ and $a < b$ (or $a \leqslant b$ in case of a closed interval).

Extending mappings 3

3.1 A Hahn–Banach-type theorem

Let us start with two definitions and a classical theorem.

3.1.1. Definition. A real functional $p\colon X \to \mathbb{R}$ acting on a vector space X is called *sublinear* if $p(x+y) \leqslant p(x) + p(y)$ for $x, y \in X$ and $p(\lambda x) = \lambda p(x)$ for $\lambda \in [0, +\infty)$ and $x \in X$. △

3.1.2. Definition. Let $T\colon A \to Y$ map a nonempty subset A of the set X into the set Y and let $\mathcal{T}$ be a family of mappings from subsets of X to Y, containing T and such that if some $\hat{T}\colon \hat{A} \to Y$ belongs to $\mathcal{T}$, then $A \subseteq \hat{A}$ and $\hat{T}|_A = T$. Then, we can introduce a partial order in $\mathcal{T}$ by defining $\hat{T}_1 \preccurlyeq \hat{T}_2$ for mappings $\hat{T}_i\colon \hat{A}_i \to Y$ $(i = 1, 2)$ belonging to $\mathcal{T}$ if $\hat{A}_1 \subseteq \hat{A}_2$ and $\hat{T}_2|_{\hat{A}_1} = \hat{T}_1$. By a *maximal mapping* (in the family $\mathcal{T}$) we will mean any maximal element in such defined partially ordered set. △

3.1.3. Theorem (Hahn, Banach)**.** *Let Y be a proper linear subspace of a vector space X. Let $p\colon X \to \mathbb{R}$ be a sublinear function and $f\colon Y \to \mathbb{R}$ a linear functional satisfying the inequality $f(y) \leqslant p(y)$ for all $y \in Y$. Then there exists a linear functional $\tilde{f}\colon X \to \mathbb{R}$ such that $\tilde{f}|_Y = f$ and $\tilde{f}(x) \leqslant p(x)$ for all $x \in X$.* △

The proof of the above theorem (or one of its numerous variants) may be found in any book on functional analysis. Here, we will sketch a proof using certain simple geometric property of the real line.

Outline of the proof. It is not difficult to prove (for instance, by a standard argument using the Kuratowski–Zorn Lemma) that there exists a maximal functional $\tilde{f}\colon \widetilde{Y} \to \mathbb{R}$ satisfying the thesis. Let us assume that $\widetilde{Y} \subsetneq X$. Let $z \in X \setminus \widetilde{Y}$ and $\widetilde{Y}_1 := \{y + \lambda z \mid y \in \widetilde{Y}, \lambda \in \mathbb{R}\}$. We will extend f to a linear functional $f_1\colon \widetilde{Y}_1 \to \mathbb{R}$ such that $f_1(y_1) \leqslant p(y_1)$ for all $y_1 \in \widetilde{Y}_1$, thus arriving at a contradiction.

It is clear that each extension of f to $\widetilde{Y}_1$ is of the form $y + \lambda z \mapsto f(y) + \lambda\zeta$, where $y \in \widetilde{Y}$, for some $\zeta \in \mathbb{R}$. We will prove the existence of a $\zeta \in \mathbb{R}$ such that $f(y) + \lambda\zeta \leqslant p(y + \lambda z)$ for any $y \in \widetilde{Y}$ and $\lambda \in \mathbb{R}$, or equivalently, $f(y) - p(y - z) \leqslant \zeta \leqslant p(y + z) - f(y)$ for $y \in \widetilde{Y}$. Denote $I_y := [f(y) - p(y - z), p(y + z) - f(y)]$ for $y \in \widetilde{Y}$; it is easily shown that $I_{y'} \cap I_{y''} \neq \emptyset$ for any $y', y'' \in \widetilde{Y}$. The existence of a ζ in question follows from the fact that any family of pairwise intersecting closed intervals in $\mathbb{R}$ has a nonempty intersection. □

Before proceeding further, let us point out a special case of the above theorem, often called also the Hahn–Banach theorem and sufficient in many applications.

3.1.4. Corollary. *Let* Y *be a linear subspace of a normed space* X *and let* $f\colon Y \to \mathbb{R}$ *be a continuous linear functional. It is then possible to extend* f *to a continuous linear functional* $\tilde{f}\colon X \to \mathbb{R}$ *with preservation of the norm, that is, in such a way that* $\|\tilde{f}\| = \|f\|$. △

Proof. It is enough to put $p(x) := \|f\| \cdot \|x\|$ in Theorem 3.1.3. □

The crucial part of the proof of the Hahn–Banach theorem was the intersection property of the real line. It is no surprise, then, that if we want to generalize it to functions with codomains other than $\mathbb{R}$, we should expect that the codomain should satisfy some kind of similar behavior. Since our goal is now to formulate a Hahn–Banach type theorem for mappings into metric spaces, we will try to define an analogous property of them. The simplest structure in an (arbitrary) metric space analogous to a closed interval in $\mathbb{R}$ is a closed ball. This suggests considering the following notion.

3.1.5. Definition. We say that a metric space posseses the *property* (P), if any family of pairwise intersecting closed balls has a nonempty intersection. △

It turns out that if what we want is a Hahn–Banach type extension theorem for metric spaces, then, well... these aren't the spaces we're looking for. The reason for this is the following. A special case of extending mappings into some set is finding retractions onto that set (since this is equivalent to extending the identity map). In order for any space to be (some kind of) absolute retract (that is, roughly speaking, a retract of anything including it – for suitable values of "anything"), it needs to be at least connected. The spaces with property (P), however, need not be connected – to see this, consider any two-point metric space. What we need – in addition to property (P) – is a property called *total convexity*.

3.1.6. Definition. A metric space $\langle X, d \rangle$ is called *totally convex* if for any two distinct points $x, y \in X$ and any decomposition $d(x, y) = \alpha + \beta$ of their distance into a sum of two numbers $\alpha, \beta > 0$, there exists a point $z \in X$ such that $d(x, z) = \alpha$ and $d(z, y) = \beta$. △

3.1.7. Remark. We can formulate the above definition in the language of ball intersections. It is easily seen that a metric space is totally convex if, and only if, any two balls $\bar{B}(x, r)$ and $\bar{B}(y, s)$ such that $d(x, y) = r + s$ have a nonempty intersection. Equivalently, we may replace the above equality with the condition $d(x, y) \leqslant r + s$. △

One may think of the above property as a way to "measure the convexity" of a metric space. A totally convex metric space is (rather unintuitively, taking into consideration the meaning of the word "total" outside mathematics) only a little bit convex. If a property similar to that described in Remark 3.1.7, but with *triples* of balls, is satisfied, the space is still not very convex, but more so than previously. Finally, if an analogous property holds for any collection of balls, however large, the space is as convex as can be – hence the following definition.

3.1.8. Definition. We call a metric space $\langle X, d\rangle$ *hyperconvex* if any family of closed balls $\{\bar{B}(x_i, r_i)\}_{i\in I}$ such that $d(x_i, x_j) \leqslant r_i + r_j$ for $i, j \in I$ has a nonempty intersection. △

A customary thing to do after introducing a new notion is to give some examples of objects satisfying its definition. We will wait for that, however, until the next chapter, since constructing many examples requires some knowledge of hyperconvex geometry. For now, let us suffice to say that the real line is hyperconvex (we will prove this in Example 4.1.1), so the notion is not empty, and that there are quite a few ways to combine hyperconvex spaces to obtain another hyperconvex spaces, so the theory is rich enough to be interesting. We will now continue examining the relation between hyperconvexity and extending mappings, especially that this was historically the main motivation for defining them in the first place.

Since most spaces we are going to deal with are totally convex, and it is easier to say "a family of pairwise intersecting balls" than "a family of balls with the property that any pair of them has centers lying within the distance equal to the sum of their radii", we will often make use of the following lemma.

3.1.9. Lemma. *A metric space is hyperconvex if, and only if, it has property* (P) *and is totally convex.* △

Proof. "Only if" being trivial, let us prove the (only slightly less trivial) "if" part. Assume that the space X has property (P) and is totally convex. Let $\{\bar{B}(x_i, r_i)\}_{i\in I}$ be any family of closed balls such that the inequality $d(x_i, x_j) \leqslant r_i + r_j$ holds for any $i, j \in I$. Total convexity of X means exactly that this family is pairwise intersecting; property (P) assures that it has a nonempty intersection. □

We are now almost ready to tackle the problem of stating and proving a Hahn–Banach type theorem for metric spaces. We begin with the definition of the notion of modulus of continuity, which will play a role analogous to the sublinear functional p in Theorem 3.1.3.

3.1.10. Definitions. A nondecreasing function $\omega\colon [0, +\infty) \to [0, +\infty]$ such that $\lim_{\delta\to 0^+} \omega(\delta) = 0$ will be called a *modulus of continuity* (or *m.o.c.* for short). A modulus of continuity ω is called *subadditive* if $\omega(\delta_1 + \delta_1) \leqslant \omega(\delta_1) + \omega(\delta_2)$ for any $\delta_1, \delta_2 > 0$. If $T\colon X \to Y$ is a mapping between metric spaces X and Y, we call ω a *modulus of continuity of the mapping* T, if it is a modulus of continuity and $d\big(T(x_1), T(x_2)\big) \leqslant \omega\big(d(x_1, x_2)\big)$ for any $x_1, x_2 \in X$. If the mapping T is uniformly continuous, we define the function

$$\omega_T(\delta) := \sup\big\{d\big(T(x_1), T(x_2)\big) \;\big|\; x_1, x_2 \in X, d(x_1, x_2) \leqslant \delta\big\},$$

called the *minimal modulus of continuity of the mapping* T. △

3.1.11. Remark. It is easy to see that a mapping between two metric spaces has a m.o.c. if, and only if, it is uniformly continuous. Moreover – as expected – the minimal m.o.c. of a uniformly continuous mapping is its m.o.c., and any its other m.o.c. is pointwise greater or equal to it. △

3.1.12. Remark. If $T\colon X \to Y$ is a uniformly continuous mapping of a totally convex space X into any metric space Y, then its minimal m.o.c. is subadditive. Indeed, let $\delta_1, \delta_2 > 0$ be arbitrary and $\varepsilon > 0$. There exist points $x_1, x_2 \in X$ such that $d_X(x_1, x_2) \leqslant \delta_1 + \delta_2$ and $d_Y\big(T(x_1), T(x_2)\big) > \omega_T(\delta_1 + \delta_2) - \varepsilon$. Since X is totally convex, there exists $z \in X$ such that $d_X(x_i, z) = \frac{\delta_i}{\delta_1+\delta_2} d_X(x_1, x_2) \leqslant \delta_i$ for $i = 1, 2$. Hence

$$\begin{aligned}
\omega_T(\delta_1 + \delta_2) - \varepsilon &< d_Y\big(T(x_1), T(x_2)\big) \\
&\leqslant d_Y\big(T(x_1), T(z)\big) + d_Y\big(T(z), T(x_2)\big) \\
&\leqslant \omega_T\big(d_X(x_1, z)\big) + \omega_T\big(d_X(z, x_2)\big) \\
&\leqslant \omega_T(\delta_1) + \omega_T(\delta_2);
\end{aligned}$$

since $\varepsilon > 0$ was arbitrary, we obtain $\omega_T(\delta_1 + \delta_2) \leqslant \omega_T(\delta_1) + \omega_T(\delta_2)$ for $\delta_1, \delta_2 > 0$ and the proof is complete. △

3.1.13. Example. The mapping $T\colon X \to Y$ between metric spaces has a m.o.c. of the form $\omega(\delta) = L\delta$ if, and only if, it satisfies the Lipschitz condition with constant L. In particular, if $T\colon X \to \mathbb{R}$ is a continuous linear functional on a normed space X, its minimal m.o.c. is given by the formula $\omega_T(\delta) = \|f\| \cdot \delta$. △

In the classical Hahn–Banach theorem, the crux of the proof lies in showing that we can extend the functional by one dimension. In the metric case, where we don't have dimensionality, we will extend the mapping by one *point* instead. Let us start with a simple application of the Kuratowski–Zorn Lemma.

3.1.14. Lemma. *Let* X, Y *be metric spaces,* $A \subseteq X$ *a nonempty subset of* X, $B \subseteq Y$ *a nonempty subset of* Y *and* $T\colon A \to Y$ *a mapping with a m.o.c.* ω. *Then, there exists a maximal extension* $\widetilde{T}\colon \widetilde{A} \to Y$ *of* T *having the same m.o.c. and such that* $\widetilde{T}(\widetilde{A} \setminus A) \subseteq B$. △

Proof. Consider the set

$$\Sigma := \{\hat{T}\colon \hat{A} \to Y | A \subseteq \hat{A} \subseteq X,\ \hat{T}|_A = T,\ \hat{T}(\hat{A} \setminus A) \subseteq B, \\ \omega \text{ is an m.o.c. of } \hat{T}\}.$$

Of course, $\Sigma \neq \emptyset$, since $T \in \Sigma$. Let us order Σ by the relation from Definition 3.1.2. Let $C := \{\hat{T}_i\}_{i \in I}$, where $\hat{T}_i\colon \hat{A}_i \to Y$, be a chain in Σ. Define $\widetilde{A} := \bigcup_{i \in I} \hat{A}_i$ and let $\widetilde{T}\colon \widetilde{A} \to Y$ be defined by the formula $\widetilde{T}(x) = \hat{T}_i(x)$ for any $i \in I$ such that $x \in \hat{A}_i$ (the fact that C is a chain guarantees that $\widetilde{T}$ is well-defined). It should be clear that $\widetilde{T}$ belongs to Σ – the only non-immediately-obvious part of the proof consists in showing the ω is a m.o.c. of $\widetilde{T}$. If $x, y \in \widetilde{A}$, then $x, y \in \hat{A}_i$ for some "large" i (here, "large" is understood with respect to the relation on the set I inherited in a natural way from the relation on Σ), so $d\big(\widetilde{T}(x), \widetilde{T}(y)\big) = d\big(\hat{T}_i(x), \hat{T}_i(y)\big) \leqslant \omega\big(d(x,y)\big)$. A standard application of the Kuratowski–Zorn Lemma ends the proof. □

In what follows now, we will in fact always set $B := Y$. The reason to include the set B in the above lemma will be apparent later, when we talk about fixed points of non-self mappings; for now, let us keep calm and read on.

We can now prove the main theorem of this section, which is the promised analogue of Theorem 3.1.3 for metric spaces.

3.1.15. Theorem. *Let* A *be a nonempty subset of a metric space* X *and let* $T\colon A \to H$ *be a mapping from* A *to a hyperconvex space* H, *possessing a sub-*

additive m.o.c. ω. *Then there exists a mapping* $\widetilde{T}\colon X \to H$ *with m.o.c.* ω *such that* $\widetilde{T}|_A = T$. △

Proof. From Lemma 3.1.14 we know that there exists a maximal extension $\widetilde{T}\colon \widetilde{A} \to H$ of the mapping T with m.o.c. ω. We will show that $\widetilde{A} = X$. If it were $X \backslash \widetilde{A} \neq \emptyset$, there would exist some $z \in X\backslash\widetilde{A}$; let $\widetilde{A}_1 := \widetilde{A} \cup \{z\}$. We will now construct a mapping $\widetilde{T}_1\colon \widetilde{A}_1 \to H$ with m.o.c. ω such that $\widetilde{T}_1|_{\widetilde{A}} = \widetilde{T}$. Let us consider the family $\{\bar{B}_x\}_{x\in\widetilde{A}}$ of closed balls $\bar{B}_x := \bar{B}\big(\widetilde{T}(x), \omega(d(x,z))\big)$ in H. For $x_1, x_2 \in \widetilde{A}$ we have $d\big(\widetilde{T}(x_1), \widetilde{T}(x_2)\big) \leqslant \omega(d(x_1,x_2)) \leqslant \omega\big(d(x_1,z)+d(z,x_2)\big) \leqslant \omega(d(x_1,z))+\omega(d(x_2,z))$. Since H is hyperconvex, we have $\bigcap_{x\in\widetilde{A}} \bar{B}_x \neq \emptyset$. Let $y \in \bigcap_{x\in\widetilde{A}} \bar{B}_x$, which means that $d\big(\widetilde{T}(x), y\big) \leqslant \omega(d(x,z))$ for $x \in \widetilde{A}$. Let us define $\widetilde{T}_1\colon \widetilde{A} \to H$ by the formulae $\widetilde{T}_1(x) := \widetilde{T}(x)$ for $x \in \widetilde{A}$ and $\widetilde{T}_1(z) := y$. As can be seen from the above inequality, ω is a m.o.c. of the mapping $\widetilde{T}_1$, and $\widetilde{T}_1|_{\widetilde{A}} = \widetilde{T}$. This, however, contradicts the maximality of $\widetilde{T}$. Hence $\widetilde{A} = X$ and the theorem is proved. □

Among the mappings having a subadditive m.o.c. there is one important class of so-called *nonexpansive* mappings. (Its importance will be already seen in the next section, and the notion will appear a few times both in the context of geometry of hyperconvex spaces and of fixed point theory.) We will now define it and formulate a version of Theorem 3.1.15 for such maps. (The reason for considering this particular case will be unveiled in a moment.)

3.1.16. Definition. A mapping $T\colon X \to Y$ between metric spaces X and Y is called *nonexpansive*, if the function $\omega(\delta) = \delta$ is its m.o.c. In other words, a mapping is nonexpansive if it satisfies the Lipschitz condition with constant 1. △

3.1.17. Corollary. $T\colon A \to H$ *be a nonexpansive mapping from a nonempty subset* A *of a metric space* X *in a hyperconvex space* H. *Then there exists a nonexpansive extension* $\widetilde{T}\colon X \to H$ *of the mapping* T. △

As we have seen, hyperconvexity is a sufficient condition for the extensibility of uniformly continuous mappings with preservation of a subadditive modulus of continuity. A natural question is: what is the *necessary* condition? It turns out that it is the same one. Even if we restrict ourselves to extending nonexpansive mappings, we can prove that any space admitting such extension for any nonexpansive mapping must be hyperconvex.

3.1.18. Theorem. *Let* Y *be a metric space such that for any triple consisting of a metric space* X*, nonempty subset* $A \subseteq X$ *and nonexpansive mapping* $T\colon A \to Y$ *there exists a nonexpansive extension* $\widetilde{T}\colon X \to Y$ *of* T*. Then, the space* Y *is hyperconvex.* △

Theorem 3.1.18 may be proved in several ways. Later, we will show a very short proof (see p. 62); now, we will first use a method of Aronszajn and Panitchpakdi, and then briefly sketch a way of considerably shortening that proof at the expense of rendering it nonconstructive. (However, as we started with a Hahn–Banach-type theorem, we may as well accept the Axiom of Choice in all subsequent proofs anyway.)
Proof I. We will start by showing the total convexity of Y. Let $\bar{B}_i := \bar{B}_Y(y_i, r_i)$ for $i = 1, 2$ be closed balls in Y such that $d(y_1, y_2) \leqslant r_1 + r_2$. We will prove that $\bar{B}_1 \cap \bar{B}_2 \neq \emptyset$. We may assume that $y_1 \neq y_2$. Let $Y_0 := \{y_1, y_2\}$ and $T\colon Y_0 \to Y$ be defined by the formula $T(y_i) := y_i$ for $i = 1, 2$. Choose $z \notin Y_0$ and put $Y_1 := Y_0 \cup \{z\}$. We can extend the metric d (inherited from Y) from Y_0 onto Y_1 by means of the formula $d(y_i, z) := \frac{r_i}{r_1+r_2} d(y_1, y_2)$ for $i = 1, 2$. By the assumption we can extend T to a nonexpansive mapping $\widetilde{T}\colon Y_1 \to Y$. This way, we get (for $i = 1, 2$) $d\big(\widetilde{T}(z), y_i\big) = d\big(\widetilde{T}(z), \widetilde{T}(y_i)\big) \leqslant d(z, y_i) = \frac{r_i}{r_1+r_2} d(y_1, y_2) \leqslant r_i$, which means that $\widetilde{T}(z) \in \bar{B}_1 \cap \bar{B}_2$.

Let us now consider a family of balls $\{\bar{B}_i\}_{i\in I}$ in Y such that $d(y_i, y_j) \leqslant r_i + r_j$ for $i, j \in I$, where $\bar{B}_i := \bar{B}_Y(y_i, r_i)$ for $i \in I$. We define $Y_0 := \{y_i\}_{i\in I}$ and again let $T\colon Y_0 \to Y$ be defined by the formula $T(y_i) := y_i$ for $i \in I$. Let $z \notin Y$ and let $Y_1 := Y_0 \cup \{z\}$. We may now define a function $g\colon Y_0 \to [0, +\infty)$ by the formula $g(y_i) := \inf\{r > 0 \mid \bar{B}_Y(y_j, r_j) \subseteq \bar{B}_Y(y_i, r) \text{ for some } j \in I\}$ for $i \in I$. We will now prove that $d(y_i, y_j) \leqslant g(y_i) + g(y_j)$ and that $g(y_i) \leqslant d(y_i, y_j) + g(y_j)$ for any $i, j \in I$.

In order to prove the former inequality, we let $\varepsilon > 0$ and choose $i', j' \in I$ such that $\bar{B}_Y(y_{i'}, r_{i'}) \subseteq \bar{B}_Y(y_i, g(y_i) + \varepsilon)$ and $\bar{B}_Y(y_{j'}, r_{j'}) \subseteq$

$\bar{B}_Y(y_j, g(y_j) + \varepsilon)$. Since $d(y_{i'}, y_{j'}) \leqslant r_{i'} + r_{j'}$, the total convexity of Y implies that $\bar{B}_Y(y_{i'}, r_{i'}) \cap \bar{B}_Y(y_{j'}, r_{j'}) \neq \emptyset$; hence also $\bar{B}_Y(y_i, g(y_i) + \varepsilon) \cap \bar{B}_Y(y_j, g(y_j) + \varepsilon) \neq \emptyset$ and $d(y_i, y_j) \leqslant g(y_i) + g(y_j) + 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, the former inequality is proved.

For the proof of the latter one, let us again choose any $\varepsilon > 0$ and an index $i' \in I$ such that $\bar{B}_Y(y_{i'}, r_{i'}) \subseteq \bar{B}_Y(y_j, g(y_j) + \varepsilon)$. From the triangle inequality we obtain the inclusion $\bar{B}_Y(y_j, g(y_j) + \varepsilon) \subseteq \bar{B}_Y(y_i, d(y_i, y_j) + g(y_j) + \varepsilon)$, which yields $g(y_i) \leqslant d(y_i, y_j) + g(y_j) + \varepsilon$. Again – by the arbitrary choice of $\varepsilon > 0$ – we get the inequality in question.

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Having the function g with the above properties, we can reason in the following way. If $y_{i_0} \in Y_0$ is a zero of g, then for any $i \in I$ we have $d(y_{i_0}, y_i) \leqslant g(y_i) \leqslant r_i$, and therefore $y_{i_0} \in \bigcap_{i \in I} \bar{B}_Y(y_i, r_i)$ and the proof is finished. Let us consider the case when g doesn't vanish anywhere. In that case, we take some $z \notin Y_0$ and let $Y_1 := Y_0 \cup \{z\}$. The above inequalities imply that we can extend the metric d (inferited from Y) from Y_0 onto Y_1 using the formula $d(y_i, z) := g(y_i)$. From the assumption we know that it is possible to extend the mapping $T\colon Y_0 \to Y$ given by the formula $T(y_i) := y_i$ for $i \in I$ to $\widetilde{T}\colon Y_1 \to Y$, preserving the nonexpansiveness. This way we obtain $d\big(f(z), y_i\big) = d\big(f(z), f(y_i)\big) \leqslant d(z, y_i) = g(y_i) \leqslant r_i$ for $i \in I$ – in other words, $T(z) \in \bigcap_{i \in I} \bar{B}_Y(y_i, r_i)$, which finishes the proof of hyperconvexity of Y. □

A sketch of proof II. Let $\{\bar{B}_Y(y_i, r_i)\}_{i \in I}$ be a family of balls in Y such that $d(y_i, y_j) \leqslant r_i + r_j$ for $i, j \in I$. Without loss of generality, we may assume that $y_i \neq y_j$ for $i \neq j$ and denote $Y_0 := \{y_i\}_{i \in I}$. Let $\mathcal{F}$ be the set of all functions $g\colon Y_0 \to [0, +\infty)$ such that $d(y_i, y_j) \leqslant g(y_i) + g(y_j)$ for $i, j \in I$. The assumption about the considered family of balls implies that the function $r\colon Y_0 \to [0, +\infty)$ defined by the formula $r(y_i) := r_i$ for $i \in I$ belongs to $\mathcal{F}$. The Kuratowski–Zorn Lemma yields the existence of a function g, pointwise minimal in $\mathcal{F}$ and (also pointwise) less than or equal to r. It can be shown that the minimality of g implies the inequality $g(y_i) \leqslant d(y_i, y_j) + g(y_j)$ for $i, j \in I$. (In fact, we will prove exactly this in part 1° of Lemma 4.6.12 and in Lemma 4.6.11.) The rest of the proof is identical to the former one. □

Notice that the former proof, while longer, avoids using the Axiom of Choice in the definition of the function g.

This way we obtained the aforementioned characterization of hyperconvex spaces.

3.1.19. Corollary. *Hyperconvexity of a metric space* Y *is equivalent to the following property: for any metric space* X, *nonempty subset* $A \subseteq X$ *and mapping* $T\colon A \to Y$ *with a subadditive m.o.c.* ω *there exists an extension* $\widetilde{T}\colon X \to Y$ *of* T *with the same m.o.c. This is also true when we restrict ourselves to nonexpansive mappings, i.e., we will only allow the m.o.c.* $\omega(\delta) = \delta$. △

Proof. It is enough to apply Theorem 3.1.15 (or Corollary 3.1.17) and Theorem 3.1.18. □

In Section 4.6 we will obtain a very similar characterization, this time expressed in the language of retracts.

Before we finish this section, let us stop for a moment and think a bit further about the analogy between the classical Hahn–Banach theorem and its metric counterpart, proved above. We have seen that in the metric case, the role of continuous linear functionals was taken by nonexpansive mappings into a hyperconvex space. This naturally leads to a question about the metric counterpart of the notion of a dual space. In other words: what can be said about the space of all nonexpansive mappings from some metric space to a hyperconvex one? It turns out that not only is this an interesting question, but it also has an interesting answer (we will have to make a few more assumptions, though, to be able to obtain it). We will not deal with this issue now, however; this must wait until Chapter 6, which contains tools necessary to provide the answer.

3.2 Hyperconvexity and retractions

This section is devoted to examine the connections between the notions of a hyperconvex metric space and a nonexpansive retract. We will start with the necessary definitions.

3.2.1. Definitions. Let A be a nonempty subset of the topological space X. A continuous mapping $R\colon X \to A$ is called a *retraction* (of the space X onto A), if $R(x) = x$ for any $x \in A$. If such a mapping exists, the set A is called a *retract* of the space X.

If X is a metric space and there exists a retraction $R\colon X \to A$, which is a nonexpansive mapping, we call it *nonexpansive retraction*. The set A is then called a *nonexpansive retract* of the metric space X. A metrizable topological space is called an *absolute retract*, if it is a retract of any metrizable topological space in which it is included as a closed subset. A metric space is called an *absolute nonexpansive retract* if it is a nonexpansive retract of any metric space in which it is included. △

(Some authors skip the word "closed" in the above definition of an absolute retract, thus obtaining a smaller class of spaces – see e.g. [2, p. 422].)

From the above definition it is not necessarily obvious whether an absolute nonexpansive retract is an absolute retract. The reason for this is that even if some space A is included (as a topological subspace) in the space X, so that their topologies agree, their *metrics* need not coincide. It turns out, however, that one can always (re)metrize X in such a way that such a coincidence holds. It follows from the following lemma, proved by Hausdorff, which we will quote without the proof.

3.2.2. Lemma. *Let A be a closed subspace of a metrizable topological space X and let d_A be a metric in A generating the topology inherited from X. The metric d_A can then be extended to a metric d_X (defined on X) such that d_X generates the original topology on X.* △

3.2.3. Corollary. *Each absolute nonexpansive retract is an absolute retract.* △

3.2.4. Corollary. *Each hyperconvex space is an absolute nonexpansive retract (and in particular, an absolute retract).* △

Proof. Let H be a hyperconvex space and let X be a metric space including H. According to Corollary 3.1.17, the identity mapping on H can be extended to a nonexpansive retraction $R\colon X \to H$. □

3.2.5. Theorem. *A nonexpansive retract of a hyperconvex space is hyperconvex.* △

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Proof. Let $R\colon H \to A$ be a nonexpansive retraction of some hyperconvex space H onto its nonempty subset A. Let $\{\bar{B}_A(x_i, r_i)\}_{i\in I}$ be a family of closed balls in A such that $d(x_i, x_j) \leqslant r_i + r_j$ for $i, j \in I$; we will show that this family has a nonempty intersection. From the hyperconvexity of H we infer that the family of closed balls $\{\bar{B}_H(x_i, r_i)\}_{i\in I}$ in H has a nonempty intersection. Let $y \in \bigcap_{i\in I} \bar{B}_H(x_i, r_i)$; then $R(y) \in A$ and for any $i \in I$ we have $d(R(y), x_i) = d\big(R(y), R(x_i)\big) \leqslant d(y, x_i) \leqslant r_i$, and hence $R(y) \in \bigcap_{i\in I} \bar{B}_A(x_i, r_i)$. □

Theorem 3.2.5 suggests the question whether any image of a hyperconvex space under a nonexpansive mapping is hyperconvex. The answer to this question, however, is negative, as the following example shows.

3.2.6. Example. Let us define the mapping $T\colon \mathbb{R} \to \mathbb{C}$ by the formula $T(x) = e^{ix}$. We will show that it is a nonexpansive mapping of a (hyperconvex) space $\mathbb{R}$ onto a set which is not hyperconvex. We have for $x, y \in \mathbb{R}$:

$$\begin{aligned}
|T(x) - T(y)| &= \left|e^{ix} - e^{iy}\right| = |\cos x + i\sin x - \cos y - i\sin y| \\
&= \left|-2\sin\tfrac{x+y}{2}\sin\tfrac{x-y}{2} + 2i\cos\tfrac{x+y}{2}\sin\tfrac{x-y}{2}\right| \\
&= 2\left|\sin\tfrac{x-y}{2}\left(i\cos\tfrac{x+y}{2} - \sin\tfrac{x+y}{2}\right)\right| \\
&= 2\left|\sin\tfrac{x-y}{2}\right| \cdot \left|i\left(\cos\tfrac{x+y}{2} + i\sin\tfrac{x+y}{2}\right)\right| \\
&= 2\left|\sin\tfrac{x-y}{2}\right| \cdot \left|ie^{i\frac{x+y}{2}}\right| = 2\left|\sin\tfrac{x-y}{2}\right| \leqslant |x - y|.
\end{aligned}$$

Let us notice that $T(\mathbb{R}) = C := \big\{z \in \mathbb{C} \;\big|\; |z| = 1\big\}$. We will now show that the set C is not hyperconvex. Let $z_k := i^k$ and $r_k := 1$ for $k = 0, 1, 2, 3$. Obviously, $z_k \in C$ and $|z_k - z_l| \leqslant r_k + r_l$ for $k, l = 0, 1, 2, 3$.

Moreover, $0 \in B := \bigcap_{k=0}^{3} \bar{B}_{\mathbb{C}}(z_k, r_k)$. If there existed some nonzero $z \in B$, we would have $\operatorname{Re} z \neq 0$ or $\operatorname{Im} z \neq 0$. For instance, let us consider the case $\operatorname{Re} z < 0$; then $|z - z_0| = |z - 1| \geqslant |\operatorname{Re}(z - 1)| = |\operatorname{Re} z - 1| = -\operatorname{Re} z + 1 > 1$, which means that $z \notin \bar{B}_{\mathbb{C}}(z_0, r_0)$. (In other cases the reasoning is analogous.) Hence $B = \{0\}$ and in consequence $\bigcap_{k=0}^{3} \bar{B}_C(z_k, r_k) = B \cap C = \emptyset$ and the set C is not hyperconvex. △

Notes and remarks

The proof of Theorem 3.1.3, explicitly using the intersection property of intervals of the real line, is quoted from [22]. Definitions 3.1.5 and 3.1.8 were first published in [2]. The majority of definitions and results from Section 3.1 are either classical, or appeared in [2] (either explicitly, like lemma 3.1.9, or implicitly, like lemma 3.1.14 for $B = Y$). Aronszajn and Panitchpakdi themselves attribute definition 3.1.6 to K. Menger, and the term "hyperconvex" to A. H. Kruse. Of course, Theorems 3.1.15 and 3.1.18 are among the main results of aforementioned paper. paper. Most results of Section 3.2 appear also in [2], with the exception of lemmas 3.2.2, which appeared in [27, s. 353, I.], 3.2.5, which can be found in [22], and 3.2.6, which is taken from the author's Master's Thesis [6].

Notice also that the paper [2] contains more results on hyperconvex spaces, which are outside the scope of this book, for instance, results on hyperconvex Banach spaces or extreme points of balls in such spaces.

Very few papers on extending mappings with values in metric spaces which have properties similar to hyperconvexity seem to exist. One exception is the paper [24], which deals with compact mappings and so-called $\aleph_0$-*hyperconvexity* (which is similar to hyperconvexity, although the relevant condition holds only for finite families of balls). The paper [2] develops some theory of $\mathfrak{m}$-hyperconvex spaces (for various cardinals $\mathfrak{m}$). In particular, it is proved there that $\aleph_0$-hyperconvexity implies hyperconvexity in case of separable spaces. Some more results on $\mathfrak{m}$-hyperconvex spaces can be found in [28].

Another way of generalizing the Hahn–Banach theorem is not dropping the linearity, but substituting an arbitrary Banach space for $\mathbb{R}$ (as the codomain). One may then ask, which Banach spaces can replace the scalar field. It turns out that the answer is more or less the same as in the metric case: the necessary and sufficient condition is for the Banach space in question be hyperconvex. (In functional analysis, hyperconvex Banach spaces – or equivalently, Banach spaces with property (P) – are called $\mathcal{P}_1$-spaces.)

Since each hyperconvex space is an absolute retract, it is natural to ask which absolute retracts can be metrized with a hyperconvex metric. This question, posed in the paper [2], remains unsolved.

Hyperconvex geometry 4

In the previous chapter, we analyzed the notion of hyperconvexity from the perspective of extending uniformly continuous mappings of metric spaces. It turns out that these spaces have some very interesting geometric properties, which will be examined in the present chapter. We will start with some very simple (though useful) facts about hyperconvex spaces. Next, we will learn a few things about a very important class of hyperconvex sets – the intersections of balls in hyperconvex spaces. This will culminate in the famous Baillon's intersection theorem. After its highly technical proof, we will relax a bit and explain how to combine "simpler" hyperconvex spaces to make some more complicated ones. Then, we will return to more sophisticated reasonings and discuss some connections between hyperconvexity and Banach spaces. In the next section, we will examine the notion of a hyperconvex hull, which is both interesting by itself and useful in some fixed point theorems. We will conclude this chapter with a somewhat lighter theme of geometric properties of complete $\mathbb{R}$-trees, which are a very special class of hyperconvex spaces.

4.1 Basic properties and examples

In this short section we will show two very simple facts. We will start with the hyperconvexity of the real line and its closed segments. Then, we will prove that hyperconvex spaces are complete.

4.1.1. Example. Any closed interval of the real line (including the whole $\mathbb{R}$) is hyperconvex. Indeed, since it is obviously totally convex, it is enough to show that any family of pairwise intersecting closed and bounded intervals has a nonempty intersection. Let $\{[a_\lambda, b_\lambda]\}_{\lambda\in\Lambda}$ be such a family. Take any $\lambda, \mu \in \Lambda$; since $[a_\lambda, b_\lambda] \cap [a_\mu, b_\mu] \neq \emptyset$, we have $a_\lambda \leqslant b_\mu$ and in particular the set $\{a_\lambda\}_{\lambda\in\Lambda}$ is bounded from above. Define $a := \sup_{\lambda\in\Lambda} a_\lambda$; for any $\lambda \in \Lambda$, we have $a_\lambda \leqslant a \leqslant b_\lambda$ and the proof is complete. △

4.1.2. Proposition. *Any hyperconvex space is complete.* △

Proof. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a Cauchy sequence in a hyperconvex space H. This means that for any $\varepsilon > 0$ there exists a k_ε such that $d(x_m, x_n) \leqslant \varepsilon$ provided that $m, n \geqslant k_\varepsilon$. Put $\bar{B}_\varepsilon := \bar{B}(x_{k_\varepsilon}, \varepsilon)$ for $\varepsilon > 0$; it is not difficult to see that for any $\varepsilon_1, \varepsilon_2 > 0$ we have $d(x_{k_{\varepsilon_1}}, x_{k_{\varepsilon_2}}) \leqslant \varepsilon_1 + \varepsilon_2$. Since H is hyperconvex, $A := \bigcap_{\varepsilon>0} \bar{B}_\varepsilon \neq \emptyset$. Obviously A is a singleton, because if $x, y \in A$, then for each $\varepsilon > 0$ we would have $x, y \in \bar{B}_\varepsilon$, so $d(x, y) \leqslant 2\varepsilon$. Denote $A = \{a\}$; we will prove that $\lim_{n\to\infty} x_n = a$. Choose $\varepsilon > 0$; then for $n \geqslant k_{\varepsilon/2}$ we have $d(x_{k_{\varepsilon/2}}, x_n) \leqslant \frac{\varepsilon}{2}$. As $a \in \bar{B}_{\varepsilon/2} := \bar{B}(x_{k_{\varepsilon/2}}, \frac{\varepsilon}{2})$, we also have $d(a, x_{k_{\varepsilon/2}}) \leqslant \frac{\varepsilon}{2}$, so $d(a, x_n) \leqslant \varepsilon$ for $n \geqslant k_{\varepsilon/2}$ and the proof is complete. □

4.2 Admissible sets

As we shall see throughout the book, a very important class of subsets of hyperconvex spaces is the class of ball intersections. The present section is devoted to study some of their properties.

4.2.1. Definition. Let X be a metric space. We call a nonempty subset $A \subseteq X$ *admissible* (*in the space* X), if it is an intersection of some family of closed balls in X. The class of all admissible subsets of the space X is denoted by $\mathcal{A}(X)$. △

4.2.2. Remark. Note that it follows directly from the definition that an intersection of admissible subsets is admissible (provided not empty). △

4.2.3. Proposition. *Every admissible subset of a hyperconvex space* H *is itself hyperconvex.* △

Proof. For the sake of formality, let us assume that the index sets I and J are disjoint. Let $A = \bigcap_{j\in J} \bar{B}_H(x_j, r_j) \in \mathcal{A}(H)$. Let $\{\bar{B}_A(x_i, r_i)\}_{i\in I} \subseteq A$ be a family of closed balls in A such that $d(x_i, x_j) \leqslant r_i + r_j$ for $i, j \in I$. Obviously A is nonempty and $x_i \in \bar{B}_H(x_j, r_j)$ for any $i \in I$ and $j \in J$, so we have $d(x_i, x_j) \leqslant r_i + r_j$ for $i, j \in I \cup J$; since H is hyperconvex, we obtain $\bigcap_{i\in I} \bar{B}_A(x_i, r_i) = A \cap \bigcap_{i\in I} \bar{B}_H(x_i, r_i) = \bigcap_{i\in J} \bar{B}_H(x_i, r_i) \cap \bigcap_{i\in I} \bar{B}_H(x_i, r_i) = \bigcap_{i\in I\cup J} \bar{B}_H(x_i, r_i) \neq \emptyset$. □

It is well-known that if B is a compact subset of a metric space X, then for any $x \in X$ there exists some $b \in B$ such that the distance from x to B is equal to $d(x, b)$. It turns out that if X is hyperconvex, one can replace “compact” with “admissible”. Let us investigate this concept further.

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4.2.4. Definition. We call a subset B of a metric space X *proximinal*, if for every $x \in X$ there exists at least one $b \in B$ such that $\mathrm{dist}(x, B) = d(x, b)$. △

Proximinal subsets of totally convex spaces have a useful property: any point closest to some point outside the set lies at its boundary. More precisely, we have the following proposition.

4.2.5. Proposition. *Let* B *be a proximinal subset of a totally convex space* X *and let* $x \in X \setminus B$. *Then, the set* $C := \bar{B}\big(x, \mathrm{dist}(x, B)\big) \cap B$ *is included in the boundary* ∂B. △

Proof. We only need to prove that if $c \in C$ and $\varepsilon > 0$, then $\bar{B}(c, \varepsilon) \nsubseteq B$. Without loss of generality we may (if needed) decrease ε so that it is less than $d(x, c)$. Let $\alpha := d(x, c) - \varepsilon$ and $\beta := \varepsilon$. Choose $y \in X$ so that $d(x, y) = \alpha$ and $d(y, c) = \beta$; then, $y \in \bar{B}(c, \varepsilon)$, but $d(x, y) < d(x, c) = \mathrm{dist}(x, B)$ and hence $y \notin B$. □

4.2.6. Proposition. *Every admissible subset of a hyperconvex metric space is proximinal.* △

Proof. Let A be an admissible subset of a hyperconvex space H and let $x \in H$ be arbitrary. Let us denote $r := \mathrm{dist}(x, A)$ and consider the family of balls $\{\bar{B}_n\}_{n\in\mathbb{N}}$, where $\bar{B}_n := \bar{B}\big(x, r + \frac{1}{n}\big)$. Also, assume that $A = \bigcap_{\lambda\in\Lambda} \bar{B}_\lambda$, where all $\bar{B}_\lambda$'s are closed balls in H. It is easy to see that the family $\{\bar{B}_n\}_{n\in\mathbb{N}} \cup \{\bar{B}_\lambda\}_{\lambda\in\Lambda}$ is pairwise intersecting (for the sake of formality, we are assuming here that $\mathbb{N} \cap \Lambda = \emptyset$) and hence there exists some point $a \in A \cap \bigcap_{n\in\mathbb{N}} \bar{B}\big(x, r + \frac{1}{n}\big) = A \cap \bar{B}(x, r)$. □

We will now prove a certain consequence of Propositions 4.2.3 and 4.2.6. It turns out that not only are admissible subsets of hyperconvex spaces their nonexpansive retracts, but there exists a nonexpansive retraction sending all points outside the set under consideration to its boundary.

4.2.7. Proposition. *Let A be an admissible subset of a hyperconvex space H. Then, there exists a nonexpansive retraction $R\colon H \to A$ such that the set $R(H \setminus A)$ is included in the boundary of A.* △

Proof. Let $\widetilde{A}$ be the maximal subset of H including A such that there exists a nonexpansive retraction $\widetilde{R}\colon \widetilde{A} \to A$ satisfying $\widetilde{R}(\widetilde{A} \setminus A) \subseteq \partial A$ (its existence follows from Lemma 3.1.14). For the sake of contradiction let us assume that $\widetilde{A} \subsetneq H$. Let $w \in H \setminus \widetilde{A}$. Let us consider the set

$$C := \bigcap_{x\in A} \bar{B}\big(x, d(x, w)\big) \cap \bar{B}\big(w, \operatorname{dist}(w, A)\big) \cap A.$$

We will show that $C \neq \emptyset$. Indeed, C is an intersection of a family of pairwise intersecting, closed balls. To see this, notice first that $w \in \bar{B}\big(x, d(x, w)\big)$ for any $x \in A$ and also $w \in \bar{B}\big(w, \operatorname{dist}(w, A)\big)$. Next, notice that $x \in \bar{B}\big(x, d(x, w)\big) \cap A$ for any $x \in A$. Finally, the proximinality of A yields the nonemptiness of $\bar{B}\big(w, \operatorname{dist}(w, A)\big) \cap A$. Since H is hyperconvex, C must be nonempty; also, Proposition 4.2.5 assures us that $C \subseteq \partial A$.

Choose any $c \in C$ and define the function $\widetilde{R}_w\colon \widetilde{A} \cup \{w\} \to A$ by the formula

$$\widetilde{R}_w(x) := \begin{cases} \widetilde{R}(x) & \text{if } x \in \widetilde{A}, \\ c & \text{if } x = w. \end{cases}$$

It is now enough to prove that $d\big(\widetilde{R}_w(x), \widetilde{R}_w(w)\big) \leqslant d(x, w)$ for any $x \in \widetilde{A}$. Indeed, $d\big(\widetilde{R}_w(x), \widetilde{R}_w(w)\big) = d\big(\widetilde{R}(x), c\big) = d\big(\widetilde{R}(x), \widetilde{R}(c)\big) \leqslant$

$d(x, c) \leqslant d(x, w)$, where in the last inequality we used the fact that $c \in \bar{B}\big(x, d(x, w)\big)$. □

Another property we will need later on (in fact, not until Chapter 6, but we include it here since it seems interesting by itself) is connected with "balls around sets" (which are called "parallel sets" of the given set by some authors).

4.2.8. Definition. Let A be a subset of a metric space X and let $r > 0$. We define the *ball centered at* A (*with radius* r) as the set $\bar{B}(A, r) := \bigcup_{a \in A} \bar{B}(a, r)$. △

4.2.9. Remark. It follows immediately from Proposition 4.2.6 that if A is an admissible subset of a hyperconvex space H, then $\bar{B}(A, r) = \{x \in H \mid \operatorname{dist}(x, A) \leqslant r\}$. However, in such a case even more can be said about what such a ball looks like. △

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4.2.10. Proposition. *Let* H *be a hyperconvex space,* $A := \bigcap_{\lambda \in \Lambda} \bar{B}(x_\lambda, r_\lambda) \in \mathcal{A}(H)$, *and* $r > 0$. *Then* $\bar{B}(A, r) = \bigcap_{\lambda \in \Lambda} \bar{B}(x_\lambda, r_\lambda + r)$. △

Proof. Let $b \in \bar{B}(A, r)$. This means that there exists some $a \in A$ such that $d(b, a) \leqslant r$. Further, $a \in A$ means that $d(a, x_\lambda) \leqslant r_\lambda$ for each $\lambda \in \Lambda$. Therefore we have $d(b, x_\lambda) \leqslant r + r_\lambda$ for any $\lambda \in \Lambda$.

Let now $b \in \bigcap_{\lambda \in \Lambda} \bar{B}(x_\lambda, r_\lambda + r)$ and fix $\lambda \in \Lambda$. Total convexity of the space H implies that there exists some $a_\lambda \in H$ such that $d(b, a_\lambda) \leqslant r$ and $d(a_\lambda, x_\lambda) \leqslant r_\lambda$. This means that the family of balls $\{\bar{B}(b, r)\} \cup \{\bar{B}(x_\lambda, r_\lambda) \mid \lambda \in \Lambda\}$ has a nonempty intersection. Let the point a belong to this intersection; this means that $\operatorname{dist}(b, A) \leqslant d(b, a) \leqslant r$, and the proof is finished. □

4.3 Baillon's intersection theorem

The main goal of this section is the proof of the Baillon's intersection theorem (Theorem 4.3.8), stating that the intersection of a chain of bounded hyperconvex sets is nonempty and hyperconvex. It is a remarkable result, interesting by itself, but also useful in fixed point theory. We will first show a simple special case of Baillon's theorem for admissible subsets of a hyperconvex space. We will then use it to prove the general case. Let us start, however, with a motivation: a simple example of two hyperconvex sets with a non-hyperconvex intersection.

4.3.1. Example. Consider the plane $\mathbb{R}^2$ with the "maximum" metric. Put $A := \{\langle 0,0\rangle, \langle 1,0\rangle\}$, fix $\alpha \in [0,1]$ and define

$$H_\alpha := \Big\{\langle x_1, x_2\rangle \in X \;\Big|\; x_1 \in \big[0, \tfrac{1}{2}\big] \text{ and } x_2 = \alpha x_1 \\ \text{or} \quad x_1 \in \big[\tfrac{1}{2}, 1\big] \text{ and } x_2 = \alpha(1 - x_1)\Big\}.$$

Let us consider the mapping $i\colon H_\alpha \to [0,1]$ given by the formula $i\big(\langle x_1, x_2\rangle\big) := x_1$. It is straightforward that i is a surjective isometry. The segment $[0,1]$ is hyperconvex (see Example 4.1.1) and so must be every set H_α (since for each $\alpha \in [0,1]$ it is a space isometric to it). Now notice that $H_0 \cap H_1 = A$; obviously, this set is not totally convex, so it cannot be hyperconvex. △

Now that we are (hopefully) convinced that the question of hyperconvexity of the intersection of hyperconvex sets may be nontrivial, let us introduce some notions we will use to show a (this time) *positive* result. It turns out that it is the *only* known result of this type, and the proof is far from simple.

4.3.2. Definitions. Let A be a nonempty bounded subset of a metric space X. Define $r_x(A) := \sup\{d(x,y) \mid y \in A\}$ for any $x \in X$. The number $r_X(A) := \inf\{r_x(A) \mid x \in X\}$ will be called the *radius of the set* A (*with respect to* X). The *diameter of the set* A is defined by the formula $\operatorname{diam} A := \sup\{d(x,y) \mid x,y \in A\}$. The *center of the set* A (*in* X) is the set $C_X(A) := \{x \in X \mid r_x(A) = r_X(A)\}$. Finally, we will denote the intersection of all closed balls in X including A by $\operatorname{cov}_X A$. If there is no doubt as to in what space is the set A embedded, we shall write $r(A)$, $C(A)$ and $\operatorname{cov} A$ instead of $r_X(A)$, $C_X(A)$ and $\operatorname{cov}_X A$, respectively. △

4.3.3. Remark. It is easy to see that $r_x(A) = \min\{r > 0 \mid A \subseteq \bar{B}(x,r)\}$ for every $x \in X$. △

Let is mention some properties of the notions defined above. For the proof, let us refer the reader to the paper [22].

4.3.4. Lemma. *For a nonempty and bounded subset* A *of a metric space* X, *the following equalities hold*:

1° $\operatorname{cov} A = \bigcap_{x \in X} \bar{B}\big(x, r_x(A)\big)$;
2° $r_x(\operatorname{cov} A) = r_x(A)$ *for* $x \in X$;
3° $r(\operatorname{cov} A) = r(A)$;
4° $C(A) = \bigcap_{x \in A} \bar{B}\big(x, r(A)\big)$.
5° *if* $B \subseteq A$, *then* $\operatorname{cov} B \subseteq \operatorname{cov} A$; *in particular, if* $A \in \mathcal{A}(X)$, *then for each* $B \subseteq A$, *also* $\operatorname{cov} B \subseteq A$. △

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It turns out that in the case of subsets of hyperconvex spaces, the radius and the diameter of the set are related by a formula that is familiar for anyone who did not forget his or her elementary geometry classes. (This is not true in general metric spaces; an easy example is any two-point metric space.) We will now show this and some other properties, which will be used later.

4.3.5. Lemma. *Let* A *be a nonempty bounded subset of a hyperconvex space* H.

1° $A \subseteq \bar{B}\big(x, \frac{1}{2} \operatorname{diam} A\big)$ *for some* $x \in H$;
2° $r(A) = \frac{1}{2} \operatorname{diam} A$;
3° $\operatorname{diam} \operatorname{cov} A = \operatorname{diam} A$;
4° $C(A) \neq \emptyset$;
5° *if* $A \subseteq C(A)$, *then* A *is a singleton;*
6° *if* H_1 *is a nonempty and hyperconvex subset of the space* H *and the set* $A \subseteq H_1$ *is nonempty and bounded, then* $C_{H_1}(A) = C_H(A) \cap H_1$. △

Proof. 1° Definition of the diameter and hyperconvexity of H imply that the intersection $\bigcap_{a \in A} \bar{B}\big(a, \frac{1}{2} \operatorname{diam} A\big)$ is not empty. Let x be any point in this intersection; then $d(x, a) \leqslant \frac{1}{2} \operatorname{diam} A$ for any $a \in A$, which means that $A \subseteq \bar{B}\big(x, \frac{1}{2} \operatorname{diam} A\big)$.
2° It follows from part 1° and Remark 4.3.3 that $r(A) \leqslant \frac{1}{2} \operatorname{diam} A$. In order to prove the opposite inequality notice that for $a, b \in A$ and $x \in H$ we have $d(a, b) \leqslant d(a, x) + d(b, x) \leqslant 2r_x(A)$. It is thus enough to take supremum over $a, b \in A$ and infimum over $x \in H$.
3° The thesis follows from part 2° above and part 3° of Lemma 4.3.4.

4° Note that for $x, y \in A$, the inequality $d(x, y) \leqslant \operatorname{diam} A = 2r(A)$ holds. We may now apply the formula from part 4° of Lemma 4.3.4 and employ the hyperconvexity of H.
5° Assume that A contains at least two points, so that $\operatorname{diam} A > 0$. Let $x, y \in A$ be two points satisfying $d(x, y) > \frac{1}{2} \operatorname{diam} A$. We have $y \in \bar{B}(x, r_x(A)) = \bar{B}(x, r(A)) = \bar{B}(x, \frac{1}{2} \operatorname{diam} A)$, where the former equality follows from the assumption that $x \in A \subseteq C(A)$ and the latter one from part 2°. We have proved in this way that $d(x, y) \leqslant \frac{1}{2} \operatorname{diam} A$, which is a contradiction.
6° Using the definition of a center of a set and part 2°, applied to the set A considered first as a subset of H and then of H_1, we infer that

$$\begin{aligned} C_H(A) \cap H_1 &= \{x \in H \mid r_x(A) = r_H(A)\} \cap H_1 \\ &= \{x \in H_1 \mid r_x(A) = \tfrac{1}{2} \operatorname{diam} A\} \\ &= \{x \in H_1 \mid r_x(A) = r_{H_1}(A)\} \\ &= C_{H_1}(A). \end{aligned}$$

□

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4.3.6. Remark. One may ask whether the property from part 1° of the previous lemma characterizes hyperconvex spaces, i.e., if the condition that each nonempty bounded subset of a given metric space X is included in a certain closed ball of radius $\frac{1}{2} \operatorname{diam} A$ is sufficient for the space X to be hyperconvex. It turns out that the answer is positive for Banach spaces (see [18]) and negative in the general case of metric spaces (see [10, Appendix]). For classes of metric spaces such as totally convex spaces or strictly convex spaces (defined e.g. in [26]) the question remains open. △

Let us now turn to the promised special case of Baillon's theorem.

4.3.7. Lemma. *An intersection of a chain of admissible subsets of a hyperconvex metric space is admissible.* △

Proof. Let $\{A_\lambda\}_{\lambda \in \Lambda}$, where $A_\lambda = \bigcap_{i \in I_\lambda} \bar{B}_H(x_{\lambda,i}, r_{\lambda,i})$ for $\lambda \in \Lambda$, be a chain of admissible subsets of a hyperconvex space H. It is enough to prove that $A := \bigcap_{\lambda \in \Lambda} A_\lambda = \bigcap_{\lambda \in \Lambda} \bigcap_{i \in I_\lambda} \bar{B}_H(x_{\lambda,i}, r_{\lambda,i}) \neq \emptyset$. Choose $\lambda_1, \lambda_2 \in \Lambda$ and $i_1 \in I_{\lambda_1}$, $i_2 \in I_{\lambda_2}$. Since $\{A_\lambda\}_{\lambda \in \Lambda}$ is a chain, either $A_{\lambda_1} \subseteq A_{\lambda_2}$ or $A_{\lambda_2} \subseteq A_{\lambda_1}$; assume $A_{\lambda_1} \subseteq A_{\lambda_2}$. We have then

$A_{\lambda_1} \subseteq \bar{B}_H(x_{\lambda_2,i_2}, r_{\lambda_2,i_2})$ and at the same time $A_{\lambda_1} \subseteq \bar{B}_H(x_{\lambda_1,i_1}, r_{\lambda_1,i_1})$ and hence $\bar{B}_H(x_{\lambda_1,i_1}, r_{\lambda_1,i_1}) \cap \bar{B}_H(x_{\lambda_2,i_2}, r_{\lambda_2,i_2}) \neq \emptyset$. Since $\lambda_1, \lambda_2 \in \Lambda$ and $i_1 \in I_{\lambda_1}$, $i_2 \in I_{\lambda_2}$ were arbitrary, hyperconvexity of H implies now that $A \neq \emptyset$. □

We are now fully equipped for the proof of the general case of Baillon's theorem. We will break it into two parts, since the latter one holds without the boundedness assumption. The proof presented here is a slight modification of the one from [22, Theorem 5.1].

4.3.8. Theorem. *Let* X *be a metric space and* $\{H_i\}_{i\in I}$ *a chain of nonempty and hyperconvex subsets of* X*, at least one of which is bounded. Then*

$$\bigcap_{i\in I} H_i \neq \emptyset. \qquad \triangle$$

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Proof. First notice that we can assume without loss of generality that all the sets H_i are bounded.

We will introduce some notation. Instead of $H_j \subseteq H_i$, where $i, j \in I$, we shall write $i \preccurlyeq j$; thus I becomes a totally ordered set and $\langle H_i \rangle_{i\in I}$ becomes a decreasing set family. For $i \in I$ and $A \subseteq X$ put $\operatorname{cov}_i A := \bigcap_{x\in H_i} \bar{B}_X(x, r_x(A))$; notice that in general $\operatorname{cov}_i A \neq \operatorname{cov}_{H_i} A$, but always $\operatorname{cov}_{H_i} A = H_i \cap \operatorname{cov}_i A$. Moreover, the operation $A \mapsto \operatorname{cov}_i A$ is monotonic in the sense that $\operatorname{cov}_i A \subseteq \operatorname{cov}_i B$ for $A \subseteq B$ (which follows from the definition of cov_i and r_x), and idempotent, i.e., $\operatorname{cov}_i(\operatorname{cov}_i A) = \operatorname{cov}_i A$ (which follows from the equality $r_x(\operatorname{cov}_i A) = r_x(A)$ for $x \in X$, analogous to item 2° of Lemma 4.3.4).

Consider the following family of subsets of the product $\prod_{i\in I} H_i$:

$$\Sigma := \left\{ \prod_{i\in I} \hat{A}_i \subseteq \prod_{i\in I} H_i \;\middle|\; \hat{A}_i \in \mathcal{A}(H_i) \text{ for } i \in I,\, \hat{A}_j \subseteq \hat{A}_i \text{ for } i \preccurlyeq j \right\},$$

partially ordered by inclusion. (Note that $\emptyset \notin \Sigma$, which follows from the definition of an admissible subset.) This family is nonempty, since $\prod_{i\in I} H_i \in \Sigma$. We will prove that there exists a minimal element in Σ. Let $\{\hat{A}_\lambda\}_{\lambda\in\Lambda}$ be a chain in Σ, where $\hat{A}_\lambda = \prod_{i\in I} \hat{A}_{\lambda,i}$ for $\lambda \in \Lambda$. Put $\hat{A} := \bigcap_{\lambda\in\Lambda} \hat{A}_\lambda = \bigcap_{\lambda\in\Lambda} \prod_{i\in I} \hat{A}_{\lambda,i} = \prod_{i\in I} \bigcap_{\lambda\in\Lambda} \hat{A}_{\lambda,i}$. Denoting $\widetilde{A}_i := \bigcap_{\lambda\in\Lambda} \hat{A}_{\lambda,i}$ for $i \in I$ we obtain $\widetilde{A} := \prod_{i\in I} \widetilde{A}_i$. From Lemma 4.3.7

we infer that the sets $\widetilde{A}_i$ are admissible for $i \in I$. Moreover, for $i \preccurlyeq j$ we have $\hat{A}_{\lambda,j} \subseteq \hat{A}_{\lambda,i}$ for each $\lambda \in \Lambda$ and hence $\widetilde{A}_j = \bigcap_{\lambda\in\Lambda} \hat{A}_{\lambda,j} \subseteq \bigcap_{\lambda\in\Lambda} \hat{A}_{\lambda,i} = \widetilde{A}_i$. It turns out that $\widetilde{A} \in \Sigma$; moreover, it is clear that $\widetilde{A}$ is a lower bound of the chain $\{\hat{A}_\lambda\}_{\lambda\in\Lambda}$. The Kuratowski–Zorn Lemma implies now the existence of a minimal element $\widetilde{A} = \prod_{i\in I} \widetilde{A}_i$ in Σ.

Fix $j_0 \in I$ and define $\widetilde{A}_1 \in \Sigma$ by the formula $\widetilde{A}_1 := \prod_{i\in I} \widetilde{A}_{1,i}$, where

$$\widetilde{A}_{1,i} := \begin{cases} \operatorname{cov}_{j_0} \widetilde{A}_{j_0} \cap \widetilde{A}_i & \text{for } i \preccurlyeq j_0, \\ \widetilde{A}_i & \text{otherwise.} \end{cases}$$

In order to prove that $\widetilde{A}_1 \in \Sigma$, note first that $\widetilde{A}_{j_0} \subseteq \widetilde{A}_i$ for $i \preccurlyeq j_0$ and hence $\operatorname{cov}_{j_0} \widetilde{A}_{j_0} \cap \widetilde{A}_i \neq \emptyset$; obviously, $\widetilde{A}_{1,i}$ is a ball intersection for each $i \in I$. Further, for $i \preccurlyeq j$ we have $\widetilde{A}_{1,j} \subseteq \widetilde{A}_{1,i}$. (Indeed, one of the cases: $j_0 \prec i \preccurlyeq j$, $i \preccurlyeq j \prec j_0$, $i \preccurlyeq j_0 \preccurlyeq j$ holds. In the first two ones the above inclusion is trivial; in the third one it follows from the equality $\operatorname{cov}_{j_0} \widetilde{A}_{j_0} \cap \widetilde{A}_{j_0} = \widetilde{A}_{j_0}$.) We have therefore $\widetilde{A}_1 \in \Sigma$ and $\widetilde{A}_1 \subseteq \widetilde{A}$; from the minimality of $\widetilde{A}$ it follows that $\widetilde{A}_1 = \widetilde{A}$. In particular, for $i \preccurlyeq j_0$ we obtain $\operatorname{cov}_{j_0} \widetilde{A}_{j_0} \cap \widetilde{A}_i = \widetilde{A}_i$, so the inclusions $\widetilde{A}_{j_0} \subseteq \widetilde{A}_i \subseteq \operatorname{cov}_{j_0} \widetilde{A}_{j_0}$ hold. Hence and from the aforementioned properties of the cov_i operation we infer that $\operatorname{cov}_{j_0} \widetilde{A}_{j_0} \subseteq \operatorname{cov}_{j_0} \widetilde{A}_i \subseteq \operatorname{cov}_{j_0} \widetilde{A}_{j_0}$. Since j_0 was arbitrary, we have $\operatorname{cov}_j \widetilde{A}_j = \operatorname{cov}_j \widetilde{A}_i$ if $i \preccurlyeq j$. This implies, among others, that $r_x(\widetilde{A}_j) = r_x(\widetilde{A}_i)$ for $i \preccurlyeq j$ and $x \in H_j$. (To see this, note that $\widetilde{A}_j \subseteq \widetilde{A}_i$ and therefore $r_x(\widetilde{A}_j) \leqslant r_x(\widetilde{A}_i)$. If the inequality $r_x(\widetilde{A}_j) < r_x(\widetilde{A}_i)$ held for some $x \in H_j$, then there would exist a $y \in \widetilde{A}_i$ such that $d(x,y) > r_x(\widetilde{A}_j)$ – but then it would be $y \notin \bar{B}_X\big(x, r_x(\widetilde{A}_j)\big)$ and hence $y \notin \operatorname{cov}_j \widetilde{A}_j := \bigcap_{x\in H_j} \bar{B}_X\big(x, r_x(\widetilde{A}_j)\big)$. At the same time, $y \in \widetilde{A}_i \subseteq \operatorname{cov}_j \widetilde{A}_i$, which yields a contradiction.) As a consequence, $r_{H_j}(\widetilde{A}_j) := \inf_{x\in H_j} r_x(\widetilde{A}_j) = \inf_{x\in H_j} r_x(\widetilde{A}_i) \geqslant \inf_{x\in H_i} r_x(\widetilde{A}_i) =: r_{H_i}(\widetilde{A}_i)$ for $i \preccurlyeq j$. On the other hand, since $i \preccurlyeq j$, we have $\widetilde{A}_j \subseteq \widetilde{A}_i$, so $r_{H_j}(\widetilde{A}_j) = \frac{1}{2} \operatorname{diam} \widetilde{A}_j \leqslant \frac{1}{2} \operatorname{diam} \widetilde{A}_i = r_{H_i}(\widetilde{A}_i)$; thus we obtain the equality $r_{H_i}(\widetilde{A}_i) = r_{H_j}(\widetilde{A}_j)$ for $i \preccurlyeq j$. We can therefore denote the common value of the radii of the sets $\widetilde{A}_i$ with respect to H_i, where $i \in I$, by r.

Let us now define the set $\widetilde{A}_2 \in \Sigma$ by the formula $\widetilde{A}_2 := \prod_{i\in I} \widetilde{A}_{2,i}$, where $\widetilde{A}_{2,i} := C_{\widetilde{A}_i}(\widetilde{A}_i)$ for $i \in I$. Nonemptiness of all the sets $\widetilde{A}_{2,i}$, where $i \in I$, follows from part 4° of Lemma 4.3.5 applied to $\widetilde{A}_i$ as the whole space. The fact that each $\widetilde{A}_{2,i}$ ($i \in I$) is an intersection of balls in H_i is a consequence of part 6° of Lemma 4.3.5 with H_i in place of H and $\widetilde{A}_i$ in place of both A and H_1 and the formula 4° of Lemma 4.3.4.

Finally, let $i \preccurlyeq j$; then $\widetilde{A}_j \subseteq \widetilde{A}_i$ and consequently

$$\begin{aligned}
\widetilde{A}_{2,j} := C_{\widetilde{A}_j}(\widetilde{A}_j) &= C_{H_j}(\widetilde{A}_j) \cap \widetilde{A}_j \\
&= \{x \in \widetilde{A}_j \mid r_x(\widetilde{A}_j) = r\} \\
&= \{x \in \widetilde{A}_j \mid r_x(\widetilde{A}_i) = r\} \\
&\subseteq \{x \in \widetilde{A}_i \mid r_x(\widetilde{A}_i) = r\} \\
&= C_{H_i}(\widetilde{A}_i) \cap \widetilde{A}_i = C_{\widetilde{A}_i}(\widetilde{A}_i) =: \widetilde{A}_{2,i}.
\end{aligned}$$

This way, we have proved that $\widetilde{A}_2 \in \Sigma$; but from its definition we have $\widetilde{A}_2 \subseteq \widetilde{A}$, so in view of the minimality of $\widetilde{A}$ we obtain $\widetilde{A}_2 = \widetilde{A}$. (A message for the reader: if you are actually reading this, congratulations – this is a long, technical proof, and getting here means that you have the patience and motivation necessary to study it in detail. Your effort will be rewarded soon: the proof is almost complete.) We have therefore $\widetilde{A}_i = C_{\widetilde{A}_i}(\widetilde{A}_i)$ for any $i \in I$, which in conjunction with part 5° of Lemma 4.3.5 implies that every set $\widetilde{A}_i$ is a singleton. Using the fact that $\widetilde{A}_j \subseteq \widetilde{A}_i$ for $i \preccurlyeq j$, we infer that there exists some point $x_0 \in X$ such that $\widetilde{A}_i = \{x_0\}$ for all $i \in I$; in particular, $x_0 \in \bigcap_{i \in I} H_i$, which completes the proof. □

4.3.9. Remark. The boundedness hypothesis in Theorem 4.3.8 is essential. To see this, put e.g. $X := \mathbb{R}$, $I := \mathbb{N}$ and $H_i := [i, +\infty)$. △

It turns out that we can apply the above theorem to prove also *hyperconvexity* of the intersection of a chain of hyperconvex sets. It is interesting – and, as we will see, important for the applications – that in this case we can omit the boundedness hypothesis.

4.3.10. Corollary. *Let* $\{H_i\}_{i \in I}$ *be a chain of hyperconvex subsets of a metric space* X *such that the intersection* $\widetilde{H} := \bigcap_{i \in I} H_i$ *is nonempty. Then* $\widetilde{H}$ *is hyperconvex.* △

Proof. Let $\{\bar{B}_{\widetilde{H}}(x_\lambda, r_\lambda) \mid \lambda \in \Lambda\}$ be a family of closed balls in $\widetilde{H}$ such that $d(x_\lambda, x_\mu) \leqslant r_\lambda + r_\mu$ for $\lambda, \mu \in \Lambda$. Hyperconvexity of H_i for $i \in I$ yields $A_i := \bigcap_{\lambda\in\Lambda} \bar{B}_{H_i}(x_\lambda, r_\lambda) \neq \emptyset$. Notice that for each $i \in I$ the set A_i is bounded and hyperconvex (as an admissible subset of a hyperconvex space H_i). Moreover, the inclusion $A_i \subseteq A_j$ holds if $H_i \subseteq H_j$. Therefore, $\{A_i\}_{i\in I}$ is a chain of nonempty, bounded and hyperconvex subsets of X. Theorem 4.3.8 implies that $\bigcap_{i\in I} A_i \neq \emptyset$. Now we have $\bigcap_{i\in I} A_i = \bigcap_{i\in I} \bigcap_{\lambda\in\Lambda} \bar{B}_{H_i}(x_\lambda, r_\lambda) = \bigcap_{\lambda\in\Lambda} \bigcap_{i\in I} \bar{B}_{H_i}(x_\lambda, r_\lambda) = \bigcap_{\lambda\in\Lambda} \bar{B}_{\widetilde{H}}(x_\lambda, r_\lambda)$, so the intersection $\bigcap_{\lambda\in\Lambda} \bar{B}_{\widetilde{H}}(x_\lambda, r_\lambda)$ is nonempty, which yields the hyperconvexity of $\widetilde{H}$. □

We will now state a trivial corollary, foreshadowing Section 4.6.

4.3.11. Corollary. *Let* $A \neq \emptyset$ *be a subset of a hyperconvex space* H. *Then* A *has a minimal* (*with respect to inclusion*) *hyperconvex superset in* H. △

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Let us conclude this section with a variant of Baillon's theorem.

4.3.12. Theorem. *Let* $\{H_\lambda\}_{\lambda\in\Lambda}$ *be a family of bounded hyperconvex spaces such that the intersection of any its finite subfamily is nonempty and hyperconvex. Then, the intersection* $\bigcap_{\lambda\in\Lambda} H_\lambda$ *is also nonempty and hyperconvex.* △

Proof. Let $\mathcal{F}$ be the family of all index sets $\hat{I} \subseteq \Lambda$ such that for any finite $J \subseteq \Lambda$, the intersection $\bigcap_{\lambda\in\hat{I}\cup J} H_\lambda$ is nonempty and hyperconvex. From the assumption we have $\emptyset \in \mathcal{F}$, so in particular $\mathcal{F} \neq \emptyset$. We will now prove the existence of a maximal element in $\mathcal{F}$. Assume that $\{\hat{I}_\alpha\}_{\alpha\in A}$ is a chain in $\mathcal{F}$, choose any finite set $J \subseteq \Lambda$ and define $\widetilde{I} := \bigcup_{\alpha\in A} \hat{I}_\alpha$. Let $\alpha, \beta \in A$ and assume without loss of generality that $\hat{I}_\alpha \subseteq \hat{I}_\beta$. This means that

$$\bigcap_{\lambda\in\hat{I}_\alpha\cup J} H_\lambda \supseteq \bigcap_{\lambda\in\hat{I}_\beta\cup J} H_\lambda.$$

Since $\alpha, \beta \in A$ were arbitrary, this means that the sets of the form $\bigcap_{\lambda\in\hat{I}_\alpha\cup J} H_\lambda$ constitute a chain of bounded hyperconvex sets. This in turn implies that their intersection is nonempty and hyperconvex; but

$$\bigcap_{\alpha\in A} \bigcap_{\lambda\in\hat{I}_\alpha\cup J} H_\lambda = \bigcap_{\lambda\in\widetilde{I}\cup J} H_\lambda,$$

and hence $\widetilde{I} \in \mathcal{F}$.

Let now $\widetilde{I}$ be the maximal element in $\mathcal{F}$. Obviously, $\widetilde{I} \cup \{\lambda\} \in \mathcal{F}$ for any $\lambda \in \Lambda$ and hence $\widetilde{I} = \Lambda$, which was to be proved. □

4.4 Making hyperconvex spaces

In the previous section, we have seen how we can obtain a hyperconvex space by means of one set-theoretical operation: taking the intersection of a certain chain of sets. It turns out that there are other ways to combine hyperconvex spaces to get new ones. We will now examine a few such ways. The first (and probably simplest) one is by taking products. Another one is to take the family of all admissible subsets of a hyperconvex space and endow it with the Hausdorff metric. Finally, we will show how one may construct a hyperconvex space by „adjoining" (for the lack of a better word) an arbitrary number of hyperconvex spaces to one such space.

4.4.1. Theorem. *Let $\{\langle X_i, d_i\rangle\}_{i\in I}$ be a family of metric spaces. Choose a point $a := \{a_i\}_{i\in I} \in \prod_{i\in I} X_i$ and define a function $d\colon X\times X \to [0,+\infty)$, where $X := \big\{\{x_i\}_{i\in I} \in \prod_{i\in I} X_i \;\big|\; \sup_{i\in I} d_i(x_i, a_i) < +\infty\big\}$, by the formula $d\big(\{x_i\}_{i\in I}, \{y_i\}_{i\in I}\big) := \sup_{i\in I} d_i(x_i, y_i)$. Then $\langle X, d\rangle$ is a metric space; moreover, if all the spaces X_i $(i \in I)$ are hyperconvex, then so is X.* △

Proof. In order to prove that the function d is a metric, it is enough to observe that for any $\{x_i\}_{i\in I}, \{y_i\}_{i\in I} \in X$ we have $d\big(\{x_i\}_{i\in I}, \{y_i\}_{i\in I}\big) := \sup_{i\in I} d_i(x_i, y_i) \leqslant \sup_{i\in I}\big(d_i(x_i, a_i) + d_i(a_i, y_i)\big) < +\infty$.

Now let us choose any point $x := \{x_i\}_{i\in I} \in X$ and $r > 0$ and notice that $\bar{B}_X(x, r) = \prod_{i\in I} \bar{B}_{X_i}(x_i, r)$. Indeed,

$$\begin{aligned}
\bar{B}_X(x, r) &= \big\{\{y_i\}_{i\in I} \in X \;\big|\; d(x, y) \leqslant r\big\} \\
&= \left\{\{y_i\}_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; \sup_{i\in I} d_i(y_i, a_i) < \infty,\;\; \sup_{i\in I} d_i(x_i, y_i) \leqslant r\right\} \\
&= \left\{\{y_i\}_{i\in I} \in \prod_{i\in I} X_i \;\middle|\; \sup_{i\in I} d_i(x_i, y_i) \leqslant r\right\} \\
&= \prod_{i\in I}\{y_i \in X_i \mid d_i(x_i, y_i) \leqslant r\} = \prod_{i\in I} \bar{B}_{X_i}(x_i, r),
\end{aligned}$$

where the third equality follows from the fact that the condition $\sup_{i\in I} d_i(x_i, y_i) \leqslant r$ implies the inequality

$$\sup_{i\in I} d_i(y_i, a_i) \leqslant \sup_{i\in I}\big(d_i(y_i, x_i) + d_i(x_i, a_i)\big) \leqslant \sup_{i\in I}\big(r + d_i(x_i, a_i)\big) < +\infty.$$

Let $\{\bar{B}_X(x_\lambda, r_\lambda)\}_{\lambda\in\Lambda}$, where $x_\lambda = \{x_{\lambda,i}\}_{i\in I}$, be a family of closed balls in X such that $d(x_{\lambda_1}, x_{\lambda_2}) \leqslant r_{\lambda_1} + r_{\lambda_2}$ for $\lambda_1, \lambda_2 \in \Lambda$. From the considerations above we infer that $\bar{B}_X(x_\lambda, r_\lambda) = \prod_{i\in I} \bar{B}_{X_i}(x_{\lambda,i}, r_\lambda)$, and hence we obtain the equalities

$$\bigcap_{\lambda\in\Lambda} \bar{B}_X(x_\lambda, r_\lambda) = \bigcap_{\lambda\in\Lambda}\prod_{i\in I} \bar{B}_{X_i}(x_{\lambda,i}, r_\lambda) = \prod_{i\in I}\bigcap_{\lambda\in\Lambda} \bar{B}_{X_i}(x_{\lambda,i}, r_\lambda);$$

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since for any $j \in I$ and $\lambda_1, \lambda_2 \in \Lambda$ the inequality $d_j(x_{\lambda_1,j}, x_{\lambda_2,j}) \leqslant \sup_{i\in I} d_i(x_{\lambda_1,i}, x_{\lambda_2,i}) = d(x_{\lambda_1}, x_{\lambda_2})$ holds, so hyperconvexity of X_i for $i \in I$ implies that each intersection of the form $\bigcap_{\lambda\in\Lambda} \bar{B}_{X_i}(x_{\lambda,i}, r_\lambda)$ is nonempty, which completes the proof. □

4.4.2. Example. Taking $I := \mathbb{N}$ and $\langle X_i, a_i\rangle := \langle \mathbb{R}, 0\rangle$ for every $i \in I$ in the previous theorem we obtain the fact that the space of bounded sequences l^∞ is hyperconvex. △

4.4.3. Example. Let $\langle v_n^-\rangle_{n=1}^\infty$ and $\langle v_n^+\rangle_{n=1}^\infty$ be two bounded real sequences such that $v_n^- \leqslant v_n^+$ for all $n \in \mathbb{N}$. Denote

$$H := \{\langle x_n\rangle_{n=1}^\infty \in l^\infty \mid v_n^- \leqslant x_n \leqslant v_n^+ \text{ for } n \in \mathbb{N}\}.$$

We will prove that H is hyperconvex. Indeed, we will define a nonexpansive retraction $R\colon l^\infty \to H$ by the formula

$$R\big(\langle x_n\rangle_{n=1}^\infty\big)_n := \begin{cases} v_n^- & \text{when } x_n < v_n^-,\\ x_n & \text{when } v_n^- \leqslant x_n \leqslant v_n^+,\\ v_n^+ & \text{when } v_n^+ < x_n.\end{cases}$$

We will now show that R is a nonexpansive retraction of l^∞ onto H. Let $\langle x_n\rangle_{n=1}^\infty, \langle y_n\rangle_{n=1}^\infty \in l^\infty$. Define

$$\begin{aligned}
A &:= \{n \in \mathbb{N} \mid v_n^- \leqslant x_n \leqslant v_n^+\},\\
A^- &:= \{n \in \mathbb{N} \mid x_n < v_n^-\},\\
A^+ &:= \{n \in \mathbb{N} \mid v_n^+ < x_n\},\\
B &:= \{n \in \mathbb{N} \mid v_n^- \leqslant y_n \leqslant v_n^+\},\\
B^- &:= \{n \in \mathbb{N} \mid y_n < v_n^-\},\\
B^+ &:= \{n \in \mathbb{N} \mid v_n^+ < y_n\}.
\end{aligned}$$

Obviously, the sequence $R\big(\langle x_n\rangle_{n=1}^\infty\big)$ is bounded for any $\langle x_n\rangle_{n=1}^\infty \in l^\infty$. Further, let us notice that $\mathbb{N} = A \cup A^- \cup A^+ = B \cup B^- \cup B^+$. It is now easy to see that for any $n \in \mathbb{N}$ and $\langle x_n\rangle_{n=1}^\infty, \langle y_n\rangle_{n=1}^\infty \in l^\infty$ we have $\big|R\big(\langle x_n\rangle_{n=1}^\infty\big)_n - R\big(\langle y_n\rangle_{n=1}^\infty\big)_n\big| \leqslant |x_n - y_n|$. For instance, consider the case when $n \in A \cap B^-$; then, $\big|R\big(\langle x_n\rangle_{n=1}^\infty\big)_n - R\big(\langle y_n\rangle_{n=1}^\infty\big)_n\big| = |x_n - v_n^-| = x_n - v_n^- < x_n - y_n \leqslant |x_n - y_n|$. In consequence, we obtain $\|R\big(\langle x_n\rangle_{n=1}^\infty\big) - R\big(\langle y_n\rangle_{n=1}^\infty\big)\|_\infty \leqslant \|\langle x_n\rangle_{n=1}^\infty - \langle y_n\rangle_{n=1}^\infty\|_\infty$, which means that R is nonexpansive. Also, by definition of R, it is a retraction of l^∞ onto H, and the proof is finished. △

4.4.4. Remark. As we will soon learn, the function spaces L^∞ are also hyperconvex. An analogous proof (which we will not repeat) shows that an order interval in L^∞ is hyperconvex as well. △

We will now prove that the family $\mathcal{A}(H)$ of admissible subsets of a hyperconvex space is itself hyperconvex. In order to do that, we will have to define a metric on this space.

4.4.5. Definitions. Let A and B be two nonempty subsets of a metric space X. Let us define $D^*(A, B) := \sup_{a \in A} \operatorname{dist}(a, B)$ and

$$D(A, B) := \max\big\{D^*(A, B), D^*(B, A)\big\}. \qquad △$$

The function D defined above may vanish for distinct sets A and B; it may also be infinite (if one of these sets is unbounded). However, it becomes a metric once we restrict the class of sets whose distances we want to measure.

4.4.6. Proposition. *The above defined function* D *is a metric on the family of the closed bounded subsets of any metric space.* △

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4.4.7. Definition. We call D the *Hausdorff metric* on the family of closed bounded sets of any metric space (or any subfamily thereof). △

We will now draw a simple, though very useful conclusion from Proposition 4.2.6; the following corollary will be used frequently in the sequel.

4.4.8. Corollary. *Let* H *be a hyperconvex space,* $A, B \in \mathcal{A}(H)$ *and* $r > 0$. *Then* $D^*(A, B) \leqslant r$ *if, and only if, for each* $a \in A$ *there exists some* $b \in B$ *such that* $d(a, b) \leqslant r$. △

We are now ready to state and prove the hyperconvexity of the family $\mathcal{A}(H)$ (for a hyperconvex space H).

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4.4.9. Theorem. *Let* H *be a hyperconvex space. The family of admissible subsets of* H*, endowed with the Hausdorff metric, is also hyperconvex.* △

Proof. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of admissible subsets of H and $\{r_\lambda\}_{\lambda\in\Lambda}$ a family of nonnegative numbers such that $D(A_\lambda, A_\mu) \leqslant r_\lambda + r_\mu$ for any $\lambda, \mu \in \Lambda$. Denote $A_\lambda = \bigcap_{i\in I_\lambda} \bar{B}(p_i, r_i)$, where we assume that all index sets I_λ are pairwise disjoint. We would like to find some admissible set $C \in \bigcap_{\lambda\in\Lambda} \bar{B}_D(A_\lambda, r_\lambda)$. Define $C := \bigcap_{\lambda\in\Lambda} \bigcap_{i\in I_\lambda} \bar{B}(p_i, r_i + r_\lambda)$. In order to prove that C is nonempty, let us consider any two balls $\bar{B}(p_i, r_i + r_\lambda)$ and $\bar{B}(p_j, r_j + r_\mu)$, where $\lambda, \mu \in \Lambda$, $i \in I_\lambda$ and $j \in I_\mu$. We may apply Corollary 4.4.8 to choose two points $s_\lambda \in A_\lambda$ and $s_\mu \in A_\mu$ such that $d(s_\lambda, s_\mu) \leqslant r_\lambda + r_\mu$. We have now

$$d(p_i, p_j) \leqslant d(p_i, s_\lambda) + d(s_\lambda, s_\mu) + d(s_\mu, p_j) \leqslant r_i + r_\lambda + r_\mu + r_j$$

and the hyperconvexity of H yields the nonemptiness of C. Thus, C is admissible.

It remains to show that C lies in the intersection of the considered family of balls. Choose any $\lambda \in \Lambda$. We will show first that for any $c \in C$, there exists some $c_\lambda \in A_\lambda$ such that $d(c, c_\lambda) \leqslant r_\lambda$. Indeed, for any $i \in I_\lambda$, we have $d(c, p_i) \leqslant r_i + r_\lambda$, so by hyperconvexity of H, the set $\bar{B}(c, r_\lambda) \cap A_\lambda$ is nonempty. Let now take any $c_\lambda \in A_\lambda$ and prove that there exists a $c \in C$ such that $d(c, c_\lambda) \leqslant r_\lambda$. We know that for any $i \in I_\lambda$, the inequality $d(c_\lambda, p_i) \leqslant r_i$ holds. Take any $\mu \in \Lambda$ and $j \in I_\mu$.

There exists a point $c_\mu \in A_\mu$ satisfying the inequality $d(c_\lambda, c_\mu) \leqslant r_\lambda + r_\mu$, so $d(c_\lambda, p_j) \leqslant d(c_\lambda, c_\mu) + d(c_\mu, p_j) \leqslant r_\lambda + r_\mu + r_j$. This means that $\bar{B}(c_\lambda, r_\lambda) \cap C \neq \emptyset$. We have now shown that $D^*(C, A_\lambda) \leqslant r_\lambda$ and $D^*(A_\lambda, C) \leqslant r_\lambda$, which completes the proof. □

We will conclude this section with the proof that one can "link" a collection of metric spaces to one "central" metric space, thus obtaining a tree-like structure with a natural metric. It turns out that if all the spaces used are hyperconvex, then the resulting one also has this property.

4.4.10. Theorem. *Let $\langle X, d_X\rangle$ be a metric space and $\{\langle X_\lambda, d_\lambda\rangle\}_{\lambda\in\Lambda}$ be a collection of pairwise disjoint metric spaces, each of which is disjoint with X. Let $f\colon \Lambda \to X$ be an arbitrary function and let $g\colon \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda$ be such that $g(\lambda) \in X_\lambda$ for any $\lambda \in \Lambda$. Denote $\check{X}_\lambda := X_\lambda \setminus \{g(\lambda)\}$ for $\lambda \in \Lambda$. Let $Y := X \cup \bigcup_{\lambda\in\Lambda} \check{X}_\lambda$. Define the function $d_Y\colon Y \times Y \to [0, +\infty)$ by the formula*

$$d_Y(x,y) := \begin{cases} d_X(x,y) & \text{if } x, y \in X, \\ d_\lambda(x,y) & \text{if } x, y \in X_\lambda, \\ d_X\big(x, f(\lambda)\big) + d_\lambda\big(g(\lambda), y\big) & \text{if } x \in X \text{ and } y \in \check{X}_\lambda, \\ d_\lambda\big(x, g(\lambda)\big) + d_X\big(f(\lambda), y\big) & \text{if } x \in \check{X}_\lambda \text{ and } y \in X, \\ d_\lambda\big(x, g(\lambda)\big) + d_X\big(f(\lambda), f(\mu)\big) & \\ \qquad\qquad + d_\mu\big(g(\mu), y\big) & \text{if } x \in \check{X}_\lambda \text{ and } y \in \check{X}_\mu, \end{cases}$$

where in each case we write "$x \in \check{X}_\lambda$" instead of "$x \in \check{X}_\lambda$ for certain $\lambda \in \Lambda$" etc. for brevity, and assume that $\lambda \neq \mu$ in the last case.

Then, d_Y is a metric on Y; moreover, if X and all the X_λ's are hyperconvex, then so is Y. △

The detailed proof of the above theorem would occupy a few pages – mostly of easy, but tedious calculations. Therefore we will only sketch its main geometric ideas.

Outline of the proof. The proof of the fact that d_Y is a metric requires a lot of perseverance and very little ingenuity and therefore we will omit it. Let us sketch the proof that Y is hyperconvex, provided all X_λ's and X are hyperconvex.

Notation. From now on, we shall write simply d instead of either d_X or d_Y and simply $\bar{B}$ instead of $\bar{B}_Y$; this will of course never give rise to ambiguity. Also, let us define the "projection" $P\colon Y \to X$ by the formula

$$P(x) := \begin{cases} x & \text{if } x \in X, \\ f(\lambda) & \text{if } x \in \check{X}_\lambda \text{ for certain } \lambda \in \Lambda. \end{cases}$$

Let $\{\bar{B}(x_i, r_i)\}_{i \in I}$ be a set of balls in Y such that $d(x_i, x_j) \leqslant r_i + r_j$ for any $i, j \in I$. Two cases are possible.

Case 1. For each $i \in I$, the ball $\bar{B}(x_i, r_i)$ intersects with X. This means that $r_i \geqslant d(x_i, P(x_i))$ for all $i \in I$. Let us denote $x_i' := P(x_i)$ and $r_i' := r_i - d(x_i, P(x_i))$ for $i \in I$. It is easy to check that $d(P(x_i), P(x_j)) \leqslant r_i' + r_j'$ for all $i, j \in I$. Hyperconvexity of X yields the nonemptiness of the intersection

$$\bigcap_{i \in I} \bar{B}_X(P(x_i), r_i').$$

Together with the fact that $\bar{B}_X(P(x_i), r_i') \subseteq \bar{B}(x_i, r_i)$ for any $i \in I$, this concludes the proof in this case.

Case 2. There exist some $k \in I$ and $\mu \in \Lambda$ such that $\bar{B}(x_k, r_k) \subseteq \check{X}_\mu$; in other words, $x_k \in \check{X}_k$ and $r_k < d(x_k, P(x_k))$. Let us define for each $i \in I$ the point x_i' and the number r_i' in the following way:

$$x_i' := \begin{cases} x_i & \text{if } x_i \in \check{X}_\mu, \\ g(\mu) & \text{otherwise;} \end{cases}$$

$$r_i' := \begin{cases} r_i & \text{if } x_i \in \check{X}_\mu, \\ r_i - d(P(x_k), x_i) & \text{otherwise.} \end{cases}$$

It is now straighforward that $d(x_i', x_j') \leqslant r_i' + r_j'$ and that $\bar{B}_{X_\mu}(x_i', r_i') \subseteq \bar{B}(x_i, r_i)$ for all $i, j \in I$ (note that we identify $f(\mu)$ with $g(\mu)$ here), which completes the proof. □

It seems interesting that the above technique may be used to examine a certain example illustrating the classical Schauder's fixed point theorem. Let us quote this theorem first.

4.4.11. Theorem (Schauder)**.** *Let* $T\colon C \to C$ *be a compact mapping of a nonempty and convex subset* C *of a Banach space into itself. Then,* T *has a fixed point.* △

In many books, the above theorem is stated with the assumption that the set C is *closed*. It turns out, however, that it is not necessary. Recall that a continuous mapping is *compact*, if its range is relatively compact (we will state this definition formally in Chapter 5). Contrary to the case of compactness, relative compactness can depend on the space the set is embedded in; for example, the open interval $(0, 1)$ is relatively compact as a subset of $\mathbb{R}$, but not as its own subset. This in turn means that even if C is not closed, the closure (in C) of the range of T is compact; for instance, if $C = (0, 1) \subseteq \mathbb{R}$, the range of T must not be "too big" (so that neither endpoint of C is its limit point).

Now one may ask whether relaxing the assumption of closedness of C really does generalize the theorem. More precisely, one might think that given some nonempty and convex set C of some Banach space and a compact mapping $T\colon C \to C$, we may always select some *closed*, convex and T-invariant subset $\hat{C} \subseteq C$ containing all the fixed points of T. This would mean that every fixed point, whose existence is guaranteed by the Schauder fixed point theorem, would be also a fixed point of a continuous self-mapping of a nonempty, *closed* and convex subset of a Banach space.

We will now sketch a proof that this hope is vain. We will describe a compact mapping T of certain convex (but not closed) subset C of the sequence space l^1 into itself such that no convex and closed subset of C contains all the fixed points of T. In other words, it turns out that it is impossible to restrict T to any nonempty, T-invariant, closed and convex subset of C without "losing" some of its fixed points.

4.4.12. Example. Let e_n, where $n \in \mathbb{N}$, denote the nth unit vector in the space l^1. Let us put $a_n := \frac{1}{n} e_n$ for $n \in \mathbb{N}$ and let $X_n := [0, a_n]$ for $n \in \mathbb{N}$. Denote $X := \bigcup_{n \in \mathbb{N}} X_n$ and $C := \operatorname{conv} X$.

The set X is compact. Indeed, if $\langle b_n \rangle_{n=1}^{\infty}$ is a seqeunce in X, then either infinitely many of its terms belong to some X_k (and the existence of its convergent subsequence follows from the compactness of a segment in $\mathbb{R}$), or infinitely many X_k's contain at least one of its terms. In such a case, there exist two sequences of natural numbers, $\langle k_l \rangle_{l=1}^{\infty}$ and $\langle n_l \rangle_{l=1}^{\infty}$ such that $b_{n_l} \in X_{k_l}$ for any $l \in \mathbb{N}$, and the sequence $\langle n_l \rangle_{l=1}^{\infty}$ is increasing (and hence injective). This way we obtain $\|b_{n_l}\| \leqslant k_l^{-1}$, so $\lim_{l \to \infty} b_{n_l} = 0$.

The set C is not closed. Indeed, let some sequence $\langle b_n \rangle_{n=1}^{\infty}$ be defined by the formula $b_n := \sum_{k=1}^{n} \frac{1}{2^k} a_k$. Obviously, $b_\infty := \left\langle \frac{1}{n 2^n} \right\rangle = \lim_{n \to \infty} b_n \in l^1$. Moreover, for $n \in \mathbb{N}$ we have

$$b_n = \left(1 - \sum_{k=1}^{n} \frac{1}{2^k}\right) \cdot 0 + \sum_{k=1}^{n} \frac{1}{2^k} \cdot a_k \in \operatorname{conv} X = C,$$

but $b_\infty \notin C$, since C contains only finite convex combinations of vectors from X (in other words, if some sequence belongs to C, it has only finitely many nonzero terms). Finally, one may apply a technique similar to that of the proof of Theorem 4.4.10 to show that X is hyperconvex. This in turn means that there exists a nonexpansive retraction $R\colon C \to X$. Clearly, R is a compact mapping with X as the fixed-point set. If there existed some closed and convex set $B \subseteq C$ including X, we would have $\overline{\operatorname{conv}}\, X \subseteq \overline{\operatorname{conv}}\, B = B \subseteq C$; but from previous considerations, we have $\overline{\operatorname{conv}}\, X = \overline{C} \not\subseteq C$. △

4.5 Hyperconvexity and Banach spaces

Since the notion of a Banach space permeates the domain of mathematical analysis, a natural question to ask is: when is a Banach space hyperconvex? It turns out that this question may be approached from different directions. One of them is giving concrete examples of hyperconvex Banach spaces (we have already seen that l^∞ is such a space). Another – more complete – is to give (if possible) a necessary and sufficient condition for a Banach space to be hyperconvex. We will pursue both paths in this section.

Since the definition of hyperconvexity is somehow related to the intuition of convexity (and in case of the closed subsets of the real line, the two notions coincide), another natural question arises. In spaces where both notions make sense (for instance, normed spaces), does any of them imply the other? It turns out that the answer is negative. Indeed, if convexity implied hyperconvexity, then every normed space would be hyperconvex, which is not the case even for finite-dimensional Banach spaces. On the other hand, it is easy to show examples of nonconvex sets which are hyperconvex; Example 4.3.1 is one possible source of them. Of course, a negative answer to one question usually spawns more questions, so let us now ask: in which normed spaces are all closed convex sets hyperconvex? In which normed spaces are all hyperconvex sets convex? The latter part of this section is devoted to answering these questions.

4.5.1. Example. The function spaces L^∞ are hyperconvex (for the proof, which is outside the scope of this book, see e.g. [42, p. 1706]). △

We will now state two necessary and sufficient conditions for a Banach space to be hyperconvex. The former will relate hyperconvexity of the space to other functional-analytic properties; the latter will make use of hyperconvexity of the unit ball. We will start with a (classical) definition.

4.5.2. Definition. We call a topological space *extremally disconnected* if the closure of every its open subset is itself open. △

4.5.3. Theorem (Nachbin, Kelley)**.** *A Banach space is hyperconvex if, and only if, it is isometrically isomorphic to a space $\mathcal{C}(K)$ of real continuous functions on an extremally disconnected compact Hausdorff space.* △

(For the proof of this theorem we refer the reader to [42, p. 1712].)

4.5.4. Theorem. *A Banach space is hyperconvex if, and only if, its* (*closed*) *unit ball is hyperconvex.* △

Proof. Hyperconvexity of any closed ball in any hyperconvex space follows from Proposition 4.2.3. Assume now that the closed unit ball (and hence any closed ball) of some Banach space E is hyperconvex. Let $\mathcal{B} := \{\bar{B}_i\}_{i\in I}$ be a family of pairwise intersecting closed balls in E. Since for any finite subfamily $\mathcal{B}'$ of $\mathcal{B}$, the union of balls from $\mathcal{B}'$ is included in some ball (which is hyperconvex), we infer that $\mathcal{B}'$ has a nonempty and admissible (and hence also hyperconvex) intersection. Theorem 4.3.12 yields the nonemptiness of the intersection of the whole $\mathcal{B}$, which completes the proof. □

We will now turn to the problem of when all closed convex subsets of a normed space are hyperconvex. Let us first notice that it makes sense only in case of normed spaces which are themselves hyperconvex (in particular, only Banach spaces), since the whole space is a closed convex set, and that the assumption of closedness is necessary due to the completeness of hyperconvex spaces.

We will make use of the following theorem.

4.5.5. Theorem. *Any nonempty, closed and convex subset of a two-dimensional normed space is a nonexpansive retract of that space.* △

4.5.6. Corollary. *In one- and two-dimensional hyperconvex Banach spaces, nonempty, closed and convex sets are hyperconvex.* △

Proof. The hyperconvexity of nonempty, closed and convex sets of a one-dimensional space follows from Example 4.1.1. In case of two-dimensional hyperconvex spaces it is enough to apply Theorem 4.5.5 and Lemma 3.2.5. □

4.5.7. Remark. Any two-dimensional hyperconvex Banach space is isometrically isomorphic to $\mathbb{R}^2$ with the "maximum" metric. It follows e.g. from [10, Theorem 4.1] or from the Nachbin–Kelley theorem (see e.g. [42]). △

It turns out that the condition from Corollary 4.5.6 is not only sufficient, but also necessary for the hyperconvexity of all nonempty, closed and convex sets. Moreover, the following theorem is true.

4.5.8. Theorem. *Let E be a hyperconvex Banach space. If E is not isometrically isomorphic to $\mathbb{R}^1$ or $\langle \mathbb{R}^2, \|\cdot\|_\infty \rangle$, then it includes a two-dimensional, non-hyperconvex linear subspace.* △

Proof. Observe first that the space of continuous real functions on a compact Hausdorff space K with cardinality at least $n \in \mathbb{N}$ includes an isometrically isomorphic copy of $\mathbb{R}^n$ with the "maximum" norm.

Indeed, it is enough to choose n points $x_1, \dots, x_n \in K$ and their pairwise disjoint neighborhoods $U_1, \dots, U_n$ and span a subspace on functions $f_1, \dots, f_n \colon K \to [0,1]$ such that each f_i vanishes outside U_i and $f_i(x_i) = 1$ for $i = 1, \dots, n$.

We may now assume – applying the Nachbin–Kelley theorem (see Theorem 4.5.3) – that $E = C(K)$ for some compact and extremally disconnected Hausdorff space K. From Remark 4.5.7 we infer that $\dim E > 2$ and hence $\operatorname{card} K > 2$. This means that the space $C(K)$ includes a copy of $\langle \mathbb{R}^3, \|\cdot\|_\infty \rangle$. It is therefore enough to prove the theorem in the case $E = \langle \mathbb{R}^3, \|\cdot\|_\infty \rangle$.

Let $V := \{\langle x_1, x_2, x_3 \rangle \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$. Let us consider the three points: $a_1 := \left\langle -\frac{4}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle, a_2 := \left\langle \frac{2}{3}, -\frac{4}{3}, \frac{2}{3} \right\rangle, a_3 := \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{4}{3} \right\rangle \in V$. The distance between any two of these points is equal to 2. Moreover, $\bar{B}_E(a_1, 1) \cap \bar{B}_E(a_2, 1) \cap \bar{B}_E(a_3, 1) = \left\{ \left\langle -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \right\rangle \right\} \not\subseteq V$, so $\bar{B}_V(a_1, 1) \cap \bar{B}_V(a_2, 1) \cap \bar{B}_V(a_3, 1) = \emptyset$. This means that the subspace V is not hyperconvex. □

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4.5.9. Corollary. *Let E be a normed space. The following conditions are equivalent*:

(a) *each nonempty, closed and convex subset of E is hyperconvex*,

(b) *E is isometrically isomorphic to $\mathbb{R}^1$ or $\langle \mathbb{R}^2, \|\cdot\|_\infty \rangle$*. △

We will now turn our attention to the last of the questions mentioned earlier: when does hyperconvexity of a closed set imply its convexity? First we will recall the notion of a strictly convex metric space.

4.5.10. Definition. We call a metric space X *strictly convex*, if for any points $x, y \in X$ and positive numbers α, β such that $\alpha + \beta = 1$ there exists exactly one point $z \in X$ such that $d(x, z) = \alpha d(x, y)$ and $d(z, y) = \beta d(x, y)$. We will denote this point by a formal expression $\alpha x + \beta y$, without giving any meaning to symbols such as αx or $x + y$. △

4.5.11. Remarks. 1° In case of a normed space, the above definition is equivalent to the classical definition of strict convexity and the symbol $\alpha x + \beta y$ has its usual meaning.

2° Let us notice that for subsets of a strictly convex metric space the notions of total and strict convexity coincide.
3° Any nonempty intersection of any family of totally (and hence strictly) convex subsets of a strictly convex metric space is also totally (and hence strictly) convex. △

It is quite easy to prove that hyperconvex subsets of a strictly convex normed space are convex. We will now show a stronger property, namely that such sets are at most one-dimensional.

4.5.12. Definition. Let $n \in \mathbb{N} \cup \{0\}$. A subset of a normed space is called n-*dimensional*, if it is included in some affine subspace of dimension n and in no affine subspace of dimension less than n. △

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4.5.13. Theorem. *In a strictly convex normed space, all hyperconvex sets are at most one-dimensional.* △

Proof. Let us assume for the sake of contradiction that there exists some hyperconvex and at least two-dimensional subset A of a strictly convex normed space E. It means that A contains some three non-collinear points a, b and c. Put $p := \frac{1}{2}(\|a-b\| + \|b-c\| + \|a-c\|)$ and $r_a := p - \|b-c\|$, $r_b := p - \|a-c\|$, $r_c := p - \|a-b\|$. It is clear that $r_a, r_b, r_c > 0$ and that $\|a-b\| = r_a + r_b$; analogous equalities hold for the remaining distances. Since the space E is strictly convex, we have $\bar{B}_E(a, r_a) \cap \bar{B}_E(b, r_b) = \left\{\frac{r_a}{r_a+r_b}a + \frac{r_b}{r_a+r_b}b\right\}$ and $\bar{B}_E(a, r_a) \cap \bar{B}_E(c, r_c) = \left\{\frac{r_a}{r_a+r_c}a + \frac{r_c}{r_a+r_c}c\right\}$. Since the points a, b, c are not collinear, we obtain $\bar{B}_E(a, r_a) \cap \bar{B}_E(b, r_b) \cap \bar{B}_E(c, r_c) = \emptyset$; the more so, $\bar{B}_A(a, r_a) \cap \bar{B}_A(b, r_b) \cap \bar{B}_A(c, r_c) = \emptyset$, contrary to the assumption that A is a hyperconvex set. □

4.5.14. Corollary. *In a strictly convex normed space all hyperconvex sets are convex.* △

Proof. Hyperconvex sets are connected (as absolute retracts), and for one-dimensional sets connectedness and convexity are equivalent. □

Once again it turns out that the above sufficient condition is also necessary. In order to show that we will need the notion of a metric segment.

4.5.15. Definition. Let X be a metric space, and b a non-negative number. Assume that the mapping $i\colon [0, b] \to X$ is an isometric embedding. We then call the set $i([0, b]) \subseteq X$ a *metric segment (in* X*) joining the points* $i(0)$ *and* $i(b)$. △

4.5.16. Theorem. *Let* E *be a normed space. If* E *is not strictly convex, then it includes a two-dimensional, nonconvex metric segment.* △

In order to prove this theorem we will need the following simple lemma.

4.5.17. Lemma. *Let* X *be a metric space and let* $a, b, c \in X$ *be such that* $d(a, c) + d(c, b) = d(a, b)$*. If there exist metric segments:* S_{ac}*, joining the points* a *and* c*, and* S_{cb}*, joining the points* c *and* b*, then* $S_{ac} \cup S_{cb}$ *is a metric segment joining the points* a *and* b*.* △

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Proof of Theorem 4.5.16. Since E is not strictly convex, there exist pairwise distinct points $a, b, c_1, c_2 \in E$ and positive numbers α, β summing up to 1 such that $\|a - c_1\| = \|a - c_2\| = \alpha\|a - b\|$ and $\|c_1 - b\| = \|c_2 - b\| = \beta\|a - b\|$. This means that $S_1 := [a, c_1] \cup [c_1, b]$ as well as $S_2 := [a, c_2] \cup [c_2, b]$ are metric segments joining the points a and b. Of course they cannot be both convex and the one which is not convex has the desired properties. □

4.5.18. Corollary. *Let* E *be a normed space. The following are equivalent:*

(a) *each nonempty and hyperconvex subset of* E *is convex,*

(b) E *is strictly convex.* △

4.6 Hyperconvex hull

In this section we will discuss the notion of *hyperconvex hull*, first examined by Isbell in the paper [30]. It is a notable fact that (like in the case of convexity) every subset of a hyperconvex space has a minimal hyperconvex superset, although (unlike in the case of convexity) it need not be unique. It is even more interesting that each *metric space* has a minimal hyperconvex superspace, and while it obviously cannot be unique, it can be shown that any two such superspaces are isometric. It also turns out that the notion of hyperconvex hull may be used to formulate some fixed point theorems. Note that, while Isbell's "construction" (the reason for the quotes will be apparent shortly) is not strictly necessary to formulate the fixed point theorems presented here, it is an important milestone in the theory of hyperconvex spaces; moreover, it will be used to establish certain properties of hyperconvex hulls.

4.6.1. Definition. Let $\langle X, d_X \rangle$ be a metric space. The triple $\langle H, d_H, e_H \rangle$, where $\langle H, d_H \rangle$ is a hyperconvex space and $e_H \colon X \to H$ an isometric embedding will be called a *hyperconvex hull* of the space X if the only hyperconvex subset of H including $e_H(X)$ is the space H itself. Instead of $\langle H, d_H, e_H \rangle$ we will usually write $\langle H, e_H \rangle$ or just H, if the form of the function d_H (and possibly e_H) is clear from the context.

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If X is a subspace of some hyperconvex space Z and $\langle H, e_H \rangle$ is its hyperconvex hull such that e_H is an identity embedding and $H \subseteq Z$, we will say that H is a *hyperconvex hull of the set* X *in the space* Z. The family of all hyperconvex hulls of the subset X in a hyperconvex space Z will be denoted by $\mathcal{H}_Z(X)$ or just $\mathcal{H}(X)$ if it does not lead to confusion. △

4.6.2. Remark. Notice that a hyperconvex hull of a subset of a hyperconvex space is simply a minimal hyperconvex superset of the given set. △

With the above definition, let us first restate Corollary 4.3.11 in the language of hyperconvex hulls.

4.6.3. Corollary. *Each subset of a hyperconvex space has a hyperconvex hull in this space.* △

The form of Definition 4.6.1 suggests that we can consider the notion of a hyperconvex hull of *any* metric space, not necessarily one that is itself a subset of some hyperconvex space. A natural question (answered by Isbell) is whether every metric space does possess a hyperconvex hull. The rest of this section will be devoted to answer this question in the affirmative. (In particular, this means that the very general first paragraph of the above definition, while convenient, is not strictly necessary, since any metric space can be embedded in a hyperconvex space anyway.)

We will start by looking at a simple example of a hyperconvex hull.

4.6.4. Example. Let us reuse the notation of Example 4.3.1. We will show that $H_\alpha \in \mathcal{H}(A)$. Let B be a subset of X such that $A \subseteq B \subsetneq H_\alpha$ and $a := \langle a_1, a_2 \rangle \in H_\alpha \setminus B$. From what we have seen in Example 4.3.1 it follows that a is the only point of H_α with the property that $d(\langle 0, 0 \rangle, a) = a_1$ and $d(a, \langle 1, 0 \rangle) = 1 - a_1$. The subspace B cannot therefore be totally convex and hence is not hyperconvex, either, and the proof is finished. △

4.6.5. Remark. It should be fairly obvious by now that a hyperconvex hull of a subset of a hyperconvex space (not to mention one of an arbitrary metric space) need not be unique. Indeed, with the same notation again, taking any two distinct $\alpha_1, \alpha_2 \in [0, 1]$ we obtain distinct hyperconvex hulls of A in X. It is true, however, that if we identify isometric spaces, the hyperconvex hull is indeed unique, although we will have to wait until we know more about hyperconvex spaces to prove it. △

4.6.6. Remark. It is easy to see that if $A \subseteq H \subseteq Z$ and both H and Z are hyperconvex, then $\mathcal{H}_H(A) \subseteq \mathcal{H}_Z(A)$. Yet another glance at Example 4.6.4 reveals that this inclusion may be proper. Moreover, if $H \in \mathcal{H}_H(A)$, then $\mathcal{H}_H(A) = \{H\}$. △

4.6.7. Remark. Any hyperconvex hull is closed (since hyperconvex spaces are complete). This simplifies some considerations in fixed point theory, since there is no need to define a separate notion of a "closed hyperconvex hull". △

Without any further ado, we will now show how a hyperconvex hull of any metric space can be constructed. (Admittedly, we are stretching the word "construct" here, since this "construction" relies heavily on the Axiom of Choice, but at this point we have lost our intuitionist readers a long, long time ago anyway.) We will start with a natural way to associate a function with each point of a metric space.

4.6.8. Definition. Let a be any point of a metric space X. Let us define the function $f_a\colon X \to [0,+\infty)$ by the formula $f_a(x) := d(a,x)$ for $x \in X$. △

4.6.9. Lemma. *Let X be a metric space. For any points $a, b, x, y \in X$, the following properties hold.*

1° *The mapping $a \mapsto f_a$ is injective.*
2° $d(x,y) \leqslant f_a(x) + f_a(y)$.
3° $f_a(x) \leqslant f_a(y) + d(x,y)$ (*in particular, f_a is nonexpansive and hence continuous*).
4° f_a *is "pointwise minimal", in the sense that if $g\colon X \to [0,+\infty)$ also satisfies properties 2° and 3° (with g in place of f_a) and $g \leqslant f_a$, then $g = f_a$.*
5° $d(a,b) = \sup|f_a - f_b|$. △

Proof. The properties 1°–3° are obvious. To prove part 4° let us assume that $g\colon X \to [0,+\infty)$ is a function such that 2° and 3° hold with g in place of f_a and $g \leqslant f_a$, but $g \neq f_a$. This means that there exists some $b \in X$ such that $g(b) < f_a(b)$ – but then $f_a(b) = d(a,b) \leqslant g(a) + g(b) < f_a(a) + f_a(b) = f_a(b)$, which is a contradiction.

In order to prove part 5°, let us notice that for any $x \in X$ we have $|f_a(x) - f_b(x)| = |d(a,x) - d(b,x)| \leqslant d(a,b)$, so $\sup|f_a - f_b| \leqslant d(a,b)$. Moreover, for $x = a$ we have $|f_a(x) - f_b(x)| = f_b(a) = d(a,b)$ and the proof is finished. □

With the above lemma at our disposal – and before further considerations – let us try to develop some intuition. Assume that we want to „enlarge" a metric space $\langle X, d\rangle$ by adding to it some point – let us call it z (of course, we assume that $z \notin X$). This means that we have to extend the metric d to the set $X \cup \{z\}$; in a sense, this means somehow defining the "distance from z" function $f_z = d(z, \cdot)\colon X \cup \{z\} \to [0,+\infty)$ such that the triangle inequality still holds, i.e., the conditions 2° and 3° with z in place of a are fulfilled. This suggests a relationship between possible extensions of a metric space X and sets of functions satisfying these inequalities. Of course, the weak point of this informal reasoning is the problem of extending X by more than one point. However, part 5° of Lemma 4.6.9 gives us a glimmer of hope: for the "distance" functions f_a of points already in X, a supremum distance between them both satisfies the triangle inequality and coincides with the metric we started with. The idea is that we may identify these functions with points, getting an isometric copy of X consisting of *functions*, then add some more functions (corresponding to the "new" points) and define the metric to be the supremum metric between those functions. It turns out that not only are these intuitions correct, but in fact they lead to a "construction" of a hyperconvex hull of the space X – at least, if we are careful about *which* exactly functions we add (perhaps not surprisingly, property 4° will play the crucial role).

4.6.10. Definition. Let X be a metric space. Let us denote by εX the set of all functions $f\colon X \to [0, +\infty)$ such that the conditions 2°–4° of Lemma 4.6.9 with f in place of f_a are satisfied; such functions will be called *extremal*. The mapping $X \ni a \mapsto f_a \in \varepsilon X$ will be denoted by $e_{\varepsilon X}$ (or e, if this does not lead to ambiguity). △

4.6.11. Lemma. *A function* $f\colon X \to [0, +\infty)$ *defined on a metric space* X *is extremal if, and only if,*

1° $d(x, y) \leqslant f(x) + f(y)$ *for* $x, y \in X$;

2° f *is pointwise minimal among nonnegative functions on* X *satisfying the condition* 1°. △

Proof. We will start by showing that a function satisfying assumptions 1°–2° is extremal. Assume that $f(a) > f(b) + d(a, b)$ for some $a, b \in X$. Let us define

$$g(x) := \begin{cases} f(x) & \text{for } x \neq a, \\ f(b) + d(a, b) & \text{for } x = a. \end{cases}$$

Then we have $g \leqslant f$ and $g \neq f$, but it is easily seen that $d(x, y) \leqslant g(x) + g(y)$ for each $x, y \in X$. Indeed, it is enough to consider the case $x = a \neq y$; then $d(x, y) \leqslant d(x, b) + d(b, y) \leqslant d(x, b) + f(b) + f(y) = g(x) + g(y)$, which contradicts the minimality of f. This means that the conditions 2°–3° from Lemma 4.6.9 with f in place of f_a are satisfied. The condition 4° from that lemma follows immediately from 2°.

Let us now assume that $f\colon X \to [0,+\infty)$ is extremal. Then, the condition 1° is satisfied by definition. We will prove condition 2°. Let $\mathcal{F}$ be a family of functions $\hat{f}\colon X \to [0,+\infty)$ satisfying 1°; this family is nonempty, since $f \in \mathcal{F}$. Let $\{\hat{f}_i\}_{i\in I}$ be a chain in $\mathcal{F}$ and let $\tilde{f}(x) := \inf_{i\in I} \hat{f}_i(x)$. Of course, $\tilde{f}(x) \in [0,+\infty)$ for $x \in X$. We will now prove that $\tilde{f} \in \mathcal{F}$. If $d(a,b) > \tilde{f}(a) + \tilde{f}(b)$ for some $a, b \in X$, there would exist a $\hat{f}_{i_1} \in \mathcal{F}$ such that

$$d(a,b) = \tilde{f}(a) + \big(d(a,b) - \tilde{f}(a) - \tilde{f}(b)\big) + \tilde{f}(b) > \hat{f}_{i_1}(a) + \tilde{f}(b)$$

and $\hat{f}_{i_2} \in \mathcal{F}$ such that

$$d(a,b) = \hat{f}_{i_1}(a) + \tilde{f}(b) + \big(d(a,b) - \hat{f}_{i_1}(a) - \tilde{f}(b)\big) > \hat{f}_{i_1}(a) + \hat{f}_{i_2}(b).$$

Since $\{\hat{f}_i\}_{i\in I}$ is a chain, we may assume that e.g. $\hat{f}_{i_1} \leqslant \hat{f}_{i_2}$. We have then $d(a,b) > \hat{f}_{i_1}(a) + \hat{f}_{i_2}(b) \geqslant \hat{f}_{i_1}(a) + \hat{f}_{i_1}(b)$, which contradicts the fact that $\hat{f}_{i_1} \in \mathcal{F}$. Kuratowski–Zorn Lemma yields now the existence of a minimal element $\tilde{f} \leqslant f$ in the set $\mathcal{F}$. From the former part of the proof it is seen that $\tilde{f}$ is extremal. Therefore, if 2° did not hold, we would have $\tilde{f} \leqslant f$ and $\tilde{f} \neq f$, which contradicts the definition of an extremal function. □

In the sequel we will use the above characterization quite often. Other basic properties of extremal functions are described below.

4.6.12. Lemma. *Let $\langle X, d\rangle$ be a metric space.*

1° *Let A be a nonempty subset of X and the function $r\colon A \to [0,+\infty)$ be such that $d(x,y) \leqslant r(x) + r(y)$ for $x, y \in A$. Then there exists its extension $\tilde{r}\colon X \to [0,+\infty)$ such that $d(x,y) \leqslant \tilde{r}(x) + \tilde{r}(y)$ for any $x, y \in X$, and an extremal function f on X less than or equal to $\tilde{r}$.*

2° *For $f \in \varepsilon X$ and $a \in X$ we have $f(a) = \sup|f - f_a|$.*

3° *For any $f \in \varepsilon X$, $\delta > 0$ and $a \in X$ there exists some point $x \in X$ such that $f(a) + f(x) < d(a,x) + \delta$.*

4° *The limit of a uniformly convergent sequence of extremal functions is an extremal function.* △

Proof. 1° Fix $a \in A$ and define

$$\tilde{r}(x) := \begin{cases} d(x,a) + r(a) & \text{for } x \in X \setminus A, \\ r(x) & \text{for } x \in A. \end{cases}$$

If $x, y \in A$, then $d(x,y) \leqslant r(x)+r(y) = \tilde{r}(x)+\tilde{r}(y)$. If $x \in X \setminus A$ and $y \in A$, then $d(x,y) \leqslant d(x,a)+d(a,y) \leqslant d(x,a)+r(a)+r(y) = \tilde{r}(x)+\tilde{r}(y)$. Finally, if $x, y \in X \setminus A$, we have $d(x,y) \leqslant d(x,a) + d(a,y) \leqslant d(x,a) + r(a) + d(y,a) + r(a) = \tilde{r}(x) + \tilde{r}(y)$. The latter part of the thesis follows from Kuratowski–Zorn Lemma as in the proof of Lemma 4.6.11.
2° Since for $a, x \in X$ we have $f_a(x) = d(a,x) \leqslant f(a) + f(x)$ and $f(x) \leqslant f(a) + d(a,x) = f(a) + f_a(x)$, therefore $|f(x) - f_a(x)| \leqslant f(a)$. Hence $\sup|f - f_a| \leqslant f(a)$. On the other hand, for $x = a$ we obtain $|f(x) - f_a(x)| = f(a)$, so $\sup|f - f_a| = f(a)$.
3° Assume for the sake of contradiction that there exist $f \in \varepsilon X$, $a \in X$ and $\delta > 0$ such that for any $x \in X$ the inequality $d(a,x)+\delta \leqslant f(a)+f(x)$ holds. We may assume that $\delta \leqslant f(a)$. Let us define – for $x \in X$ – the function

$$g(x) := \begin{cases} f(x) & \text{for } x \neq a, \\ f(a) - \delta & \text{for } x = a. \end{cases}$$

For $x, y \in X$ we have $d(x,y) \leqslant g(x) + g(y)$. (It is again enough to consider the case $x = a \neq y$; we have then $d(x,y) \leqslant f(x) + f(y) - \delta = g(x) + g(y)$.) Moreover, $0 \leqslant g \leqslant f$ and $g \neq f$, which contradicts the minimality of f.
4° Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of extremal functions on X, uniformly convergent to some function $f \colon X \to [0,+\infty)$. Obviously, $d(x,y) \leqslant f(x) + f(y)$ for $x, y \in X$. Assume that f is not pointwise minimal; this means that there exists some function $g \colon X \to [0,+\infty)$ different from f such that $g \leqslant f$ and $d(x,y) \leqslant g(x) + g(y)$ for $x, y \in X$. Let $a \in X$ be the point where $g(a) < f(a)$ and let $\delta := f(a) - g(a)$. Choose $N \in \mathbb{N}$ such that $\sup|f_N - f| < \frac{\delta}{3}$. From part 3° we know that there exists a point $x \in X$ such that $f_N(a) + f_N(x) < d(a,x) + \frac{\delta}{3}$. Thus we obtain $g(a)+g(x) = f(a)-\delta+g(x) \leqslant f(a)+f(x)-\delta \leqslant f_N(a)+\frac{\delta}{3}+f_N(x)+\frac{\delta}{3} - \delta = f_N(a) + f_N(x) - \frac{\delta}{3} < d(a,x)$, which is a contradiction. □

4.6.13. Remark. Notice that part 2° of the above lemma is a generalization of the formula 5° from Lemma 4.6.9. △

By far we have only examined the properties of isolated extremal functions. Now the time has come to introduce a metric in the set of all of them and prove some properties of the space of extremal functions.

4.6.14. Definition. Let X be a metric space. The distance between two extremal functions $f, g \in \varepsilon X$ will be defined by the formula $d(f, g) := \sup |f - g|$. △

4.6.15. Remark. The fact that the abovementioned function d is actually a metric is obvious. The only nontrivial part is the *finiteness* of d (recall that extremal functions need not be bounded!). This fact – and some other properties of εX treated as a metric space – is the content of the next lemma. △

4.6.16. Lemma. *Let* X *be a metric space.*

1° *The function* $d\colon \varepsilon X \times \varepsilon X \to [0, +\infty)$ *from Definition 4.6.14 is a metric on* εX *and the mapping* $e\colon X \to \varepsilon X$ (*see Definition 4.6.10*) *is an isometric embedding.*

2° *If* $f \in \varepsilon X$, *then* $f \leqslant \operatorname{diam} X$; *in particular,* $\operatorname{diam} \varepsilon X = \operatorname{diam} X$.

3° *If* X *is compact, then so is* εX.

4° *If* s *is an extremal function on* εX, *then* $s \circ e$ *is an extremal function on* X. △

Proof. 1° It is enough to show that the supremum in the definition of d cannot be infinite. Let $f, g \in \varepsilon X$. Choose $a \in X$. By part 2° of Lemma 4.6.12 we have: $d(f, g) = \sup |f - g| \leqslant \sup(|f - f_a| + |f_a - g|) \leqslant \sup |f - f_a| + \sup |g - f_a| = f(a) + g(a) < +\infty$. (The latter part of the thesis is equivalent to part 5° of Lemma 4.6.9.)

2° If X is unbounded, there is nothing to prove. Assume therefore that X is bounded. If it were true that $f(a) > \operatorname{diam} X$ for some $a \in X$, then by putting $h(x) := \min\{f(x), \operatorname{diam} X\}$ for $x \in X$ we would obtain $h \leqslant f$, $h \neq f$ and $d(x, y) \leqslant h(x) + h(y)$ for any $x, y \in X$, which contradicts the extremality of f. Hence for any $f \in \varepsilon X$, $x \in X$ we have $f(x) \in [0, \operatorname{diam} X]$ and consequently $\operatorname{diam} \varepsilon X \leqslant \operatorname{diam} X$. The

inequality in the opposite direction follows from the latter part of item 1°.
3° Each function $f \in \varepsilon X$ is nonexpansive, so εX is a family of uniformly equicontinuous functions. From part 2° it follows that all the extremal functions on X are uniformly bounded. Using the Arzelà–Ascoli Theorem and part 4° of Lemma 4.6.12 we obtain the compactness of εX in $C(X)$.
4° Let s be an extremal function on εX (that is, $s \in \varepsilon\varepsilon X$). For $x, y \in X$ we have $d(x,y) = d(f_x, f_y) \leqslant s(f_x) + s(f_y) = (s \circ e)(x) + (s \circ e)(y)$. It remains to show that $s \circ e$ is minimal. Assume that there exists an extremal function $h \in \varepsilon X$ less than or equal to $s \circ e$ and a point $a \in X$ such that $h(a) < (s \circ e)(a)$. Let us define the function $t\colon \varepsilon X \to [0, +\infty)$ by the formula:

$$t(f) := \begin{cases} s(f) & \text{for } f \neq f_a, \\ h(a) & \text{for } f = f_a. \end{cases}$$

It is easily seen that $t \leqslant s$ and $t \neq s$. Let us show that $d(f,g) \leqslant t(f) + t(g)$ for $f, g \in \varepsilon X$, thus arriving at a contradiction with the extremality of s.
It is enough to consider the case $f \neq f_a = g$. Choose any $\delta > 0$. From part 3° of Lemma 4.6.12 we know that there exists some $b \in X$ such that $f(a) + f(b) < d(a,b) + \delta$. Let us now consider two cases.

a) If $a = b$, from the above inequality we get $f(a) < \frac{\delta}{2}$; then $d(f, f_a) = f(a) \leqslant t(f) + t(f_a) + \frac{\delta}{2}$.
b) If $a \neq b$, then $d(f, f_a) + f(b) - \delta = f(a) + f(b) - \delta < d(a,b) \leqslant h(a) + h(b) \leqslant h(a) + (s \circ e)(b) = t(f_a) + t(f_b)$. Since s is an extremal function, we have $t(f_b) = s(f_b) \leqslant s(f) + d(f, f_b) = t(f) + f(b)$. Adding the sides of both inequalities we see that $d(f, f_a) + f(b) - \delta + t(f_b) < t(f_a) + t(f_b) + t(f) + f(b)$, or equivalently, $d(f, f_a) < t(f) + t(f_a) + \delta$.

In both cases – as $\delta > 0$ – we arrive at the thesis. □

In the sequel we will see that the property 3° of the above lemma is a special case of part 4° of Lemma 5.4.5. It is interesting, since it is an exact analogue of the Mazur's Theorem known from functional analysis.

The main result of this section is the following theorem, proved by Isbell. It states the affirmative answer to the problem of existence of a hyperconvex hull of an arbitrary metric space.

4.6.17. Theorem. *For any metric space* X*, the space* εX *is a hyperconvex hull of* X*.* △

Proof. We will show first that εX is a hyperconvex space. Let $\{f_i\}_{i\in I}$ be some subset of εX and $\{r_i\}_{i\in I}$ a set of nonnegative numbers such that $d(f_i, f_j) \leqslant r_i + r_j$ for $i, j \in I$. Let us define the mapping $r\colon \{f_i\}_{i_I} \to [0, +\infty)$ by the formula $r(f_i) := r_i$. Applying part 1° of Lemma 4.6.12 we obtain the existence of an extension $\tilde{r}\colon \varepsilon X \to [0, +\infty)$ of the function r satisfying the condition $d(f, g) \leqslant \tilde{r}(f) + \tilde{r}(g)$ for $f, g \in \varepsilon X$ and an extremal function $s \in \varepsilon\varepsilon X$ less than or equal to $\tilde{r}$. From part 4° of Lemma 4.6.16 we infer that $s \circ e \in \varepsilon X$. Let $f \in \varepsilon X$ be arbitrary. For any $x \in X$ we have: $(s \circ e)(x) - f(x) = s(f_x) - d(f, f_x) \leqslant s(f) \leqslant \tilde{r}(f)$ and $f(x) - (s \circ e)(x) = d(f, f_x) - s(f_x) \leqslant s(f) \leqslant \tilde{r}(f)$, and hence $|(s \circ e)(x) - f(x)| \leqslant \tilde{r}(f)$. Since $x \in X$ was arbitrary, we obtain $d(s \circ e, f) = \sup |s \circ e - f| \leqslant \tilde{r}(f)$. Therefore, $s \circ e \in \bar{B}\big(f, \tilde{r}(f)\big)$ for every function $f \in \varepsilon X$ and in particular $s \circ e \in \bigcap_{i\in I} \bar{B}(f_i, r_i)$; hence this intersection is nonempty.

Assume now that Y is a hyperconvex space such that $e(X) \subseteq Y \subseteq \varepsilon X$. From Corollary 3.2.4 we know that there exists a nonexpansive retraction $R\colon \varepsilon X \to Y$. Let $f \in \varepsilon X$. We have $R(f)(x) = d\big(R(f), f_x\big) = d\big(R(f), R(e(x))\big) \leqslant d\big(f, e(x)\big) = d(f, f_x) = f(x)$ for any $x \in X$. Hence $R(f) = f$ for each $f \in \varepsilon X$, which means that the retraction $R\colon \varepsilon X \to Y$ is an identity mapping and $Y = \varepsilon X$. □

4.6.18. Remark. We will now see how this fact in a sense renders our proofs of Theorem 3.1.18 (that spaces that allow extending nonexpansive mappings are hyperconvex) obsolete: equipped with our knowledge about hyperconvex hulls, we may prove it in just a few lines. △

Proof III of Theorem 3.1.18. Obviously, any space Y satisfying the assumptions is an ANR. This means that there exists a nonexpansive retraction from a hyperconvex space εY onto Y, and hence Y is also hyperconvex. □

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4.6.19. Remark. Let us notice that $e_{\varepsilon X}\colon \varepsilon X \to \varepsilon\varepsilon X$ is an isometry. Indeed, by part 1° of Lemma 4.6.16 it is enough to show that $e_{\varepsilon X}$ is a surjection – but this is an immediate consequence of the definition of a hyperconvex hull and the hyperconvexity of εX. △

4.6.20. Corollary. *The class of hyperconvex metric spaces coincides with the class of absolute nonexpansive retracts.* △

Proof. Let X be an absolute nonexpansive retract; the subspace $e(X)$ of the space εX obviously has the same property. There must exist a nonexpansive retraction $R\colon \varepsilon X \to e(X)$ of the hyperconvex space εX onto $e(X)$. Theorem 3.2.5 yields the hyperconvexity of $e(X)$ – and hence X. On the other hand, Corollary 3.2.4 shows that each hyperconvex space is an absolute nonexpansive retract. □

We have already seen a simple example of a hyperconvex hull (see Example 4.6.4). We will finish this section with another one, this time less trivial. Let us consider the set c_0 of real sequences convergent to zero, considered as a subset of the space l^∞ of bounded real sequences.

4.6.21. Example. There exists only one hyperconvex hull of c_0 in l^∞ – the whole l^∞. To prove this, let us choose any point $a \in l^\infty$ and construct a family of balls in the following way. For each $x \in c_0$, we consider the ball $\bar{B}_{l^\infty}(x, \|a - x\|_\infty)$. The triangle inequality implies that the distance of the centers of any two balls from this family does not exceed the sum of their radii. Define now the set

$$J(a) := \bigcap_{x \in c_0} \bar{B}_{l^\infty}(x, \|a - x\|_\infty).$$

It is obviously nonempty, since $a \in J(a)$. What is more important, any hyperconvex hull H of c_0 in l^∞ must contain at least one point in $J(a)$, for otherwise we would have a family of pairwise intersecting balls in H with an empty intersection. We will now proceed to show that the set $J(a)$ is a singleton for any $a \in l^\infty$, which will complete the proof.

Let m be a natural number. For each $n \in \mathbb{N}$ we define

$$l_{m,n} := \begin{cases} a_n - \|a\|_\infty & \text{if } n \leqslant m, \text{ and} \\ 0 & \text{otherwise;} \end{cases}$$

$$u_{m,n} := \begin{cases} a_n + \|a\|_\infty & \text{if } n \leqslant m, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $l_m := \langle l_{m,n} \rangle_{n=1}^\infty$ and $u_m := \langle u_{m,n} \rangle_{n=1}^\infty$ both belong to c_0. Also, $\|a - l_m\|_\infty = \|a - u_m\|_\infty = \|a\|_\infty$. Let us consider the set

$$A := \bigcap_{m=1}^\infty \left[\bar{B}\big(l_m, \|a\|_\infty\big) \cap \bar{B}\big(u_m, \|a\|_\infty\big) \right].$$

It is clear that $J(a) \subseteq A$. Take any sequence $b := \langle b_n \rangle_{n=1}^\infty \in A$ and fix $m = n \in \mathbb{N}$ for a moment. Since $b \in \bar{B}\big(l_m, \|a\|_\infty\big)$, we have

$$\big| b_n - a_n + \|a\|_\infty \big| \leqslant \|a\|_\infty,$$

and since $b \in \bar{B}\big(u_m, \|a\|_\infty\big)$, we have

$$\big| b_n - a_n - \|a\|_\infty \big| \leqslant \|a\|_\infty.$$

These two inequalities imply that $b_n = a_n$. Since n was arbitrary, we have proved that $b = a$ and hence $a \in J(a) \subseteq A = \{a\}$ and the proof is finished. △

We will finish this section with a proof of the fact that that hyperconvex hulls are unique up to an isometry, hinted at in Remark 4.6.5, and a corollary regarding extremal functions.

4.6.22. Theorem. *Let X be a metric space and $\langle H, e_H \rangle$ any of its hyperconvex hulls (in the sense of Definition 4.6.1). There exists an isometry $i \colon \varepsilon X \to H$ such that $e_H^{-1} \circ i \circ e$ is an identity mapping on X.* △

Proof. Let $i_1 \colon e(X) \to H$ be defined by the formula $i_1\big(e(x)\big) := e_H(x)$ for $x \in X$; since $e \colon X \to \varepsilon X$ is injective, i_1 is well defined. Moreover, i_1 is an isometric embedding, since for $x_1, x_2 \in X$ we have the equalities $d\big(i_1(e(x_1)), i_1(e(x_2))\big) = d\big(e_H(x_1), e_H(x_2)\big) = d(x_1, x_2) = d\big(e(x_1), e_H(x_2)\big)$; in particular, the mapping i_1 is nonexpansive. From Corollary 3.1.17 there exists an extension of i_1 to a nonexpansive mapping $\tilde{i}_1 \colon \varepsilon X \to H$. In a similar way we show the existence of a nonexpansive mapping $\tilde{i}_2 \colon H \to \varepsilon X$ with the property that $\tilde{i}_2|_{e_H(X)} = e \circ e_H^{-1}$. Let us consider the mapping $T \colon \varepsilon X \to \varepsilon X$ defined by the formula

$T := \tilde{\imath}_2 \circ \tilde{\imath}_1$. Of course, it is a nonexpansive mapping of the metric space εX into itself; moreover, the restriction $T|_{e(X)}$ is the identity mapping on $e(X)$ (indeed, for $y_1 \in e(X)$ we have $T(y_1) = \tilde{\imath}_2 \circ \tilde{\imath}_1(y_1) = \tilde{\imath}_2 \circ i_1(y_1) = \tilde{\imath}_2 \circ e_H \circ e^{-1}(y_1) = e \circ e_H^{-1} \circ e_H \circ e^{-1}(y_1) = y_1$).

We claim that T is the identity mapping of εX. Indeed, let $f \in \varepsilon X$ and $g := T(f)$. For any point $x \in X$ we have $g(x) = d\big(g, f_x\big) = d\big(T(f), T(f_x)\big) \leqslant d(f, f_x) = f(x)$. Since x was arbitrary, we arrive at the conclusion that g is an extremal function less than or equal to f, and hence $g = f$.

Let us now put $i := \tilde{\imath}_1$. It is easily seen that $e_H^{-1} \circ i \circ e$ is the identity on X. If i were not an isometry, then for some $x_1, x_2 \in X$ we would have $d\big(i(x_1), i(x_2)\big) < d(x_1, x_2)$ (since i is nonexpansive); but then, as $\tilde{\imath}_2$ is also nonexpansive, we would have $d\big(T(x_1), T(x_2)\big) = d\big(\tilde{\imath}_2(i(x_1)), \tilde{\imath}_2(i(x_2))\big) \leqslant d\big(i(x_1), i(x_2)\big) < d(x_1, x_2)$ – this contradicts the fact that T is the identity and the proof is finished. □

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The above Corollary has one especially nice consequence. It will not be used in the sequel, although it seems elegant enough to justify its inclusion here.

4.6.23. Proposition. *Let* A *be a subset of a hyperconvex space* H *and let* B *be a hyperconvex hull of* A *in* H. *Then, for each* $b \in B$, *the function* $f_b|_A := d(b, \cdot\,)$ *is an extremal function on* A, *and all extremal functions on* A *are of such form.* △

Proof. Obviously, $d(x, y) \leqslant f_b(x) + f_b(y)$ for all $x, y \in A$, so we only have to prove the minimality of $f_b|_A$. From Theorem 4.6.22 we know that there exists an isometry $i \colon \varepsilon A \to B$ such that $i \circ e = I_A$. Consider now the mapping $f_b \circ i \colon \varepsilon A \to [0, +\infty)$. Let $g, h \in \varepsilon A$ be arbitrary; clearly, $\|g - h\|_\infty = d\big(i(g), i(h)\big) \leqslant f_b(i(g)) + f_b(i(h)) = f_b \circ i(g) + f_b \circ i(h)$. Assume that $f_b \circ i$ is not minimal; then there exists some $u \colon \varepsilon A \to [0, +\infty)$ satisfying $\|g - h\|_\infty \leqslant u(g) + u(h)$ for all $g, h \in \varepsilon A$ and $u \leqslant f_b \circ i$ while $u \neq f_b \circ i$. Let $\tilde{f}_b := u \circ i^{-1} \colon B \to [0, +\infty)$. For any $x, y \in B$; denote $g := i^{-1}(x) \in \varepsilon A$ and $h := i^{-1}(y) \in \varepsilon A$. It is obvious that $d(x, y) = \|g - h\|_\infty \leqslant u(g) + u(h) = \tilde{f}_b(x) + \tilde{f}_b(y)$. Since f_b is minimal on B, it must be the case that $\tilde{f}_b \geqslant f_b$. On the other hand,

$\tilde{f}_b = u \circ i^{-1} \leqslant f_b \circ i \circ i^{-1} = f_b$ and – as $u \neq f_b \circ i$ – this inequality must be strict at some point. Thus we have arrived at a contradiction.

Now that we know that $f_b \circ i$ is extremal on εA, we apply part 4° of Lemma 4.6.16 and see that $(f_b \circ i) \circ e = f_b \circ (i \circ e) = f_b|_A$ is also extremal.

Assume now that there exists an extremal function $f \colon A \to [0, +\infty)$ not of the form $f_b|_A$ for some $b \in B$. This would mean that $e(A) \subseteq e(B) \subsetneq \varepsilon A$, which would contradict Definition 4.6.1. □

4.6.24. Corollary. *Every extremal function on a hyperconvex space* H *is of the form* f_a *for some point* $a \in H$. △

4.7 ℝ-trees

In this section we will discuss an interesting class of hyperconvex spaces. ℝ-trees are a natural generalization of more "discrete" trees known from graph theory and seem to appear in a few branches of mathematics. It turns out that complete ℝ-trees are hyperconvex, and that they are precisely these hyperconvex spaces where hyperconvex hulls are unique. Let us start with the definitions.

4.7.1. Definition. We will say that a metric space $\langle X, d\rangle$ has *unique metric segments*, if for any points $p, q \in X$ there exists exactly one metric segment in X joining the points p and q. We will then denote it by $[p, q]_d$. △

4.7.2. Definition. A metric space $\langle T, d\rangle$ is called an ℝ-*tree*, if

1° T has unique metric segments,

2° for any points $p, q, r \in T$ there exists some $s \in T$ such that $[p, q]_d \cap [p, r]_d = [p, s]_d$, and

3° if $p, q, r \in T$ and $[p, q]_d \cap [q, r]_d = \{q\}$, then $[p, q]_d \cup [q, r]_d = [p, r]_d$. △

In 1998 W. A. Kirk gave the following characterization.

4.7.3. Theorem. *Let* X *be a metric space. The following conditions are equivalent*:

(a) X *is a complete* $\mathbb{R}$*-tree,*
(b) X *is hyperconvex and has unique metric segments.* △

In order to prove the above theorem, we shall need a few more notions and a couple of lemmas (*cf.* [37]).

4.7.4. Definitions. A family of subsets $\mathcal{F}$ of some metric space X is called *normal*, if for any set $A \in \mathcal{F}$ with positive diameter, the inequality $r_A(A) < \operatorname{diam}(A)$ holds. If $r_A(A) < c \operatorname{diam} A$ for some constant $c \in \left[\frac{1}{2}, 1\right)$ and all A's with positive diameter, we call $\mathcal{F}$ *uniformly normal*. A family of sets is said to have the *finite intersection property* if any its finite subfamily has a nonempty intersection. Lastly, the family $\mathcal{F}$ is called (*countably*) *compact*, if every (countable) family of nonempty sets in $\mathcal{F}$ having the finite intersection property has a nonempty intersection. △

4.7.5. Lemma. *Let* X *be a complete metric space for which* $\mathcal{A}(X)$ *is uniformly normal. Then,* $\mathcal{A}(X)$ *is countably compact.* △

This is easily deduced from [33, p. 726, Theorem 13]; since the proof has little to do with the theory of hyperconvex spaces, we will omit it.

4.7.6. Lemma. *Let* X *be a metric space such that the family* $\mathcal{A}(X)$ *of its admissible subsets is countably compact and normal. Then, the family* $\mathcal{A}(X)$ *is compact.* △

4.7.7. Lemma. $\mathbb{R}$*-trees are strictly convex.* △
Proof. It is enough to apply the uniqueness of metric segments and Lemma 4.5.17. □

Proof of Theorem 4.7.3. Let X be a complete $\mathbb{R}$-tree. We will start with a proof that the family of admissible subsets of an $\mathbb{R}$-tree is uniformly normal (and hence compact). Choose $\varepsilon \in (0,1)$ and $D \in \mathcal{A}(X)$ such that $\delta := \operatorname{diam} D > 0$. Fix $u, v \in D$ such that $d(u,v) > (1-\varepsilon)\delta$ and pick any $x \in D$. Denote by w the point in X satisfying the equality $[u,v]_d \cap [u,x]_d = [u,w]_d$. Then $d(x,u) = d(x,w) + d(w,u) \leqslant \delta$, and –

since $[w,v]_d \cap [w,x]_d = \{w\}$ and hence $[x,v]_d = [x,w]_d \cup [w,v]_d$ – also $d(x,v) = d(x,w) + d(w,v) \leqslant \delta$. Let m be the midpoint of the segment $[u,v]_d$, i.e., the unique point $m \in [u,v]_d$ such that $d(m,u) = d(m,v)$. Now either $w \in [u,m]_d$, or $w \in [m,v]_d$. In the former case, we have $\delta \geqslant d(x,v) = d(x,m) + d(m,v) > d(x,m) + \frac{1}{2}(1-\varepsilon)\delta$, so $d(x,m) < \frac{1}{2}(1+\varepsilon)\delta$. In the latter case, $\delta \geqslant d(x,u) = d(x,m) + d(m,u) > d(x,m) + \frac{1}{2}(1-\varepsilon)\delta$ and again $d(x,m) < \frac{1}{2}(1+\varepsilon)\delta$. Either way, we conclude that $D \subseteq \bar{B}\big(m, \frac{1}{2}(1+\varepsilon)\delta\big)$. Since balls in $\mathbb{R}$-trees are totally convex (this follows directly from parts 2° and 3° of Definition 4.7.2) and $D \in \mathcal{A}(X)$, we have $m \in D$, so $r_D(D) \leqslant \frac{1}{2}(1+\varepsilon)\delta$ for any $\varepsilon \in (0,1)$. This means that $r_D(D) \leqslant \frac{1}{2}\delta$ – in other words, $\mathcal{A}(X)$ is uniformly normal as claimed.

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Because of Lemmas 4.7.5, 4.7.6 and 4.7.7, it is now enough to prove that $\mathcal{A}(X)$ has the finite intersection property. Obviously, any two-element, pairwise intersecting ball family has a nonempty intersection. Assume that it is true also for any collection of n balls. Take any family $\{\bar{B}(x_i, r_i)\}_{i=1}^{n+1}$ of pairwise intersecting balls. Let $S := \bigcap_{i=1}^{n} \bar{B}(x_i, r_i)$ – by the inductive hypothesis, S is nonempty – and pick $p \in S$. Of course, $d(x_{n+1}, p) > r_{n+1}$. Choose $t \in [x_{n+1}, p]_d$ such that $d(x_{n+1}, t) = r_{n+1}$.

Let $i \in \{1, \ldots, n\}$. It might happen that $t \in [x_i, p]_d$. Then we would have $d(x_i, t) \leqslant d(x_i, p) \leqslant r_i$ and $t \in \bar{B}(x_i, r_i) \cap \bar{B}(x_{n+1}, r_{n+1})$. On the other hand, if $t \notin [x_i, p]_d$, then $[x_i, t]_d \cap [t, x_{n+1}]_d = \{t\}$, so $[x_i, t]_d \cup [t, x_{n+1}]_d = [x_i, x_{n+1}]_d$ and hence $t \in [x_{n+1}, x_i]_d$. This would mean that $d(x_i, t) = d(x_i, x_{n+1}) - r_{n+1} \leqslant r_i$ and again $t \in \bar{B}(x_i, r_i) \cap \bar{B}(x_{n+1}, r_{n+1})$. Since i was arbitrary, either way we conclude that $\bigcap_{i=1}^{n+1} \bar{B}(x_i, r_i) \neq \emptyset$.

Assume now that X is a hyperconvex metric space with unique metric segments. We will first prove condition 2° from Definition 4.7.2. Let $p, q, r \in X$ and define w to be the point of $[p,q]_d \cap [p,r]_d$ nearest to q (its existence follows from a usual compactness argument and the uniqueness from the properties of the real line and the observation that $[p,q]_d$ – and hence also $[p,q]_d \cap [p,r]_d$ – may be viewed as a subset of $\mathbb{R}$). From the uniqueness of metric segments and the fact that $w \in [p,q]_d \cap [p,r]_d$, we infer that $[p,w]_d \subseteq [p,q]_d \cap [p,r]_d$; from the choice of w we know that actually $[p,w]_d = [p,q]_d \cap [p,r]_d$.

It remains to show that the condition 3° from Definition 4.7.2 holds. Let us suppose again that $p, q, r \in X$, but this time, $[p, q]_d \cap [q, r]_d = \{q\}$. Without loss of generality we may assume that $d(q, r) \leqslant d(q, p)$. Let s be the point of $[r, p]_d \cap [r, q]_d$ nearest to q (in view of the preceding paragraph, this means that $[r, p]_d \cap [r, q]_d = [r, s]_d$).

Assume now that the equality $[p, s]_d = [p, q]_d \cup [q, s]_d$ is true. Then,

$$[p, r]_d = [p, s]_d \cup [s, r]_d = [p, q]_d \cup [q, s]_d \cup [s, r]_d = [p, q]_d \cup [q, r]_d.$$

On the other hand, imagine that $[p, s]_d \neq [p, q]_d \cup [q, s]_d$ (in particular, $s \neq q$). Then we have $d(p, s) < d(p, q) + d(q, s)$ (for if this were not the case, we would arrive at a contradiction with Lemma 4.7.7). Denote $\eta := d(p, q) + d(q, s) - d(p, s) > 0$ and let m be the midpoint of the segment $[q, s]_d$. Consider the following three closed balls:

$$\begin{aligned} \bar{B}_1 &:= \bar{B}\big(q, \tfrac{1}{2}d(q, s)\big), \\ \bar{B}_2 &:= \bar{B}\big(s, \tfrac{1}{2}d(q, s)\big), \\ \bar{B}_3 &:= \bar{B}\big(p, d(p, q) - \tfrac{1}{2}d(q, s)\big). \end{aligned}$$

Again by Lemma 4.7.7, $\bar{B}_1 \cap \bar{B}_2 = \{m\}$ and $\bar{B}_1 \cap \bar{B}_3 = \{u\}$ for some $u \in [p, q]_d$. Moreover, $\bar{B}_2 \cap \bar{B}_3 \neq \emptyset$. Indeed, if $\bar{B}_2$ intersected with $\bar{B}_3$, then by the hyperconvexity of X we would have $\bar{B}_1 \cap \bar{B}_2 \cap \bar{B}_3 \neq \emptyset$ and hence $m = u \in [p, q]_d \cap [q, s]_d \subseteq [p, q]_d \cap [q, r]_d = \{q\}$, so $m = q$, which is a contradiction.

As $\bar{B}_2 \cap \bar{B}_3 = \emptyset$, it follows that

$$d(p, s) > \tfrac{1}{2}d(q, s) + \big(d(p, q) - \tfrac{1}{2}d(q, s)\big) = d(p, q).$$

Again we consider three balls in X:

$$\begin{aligned} \bar{B}'_1 &:= \bar{B}(q, \eta), \\ \bar{B}'_2 &:= \bar{B}\big(s, d(p, s) - d(p, q)\big), \\ \bar{B}'_3 &:= \bar{B}\big(p, d(p, q)\big). \end{aligned}$$

We have $\bar{B}'_2 \cap \bar{B}'_3 = \{z_1\} \subseteq [p, s]_d$. Also, $\bar{B}'_1 \cap \bar{B}'_2 = \{z_2\} \subseteq [q, s]_d$. Finally, $q \in \bar{B}'_1 \cap \bar{B}'_3$. Therefore, $\bar{B}'_1 \cap \bar{B}'_2 \cap \bar{B}'_3 \neq \emptyset$, so $z_1 = z_2 =: z$ and we have $z \in [s, p]_d \cap [s, q]_d \subseteq [r, p]_d \cap [r, q]_d$. By choice of s, it must be $z = s$; but $z \in \bar{B}'_3$, so $d(p, z) \leqslant d(p, q) < d(p, s)$, which is a contradiction. $\square$

4

Let us now turn to the characterization of the hyperconvex spaces in which each subset has a unique hyperconvex hull as $\mathbb{R}$-trees.

4.7.8. Definition. A nonempty subset of an $\mathbb{R}$-tree T is called a *sub-$\mathbb{R}$-tree* (*of the* $\mathbb{R}$*-tree* T) if it is an $\mathbb{R}$-tree as a subspace. △

4.7.9. Lemma. *Let* S *be a nonempty subset of some* $\mathbb{R}$*-tree* T*. Then* S *is a sub-*$\mathbb{R}$*-tree of* T *if, and only if, for any two points* $p, q \in S$*, the metric segment* $[p, q]_d$ *is included in* S*.* △

Proof. The necessity is an obvious consequence of the definition of an $\mathbb{R}$-tree. Let us prove the sufficiency. Existence of metric segments follows from the assumption about S, and their uniqueness from the assumption that T is an $\mathbb{R}$-tree. The existence of the point s from condition 2° of Definition 4.7.2 is inferred from the fact that if $p, q, r \in S$, then the point $s \in T$, existing because of the assumption on T, belongs to $[p, q]_d \cap [p, r]_d \subseteq S$. Finally, the condition 3° is satisfied because $[p, q]_d \subseteq S$ for $p, q \in S$. □

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4.7.10. Lemma. *Let* H *be a hyperconvex space and* $x, y \in H$ *two of its points. Then each hyperconvex hull of the set* $\{x, y\}$ *in the space* H *is a metric segment joining the points* x *and* y*.* △

Proof. Let $Y_1 := [0, d(x, y)] \subseteq \mathbb{R}$. Choose $\langle Y_2, e_2\rangle \in \mathcal{H}_H\{x, y\}$; in particular, this means that $e_2(x) = x$ and $e_2(y) = y$. Define $e_1 \colon \{x, y\} \to Y_1$ by the formulae $e_1(x) := 0$ and $e_1(y) := d(x, y)$. Using Example 4.1.1 we infer that $\langle Y_1, e_1\rangle$ is a hyperconvex hull of $\{x, y\}$. Theorem 4.6.22 guarantees the existence of an isometry $i \colon Y_1 \to Y_2$ such that $e_2^{-1} \circ i \circ e_1$ is an identity on $\{x, y\}$. This means that i is an isometry of the interval $[0, d(x, y)]$ onto the set Y_2 such that $i(0) = e_2(x) = x$ and $i(d(x, y)) = e_2(y) = y$ or $i(0) = e_2(y) = y$ and $i(d(x, y)) = e_2(x) = x$. In either case the set Y_2 is a metric segment in H joining the points x and y. □

4.7.11. Lemma. *Let* C *be a nonempty subset of an* $\mathbb{R}$*-tree* T*. The following conditions are equivalent*:

(a) C *is hyperconvex*,

(b) C *is closed and totally convex.* △

Proof. The necessity follows immediately from Lemma 3.1.9 and Proposition 4.1.2. In order to prove the sufficiency, let us notice that every closed and totally convex subset $C \subseteq T$ is a complete sub-$\mathbb{R}$-tree of T. In fact, in view of Lemma 4.7.9 it is enough to prove that for any $x, y \in C$, the metric segment $[x, y]_d$ is included in C. From the strict convexity of T we infer that $[x, y]_d = \{\alpha x + \beta y | \alpha, \beta \geqslant 0, \alpha + \beta = 1\} \subseteq C$. It remains now to apply Theorem 4.7.3. □

4.7.12. Theorem. *Let* X *be a hyperconvex space. The following conditions are equivalent*:

(a) *for any nonempty subset* $A \subseteq X$ *there exists exactly one hyperconvex hull of* A *in* X,

(b) X *is an* $\mathbb{R}$*-tree.* △

Proof. The necessity follows from Theorem 4.7.3 and Lemma 4.7.10. For the proof of sufficiency let us observe that

$$\begin{aligned}\Sigma &:= \{C \subseteq X \mid A \subseteq C \text{ and } C \text{ is hyperconvex}\}\\ &= \{C \subseteq X \mid A \subseteq C \text{ and } C \text{ is closed and totally convex}\}\\ &= \{C \subseteq X \mid A \subseteq C \text{ and } C \text{ is closed and strictly convex}\},\end{aligned}$$

where we use subsequently Lemma 4.7.11 and part 2° of Remarks 4.5.11. Now part 3° of those Remarks, together with Lemma 4.7.11, show that $\bigcap \Sigma$ is the smallest (and hence the unique minimal) hyperconvex superset of A. □

Notes and remarks

A few things in this chapter – like the hyperconvexity of the real line (Example 4.1.1), or the fact that arbitrary intersections do not preserve hyperconvexity (Example 4.3.1) – are folklore. Completeness of hyperconvex spaces (Proposition 4.1.2) was established in [2]. The notion of admissible subset is also contained (although implicitly) in [2]. Propositions 4.2.3 and 4.2.6 (even for a more general class of subsets than the admissible ones, so-called *externally hyperconvex subsets*) are proved in [2]. Proposition 4.2.5 is implicitly contained e.g. in the proofs in [23]. Proposition 4.2.7 is also proved (using a slightly different technique) in [23]. Proposition 4.2.10 is proved in [38].

The results of Section 4.3 are (obviously) proved in [3], though the proofs given here are mostly based on excellent exposition in [22].

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Theorems gathered in Section 4.4 come from many sources. Theorem 4.4.1 and Example 4.4.2 are copied from [22]. Theorem 4.4.9 is proved in [39]. The linking construction is described in [1] and is a bit more general than a similar construction shown a few years earlier in [10] and inspired by the paper [17]; see also [9] for similar metrics. Example 4.4.12 (not published before) is taken from the author's doctoral dissertation [7].

The relations between the theory of hyperconvex metric spaces and Banach spaces examined in Section 4.5. The first part of this section, which is devoted to the question of hyperconvexity of a Banach space, is due to Nachbin and Kelley (Theorem 4.5.3, see [42] and the references therein) and Cianciaruso and De Pascale, who proved Theorem 4.5.4 (see [16]). Theorem 4.5.5 is quoted from [32, p. 474, Theorem 1]. The rest is mostly the work of the author and is based on his doctoral dissertation. The notion of a strictly convex metric space seems to be folklore; such metric spaces are also called *strongly convex*. The observation that for Banach spaces, strict convexity (in the metric sense) and the classical notion of strict convexity coincide can be found in [26]. Theorem 4.5.13 was suggested to the author by Jerzy Grzybowski. Lemma 4.5.17 is taken from the classical book [5, p. 44, Lemma 15.1].

Section 4.6, devoted to Isbell's hyperconvex hull, is based partly on Isbell's paper [30] and partly on [22]. Example 4.6.21 is taken from the paper [38] of Sine. Theorem 4.6.22 is proved in [30] in a (a bit) weaker version (without the part about the isometry being essentially an identity on X), and also in [22], though the proof there makes use of Baillon's fixed point theorem (Theorem 5.1.1 in this book).

Section 4.7 gathers results from a few sources. The definition of $\mathbb{R}$-tree and Theorem 4.7.3 is taken from [34]. Let us mention that the method used in the proof of Theorem 4.4.10 can be applied to establish hyperconvexity of $\mathbb{R}$-trees representable as a union of finitely many metric segments, which is another way of showing that the family of admissible subsets of an $\mathbb{R}$-tree has the finite intersection property. This approach is examined in [11].

Theorem 4.7.12 is proved in [11]; the lemmas preceding are also proved there, though they are probably folklore.

Fixed points 5

5.1 Baillon's fixed point theorem

The first fixed-point theorem proved for hyperconvex spaces was the Baillon's theorem. It may be viewed as a hyperconvex analogue of the Banach contraction principle. In case of Banach's theorem, we have a contraction of a complete space, and a unique fixed point. In case of Baillon's theorem, we have a nonexpansive mapping on a bounded hyperconvex space; obviously, we lose the uniqueness of the fixed point, but it turns out that the fixed point set is itself hyperconvex.

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5.1.1. Theorem. *Let* H *be a bounded hyperconvex space and* $T\colon H \to H$ *a nonexpansive mapping. Then* T *has a fixed point.* $\triangle$

Proof. Let $\Sigma := \{A \subseteq H \mid A \in \mathcal{A}(H), T(A) \subseteq A\}$. Since H is bounded, $H \in \mathcal{A}(H)$ and therefore $H \in \Sigma$, so the family Σ is nonempty. If $\{A_\lambda\}_{\lambda\in\Lambda}$ is a chain of sets in Σ, then their intersection $\widetilde{A}$ is admissible (in particular, nonempty) by Lemma 4.3.7. Moreover, since if $x \in T(\widetilde{A})$, then $x \in T(A_\lambda)$ for any $\lambda \in \Lambda$, so $x \in A_\lambda$ and hence $x \in \widetilde{A}$. This means that $\widetilde{A} \in \Sigma$ and we can apply the Kuratowski–Zorn Lemma. Let $\widetilde{A}$ be a minimal set in Σ. We will show that $\widetilde{A} = \operatorname{cov} T(\widetilde{A})$. Since $T(\widetilde{A}) \subseteq \widetilde{A}$ and $\operatorname{cov} T(\widetilde{A})$ is the smallest admissible subset including $T(\widetilde{A})$, we have $\operatorname{cov} T(\widetilde{A}) \subseteq \widetilde{A}$. Hence $T\big(\operatorname{cov} T(\widetilde{A})\big) \subseteq T(\widetilde{A}) \subseteq \operatorname{cov} T(\widetilde{A})$, and consequently $\operatorname{cov} T(\widetilde{A}) \in \Sigma$. From the minimality of $\widetilde{A}$ we infer that $\operatorname{cov} T(\widetilde{A}) = \widetilde{A}$.

Notice that for each $x \in H$ we have $r_x(\widetilde{A}) = r_x\big(T(\widetilde{A})\big)$. Indeed, $T(\widetilde{A}) \subseteq \widetilde{A}$, so $r_x\big(T(\widetilde{A})\big) \leqslant r_x(\widetilde{A})$. If for some point $x \in H$ there were $r_x\big(T(\widetilde{A})\big) < r_x(\widetilde{A})$, there would exist some $y \in \widetilde{A}$ such that $r_x\big(T(\widetilde{A})\big) < d(x,y)$, and hence $y \notin \bar{B}\big(x, r_x\big(T(\widetilde{A})\big)\big)$ and $y \notin \operatorname{cov} T(\widetilde{A})$.

Let $C := C_{\widetilde{A}}(\widetilde{A}) = \bigcap_{x\in\widetilde{A}} \bar{B}_{\widetilde{A}}\big(x, r(\widetilde{A})\big)$. Since $\widetilde{A}$ is hyperconvex, $C \neq \emptyset$ (see part 4° of Lemma 4.3.5). The set C is admissible in H as an intersection of admissible sets: $C = \widetilde{A} \cap \bigcap_{x\in\widetilde{A}} \bar{B}_H\big(x, r(\widetilde{A})\big)$. We will now prove that $T(C) \subseteq C$. Let $y \in T(C)$, then $y = T(x)$ for some $x \in C$. We have then: $r_y(\widetilde{A}) = r_y\big(T(\widetilde{A})\big) = \sup_{z\in T(\widetilde{A})} d(y,z) = \sup_{w\in\widetilde{A}} d\big(T(x),T(w)\big) \leqslant \sup_{w\in\widetilde{A}} d(x,w) = r_x(\widetilde{A}) = \frac{1}{2}\operatorname{diam}\widetilde{A} = r_{\widetilde{A}}(\widetilde{A})$. On the other hand, $r_y(\widetilde{A}) \geqslant r_{\widetilde{A}}(\widetilde{A})$; hence $r_y(\widetilde{A}) = r_{\widetilde{A}}(\widetilde{A})$ and in consequence $y \in C$. We have therefore $C \in \Sigma$ and at the same time $C \subseteq \widetilde{A}$; but $\widetilde{A}$ is minimal in Σ, so $C = \widetilde{A}$. By the hyperconvexity of $\widetilde{A}$ and part 5° of Lemma 4.3.5, the set $\widetilde{A}$ is a singleton and the proof is finished. □

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We will now prove an interesting corollary of the above theorem. Contrary to the paper [3], we did not include it in that theorem, since it does not require the boundedness assumption.

5.1.2. Corollary. *Let* $T\colon H \to H$ *be a nonexpansive mapping of a hyperconvex space* H *into itself. If the fixed-point set of* T *is nonempty, it is hyperconvex.* △

Proof. Let $F := \{x \in H \mid x = T(x)\}$. Let $\{\bar{B}_F(x_i, r_i)\}_{i\in I}$ be a family of closed balls in F such that $d(x_i, x_j) \leqslant r_i + r_j$ for $i, j \in I$. Let $A := \bigcap_{i\in I} \bar{B}_H(x_i, r_i)$; the hyperconvexity of H implies that A is nonempty and hence hyperconvex (as an admissible subset of H). Of course, A is also bounded. Notice that since every point x_i is a fixed point of the mapping T, therefore for any $y \in A$ and $i \in I$ we have $d\big(x_i, T(y)\big) = d\big(T(x_i), T(y)\big) \leqslant d(x_i, y) \leqslant r_i$, which means that $T(y) \in \bar{B}_H(x_i, r_i)$ for every $i \in I$; this in turn means that $T(y) \in A$. Hence the restricton $T|_A$ is a nonexpansive mapping of a bounded hyperconvex space A into itself. From Theorem 5.1.1 we infer that it has a fixed point, or in other words, $A \cap F \neq \emptyset$. Since $A \cap F = \bigcap_{i\in I} \bar{B}_F(x_i, r_i)$, the proof is finished. □

We will conclude this section with a variant of Baillon's theorem for non-self-mappings. Let us consider a nonexpansive mapping from some subset of a hyperconvex space to the *whole* space. As simple examples of translations in $\mathbb{R}$ show, such a mapping need not have a fixed point. It may be asked, however, whether one can assure the existence of a fixed point by restricting the possible behavior of the mapping in question on the boundary of its domain. It turns out that the answer is in the affirmative, at least when the domain is admissible.

5.1.3. Theorem. *Let* A *be an admissible subset of a hyperconvex space* H. *Any nonexpansive mapping* $T\colon A \to H$ *satisfying the condition* $T(\partial A) \subseteq A$ *has a fixed point.* △

Proof. Proposition 4.2.7 yields the existence of a nonexpansive retraction $R\colon H \to A$ such that $R(H \backslash A) \subseteq \partial A$. Since T is nonexpansive and A is admissible, $T(A)$ is bounded. Let C be any hyperconvex hull of $T(A)$ in H; from part 2° of Lemma 4.6.16 we know that C is also bounded. The mapping $T \circ R|_C\colon C \to C$ is clearly a nonexpansive mapping of a bounded hyperconvex space into itself, and hence it has a fixed point x_0 by Theorem 5.1.1. If x_0 were outside the set A, we would have $R(x_0) \in \partial A$ and hence $x_0 = T \circ R(x_0) \in A$, which is a contradiction. Since $x_0 \in A$ and R is a retraction, we have $x_0 = T\big(R(x_0)\big) = T(x_0)$ and the proof is finished. □

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5.2 Schauder-type fixed-point theorem

There are a few ways to prove a Schauder-type theorem for hyperconvex spaces. One of them uses the fact that hyperconvex spaces are absolute retracts, and utilizes a version of Schauder theorem for them. In turns out, however, that it is enough to know the classical Schauder theorem for convex sets in Banach spaces. Again, Isbell's hyperconvex hull will come to rescue; but first – for completeness – let us state the classical definition of a compact mapping.

5.2.1. Definition. A continuous mapping $T\colon X \to Y$ is called *compact* if $T(X)$ is relatively compact in Y. △

5.2.2. Theorem. *A compact mapping of a hyperconvex space into itself has a fixed point.* △

Proof. Let $T\colon H \to H$ be a compact mapping of a hyperconvex space H into itself. From Corollary 4.6.3 we know that $\overline{T(H)}$ has a hyperconvex hull K in H. Theorem 4.6.22 and part 3° of Lemma 4.6.16 imply that K is compact. From Theorem 4.6.17 we see that the mapping $e\colon K \to \varepsilon K$ is surjective – and hence an isometry and in particular a homeomorphism – so εK is also compact. Let the mapping $T_{\varepsilon K}\colon \varepsilon K \to \varepsilon K$ be defined by the formula $T_{\varepsilon K} := e \circ T \circ e^{-1}$. It is obviously uniformly continuous (as a continuous mapping defined on a compact space); from Proposition 3.1.9 and Remark 3.1.12 we infer that its minimal m.o.c. is subadditive. Theorem 3.1.15 lets us extend the mapping $T_{\varepsilon K}$ continuously to a mapping $\widetilde{T}_{\varepsilon K}\colon \overline{\mathrm{conv}}_{\mathcal{C}(K)}\, \varepsilon K \to \varepsilon K \subseteq \overline{\mathrm{conv}}_{\mathcal{C}(K)}\, \varepsilon K$, where $\overline{\mathrm{conv}}_{\mathcal{C}(K)}\, \varepsilon K$ denotes the closed convex hull of the set εK in the Banach space $\mathcal{C}(K)$ of real continuous functions on the compact space K. It is obvious that $\widetilde{T}_{\varepsilon K}$ is a compact mapping; the classical Schauder theorem implies that it has a fixed point $x_0 \in \varepsilon K = e(K)$. It remains to prove that $e^{-1}(x_0)$ is a fixed point of the mapping T:

$$\begin{aligned} T\big(e^{-1}(x_0)\big) &= e^{-1} \circ e \circ T \circ e^{-1}(x_0) \\ &= e^{-1}\big(T_{\varepsilon K}(x_0)\big) = e^{-1}\big(\widetilde{T}_{\varepsilon K}(x_0)\big) = e^{-1}(x_0). \end{aligned}$$ □

5.2.3. Corollary. *A continuous mapping of a compact hyperconvex space into itself has a fixed point.* △

Proof. This Corollary is obvious from the preceding theorem and the definition of a compact mapping. □

It turns out that this theorem can be generalized in several possible ways. (In fact, the remainder of this chapter can be viewed as a survey of a few of them.) It would be tempting to give here a generalization for non-self mappings in the spirit of Theorem 5.1.3. We will resist this temptation and wait until Section 5.4, which contains a result more general than Theorem 5.2.2.

5.3 Krasnoselskii-type fixed-point theorem

It is well known that the Banach Contraction Principle can be combined with the Schauder fixed-point theorem to obtain the Krasnoselskii theorem on a fixed point of a sum of a contraction and a compact mapping in a Banach space. Therefore, it is not surprising that a similar result holds for hyperconvex spaces. Of course, we may replace a contraction with a nonexpansive mapping; unfortunately, we need an additional assumption of the Palais–Smale type.

5.3.1. Theorem. *Let* $H \neq \emptyset$ *be a hyperconvex and bounded subset of a normed space* E. *Let us assume that*:

1° $T_1 \colon H \to E$ *is nonexpansive,*
2° $T_2 \colon H \to E$ *is compact,*
3° $T(x) := T_1(x) + T_2(x) \in H$ *for any* $x \in H$,
4° *every sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *of points in* H *satisfying the Palais–Smale-type condition* $\lim_{n\to\infty} (x_n - T(x_n)) = 0$ *has a limit point.*

Then the mapping T *has a fixed point.* △

Instead of directly proving the above result, we will state its simple generalization and show how Theorem 5.3.1 follows from it.

5.3.2. Theorem. *Let* $H \subseteq E$ *be a bounded hyperconvex subset of a normed space* E *and let* Y *be a metric space. Assume that* $f \colon H \to Y$ *is a compact mapping and* $g \colon H \times \overline{f(H)} \to H$ *a continuous function satisfying for any* $x_1, x_2 \in E$ *and* $y \in \overline{f(H)}$ *the condition* $\|g(x_1, y) - g(x_2, y)\| \leqslant \|x_1 - x_2\|$. *Further assume that any sequence* $\langle x_n \rangle_{n=1}^{\infty} \subseteq H$ *such that* $\lim_{n\to\infty} \|x_n - g(x_n, f(x_n))\| = 0$ *has a limit point in* H. *Then there exists a point* $x \in H$ *such that* $x = g(x, f(x))$. △

Proof. Let $R \colon E \to H$ be a nonexpansive retraction from E onto H. Fix $q > 1$ and $y \in \overline{f(H)}$. The mapping $x \mapsto R\big(\frac{1}{q} g(x, y)\big) \colon H \to H$ is a contraction of a complete metric space H into itself, so it has exactly one fixed point $u_q(y)$. We have therefore

$$u_q(y) = R\big(\tfrac{1}{q} g(u_q(y), y)\big).$$

We will show that such defined mapping $u_q \colon \overline{f(H)} \to H$ is continuous.

For $y_1, y_2 \in \overline{f(H)}$ we have

$$\begin{aligned}\|u_q(y_1) - u_q(y_2)\| &= \big\|R\big(\tfrac{1}{q}g(u_q(y_1), y_1)\big) - R\big(\tfrac{1}{q}g(u_q(y_2), y_2)\big)\big\| \\ &\leqslant \tfrac{1}{q}\|g(u_q(y_1), y_1) - g(u_q(y_2), y_2)\| \\ &\leqslant \tfrac{1}{q}\big(\|g(u_q(y_1), y_1) - g(u_q(y_2), y_1)\| \\ &\qquad + \|g(u_q(y_2), y_1) - g(u_q(y_2), y_2)\|\big) \\ &\leqslant \tfrac{1}{q}\|u_q(y_1) - u_q(y_2)\| \\ &\qquad + \tfrac{1}{q}\|g(u_q(y_2), y_1) - g(u_q(y_2), y_2)\|,\end{aligned}$$

and hence $\|u_q(y_1) - u_q(y_2)\| \leqslant \frac{q}{q-1}\|g(u_q(y_2), y_1) - g(u_q(y_2), y_2)\|$, which – together with the continuity of g – yields the continuity of u_q.

Let us now consider the function $u_q \circ f \colon H \to H$. Notice that $u_q \circ f(H) = u_q\big(f(H)\big) \subseteq u_q\big(\overline{f(H)}\big)$ and the last set is compact as a continuous image of a compact set. Applying Theorem 5.2.2 to the mapping $u_q \circ f$ we infer that for any $q > 1$ there exists some $x_q \in H$ such that $x_q = u_q(f(x_q))$.

Choose a decreasing sequence $\langle q_n \rangle_{n=1}^{\infty}$ of real numbers convergent to 1. For $n \in \mathbb{N}$ we have

$$\begin{aligned}x_{q_n} &= u_{q_n}(f(x_{q_n})) \\ &= R\big(\tfrac{1}{q_n}g\big(u_{q_n}(f(x_{q_n})), f(x_{q_n})\big)\big) \\ &= R\big(\tfrac{1}{q_n}g(x_{q_n}, f(x_{q_n}))\big),\end{aligned}$$

and hence

$$\begin{aligned}\|x_{q_n} - g(x_{q_n}, f(x_{q_n}))\| &= \big\|R\big(\tfrac{1}{q_n}g(x_{q_n}, f(x_{q_n}))\big) - R\big(g(x_{q_n}, f(x_{q_n}))\big)\big\| \\ &\leqslant \big\|\tfrac{1}{q_n}g(x_{q_n}, f(x_{q_n})) - g(x_{q_n}, f(x_{q_n}))\big\| \\ &= \big(1 - \tfrac{1}{q_n}\big)\|g(x_{q_n}, f(x_{q_n}))\| \to 0.\end{aligned}$$

By our assumption the sequence $\langle x_{q_n} \rangle_{n=1}^{\infty}$ has a limit point. We may assume (passing to a subsequence if necessary) that $\lim_{n\to\infty} x_{q_n} = x_\infty \in H$. Therefore, on the one hand, $x_{q_n} \to x_\infty$, and on the other, $x_{q_n} = R\big(\frac{1}{q_n}g(x_{q_n}, f(x_{q_n}))\big) \to R\big(g(x_\infty, f(x_\infty))\big) = g(x_\infty, f(x_\infty))$ and the proof is finished. □

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5.3.3. Remark. Notice that Theorem 5.3.1 is a corollary of Theorem 5.3.2. In fact, let us assume that the assumptions of Theorem 5.3.1 are satisfied. Put $Y := E$, $f := f_2$ and $g(x,y) := f_1(x) + y$ for $x \in H$, $y \in \overline{f_2(H)}$. It is easy to see that all the assumptions of Theorem 5.3.2 are satisfied and $g(x, f(x)) = f_1(x) + f_2(x)$. △

5.4 Darbo–Sadovskii-type fixed-point theorem

In 1955, Darbo proved yet another variant of Schauder theorem. This time, we relax the compactness assumption and require only that the images of bounded sets are "more compact" than these sets themselves. This imprecise intuition was formalized in the notion of a set contraction. A bit later, in 1967, Sadovskii showed (using a different technique) that the assumption of being a set contraction may be further relaxed to a weaker notion of a condensing mapping. In 1996, Espínola proved an analogue of those results for mappings of hyperconvex spaces. Before we can state and prove it, we need to formulate the necessary, classical definitions.

5

5.4.1. Definitions. Let A be a subset of a metric space X. The infimum of the set of all positive numbers ε such that there exists a finite covering of A by closed balls in X with radii not greater than ε is called the *Hausdorff measure of noncompactness of the set* A (*relative to the space* X) and denoted $\chi_X(A)$. The infimum of the set of all positive numbers ε such that A may be covered by finitely many sets of diameter not greater than ε is called the *Kuratowski measure of noncompactness* and denoted by $\alpha_X(A)$. As usual, we will omit the index X if there would not be any doubts as to in what space are the considered sets included. Also, in the sequel we will use the symbol μ to mean either α or χ if the result discussed holds for both measures of noncompactness. △

5.4.2. Remark. We will use the obvious fact that if $i\colon X \to Y$ is an isometry of metric spaces X and Y and $i(A) = B$ for some $A \subseteq X$, $B \subseteq Y$, then $\mu_X(A) = \mu_Y(B)$ for $\mu = \chi$ or $\mu = \alpha$. △

The following lemma describes the relationships between both measures of noncompactness.

5.4.3. Lemma. *Let* A *be a subset of a metric space* X.

1° $\chi(A) \leqslant \alpha(A) \leqslant 2\chi(A)$,

2° *if* X *is hyperconvex, then* $\alpha(A) = 2\chi(A)$,

3° $\alpha(\varepsilon X) \leqslant 2\chi_{\varepsilon X}(e(X))$. △

Proof. 1° The former inequality is a simple consequence of the definition of a diameter of a set, and the latter one follows immediately from the triangle inequality.

2° It follows from part 1° of Lemma 4.3.5 that $2\chi(A) \leqslant \alpha(A)$.

3° If the space X is unbounded, there is nothing to prove. Assume now that X is bounded. Let $d := \operatorname{diam} X = \operatorname{diam} \varepsilon X$ (see Lemma 4.6.16, part 2°). Choose $\varepsilon > 0$; there exists a finite set $\{f_i\}_{i\in I} \subseteq \varepsilon X$ such that $e(X) \subseteq \bigcup_{i\in I} \bar{B}_{\varepsilon X}\big(f_i, \chi_{\varepsilon X}(e(X)) + \frac{1}{4}\varepsilon\big)$. Moreover, the compactness of the interval $[0, d]$ guarantees the existence of a finite set $\{c_j\}_{j\in J} \subseteq [0, d]$ with the property that for any $x \in [0, d]$ there exists some $j \in J$ such that $|x - c_j| \leqslant \frac{1}{4}\varepsilon$. Let J^I be the set of all mappings $\phi\colon I \to J$. For any $\phi \in J^I$ put $A_\phi := \{s \in \varepsilon\varepsilon X \mid \sup_{i\in I}|s(f_i) - c_{\phi(i)}| \leqslant \frac{1}{4}\varepsilon\}$; notice that $\varepsilon\varepsilon X = \bigcup_{\phi\in J^I} A_\phi$. Indeed, let $s \in \varepsilon\varepsilon X$ and $i \in I$; there exists some index $j \in J$ such that $|s(f_i) - c_j| \leqslant \frac{1}{4}\varepsilon$. Putting $\phi(i) := j$ and repeating that reasoning for all $i \in I$ we obtain a function $\phi \in J^I$ such that $s \in A_\phi$.

We will now estimate the diameter of the set A_ϕ. Choose any $s, u \in A_\phi$. Using Remark 4.6.19 we infer that $s = e_{\varepsilon X}(g)$ and $u = e_{\varepsilon X}(h)$ for some $g, h \in \varepsilon X$, so $d(s, u) = d\big(e_{\varepsilon X}(g), e_{\varepsilon X}(h)\big) = d(g, h)$. Let $x \in X$ be arbitrary; then there exists some $i \in I$ such that $d(f_x, f_i) \leqslant \chi_{\varepsilon X}\big(e(X)\big) + \frac{1}{4}\varepsilon$. We have

$$\begin{aligned}
|g(x) - h(x)| &= |d(g, f_x) - d(h, f_x)| = |s(f_x) - u(f_x)| \\
&\leqslant |s(f_x) - s(f_i)| + |s(f_i) - c_{\phi(i)}| + |c_{\phi(i)} - u(f_i)| + |u(f_i) - u(f_x)| \\
&\leqslant d(f_x, f_i) + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + d(f_i, f_x) \leqslant 2\chi_{\varepsilon X}\big(e(X)\big) + \varepsilon,
\end{aligned}$$

which implies that $d(g, h) \leqslant 2\chi_{\varepsilon X}\big(e(X)\big) + \varepsilon$. This means that

$$\operatorname{diam} A_\phi \leqslant 2\chi_{\varepsilon X}\big(e(X)\big) + \varepsilon$$

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for all $\phi \in J^I$. We have therefore $\alpha(\varepsilon X) \leqslant 2\chi_{\varepsilon X}\big(e(X)\big) + \varepsilon$ and – since the choice of $\varepsilon > 0$ was arbitrary – we obtain the inequality $\alpha(\varepsilon X) \leqslant 2\chi_{\varepsilon X}\big(e(X)\big)$. □

5.4.4. Remarks. Let us notice that $\chi_Y(A) \leqslant \chi_X(A)$, if $A \subseteq X \subseteq Y$. In fact, if $\{\bar{B}_X(x_i, r_i) \mid i \in I\}$ is a finite covering of A by closed balls in X, then $\{\bar{B}_Y(x_i, r_i) \mid i \in I\}$ is a covering of A by finitely many closed balls in Y. (It can also be shown that in general, the inequality here cannot be replaced by an equality.) Moreover, we also have $\alpha_Y(A) = \alpha_X(A)$; thus, we can (and will) always write $\alpha(A)$ instead of $\alpha_X(A)$. Finally, if $A \subseteq H_1$ and $A \subseteq H_2$, where H_1 and H_2 are any hyperconvex spaces, then $\chi_{H_1}(A) = \chi_{H_2}(A)$ by the above considerations and part 2° of Lemma 5.4.3. △

5.4.5. Lemma. *Let μ be the Hausdorff or Kuratowski measure of noncompactness, X a metric space and A, B some subsets of X. Then,*

1° *if $A \subseteq B$, then $\mu(A) \leqslant \mu(B)$,*

2° $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ (*in particular, for any $b \in X$ we have* $\mu(A \cup \{b\}) = \mu(A)$),

3° $\mu(A) = 0$ *if, and only if, A is precompact,*

4° *if the space X is hyperconvex and $H \in \mathcal{H}(A)$, then $\mu(A) = \mu(H)$.* △

Proof. The properties 1°–3° are well-known and not related to the theory of hyperconvex spaces, so we will omit their proof; an interested reader may find details e.g. in [4].

Let us prove the property 4°. We have $2\chi_X(A) = \alpha(A) \leqslant \alpha(H) \leqslant 2\chi_H(A) = 2\chi_X(A)$, where we applied first part 2° of Lemma 5.4.3, then part 1° of Lemma 5.4.5, part 3° of Lemma 5.4.3 and finally Remarks 5.4.2 and 5.4.4, which yields the thesis for $\mu = \alpha$. The case $\mu = \chi$ follows now from the property 2° of Lemma 5.4.3. □

Armed with the machinery of measures of noncompactness and their basic properties we can now start aiming at some fixed point results. We will state two more (classical) definitions.

5.4.6. Definitions. Let $T\colon X \to Y$ be a continuous mapping between metric spaces and μ a measure of noncompactness. We will call T a *set* *contraction* (*with respect to the measure of noncompactness* μ), if there exists a constant $k \in (0,1)$ such that $\mu\big(T(A)\big) \leqslant k\mu(A)$ for any bounded set $A \subseteq X$. We will call T a *condensing mapping* (*with respect to* μ), if $\mu\big(T(A)\big) < \mu(A)$ for any bounded set $A \subseteq X$ such that $\mu(A) > 0$. △

5.4.7. Remark. It is clear that all contractions and compact mappings are also set contractions (with the same constant in case of contractions). Also, any set contraction is a condensing mapping. In fact, we will not use the notion of a set contraction now – but we will come back to it later. △

5.4.8. Theorem. *Let* μ *denote the Hausdorff or Kuratowski measure of noncompactness. Any continuous and* μ*-condensing mapping* T *of a bounded hyperconvex space* H *into itself has a fixed point.* △

Proof. Choose any point $a \in H$ and define

$$\Sigma := \{\hat{H} \subseteq H \mid a \in \hat{H}, \hat{H} \text{ hyperconvex}, T(\hat{H}) \subseteq \hat{H}\}.$$

Of course $\Sigma \neq \emptyset$, since $H \in \Sigma$; Corollary 4.3.10 and the Kuratowski–Zorn Lemma assure us that there exists a minimal element $\widetilde{H}$ in Σ. Let $C \in \mathcal{H}_{\widetilde{H}}\big(T(\widetilde{H}) \cup \{a\}\big)$; clearly $a \in C \subseteq H$, the set C is hyperconvex and $T(C) \subseteq C$ and hence $C \in \Sigma$; but $C \subseteq \widetilde{H}$ and the minimality of $\widetilde{H}$ in Σ yields $C = \widetilde{H}$. We have now

$$\mu(\widetilde{H}) = \mu(C) = \mu\big(T(\widetilde{H}) \cup \{a\}\big) = \mu\big(T(\widetilde{H})\big).$$

Since T is μ-condensing and $\widetilde{H} \subseteq H$ must be bounded, we infer that $\mu(\widetilde{H}) = 0$ – in other words, $\widetilde{H}$ is compact (as a complete, and hence closed, relatively compact set). The existence of a fixed point of the mapping $T|_{\widetilde{H}} \colon \widetilde{H} \to \widetilde{H}$ follows now from Theorem 5.2.2, which finishes the proof. □

Interestingly, it turns out that the assumption of boundedness – even though it seems essential in the above proof – can be relaxed. This, however, is shown by a completely different technique (also using an assumption more general than condensingness), which we shall analyse in the next section. For now, let us state and prove a generalization analogous to Theorem 5.1.3.

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5.4.9. Theorem. *Let* μ *denote the Hausdorff or Kuratowski measure of noncompactness and let* A *be an admissible subset of a bounded hyperconvex space* H. *Any* μ*-condensing mapping* $T\colon A \to H$ *satisfying the condition* $T(\partial A) \subseteq A$ *has a fixed point.* △

Proof. As in the proof of Theorem 5.1.3, let R be a nonexpansive retraction $R\colon H \to A$ such that $R(H \setminus A) \subseteq \partial A$. The mapping $T \circ R\colon H \to H$ is then a μ-condensing mapping of a bounded hyperconvex space into itself, so it has a fixed point x_0 by Theorem 5.4.8. Again, if the point x_0 were outside the set A, we would have $R(x_0) \in \partial A$ and hence $x_0 = T \circ R(x_0) \in A$, which contradicts this assumption, and again x_0 is a fixed point of T. □

5.5 Mönch-type fixed-point theorem

This short section is devoted to prove a surprising generalization of Theorem 5.4.8. The surprise comes from the fact that the assumptions are much weaker here: not only do we consider a larger class of mappings than condensing ones, but also we drop the boundedness condition. Unsurprisingly, however, the technique from the previous section does not work in this case.

5.5.1. Theorem. *Let* H *be a hyperconvex space and* $a \in H$ *any point. Assume that* $T\colon H \to H$ *is continuous and that any subset* $V \subseteq H$ *such that* $T(V) \cup \{a\} = V$ *or* $V \in \mathcal{H}_H\big(T(V)\big)$ *is relatively compact in* H. *Then* T *has a fixed point.* △

Proof. We will start by proving the existence of a certain nonempty subset $Z \subseteq H$ invariant with respect to T, i.e., such that $Z \subseteq T(Z)$. Let $A := \{a, T(a), T^2(a), \dots\}$. If A is a finite set, then for some $n, k \geqslant 0$ we have $T^n(a) = T^{n+k}(a)$ and it is enough to put

$$Z := \{T^n(a), T^{n+1}(a), \dots, T^{n+k}(a)\}.$$

Otherwise, let us define Z as the set of limit points of A; of course, $A = T(A) \cup \{a\}$, so A is relatively compact in H and hence $Z \neq \emptyset$. Choose now any $y \in Z$. There exists a sequence of pairwise distinct points $\langle y_n \rangle_{n=1}^{\infty}$ in $A \setminus \{y, a\}$ convergent to y. For any $n \in \mathbb{N}$ there exists an $x_n \in A$ such that $y_n = T(x_n)$. From the relative compactness of A in H we infer that the sequence $\langle x_n \rangle_{n=1}^{\infty}$ has a subsequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$ convergent to some $x \in H$. Since we have assumed that $\langle y_n \rangle_{n=1}^{\infty}$ is injective, the same can be said about $\langle x_{n_k} \rangle_{k=1}^{\infty}$; in particular, x does not appear in $\langle x_{n_k} \rangle_{k=1}^{\infty}$ infinitely many times and hence $x \in Z$. In consequence, $y = \lim_{n\to\infty} y_n = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} T(x_{n_k}) = T(\lim_{k\to\infty} x_{n_k}) = T(x)$ and $y \in T(Z)$, and since $y \in Z$ was arbitrary, the inclusion $Z \subseteq T(Z)$ is proved.

Let us now denote

$$\Sigma := \{\hat{H} \subseteq H \mid Z \subseteq \hat{H}, \hat{H} \text{ hyperconvex}, T(\hat{H}) \subseteq \hat{H}\}.$$

Of course $H \subseteq \Sigma$, so Σ is nonempty. Corollary 4.3.10 and the Kuratowski–Zorn Lemma yield the existence of a minimal element $\widetilde{H}$ in Σ. Let $C \subseteq \mathcal{H}_{\widetilde{H}}\big(T(\widetilde{H})\big)$. Obviously, $Z \subseteq T(Z) \subseteq T(\widetilde{H}) \subseteq C$ and $T(C) \subseteq T(\widetilde{H}) \subseteq C$, hence $C \in \Sigma$; but $C \subseteq \widetilde{H}$, so from the minimality of $\widetilde{H}$ we have $C = \widetilde{H}$. Therefore, $\widetilde{H} \in \mathcal{H}_{\widetilde{H}}\big(T(\widetilde{H})\big) \subseteq \mathcal{H}_H\big(T(\widetilde{H})\big)$ and by the assumption the set $\widetilde{H}$ is relatively compact and hence compact (since hyperconvex subsets of any metric space are closed as complete subspaces). Applying Theorem 5.2.2 to the mapping $T|_{\widetilde{H}} \colon \widetilde{H} \to \widetilde{H}$ we obtain the thesis. □

5.5.2. Remarks. It is easily seen that each μ-condensing mapping of a hyperconvex space into itself satisfies the assumptions of the above theorem. Indeed, let $T \colon H \to H$ be such a mapping and let $T(V) \cup \{a\} = V$ for some point $a \in H$ and subset $V \subseteq H$; if $\mu(V) > 0$, we would have $\mu(V) = \mu\big(T(V) \cup \{a\}\big) = \mu\big(T(V)\big) < \mu(V)$ – contradiction. If, on the other hand, $V \in \mathcal{H}_H\big(T(V)\big)$, then $\mu(V)$ cannot be positive, since then it would be $\mu(V) = \mu\big(T(V)\big)$ by part 4° of Lemma 5.4.5. In either case it turns out that V is precompact and hence relatively compact in the complete space H. Theorem 5.5.1 is therefore a generalization of Theorem 5.4.8. △

5.6 Leray–Schauder-type fixed-point theorem

In this section we are going to prove a Leray–Schauder-type theorem for mappings of hyperconvex spaces and state a few of its corollaries. Probably the most important of them are Corollaries 5.6.5–5.6.7, which are analogues of the so-called nonlinear alternative (see [19, p. 61, Theorem 5.1]).

5.6.1. Theorem. *Let* Ω *be a nonempty and open subset of a hyperconvex space* H. *Let the homotopy* $h\colon [0,1] \times \overline{\Omega} \to H$ *satisfy the following conditions*:

1° *the mapping* $h(0,\cdot\,)$ *has a subadditive m.o.c. and the set* $h(\{0\} \times \overline{\Omega})$ *is included in some compact and hyperconvex set* $V \subseteq \overline{\Omega}$;

2° *none of the mappings* $h(t,\cdot\,)$, *where* $t \in [0,1)$, *has any fixed points in the set* $\partial\Omega$;

3° *each subset* $C \subseteq \Omega$ *with the property that* $C = \Omega \cap P$ *for some set* $P \in \mathcal{H}_H\big(h([0,1] \times C) \cup V\big)$ *is relatively compact.*

Then the mapping $h(1,\cdot\,)$ *has a fixed point in* $\overline{\Omega}$. △

Proof. Step I. The existence of a compact subset $C \subseteq H$ *satisfying the condition* $C \in \mathcal{H}_H\big(h\big([0,1] \times (C \cap \Omega)\big) \cup V\big)$. Let us consider the family

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$$\Sigma := \{\hat{C} \subseteq H \mid V \subseteq \hat{C}, \hat{C} \text{ hyperconvex}, h\big([0,1] \times (\hat{C} \cap \Omega)\big) \subseteq \hat{C}\}.$$

We will show that it contains a minimal element. Obviously Σ is nonempty, since $H \in \Sigma$. Let $\{\hat{C}_\lambda\}_{\lambda\in\Lambda}$ be a chain in Σ. Denote $\hat{C} := \bigcap_{\lambda\in\Lambda} \hat{C}_\lambda$. It is clear that $V \subseteq \hat{C}$. The Baillon's Theorem (see Theorem 4.3.8 and Corollary 4.3.10) yields the hyperconvexity of the set $\hat{C}$. Finally, for any $\lambda \in \Lambda$ we have $\hat{C} \subseteq \hat{C}_\lambda$, so $h\big([0,1] \times (\hat{C} \cap \Omega)\big) \subseteq h\big([0,1] \times (\hat{C}_\lambda \cap \Omega)\big) \subseteq \hat{C}_\lambda$; hence $h\big([0,1] \times (\hat{C} \cap \Omega)\big) \subseteq \hat{C}$ and therefore $\hat{C} \in \Sigma$. Now, our favorite Kuratowski–Zorn Lemma assures us that there exists a minimal element $\widetilde{C}$ in Σ (even though most probably nobody could ever find it...). Choose $C \in \mathcal{H}_{\widetilde{C}}\big(h\big([0,1] \times (\widetilde{C} \cap \Omega)\big) \cup V\big)$. From the definition of C it follows that $V \subseteq C$ and that C is hyperconvex. Further, $C \subseteq \widetilde{C}$, so $h\big([0,1] \times (C \cap \Omega)\big) \subseteq h\big([0,1] \times (\widetilde{C} \cap \Omega)\big) \subseteq C$. We have therefore $C \in \Sigma$ and $C \subseteq \widetilde{C}$, which – together with the minimality of $\widetilde{C}$ – means that $C = \widetilde{C}$; in particular, $C \in \mathcal{H}_H\big(h\big([0,1] \times (C \cap \Omega)\big) \cup V\big)$.

It remains to show that C is compact. From the condition 3° we know that the set $C' := C \cap \Omega$ is relatively compact. We have

$$h([0,1] \times \overline{C'}) \cup V = h(\overline{[0,1] \times C'}) \cup V = \\ = \overline{h([0,1] \times C')} \cup \overline{V} = \overline{h([0,1] \times C') \cup V},$$

where the second equality is a consequence of the fact that the sets $\overline{[0,1] \times C'}$ and V are compact. Applying Remarks 4.6.6 and 4.6.7, we obtain

$$C \in \mathcal{H}_{\widetilde{C}}\big(h([0,1] \times C') \cup V\big) = \\ = \mathcal{H}_{\widetilde{C}}\big(\overline{h([0,1] \times C') \cup V}\big) \subseteq \mathcal{H}_H\big(h([0,1] \times \overline{C'}) \cup V\big).$$

From the compactness of the set C' and part 3° of Lemma 4.6.16 we infer that the set C is also compact.

Step II. The auxiliary function τ. Let us denote the set of all fixed points of the mappings of the form $h(t, \cdot\,)$ for $t \in [0,1]$ belonging to $C \cap \overline{\Omega}$ by S, i.e., $S := \bigcup_{t \in [0,1]} \operatorname{Fix} h(t, \cdot\,)|_{C \cap \overline{\Omega}}$. If $h(1, \cdot\,)$ has a fixed point at the boundary $\partial\Omega$, the theorem is true. If not, we have $S \cap \partial\Omega = \emptyset$. Let us consider the mapping

$$\langle t, x \rangle \mapsto \langle x, h(t,x) \rangle \colon [0,1] \times (C \cap \overline{\Omega}) \to H \times H.$$

It is continuous, so the preimage of the diagonal $\{\langle x, x \rangle \mid x \in H\}$ is a closed subset of the compact set $[0,1] \times (C \cap \overline{\Omega})$, and therefore is compact. Since S is the image of that set with respect to the projection onto the second factor, the set S is also compact, and in particular closed in H.

By definition of the set S we have $S \subseteq \overline{\Omega}$; but $S \cap \partial\Omega = \emptyset$, so $S \subseteq \Omega$. The sets $H \setminus \Omega$ and S are therefore disjoint closed subsets of a metric space H. This means that there exists a continuous function $\tau \colon H \to [0,1]$ equal to 1 on S and vanishing on $H \setminus \Omega$.

Step III. The auxiliary mapping $F \colon C \to C$ *and its fixed point.* Denote $h_0 := h(0, \cdot\,)|_{C \cap \overline{\Omega}}$. This mapping transforms the set $C \cap \overline{\Omega}$ into a hyperconvex space V and (by assumption) has a subadditive m.o.c. Using Theorem 3.1.15 we know that it has a continuous extension $\widetilde{h_0} \colon C \to V$.

Let us define a mapping $F\colon C \to H$ by the formulae

$$F(x) := \begin{cases} h(\tau(x), x) & \text{for } x \in C \cap \overline{\Omega}; \\ \widetilde{h}_0(x) & \text{for } x \in C \setminus \Omega. \end{cases}$$

Since both $C \cap \overline{\Omega}$ and $C \setminus \Omega$ are closed in C and their union is the whole C, we see that in order to show that F is well defined and continuous it is enough to notice that for $x \in (C \cap \overline{\Omega}) \cap (C \setminus \Omega)$ we have $h(\tau(x), x) = h(0, x) = h_0(x) = \widetilde{h}_0(x)$.

We will now prove that F maps the set C into itself. In fact, since $C \in \Sigma$, for $x \in C \cap \Omega$ we have $F(x) = h(\tau(x), x) \in C$. If $x \in C \setminus \Omega$, we have $F(x) = \widetilde{h}_0(x) \in V \subseteq C$. We have proved that F is a continuous mapping of a nonempty, compact and hyperconvex set C into itself; by the Schauder-type theorem for hyperconvex spaces (Theorem 5.2.2) it has a fixed point x_0.

Step IV. The fixed point of the mapping $h(1, \cdot)$. Let us observe that it is impossible that $x_0 \in C \setminus \Omega$, since then we would have $x_0 = F(x_0) = \widetilde{h}_0(x_0) \in V \subseteq \Omega$ – contradiction. Hence $x_0 \in \Omega$, and this means that $x_0 = h(\tau(x_0), x_0)$; but this, together with the definition of the set S, implies that $\tau(x_0) = 1$, so that $x_0 = h(1, x_0)$ and the proof is finished. □

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We will now state a few corollaries of Theorem 5.6.1. First we will prove that the condition 3° from that theorem holds for homotopies satisfying a Sadovskii-type condition. The subsequent corollaries show how it is sometimes possible to construct a homotopy satisfying the assumptions of Theorem 5.6.1 and such that the end of that homotopy conincides with a given mapping. Finally we will show how some estimations of the norm of the image $T(x)$ may yield the condition 2°.

5.6.2. Remark. Let Ω be a nonempty and open subset of a hyperconvex space H. Assume that there exists a homotopy $h\colon [0,1] \times \overline{\Omega} \to H$ satisfying the following Sadovskii-type condition:

$$\alpha\big(h([0,1] \times C)\big) < \alpha(C) \quad \text{for } C \subseteq \Omega \text{ such that } \alpha(C) > 0, \tag{S}$$

where α is the Kuratowski measure of noncompactness. Then the Mönch-type condition 3° from Theorem 5.6.1 holds. Indeed, if $C \subseteq \Omega$ can be written down as $\Omega \cap P$, where $P \in \mathcal{H}_H\big(h([0,1] \times C) \cup V\big)$, then

$$\alpha(C) = \alpha(\Omega \cap P) \leqslant \alpha(P) = \alpha\big(h([0,1] \times C) \cup V\big) = \alpha\big(h([0,1] \times C)\big).$$

This is only possible if $\alpha(C) = 0$, or in other words, if C is relatively compact. △

5.6.3. Corollary. *Let E be a linear metric space, H a hyperconvex subset of E and Ω a nonempty open subset of H. Assume that $R\colon E \to H$ is a retraction of the space E onto H such that $R(0) \in \overline{\Omega}$ and that $T\colon \overline{\Omega} \to E$ is continuous. Moreover, assume that $R(tT(x)) \neq x$ for any $x \in \partial\Omega$ and $t \in [0,1)$ and that every set $C \subseteq \Omega$ which can be represented as $\Omega \cap P$, where $P \in \mathcal{H}_H\, R\big(\bigcup_{t\in[0,1]} tT(C)\big)$, is relatively compact. Then the superposition $R \circ T$ has a fixed point in $\overline{\Omega}$.* △

Proof. It is easy to check that the assumptions of Theorem 5.6.1 are satisfied with $V := \{0\}$ and $h(t,x) := R(tT(x))$. □

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5.6.4. Remark. It is obvious that we can impose additional conditions on the mapping T in order to get rid of the retraction R in some places of Corollary 5.6.3. For instance, if we assume that $T(x) \in H$ for any $x \in \overline{\Omega}$, we will obtain the existence of a fixed point of the mapping T. If also $tT(x) \in H$ for any $t \in [0,1]$ and $x \in \overline{\Omega}$ (for example, we may assume that $T(\overline{\Omega}) \subseteq H$ and the set H is star-shaped), we may completely leave out the retraction R in the statement of Corollary 5.6.3. △

Using Theorem 5.6.1 we can also prove a hyperconvex analogue of the known topological nonlinear alternative (see [19, p. 61, Theorem 5.1]). This time we can also state its various versions, depending on whether we want to assume more or less about the mapping T (and whether we accept that the retraction onto a hyperconvex subset appears in more or less places). Let us now state three such corollaries.

5.6.5. Corollary. *Let E be a linear metric space, H its hyperconvex subset and Ω a nonempty set open in H. Let $R\colon E \to H$ be a nonexpansive retraction of the space E onto the set H such that $R(0) \in \overline{\Omega}$. Then for each compact mapping $T\colon \overline{\Omega} \to E$ at least one of the following conditions is true*:

i. $R \circ T$ *has a fixed point*,

ii. *there exist* $x \in \partial\Omega$ *and* $t \in [0,1)$ *such that* $x = R(tT(x))$. △

Proof. Assume that the condition ii does not hold. Put $h(t,x) := R(tT(x))$ for $(t,x) \in [0,1] \times \overline{\Omega}$ and $V := \{R(0)\}$. It is clear that the assumptions 1° and 2° of Theorem 5.6.1 are satisfied. Let $C \subseteq \Omega$ be such that $C = \Omega \cap P$ for some $P \in \mathcal{H}_H\big(h([0,1] \times C) \cup V\big)$. Notice that

$$\mathcal{H}_H\big(h([0,1] \times C) \cup V\big) = \mathcal{H}_H\Bigg(\bigcup_{t \in [0,1]} R\big(tT(C)\big)\Bigg) = \mathcal{H}_H\Bigg(R\Bigg(\bigcup_{t \in [0,1]} tT(C)\Bigg)\Bigg),$$

and that the set $\bigcup_{t \in [0,1]} tT(C)$ is included in the relatively compact set $\operatorname{conv}\big(\{0\} \cup T(C)\big)$. Therefore, its image with respect to the nonexpansive mapping R – and its hyperconvex hull – are also relatively compact. In consequence, the set $C = \Omega \cap P$ is relatively compact and the proof is finished. □

5.6.6. Corollary. *With the assumptions of Corollary 5.6.5, if additionally* $T(x) \in H$ *for* $x \in \overline{\Omega}$, *then at least one of the following conditions is true*:

i. T *has a fixed point*,

ii. *there exist* $x \in \partial\Omega$ *and* $t \in [0,1)$ *such that* $x = R(tT(x))$. △

Proof. The thesis follows immediately from Corollary 5.6.5. □

5.6.7. Corollary. *With the assumptions of Corollary 5.6.6, if additionally* $tT(x) \in H$ *for each* $t \in [0,1]$ *and* $x \in \overline{\Omega}$, *then at least one of the following conditions is true*:

i. T *has a fixed point*,

ii. *there exist* $x \in \partial\Omega$ *and* $t \in [0,1)$ *such that* $x = tT(x)$. △

Proof. The thesis follows immediately from Corollary 5.6.6. □

In the last corollary in this section we will demonstrate how one can assure that the condition ii from Corollary 5.6.6 is not satisfied, using a certain estimate on the norm of the image $T(x)$ (it is a special case of the so-called Altman condition, see [19, p. 61]).

5.6.8. Corollary. *Let* H *be a hyperconvex subset of a normed space* E, Ω *an open subset of* H *containing* 0. *Let* $T\colon \overline{\Omega} \to H$ *be a compact mapping satisfying the condition*

$$\|T(x)\|^2 \leqslant \|x\|^2 + \|x - T(x)\|^2 \quad \textit{for } x \in \partial\Omega.$$

Then T *has a fixed point.* △

Proof. We will prove that the situation described in item ii of Corollary 5.6.6 cannot happen. Let $R\colon E \to H$ be a nonexpansive retraction of the space E onto the set H. Assume for the sake of contradiction that $x = R(tT(x))$ for some $x \in \partial\Omega$ and $t \in [0, 1)$. Since $R(0) = 0$ belongs to the open set Ω, it must be $t \neq 0$ and $T(x) \neq 0$. Therefore, we have:

$$\begin{aligned} \|T(x)\|^2 &\leqslant \|x\|^2 + \|x - T(x)\|^2 \\ &= \|R(tT(x)) - R(0)\|^2 + \|R(tT(x)) - R(T(x))\|^2 \\ &\leqslant \|tT(x)\|^2 + \|(1-t)T(x)\|^2 \\ &= \big(t^2 + (1-t)^2\big)\|T(x)\|^2 < \|T(x)\|^2, \end{aligned}$$

which is impossible. □

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5.6.9. Remark. Let us notice that the inequality of the form $\|T(x)\|^2 \leqslant \|x\|^2 + \|x - T(x)\|^2$ follows from any of the following estimates: $\|T(x)\| \leqslant \|x\|$ and $\|T(x)\| \leqslant \|x - T(x)\|$. △

Notes and remarks

Theorem 5.1.1 and Corollary 5.1.2 were proved in [3]. Theorem 5.1.3 was proved in [23]. Theorem 5.2.2 was proved by Espínola in [21]. Theorem 5.3.1 is taken from [13], with slightly stronger assumptions: it is assumed there that for $\alpha \in [0, 1]$ and $x \in H$ it is always the case that $\alpha x \in H$. The way of omitting this restriction, due to Espínola, was communicated to me by Bugajewski. Theorem 5.3.2, proved in [12], is inspired by the paper [31] by G. L. Karakostas.

The results of section 5.4 are mainly due to Espínola. For the proofs of the properties of measures of noncompactness, related to hyperconvexity, see e.g. [22]. Theorem 5.4.8 was proved in [21] and its variant for non-self mappings, Theorem 5.4.9, in [23].

Theorem 5.5.1 was proved by Bugajewski and Grzelaczyk in [15]. Theorem 5.6.1, the main theorem of section 5.6, is an analogue of the result of the paper [40] for hyperconvex spaces; it was proved (together with the rest of the results of that section) in [8].

For a survey of most recent results in fixed point theory in hyperconvex metric spaces, we refer the reader to the paper [25].

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Multivalued mappings in hyperconvex spaces 6

6.1 Basic notions

So far, we have only discussed single-valued mappings, i.e. mappings whose values were points. In this chapter we will examine some properties of multi-valued mappings in hyperconvex spaces. Let us start with a definition (and notation).

6.1.1. Definition. Let X and Y be nonempty sets. If F is a mapping from X to the family of nonempty subsets of Y, we will call it a *multi-valued mapping* (*from* X *to* Y), or a *multifunction*, and denote this fact by the symbol $F\colon X \multimap Y$. The set $\{(x, y) \in X \times Y \mid y \in F(x)\}$ will be called the *graph* of the multifunction F. △

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Most commonly, the possible values of multivalued mappings are restricted to some family of subsets of Y. When Y is a normed space, for instance, it is often assumed that the values are compact and convex, or bounded, closed and convex. In the hyperconvex setting, it should be no surprise that an often appearing restriction will be that the values are admissible.

Even though the theory of multi-valued mappings in hyperconvex spaces is broad and well-developed, we will concentrate on two types of results. The former one is *selection theorems*: given some multivalued mapping $F\colon X \multimap Y$ satisfying some properties (usually connected with continuity), we seek a single-valued *selection* of F (also satisfying some continuity properties). The latter one is fixed point theorems; of course, in the multi-valued case, a fixed point is understood in a little different (though natural) way. For the sake of formality, let us include the relevant definitions here.

6.1.2. Definitions. Let $F\colon X \multimap Y$ be a multifunction from a set X to Y. A function $f\colon X \to Y$ is called a *selection* of F if $f(x) \in F(x)$ for all $x \in X$. A point $x_0 \in X$ is called a (*multi-valued*) *fixed point* of F if $x_0 \in F(x_0)$. (As usual, we will denote the set of all fixed points of F by $\operatorname{Fix} F$.) △

6.2 Selection theorems

We are now going to state a hyperconvex version of the famous Michael selection theorem for lower semicontinuous functions. Let us start with a definition and one earlier result.

6.2.1. Definition. Let $F\colon X \multimap Y$ be a multivalued mapping between metric spaces. We call F *quasi-lower semicontinuous*, if for each $x \in X$ and $\varepsilon > 0$ there exists some point $z \in F(x)$ and $\delta > 0$ such that if $y \in \bar{B}(x, \delta)$, then $F(y) \cap \bar{B}(z, \varepsilon) \neq \emptyset$. △

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6.2.2. Theorem. *Let $F\colon X \to \mathcal{A}(H)$ be a quasi-lower semicontinuous multifunction from a metric space X to a hyperconvex space H with admissible values. Then for each $\varepsilon > 0$ there exists a continuous mapping $f\colon X \to H$ such that $F(x) \cap \bar{B}\big(f(x), \varepsilon\big) \neq \emptyset$ for all $x \in X$.* △

6.2.3. Lemma. *Let X be a metric space, H a hyperconvex metric space, $F\colon X \to \mathcal{A}(H)$ a quasi-lower semicontinuous multivalued mapping, $\eta > 0$ a positive number and $g\colon X \to H$ a continuous mapping such that for each $x \in X$, $\bar{B}\big(g(x), \eta\big) \cap F(x) \neq \emptyset$. Then the mapping $\bar{B}\big(g(\cdot), \eta\big) \cap F(\cdot)$ is a quasi-lower semicontinuous mapping from X to $\mathcal{A}(H)$.* △

Proof. Let $x \in X$ and $\varepsilon > 0$. There exists a $\delta > 0$ and $z \in F(x)$ such that for each $y \in U := \bar{B}(x, \delta)$, the intersection $\bar{B}(z, \varepsilon) \cap F(y)$ is nonempty and $D\big(\bar{B}(g(y), \eta), \bar{B}(g(x), \eta)\big) \leqslant d\big(g(x), g(y)\big) < \frac{1}{2}\varepsilon$. Let us define a multifunction $G\colon U \multimap H$ in the following way. Choose any $y \in U$;

if $y = x$, put $G(y) := G(x) := F(x)$. If $y \neq x$, pick any $w_1(y) \in \bar{B}(z,\varepsilon) \cap F(y)$ and $w_2(y) \in \bar{B}\big(g(y),\eta\big) \cap F(y)$, and define $G(y) := \operatorname{cov}\{w_1(y), w_2(y)\} \subseteq F(y)$. Consider the following ball intersection in the space H:

$$\Big[\bigcap_{y\in U} \bar{B}\big(g(y),\varepsilon+\eta\big)\Big] \cap \Big[\bigcap_{y\in U} \bar{B}\big(G(y),\varepsilon\big)\Big] \cap \bar{B}\big(g(x),\eta\big) \cap F(x). \qquad (6.1)$$

We are going to prove that it is nonempty. Let $y \in U$. First of all, notice that $\bar{B}\big(g(x),\eta\big) \subseteq \bar{B}\big(g(y),\varepsilon+\eta\big)$, so any two balls from the first term in (6.1) intersect. Further, for any $y \in U$, we have $w_1(y) \in G(y)$ and $d\big(w_1(y),z\big) \leqslant \varepsilon$, so $z \in \bar{B}\big(G(y),\varepsilon\big)$ and any two balls from the second term in (6.1) also intersect. Now, if $y_1, y_2 \in U$, the intersection $\bar{B}\big(g(y_1),\varepsilon+\eta\big) \cap \bar{B}\big(G(y_2),\varepsilon\big)$ is also nonempty, since $G(y_2) \cap \bar{B}\big(g(y_2),\eta\big) \neq \emptyset$ and $\bar{B}\big(g(y_2),\eta\big) \subseteq \bar{B}\big(g(y_1),\varepsilon+\eta\big)$. The facts that $\bar{B}\big(g(x),\eta\big)$ intersects with each ball in the first term and that $F(x)$ intersects with each ball in the second term are obvious – the former because of proximity of $g(x)$ and $g(y)$ and hyperconvexity of H, and the latter because z is contained in $F(x)$, the distance $d\big(w_1(y),z\big)$ is less or equal to ε, and $w_1(y) \in G(y)$ for $y \in U$. Further, consider the intersection $\bar{B}\big(G(y),\varepsilon\big) \cap \bar{B}\big(g(x),\eta\big)$ for $y \in U$; it is also nonempty, because $w_2(y) \in G(y)$ (so the former term is a superset of $\bar{B}\big(w_2(y),\varepsilon\big)$) and $d\big(g(x),w_2(y)\big) \leqslant \frac{1}{2}\varepsilon+\eta \leqslant \varepsilon+\eta$. Finally, let us look at the intersection $\bar{B}\big(g(y),\varepsilon+\eta\big) \cap F(x)$ for any $y \in U$. Choose any point $v \in \bar{B}\big(g(x),\eta\big) \cap F(x)$ and note that $d(v,g(y)) \leqslant d(v,g(x)) + d(g(x),g(y)) \leqslant \frac{1}{2}\varepsilon+\eta \leqslant \varepsilon+\eta$, so this intersection is also nonempty. The hyperconvexity of H yields now the nonemptiness of the whole intersection (6.1).

The remainder of the proof is now straightforward: take any v in the intersection (6.1) and observe that $v \in \bar{B}\big(g(x),\eta\big) \cap F(x)$ and at the same time, $\bar{B}\big(g(y),\eta\big) \cap G(y) \cap \bar{B}(v,\varepsilon) \neq \emptyset$ for all $y \in U$, which was exactly our claim. □

6.2.4. Theorem. *Every quasi-lower semicontinuous multifunction* $F\colon X \to \mathcal{A}(H)$ *from a metric space* X *to a hyperconvex space* H *with admissible values has a continuous selection.* △

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Proof. We shall construct a sequence of continuous functions uniformly convergent to the desired selection. By Theorem 6.2.2, there exists an $f_1 \colon X \to H$ such that for each point $x \in X$, the inequality $\operatorname{dist}\big(f_1(x), F(x)\big) \leqslant \frac{1}{2}$ is satisfied. Let us also define G_1 to be equal to F. Having defined $f_1, \dots, f_n$ and $G_1, \dots, G_n$, put $G_{n+1}(x) := \bar{B}\big(f_n(x), \frac{1}{2^n}\big) \cap G_n(x)$. By Lemma 6.2.3, the function G_{n+1} is quasi-lower semicontinuous and hence there exists a function $f_{n+1} \colon X \to H$ such that $\operatorname{dist}\big(f_{n+1}(x), G_{n+1}(x)\big) \leqslant \frac{1}{2^{n+1}}$ for $x \in X$. This means that $\operatorname{dist}\big(f_{n+1}(x), F(x)\big) \leqslant \frac{1}{2^{n+1}}$. By the triangle inequality we have $d\big(f_{n+1}(x), f_n(x)\big) < \frac{1}{2^{n+1}} + \frac{1}{2^n} < \frac{1}{2^{n-1}}$. Combining this with the previous inequality, we conclude that there exists a function $g_n \colon X \to H$ such that $g_n(x) \in F(x)$ and $d\big(f_n(x), g_n(x)\big) < \frac{1}{2^n}$ for all $x \in X$. Further, we have $d\big(g_{n+1}(x), g_n(x)\big) \leqslant \frac{1}{2^{n+1}} + \frac{1}{2^{n-1}} + \frac{1}{2^n} < \frac{1}{2^{n-2}}$ and hence $d\big(g_{n+k}(x), g_n(x)\big) < \frac{1}{2^{n-3}}$ for each $x \in X$ and $k, n \in \mathbb{N}$. Since each $F(x)$ is admissible and hence complete, the sequence g_n is uniformly convergent to some $f \colon X \to H$ satisfying $f(x) \in F(x)$ for all $x \in X$. It is now enough to observe that since $d\big(f_n(x), g_n(x)\big) < \frac{1}{2^n}$ and $d\big(g_n(x), f(x)\big) < \frac{1}{2^{n-3}}$ for all $x \in X$ and $n \in \mathbb{N}$, the sequence $\langle f_n \rangle_{n=1}^{\infty}$ is also uniformly convergent to f, which yields continuity of f and the proof is finished. □

6.2.5. Theorem. *Let* $F \colon X \multimap H$ *be a nonexpansive mapping of a metric space* X *with values admissible in a hyperconvex space* X. *Then* F *has a nonexpansive selection.* △

Proof. Let Σ be the family of all nonexpansive, admissible-valued multifunctions $\hat{F} \colon X \multimap H$ such that $\hat{F}(x) \subseteq F(x)$ for all $x \in X$. Obviously $\Sigma \neq \emptyset$, since $F \in \Sigma$. We partially order Σ by inclusion of graphs in $X \times H$. Let $\{\hat{F}_\lambda\}_{\lambda \in \Lambda}$ be a chain in Σ and $\widetilde{F} \colon X \multimap H$ be defined by the formula $\widetilde{F}(x) := \bigcap_{\lambda \in \Lambda} \hat{F}_\lambda(x)$. We want to show that $\widetilde{F} \in \Sigma$. From Lemma 4.3.7 we know that $\widetilde{F}(x)$ is nonempty (and hence admissible) for each $x \in X$. To prove that $\widetilde{F}$ is nonexpansive, take any $x, y \in X$; we have $\widetilde{F}(x) \subseteq F_\lambda(x) \subseteq \bar{B}\big(F_\lambda(y), d(x, y)\big)$ for any $\lambda \in \Lambda$, and taking the intersection over all $\lambda \in \Lambda$, we get $\widetilde{F}(x) \subseteq \bigcap_{\lambda \in \Lambda} \bar{B}\big(F_\lambda(y), d(x, y)\big) = \bar{B}\big(\widetilde{F}(x), d(x, y)\big)$, where the last equality follows from Lemma 4.2.10. This means that $D^*\big(\widetilde{F}(x), \widetilde{F}(y)\big) \leqslant d(x, y)$.

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Analogously, $D^*\big(\widetilde{F}(y), \widetilde{F}(x)\big) \leqslant d(x,y)$, which completes the proof of nonexpansiveness of $\widetilde{F}$. The Kuratowski–Zorn lemma yields the existence of a multifunction $\widetilde{F}$ minimal in Σ.

We will now prove that the values of $\widetilde{F}$ are actually singletons, which means that $\widetilde{F}$ is the desired selection (notice that for singletons, the Hausdorff distance coincides with the usual distance between the coresponding points). Let $y \in X$ be arbitrary and let $w \in \widetilde{F}(y)$. Define another multifunction $F^* \colon X \multimap H$ by the formula

$$F^*(x) := \widetilde{F}(x) \cap \bar{B}\big(w, d(x,y)\big)$$

for all $x \in X$. Obviously, $F^*(x) \subseteq \widetilde{F}(x) \subseteq F(x)$ for any $x \in X$. Also for any $x \in X$, we have $w \in F^*(y) \subseteq \bar{B}\big(F^*(x), d(x,y)\big)$, so we can see that $F^*(x) \cap \bar{B}\big(w, d(x,y)\big) \neq \emptyset$ and hence F^* has admissible values. It remains to show that F^* is nonexpansive. In order to see this, take any $x_1, x_2 \in X$ and let $r := d(x_1, x_2)$. Let $c_1 \in F^*(x_1)$. We will show that there exists some $c_2 \in F^*(x_2)$ such that $d(c_1, c_2) \leqslant r$. Denote $\widetilde{F}(x_2) = \bigcap_{\lambda \in \Lambda} \bar{B}_\lambda$, where all $\bar{B}_\lambda$'s are closed balls. We want to prove that

$$\bigcap_{\lambda \in \Lambda} \bar{B}_\lambda \cap \bar{B}\big(w, d(x_2, y)\big) \cap \bar{B}(c_1, r) \neq \emptyset.$$

Since $\widetilde{F}$ is nonexpansive, $\bar{B}(c_1, r) \cap \bar{B}_\lambda \neq \emptyset$ for any $\lambda \in \Lambda$. As $F^*(x_2) \neq \emptyset$, we know that $\bar{B}_\lambda \cap \bar{B}\big(w, d(x_2, y)\big) \neq \emptyset$. Finally, by definition of F^*, we know that $d(c_1, w) \leqslant d(x_1, y)$, so $d(c_1, w) \leqslant r + d(x_2, y)$. Our claim follows now from the hyperconvexity of H, so $D^*\big(F^*(x_1), F^*(x_2)\big) \leqslant r$. By a symmetrical argument, also $D^*\big(F^*(x_2), F^*(x_1)\big) \leqslant r$ and hence F^* is nonexpansive. This means that $F^* \in \Sigma$ and from the minimality of $\widetilde{F}$ we infer that $F^* = \widetilde{F}$. But the definition of F^* implies that $\widetilde{F}(y) = F^*(y) \subseteq \{w\}$; since $\widetilde{F}$ has nonempty values, $\widetilde{F}(y) = \{w\}$ and the proof is complete. □

6.3 Fixed point theorems

We will start this section with a simple consequence of Theorem 6.2.4.

6.3.1. Theorem. *Let* H *be a hyperconvex space and* $A \in \mathcal{A}(H)$ *a compact admissible subset of* H*. Then, every quasi-lower semicontinuous mapping* $F\colon A \multimap \mathcal{A}(A)$ *has a fixed point.* $\triangle$

Proof. Theorem 6.2.4 yields the existence of a continuous selection of F. It is now enough to apply Theorem 5.2.2. $\square$

There is also a version of the above result for the case where the codomain is larger than the domain.

6.3.2. Theorem. *Let* H *be a hyperconvex space and* $A \in \mathcal{A}(H)$ *a compact admissible subset of* H*. Assume that* $F\colon A \multimap \mathcal{A}(H)$ *is a quasi-lower semicontinuous multifunction satisfying the condition* $F(x) \cap A \neq \emptyset$ *for all* $x \in A$*. Then,* F *has a fixed point.* $\triangle$

Proof. We proceed similarly as in the proof of Lemma 6.2.3. For any point $x \in A$ and $\varepsilon > 0$, there exist: a point $z \in F(x)$ and a number $\delta > 0$ such that $F(y) \cap \bar{B}(z, \varepsilon) \neq \emptyset$ for any $y \in U := \bar{B}(x, \delta)$. Again, let $G(x) := F(x)$ and $G(y) := \operatorname{cov}\{w_1(y), w_2(y)\}$ for $y \neq x$, where $w_1(y) \in \bar{B}(z, \varepsilon) \cap F(y)$ and $w_2(y) \in A \cap F(y)$. We claim that the intersection

$$\left[\bigcap_{y \in U} \bar{B}\big(G(y), \varepsilon\big)\right] \cap F(x) \cap A \tag{6.2}$$

is nonempty. Indeed, $z \in \left[\bigcap_{y \in U} \bar{B}\big(G(y), \varepsilon\big)\right] \cap F(x)$, the intersection $F(x) \cap A$ is nonempty by assumption and $w_2(y) \in G(y) \cap A$ for each $y \in U$. Since H is hyperconvex, the whole set (6.2) is also nonempty. Choose any u in that intersection. Applying again the hyperconvexity of H, we have $G(y) \cap A \cap \bar{B}(u, \varepsilon) \neq \emptyset$ for all $y \in U$; in other words, the mapping $F(\cdot) \cap A$ is quasi-lower semicontinuous. Hence it has a continuous selection, which has a fixed point by Theorem 5.2.2. $\square$

In a similar way, Theorem 6.2.5 yields another fixed-point result.

6.3.3. Theorem. *Any nonexpansive admissible-valued multifunction on a bounded hyperconvex space has a fixed point.* △

Proof. It is enough to apply Theorems 6.2.5 and 5.1.1. □

In fact, it turns out that – just like in the single-valued case – more can be said.

6.3.4. Theorem. *Let* $F\colon H \multimap H$ *be a nonexpansive admissible-valued multifunction on a hyperconvex space* H*. If the fixed point set of* F *is nonempty, it is hyperconvex.* △

Proof. The idea of the proof is to construct the selection as in the proof of Theorem 6.2.5, but in such a way as to preserve the fixed point set. Let us start with defining Σ as the family of all admissible-valued, nonexpansive multifunctions $\hat{F}$ on H such that $\hat{F}(x) \subseteq F(x)$ for all $x \in H$ and $\operatorname{Fix}\hat{F} = \operatorname{Fix} F$. Very much like in the proof of Theorem 6.2.5 we show that there exists a minimal element $\widetilde{F}$ in Σ. Again we are going to show that $\widetilde{F}$ is singleton-valued. To prove this, let $y \in H$ be arbitrary. Let us define w in the following way. If $y \in \operatorname{Fix} F = \operatorname{Fix}\widetilde{F}$, we put $w := y$. In the opposite case, notice that the intersection of closed balls given by the formula

$$\widetilde{F}(y) \cap \bigcap_{z \in \operatorname{Fix}\widetilde{F}} \bar{B}\big(z, d(z,y)\big)$$

is nonempty. (Indeed, any of the balls $\bar{B}\big(z, d(z,y)\big)$, where $z \in \widetilde{F}(z)$, intersects with $\widetilde{F}(y)$ because of nonexpansivity of $\widetilde{F}$; also, every such ball contains y, so any two balls from the above formula intersect.) Choose any w in that intersection. In either case, we obtain a point $w \in \widetilde{F}(y)$ satisfying the inequality $d(z,w) \leqslant d(z,y)$ for any $z \in \operatorname{Fix}\widetilde{F}$.

Define now a multifunction $F^*\colon H \multimap H$ by the familiar formula

$$F^*(x) := \widetilde{F}(x) \cap \bar{B}\big(w, d(x,y)\big)$$

for all $x \in X$. Repeating the reasoning from the proof of Theorem 6.2.5, we can infer that F^* has nonempty (and hence admissible) values, is

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nonexpansive and $F^* \subseteq \widetilde{F} \subseteq F$ pointwise. From that last inclusion we also see that $\operatorname{Fix} F^* \subseteq \operatorname{Fix} \widetilde{F} = \operatorname{Fix} F$; we will now show the opposite inclusion. Let $z \in \operatorname{Fix} F = \operatorname{Fix} \widetilde{F}$, in other words, $z \in \widetilde{F}(z)$. We have chosen w so that $d(z, w) \leqslant d(z, y)$, and therefore $z \in \bar{B}\big(w, d(z, y)\big)$. Combining these, we obtain $z \in F^*(z)$.

We have proved that $F^*(x) \subseteq \widetilde{F}(x)$ for each $x \in H$; the minimality of $\widetilde{F}$ implies that $F^* = \widetilde{F}$. It is now enough to observe that $\widetilde{F}(y) = F^*(y) = \{w\}$ and apply Theorem 5.1.1. □

We will now turn to two theorems whose statements seem to belong rather to Section 4.4; the reason we include them here will be apparent in a moment from their proofs.

6.3.5. Theorem. *The space of all nonexpansive (single-valued) mappings from a metric space* X *to a bounded hyperconvex space* H*, endowed with the "maximum" metric, is hyperconvex.* △

Proof. Let $\{\bar{B}(f_i, r_i) \mid i \in I\}$ be a family of closed balls in the space of nonexpansive maps from X to H such that $d_{\max}(f_i, f_j) \leqslant r_i + r_j$ for all $i, j \in I$. We are going to construct (this time for real, no axiom of choice involved!) a certain multifunction $F\colon X \multimap H$. Define $F(x) := \bigcap_{i \in I} \bar{B}(f_i(x), r_i)$ for $x \in X$. Clearly, F has admissible values (notice that if $x \in X$, we have $d\big(f_i(x), f_j(x)\big) \leqslant d_{\max}(f_i, f_j) \leqslant r_i + r_j$ for $i, j \in I$, so hyperconvexity of H yields nonemptiness of $F(x)$).

We will now prove that F is nonexpansive with respect to the Hausdorff metric. Let $x, y \in X$ and denote $r := d(x, y)$. Pick any $c \in F(x)$ and let $i \in I$ be arbitrary; we know that $d(c, f_i(x)) \leqslant r_i$, so

$$d(c, f_i(y)) \leqslant d(c, f_i(y)) + d\big(f_i(y), f_i(x)\big) \leqslant r_i + r;$$

this means that $\bar{B}(f_i(y), r_i) \cap \bar{B}(c, r) \neq \emptyset$ for each $i \in I$. In other words, there exists some $c' \in \bigcap_{i \in I} \bar{B}(f_i(y), r_i)$ such that $d(c, c') \leqslant r$ and hence F is nonexpansive.

We can now apply Theorem 6.2.5 to obtain a nonexpansive selection f of F. Now, for any $i \in I$ and $x \in X$, we have $f(x) \in \bar{B}\big(f_i(x), r_i\big)$, so $d\big(f(x), f_i(x)\big) \leqslant r_i$ for any $x \in X$ and hence $d_{\max}(f, f_i) \leqslant r_i$. This in turn means that $f \in \bigcap_{i \in I} \bar{B}(f_i, r_i)$ and the proof is complete. □

6.3.6. Theorem. *The space of all nonexpansive retractions from a metric space* X *onto its bounded hyperconvex subset* H*, endowed with the "maximum" metric, is hyperconvex.* △

Proof. We proceed very similarly as in the proof of the previous theorem (although this time we will need the axiom of choice). Consider a family $\{\bar{B}(R_i, r_i) \mid i \in I\}$ of balls in the considered space such that $d_{\max}(R_i, R_j) \leqslant r_i + r_j$ for $i, j \in I$. Again, let $F(x) := \bigcap_{i \in I} \bar{B}_H(R_i(x), r_i)$ for $x \in X$; of course, $F(x)$ is nonempty for all $x \in X$. Let Σ be the family of all nonexpansive extensions of the identity mapping on H to supersets of H (i.e., nonexpansive retractions of subsets of X onto H) with the property that each such extension is a selection of an appropriate restriction of F. A reasoning analogous to that of Lemma 3.1.14 yields the existence of a maximal element $\widetilde{R}$ in Σ. If the domain $\mathrm{dom}_{\widetilde{R}}$ of $\widetilde{R}$ were not the whole X, we could select some point $s \in X \setminus \mathrm{dom}_{\widetilde{R}}$ and define

$$A := F(s) \cap \bigcap_{x \in \mathrm{dom}_{\widetilde{R}}} \bar{B}_H\big(\widetilde{R}(x), d(x,s)\big).$$

We will prove that A is admissible in H. We only need to show its nonemptiness. Notice that for any $x_1, x_2 \in \mathrm{dom}_{\widetilde{R}}$ we have

$$d\big(\widetilde{R}(x_1), \widetilde{R}(x_2)\big) \leqslant d(x_1, x_2) \leqslant d(x_1, s) + d(x_2, s),$$

and for any $x \in X$,

$$d\big(\widetilde{R}(x), R_i(s)\big) \leqslant d\big(\widetilde{R}(x), \widetilde{R}(s)\big) + d\big(\widetilde{R}(s), R_i(s)\big) \leqslant d(x,s) + r_i,$$

so hyperconvexity of H proves our claim. Choose now any $a \in A$ and define $\widetilde{R}_1\colon\ \mathrm{dom}_{\widetilde{R}} \cup \{s\} \to H$ by the formula

$$\widetilde{R}_1(x) := \begin{cases} \widetilde{R}(x) & \text{if } x \in \mathrm{dom}_{\widetilde{R}}, \\ a & \text{if } x = s. \end{cases}$$

We are going to prove that $\widetilde{R}_1$ is nonexpansive. It is enough to notice that for any $x \in \mathrm{dom}_{\widetilde{R}}$, we have $d\big(\widetilde{R}_1(x), \widetilde{R}_1(s)\big) = d(\widetilde{R}(x), a) \leqslant d(x,s)$. It turns out that $\widetilde{R}_1$ is a nonexpansive retraction onto H and at the same time a selection of the restriction $F|_{\mathrm{dom}_{\widetilde{R}} \cup \{s\}}$, which contradicts

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the maximality of $\widetilde{R}$. This way, we have shown that $\widetilde{R}\colon X \to H$ is a nonexpansive retraction and a selection of F.

The rest is now straightforward: take an arbitrary $i \in I$, see that $\widetilde{R}(x) \in \bar{B}_H\big(R_i(x), r_i\big)$ for any $x \in X$, which means that $d_{max}(\widetilde{R}, R_i) \leqslant r_i$, and conclude that $\widetilde{R} \in \bigcap_{i \in I} \bar{B}(R_i, r_i)$. □

Notes and remarks

Definition 6.2.1 and its relation to the better known notion of *lower semi-continuous* multifunction can be found in [36] and in the references therein. Theorem 6.2.2 was originally proved in [41] for the case when X is a paracompact topological space, H is a so-called *l.c.-space* and F has so-called H-*convex* values. However, C. D. Horvath proved in [29] that hyperconvex spaces are l.c.-spaces and admissible subsets are H-convex, and therefore we could apply the theorem in our setting. Lemma 6.2.3 and Theorem 6.2.4 are also proved in [36] in slightly more general form, using the notion of *sub-admissible* sets, of which admissible sets are a special case. Theorem 6.2.5 is taken from [39].

Theorems 6.3.1 and 6.3.2 are also proved in [36]. Theorems 6.3.3 and 6.3.4 are (rather unsurprisingly) proved in [39], as well as their corollaries, Theorem 6.3.5 and 6.3.6.

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