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VALUE FUNCTIONS IN CONTROL SYSTEMS AND DIFFERENTIAL GAMES: A VIABILITY METHOD

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Sławomir Plaskacz

Abstract

The present work is the habilitation dissertation of the author written in the Faculty of Mathematics and Computer Science of the Nicolaus Copernicus University in Toruń, Poland.

The paper, divided into 6 Chapters, is devoted to the study of properties of value functions in deterministic control systems and differential games. The main goal is to characterize the value functions as the unique solution of the Hamilton–Jacobi equations. New definitions of weak solutions are introduced and uniqueness as well as existence results are obtained. Viability and invariance theorems for differential inclusions and differential games are the main tools to study invariance properties of the epigraph and/or hypograpf of the value functions.

Viability (invariance) problem is considered for time-dependent constraint sets (tubes) and measurable in time differential inclusions. The tangential condition necessary and sufficient to viability (invariance) of the tube is assumed to hold true for almost all t. The measurable viability and invariance theorems presented in Chapter 2 are the main tools to obtain the characterization of the value function in the Mayer problem with dynamics measurable in time and a lower semicontinuous terminal cost function. The value function is proved to be the semicontinuous solution such that the equality in the Hamilton–Jacobi equation holds true for almost all t and every subgradient.

Control systems with state constraints are considered in Chapter 3. State constraints are given by a closed, not necessarily smooth, set. The value functions in the Bolza problem and in the infinite horizon problem are the unique solutions of the corresponding Hamilton–Jacobi–Bellman equations. Set-valued analysis tools, like paratingent cones, play crucial role in the formulation of controllability assumptions at the boundary of the set of constraints. This assumptions do not imply the continuity of the value function.

Zero sum differential games are considered in the framework of non-anticipative strategies. It is shown that the upper and lower values coincide for games with dynamics measurable in time. In Chapter 4, viability theory is extended to differential games. Discriminating and leadership tubes in differential games play a role similar to the one played by viability and invariance tubes

in differential inclusions. The proof of the discrimination theorem makes use of a non-expansive selection theorem in ultrametric spaces. To prove the existence of value in the Mayer problem for differential games with semicontinuous terminal cost several methods have been used. The viability method was combined with inf-convolutions approximation and some stability properties of viscosity solutions.

Oleinik–Lax explicit formula for the solution of Hamilton–Jacobi equation is generalized in Chapter 6 to the case when Hamiltonians depend on time and on u, where $u(t, x)$ denotes the solution. It is used together with a commutation property of reachable maps of differential inclusions to establish the existence and uniqueness of solutions to some overdetermined systems of the Hamilton–Jacobi equations.

The work is based on some earlier papers of the author.

INTRODUCTION

We consider value functions for deterministic control systems and zero-sum differential games with dynamics governed by ordinary differential equations. Value functions satisfy the Dynamic Programming Property. If the value function is smooth then the Dynamic Programming Property leads to a first order PDE which is called Hamilton–Jacobi–Bellman's equation in control theory and Hamilton–Jacobi–Isaacs' equation in games. One of the main difficulties consists in the fact that the value function usually is not smooth. In the early 80' Crandall and Lions introduced a notion of weak solutions to the Hamilton–Jacobi equations – called viscosity solutions [35], [34]. Another concepts of weak solutions to first order PDE's are minimax solutions introduced by Subbotin [93], [95] in the framework of positional differential games and contingent solutions introduced by Aubin to study Lyapunov functions [5]. These concepts of weak solutions base on some tools of nonsmooth analysis. If the value function is continuous then it is a unique viscosity solution of the corresponding Hamilton–Jacobi equation, as well as a unique minimax or contingent solution.

In many control problems, especially with state constraints, the value function is discontinuous. In order to describe the value function as a unique solution to the corresponding Hamilton–Jacobi equation it was necessary to modify the notion of a weak solution. It was done by Barron–Jensen [14] and Frankowska [44] by means of different methods.

Viability approach to the problem of the description of a discontinuous value function was initiated by Frankowska in [44]. This method is based on the fact that the value function is uniquely determined by invariance properties of its epigraph with respect to an appropriate dynamical system. In the Mayer problem for control systems the epigraph of value function is forward (in time) viable and backward invariant. These two properties of the epigraph and a terminal condition uniquely characterize the value function. Viability theory provides geometric conditions which are equivalent to viability or invariance properties. These conditions can be expressed with contingent cones or with normal cones.

In the dissertation, we present some generalizations of viability and invariance theorems. Our generalizations are done in the direction determined by our future applications. We obtained viability and invariance theorems for differential inclusions with the right-hand side measurable in time and for varying in time state constraints (tube). We assume that the tube is (left) absolutely continuous. The regularity assumption on the tube is justified by the fact that the epitube corresponding to the value function satisfies it. The obtained viability and invariance results allow to characterize value function as the unique solution of the corresponding Hamilton–Jacobi equation. Solutions are understood in the meaning given in the dissertation, i.e. the Hamilton–Jacobi equation holds true for almost all t. The uniqueness results justify our definition of solution.

We use a similar scheme to study value functions for differential games. Discriminating and leadership theorems play the role of viability and invariance. Differential games are considered in the framework of nonanticipative strategies. The proof of existence of a nonanticipative strategy in discriminating theorem base on a selection lemma about existence of a nonexpansive selection in ultrametric spaces. The lemma, in the author's opinion, is a convenient tool to reduce some differential games problems to differential inclusion.

Viability approach to optimal control is especially useful for problems with state constraints. We describe value function as an appropriate solution to the corresponding Hamilton–Jacobi–Bellman equation under assumptions that a priori exclude continuity of value. Thus other methods seem to be difficult to apply.

We also use viability approach jointly with some other methods. In the Mayer problem for differential games with semicontinuous terminal cost we prove existence of value using discriminating theorem and stability of viscosity supersolutions. We show that the value function in this problem is a unique generalized solution (in the meaning similar to envelope solutions) to the corresponding Hamilton–Jacobi equation.

In Chapter 2, we apply viability approach for the Mayer problem with dynamics measurable in time and a lower semicontinuous terminal cost function. We generalize viability and invariance theorems to the case when the right-hand side of differential inclusion is measurable in time and the set of constraints depends upon the time (Theorems 2.2.5, 2.2.2, 2.2.6). Our key observation is that tangency conditions have to be satisfied almost everywhere with respect to t. We assume that the tube of constraints is absolutely continuous. Another measurable viability theorem was obtained by Bothe [20]. The main difference between our Theorem 2.2.5 and Bothe's viability results is that our tangency condition is formulated in a weaker way involving the convexification of Bouligand's tangent cone. Thanks to this the tangency conditions in viability and invariance theorems can be equivalently formulated in a dual way involving normal cones. Next, we apply our measurable viability and invariance theorems to prove that the value function is the unique weak solution to the corresponding

Hamilton–Jacobi–Bellman equation in the meaning that the equality in the equation holds for almost all t. If the dynamics is measurable in time, then so is the Hamiltonian H. Viscosity solutions in this case were also studied in [72], [13] using super- and subdifferentials involving L^1 test function. The main result of the chapter is Theorem 2.3.4.

In Chapter 3, we consider control systems with degenerate state constraints. The set of constraints K is the closure $K = \operatorname{cl} D$ of an arbitrary subset $D \subset \mathbb{R}^n$. We present two types of control problems: the Bolza problem and infinite horizon problem. In both cases we pose minimal assumptions to apply the viability approach. In both cases, the main difficulty appears in the proof that if a function W is a backward invariance domain of a corresponding differential inclusion then W is dominated by the value function (comp. Propositions 3.1.6, 3.2.11). The key point of the proof is the construction of a control $u(\cdot)$ and a corresponding trajectory $x_u(\cdot)$ that are close to a given pair $(\overline{u}, x_{\overline{u}})$ ($x_{\overline{u}}(t) \in K$ for every $t \in [t_0, T]$) and moreover the trajectory $x_{\overline{u}}$ remains in D. The crucial assumptions to perform the construction of (u, x_u) are (3.4), (3.5) for nonautonomous systems and (3.32), (3.33) for autonomous one. To formulate the assumptions, we use paratingent cones. If the set D is an open set Ω with a smooth boundary $\partial\Omega$ then our assumptions are equivalent to the following condition

$$\forall t,\ \forall x \in \partial\Omega,\ \exists u, \quad \langle f(t,x,u), n(x)\rangle > 0$$

where $n(x)$ is an exterior normal.

The control problems with state constraints were considered in [92], [23], [62]. In those papers authors assume that the dynamics of the system satisfies an opposite condition (called the Soner condition)

$$\exists \varepsilon > 0,\ \forall t,\ \forall x \in \partial\Omega,\ \exists u \in U, \quad \langle f(t,x,u), n(x)\rangle < -\varepsilon.$$

Under Soner's condition the value function is continuous. We provide examples of control systems for which the value function is discontinuous but despite of this fact it is the unique solution (in the sense proposed in the chapter) of the corresponding Hamilton–Jacobi equation.

A different approach to the Mayer control problems with state constraints is provided in Section 3.3. Using results of Chapter 5 we obtain a general description of a value function in the Mayer problems with totally discontinuous terminal cost function. Next adopting the classical method of adding an extra variable (usually used to reduce the Bolza problem to the Mayer one) and the technique of penalty function we characterize the value function in the Mayer control problem with state constraints as a generalized solution of the corresponding Hamilton–Jacobi equation.

The main results of the chapter are Theorems 3.1.7, 3.2.1 and 3.3.2.

In the second part of the dissertation we consider zero-sum differential games with dynamics given by $x'(t) = f(t, x(t), y, z)$. By $x(\,\cdot\,; t_0, x_0, y(\,\cdot\,), z(\,\cdot\,))$ we denote the solution of the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t), y(t), z(t)) & \text{for a.e. } t \in [0, T], \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $y\colon [0, T] \to Y$, $z\colon [0, T] \to Z$ are measurable controls (open loops) of player I and II, respectively and Y, Z are compact metric spaces.

Let $M_t = \{y\colon [t, T] \to Y : y \text{ is measurable}\}$ and $N_t = \{z\colon [t, T] \to Z : z \text{ is measurable}\}$. We say that a map $\alpha\colon N_t \to M_t$ is a nonanticipative strategy of the first player if for every control z_1, $z_2 \in N_t$ such that

$$z_1(s) = z_2(s) \quad \text{for almost all } s \in [t, \tau]$$

we have

$$\alpha(z_1)(s) = \alpha(z_2)(s) \quad \text{for almost all } s \in [t, \tau].$$

We say that a map $\beta\colon M_t \to N_t$ is a nonanticipative strategy of the second player if for every control y_1, $y_2 \in M_t$ such that

$$y_1(s) = y_2(s) \quad \text{for almost all } s \in [t, \tau]$$

we have

$$\beta(y_1)(s) = \beta(y_2)(s) \quad \text{for almost all } s \in [t, \tau].$$

Let Γ_t, Δ_t denote the set of all nonanticipative strategies of the first and of the second player, respectively.

We shall consider a terminal time payoff functional

$$Q(y, z) = Q_{t_0 x_0}(y, z) = g(x(T, t_0, x_0, y, z)),$$

where $g\colon \mathbb{R}^n \to \mathbb{R}$ is a terminal cost function, $y \in M_{t_0}$, $z \in N_{t_0}$. The aim of the first player is to maximize the payoff, the aim of the second player is to minimize it.

The value function of the first player is given by

$$U^+(t_0, x_0) = \sup_{\alpha \in \Gamma_{t_0}} \inf_{z \in N_{t_0}} Q_{t_0 x_0}(\alpha(z), z). \tag{2}$$

The value function of the second player is given by

$$U^-(t_0, x_0) = \inf_{\beta \in \Delta_{t_0}} \sup_{y \in M_{t_0}} Q_{t_0, x_0}(y, \beta(y)). \tag{3}$$

The value of the first player U^+ is also called an upper value and U^- is called a lower value. If the upper value is equal to the lower value then we say the game has a value. The main problem in zero sum differential games is the existence of value. It has been considered by many authors. A pioneering work was that of Isaacs [60]. He introduced condition (4) which provides the existence of the value in the case where both values are smooth,

$$\max_{y\in Y}\min_{z\in Z}\langle f(t,x,y,z),p\rangle = \min_{z\in Z}\max_{y\in Y}\langle f(t,x,y,z),p\rangle \quad \text{for every } t,\ x \text{ and } p\in\mathbb{R}^n. \tag{4}$$

Later on several concepts of strategies appeared (see [39], [37], [67]). For these concepts of strategy technical proofs of the existence of value were provided. Evans and Souganidis in [38] proved that if g is Lipschitz continuous and f is continuous and Lipschitz continuous with respect to x then the upper value U^+ is the viscosity solution of the upper Isaacs equation

$$\begin{cases} U_t + H^+(t,x,U_x) = 0 & (0\le t\le T,\ x\in\mathbb{R}^n),\\ U(T,x) = g(x) & (x\in\mathbb{R}^n), \end{cases} \tag{5}$$

where the upper Hamiltonian H^+ is given by

$$H^+(t,x,p) = \min_{z\in Z}\max_{y\in Y}\langle f(t,x,y,z),p\rangle$$

and the lower value U^- is the viscosity solution to the lower Isaacs equation

$$\begin{cases} U_t + H^-(t,x,U_x) = 0 & (0\le t\le T,\ x\in\mathbb{R}^n),\\ U(T,x) = g(x) & (x\in\mathbb{R}^n), \end{cases} \tag{6}$$

where the lower Hamiltonian H^- is defined by

$$H^-(t,x,p) = \max_{y\in Y}\min_{z\in Z}\langle f(t,x,y,z),p\rangle.$$

The Isaacs condition (4) says that $H^- = H^+$. Thus the upper and the lower Isaacs equations are the same. A direct conclusion from uniqueness of viscosity solutions to (5) and to (6) is that the value of the game exists.

In Chapter 4 we prove the existence of value for a game with dynamics given by a right-hand side $f(t,x,u,v)$ measurable in t. Our scheme of proof follows the same arguments as in [38]. We prove that the upper and the lower values are generalized solutions to the same Isaacs equation (under Isaacs condition). Our approach is based on the notions of discriminating and leadership domains. Briefly speaking, we say that a tube $P(t)$ has discriminating property for the first player if for every initial condition at the tube there exists a strategy of the first player such that whatever control is chosen by the second player the corresponding trajectory remains in the tube P. In Theorem 4.2.2 we give a geometric sufficient and necessary condition (4.3) to discriminating property for games with dynamics measurable in time and absolutely continuous tubes. In the continuous case analogous result was obtained by Cardaliaguet in [24]. The crucial role in the proof of Theorem 4.2.2 is played by a nonexpansive selection theorem in ultrametric spaces (see Lemma 4.1.1) and measurable Viability Theorem 2.2.2. This lemma became a standard tool for reducing some differential games problems to differential inclusions.

In Chapter 5 we consider the existence of value for a game where the terminal cost function g is an extended lower semicontinuous function. To prove the existence of value we have to modify the notion of upper and lower value (see (5.1)). Since the value function is discontinuous and the Hamiltonian is not convex with respect to the last variable then the known uniqueness results for viscosity solutions of PDE cannot be used. In the proof of the existence of a value we use several methods. Firstly, we use viability theory (discriminating theorem in the version of Theorem 4.2.3) to show that any supersolution (subsolution) is greater (lower) or equal to the lower (upper) value. Secondly, we apply inf-convolutions to approximate a lower semicontinuous function by an increasing sequence of Lipschitz continuous functions. Thirdly, we base on the Barles–Perthame stability result [10] and the Evans–Souganidis [38] existence of value result in the Lipschitz case. Our studies were motivated by Subbotin's existence of a value result for differential games with positional strategies [94].

In [73], P. L. Lions and J. C. Rochet obtained existence and uniqueness of viscosity solutions to some overdetermined systems of the Hamilton–Jacobi equation. In Chapter 6, we generalize this result to a wider class of Hamiltonians. In the proof we use a commutation property of reachable maps of differential inclusions and the Oleinik–Lax type explicit formula of solution to the Hamilton–Jacobi equation. We obtain an explicit representation formula of the value function which generalizes some result obtained recently by Barron–Jensen–Liu [16] and Alvarez–Barron–Ishii [1].

The dissertation is mainly based on some earlier works of the author.

Chapter 2 contains some results obtained with H. Frankowska and T. Rzeżuchowski and previously published in [50], [51], [46].

Results of Chapter 3 have been obtained with H. Frankowska and M. Quincampoix and come from [47]–[49], [80].

Results obtained with P. Cardaliaguet and published in [26], [25] are collected in Chapter 4.

The contents of Chapter 5 appears in a joint paper with M. Quincampoix [79]. Chapter 6 is based on results obtained with M. Quincampoix [81].

CHAPTER 1

PRELIMINARIES

In this chapter we shall acquaint the reader with set valued analysis, viability theory and viscosity solutions of Hamilton–Jacobi equations. Comprehensive treatment of these topics can be found in [7], [87], [3], [34], [9]. So the only properties relevant to our purposes will be listed here, mostly without proofs.

1.1. Set limits

Let $\mathcal{T}$ be a metric space and $\{A_\tau\}_{\tau\in\mathcal{T}}$ be a family of subsets of a metric space X. The upper limit Lim sup and the lower limit Lim inf of A_τ at $\tau_0 \in T$ are closed sets defined by

$$\begin{aligned}
\operatorname*{Lim\,sup}_{\tau\to\tau_0} A_\tau &= \{v \in X : \liminf_{\tau\to\tau_0} \operatorname{dist}(v, A_\tau) = 0\},\\
\operatorname*{Lim\,inf}_{\tau\to\tau_0} A_\tau &= \{v \in X : \limsup_{\tau\to\tau_0} \operatorname{dist}(v, A_\tau) = 0\}.
\end{aligned}$$

A subset $A \subset X$ is said to be the limit of A_τ if

$$A = \operatorname*{Lim\,inf}_{\tau\to\tau_0} A_\tau = \operatorname*{Lim\,sup}_{\tau\to\tau_0} A_\tau =: \operatorname*{Lim}_{\tau\to\tau_0} A_\tau.$$

1.2. Tangent and normal cones

Let $K \subset \mathbb{R}^n$ be a nonempty subset and $x \in K$. The contingent cone $T_K(x)$ to K at x is defined by

$$v \in T_K(x) \Leftrightarrow \liminf_{h\to 0^+} \frac{\operatorname{dist}(x + hv, K)}{h} = 0.$$

The proximal normal cone $N_K(x)$ to K at x is defined by

$$N_K(x) = \{v \in \mathbb{R}^n : \exists \alpha > 0,\ \operatorname{dist}(x + \alpha v, K) = |\alpha v|\}.$$

The negative polar cone T^- to a subset $T \subset \mathbb{R}^n$ is given by

$$T^- = \{v \in \mathbb{R}^n : \forall w \in T,\ \langle v, w\rangle \le 0\}.$$

We set $N^0_K(x) = T^-_K(x)$ and say that $N^0_K(x)$ is the normal cone to K at $x \in K$.

It is well known that $N_K(x) \subset N^0_K(x)$. If K is the graph of $y = x^\alpha$, $1 < \alpha < 2$ then $N^0_K(0,0) = \{(v_1, v_2) : v_1 = 0\} \neq N_K(0,0) = \{(v_1, v_2) : v_1 = 0,\ v_2 \leq 0\}$.

Ioffe proved in [59] that

$$\operatorname*{Lim\,sup}_{K \ni y \to x} N^0_K(y) = \operatorname*{Lim\,sup}_{K \ni y \to x} N_K(y). \tag{1.1}$$

We denote by $M_K(x)$ the cone $\limsup_{K \ni y \to x} N^0_K(y)$. For the first time it was considered by Mordukhovich in [75]. If $K = \{1/n : n = 1, 2, \dots\} \cup \{0\}$ then $M_K(0) = \mathbb{R} \neq N^0_k(0) = (-\infty, 0]$.

It was proved by Cornet [33] (see also [7, p. 130]) that

$$\operatorname*{Lim\,inf}_{K \ni y \to x} T_K(y) = \operatorname*{Lim\,inf}_{K \ni y \to x} \overline{\mathrm{co}}(T_K(y)) = C_K(x) \subset T_K(x), \tag{1.2}$$

where $C_K(x)$ denotes Clarke tangent cone to K at x

$$C_K(x) = \operatorname*{Lim\,inf}_{K \ni y \to x,\, h \to 0^+} \frac{K - y}{h}.$$

The paratingent cone $P^L_K(x)$ to K, relative to $L \subset K$, at $x \in L$ is defined by

$$v \in P^L_K(x) \Leftrightarrow \exists h_n \to 0^+,\ \exists v_n \to v,\ \exists x_n \to x,\ x_n \in L \text{ and } x_n + h_n v_n \in K.$$

Proposition 1.2.1 (see Bouligand [21] and Choquet [30]). *Let L be a subset of a closed set $K \subset \mathbb{R}^n$, $x_0 \in L$ and $w \notin P^L_K(x_0)$. Then there is an $\varepsilon > 0$ such that for every $x \in K \cap B(x_0, \varepsilon)$*

$$(x + (0, \varepsilon] B(-w, \varepsilon)) \cap K \subset (K \setminus L).$$

Proof. Otherwise there are $x_n \to x_0$, $x_n \in K$, $h_n \to 0^+$, $w_n \to w$ such that $y_n := x_n - h_n w_n \in L$. Then $y_n \to x_0$, $y_n \in L$, $y_n + h_n w_n \in K$, so $w \in P^L_K(x_0)$. Hence we get a contradiction. $\square$

Remark. (a) Consider an open subset $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary and let $K = \overline{\Omega}$, $L = \partial\Omega$, $x \in \partial\Omega$. Denote by $n(x)$ the outer normal to Ω at $x \in \partial\Omega$. We assume that $n(\cdot)$ is continuous on $\partial\Omega$. Let $v \notin P^L_K(x)$. So for all sequences $x_n \xrightarrow{\partial\Omega} x$, $h_n \to 0^+$, $v_n \to v$ we have $x_n + h_n v_n \notin K$ for n large enough. In particular, taking $x_n = x$, we deduce that for some $\varepsilon > 0$

$$(x +]0, \varepsilon] B(v, \varepsilon)) \cap K = \emptyset. \tag{1.3}$$

Hence $\langle n(x), v \rangle > 0$. Conversely, if for some $v \in \mathbb{R}^n$, $\langle n(x), v \rangle > 0$, then $v \notin P^L_K(x)$.

(b) If K is a smooth submanifold of $\mathbb{R}^n$ with the boundary $L = \partial K$, then the paratingent cone $P^L_K(x)$ at a point $x \in L$ is equal to the tangent half space $H_x K = \{v \in T_x K : \langle v, n(x) \rangle \leq 0\}$, where $n(x)$ is an exterior normal to ∂K at x with respect to K.

Systematic presentation of different concepts of tangent cones can be found in Chapter 4 in [7].

1.3. Some elements of nonsmooth analysis

Let us recall generalizations of notions of directional derivatives and gradients for nonsmooth functions.

Definition 1.3.1. Consider an extended function $\varphi\colon \mathbb{R}^n \mapsto \mathbb{R} \cup \{\pm\infty\}$.

(a) The domain of φ, $\mathrm{Dom}(\varphi)$, is the set of all x_0 such that $\varphi(x_0) \neq \pm\infty$.

(b) The subdifferential and the superdifferential of φ at $x_0 \in \mathrm{Dom}(\varphi)$ are respectively given by

$$\partial_- \varphi(x_0) = \left\{ p \in \mathbb{R}^n : \liminf_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \right\}$$

and

$$\partial_+ \varphi(x_0) = \left\{ p \in \mathbb{R}^n : \limsup_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \leq 0 \right\}.$$

(c) The contingent epiderivative and the contingent hypoderivative of φ at $x_0 \in \mathrm{Dom}(\varphi)$ in the direction $u \in \mathbb{R}^d$ are respectively defined by

$$D_\uparrow \varphi(x_0)(u) = \liminf_{h \to 0+,\, u' \to u} \frac{\varphi(x_0 + hu') - \varphi(x_0)}{h}$$

and

$$D_\downarrow \varphi(x_0)(u) = \limsup_{h \to 0+,\, u' \to u} \frac{\varphi(x_0 + hu') - \varphi(x_0)}{h}.$$

It was shown in [7, p. 226] that for all $x_0 \in \mathrm{Dom}(\varphi)$

$$\mathcal{E}pi(D_\uparrow \varphi(x_0)) = T_{\mathcal{E}pi(\varphi)}(x_0, \varphi(x_0)), \tag{1.4}$$

where $\mathcal{E}pi$ denotes the epigraph. Similarly

$$\mathcal{H}yp(D_\downarrow \varphi(x_0)) = T_{\mathcal{H}yp(\varphi)}(x_0, \varphi(x_0)),$$

where $\mathcal{H}yp$ denotes the hypograph.

From [43] (see also [7, pp. 249, 253]) we know that

Proposition 1.3.2. *Let $\varphi\colon \mathbb{R}^d \mapsto \mathbb{R} \cup \{\pm\infty\}$ and $x_0 \in \mathrm{Dom}(\varphi)$. Then*

$$\begin{aligned} p \in \partial_- \varphi(x_0) &\Leftrightarrow \forall u \in \mathbb{R}^d,\ \langle p, u \rangle \leq D_\uparrow \varphi(x_0)(u) \\ &\Leftrightarrow (p, -1) \in [T_{\mathcal{E}pi(\varphi)}(x_0, \varphi(x_0))]^- \quad (\textit{the negative polar cone}) \end{aligned}$$

and

$$\begin{aligned} p \in \partial_+ \varphi(x_0) &\Leftrightarrow \forall u \in \mathbb{R}^d,\ \langle p, u \rangle \geq D_\downarrow \varphi(x_0)(u) \\ &\Leftrightarrow (p, -1) \in [T_{\mathcal{H}yp(\varphi)}(x_0, \varphi(x_0))]^+ \quad (\textit{the positive polar cone}). \end{aligned}$$

Proposition 1.3.3. *Let* $\varphi\colon \mathbb{R}^d \mapsto \mathbb{R}\cup\{\pm\infty\}$ *and* $x_0 \in \mathrm{Dom}(\varphi)$. *Then*

$$\partial_-\varphi(x_0) \neq \emptyset \Rightarrow [T_{\mathcal{E}pi(\varphi)}(x_0,\varphi(x_0))]^- = \overline{\bigcup_{\lambda\geq 0} \lambda(\partial_-\varphi(x_0),-1)}$$

and

$$\partial_+\varphi(x_0) \neq \emptyset \Rightarrow [T_{\mathcal{H}yp(\varphi)}(x_0,\varphi(x_0))]^+ = \overline{\bigcup_{\lambda\geq 0} \lambda(\partial_+\varphi(x_0),-1)}.$$

Proof. We only prove the first statement. By Proposition 1.3.2

$$[T_{\mathcal{E}pi(\varphi)}(x_0,\varphi(x_0))]^- \supset \overline{\bigcup_{\lambda\geq 0} \lambda(\partial_-\varphi(x_0),-1)}.$$

Fix any $(p,q) \in [T_{\mathcal{E}pi(\varphi)}(x_0,\varphi(x_0))]^-$. Since $\{0\}\times\mathbb{R}_+ \subset T_{\mathcal{E}pi(\varphi)}(x_0,\varphi(x_0))$ we deduce that $q \leq 0$. If $q<0$, then $(p/|q|,-1) \in [T_{\mathcal{E}pi(\varphi)}(x_0,\varphi(x_0))]^-$ and by Proposition 1.3.2, $p/|q| \in \partial_-\varphi(x_0)$. Hence $(p,q) \in \bigcup_{\lambda\geq 0}\lambda(\partial_-\varphi(x_0),-1)$. It remains to consider the case $q=0$. Let $\overline{p} \in \partial_-\varphi(x_0)$. Then $(\overline{p},-1) \in [T_{\mathcal{E}pi(\varphi)}(x_0,\varphi(x_0))]^-$ and from convexity of the polar cone, for all $\mu \in]0,1[$

$$(\mu\overline{p}+(1-\mu)p,-\mu) \in [T_{\mathcal{E}pi(\varphi)}(x_0,\varphi(x_0))]^-.$$

By the first part of the proof, $(\mu\overline{p}+(1-\mu)p,-\mu) \in \bigcup_{\lambda\geq 0}\lambda(\partial_-\varphi(x_0),-1)$. Taking the limit when $\mu \to 0+$ we end the proof. □

The following Rockafellar's result (see [87]) gives more information about the connection beetwen subgradients and normals to epigraph.

Lemma 1.3.4. *Let* $\varphi\colon\mathbb{R}^n \mapsto \mathbb{R}\cup\{+\infty\}$ *be an extended lower semicontinuous function. If* $(p,0) \in [T_{\mathcal{E}pi(\varphi)}(x_0,\varphi(x_0))]^-$ *then there exist* $x_n \to x_0$, $p_n \to p$, $q_n \to 0$, $q_n < 0$ *such that*

$$(p_n,q_n) \in [T_{\mathcal{E}pi(w)}(x_n,w(x_n))]^- \quad \textit{and} \quad \varphi(x_n) \to \varphi(x_0).$$

1.4. Viability and invariance for fixed set of state constraint

A subset $K \subset \mathbb{R}^n$ is locally compact if for every $x \in K$ there exists a closed ball $\overline{B}(x,r)$ centered at x with a radius $r>0$ such that the set $D\cap\overline{B}(x,r)$ is closed. A locally compact subset K of $\mathbb{R}^n$ is a viability domain of a set-valued map $G\colon\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ if for every $x \in K$

$$G(x)\cap T_K(x) \neq \emptyset.$$

Define the Hamiltonian $H\colon\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ by

$$H(x,p) = \sup_{g\in G(x)} \langle g,p\rangle$$

The following formulation summarizes several versions of the viability theorem (comp. [4], [3]).

Theorem 1.4.1 (Viability). *Suppose that $G\colon \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is an upper semicontinuous map with compact convex values. For a locally compact subset $K \subset \mathbb{R}^n$ the following conditions are equivalent:*

(a) *K is a viability domain of G;*
(b) *$G(x) \cap \overline{\mathrm{co}}\, T_K(x) \neq \emptyset$ for every $x \in K$;*
(c) *$H(x, -n) \geq 0$ for every $x \in K$ and every $n \in [T_K(x)]^-$;*
(d) *$H(x, -n) \geq 0$ for every $x \in K$ and every $n \in N_K(x)$;*
(e) *$H(x, -n) \geq 0$ for every $x \in K$ and every $n \in M_K(x)$;*
(f) *for every $x_0 \in K$ there is $T > 0$ and a solution $x\colon [0, T] \to K$ of the Cauchy problem*

$$\begin{cases} x'(t) \in G(x(t)), \\ x(0) = x_0; \end{cases} \tag{1.6}$$

(g) *for every $x_0 \in K$ there is $T > 0$ such that for every $h \in (0, T)$, $\varepsilon > 0$ there is a solution $x\colon [0, h] \to \mathbb{R}^n$ of* (1.6) *such that*

$$\operatorname{dist}(x(h), K) < \varepsilon.$$

If K is closed and G is of linear growth the above conditions (a)–(g) *are equivalent to*

(h) *for every $x_0 \in K$ there is a solution $x\colon [0, \infty) \to K$ of the Cauchy problem* (1.6)

Proof. We sketch the proof for the reader's convenience. Since G is an upper semicontinuous set-valued map, it follows that the Hamiltonian H is an upper semicontinuous function. Thus, by (1.1) we obtain equivalence of (c)–(e). The implication (c) $\Rightarrow$ (b) follows from the separation theorem. In [55], the equivalence (a) $\Leftrightarrow$ (b) was proved. The fact that (a) $\Leftrightarrow$ (f) can be found in Aubin–Cellina [5] or Aubin [4], [3]. Below we prove that (g) $\Rightarrow$ (a).

Fix $x_0 \in K$, $\varepsilon > 0$ and choose $\delta > 0$ such that for $|x - x_0| \leq \delta$ we have $G(x) \subset G(x_0) + \varepsilon B$. There is $\theta > 0$ such that every solution $x(\,\cdot\,)$ to (1.6) satisfies $|x(t) - x_0| \leq \delta$ for $t \in [0, \theta]$. Thus $x(t) - x_0 = \int_0^t x'(s)\, ds$ and $x'(s) \in G(x_0) + \varepsilon B$ for almost all $s \in [0, \theta]$. Then $(x(t) - x_0)/t \in G(x_0) + \varepsilon B$. It follows that

$$\operatorname*{Lim\,sup}_{h \to 0^+} \left\{ \frac{x(h) - x_0}{h} : x(\,\cdot\,) \text{ is a solution of (1.6)} \right\} \subset G(x_0). \tag{1.7}$$

Now, we choose $h_n \to 0^+$ and $\varepsilon_n \to 0^+$ such that $\varepsilon_n / h_n \to 0$. For every n there is a solution $x_n\colon [0, h_n] \to \mathbb{R}^n$ of (1.6) such that

$$\operatorname{dist}(x_n(h_n), K) \leq \varepsilon_n.$$

Passing to a subsequence (again denoted by x_n, h_n) we can obtain

$$\lim_{n \to \infty} \frac{x_n(h_n) - x_0}{h_n} = v \in G(x_0).$$

Since

$$\frac{\operatorname{dist}(x_0+h_nv,K)}{h_n} \le \frac{|x_0+h_nv-x_n(h_n)|}{h_n} + \frac{\operatorname{dist}(x_n(h_n),K)}{h_n},$$

we obtain $v \in T_K(x_0)$.

It remains to prove that (f) $\Rightarrow$ (h). Since G is of linear growth then a solution of (1.6) does not escape to infinity in finite time. If x is a solution on the interval $[0,T)$ then the limit $\lim_{t\to T^-} x(t)$ exists and belongs to K. So, the condition (f) (existence of local viable solution) implies (h) (existence of global viable solution). □

A short and elementary proof of (d) $\Rightarrow$ (f) is given in [32].

If K is an open subset then it is locally compact. If K is open and G is an arbitrary map then K is a viability domain to G. Obviously (a) does not imply (h) in this case.

We say that a locally compact subset $D \subset \mathbb{R}^n$ is a backward invariance domain of a set-valued map $G: \mathbb{R}^n \mapsto \mathbb{R}^n$ if for every $x \in D$

$$-G(x) \subset T_D(x).$$

To make the presentation self-contained we recall the following formulation of the invariance theorem.

Theorem 1.4.2 (Invariance). *Assume that $G: \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a locally Lipschitz continuous map with nonempty compact values and D is a locally compact subset of $\mathbb{R}^n$. Then the following conditions are equivalent:*

(a) *D is a backward invariance domain of G;*
(b) *$-G(x) \subset \mathrm{co}T_D(x)$ for every $x \in D$;*
(c) *$H(x,-n) \le 0$ for every $x \in D$, $n \in [T_D(x)]^-$;*
(d) *$H(x,-n) \le 0$ for every $x \in D$, $n \in N_D(x)$;*
(e) *$H(x,-n) \le 0$ for every $x \in D$, $n \in M_D(x)$;*
(f) *for every $x_0 \in D$ there exists $T > 0$ such that every solution $x(\,\cdot\,)$ to (1.6) satisfies $x(t) \in D$ for $t \in [-T,0]$.*

Proof. The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious. Since the Hamiltonian H is lower semicontinuous, it follows, from (1.1) that (c)–(e) are equivalent. (c) $\Rightarrow$ (b) by separation theorem. By (1.2), (b) $\Rightarrow$ (a). (a) $\Leftrightarrow$ (f) by Aubin–Cellina [5]. □

Considering control systems with state constraint we shall use the following

Proposition 1.4.3. *Assume that $f: \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz continuous map and D is a locally compact subset of $\mathbb{R}^n$. Let $K := cl(D)$, $M := K \setminus D$. Suppose that*

$$\forall x \in D, \quad -f(x) \in T_D(x), \tag{1.8}$$

$$\forall x \in M, \quad f(x) \notin P_K^M(x). \tag{1.9}$$

Then for every $x_0 \in M$ *there exists* $\delta > 0$ *such that for every* $y_0 \in K$, $|y_0 - x_0| < \delta$ *we have*

$$x(t; y_0) \in D \quad \textit{for every } t \in (-\delta, 0),$$

where $x(\,\cdot\,; y_0)$ *denotes the solution of the Cauchy problem*

$$\begin{cases} x'(t) = f(x(t)), \\ x(0) = y_0. \end{cases}$$

Proof. Let $w := f(x_0)$. By Proposition 1.2.1 and (1.9), there exists $\varepsilon > 0$ such that for every $y_0 \in K \cap B(x_0, \varepsilon)$ we have

$$y_0 + (0, \varepsilon)B(-w, \varepsilon) \cap K \subset D. \tag{1.10}$$

Let $C \geq 1$, $l \geq 1$ be an upper bound and a Lipschitz constant of f on the ball $B(x_0, \varepsilon)$, respectively. We set $\delta = \varepsilon/2Cl$.

Fix $y_0 \in K$ such that $|y_0 - x_0| < \delta$. For $|t| < \delta$ we have

$$|x(t; y_0) - (y_0 + tw)| \leq |t|(l|x_0 - y_0| + lC|t|/2).$$

Hence, for $-\delta < t < 0$ we have

$$x(t; y_0) \in y_0 + (0, \varepsilon)B(-w, \varepsilon). \tag{1.11}$$

First we consider the case $y_0 \in D$. We define $t_1 := \inf\{t \in (-\delta, 0) : x(s, y_0) \in D \text{ for } s \in (t, 0)\}$. By Invariance Theorem 1.4.2 and (1.8), we have $t_1 < 0$. We claim that $t_1 = \delta$. Suppose to the contrary that $t_1 > -\delta$. By (1.10), (1.11), we obtain $x(t_1; y_0) \in D$. By Invariance Theorem 1.4.2 and (1.8), there exists $t_2 < t_1$ such that $x(s, y_0) \in D$ for $s \in (t_2, t_1)$, which contradicts the definition of t_1.

Next, we consider the case $y_0 \in K$, $|y_0 - x_0| < \delta/2$. We choose a sequence $(y_n) \subset D$ such that $y_n \to y_0$ and $|y_n - x_0| < \delta$. By the previous part of the proof we have $x(t; y_n) \in D$ for $t \in (-\delta, 0)$. Thus, taking the limit, we deduce that $x(t; y_0) \in K$ for $t \in (-\delta, 0)$. Combining it with (1.10) and (1.11) we get $x(t; y_0) \in D$ for $t \in (-\delta, 0)$. $\square$

1.5. Regularity of tubes

If the set of state constrains $P(t)$ depends on a real variable t (time) we shall call it a tube. Tubes considered here and in the sequel are assumed to have closed values in $\mathbb{R}^n$.

Definition 1.5.1. Let $P\colon [0, T] \rightsquigarrow \mathbb{R}^d$ be a nonempty valued map. We say that P is *left* (*right*) *absolutely continuous on* $[0, T]$ if for every $R > 0$ there exists an integrable function $\mu\colon [0, T] \to [0, +\infty)$ such that for every $t_1 < t_2$ ($t_2 < t_1$) we have

$$P(t_1) \cap B(0, R) \subset P(t_2) + B\left(0, \left| \int_{t_1}^{t_2} \mu(s)\, ds \right|\right) \tag{1.12}$$

where $B(x_0, r)$ denotes the ball in $\mathbb{R}^n$ centered at x_0 with radius r and $A + D = \{a + d : a \in A \text{ and } d \in D\}$ for $A, D \subset \mathbb{R}^n$.

We say that P is *absolutely continuous* if it is left absolutely continuous and right absolutely continuous.

If a tube P is left absolutely continuous then the following property holds:

$$\begin{cases} \forall \varepsilon > 0,\ \forall \text{ compact } K \subset \mathbb{R}^d,\ \exists \delta > 0,\ \forall \Lambda \subset N, \\ \forall \{t_i, \tau_i : t_i < \tau_i,\ i \in \Lambda\} \text{ with }]t_i, \tau_i[\cap]t_j, \tau_j[= \emptyset \text{ for } i \neq j, \\ \sum (\tau_i - t_i) \le \delta \Rightarrow \sum e(P(t_i) \cap K, P(\tau_i)) \le \varepsilon, \end{cases} \tag{1.13}$$

where $e(U, V) = \inf\{\varepsilon > 0 : U \subset V + \varepsilon B\}$ and N is the set of natural numbers (we choose $\delta > 0$ in such a way that for every measurable subset $C \subset [0, T]$ if $m(C) < \delta$ then $\int_C \mu < \varepsilon$; m denotes the Lebesque measure).

In general, an absolutely continuous tube P does not have an absolutely continuous selection passing through each point of its graph. For instance, one can check that the tube $P\colon [0, 1] \to \mathbb{R}$ given by

$$P(t) = \begin{cases} \{1/(1 - t), 0\} & \text{if } t \in [0, 1), \\ \{0\} & \text{if } t = 1, \end{cases}$$

is absolutely continuous. However, there is no absolutely continuous function $x\colon [0, 1] \to \mathbb{R}$ such that $x(0) = 1$ and $x(t) \in P(t)$ for every $t \in [0, 1]$.

A different definition of $\delta(\,\cdot\,)$-absolutely continuous tubes is given in [83]. A tube P is called δ-absolutely continuous (where $\delta\colon (0, \infty) \to (0, \infty)$ is an arbitrary function) if for every $\varepsilon > 0$ and for arbitrary $0 \le t_1 < \tau_1 \le \ldots \le t_m \le \tau_m \le T$

$$\sum_i (\tau_i - t_i) < \delta(\varepsilon) \Rightarrow \sum_i H(P(t_i), P(\tau_i)) \le \varepsilon$$

where H denotes the Hausdorff distance.

Lemma 1 in [83] states that if P is $\delta(\,\cdot\,)$-absolutely continuous, then for every $t_0 \in [0, T]$ and $x_0 \in P(t_0)$ there is a $\delta(\,\cdot\,)$-absolutely continuous function $x\colon [0, T] \to \mathbb{R}^n$ such that $x(t_0) = x_0$ and $x(t) \in P(t)$ for every $t \in [0, T]$. In conclusion, absolutely continuous tubes represent a larger family then $\delta(\,\cdot\,)$-absolutely continuous tubes.

The contingent derivative $DP(\tau, y)$ of P at $(\tau, y) \in \operatorname{Graph}(P)$ is defined as the set-valued map from $\mathbb{R}$ to $\mathbb{R}^d$ whose graph is described by

$$\operatorname{Graph}(DP(\tau, y)) = T_{\operatorname{Graph}(P)}(\tau, y).$$

It is not difficult to prove, using Proposition 5.1.4. from [7], that

$$v \in DP(\tau, y)(1) \Leftrightarrow \liminf_{h \to 0+} \operatorname{dist}\left(v, \frac{P(\tau + h) - y}{h}\right) = 0. \tag{1.14}$$

1.6. Viscosity solutions of Hamilton–Jacobi equations

The theory of viscosity solution is a topic to broad to cover it in this section. Below we provide only some basic definitions and selected facts which we will need later on. For a more complete presentation we refer to [34], [14], [10], [12], [95].

First, we recall the definition of viscosity sub- and supersolution for the class of the Hamilton–Jacobi equations that most often appear in the paper

$$u_t + H(t, x, u_x) = 0, \tag{1.15}$$

where $H\colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function. A smooth function $u\colon (0, T) \times \mathbb{R}^n \to \mathbb{R}$ is a classical solution of if (1.15) holds true for every $(t, x) \in (0, T) \times \mathbb{R}^n$. (The function H is called Hamiltonian.)

We say that a lower semicontinuous function $\psi\colon (0, T] \times \mathbb{R}^n \to \mathbb{R}$ is a supersolution of (1.15) if

$$\forall (t, x) \in (0, T) \times \mathbb{R}^n,\ \forall (p_t, p_x) \in \partial_- \psi(t, x), \quad p_t + H(t, x, p_x) \le 0.$$

An upper semicontinuous function $\phi\colon (0, T] \times \mathbb{R}^n \to \mathbb{R}$ is a subsolution of (1.15) if

$$\forall (t, x) \in (0, T) \times \mathbb{R}^n,\ \forall (p_t, p_x) \in \partial_+ \phi(t, x), \quad p_t + H(t, x, p_x) \ge 0.$$

A continuous function $u\colon (0, T0 \times \mathbb{R}^n \to \mathbb{R}$ is a viscosity solution of (1.15) if it is a super- and a subsolution.

If H is positively homogeneous with respect to the last variable, that is

$$H(t, x, \alpha p) = \alpha H(t, x, p)$$

for every $\alpha \ge 0$, then supersolutions (subsolutions) can be defined equivalently using normal cones to epigraph (hypograph) instead of subdifferentials (superdifferentials).

Proposition 1.6.1. *Let $H\colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be continuous and positively homogeneous with respect to the last variable.*

(a) *A lower semicontinuous function $\psi\colon (0, T] \times \mathbb{R}^n \to \mathbb{R}$ is a supersolution of* (1.15) *if and only if for every $(t, x) \in (0, T) \times \mathbb{R}^n$*

$$\forall (n_t, n_x, n_u) \in N^0_{\mathcal{E}pi(\psi)}(t, x, \psi(t, x)), \quad n_t + H(t, x, n_x) \le 0. \tag{1.16}$$

(b) *An upper semicontinuous function $\phi\colon (0, T] \times \mathbb{R}^n \to \mathbb{R}$ is a subsolution of* (1.15) *if and only if for every $(t, x) \in (0, T) \times \mathbb{R}^n$*

$$\forall (n_t, n_x, n_u) \in N^0_{\mathcal{H}ypo(\phi)}(t, x, \phi(t, x)), \quad -n_t + H(t, x, -n_x) \ge 0. \tag{1.17}$$

Proof. We provide the proof for (a) only. By Proposition 1.3.2, if (1.16) holds true for every (t,x) then ψ is a supersolution. To prove the opposite, let us assume that ψ is a supersolution and let $(n_t,n_x,n_u) \in N^0_{\mathcal{E}pi(\Psi)}(t,x,\psi(t,x))$.

If $n_u < 0$ then $(n_t/-n_u, n_x/-n_u) \in \partial_-\psi(t,x)$. As $H(t,x,\,\cdot\,)$ is homogeneous, we are done.

Let $n_u = 0$. By Proposition 1.3.4, there exist sequences $(t_k,x_k) \to (t,x)$ and $(n_{tk},n_{x,k},n_{uk}) \to (n_t,n_x,n_u)$ such that

$$(n_{tk},n_{x,k},n_{uk}) \in N^0_{\mathcal{E}pi(\Psi)}(t_k,x_k,\psi(t_k,x_k)) \text{ and } n_u < 0.$$

By the previous part of the proof we have $n_{tk} + H(t_k,x_k,n_{xk}) \leq 0$. As H is continuous, we obtain $n_t + H(t,x,n_x) \leq 0$. □

Proposition 1.6.1 is a special version of the results presented in [95], [31].

Below we recall – in an adapted version – a result proved in [10] (cf. Theorem 4.1 in [10]).

Lemma 1.6.2. *Assume that $H\colon (0,T)\times\mathbb{R}^{2n} \to \mathbb{R}$ is a continuous Hamiltonian. If $w_n\colon (0,T)\times\mathbb{R}^n \to \mathbb{R}$ is an increasing sequence of uniformly locally bounded supersolutions of the Hamilton–Jacobi equation* (1.15) *and $w\colon (0,T)\times\mathbb{R}^n \to \mathbb{R}$ is a pointwise limit of w_n, then w is a supersolution of* (1.15).

1.7. Value function of generalized Bolza problem

We apply Frankowska's method (see [44]) to characterize the value function of the generalized Bolza problem as a lower semicontinuous solution of the corresponding Hamilton–Jacobi equation.

Suppose that $L\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n \to [0,+\infty)$ satisfies:

$$\begin{cases} L(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,) \text{ is locally Lipschitz continuous,} \\ L(t,x,u,\,\cdot\,) \text{ is convex,} \\ L(t,x,\,\cdot\,,v) \text{ is nonincreasing,} \end{cases} \tag{1.18}$$

and let a set-valued map $F\colon [0,T]\times\mathbb{R}^n\times\mathbb{R} \to \mathbb{R}^n$ be as follows

$$\begin{cases} F(t,x,u) \text{ is nonempty convex compact for every } (t,x,u), \\ F \text{ is locally Lipschitz,} \\ F(t,x,u_1) \subset F(t,x,u_2) \text{ for } u_1 < u_2, \\ \forall u,\ \exists C,\ \forall t,x,\ |F(\,\cdot\,,\,\cdot\,,u)| \leq C(1+|x|). \end{cases} \tag{1.19}$$

We define a new Lagrangian $L_F\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n \to [0,+\infty]$ by

$$L_F(t,x,u,v) = \begin{cases} L(t,x,u,v) & \text{if } v \in F(t,x,u), \\ +\infty & \text{elsewhere.} \end{cases} \tag{1.20}$$

We shall consider a dynamical system

$$\begin{cases} x'(t) \in F(t, x(t), u(t)), \\ u'(t) \leq -L(t, x(t), u(t), x'(t)), \end{cases} \tag{1.21}$$

where $x(\cdot)$, $u(\cdot)$ are absolutely continuous functions with values in $\mathbb{R}^n$, $\mathbb{R}$, respectively. A pair $(x(\cdot), u(\cdot))$ is a solution of (1.21) if and only if it is a solution of the differential inclusion

$$(x'(t), u'(t)) \in \widetilde{Q}(t, x(t), u(t)) \quad \text{for almost all } t \in [t_0, T], \tag{1.22}$$

where $\widetilde{Q}\colon [0, T] \times \mathbb{R}^n \times \mathbb{R} \rightsquigarrow \mathbb{R}^n \times \mathbb{R}$ denotes the following set-valued map

$$\widetilde{Q}(t, x, u) = \{(f, -\eta) : \eta \geq L(t, x, u, f)\}. \tag{1.23}$$

We define a value function $V\colon [0, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ corresponding to L, F and a terminal cost function $g\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned} V(t_0, x_0) = \inf\{u(t_0) : (x(\cdot), u(\cdot)) &\text{ solves (1.21)} \\ &\text{and } x(t_0) = x_0,\ u(T) \geq g(x(T))\} \end{aligned} \tag{1.24}$$

(we set $\inf \emptyset = +\infty$). When L does not depend on u, the definition of V reduces to the value function

$$V(t_0, x_0) = \min_{x(\cdot) \in W^{1,1}[t_0, T],\, x(t_0) = x_0} g(x(T)) + \int_{t_0}^{T} L_F(s, x(s), x'(s))\, ds \tag{1.25}$$

associated with the following control system

$$x'(t) \in F(t, x(t)).$$

To a pair $L(t, x, u, f)$, $F(t, x, u)$ we associate a Hamiltonian $H(t, x, u, p)$

$$H(t, x, u, p) = \inf_{f \in F(t, x, u)} \langle f, p\rangle + L(t, x, u, f) \tag{1.26}$$

which can be viewed as the Legendre–Fenchel transform of L_F.

If the value function $V(t, x)$ is smooth then it is a classical solution of the Hamilton–Jacobi equation

$$\begin{cases} \dfrac{\partial U}{\partial t} + H\left(t, x, U, \dfrac{\partial U}{\partial x}\right) = 0, & (t, x) \in \,]0, T[\, \times \mathbb{R}^n, \\ U(T, x) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.27}$$

Definition 1.7.1. An extended lower semicontinuous function $U\colon (0,T]\times\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is a *lower semicontinuous solution* of (1.27) if for every $(t,x)\in\mathrm{Dom}(U)$, $t<T$

$$\forall (p_t,p_x)\in\partial_-U(t,x),\quad p_t+H(t,x,U(t,x),p_x)=0$$

and

$$\text{(1.28)}\qquad \begin{cases} \text{for every } x\in\mathrm{Dom}(U(T,\,\cdot\,) \text{ there exist } x_n\to x,\ t_n\to T^- \\ \text{such that } \lim\limits_{n\to\infty} U(t_n,x_n)=U(T,x). \end{cases}$$

Below we recall a result about the existence and uniqueness of lower semicontinuous solution of (1.27), which also gives the exact representation of the solution. The result comes from [82], where, in fact, a more general class of Lagrangians was considered. In the proof of Theorem 1.7.2 it would be enough to apply the technics of proof used for the first time in [44].

Theorem 1.7.2. *Assume that L_F is given by* (1.20)*, where L, F satisfy* (1.19)*,* (1.18) *and $g\colon\mathbb{R}^n\to\mathbb{R}$ is a lower semicontinuous function bounded from below. Then the value funtion V given by* (1.24) *is the unique bounded from below lower semicontinuous solution of the corresponding Hamilton–Jacobi equation.*

1.8. Inf-convolution

The aim of the section is to provide a tool to approximate a lower semicontinuous function by Lipschitz continuous functions. The suitable one is so called inf-convolution. We consider it in a special case. A general approach can be found in [87, Chapter 1].

Suppose that $\phi\colon\mathbb{R}^n\to\mathbb{R}$ is a function bounded from below and $L>0$. We define the inf-convolution $\phi_L\colon\mathbb{R}^n\to\mathbb{R}$ by

$$\phi_L(x)=\inf_{y\in\mathbb{R}^n}\phi(y)+L\|y-x\|.$$

Proposition 1.8.1. *Suppose that $\phi\colon\mathbb{R}^n\to\mathbb{R}$ is a lower semicontinuous function bounded from below and L_n is an increasing sequence converging to $+\infty$. Then $\phi_n=\phi_{L_n}$ is Lipschitz continuous with constant L_n and $\phi_n(x)$ is nondecreasing and convergent to $\phi(x)$ for every $x\in\mathbb{R}^n$.*

For the readers' convenience we list some elementary properties of inf-convolution.

- If ϕ is bounded from below by a constant c then $c\le\phi_L(x)\le\phi(x)$ for every x.
- $\phi_L(y)-\phi_L(x)\le L\|y-x\|$ for every x, y (ϕ_L is L-Lipschitz continuous).
- If ϕ is L-Lipschitz continuous, then $\phi_L=\phi$.
- If ψ, ϕ are bounded from below and $\psi\le\phi$, then $\psi_L\le\phi_L$.

- If ψ is L-Lipschitz continuous and $\psi \leq \phi$, then $\psi \leq \phi_L$. Thus, ϕ_L is the greatest L-Lipschitz continuous function dominated by ϕ.
- If $L_1 \leq L_2$, then $\phi_{L_1} \leq \phi_{L_2}$ and $(\phi_{L_2})_{L_1} = \phi_{L_1}$.

Proof of Proposition 1.8.1. It remains to show the pointwise convergence of ϕ_n to ϕ. Fix x. If $\|y - x\| > (\phi(x) - c)/L := R_L$, then

$$\phi(y) + L\|y - x\| \geq c + L\|y - x\| > c + \phi(x) - c.$$

Thus, there exists y_L such that $\phi_L(x) = \phi(y_L) + L\|y_L - x\|$ and $\|y_L - x\| \leq R_L$. So,

$$\liminf_{n\to\infty} \phi_{L_n}(x) \geq \liminf_{n\to\infty} \phi(y_{L_n}) \geq \phi(x). \qquad \square$$

CHAPTER 2

TIME MEASURABLE CONTROL SYSTEMS

In the chapter we consider the Mayer problem with an extended lower semicontinuous terminal cost function and dynamics given by a differential inclusion with right-hand side measurable in time and Lipschitz continuous in space variable. To describe the value function as the unique solution of the corresponding Hamilton–Jacobi equation we developed viability theory and adapted it to the case where the right-hand side is as above and state constraints vary in time. We assume that the tube of constraints is absolutely continuous. We show that the necessary and sufficient condition for viability is the Nagumo type one for almost all t. In the Nagumo type condition we can take the convexification of the Bouligand tangent cone to the tube. An analogous result is obtained for invariance. To investigate viability and invariance problems in measurable case we use an extension of the Scorza–Dragoni theorem to set-valued maps from [64] and [88]. Viability problems for nonautonomous case were considered in [20] and [96], where Nagumo type conditions are in a stronger version, i.e. the right-hand side of the differential inclusion is supposed to have a nonempty intersection with the Bouligand tangent cone to the tube. The stronger Nagumo condition cannot be expressed in an equivalent dual version involving normal cones, what is the key point in applications to the Hamilton–Jacobi–Bellman equations.

2.1. Infinitesimal generators of reachable maps

2.1.1. Scorza–Dragoni type properties for set-valued maps. We shall use in the sequel several consequences of the so called Scorza–Dragoni type theorems. We first recall a theorem for set-valued maps which are upper semicontinuous with respect to one of its variables. Next we show how some other related properties can be deduced from it.

We denote by m the Lebesgue measure on $[0, T]$ and by B the closed unit ball in $\mathbb{R}^d$. If $\varphi\colon X \rightsquigarrow Y$ is a set-valued map from a set X to a set Y, then by

Graph(φ) we denote the graph of φ given by

$$\mathrm{Graph}(\varphi) = \{(x, y) \in X \times Y : y \in \varphi(x)\}.$$

When $Y = \mathbb{R}^d$ we set $\|\emptyset\| = -\infty$ and if $\varphi(x) \neq \emptyset$

$$\|\varphi(x)\| = \sup\{\|y\| : y \in \varphi(x)\}.$$

Let X, Y be separable metric spaces.

Theorem 2.1.1 ([88]). *Let $F\colon [0,T] \times X \rightsquigarrow Y$ be a set-valued map such that* Graph$(F(t,\,\cdot\,))$ *is a closed subset of $X \times Y$ for almost all $t \in [0,T]$. Then there exists a closed-valued map $\widehat{F}\colon [0,T]\times X \rightsquigarrow \mathbb{R}^d$ satisfying the following conditions:*

(a) *For almost all $t \in [0,T]$ and for all $x \in X$, $\widehat{F}(t,x) \subset F(t,x)$.*
(b) *For every measurable set $\Lambda \subset [0,T]$ and all measurable maps $u\colon \Lambda \to X$, $v\colon \Lambda \to \mathbb{R}^d$ such that $v(t) \in F(t,u(t))$ almost everywhere (a.e.) in Λ we have $v(t) \in \widehat{F}(t,u(t))$ a.e. in Λ.*
(c) *For any $\varepsilon > 0$ there is a closed set $A_\varepsilon \subset [0,T]$ such that $m([0,T] \setminus A_\varepsilon) < \varepsilon$ and $\widehat{F}|_{A_\varepsilon \times X}$ has a closed graph.*

The proof of Theorem 2.1.1 given in [88] was based on Lusin's Theorem.

Corollary 2.1.2 ([63]). *Suppose that $F\colon [0,T] \times X \rightsquigarrow \mathbb{R}^d$ has convex closed values and:*

- *for almost all $t \in [0,T]$, $F(t,\,\cdot\,)$ is upper semicontinuous, F is measurably bounded, i.e. there is a measurable function $\mu\colon [0,T] \to \mathbb{R}$ such that for almost all $t \in [0,T]$ and every $x \in X$, $\|F(t,x)\| \le \mu(t)$.*

Then there exists a set-valued map $\widetilde{F}\colon [0,T] \times X \rightsquigarrow \mathbb{R}^d$ with closed convex values satisfying the following conditions:

(a) *for almost all $t \in [0,T]$ and for all $x \in X$, $\widetilde{F}(t,x) \subset F(t,x)$,*
(b) *for every measurable set $\Lambda \subset [0,T]$ and every $u\colon \Lambda \to X$, $v\colon \Lambda \to \mathbb{R}^d$ measurable maps such that $v(t) \in F(t,u(t))$ a.e. in Λ we have $v(t) \in \widetilde{F}(t,u(t))$ a.e. in Λ,*
(c) *for any $\varepsilon > 0$ there is a closed set $A_\varepsilon \subset [0,T]$ such that $m([0,T]\setminus A_\varepsilon) < \varepsilon$ and $\widetilde{F}\,|_{A_\varepsilon \times X}$ is an upper semicontinuous map.*

Remark. The set valued maps $\widehat{F}$, $\widetilde{F}$ can also have empty values at some points. We would like to underline that in (b) the condition $v(t) \in \widehat{F}(t,u(t))$ holds true "only" for almost all $t \in \Lambda$ and the sets of measure zero on which it does not hold can be different for distinct u, v.

We shall use in the proof of the following theorem and in the sequel the notion of points of density. We recall it now as well as the one of the Lebesgue points of measurable functions.

A point $x \in \mathbb{R}^d$ is a point of density of a measurable set $U \subset \mathbb{R}^d$ if

$$\lim_{\delta(Q)\to 0,\, x\in Q} \frac{m_d(U\cap Q)}{m_d(Q)} = 1$$

where Q denotes d-dimensional cubes, m_d the Lebesgue measure in $\mathbb{R}^d$ and $\delta(Q)$ is the diameter of Q, i.e. $\delta(Q) = \sup\{|x-y| : x, y \in Q\}$.

Let V be the set of all density points of U. It is well known that $m_d(U \setminus V) = 0$. For an arbitrary set $M \subset \mathbb{R}^d$ of measure zero the equality $\overline{V} = \overline{V \setminus M}$ holds true. If the original set U was closed, then $V \subset U$ and, of course, $\overline{V} \subset U$.

A point $x \in \mathbb{R}^d$ is the Lebesgue point of a measurable function $f\colon \mathbb{R}^d \to \mathbb{R}$ if

$$\lim_{\delta(Q)\to 0,\, x\in Q} \frac{1}{m_d(Q)} \int_Q \|f(u) - f(x)\|\, du = 0.$$

The set of points which are not the Lebesgue points of f has measure 0.

Here is a version of the original Theorem of Scorza–Dragoni [90]. We give it, as well as the next theorem, not in the most general possible form but sufficient for our purposes.

Theorem 2.1.3. *Let $f\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be measurable with respect to $t \in [0,T]$ and continuous with respect to $x \in \mathbb{R}^d$. Then, for every $\varepsilon > 0$, there is a closed set $A_\varepsilon \subset [0,T]$ such that $m([0,T] \setminus A_\varepsilon) < \varepsilon$ and the restriction $f|_{A_\varepsilon \times \mathbb{R}^d}$ is continuous.*

Proof. Fix $\varepsilon > 0$, $k \in N$ and consider the restriction $f_k = f|_{[0,T]\times kB}$ and the function $\mu_k(t) = \sup\{\|f(t,x)\| : x \in kB\}$. The function μ_k has finite values. We apply Theorem 2.1.1 to the map $F_k(t,x) = \{f_k(t,x)\}$ with the domain $[0,T]\times kB$ and get a corresponding map $\widehat{F}_k$. We take $A_k \subset [0,T]$ for which $m([0,T] \setminus A_k) < \varepsilon/2^{k+1}$, $\widehat{F}_k|_{A_k\times kB}$ has closed graph and $\widehat{F}_k(t,x) \subset F_k(t,x)$ for all $(t,x) \in A_k \times kB$. Let $\widetilde{A}_k \subset A_k$ be a closed set for which $m(A_k \setminus \widetilde{A}_k) < \varepsilon/2^{k+1}$ and $\sup_{t\in\widetilde{A}_k} \mu_k(t) < \infty$. Finally, let U_k be the set of density points of $\widetilde{A}_k$. For any fixed $x \in kB$ there is (by (b) of Theorem 2.1.1) a set $J_x \subset [0,T]$ of measure zero such that $f_k(t,x) \in \widehat{F}_k(t,x)$, for $t \in A_k \setminus J_x$. By the definition of A_k we have, for those t, the equality $\widehat{F}(t,x) = \{f(t,x)\}$. Since the graph of $\widehat{F}_k$ restricted to $\overline{U_k} \times \{x\}$ is compact and $\overline{U_k} = \overline{U_k \setminus J_x}$ so the equality $\widehat{F}(t,x) = \{f(t,x)\}$ holds for all $t \in \overline{U_k}$. The graph of the restriction of f_k to $\overline{U_k} \times kB$ is compact so this restriction is continuous with respect to (t,x). This implies the required continuity of f restricted to $(\bigcap_{k=1}^\infty \overline{U_k}) \times \mathbb{R}^d$ and we have, of course, $m([0,T] \setminus \bigcap_{k=1}^\infty \overline{U_k}) \le \varepsilon$.$\square$

The following theorem was proved by N. Kikuchi in [65] for convex valued maps.

Theorem 2.1.4. *Let the compact-valued map $F\colon [0,T] \times \mathbb{R}^d \rightsquigarrow \mathbb{R}^{d_1}$ be measurable with respect to $t \in [0,T]$ and continuous with respect to $x \in \mathbb{R}^d$. Then,* *for every $\varepsilon > 0$, there exists a closed set $A_\varepsilon \subset [0,T]$ for which $m([0,T] \setminus A_\varepsilon) < \varepsilon$ and the restriction of F to $A_\varepsilon \times \mathbb{R}^d$ is continuous.*

Proof. Consider the function $f(t,x,z) = \operatorname{dist}(z, F(t,x))$. It is measurable in t and continuous in (x,z) so we get A_ε, as in Theorem 2.1.3, for which the restriction of f to $A_\varepsilon \times \mathbb{R}^d \times \mathbb{R}^{d_1}$ is continuous with respect to (t,x,z). This implies that the restriction of F to the set $A_\varepsilon \times \mathbb{R}^d$ is lower semicontinuous and has closed graph. For every $k > 0$ the function $\mu_k(t) = \sup\{\|y\| : y \in F(t,x), \|x\| \le k\}$ is measurable. Hence we can choose a closed subset $A_{\varepsilon k} \subset A_\varepsilon$ such that the graph of the restriction $F|_{A_{\varepsilon k} \times kB}$ is compact and $m(A_\varepsilon \setminus A_{\varepsilon k}) < \varepsilon/2^k$. Thus $F|_{(\bigcap_{k\ge1} A_{\varepsilon k}) \times \mathbb{R}^d}$ is both upper and lower semicontinuous and $m([0,T] \setminus \bigcap_{k\ge1} A_{\varepsilon k}) < 2\varepsilon$. □

2.1.2. Infinitesimal behaviour of reachable maps. Let V be a separable, metric space and $F\colon [0,T] \times \mathbb{R}^d \times V \rightsquigarrow \mathbb{R}^d$ a convex-valued map. We consider a family of differential inclusions

$$\begin{cases} x'(t) \in F(t, x(t), v), \\ x(\tau) = x_\tau, \end{cases} \tag{2.1}$$

where $\tau \in [0,T]$, $x_\tau \in \mathbb{R}^d$, $v \in V$. The family of solutions of this problem, defined on some interval contained in $[0,T]$, for some fixed v, will be denoted by $\operatorname{Sol}(F, v, \tau, x_\tau)$. The reachable set of (2.1) at time $t \in [0,T]$ is given by

$$R_v(t,\tau)(x_\tau) = \{x(t) : x \in \operatorname{Sol}(F, v, \tau, x_\tau)\}.$$

We study here the sets

$$\operatorname*{Lim\,sup}_{t\to\tau} \frac{R_v(t,\tau)(x_\tau) - x_\tau}{t-\tau} \quad \text{and} \quad \operatorname*{Lim\,inf}_{t\to\tau} \frac{R_v(t,\tau)(x_\tau) - x_\tau}{t-\tau}.$$

Theorem 2.1.5. *Let $F\colon [0,T] \times \mathbb{R}^d \times V \rightsquigarrow \mathbb{R}^d$ have closed convex values and satisfy:*

- *$(x,v) \rightsquigarrow F(t,x,v)$ is continuous for almost all $t \in [0,T]$;*
- *$t \rightsquigarrow F(t,x,v)$ is measurable for all $(x,v) \in \mathbb{R}^d \times V$;*
- *$\|F(t,x,v)\| \le \mu(t)$ for almost all $t \in [0,T]$ and for all $(x,v) \in \mathbb{R}^d \times V$, where $\mu(\cdot)$ is integrable.*

Then there exists a set $A \subset [0,T]$ of full measure, i.e. $m([0,T] \setminus A) = 0$, such that for every $(\tau, x_\tau, v) \in A \times \mathbb{R}^d \times V$

$$\operatorname*{Lim}_{t\to\tau} \frac{R_v(t,\tau)(x_\tau) - x_\tau}{t-\tau} = F(\tau, x_\tau, v).$$

The proof will follow from the properties given below.

Lemma 2.1.6. *Assume that* $F\colon [0,T] \times \mathbb{R}^d \times V \rightsquigarrow \mathbb{R}^d$ *have closed convex values and*

(a) *the graphs* $\mathrm{Graph}(F(t,\,\cdot\,,\,\cdot\,))$ *are closed for almost all* $t \in [0,T]$*;*
(b) $\|F(t,x,v)\| \le \mu(t)$ *for almost all* $t \in [0,T]$ *and for all* $(x,v) \in \mathbb{R}^d \times V$*, where* $\mu(\,\cdot\,)$ *is integrable;*
(c) *for all* $(x,v) \in \mathbb{R}^d \times V$ *the set-valued map* $F(\,\cdot\,,x,v)$ *is measurable.*

Then there exists a set $A \subset [0,T]$ *of full measure such that*

$$\forall (\tau, x_\tau, v) \in A \times \mathbb{R}^d \times V,\ \forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x \in \mathrm{Sol}(F, v, \tau, x_\tau),\ \forall 0 < |h| < \delta$$
$$\frac{1}{h}(x(\tau + h) - x_\tau) \in F(\tau, x_\tau, v) + \varepsilon B.$$

In particular, for every $(\tau, x_\tau, v) \in A \times \mathbb{R}^d \times V$

$$\emptyset \neq \operatorname*{Lim\,sup}_{t \to \tau} \frac{R_v(t,\tau)(x_\tau) - x_\tau}{t - \tau} \subset F(\tau, x_\tau, v).$$

Proof. We give a slight modification of the proof from [89] where the case without a parameter was considered. The method used in [89] was, in turn, based on [77] (see also [64]).

Let $\widetilde{F}$ be as in Corollary 2.1.2 with $X = \mathbb{R}^d \times V$. Then, for every $v \in V$, solutions of differential inclusions $x' \in F(t,x,v)$ and $x' \in \widetilde{F}(t,x,v)$ coincide.

Fix $\gamma > 0$ and choose a closed set $A_\gamma \subset [0,T]$ for which $m([0,T] \setminus A_\gamma) < \gamma$, $\widetilde{F}|_{A_\gamma \times \mathbb{R}^d \times V}$ is upper semicontinuous with respect to (t,x,v) and $\widetilde{F}(t,x,v) \subset F(t,x,v)$ for all $(t,x,v) \in A_\gamma \times \mathbb{R}^d \times V$.

Let $\widetilde{A}_\gamma \subset A_\gamma$ be the set of density points of A_γ which also are the Lebesgue points of the function $\mu(\,\cdot\,) \cdot \chi_{[0,T] \setminus A_\gamma}(\,\cdot\,)$ – note that $m(\widetilde{A}_\gamma) = m(A_\gamma)$. We fix now a $\tau \in \widetilde{A}_\gamma$ and an arbitrary $(x_\tau, v) \in \mathbb{R}^d \times V$. To make the notations simpler we shall suppose that $t > \tau$. The case $t < \tau$ follows by the same arguments.

Due to assumptions (a), (b) there is $\delta_1 > 0$ such that if $t - \tau < \delta_1$ and $t \in A_\gamma$ then for every $x(\,\cdot\,) \in \mathrm{Sol}(F, v, \tau, x_\tau)$

$$\widetilde{F}(t, x(t), v) \subset \widetilde{F}(\tau, x_\tau, v) + \frac{\varepsilon}{3} B.$$

There also is $\delta_2 > 0$ such that if $0 < t - \tau < \delta_2$ then

$$\frac{m([\tau,t] \cap A_\gamma)}{t - \tau} F(\tau, x_\tau, v) \subset F(\tau, x_\tau, v) + \frac{\varepsilon}{3} B.$$

Next, for some $\delta_3 > 0$, if $0 < t - \tau < \delta_3$ then

$$\frac{1}{t - \tau} \int_{[\tau,t] \setminus A_\gamma} \mu(s)\, ds < \frac{\varepsilon}{3}.$$

Let now $0 < t - \tau < \delta = \min\{\delta_1, \delta_2, \delta_3\}$ and $x(\,\cdot\,) \in \mathrm{Sol}(F, v, \tau, x_\tau)$. We get then

$$\begin{aligned}\frac{x(t) - x_\tau}{t - \tau} &\in \frac{1}{t-\tau}\int_{[\tau,t]\cap A_\gamma} \widetilde{F}(s, x(s), v)\,ds + \frac{1}{t-\tau}\int_{[\tau,t]\setminus A_\gamma} \widetilde{F}(s, x(s), v)\,ds \\ &\subset \frac{m([\tau,t]\cap A_\gamma)}{t-\tau}\left(F(\tau, x_\tau, v) + \frac{\varepsilon}{3}B\right) + \frac{1}{t-\tau}\int_{[\tau,t]\setminus A_\gamma} \mu(s)\,ds \cdot B \\ &\subset F(\tau, x_\tau, v) + \frac{2\varepsilon}{3}B + \frac{\varepsilon}{3}B = F(\tau, x_\tau, v) + \varepsilon B.\end{aligned}$$

To finish the proof it is enough to consider the set $A = \bigcup_{n=1}^\infty \widetilde{A}_{1/n}$. □

Lemma 2.1.6 yields the following property which was proved in [89] without the parameter v.

Corollary 2.1.7. *Under the assumptions of Lemma* 2.1.6 *there is a set $A \subset [0, T]$ of full measure such that for every $\tau \in A$, $x_\tau \in \mathbb{R}^d$, $v \in V$ and $x \in \mathrm{Sol}(F, v, \tau, x_\tau)$, we have*

$$\emptyset \neq \operatorname*{Lim\,sup}_{h\to 0^+}\left\{\frac{x(\tau + h) - x_\tau}{h}\right\} \subset F(\tau, x_\tau, v),$$
$$\emptyset \neq \operatorname*{Lim\,sup}_{h\to 0^+}\left\{\frac{x(\tau - h) - x_\tau}{-h}\right\} \subset F(\tau, x_\tau, v).$$

The following property, which has been proved for the first time in [91] without the parameter v, can be deduced from Corollary 2.1.7.

Corollary 2.1.8. *Let $f\colon [0, T] \times \mathbb{R}^d \times V \to \mathbb{R}^d$ satisfy the following conditions:*

- *$(x, v) \to f(t, x, v)$ is continuous for almost all $t \in [0, T]$;*
- *$t \to f(t, x, v)$ is measurable for all (x, v);*
- *$\|f(t, x, v)\| \le \mu(t)$ for almost all $t \in [0, T]$ and all (x, v), where $\mu(\,\cdot\,)$ is integrable.*

Then there exists a set $A \subset [0, T]$ of full measure such that for every $\tau \in A$, $v \in V$ and solution of $x'(t) = f(t, x(t), v)$ the derivative $x'(\tau)$ exists and $x'(\tau) = f(\tau, x(\tau), v)$.

Theorem 2.1.9. *Let $F\colon [0, T] \times \mathbb{R}^d \times V \rightsquigarrow \mathbb{R}^d$ have closed convex values and satisfy:*

- *$(x, v) \rightsquigarrow F(t, x, v)$ is continuous for almost all t;*
- *$t \rightsquigarrow F(t, x, v)$ is measurable for all (x, v);*
- *$\|F(t, x, v)\| \le \mu(t)$ for all (x, v), where $\mu(\,\cdot\,)$ is integrable.*

Under these assumptions there exists a set $A \subset [0, T]$ of full measure such that for every $(\tau, x_\tau, v) \in A \times \mathbb{R}^d \times V$ and $u \in F(\tau, x_\tau, v)$ the problem

$$\begin{cases} x'(t) \in F(t, x(t), v), \\ x(\tau) = x_\tau,\ x'(\tau) = u, \end{cases} \tag{2.2}$$

has a solution defined on $[0,T]$*. In particular, for every* $(\tau, x_\tau, v) \in A \times \mathbb{R}^d \times V$

$$F(\tau, x_\tau, v) \subset \operatorname*{Lim\,inf}_{t\to\tau} \frac{R_v(t,\tau)(x_\tau) - x_\tau}{t-h}.$$

Proof. By Theorem 9.6.2 of Łojasiewicz [7] there is a function $f\colon [0,T]\times\mathbb{R}^d\times V\times B \to \mathbb{R}^d$ for which:

- $f(t,x,v,B) = F(t,x,v)$;
- $t \to f(t,x,v,b)$ is measurable for all (x,v,b);
- $(x,v) \to f(t,x,v,b)$ is continuous for all (t,b);
- $\|f(t,x,v,b_1) - f(t,x,v,b_2)\| \le c\mu(t)\|b_1 - b_2\|$, where c is a constant depending only on the dimension d.

The last two properties imply that for a fixed t the function $(x,v,b) \to f(t,x,v,b)$ is continuous. We can apply Corollary 2.1.8, where the role of parameter will be played now by $(v,b) \in V \times B$.

So let A be as in Corollary 2.1.8, $\tau \in A$, $(x_\tau, v) \in \mathbb{R}^d \times V$, $u \in F(\tau, x_\tau, v)$ and $b \in B$ be such that $u = f(\tau, x_\tau, v, b)$. By Corollary 2.1.8 we get a solution of the problem

$$\begin{cases} x'(t) = f(t, x(t), v, b), \\ x(\tau) = x_\tau,\ x'(\tau) = u, \end{cases}$$

which is also a solution of (2.2). □

Now, we are in a position to deduce Theorem 2.1.5 from Lemma 2.1.6 and Theorem 2.1.9.

2.2. Viability and Invariance Theorems for tubes

Consider $T > 0$, a set-valued map $F\colon [0,T]\times\mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ and the differential inclusion

$$x'(t) \in F(t, x(t)). \tag{2.3}$$

Denote by $S_{[t_0,T]}(x_0)$ the set of absolutely continuous solutions of (2.3) defined on $[t_0,T]$ and satisfying the initial condition $x(t_0) = x_0$.

We are interested in the existence of solutions of the differential inclusion (2.3) satisfying constrains of the type $x(t) \in P(t)$, where $P\colon [0,T] \rightsquigarrow \mathbb{R}^d$ is a set-valued map (we shall call it a tube). The tube $P(\,\cdot\,)$ is said to have a viability property if for every $t_0 \in [0,T]$, $x_0 \in P(t_0)$ there is a solution $x \in S_{[t_0,T]}(x_0)$ satisfying $x(t) \in P(t)$ for every $t \in [t_0,T]$. The tube $P(\,\cdot\,)$ is called invariant for F if for every $t_0 \in [0,T]$ every solution $x \in S_{[t_0,T]}(x_0)$ starting in the tube (i.e. $x_0 \in P(t_0)$) satisfies $x(t) \in P(t)$ for every $t \in [t_0,T]$.

2.2.1. Viability Theorem in the Lipschitz case.

Definition 2.2.1 (Viability tube)**.** A tube $P\colon [0,T] \rightsquigarrow \mathbb{R}^n$ is a *viability tube for* $F\colon [0,T]\times\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ if there exists a full measure set $C \subset [0,T]$ such that for every $t \in C$ and every $x \in P(t)$ we have

$$(\{1\} \times F(t,x)) \cap \overline{\mathrm{co}}(T_{\mathrm{Graph}(P)}(t,x)) \neq \emptyset. \tag{2.4}$$

Theorem 2.2.2. *Assume that a tube $P\colon [0,T] \rightsquigarrow \mathbb{R}^d$ is left absolutely continuous, F has convex compact images and satisfies*

$$t \rightsquigarrow F(t,x) \text{ is measurable for every } x \in \mathbb{R}^d; \tag{2.5}$$

$$\begin{cases} \|F(t,x)\| \leq \mu(t) \text{ for almost all } t \in [0,T] \text{ and all } x \in \mathbb{R}^n, \\ \text{where } \mu \text{ is integrable}; \end{cases} \tag{2.6}$$

$$\begin{cases} \forall k > 0,\ \exists c_k \in L^1(0,T) \text{ such that for almost all } t \in [0,T] \\ F(t,\,\cdot\,) \text{ is } c_k(t)\text{-Lipschitz continuous on } kB. \end{cases} \tag{2.7}$$

Then the following three statements are equivalent:

(a) *There exists a set $A \subset [0,T]$ of full measure such that*

$$\forall t \in A,\ \forall x \in P(t), \quad F(t,x) \cap DP(t,x)(1) \neq \emptyset;$$

(b) *The tube P is a viability tube for F;*

(c) *For every $t_0 \in [0,T]$ and $x_0 \in P(t_0)$ there exists $x \in S_{[t_0,T]}(x_0)$ such that $x(t) \in P(t)$ for every $t \in [t_0,T]$.*

Proof. By Lemma 2.1.6 (c) $\Rightarrow$ (a). Clearly (a) yields (b). So it remains to show (b) $\Rightarrow$ (c).

We recall that the reachable set of (2.3) from an initial condition (t_0,x_0) at time $t \geq t_0$ is defined by

$$R(t,t_0)(x_0) := \{x(t) : x \in S_{[t_0,T]}(x_0)\}.$$

We proceed in several steps. First we show that for all $x_0 \in P(t_0)$, the map

$$[t_0,T] \ni t \mapsto g(t) := \mathrm{dist}(P(t), R(t,t_0)(x_0)) \tag{2.8}$$

is of bounded variation and the Gronwall inequality holds true for g. Then we check that from (b) it follows that for some $c \in L^1(t_0,T)$ $g'(t) \leq c(t)g(t)$ a.e. in $[t_0,T]$. In this way $g \equiv 0$, which in turn implies that (2.3) has a solution viable in the tube $P(\,\cdot\,)$.

Lemma 2.2.3. *If P is left absolutely continuous then:*

(a) $g(t_2) \leq g(t_1) + 2\int_{t_1}^{t_2} \mu(s)\,ds$ *for* $t_0 \leq t_1 < t_2 \leq T$*;*

(b) *g has bounded variation, in particular g is differentiable a.e. in $[t_0,T]$;*

(c) *if there exists $c \in L^1(t_0,T)$ such that $g'(t) \leq c(t)g(t)$, a.e. in $[t_0,T]$, then $g \equiv 0$.*

Proof. Fix $t_1 < t_2$. Since $S_{[t_0,T]}(x_0)$ is compact, there exists $x \in S_{[t_0,T]}(x_0)$ such that $g(t_1) = \operatorname{dist}(x(t_1), P(t_1))$. Since F is integrably bounded there exist $R > 0$ such that $|x(t)| \leq R$ for every $x \in S_{[t_0,T]}(x_0)$. We choose an integrable function μ such that (2.6) and (1.12) hold true. We have

$$\begin{aligned} g(t_2) &\leq \operatorname{dist}(x(t_2), P(t_2)) \\ &\leq \|x(t_2) - x(t_1)\| + \operatorname{dist}(x(t_1), P(t_1)) + \sup\{\operatorname{dist}(y, P(t_2) : y \in P(t_1)\} \\ &\leq \int_{t_1}^{t_2} \mu(s)\, ds + g(t_1) + \int_{t_1}^{t_2} \mu(s), \end{aligned}$$

which is our assertion (a).

To estimate the variation of g on $[t_0, T]$ we take a partition $t_0 < \ldots < t_k = T$. Let $S = \{i \in \{1, \ldots, k\} : g(t_i) - g(t_{i-1}) \geq 0\}$ and $S' = \{1, \ldots, k\} \setminus S$. We have

$$g(t_k) - g(t_0) = \sum_{i=1}^{k} g(t_i) - g(t_{i-1}) = \sum_{i \in S} |g(t_i) - g(t_{i-1})| - \sum_{i \in S'} |g(t_i) - g(t_{i-1})|.$$

Thus

$$\sum_{i=1}^{k} |g(t_i) - g(t_{i-1})| = 2 \sum_{i \in S} |g(t_i) - g(t_{i-1})| + g(t_k) - g(t_0) \leq 6 \int_{t_0}^{T} \mu(s)\, ds,$$

which gives us (b).

Let us set $h(t) = \sup\{g(s) : s \in [t_0, t]\}$. The function h is nonnegative, nondecreasing, $g(t) \leq h(t)$ for $t \in [t_0, T]$ and

$$h(t_2) - h(t_1) \leq 2 \int_{t_1}^{t_2} \mu(s)\, ds,$$

for $t_1 < t_2$. Hence h is absolutely continuous. Moreover, we have

$$\limsup_{\tau \to 0^+} \frac{h(t+\tau) - h(t)}{\tau} \leq \max\left(\limsup_{\tau \to 0^+} \frac{g(t+\tau) - g(t)}{\tau}, 0 \right).$$

Thus, for almost all $t \in [t_0, T]$, we have $h'(t) \leq c(t)h(t)$. The Gronwall inequality now yields $h \equiv 0$, which completes the proof of Lemma 2.2.3. □

Lemma 2.2.4. *Under all assumptions of Theorem* 2.2.2*,* $g \equiv 0$.

Proof. Indeed, assume for a moment that for some $t_2 > t_0$, $g(t_2) > 0$. Set $t_1 = \sup\{t < t_2 : g(t) = 0\}$. Thus $g > 0$ on $]t_1, t_2]$ and $g(t_1) = 0$. By Theorem 2.1.9 and Lemma 2.1.6 there exists a subset $A \subset [0, T]$ of full measure such that for all $t \in A$, $x \in \mathbb{R}^d$ the following two properties are verified:

(a) For every $v \in F(t, x)$ there exists a solution of the problem

$$y' \in F(t, y), \quad y(t) = x, \quad y'(t) = v.$$

(b) For every $t_1 < t$ and $y \in S_{[t_1,t]}$ satisfying $y(t) = x$ and for every sequence $h_i \to 0+$ we have

$$\emptyset \neq \operatorname*{Lim\,sup}_{i\to\infty} \frac{y(t-h_i) - x}{h_i} \subset -F(t,x).$$

Consider $t \in A$ such that g is differentiable at t, $F(t,\cdot)$ is $c_k(t)$-Lipschitz continuous on kB and let $z \in R(t)$, $y \in P(t)$ satisfy $g(t) = \|z-y\|$. Put

$$p = \frac{z-y}{\|z-y\|}.$$

Fix $(u,w) \in T_{\mathrm{Graph}(P)}(t,y)$ and $h_i \to 0+$, $u_i \to u$, $w_i \to w$ satisfying $y + h_i w_i \in P(t + h_i u_i)$.

Case 1. There exists a subsequence $\{u_{i_k}\}_{k\geq 1}$ with $u_{i_k} \geq 0$ for all k.

Let $v \in F(t,z)$ and $x \in S_{[t,T]}(z)$ be such that $x'(t) = v$. Thus

$$g(t + h_{i_k}u_{i_k}) - g(t) \leq \|x(t + h_{i_k}u_{i_k}) - y - h_{i_k}w_{i_k}\| - \|z-y\|.$$

Dividing by h_{i_k} and taking the limit we get $g'(t)u \leq \langle p, uv - w\rangle$.

Case 2. For all i large enough $u_i < 0$.

Consider $\overline{x} \in S_{[t_0,t]}(x_0)$ satisfying $\overline{x}(t) = z$ and i_k, $\overline{v} \in F(t,z)$ such that

$$\lim_{k\to\infty} \frac{\overline{x}(t + h_{i_k}u_{i_k}) - z}{h_{i_k}} = u\overline{v}.$$

Hence for all k large enough

$$g(t + h_{i_k}u_{i_k}) - g(t) \leq \|\overline{x}(t + h_{i_k}u_{i_k}) - y - h_{i_k}w_{i_k}\| - \|z-y\|,$$

dividing by h_{i_k} and taking the limit we get $g'(t)u \leq \langle p, u\overline{v} - w\rangle$.

Therefore we have shown that for all $(u,w) \in T_{\mathrm{Graph}(P)}(t,y)$

$$\begin{cases} u \geq 0 \Rightarrow \forall v \in F(t,z), \quad g'(t)u \leq \langle p, uv - w\rangle, \\ u < 0 \Rightarrow \exists \overline{v} \in F(t,z), \quad g'(t)u \leq \langle p, u\overline{v} - w\rangle. \end{cases} \tag{2.9}$$

Consider $\lambda_j \geq 0$, $(u_j, w_j) \in T_{\mathrm{Graph}(P)}(t,y)$, $j = 0,\ldots,d$ such that

$$\sum_{j=0}^{d} \lambda_j = 1 \quad \text{and} \quad u := \sum_{j=0}^{d} \lambda_j u_j > 0.$$

By reordering we may always assume that for some $0 \leq r \leq d$ and all $j \leq r$ we have $u_j \geq 0$ and for all $j > r$, $u_j < 0$. Consequently,

$$\sum_{j>r} \lambda_j |u_j| = \Big|\sum_{j>r} \lambda_j u_j\Big| < \sum_{j=0}^{r} \lambda_j u_j. \tag{2.10}$$

By (2.9) for every $j > r$ there exists $\overline{v}_j \in F(t,z)$ such that

$$g'(t)u_j \leq \langle p, u_j\overline{v}_j - w_j\rangle. \tag{2.11}$$

Thus, from (2.9), (2.11) it follows that for all $v \in F(t,z)$

$$g'(t)\Big(\sum_{j=0}^{r} \lambda_j u_j\Big) \le \Big\langle p, \sum_{j=0}^{r} \lambda_j u_j v - \sum_{j=0}^{r} \lambda_j w_j \Big\rangle \tag{2.12}$$

and

$$g'(t)\Big(\sum_{j>r} \lambda_j u_j\Big) \le \Big\langle p, \sum_{j>r} \lambda_j u_j \overline{v}_j - \sum_{j>r} \lambda_j w_j \Big\rangle. \tag{2.13}$$

On the other hand, by (2.10) for

$$\mu = \frac{\sum_{j>r} \lambda_j |u_j|}{\sum_{j=0}^{r} \lambda_j u_j} \quad \text{and} \quad \mu_j = \frac{\lambda_j |u_j|}{\sum_{j=0}^{r} \lambda_j u_j}$$

we have $\mu < 1$, $(1-\mu) + \sum_{j>r} \mu_j = 1$. Hence, by convexity of $F(t,z)$,

$$\forall v_z \in F(t,z), \quad v := (1-\mu)v_z + \sum_{j>r} \mu_j \overline{v}_j \in F(t,z).$$

Adding (2.12), (2.13) with such a v we obtain

$$\begin{aligned} g'(t)u &\le \Big\langle p, \Big(\sum_{j=0}^{r} \lambda_j u_j - \sum_{j>r} \lambda_j |u_j|\Big) v_z + \sum_{j>r} \lambda_j (|u_j| + u_j)\overline{v}_j - \sum_{j=0}^{d} \lambda_j w_j \Big\rangle \\ &= \Big\langle p, u v_z - \sum_{j=0}^{d} \lambda_j w_j \Big\rangle \end{aligned}$$

and prove that

$$\forall (u,w) \in \mathrm{co}(T_{\mathrm{Graph}(P)}(t,y)) \text{ with } u > 0,\ \forall v_z \in F(t,z), \quad g'(t)u \le \langle p, u v_z - w\rangle.$$

The above inequality holds also true with *co* replaced by $\overline{\mathrm{co}}$.

On the other hand, by our assumptions, there exists $v_y \in F(t,y)$ such that $(1, v_y) \in \overline{\mathrm{co}}(T_{\mathrm{Graph}(P)}(t,y))$. Thus

$$g'(t) \le \Big\langle \frac{z-y}{\|z-y\|}, v_y - v_y \Big\rangle + c_k(t)\|z-y\| = c_k(t)g(t).$$

This and the Gronwall inequality yield the result. □

To finish the proof of our theorem it is enough to show that Lemma 2.2.4 implies existence of a viable trajectory for any initial condition $x_0 \in P(t_0)$. Fix t_0, x_0. Set $r = \|x_0\| + \int_{t_0}^{T} \mu(s)\, ds$ and

$$k := r + \max\{\mathrm{dist}(x, P(t)) : t \in [t_0, T],\ x \in rB\}.$$

Let $K = kB$. We first show that for every $\varepsilon > 0$ there exists $x \in S_{[t_0,T]}(x_0)$ such that $\sup_{t\in[t_0,T]} \mathrm{dist}(x(t), P(t)) \le \varepsilon$.

Pick any $\varepsilon > 0$ and let $\delta > 0$ be such that for all $A \subset [0,T]$ with $m(A) \leq \delta$ we have $\int_A \mu(s)ds < \varepsilon/2$ and

$$\forall t_0 \leq t < \tau \leq T, \quad \tau - t < \delta \Rightarrow P(t) \cap K \subset P(\tau) + \frac{\varepsilon}{2}B.$$

Let $t_0 < \ldots < t_n = T$ be such that $t_{i+1} - t_i < \delta$. We claim that there exists $x_\varepsilon \in S_{[t_0,T]}(x_0)$ such that, for every i, $x_\varepsilon(t_i) \in P(t_i)$. We proceed by the induction. Assume that for some $j \geq 0$ there exists $x_\varepsilon \in S_{[t_0,t_j]}$ such that $x_\varepsilon(t_i) \in P(t_i)$ for all $i \leq j$. Set $g(t) = \operatorname{dist}(P(t), \{x(t) : x \in S_{[t_j,T]}(x_\varepsilon(t_j))\})$. By Lemma 2.2.4 applied with (t_0, x_0) replaced by $(t_j, x_\varepsilon(t_j))$ there exists $\overline{x} \in S_{[t_j,T]}(x(t_j))$ such that $\overline{x}(t_{j+1}) \in P(t_{j+1})$. Thus we can extend x_ε on the time interval $[t_j, t_{j+1}]$ by setting $x_\varepsilon(t) = \overline{x}(t)$ for all $t \in [t_j, t_{j+1}]$. This and the induction argument finish the proof of our claim.

Fix $t \in [t_0, T]$ and let i be such that $t_i \leq t < t_{i+1}$. Then, by the choice of δ,

$$\operatorname{dist}(x_\varepsilon(t), P(t)) \leq \operatorname{dist}(x_\varepsilon(t_i), P(t)) + \|x_\varepsilon(t_i) - x_\varepsilon(t)\| \leq \varepsilon.$$

Consider a subsequence x_{ε_i} converging weakly in $W^{1,1}(t_0,T)$ to some x, where $\varepsilon_i \to 0$. Then $x \in S_{[t_0,T]}(x_0)$ and $x(t) \in P(t)$ for all $t \in [t_0, T]$. Hence $x(\,\cdot\,)$ is the viable solution we were looking for. $\square$

2.2.2. Viability Theorem in the upper semicontinuous case.

Theorem 2.2.5. *Assume that a tube $P\colon [0,T] \rightsquigarrow \mathbb{R}^d$ is absolutely continuous, that F has convex compact images and satisfies* (2.6) *and*

(2.14) $\quad x \rightsquigarrow F(t,x)$ *is uppersemicontinuous for almost all t,*

(2.15) $\quad F(\,\cdot\,,\,\cdot\,)$ *is $\mathcal{L} \times \mathcal{B}$ measurable.*

Then the following three statements are equivalent:

(a) *There exists a set $A \subset [0,T]$ of full measure such that*

$$\forall t \in A,\ \forall x \in P(t), \quad F(t,x) \cap DP(t,x)(1) \neq \emptyset;$$

(b) *The tube P is a viability tube for F;*

(c) *For every $t_0 \in [0,T]$ and $x_0 \in P(t_0)$ there exists $x \in S_{[t_0,T]}(x_0)$ such that $x(t) \in P(t)$ for every $t \in [t_0,T]$.*

Proof. By Lemma 2.1.6 (c) $\Rightarrow$ (a). Clearly (a) yields (b). So it remains to show that (b) $\Rightarrow$ (c).

Without loss of generality we can assume that $t_0 = 0$.

Step 1. We construct an increasing sequence $\{K_k\}$ of closed subsets of $[0,T]$ such that:

(i) $\bigcup_1^\infty K_k$ is of full measure;

(ii) for every k, the restriction $F|_{K_k \times \mathbb{R}^d}$ is upper semicontinuous;

(iii) a function $\nu\colon [0,t] \to [0,\infty)$ given by

$$\nu = \sum_{k=1}^{\infty} \sup\{\nu(t) : t \in K_k\} \chi_{(K_k \setminus K_{k-1})}$$

is integrable, where χ_A denotes the characteristic function of a set A.

By Theorem 2.4 in [78], there exists an increasing sequence $\{A_n\}$ of closed subsets of $[0,T]$ such that $\lim_{n\to\infty} m([0,T] \setminus A_n) = 0$ and $F|_{A_n \times \mathbb{R}^d}$ is upper semicontinuous, for every n. Let

$$C_k = \{t \in [0,T] : \nu(t) \geq k\}.$$

Since $\nu \in L^1$, $\lim_{k\to\infty} m(C_k) = 0$.

If there is k_0 such that $m(C_{k_0}) = 0$, then we choose a decreasing sequence $\{D_m\}$ of open subsets of $[0,T]$ such that $\lim_{m\to\infty} m(D_m) = 0$, $C_{k_0} \subset D_m$, for every m, and finally define $K_k = A_k \setminus D_k$, $K_0 = \emptyset$.

It remains to investigate the case $M(C_k) > 0$, for all k. Consider a subsequence $\{A_{n_k}\}$ such that

$$m([0,T] \setminus A_{n_k}) < \frac{1}{(k+1)^3} m(C_k)$$

and a decreasing sequence of open subsets O_k of $[0,T]$ such that $C_k \subset O_k$ and

$$m(O_k \setminus C_k) + m([0,T] \setminus A_{n_k}) < \frac{1}{(k+1)^3} m(C_k). \tag{2.16}$$

Then $D_k := A_{n_k} \setminus O_k$ is a closed subset of A_{n_k} such that $D_k \cap C_k = \emptyset$ and

$$m([0,T] \setminus D_k) < \left(1 + \frac{1}{(k+1)^3}\right) m(C_k).$$

We set $K_k = \bigcup_{l=1}^{k} D_l$, $K_0 = \emptyset$. Conditions (i), (ii) are obvious. To show that ν is an integrable function observe that (2.16) yields

$$\begin{aligned}
\int_0^T \nu &\leq \sum_{k=1}^{\infty} k m(K_k \setminus K_{k-1}) \leq T + \sum_{k=1}^{\infty} (k+1) m(K_{k+1} \setminus K_k) \\
&\leq T + \sum_{k=1}^{\infty} (k+1)\{m([0,T] \setminus (C_k \cup K_k)) + m(C_k \setminus C_{k+1})\} \\
&\leq T + \sum_{k=1}^{\infty} \frac{T}{(k+1)^2} + \sum_{k=1}^{\infty} (k+1) m(C_k \setminus C_{k+1}) \\
&\leq 2T + \sum_{k=1}^{\infty} \frac{T}{(k+1)^2} + \int_0^T \mu
\end{aligned}$$

which completes the proof of step one.

Step 2. Fix k. By theorem 1.13.1 in [5] there exists a sequence $\{F_m^k\}_{m=1}^{\infty}$ of nonempty compact convex valued maps from $K_k \times \mathbb{R}^d$ into $\mathbb{R}^d$ such that

(a) $\forall t \in K_k,\ \forall x \in \mathbb{R}^d,\ \forall m,\ F_{m+1}^k(t,x) \subset F_m^k(t,x)$;
(b) $\forall t \in K_k,\ \forall x \in \mathbb{R}^d,\ F(t,x) = \bigcap_{m=1}^{\infty} F_m^k(t,x)$;
(c) $\forall m,\ F_m^k$ is locally Lipschitz continuous;
(d) $\forall t \in K_k,\ \forall x \in \mathbb{R}^d,\ \forall m,\ F_m^k(t,x) \subset \overline{\mathrm{co}} F(K_k \times \mathbb{R}^d) \subset \sup_{t\in K_k} \mu(t)B$.

We define the set-valued map $F_k\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$F_k(t,x) = \begin{cases} \nu(t)B & \text{if } t \notin K_k, \\ F_k^m(t,x) & \text{if } t \in K_m \setminus K_{m-1} \text{ and } m \in \{1,\dots,k\}. \end{cases}$$

Denote by $\mathcal{S}_k$ the set of solutions of the following viability problem

$$\begin{cases} x'(t) \in F_k(t,x(t)) & \text{a.e. in } [0,T], \\ x(0) = x_0, \\ x(t) \in P(t) & \text{for all } t \in [0,T]. \end{cases}$$

It is easy to check that F_k satisfies assumptions of Theorem 2.2.2. Thus the set $\mathcal{S}_k$ is nonempty and compact in $C([0,T])$.

It follows directly from the construction that

$$\begin{cases} F_{k+1}(t,x) \subset F_k(t,x) & \forall (t,x) \in [0,T] \times \mathbb{R}^d, \\ F(t,x) = \displaystyle\bigcap_{k=1}^{\infty} F_k(t,x) & \forall t \in \displaystyle\bigcup_{k=1}^{\infty} K_k,\ \forall x \in \mathbb{R}^d. \end{cases}$$

Thus $\mathcal{S}_{k+1} \subset \mathcal{S}_k$, which in turn implies that $\bigcap_{k=1}^{\infty} \mathcal{S}_k$ is nonempty, which yields the result. □

2.2.3. Invariance Theorem for tubes.

Theorem 2.2.6 (Invariance)**.** *Assume that a tube $P\colon [0,T] \rightsquigarrow \mathbb{R}^d$ is absolutely continuous, that F has convex compact images and satisfies* (2.5)–(2.7)*. Then the following three statements are equivalent:*

(a) *There exists a set $A \subset [0,T]$ of full measure such that for every $t \in A$ and all $x \in P(t)$ we have*

$$F(t,x) \subset DP(t,x)(1);$$

(b) *There exists a set $C \subset [0,T]$ of full measure such that for every $t \in C$ and all $x \in P(t)$ we have*

$$\{1\} \times F(t,x) \subset \overline{\mathrm{co}}(T_{\mathrm{Graph}(P)}(t,x));$$

(c) *For all $t_0 \in [0,T]$ and $x_0 \in P(t_0)$ every $x \in S_{[t_0,T]}(x_0)$ satisfies $x(t) \in P(t)$ for all $t \in [t_0,T]$.*

Proof. Clearly (a) yields (b). From [7, p. 380] there exists a measurable with respect to the first variable function $f\colon [0,T] \times \mathbb{R}^d \times B \mapsto \mathbb{R}^d$, such that

- $\forall (t,x) \in [0,T] \times \mathbb{R}^d,\ F(t,x) = f(t,x,B)$,
- $\forall u \in B,\ f(t,\cdot,u)$ is $5dc_k(t)$-Lipschitz on kB for a.e. $t \in [0,T]$,
- $\forall u,v \in B,\ \|f(t,x,u) - f(t,x,v)\| \le 5d(\sup_{y\in F(t,x)} \|y\|)\|u - v\|$.

Consider a countable family $u_i \in B,\ i \ge 1$, dense in the unit ball anf fix i. Then if (b) holds true, for all $t \in C$ and $x \in P(t)$

$$\{1\} \times f(t,x,u_i) \in \overline{\mathrm{co}}(T_{\mathrm{Graph}(P)}(t,x)).$$

By Theorem 2.2.2 there exists a set $A_i \subset [0,T]$ of full measure such that for all $t \in A_i$ and $x \in P(t)$, $f(t,x,u_i) \in DP(t,x)(1)$. Define $A = \bigcap_{i\ge1} A_i$. Then for all $t \in A$ and $x \in P(t)$ we have $\bigcup_{i\ge1} f(t,x,u_i) \subset DP(t,x)(1)$. Since $\{u_i\}_{i\ge1}$ is dense in the unit ball we finally obtain that for almost all $t \in [0,T]$ and all $x \in P(t)$, $F(t,x) \subset DP(t,x)(1)$.

We show next that (a) $\Leftrightarrow$ (c). Fix $t_0 \in [0,T]$, $x_0 \in P(t_0)$ and $x \in S_{[t_0,T]}(x_0)$. By [7, p. 316] for a measurable $u\colon [t_0,T] \mapsto B$, it holds $x'(t) = f(t,x(t),u(t))$ almost everywhere. On the other hand, $x(\cdot)$ is the only solution of

$$\begin{cases} x'(t) = f(t,x(t),u(t)), \\ x(t_0) = x_0. \end{cases} \tag{2.17}$$

By (a) we know that for all $t \in A$

$$\forall x \in P(t), \quad f(t,x,u(t)) \in DP(t,x)(1).$$

The map $(t,x) \mapsto f(t,x,u(t))$ is measurable in t and $5d\mu(t)$-Lipschitz continuous in x, hence the Viability Theorem 2.2.2 implies that

$$\forall t \in [t_0,T], \quad x(t) \in P(t).$$

Conversely, assume that (iii) is satisfied. Consider a dense family $u_i \in B$, $i \ge 1$ and the equations

$$x' = f(t,x,u_i).$$

By the assumption (c), for every fixed i, $t_0 \in [0,T]$, $x_0 \in P(t_0)$ the solution of the above equation verifies $x(t) \in P(t)$ for all $t \in [t_0,T]$. By Theorem 2.2.2 there exists a set $A_i \subset [0,T]$ of full measure such that for all $t \in A_i$ and $x \in P(t)$, $f(t,x,u_i) \in DP(t,x)(1)$. Define the set $A = \bigcap_{i\ge1} A_i$ of full measure. Then for all $t \in A$ and $x \in P(t)$, $\bigcup_{i\ge1} f(t,x,u_i) \subset DP(t,x)(1)$. Since $\{u_i\}_{i\ge1}$ is dense in the unit ball we finally obtain that for almost all $t \in [0,T]$ and all $x \in P(t)$ we have $F(t,x) \subset DP(t,x)(1)$. $\square$

2.3. Application: Hamilton–Jacobi–Bellman theory

2.3.1. Value function of Mayer's problem. Let an extended function $g: \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\}$ be given. Consider the minimization problem (called *Mayer's problem*):

$$\min\{g(x(T)) : x \in S_{[t_0,T]}(x_0)\}. \tag{2.18}$$

The value function $\mathcal{V}: [0,T] \times \mathbb{R}^d \mapsto \mathbb{R} \cup \{\pm\infty\}$ is defined by:

$$\forall (t_0, x_0) \in [0,T] \times \mathbb{R}^d, \quad \mathcal{V}(t_0, x_0) = \inf\{g(x(T)) : x \in S_{[t_0,T]}(x_0)\}. \tag{2.19}$$

We assume that

$$\begin{cases} \bullet\ F \text{ has nonempty convex compact images,} \\ \bullet\ \forall x \in \mathbb{R}^d,\ F(\,\cdot\,, x) \text{ is measurable,} \\ \bullet\ \exists \mu \in L^1(0,T) \text{ such that for almost all } t \in [0,T] \text{ we have} \\ \quad \forall x \in \mathbb{R}^d,\ \|F(t,x)\| \le \mu(t), \\ \bullet\ g \text{ is lower semicontinuous.} \end{cases} \tag{2.20}$$

Proposition 2.3.1. *If* (2.20) *holds true and for almost all* $t \in [0,T]$, $F(t,\cdot)$ *is continuous, then* $\mathcal{V}$ *is lower semicontinuous and*

$$\forall (t_0, x_0) \in [0,T] \times \mathbb{R}^d, \quad \mathcal{V}(t_0, x_0) = \min\{g(x(T)) : x \in S_{[t_0,T]}(x_0)\}. \tag{2.21}$$

Furthermore, the set-valued map

$$t \rightsquigarrow P(t) = \{(x,r) \in \mathbb{R}^d \times \mathbb{R} : r \ge \mathcal{V}(t,x)\} \text{ is absolutely continuous} \tag{2.22}$$

and

$$\begin{cases} \exists A \subset\,]0,T[\text{ of full measure such that } \forall (t,x) \in \operatorname{Dom}(\mathcal{V}) \cap A \times \mathbb{R}^d, \\ \inf\limits_{v \in F(t,x)} D_\uparrow \mathcal{V}(t,x)(1,v) \le 0, \quad \sup\limits_{v \in F(t,x)} D_\uparrow \mathcal{V}(t,x)(-1,-v) \le 0. \end{cases}$$

Remark. We observe that $\operatorname{Graph}(P)$ is equal to the epigraph $\mathcal{E}pi(\mathcal{V})$ of $\mathcal{V}$ and (2.22) yields the following relations: for every $\overline{x} \in \mathbb{R}^d$

$$g(\overline{x}) = \mathcal{V}(T, \overline{x}) = \liminf_{t \to T-,\, x \to \overline{x}} \mathcal{V}(t,x), \quad \mathcal{V}(0,\overline{x}) = \liminf_{t \to 0+,\, x \to \overline{x}} \mathcal{V}(t,x). \tag{2.23}$$

Proof. The first two statements follow by exactly the same arguments as in the proof of [44, Proposition 2.1]. Fix any $0 \le t_0 < t_1 \le T$. Let $(x_0, r_0) \in P(t_0)$ and consider $\overline{x} \in S_{[t_0,T]}(x_0)$ such that $\mathcal{V}(t, \overline{x}(t)) = g(\overline{x}(T))$ for all $t \in [y_0, T]$. Thus $(x(t_1), r_0) \in P(t_1)$ and so for all $r \ge \mathcal{V}(t_0, x_0)$, $(x_0, r) \in P(t_1) + (\int_{t_0}^{t_1} \mu(s)\, ds)B$, where B denotes at the moment the unit ball in $\mathbb{R}^{d+1}$. Hence

$$P(t_0) \subset P(t_1) + \left(\int_{t_0}^{t_1} \mu(s)\, ds\right) B.$$

Consider next $(x_1, r_1) \in P(t_1)$ and let $x \in S_{[t_0,t_1]}$ be such that $x(t_1) = x_1$. Then $\mathcal{V}(t_0, x(t_0)) \leq \mathcal{V}(t_1, x_1)$ and therefore we have

$$(x_1, r_1) \in P(t_0) + \left(\int_{t_0}^{t_1} \mu(s)\, ds \right) B.$$

So,

$$P(t_1) \subset P(t_0) + \left(\int_{t_0}^{t_1} \mu(s)\, ds \right) B.$$

To prove the latest statement it is enough to apply Theorem 2.1.5 and the fact that $\mathcal{V}$ is nondecreasing along solutions of (2.3) and is constant along optimal solutions. □

In [97, Section 3] the following Proposition is shown:

Proposition 2.3.2. *If $g: \mathbb{R}^d \mapsto \mathbb{R}$ is a locally Lipschitz continuous function and* (2.7), (2.20) *hold true, then the value function $\mathcal{V}$ has the following properties:*

$$(2.24) \quad \begin{cases} \text{(a) } \mathcal{V}(t, \cdot\,) \textit{ is locally Lipschitz continuous for almost all } t \in [0, T], \\ \text{(b) } \textit{for every compact } K \subset \mathbb{R}^d \textit{ there is an absolutely continuous} \\ \quad \textit{function } \alpha: [0, T] \mapsto \mathbb{R} \textit{ such that} \\ \quad \forall x \in K,\ \forall t_1, t_2 \in [0, T],\ |\mathcal{V}(t_1, x) - \mathcal{V}(t_2, x)| \leq |\alpha(t_1) - \alpha(t_2)|. \end{cases}$$

We shall need in the sequel the following result.

Proposition 2.3.3. *If a function $V: [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$ satisfies* (2.24)*, then*

$$m\{t \in [0, T] : \exists x \in \mathbb{R}^d,\ [T_{\mathcal{E}pi(V)}(t, x, V(t, x))]^- \cap \mathbb{R} \times \mathbb{R}^d \times \{0\} \neq \{0\}\} = 0.$$

Proof. Let α_k be choosen for $K = \{x \in \mathbb{R}^d : \|x\| \leq k\}$ and $A \subset [0, T]$ be a set of full measure such that for every $t \in A$ the map $V(t, \cdot\,)$ is locally Lipschitz continuous and for every k the function α_k is differentiable at t.

Fix $t \in A$ and $x \in \mathbb{R}^d$. We show that

$$\forall (u_t, u_x) \in \mathbb{R} \times \mathbb{R}^d, \quad D_\uparrow V(t, x)(u_t, u_x) < +\infty.$$

Let $s_n \to u_t$, $u_n \to u_x$, $h_n \to 0^+$. Then

$$\begin{aligned} &\frac{V(t + h_n s_n, x + h_n u_n) - V(t, x)}{h_n} \\ &\leq \frac{|V(t + h_n s_n, x + h_n u_n) - V(t, x + h_n u_n)|}{h_n} + \frac{|V(t, x + h_n u_n) - V(t, x)|}{h_n} \\ &\leq \frac{|\alpha_k(t + h_n s_n) - \alpha_k(t)|}{h_n} + l\|u_n\|, \end{aligned}$$

where k is choosen sufficiently large and l is a Lipschtz constant of $V(t, \cdot\,)$. So, $[T_{\mathcal{E}pi(V)}(t, x, V(t, x))]^- \cap \mathbb{R} \times \mathbb{R}^d \times \{0\} = \{0\}$. □

Assume that F has nonempty compact images and define the Hamiltonian $H\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\mapsto\mathbb{R}$ by

$$H(t,x,p)=\max_{v\in F(t,x)}\langle p,v\rangle. \tag{2.25}$$

Then $H(t,x,\cdot\,)$ is convex and positively homogeneous. Furthermore, if $F(t,\cdot\,)$ is upper semicontinuous (resp. lower semicontinuous), then so is $H(t,\cdot\,,p)$ and if $F(\,\cdot\,,x)$ is measurable, then $H(\,\cdot\,,x,p)$ is also measurable.

Consider an extended function $V\colon[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}\cup\{+\infty\}$. We may always assume that V is defined on $\mathbb{R}\times\mathbb{R}^d$ by setting $V(t,x)=+\infty$, whenever $t\notin[0,T]$. In theorem below we use Definition 1.3.1 with such extension of V.

Theorem 2.3.4. *Assume* (2.7), (2.20) *and let* $V\colon[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}\cup\{+\infty\}$ *be an extended lower semicontinuous function. Consider the set-valued map*

$$[0,T]\ni t\rightsquigarrow P(t)=\{(x,r)\in\mathbb{R}^d\times\mathbb{R}:r\geq V(t,x)\}. \tag{2.26}$$

Then the following three statements are equivalent:

(a) V *is the value function, i.e.* $V=\mathcal{V}$,

(b) $\exists A\subset\ ]0,T[$ *of full measure such that* $\forall(t,x)\in\mathrm{Dom}(V)\cap A\times\mathbb{R}^d$, $\inf_{v\in F(t,x)}D_{\uparrow}V(t,x)(1,v)\leq 0$, $\sup_{v\in F(t,x)}D_{\uparrow}V(t,x)(-1,-v)\leq 0$, $P(\cdot\,)$ *is absolutely continuous and* $V(T,\cdot\,)=g(\cdot\,)$,

(c) $\exists C\subset\ ]0,T[$ *of full measure such that* $\forall(t,x)\in\mathrm{Dom}(V)\cap C\times\mathbb{R}^d$, $\forall(p_t,p_x,q)\in[T_{\mathcal{E}pi(V)}(t,x,V(t,x))]^-$, $-p_t+H(t,x,-p_x)=0$, $P(\cdot\,)$ *is absolutely continuous and* $V(T,\cdot\,)=g(\cdot\,)$.

Remark. If a function $V\colon[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}$ is locally uniformly absolutely continuous in the sense of [97], then the tube P given by (2.26) is always absolutely continuous.

Theorem 2.3.5. *Under all assumptions of Theorem* 2.3.4 *let* $\mathrm{Dom}(V)$ *be closed, the restriction of V to its domain be continuous and the maps*

$$\begin{aligned}&t\rightsquigarrow\{(x,r)\in\mathbb{R}^d\times\mathbb{R}:r\geq V(t,x)\},\\&t\rightsquigarrow\{(x,r)\in\mathbb{R}^d\times\mathbb{R}:r\leq V(t,x)\neq+\infty\},\end{aligned}$$

be absolutely continuous. Then V is the value function if and only if

$$\begin{cases}V(T,\cdot\,)=g(\cdot\,),\ \exists D\subset[0,T]\ \text{of full measure}\ \forall(t,x)\in D\times\mathbb{R}^d,\\ \forall(p_t,p_x,q)\in[T_{\mathcal{E}pi(V)}(t,x,V(t,x))]^-,\ -p_t+H(t,x,-p_x)\geq 0,\\ \forall(p_t,p_x,q)\in[T_{\mathcal{H}yp(V)}(t,x,V(t,x))]^+,\ -p_t+H(t,x,-p_x)\leq 0.\end{cases} \tag{2.27}$$

Proofs of Theorems 2.3.4, 2.3.5 are given at the end of the section.

From Theorem 2.3.4 and Proposition 1.3.3 we obtain

Corollary 2.3.6. *Under all assumptions of Theorem* 2.3.4 *suppose that*

$$m\{t \in [0,T] : \exists x \in \mathbb{R}^d,\ (t,x) \in \mathrm{Dom}(V) \text{ and } \{0\} \neq [T_{\mathcal{E}pi(V)}(t,x,V(t,x))]^- \subset \mathbb{R} \times \mathbb{R}^d \times \{0\}\} = 0. \tag{2.28}$$

Then V is the value function if and only if

$$\begin{cases} \exists C \subset \,]0,T[\text{ of full measure such that } \forall (t,x) \in C \times \mathbb{R}^d, \\ \forall (p_t,p_x) \in \partial_- V(t,x),\ -p_t + H(t,x,-p_x) = 0, \\ P(\,\cdot\,) \text{ is absolutely continuous and } V(T,\,\cdot\,) = g(\,\cdot\,). \end{cases} \tag{2.29}$$

Corollary 2.3.7. *Under all assumptions of Theorem* 2.3.4 *suppose that a function $V : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}$ satisfies* (2.24)*. Then V is the value function if and only if*

$$\begin{aligned} &\exists C \subset \,]0,T[\text{ of full measure such that } \forall (t,x) \in C \times \mathbb{R}^d, \\ &\qquad \forall (p_t,p_x) \in \partial_- V(t,x), \quad -p_t + H(t,x,-p_x) = 0 \text{ and } V(T,\,\cdot\,) = g(\,\cdot\,). \end{aligned}$$

Corollary 2.3.8. *Under all assumptions of Theorem* 2.3.5 *suppose* (2.28) *and*

$$m\{t \in [0,T] : \exists x \in \mathbb{R}^d,\ (t,x) \in \mathrm{Dom}(V) \text{ and } \{0\} \neq [T_{\mathcal{H}yp(V)}(t,x,V(t,x))]^- \subset \mathbb{R} \times \mathbb{R}^d \times \{0\}\} = 0. \tag{2.30}$$

Then V is the value function if and only if

$$\begin{cases} V(T,\,\cdot\,) = g(\,\cdot\,),\ \exists D \subset [0,T] \text{ of full measure } \forall (t,x) \in D \times \mathbb{R}^d, \\ \forall (p_t,p_x) \in \partial_- V(t,x),\ -p_t + H(t,x,-p_x) \geq 0, \\ \forall (p_t,p_x) \in \partial_+ V(t,x),\ -p_t + H(t,x,-p_x) \leq 0. \end{cases}$$

Remark. It was shown in [44] that when in addition H is continuous with respect to t, then in the above corollaries stated with $C = \,]0,T[\, = D$ assumptions of absolute continuity, (2.28) and (2.30) can be omitted. Corollary 2.3.6 extends the results from [44] to the measurable case. Corollary 2.3.8 is the uniqueness result for viscosity solutions of the associated Hamilton–Jacobi–Bellman equation with data measurable in time.

Corollary 2.3.9. *Under all assumptions of Theorem* 2.3.5 *suppose* (2.24)*. Then V is the value function if and only if*

$$\begin{cases} V(T,\,\cdot\,) = g(\,\cdot\,),\ \exists D \subset [0,T] \text{ of full measure } \forall (t,x) \in D \times \mathbb{R}^d, \\ \forall (p_t,p_x) \in \partial_- V(t,x),\ -p_t + H(t,x,-p_x) \geq 0, \\ \forall\ (p_t,p_x) \in \partial_+ V(t,x),\ -p_t + H(t,x,-p_x) \leq 0. \end{cases}$$

Theorem 2.3.10. *Let* $V\colon [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}\cup\{+\infty\}$ *be an extended lower semicontinuous function such that the set-valued map* P *given by* (2.26) *is absolutely continuous. Assume that* $F(t,x)$ *satisfies assumptions of Theorem* 2.2.2. *Then the following statements are equivalent:*

(a1) *There exists a subset* $A \subset \,]0,T[$ *of full measure such that*

$$\forall (t,x)\in \mathrm{Dom}(V)\cap A\times\mathbb{R}^d,\qquad \inf_{v\in F(t,x)} D_{\uparrow}V(t,x)(1,v)\le 0.$$

(b1) *There exists a subset* $C\subset\,]0,T[$ *of full measure such that for all* $(t,x)\in \mathrm{Dom}(V)\cap C\times\mathbb{R}^d$

$$\forall (p_t,p_x,q)\in [T_{\mathcal{E}pi(V)}(t,x,V(t,x))]^-,\quad -p_t+H(t,x,-p_x)\ge 0.$$

(c1) $\forall (t_0,x_0)\in[0,T]\times\mathbb{R}^d$, $\exists\,\overline{x}\in S_{[t_0,T]}(x_0)$, $\forall t\in[t_0,T]$, $\quad V(t,\overline{x}(t))\le V(t_0,x_0)$.

Corollary 2.3.11. *Under all assumptions of Theorem* 2.3.10 *suppose that* $V(T,\cdot\,)=g(\cdot\,)$. *If* V *satisfies* (a1) *or* (b1), *then* $V\ge\mathcal{V}$.

Proof. Fix $(t_0,x_0)\in[0,T]\times\mathbb{R}^d$ and let $\overline{x}$ be as in (b1). Then

$$\mathcal{V}(t_0,x_0)\le g(\overline{x}(T)) = V(T,\overline{x}(T))\le V(t_0,x_0). \qquad \square$$

Proof of Theorem 2.3.10. Observe that conditions (a1)–(c1) in Theorem 2.3.10 are equivalent to conditions (a)–(c) in Theorem 2.2.2, respectively. To see (a1) $\Rightarrow$ (a) let us fix $(t_0,x_0)\in\mathrm{Dom}(V)$. Then for all $t\in A$, $(x,z)\in P(t)$ we have

$$\begin{aligned}\{(v,u)\in\mathbb{R}^{d+1} : u\ge D_{\uparrow}V(t,x)(1,v)\}&\\ = DP(t,x,V(t,x))(1)\subset DP(t,x,z)(1).&\end{aligned}\tag{2.31}$$

Consider the viability problem

$$\begin{cases} (x,z)'\in F(t,x)\times\{0\},\\ (x,z)(t_0)=(x_0,V(t_0,x_0)),\\ (x,z)(t)\in P(t).\end{cases}\tag{2.32}$$

By (a1) and (2.31), for all $t\in A$

$$\forall (x,z)\in P(t),\quad F(t,x)\times\{0\}\cap DP(t,x,z)(1)\ne\emptyset.\tag{2.33}$$

The opposite implication is obvious.

To obtain the implication (b1) $\Rightarrow$ (b) we have to use the separation theorem. $\square$

Theorem 2.3.12. *Let* $V : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\}$ *be an extended lower semicontinuous function. Assume that* F *satisfies* (2.6), (2.20) *and that the set-valued map* P *defined by* (2.26) *is absolutely continuous. Then the following two statements are equivalent:*

(a) *There exists a subset* $A \subset]0, T[$ *of full measure such that*

$$\forall (t, x) \in \mathrm{Dom}(V) \cap A \times \mathbb{R}^d, \qquad \sup_{v \in F(t,x)} D_\uparrow V(t, x)(-1, -v) \leq 0.$$

(b) $\forall (t_0, x_0) \in [0, T] \times \mathbb{R}^d,\ \forall x \in S_{[t_0,T]}(x_0),\ \forall t \in [t_0, T],\ V(t_0, x_0) \leq V(t, x(t))$.

Corollary 2.3.13. *Under all assumptions of Theorem* 2.3.12 *suppose that* $V(T, \cdot) = g(\cdot)$. *If* V *satisfies* (a), *then* $V \leq \mathcal{V}$.

Proof. Fix $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ and let $\overline{x} \in S_{[t_0,T]}(x_0)$ be such that $V(t_0, x_0) = g(\overline{x}(T))$. Then, by (b),

$$\mathcal{V}(t_0, x_0) = g(\overline{x}(T)) = V(T, \overline{x}(T)) \geq V(t_0, x_0). \qquad \square$$

Proof of Theorem 2.3.12. Assume (a) and fix $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$, $x \in S_{[t_0,T]}(x_0)$. Since (a) does not involve T, it is sufficient to prove the inequality in (b) for $t = T$ and it is enough to consider the case $V(T, x(T)) < \infty$. Set $\widehat{P}(t) = P(T - t)$. Then $\widehat{P}$ is absolutely continuous and for all $(x, z) \in \widehat{P}(t)$

$$\begin{aligned} \{(-v, u) \in \mathbb{R}^{d+1} : u \geq D_\uparrow V(T - t, x)(-1, -v)\} \\ = D\widehat{P}(t, x, V(T - t, x))(1) \\ = DP(T - t, x, V(T - t, x))(-1) \subset D\widehat{P}(t, x, z)(1). \end{aligned} \tag{2.34}$$

By (a) we know that for almost all $t \in [0, T]$,

$$\forall (x, z) \in \widehat{P}(t), \quad -F(T - t, x) \times \{0\} \subset D\widehat{P}(t, x, z)(1). \tag{2.35}$$

Since the map $t \mapsto (x(T - t), V(T, x(T)))$ solves the differential inclusion

$$(y, z)' \in -F(T - t, y) \times \{0\}, \quad y(0) = x(T), \quad z(0) = V(T, x(T))$$

by Theorem 2.2.6, for all $0 \leq t \leq T - t_0$, $V(T, x(T)) \geq V(T - t, x(T - t))$. In particular, $V(t_0, x(t_0)) \leq V(T, x(T))$.

If (b) is verified, then by Theorem 2.2.6 there exists a set $A \subset [0, T]$ of full measure such that for all $t \in A$ inclusion (2.35) holds true. This and (2.34) yield (a). $\square$

Proof of Theorem 2.3.4. The equivalence (a) $\Leftrightarrow$ (b) follows from Corollaries 2.3.11 and 2.3.13.

We show next that (b) $\Leftrightarrow$ (c). Let A be as in (b). Fix $t \in A$ and let $(p_t, p_x, q) \in [T_{\mathcal{E}pi(V)}(t, x, V(t, x))]^-$. From (1.4) and (b)

$$p_t + \inf_{v \in F(t,x)} \langle p_x, v \rangle \leq 0 \quad \text{and} \quad - p_t + \sup_{v \in F(t,x)} \langle p_x, -v \rangle \leq 0.$$

These two inequalities yield (c).

Conversely, assume (c). By the separation theorem for all $(t, x) \in \mathrm{Dom}(V) \cap C \times \mathbb{R}^d$ we have

$$\{1\} \times F(t, x) \times \{0\} \cap \overline{\mathrm{co}}(T_{\mathcal{E}pi(V)}(t, x, V(t, x))) \neq \emptyset$$

and

$$\{-1\} \times -F(t, x) \times \{0\} \subset \overline{\mathrm{co}}(T_{\mathcal{E}pi(V)}(t, x, V(t, x))).$$

These two relations, (2.31), (2.34) and Theorems 2.2.2, 2.2.6 imply (b) and finish the proof of Theorem 2.3.4. $\square$

Proof of Theorem 2.3.5. Assume that V is the value function. Then (c) of Theorem 2.3.4 is verified. Thus for all $(t, x) \in \mathrm{Dom}(V) \cap C \times \mathbb{R}^d$

$$\forall (p_t, p_x, q) \in [T_{\mathcal{E}pi(V)}(t, x, V(t, x))]^-, \quad -p_t + H(t, x, -p_x) \geq 0.$$

On the other hand by Theorem 2.1.9 there exists a set $D \subset C$ of full measure such that for all $t \in D$, $x \in \mathbb{R}^d$, $v \in F(t, x)$ the differential inclusion

$$\begin{cases} x'(s) \in F(s, x(s)) & \text{for almost all } s \in [0, T], \\ x(t) = x, \ x'(t) = v, \end{cases}$$

has a solution. Furthermore, for all $h > 0$, $V(t + h, x(t + h)) - V(t, x) \geq 0$. Dividing by h and taking the limit we get

$$D_{\downarrow} V(t, x)(1, v) \geq 0 \Leftrightarrow (1, v, 0) \in T_{\mathcal{H}yp(V)}(t, x, V(t, x)).$$

Since $v \in F(t, x)$ is arbitrary,

$$\forall (p_t, p_x, q) \in [T_{\mathcal{H}yp(V)}(t, x, V(t, x))]^+, \quad -p_t + H(t, x, -p_x) \leq 0$$

This proves (2.27).

Conversely, assume (2.27). It is enough to prove (b) of Theorem 2.3.4. By the separation theorem for all $(t, x) \in \mathrm{Dom}(V) \cap D \times \mathbb{R}^d$

$$\begin{cases} \{1\} \times F(t, x) \times \{0\} \cap \overline{\mathrm{co}}(T_{\mathcal{E}pi(V)}(t, x, V(t, x))) \neq \emptyset, \\ \{1\} \times F(t, x) \times \{0\} \subset \overline{\mathrm{co}}(T_{\mathcal{H}yp(V)}(t, x, V(t, x))). \end{cases} \tag{2.36}$$

From the first relation, (2.31) and Theorem 2.2.2 we deduce the first inequality in (b). Applying Theorem 2.2.6 to the absolutely continuous set-valued map

$$t \rightsquigarrow \mathcal{P}(t) = \{(x, z) : +\infty \neq V(t, x) \geq z\}$$

we deduce that for a subset $A \subset [0,T]$ of full measure and all $(x,z) \in \mathcal{P}(t)$, $F(t,x) \times \{0\} \subset D\mathcal{P}(t,x,z)(1)$. Fix $t_0 \in [0,T]$ and $x \in S_{[t_0,T]}(x_0)$. By Theorem 2.2.6, $(x(t), V(t_0,x_0)) \in P(t)$ for all $t \in [t_0,T]$. In particular this yields $V(t_0,x_0) \le V(t,x(t))$. Hence, by Theorem 2.3.12, the second inequality in (b) holds true as well. □

2.3.2. Solutions of the Hamilton–Jacobi–Bellman equation with the Hamiltonian measurable in time. Consider $H\colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ and the Hamilton–Jacobi–Bellman equation

$$-\frac{\partial V}{\partial t}(t,x) + H\left(t, x, -\frac{\partial V}{\partial x}(t,x)\right) = 0. \tag{2.37}$$

We assume:

$$\begin{cases} \text{(a) } \forall t \in [0,T],\ H(t,\cdot,\cdot) \text{ is continuous,} \\ \text{(b) } \forall (x,p) \in \mathbb{R}^d \times \mathbb{R}^d,\ H(\cdot,x,p) \text{ is measurable,} \\ \text{(c) } H(t,x,\cdot) \text{ is convex,} \\ \text{(d) } \exists \mu \in L^1(0,T),\ \forall p \in B,\ |H(t,x,p)| \le \mu(t), \\ \text{(e) } \forall k > 0,\ \exists c_k \in L^1(0,T) \text{ such that for almost all } t \in [0,T], \\ \quad\ \forall p \in B,\ H(t,\cdot,p) \text{ is } c_k(t)\text{-Lipschitz on } kB, \\ \text{(f) } H(t,x,\cdot) \text{ is positively homogeneous.} \end{cases} \tag{2.38}$$

where B denotes the closed unit ball in $\mathbb{R}^d$.

Remark. Assumption (f) may be replaced by the Lipschitz continuity of $H(t,x,\cdot)$ together with modified, with respect to p, conditions (d), (e). Then it is possible to study solutions of (2.37) via a Hamilton–Jacobi–Bellman equation with the new (conjugate) Hamiltonian meeting assumptions (2.38) (as it was done for instance in [14]).

Define $F\colon [0,T] \times \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ by

$$F(t,x) = \bigcap_{\|p\|=1} \{v \in \mathbb{R}^d : \langle p, v\rangle \le H(t,x,p)\}. \tag{2.39}$$

Proposition 2.3.14. *If* (2.38) *holds true, then F satisfies* (2.6)*,* (2.20) *and*

$$\forall p \in \mathbb{R}^d, \qquad \sup_{v \in F(t,x)} \langle p, v\rangle = H(t,x,p).$$

Proof. Fix $x \in \mathbb{R}^d$ and consider a dense subset $\{p_i\}_{i \ge 1}$ of the unit sphere in $\mathbb{R}^d$. For every $i \ge 1$ define the set-valued map $\mathcal{P}_i\colon [0,T] \rightsquigarrow \mathbb{R}^d$ by

$$\mathcal{P}_i(t) = \{\ v \in \mathbb{R}^d : \langle p_i, v\rangle \le H(t,x,p_i)\}.$$

From the separation theorem and continuity of $H(t,x,\,\cdot\,)$ it follows that

$$F(t,x)=\bigcap_{i\geq 1}\mathcal{P}_i(t).$$

By [7, Theorem 8.2.9], $\mathcal{P}_i$ is measurable. Thus by [7, Theorem 8.2.4] the set-valued map

$$t \rightsquigarrow \bigcap_{i\geq 1}\mathcal{P}_i(t)=F(t,x)$$

is also measurable. The remaining properties of F were checked in the proof of Proposition 7.1 of [44]. □

Consider the differential inclusion

$$x'(t)\in F(t,x(t)) \quad \text{almost everywhere} \tag{2.40}$$

and let $S_{[t_0,T]}(x_0)$ have the same meaning as before. From Theorems 2.3.4, 2.3.5 we immediately deduce

Theorem 2.3.15. *Assume* (2.38) *and consider an extended lower semicontinuous function* $V\colon[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}\cup\{+\infty\}$. *Set* $g(\,\cdot\,)=V(T,\,\cdot\,)$. *Then the following two statements are equivalent:*

(a) *The set-valued map* $t\rightsquigarrow\{(x,r):r\geq V(t,x)\}$ *is absolutely continuous and there exists* $A\subset[0,T]$ *of full measure such that for all* $(t,x)\in A\times\mathbb{R}^d$

$$\forall(p_t,p_x,q)\in\big[T_{\mathcal{E}pi(V)}(t,x,V(t,x))\big]^-,\quad -p_t+H(t,x,-p_x)=0. \tag{2.41}$$

(b) *For all* $(t_0,x_0)\in[0,T]\times\mathbb{R}^d$,

$$V(t_0,x_0)=\inf\{g(x(T)):x\in S_{[t_0,T]}(x_0)\}. \tag{2.42}$$

Corollary 2 .3.16 (Maximum Principle)**.** *Assume* (2.38) *and let* V_1, V_2 *be extended lower semicontinuous functions from* $[0,T]\times\mathbb{R}^d$ *into* $\mathbb{R}\cup\{+\infty\}$ *satisfying* (a) *of Theorem* 2.3.15. *If* $V_1(T,\,\cdot\,)\geq V_2(T,\,\cdot\,)$, *then* $V_1\geq V_2$.

Proof. By Theorem 2.3.15, V_i is given by (2.42) with $g(\,\cdot\,)=V_i(T,\,\cdot\,)$. □

Results of Section 5.2 imply different equivalent formulations of statement (b) of Theorem 2.315 linking V to viscosity solutions. For instance Corollary 2.3.9 yields

Corollary 2.3.17. *Assume* (2.38) *and consider a locally Lipschitz continuous function* $V\colon[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}$. *Set* $g(\,\cdot\,)=V(T,\,\cdot\,)$. *Then the following three statements are equivalent:*

(a) *There exists a set* $A\subset[0,T]$ *of full measure such that*

$$\forall(t,x)\in A\times\mathbb{R}^d,\ \forall(p_t,p_x)\in\partial_-V(t,x),\quad -p_t+H(t,x,-p_x)=0.$$

(b) *For all* $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$,

$$V(t_0, x_0) = \inf\{g(x(T)) : x \in S_{[t_0,T]}(x_0)\}.$$

(c) *There exists* $C \subset [0, T]$ *of full measure such that for all* $(t, x) \in C \times \mathbb{R}^d$

$$\begin{cases} \forall (p_t, p_x) \in \partial_- V(t, x), & -p_t + H(t, x, -p_x) \geq 0, \\ \forall (p_t, p_x) \in \partial_+ V(t, x), & -p_t + H(t, x, -p_x) \leq 0. \end{cases}$$

CHAPTER 3

CONTROL PROBLEMS WITH STATE CONSTRAINTS

The value function for control problems with state constraints was considered in [92], [23], [62] (see also Chapter IV in [2]). In the cited papers under hypotheses including an inward-pointing condition it is shown that the value function is continuous and is a viscosity solution of the corresponding Hamilton–Jacobi–Bellman equation. In this chapter we consider the Bolza problem and an infinite horizon control problem under an outward-pointing condition (see (3.5), (3.33)). Under hypotheses of the main results of the chapter: Theorems 3.1.7 and 3.2.1, the value function can be discontinuous but despite of this fact it is still the unique solution of the corresponding Hamilton–Jacobi–Bellman equation in the weak sense proposed in the chapter. The method of proof is attributed to Frankowska [44]. The presence of constraints needs some crucial improvements in the construction of backward invariant solutions.

3.1. Bolza problem

Let a nonempty subset $D \subset \mathbb{R}^n$ be locally compact. Set $K := \overline{D}$ (closure of D), $M := K \setminus D$. Then M is closed. Consider a complete separable metric space U and let

$$f\colon [0,T] \times \mathbb{R}^n \times U \mapsto \mathbb{R}^n,\ L\colon [0,T] \times \mathbb{R}^n \times U \mapsto [0,\infty) \quad \text{be bounded continuous maps;} \tag{3.1}$$

such that

$$f,\ L \text{ are locally Lipschitz continuous with respect to } (t,x) \text{ uniformly in } u; \tag{3.2}$$

in the following sense

$$\forall r > 0,\ \exists\, l_r > 0 \text{ such that } \forall u \in U,$$
$$f(\,\cdot\,,\,\cdot\,,u) \text{ and } L(\,\cdot\,,\,\cdot\,,u) \text{ are } l_r\text{-Lipschitz on } [0,T] \times B(0,r).$$

We also assume that

$$\{(f(t,x,u),L(t,x,u)+r)\in\mathbb{R}^n\times\mathbb{R} : u\in U,\ r\geq 0\}\quad\text{is closed and convex for every } t\in[0,T],\ x\in K, \tag{3.3}$$

$$\forall (t,x)\in\,]0,T]\times D,\ \forall u\in U,\quad -f(t,x,u)\in T_D(x), \tag{3.4}$$

$$\forall (t,x)\in\,]0,T]\times M,\ \exists u\in U,\quad f(t,x,u)\notin P_K^M(x). \tag{3.5}$$

Remark. If $D=\Omega$ is an open subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = M$ then (3.4) is always satisfied ($T_\Omega(x)=\mathbb{R}^n$) and (3.5) means

$$\forall (t,x)\in\,]0,T]\times\partial\Omega,\ \exists u\in U,\quad \langle f(t,x,u),n(x)\rangle>0$$

where $n(x)$ is an outer normal to Ω at x.

Let

$$g\colon K\mapsto\mathbb{R}\cup\{+\infty\}\ \text{be proper and lower semicontinuous} \tag{3.6}$$

(Proper means here not identically equal to $+\infty$). For any measurable control $u\colon[0,T]\mapsto U$ and $t_0\in[0,T]$ denote by $x(\,\cdot\,;t_0,x_0,u)$ the unique solution of

$$\begin{cases} x'(t)=f(t,x(t),u(t)),\\ x(t_0)=x_0,\end{cases}$$

defined on the interval $[0,T]$. Let us denote by

$$A(t_0,x_0)=\{u\colon[t_0,T]\mapsto U : x(t;\ t_0,x_0,u)\in K \text{ for every } t\in[t_0,T]\}$$

the set of all admissible controls from the initial condition $(t_0,x_0)\in[0,T]\times K$. The value function $V\colon[0,T]\times K\mapsto\mathbb{R}\cup\{+\infty\}$ for the Bolza problem (P) is given by

$$V(t_0,x_0)=\inf_{u\in A(t_0,x_0)}\int_{t_0}^T L(s,x(s;t_0,x_0,u),u(s))\,ds+g(x(T;t_0,x_0,u)).$$

If the set $A(t_0,x_0)$ is empty, then we set $V(t_0,x_0)=+\infty$.

Define a set-valued map $F\colon[0,T]\times\mathbb{R}^n\times\mathbb{R}\rightsquigarrow\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ by

$$F(t,x,v)=\{(1,f(t,x,u),-L(t,x,u)-r) : u\in U,\ r\in[0,C-L(t,x,u)]\}$$

where C is a bound on L. If (3.1)–(3.3) hold true then F is locally Lipschitz continuous bounded set-valued map and it has convex compact values.

Denote by $S(t_0,x_0)$ the set of all solutions of

$$\begin{cases} z'(t)\in F(z(t)),\\ z(t_0)=(t_0,x_0,0),\end{cases}$$

defined on the interval $[t_0,T]$. Let

$$S_v(t_0,x_0)=\{z\in S(t_0,x_0) : z(t)\in[t_0,T]\times K\times\mathbb{R} \text{ for every } t\in[t_0,T]\}.$$

It is easy to check that $z \in S_v(t_0, x_0)$ if and only if there exist $u \in A(t_0, x_0)$ and a measurable η such that $L(s, x(s; t_0, x_0, u), u(s)) + \eta(s) \le C$ and

$$z(t) = \left(t, x(t; t_0, x_0, u), \int_t^T -(L(s, x(s; t_0, x_0, u), u(s)) + \eta(s))\, ds\right).$$

Proposition 3.1.1. *Assume that* (3.1)–(3.3) *and* (3.6) *hold true. Then the value function* $V\colon [0,T] \times K \mapsto \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous. Moreover,*

$$\liminf_{t_n \to 0^+,\, y_n \to y,\, y_n \in K} V(t_n, y_n) = V(0, y)$$

for every $y \in K$.

Proof. The first statement is well known. To prove the second one, fix $y \in K$ such that $(0, y) \in \mathrm{Dom}(V)$. Let $\overline{u}\colon [0,T] \mapsto U$ be an optimal control for this initial conditions $(0, y)$. Then

$$V(s, x(s; 0, y, \overline{u})) = V(0, y) - \int_0^s L(\tau, x(\tau; 0, y, \overline{u}), \overline{u}(\tau))\, d\tau.$$

Clearly $\lim_{s \to 0^+} V(s, x(s;\, 0, y, \overline{u})) = V(0, y)$. This and the lower semicontinuity of V end the proof. $\square$

Proposition 3.1.2. *Assume that* (3.1)–(3.3) *and* (3.6) *hold true. Then the epigraph of the value function over* $[0,T) \times K$*, i.e. the set* $\mathcal{K} := \{(t, x, v) : t \in [0,T),\ x \in K,\ v \ge V(t,x)\}$ *is a viability domain for the set-valued map* F*.*

Proof. Fix $(t_0, x_0, v_0) \in \mathcal{K}$. Consider an optimal control $\overline{u} \in A(t_0, x_0)$, i.e. $V(t_0, x_0) = \int_{t_0}^T L(s, \overline{x}(s), \overline{u}(s))\, ds + g(\overline{x}(T))$, where $\overline{x}(\,\cdot\,) = x(\,\cdot\,; t_0, x_0, \overline{u})$. Then the function

$$z(t) = \left(t, \overline{x}(t), v_0 + \int_{t_0}^t -L(s, \overline{x}(s), \overline{u}(s))\, ds\right)$$

is a solution of $z' \in F(z)$. Observe that

$$v_0 - \int_{t_0}^t L(s, \overline{x}(s), \overline{u}(s))\, ds \ge \int_t^T L(s, \overline{x}(s), \overline{u}(s))\, ds + g(\overline{x}(T)) \ge V(t, \overline{x}(t)).$$

Thus, $z(t) \in \mathcal{K}$ for $t \in [t_0, T)$. By the Viability Theorem (Theorem 1.4.1), we obtain the desired conclusion. $\square$

Proposition 3.1.3. *Assume that* (3.1)–(3.3) *and* (3.6) *hold true. Then the epigraph*

$$D_T = \{(t, x, v) : t \in (0,T],\ x \in D \text{ and } v \ge V(t,x)\}$$

of the function V *restricted to* $(0,T] \times D$ *is a backward invariance domain for* F*.*

Proof. Let $t_0 \in (0,T]$, $x_0 \in D$ and $+\infty > v_0 \ge V(t_0, x_0)$. There is $\varepsilon > 0$ such that for every measurable $u\colon [t_0 - \varepsilon, t_0] \mapsto U$ the solution $x_u\colon [t_0 - \varepsilon, t_0] \mapsto \mathbb{R}^n$ to

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \\ x(t_0) = x_0, \end{cases}$$

satisfies $x_u(t) \in D$. Let $v_u(t) = v_0 + \int_{t_0}^{t} -L(s, x_u(s), u(s))\,ds$ and $\overline{u} \in A(t_0, x_0)$ be an optimal control. Fix $t_1 \in [t_0 - \varepsilon, t_0]$ and let $x_1 = x_u(t_1)$. Define

$$u_1(s) = \begin{cases} u(t) & \text{for } t \in [t_1, t_0], \\ \overline{u}(t) & \text{for } t > t_0. \end{cases}$$

We have $u_1 \in A(t_1, x_1)$ and

$$\begin{aligned} V(t_1, x_1) &\le \int_{t_1}^{T} L(s, x(s; t_1, x_1, u_1), u_1(s))\,ds + g(x(T; t_1, x_1, u_1)) \\ &= \int_{t_1}^{t_0} L(s, x(s; t_1, x_1, u_1), u_1(s))\,ds + V(t_0, x_0) \le v_u(t_1). \end{aligned} \tag{3.7}$$

If $z(t)$ is a solution of the differential inclusion $z'(t) \in F(z(t))$ defined on the interval $[t_0-\varepsilon, t_0]$ and $z(t_0) = (t_0, x_0, v_0)$, then there is a control $u: [t_0-\varepsilon, t_0] \mapsto U$ and a measurable function $\eta: [t_0-\varepsilon, t_0] \mapsto [0, C]$ such that $z(t) = (t, x_u(t), v_u(t) + \int_{t_0}^{t} \eta(s)\,ds)$. By (3.7), we obtain $z(t) \in \mathrm{Epi}(V)$. From Theorem 1.4.2, we get the conclusion. □

Proposition 3.1.4. *Suppose that* (3.1)–(3.3) *and* (3.6) *hold true and let* $W: [0, T] \times K \mapsto \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function. If* $W(T, x) \ge g(x)$ *for* $x \in K$ *and* $\mathcal{K} := \{(t, x, v) : t \in [0, T),\ x \in K,\ v \ge W(t, x)\}$ *is a viability domain of* F*, then*

$$W(t_0, x_0) \ge V(t_0, x_0)$$

for every $(t_0, x_0) \in [0, T] \times K$.

Proof. By Theorem 1.4.1 for $(t_0, x_0) \in \mathrm{Dom}(W)$, there is a $\mathcal{K}$-viable solution $z: [t_0, t_1) \mapsto \mathbb{R}^{n+2}$ of the Cauchy problem $z' \in F(z)$, $z(t_0) = (t_0, x_0, W(t_0, x_0))$. There is a control $u \in A(t_0, x_0)$ such that $z(t) = (t, x_u(t), v_u(t))$, where $x_u(t) = x(t; t_0, x_0, u)$ and $v_u(t) = W(t_0, x_0) - \int_{t_0}^{t} (L(s, x_u(s), u(s)) + \eta(s))\,ds$ for a non-negative measurable function η. Setting $\eta = 0$ we get another $\mathcal{K}$-viable solution denoted again by z. We have

$$v_1 := \lim_{t \to t_1^-} v_u(t) \ge \liminf_{t \to t_1^-} W(t, x_u(t)) \ge W(t_1, x_u(t_1)).$$

This solution $z(\,\cdot\,)$ can be extended to the interval $[t_0, T]$ and

$$W(t_0, x_0) - \int_{t_0}^{T} L(s, x_u(s), u(s))\,ds \ge W(T, x_u(T)) \ge g(x_u(T)).$$

Hence $V(t_0, x_0) \le W(t_0, x_0)$, which completes the proof. □

Corollary 3.1.5. *Assume that* (3.1)–(3.6) *hold true. Then for every* $(t_0, y_0) \in (0, T] \times K$ *there is a sequence* (t_n) *converging to* t_0 *from the left and a sequence* $(y_n) \subset D$ *converging to* y_0 *such that* $\lim_{n\to\infty} V(t_n, y_n) = V(t_0, y_0)$.

Proof. By Proposition 3.1.3 and the lower semicontinuity of V it is enough to consider $y_0 \in M$. By (3.5), there is $\overline{u} \in U$ such that $f(t_0, y_0, \overline{u}) \notin P_K^M(y_0)$.

Using Theorem 1.4.2, Proposition 1.2.1 and (3.4) it is not difficult to realize that there is $\varepsilon > 0$ such that $\overline{x}(t) \in D$ for $t \in (t_0 - \varepsilon, t_0)$, where $\overline{x}$ is a solution of

$$\begin{cases} x'(t) = f(t, x(t), \overline{u}), \\ x(t_0) = y_0. \end{cases}$$

Setting $y_n = \overline{x}(t_n)$, where $t_n \to t_0^-$, we have

$$V(t_n, y_n) \leq V(t_0, y_0) + \int_{t_n}^{t_0} L(s, \overline{x}(s), \overline{u})\, ds.$$

This and the lower semicontinuity of V yield $\lim_{n\to\infty} V(t_n, y_n) = V(t_0, y_0)$. $\square$

Proposition 3.1.6. *Assume that* (3.1)–(3.6) *hold true. Let* $W\colon [0,T] \times K \mapsto \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function such that*

$$\liminf_{t_n \to T^-,\, y_n \to y_0,\, y_n \in D} W(t_n, y_n) = g(y_0) \quad \textit{for every } y_0 \in K.$$

If the set $\{(t, x, v) : t \in (0, T),\ x \in D \text{ and } v \geq W(t, x)\}$ *is a backward invariance domain of* F*, then*

$$W(t_0, x_0) \leq V(t_0, x_0) \quad \textit{for every } (t_0, x_0) \in [0, T] \times K.$$

Proof. Fix $t_0 \in [0, T)$, $x_0 \in K$ such that $(t_0, x_0) \in \mathrm{Dom}(V)$. Let $u \in A(t_0, x_0)$, $x_u(s) = x(s; t_0, x_0, u)$ and $w_u\colon [t_0, T] \mapsto \mathbb{R}$ solves the Cauchy problem

$$\begin{cases} w'(t) = -L(t, x_u(t), u(t)), \\ w(T) = g(x_u(T)). \end{cases}$$

It is enough to show that

$$W(t_0, x_u(t_0)) \leq w_u(t_0). \tag{3.8}$$

Let $l > 1$ be the Lipschitz constant of $f(\,\cdot\,, \,\cdot\,, u)$, $L(\,\cdot\,, \,\cdot\,, u)$ on the set $]0, T] \times B(x_0, CT + 1)$, where C is an upper bound f. By the assumption (3.5) and Proposition 1.2.1 for every $(t, x) \in\,]0, T] \times M$ there exist $u_{t,x} \in U$, $\varepsilon_{t,x} > 0$ such that for every $y \in K \cap B(x, \varepsilon_{t,x})$ we have

$$(y + (0, \varepsilon_{t,x}] B(-f(t, x, u_{t,x}), \varepsilon_{t,x})) \cap K \subset D.$$

If $|t' - t| < \varepsilon_{t,x}/4l$, $|x' - x| < \varepsilon_{t,x}/4l$ then

$$B\biggl(- f(t', x', u_{t,x}), \frac{\varepsilon_{t,x}}{2} \biggr) \subset B(-f(t, x, u_{t,x}), \varepsilon_{t,x}).$$

Hence, for every $(t, x) \in]0, T] \times M$ there are $u_{t,x} \in U$, $R_{t,x} > 0$ such that for every $t' \in [0, T]$, $x', y \in K$ satisfying $|y - x'| < R_{t,x}$, $|x' - x| < R_{t,x}$, $|t - t'| < R_{t,x}$ we have

$$(y + (0, R_{t,x}] B(-f(t', x', u_{t,x}), R_{t,x}) \cap K \subset D.$$

Since the set M is closed,

$$\begin{aligned}(3.9)\quad &\exists 0 < R < 1,\ \forall t' \in [t_0, T],\ \forall x' \in K \cap B(M, R) \cap B(x_0, CT + 1),\ \exists u' \in U,\\ &\forall y \in K \cap B(x', R), \quad (y + (0, R]B(-f(t', x', u'), R))) \cap K \subset D.\end{aligned}$$

We choose $\Delta\tau > 0$ such that

$$(C + l + 2)(e^{l\Delta\tau} - 1) < \frac{R}{4} \quad \text{and} \quad \Delta\tau\, C < \frac{R}{2}. \tag{3.10}$$

Let N be a natural number such that $N\Delta\tau > T$. We set

$$C_2 := \frac{4}{R}, \quad C_1 := (1 + C_2(2C + (l + C)T))^N. \tag{3.11}$$

Now, we choose a sequence $(t_k, y_k) \in (0, T) \times D$ such that $t_k \to T^-$, $y_k \to x_u(T)$ and $W(t_k, y_k) \to W(T, x_u(T))$. Without loss of generality we can assume that for every k

$$|y_k - x_u(t_k)| \le \frac{d}{C_1 e^{lT}}$$

where

$$d := \frac{R^2}{8l(C + 1)}. \tag{3.12}$$

We next describe how to construct controls $v_k\colon [t_0, t_k] \mapsto U$ and corresponding trajectories $x_k\colon [t_0, t_k] \mapsto R^n$ which satisfy:

$$\begin{cases} \bullet\ x_k(t_k) = y_k \text{ and } x_k(t) \in D \text{ for } t \in [t_0, t_k];\\ \bullet\ |x_k(t) - x_u(t)| \le C_1 e^{l(t_k - t)}|y_k - x_u(t_k)| \text{ for } t \in [t_0, t_k];\\ \bullet\ \text{The interval } [t_0, t_k] \text{ is divided into subintervals } [\tau^k_{i+1}, \tau^k_i],\\ \quad i = 0, \dots, n_k,\ n_k \le N. \text{ Moreover, } v_k(t) = u(t + \varepsilon^k_i)\\ \quad \text{for every } t \in [\tau^k_{i+1}, \tau^k_i - \varepsilon^k_i],\ i = 0, \dots, n_k - 1,\\ \quad \text{where } \varepsilon^k_i = C_2|x_k(\tau_k) - x_u(\tau_k)|. \end{cases} \tag{3.13}$$

Only after having the control v_k and the corresponding trajectory defined on the interval $[\tau^k_i, \tau^k_0]$ we choose the left end τ^k_{i+1} and extend v_k to $[\tau^k_{i+1}, \tau^k_i[$. The construction is based on the assumption that the initial condition $t' = \tau^k_i$, $x' = x_k(\tau^k_i)$ satisfies

$$x' \in D, \quad |x_u(t') - x'| < d, \quad \operatorname{dist}(x', M) < R. \tag{3.14}$$

Notice that by (3.9), there exists $u' \in U$ such that

$$(y +]0, R]B(-f(t', x', u'), R)) \cap K \subset D$$

for every $y \in K$, $|y - x'| < R$. So, if $z \in K$, $y \in K$, $|x' - y| < R$, $0 < r < R$ and

$$|z - (y - rf(t', x', u'))| < rR \Rightarrow z \in D. \tag{3.15}$$

Set $\varepsilon' := \min(C_2|x_u(t') - x'|, \Delta\tau)$ and define a control $w\colon [t_0, t'] \mapsto U$ by

$$w(t) = \begin{cases} u' & \text{for } t \in [t' - \varepsilon', t'], \\ u(t + \varepsilon') & \text{for } t \in [t_0, t' - \varepsilon'). \end{cases}$$

We claim that the trajectory x_w corresponding to the control w and to the initial condition $x_w(t') = x'$ satisfies $x_w(t) \in D$ for $t \in [t' - \Delta\tau, t']$.

First consider the case $t \in [t' - \varepsilon', t']$. We have

$$|x_w(t) - (x' - (t' - t) f(t', x', u'))| < R(t' - t). \tag{3.16}$$

Let $t_1 = \inf\{t \in [t' - \varepsilon', t'] : x_w(s) \in D \text{ for } s \in [t, t']\}$. By (3.4) and the Invariance Theorem, we obtain $t_1 < t'$. By (3.15), (3.16), we have $x_w(s) \in D$ for $s \in (t_1, t']$. Since $x_w(t_1)$ belongs to K and (3.16) holds true for $t = t_1$ we obtain $x_w(t_1) \in D$. We claim that $t_1 = t' - \varepsilon'$. Suppose to the contrary that $t_1 > t' - \varepsilon'$. By (3.4) and the Invariance Theorem, there exists $t_2 < t_1$ such that $x_w(s) \in D$ for $s \in [t_2, t_1]$ contradicting the definition of t_1.

Next consider the case $t \in [t' - \Delta\tau, t' - \varepsilon']$. We have

$$\begin{aligned} |x_w(t) &- (x_u(t + \varepsilon') - \varepsilon' f(t', x', u'))| \\ &\le |x' - x_u(t')| + \left| \int_{t' - \varepsilon'}^{t'} (f(t', x', u') - f(s, x_w(s), u'))\, ds \right| \\ &\quad + \int_{t + \varepsilon'}^{t'} |f(s, x_u(s), u(s)) - f(s - \varepsilon', x_w(s - \varepsilon'), u(s))|\, ds \\ &\le |x' - x_u(t')| + \frac{l(C+1)}{2} (\varepsilon')^2 + \varepsilon' (e^{l\Delta\tau} - 1) \left(C + l + 1 + \frac{|x_u(t') - x'|}{\varepsilon'} \right). \end{aligned}$$

Thus

$$\begin{aligned} &|x_w(t) - (x_u(t + \varepsilon') - \varepsilon' f(t', x', u'))| \\ &\le \left(\frac{|x' - x_u(t')|}{\varepsilon'} + \varepsilon' \frac{l(C+1)}{2} + (e^{l\Delta\tau} - 1) \left(C + l + 1 + \frac{|x_u(t') - x'|}{\varepsilon'} \right) \right) \varepsilon'. \end{aligned}$$

By (3.10)–(3.12), we obtain

$$\frac{|x' - x_u(t')|}{\varepsilon'} + \varepsilon' \frac{l(C+1)}{2} + (e^{l\Delta\tau} - 1) \left(C + l + 1 + \frac{|x_u(t') - x'|}{\varepsilon'} \right) \le \frac{3R}{4}.$$

Moreover,

$$\varepsilon' \le 4 \frac{|x_u(t') - x'|}{R} < 4 \frac{d}{R} < R$$

and

$$|x_u(t + \varepsilon') - x'| \le |x_u(t + \varepsilon') - x_u(t')| + |x_u(t') - x'| < C\Delta\tau + d < R.$$

Since $x_u(t + \varepsilon') \in K$, by (3.15) as long as $x_w(t) \in K$ we also have $x_w(t) \in D$. Using the same arguments as in the first case we conclude that $x_w(t) \in D$ for $t \in [t' - \Delta\tau, t' - \varepsilon']$.

We set $t'' := \inf\{t \in [t_0, t') : \text{for all } s \in (t,t'),\ x_w(s) \in D\}$. We have just proved that $t'' \le t' - \Delta\tau$. Obviously $x_w(t'') \in K$. We claim that if $t'' > t_0$ then $x_w(t'') \in M$. Assume to the contrary that $x_w(t'') \in D$. By assumption (3.4) and invariance theorem there exists $\delta > 0$ such that $x_w(s) \in D$ for $s \in (t' - \delta, t')$, which contradicts the definition of t''. So, we can choose $\bar t \in [t_0, t')$ in such a way that:

- $t' - \bar t < \Delta\tau$;
- $x(s) \in D$ for $s \in [\bar t, t']$;
- $\bar t = t_0$ or $x_w(\bar t) \in M + B(0,R)$.

To repeat the same arguments in the next step, we have to know that (3.14) holds true for $t' = \bar t$, $x' = x_w(\bar t)$. It remains to prove that

$$|x_u(\bar t) - x_w(\bar t)| < d. \tag{3.17}$$

To prove this last inequality let us fix k and set $b := t_k$, $y := y_k$. Define successive subintervals $[\tau_{i+1}, \tau_i]$ and a control $v(\,\cdot\,)$ on $[\tau_{i+1}, \tau_i]$ inductively:

(1) We set $\varepsilon_0 = C_2|x_u(b) - y|$. Assume that $\tau_1 < \tau_0 - \varepsilon_0$ is chosen arbitrarily and define $v(\,\cdot\,)$ on $[\tau_1, \tau_0]$ by

$$v(t) = \begin{cases} \text{arbitrary} & \text{for } t \in\,]\tau_0 - \varepsilon_0, \tau_0], \\ u(t + \varepsilon_0) & \text{for } t \in [\tau_1, \tau_0 - \varepsilon_0]. \end{cases}$$

Consider the trajectory $x_v\colon [\tau_1, \tau_0] \mapsto \mathbb{R}^n$ which corresponds to the control v and satisfies $x_v(\tau_0) = y$.

(2) If v, x_v are defined on the interval $[\tau_i, \tau_0]$ we choose $\tau_{i+1} < \tau_i - \varepsilon_i$, where $\varepsilon_i := C_2|x_u(\tau_i) - x_v(\tau_i)|$ and extend v onto $[\tau_{i+1}, \tau_i)$ by

$$v(t) = \begin{cases} \text{arbitrary} & \text{for } t \in [\tau_i - \varepsilon_i, \tau_i), \\ u(t + \varepsilon_i) & \text{for } t \in [\tau_{i+1}, \tau_i - \varepsilon_i). \end{cases}$$

By the Gronwall Lemma we obtain an estimation

$$\begin{aligned} &|x_u(\tau_{i+1}) - x_v(\tau_{i+1}| \\ &\qquad \le (|x_u(\tau_i) - x_v(\tau_i)| + 2\varepsilon_i C + (l + C)\varepsilon_i(\tau_i - \varepsilon_i - \tau_{i+1})e^{l(\tau_i - \varepsilon_i - \tau_{i+1})}. \end{aligned}$$

By the definition of ε_i we get

$$\begin{aligned} &|x_u(\tau_{i+1}) - x_v(\tau_{i+1})| \\ &\qquad \le |x_u(\tau_i) - x_v(\tau_i)|(1 + C_2(2C + (l + C)(\tau_i - \tau_{i+1})))e^{l(\tau_i - \tau_{i+1})}. \end{aligned}$$

We set $q := 1 + C_2(2C + (l + C)(\tau_i - \tau_0))$. We can inductively prove that

$$|x_u(\tau_i) - x_v(\tau_i)| \le q^i e^{l(\tau_0 - \tau_i)}|x_u(b) - y|.$$

If $t \in [\tau_i - \varepsilon_i, \tau_i]$, then

$$\begin{aligned} |x_u(t) - x_v(t)| &\le |x_u(\tau_i) - x_v(\tau_i)| + 2(\tau_i - t)C \\ &\le q^i e^{l(\tau_0 - \tau_i)}|x_u(b) - y|(1 + 2C_2C) \le q^{i+1}e^{l(\tau_0 - t)}|x_u(b) - y|. \end{aligned}$$

If $t \in [\tau_{i+1}, \tau_i - \varepsilon_i)$, then

$$\begin{aligned}|x_u(t) - x_v(t)| &\le (|x_u(\tau_i) - x_v(\tau_i)| + \varepsilon_i(2C + (l + C)(\tau_i - t)))e^{l(\tau_i - t)}\\ &\le q^{i+1}|x_u(b) - y|e^{l(\tau_0 - t)}.\end{aligned}$$

Since the number of subintervals n is bounded by N and the length of the interval $[\tau_n, \tau_0]$ is bounded by T, then we have for $t \in [\tau_n, \tau_0]$,

$$|x_u(t) - x_v(t)| \le C_1 e^{l(\tau_0 - t)}|x_u(\tau_0) - y|$$

which implies (3.17) and finish the construction of controls v_k satisfying (3.13).

Let $w_k \colon [t_0, t_k] \mapsto \mathbb{R}$ be a solution of:

$$\begin{cases} w_k'(t) = -L(t, x_k(t), v_k(t)),\\ w_k(t_k) = W(t_k, y_k).\end{cases}$$

By (3.13) and the assumption that the set $\{(t, x, w) : t \in (0, T),\ x \in D,\ w \ge W(t, x)\}$ is the backward invariance domain of F we obtain

$$w_k(t_0) \ge W(t_0, x_k(t_0)).$$

Obviously

$$w_k(t_0) = \int_{t_0}^{t_k} L(t, x_k(t), v_k(t))\, dt + W(t_k, y_k).$$

On the other hand

$$\begin{aligned}&\left| \int_{t_0}^{t_k} (L(t, x_k(t), v_k(t)) - L(t, x_u(t), u(t)))\, dt \right|\\ &\quad\le \sum_{i=0}^{n_k - 1} \int_{\tau_{i+1}^k}^{\tau_i^k - \varepsilon_i^k} |L(t, x_k(t), u(t + \varepsilon_i^k)) - L(t + \varepsilon_i^k, x_u(t + \varepsilon_i^k), u(t + \varepsilon_i^k))|\, dt\\ &\quad\quad + 2C \sum_{i=0}^{n_k - 1} \varepsilon_i^k\\ &\quad\le \sum_{i=0}^{n_k - 1} \int_{\tau_{i+1}^k}^{\tau_i^k - \varepsilon_i^k} l(\varepsilon_i^k + C_1 e^{l(t_k - t)}|y_k - x_u(t_k)|)\, dt + 2C \sum_{i=0}^{n_k - 1} \varepsilon_i^k.\end{aligned}$$

Since $\varepsilon_i^k \le C_2 C_1 e^{lT}|y_k - x_u(t_k)|$ and $n_k < N$,

$$\lim_{k \to \infty} \int_{t_0}^{t_k} L(t, x_k(t), v_k(t))\, dt = \int_{t_0}^{T} L(t, x_u(t), u(t))\, dt.$$

Thus $\lim_{k \to \infty} w_k(t_0) = w_u(t_0)$. Combining it with the lower semicontinuity of W we obtain

$$W(t_0, x_u(t_0)) \le \liminf_{k \to \infty} W(t_0, x_k(t_0)) \le \lim_{k \to \infty} w_k(t_0) = w_u(t_0). \qquad \square$$

Define $H\colon [0,T]\times K\times\mathbb{R}^n\mapsto\mathbb{R}$ by

$$H(t,x,p)=\sup_{u\in U}(\langle f(t,x,u),p\rangle - L(t,x,u)).$$

Let us summarize the obtained results in

Theorem 3.1.7. *Assume that* (3.1)–(3.6) *hold true. Then for a function* $W\colon[0,T]\times K\mapsto\mathbb{R}\cup\{+\infty\}$ *the following conditions are equivalent:*

(a) $W=V$;

(b) *W is a lower semicontinuous function such that* $W(T,\,\cdot\,)=g(\,\cdot\,)$,

$$\liminf_{t_n\to T^-,\, y_n\to y,\, y_n\in D} W(t_n,y_n)=g(y),\quad \textit{for all } y\in K$$

and

$$\{(t,x,v): t\in[0,T),\ x\in K,\ ,v\geq W(t,x)\}\ \textit{is a viability domain of } F, \tag{3.18}$$

$$\{(t,x,v): t\in(0,T],\ x\in D,\ v\geq W(t,x)\}\ \textit{is a backward invariance domain of } F; \tag{3.19}$$

(c) *W is a lower semicontinuous function such that* $W(T,\,\cdot\,)=g(\,\cdot\,)$ *and for all* $y\in K$

$$\begin{aligned}\liminf_{t_n\to T^-,\, y_n\to y,\, y_n\in D} W(t_n,y_n) &= g(y),\\ \liminf_{t_n\to 0^+,\, y_n\to y,\, y_n\in K} W(t_n,y_n) &= W(0,y),\end{aligned}$$

and

$$\forall (t,x)\in(0,T)\times M,\ \forall(p_t,p_x)\in\partial_-W(t,x),\quad -p_t+H(t,x,-p_x)\geq 0; \tag{3.20}$$

$$\forall (t,x)\in(0,T)\times D,\ \forall(p_t,p_x)\in\partial_-W(t,x),\quad -p_t+H(t,x,-p_x)= 0. \tag{3.21}$$

Proof. By Propositions 3.1.1–3.1.3 and Corollary 3.1.5, we obtain the implication (a) $\Rightarrow$ (b). Assume (b). By Propositions 3.1.4 and 3.1.6, we get $W=V$.

We prove next that (a) $\Rightarrow$ (c). From Proposition 3.1.1 and Corollary 3.1.5 we obtain the desired regularity of W. The remaining properties of W follow from (b) (which holds true by the previous part of the proof). Fix $t\in(0,T)$ and $x\in K$. If $(p_t,p_x)\in\partial_-W(t,x)$ then $(p_t,p_x,-1)\in[T_{\mathcal{E}pi(W)}(t,x,W(t,x))]^-$. Viability Theorem 1.4.1 (a) $\Rightarrow$ (b) and (3.18) yield

$$-p_t+H(t,x,-p_x)\geq 0,\quad (t,x)\in(0,T)\times K, (p_t,p_x)\in\partial_-W(t,x). \tag{3.22}$$

From Invariance Theorem 1.4.2 and (3.19) we obtain

$$-p_t+H(t,x,-p_x)\leq 0,\quad (t,x)\in(0,T)\times D, (p_t,p_x)\in\partial_-W(t,x). \tag{3.23}$$

Combining it with (3.22) we get (3.20), (3.21). It remains to prove that (c) $\Rightarrow$ (a).

Step 1. We show that (c) implies

$$K = \{(t,x,v) : t \in (0,T),\ x \in K,\ v \geq W(t,x)\} \tag{3.24}$$
is a viability domain of F,

$$D = \{(t,x,v) : t \in (0,T),\ x \in D,\ v \geq W(t,x)\} \tag{3.25}$$
is a backward invariance domain of F.

First observe that (3.22), (3.23) hold true. To prove (3.24) it is enough to verify the condition (b) in Viability Theorem 1.4.1, i.e.

$$\sup_{u\in U}\langle (1, f(t,x,u), -L(t,x,u)), (-p_t, -p_x, -p_v)\rangle \geq 0 \tag{3.26}$$

for $(p_t, p_x, p_v) \in [T_{\mathcal{E}pi(W)}(t,x,W(t,x))]^-$ and $(t,x) \in (0,T) \times K$. If $p_v < 0$, then $(p_t/{-p_v}, p_x/{-p_v}) \in \partial_- W(t,x)$. From (3.22) we obtain

$$\frac{-p_t}{-p_v} + H\left(t, x, \frac{-p_x}{-p_v}\right) \geq 0$$

which implies (3.26).

If $p_v = 0$, then by Lemma 1.3.4, there are $t_n \to t$, $x_n \to x$, $v_n \to v$, $p_{t,n} \to p_t$, $p_{x,n} \to p_x$, $p_{v,n} \to 0$, $p_{v,n} < 0$ such that

$$(p_{t,n}, p_{x,n}, p_{v,n}) \in [T_{\mathcal{E}pi(W)}(t_n, x_n, v_n)]^-.$$

By (3.22), we have

$$\sup_{u\in U}\langle (1, f(t_n,x_n,u), -L(t_n,x_n,u)), (-p_{t,n}, -p_{x,n}, -p_{v,n})\rangle \geq 0.$$

This and assumptions (3.1), (3.3) imply (3.26).

To obtain (3.25) we have to verify the statement (b) in Invariance Theorem 1.4.2, i.e.

$$\sup_{u\in U}\langle (1, f(t,x,u), -L(t,x,u)), (-p_t, -p_x, -p_v)\rangle \leq 0 \tag{3.27}$$

for $(p_t, p_x, p_v) \in [T_{\mathcal{E}pi(W)}(t,x,W(t,x))]^-$, $(t,x) \in (0,T) \times D$.

If $p_v < 0$, then $(p_t/{-p_v}, p_x/{-p_v}) \in \partial_- W(t,x)$. By (3.23), we obtain

$$\frac{-p_t}{-p_v} + H\left(t, x, \frac{-p_x}{-p_v}\right) \leq 0$$

which yields (3.27).

If $p_v = 0$, then by Lemma 1.3.4, there are $t_n \to t$, $x_n \to x$, $v_n \to v$, $p_{t,n} \to p_t$, $p_{x,n} \to p_x$, $p_{v,n} \to 0$, $p_{v,n} < 0$ such that

$$(p_{t,n}, p_{x,n}, p_{v,n}) \in [T_{Epi(W)}(t_n, x_n, v_n)]^-.$$

By (3.23), we have

$$\sup_{u\in U}\langle(1, f(t_n, x_n, u), -L(t_n, x_n, u)), (-p_{t,n}, -p_{x,n}, -p_{v,n})\rangle \le 0.$$

We fix u and pass to the limit with n. Using (3.1), (3.3) we obtain (3.27).

Step 2. Applying Propositions 3.1.4, 3.1.6 with the time interval $[0, T]$ replaced by $[t, T]$ with $t > 0$ we get

$$W(t, x) = V(t, x) \quad \text{for } (t, x) \in (0, T] \times K.$$

For $t = 0$ and $y \in K$ we have

$$W(0, y) = \liminf_{t\to 0^+,\, x\to y,\, x\in K} W(t, x) = \liminf_{t\to 0^+,\, x\to y,\, x\in K} V(t, x) = V(0, y)$$

which completes the proof. □

Example 1. $T = 1$, $\Omega = \{(x, y) : x < 0 \text{ or } y < 0\}$, $U = [0, 1]$, $f(t, x, y, u) = (u, 1 - u)$, $L(t, x, y, u) = u$ and $g = 0$. Setting $D := \Omega$, $K := \overline{\Omega}$, $M = \partial\Omega$ we obtain that (3.4)–(3.5) hold true. Indeed, $P^M_K(x, 0) = \{(v_1, v_2) : v_2 \le 0\}$ for $x > 0$, $P^M_K(0, y) = \{(v_1, v_2) : v_1 \le 0\}$ for $y > 0$ and $P^M_K(0, 0) = \{(v_1, v_2) : v_1 \le 0 \text{ or } v_2 \le 0\}$. The value function $V: [0, 1] \times K \mapsto R$ is given by

$$V(t, x, y) = \begin{cases} 1 - t + y & \text{if } x > 0 \text{ and } t - 1 < y \le 0, \\ 0 & \text{elsewhere in } [0, 1] \times \overline{\Omega}. \end{cases}$$

The function V is the unique discontinuous solution of the Hamilton–Jacobi equation

$$-V_t + H(t, (x, y), -(V_x, V_y)) = 0$$

where

$$H(t, (x, y), (p_1, p_2)) = \begin{cases} p_2 & \text{if } p_2 - p_1 + 1 \ge 0, \\ p_1 - 1 & \text{if } p_2 - p_1 + 1 < 0, \end{cases}$$

satisfying the terminal condition $V(1, x) = 0$.

Example 2. $T = 1$, $U = \{(u_1, u_2) : u_1 \ge 1,\ u_1^2 + u_2^2 \le 1\}$, $\Omega = \{(x_1, x_2) : x_1 \le x_2^2\}$, $f(t, (x_1, x_2), (u_1, u_2)) = (u_1, u_2)$, $L(t, (x_1, x_2), (u_1, u_2)) = u_2 + 1$, $g(x_1, x_2) = 0$. The control system satisfies (3.1)–(3.6). Moreover, the set of state constraints $\overline{\Omega}$ is a smooth submanifold with boundary. The discontinuous value function $V: [0, 1] \times \overline{\Omega} \mapsto \mathbb{R}$ is given by

$$V(t, x) = \begin{cases} 0 & \text{if } x_1 \le 0, \\ 0 & \text{if } x_1 > 0 \text{ and } x_2 \le -\sqrt{x_1}, \\ 0 & \text{if } x_1 > 0 \text{ and } x_2 > \sqrt{x_1} + 1 - t, \\ 1 - t - x_2 + \sqrt{x_1} & \text{if } x_1 > 0 \text{ and } -\sqrt{x_1} \le x_2 \le \sqrt{x_1} + 1 - t. \end{cases}$$

3.2. Infinite horizon control problem

We assume that $D \subset \mathbb{R}^n$ is locally compact, $K := \mathrm{cl}(D)$, $M = K \setminus D$, U is a complete separable metric space and

(3.28) $f\colon \mathbb{R}^n \times U \to \mathbb{R}^n,\ L\colon \mathbb{R}^n \times U \to [0,\infty)$ are bounded continuous maps;

(3.29) $f(x,u),\ L(x,u)$ are x-locally Lipschitz continuous uniformly in u;

in the following sense

$$\forall r > 0,\ \exists l_r > 0,\ \forall u \in U,\ f(\,\cdot\,,u),\quad L(\,\cdot\,,u) \text{ are } l_r\text{-Lipschitz on } B(0,r),$$

(3.30) $\{(f(x,u), L(x,u)+r) \in \mathbb{R}^n \times \mathbb{R} : u \in U,\ r \geq 0\}$
is closed and convex for every $x \in K$,

$$\text{for every } x \in K \text{ there is } u \in U \text{ such that } f(x,u) \in T_K(x), \tag{3.31}$$

$$-f(x,U) \subset T_D(x) \quad \text{for every } x \in D, \tag{3.32}$$

$$\forall x \in M,\ \exists u \in U,\quad f(x,u) \notin P_K^M(x). \tag{3.33}$$

Remark. The assumption in (3.28) that L is nonnegative may be replaced by the hypothesis

$$\exists \gamma > 0,\ \forall x \in K,\ \forall u \in U,\quad L(x,u) \geq -\gamma.$$

We denote by $x(\,\cdot\,;\ t_0, x_0, u)$ the unique solution of the Cauchy problem

$$\begin{cases} x'(t) = f(x(t), u(t)), \\ x(t_0) = x_0, \end{cases}$$

defined on the interval (a,b) containing t_0, where (a,b) is the domain of a measurable control $u\colon (a,b) \to U$. If $t_0 = 0$ then we write simply $x(\,\cdot\,; x_0, u)$. Let us denote by $A(x_0) = \{u\colon [0,\infty) \to U : x(t; x_0, u) \in K \text{ for every } t \in [0,\infty)\}$ the set of admissible controls for the point $x_0 \in K$. The value function $V\colon K \to \mathbb{R}$ for the discounted cost in the infinite horizon problem is given by

$$V(x_0) = \inf_{u \in A(x_0)} \int_0^\infty e^{-s} L(x(s; x_0, u), u(s))\, ds. \tag{3.34}$$

Let $W\colon K \to \mathbb{R}$ be a lower semicontinuous function. We extend W to $\widetilde{W}$ defined on $\mathbb{R}^n$ by setting $\widetilde{W}(x) = +\infty$ for $x \notin K$. The subdifferential of $\widetilde{W}$ at $x_0 \in K$ is defined by

$$\partial_- \widetilde{W}(x_0) = \left\{ p \in \mathbb{R}^n : \liminf_{x \to x_0} \frac{\widetilde{W}(x) - \widetilde{W}(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \geq 0 \right\}.$$

The subdifferential of W at $x_0 \in K$ relative to K is given by

$$\partial_- W(x_0) = \left\{ p \in \mathbb{R}^n : \liminf_{x \to x_0,\, x \in K} \frac{W(x) - W(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \geq 0 \right\}.$$

Obviously $\partial\widetilde{W}(x_0) = \partial W(x_0)$. Moreover, $p \in \partial_-\widetilde{W}(x_0)$ if and only if $(p,-1) \in [T_{\mathcal{E}pi(W)}(x_0, W(x_0))]^-$.

We define the Hamiltonian $H: K \times \mathbb{R}^n \to \mathbb{R}$ by

$$H(x,p) = \sup_{u\in U}\langle f(x,u), p\rangle - L(x,u).$$

The main result of this section is the following one.

Theorem 3.2.1. *Assume that* (3.28)–(3.33) *hold true. Let D be a locally compact subset of $\mathbb{R}^n$, $K := \mathrm{cl}(D)$ and $W: K \to \mathbb{R}_+$ be a lower semicontinuous bounded function. Then the following conditions are equivalent:*

(a) *$W = V$;*

(b) *For F defined by* (3.40)

$$\mathcal{E}pi(W) \text{ is a viability domain of } F, \tag{3.35}$$

$$\{(x,v) : v \geq W(x) \text{ and } x \in D\} \text{ is a backward invariance domain of } F, \tag{3.36}$$

and

$$\text{for every } x \in K \setminus D \text{ and every } p \in \partial_- W(x) \quad W(x) + \sup_{f(x,u)\notin P_K^M(x)} (\langle -p, f(x,u)\rangle + L(x,u)) \leq 0. \tag{3.37}$$

(c) *W solves the Hamilton–Jacobi equation*

$$W(x) + H(x, DW(x)) = 0$$

in the following sense

$$\forall x \in D,\ \forall n \in \partial_- W(x), \quad W(x) + H(x,-n) = 0, \tag{3.38}$$

$$\forall x \in M,\ \forall n \in \partial_- W(x), \quad W(x) + H(x,-n) \geq 0, \tag{3.39}$$

and (3.37) *holds true.*

Let C be an upper bound of L. In the sequel we shall use the set valued map $F_{f,l,C}: \mathbb{R}^n \times R \mapsto \mathbb{R}^n \times \mathbb{R}$ given by

$$F_{f,l,C}(x,v) = \{(f(x,u), v - L(x,u) - r) : u \in U \text{ and } r \in [0, C - L(x,u)]\}. \tag{3.40}$$

If we assume that (3.28)–(3.31) hold true then the map $F_{f,l,C}$ is Lipschitz continuous, bounded and it has convex compact values. To simplify the notation we shall skip subscripts f, l, C, i.e.

$$F(x,v) = F_{f,l,C}(x,v).$$

Lemma 3.2.2. *If $l\colon [0,\infty) \to \mathbb{R}$ is a bounded measurable function, then a function $w(t) = \int_0^\infty e^{-s} l(s+t)\,ds$ is the unique bounded solution of*

$$w'(t) = w(t) - l(t).$$

Proof. We observe that

$$w(t+h) = e^h \int_0^\infty e^{-s-h} l(s+t+h)\,ds = e^h \int_h^\infty e^{-s} l(s+t)\,ds.$$

Thus

$$\frac{w(t+h)-w(t)}{h} = e^h \Big(-\frac{1}{h} \int_0^h e^{-s} l(s+t)\,ds + \frac{e^h - 1}{h} \int_0^\infty e^{-s} l(s+t)\,ds \Big).$$

Passing to the limit we obtain

$$w'(t) = -l(t) + w(t)$$

for almost all t. If a function v is a solution of $v'(t) = v(t) - l(t)$, then there is a constant C such that $v(t) = w(t) + Ce^t$. If v, w are bounded then $C = 0$. $\square$

Proposition 3.2.3. *Assume that* (3.28)–(3.31) *hold true. Then the value function $V\colon K \to \mathbb{R}$ is lower semicontinuous.*

Proof. Fix $x_0 \in K$ and a sequence $x_n \in K$ convergent to x_0. Choose $u_n \in A(x_n)$ such that

$$\liminf_{n\to\infty} V(x_n) = \liminf_{n\to\infty} \int_0^\infty e^{-s} L(z_n(s), u_n(s))\,ds = v_0$$

where $z_n(\cdot) = x(\cdot, x_n, u_n)$. Define $w_n(t) = \int_0^\infty e^{-s} L(z_n(s+t), u_n(s+t))\,ds$. By Lemma 3.2.2, we have $w_n(t)' = w_n(t) - L(z_n(t), u_n(t))$ for a.a. $t \geq 0$. Thus $(z_n'(t), w_n'(t)) \in F(z_n(t), w_n(t))$. Passing to the subsequence, if necessary, (denoted again by (z_n, w_n)) we obtain

$$\lim_{n\to\infty} z_n(0) = x_0, \quad \lim_{n\to\infty} w_n(0) = v_0.$$

Fix $\varepsilon > 0$ and $T > 0$. Let $S_F(\overline{x}, \overline{w})$ denote the set of solutions $(x(\cdot), w(\cdot))\colon [0,T] \to K \times \mathbb{R}$ to the differential inclusion $(x', w') \in F(x, w)$ satisfying the initial condition $(x(0), w(0)) = (\overline{x}, \overline{w})$. The set-valued map S_F is upper semicontinuous and has nonempty compact values. So we can choose a subsequence (denoted again by (z_n, w_n)) such that (z_n, w_n) converges uniformly on $[0,T]$ to a solution $(z_T(\cdot), w_T(\cdot)) \in S_F(x_0, v_0)$. We apply the above procedure for an increasing sequence T_n converging to ∞. For T_1 we choose a subsequence z_{n_k}, w_{n_k} uniformly converging on $[0,T_1]$ to a solution $(z_{T_1}(\cdot), w_{T_1}(\cdot))$. For T_2 we choose a subsequence $(z_{n_{k_l}}(\cdot), w_{n_{k_l}}(\cdot))$ uniformly converging on $[0,T_2]$ to a solution (z_{T_2}, w_{T_2}). For $t \in [0,T_1]$ we have $(z_{T_1}(t), w_{T_1}(t)) = (z_{T_2}(t), w_{T_2}(t))$. Iterating the procedure we obtain a solution $(z(\cdot), w(\cdot))$ to the differential inclusion $(z', w') \in F(z, w)$ such that for every T_n there is a subsequence of the sequence (z_n, w_n) uniformly converging on $[0,T_n]$ to $(z(\cdot),w(\cdot))$. Every function w_n is bounded by C. Thus w is a bounded function. By the measurable selection theorem there exist measurable functions $u\colon[0,\infty)\to U$, $r\colon[0,\infty)\to\mathbb{R}$ such that

$$z'(t)=f(z(t),u(t));\quad w'(t)=w(t)-L(z(t),u(t))-r(t),$$

and $r(t)\in[0,\, C-L(z(t),u(t))]$. By Lemma 3.2.2, we have

$$w(t)=\int_0^\infty e^{-s}(L(z(s+t),u(s+t))+r(t+s))\,ds.$$

Thus

$$\begin{aligned}v_0=w(0)\;&=\int_0^\infty e^{-s}L(z(s),u(s))\,ds+\int_0^\infty e^{-s}r(s)\,ds\\&\geq V(z(0))+\int_0^\infty e^{-s}r(s)\,ds\end{aligned}$$

and therefore $v_0\geq V(z(0))$. □

Proposition 3.2.4. *Assume that* (3.28)–(3.31) *hold true. Then for every $x_0\in K$ there is $\overline{u}\in A(x_0)$ such that $V(x_0)=\int_0^\infty e^{-s}L(\overline{x}(s),\overline{u}(s))\,ds$, where $\overline{x}(s)=x(s;x_0,\overline{u})$, i.e. there exists an optimal control $\overline{u}$.*

Proof. Fix $x_0\in K$ and choose $u_n\in A(x_0)$ such that

$$\lim_{n\to\infty}\int_0^\infty e^{-s}L(x_n(s),u_n(s))\,ds=V(x_0),$$

where $x_n(s)=x(s;x_0,u_n)$. Setting $v_n(t)=\int_0^\infty e^{-s}L(x_n(s+t),u_n(s+t))\,ds$ we obtain a solution $(x_n(\cdot),v_n(\cdot))$ to a differential inclusion

$$(x'(s),v'(s))\in F(x(s),v(s)). \tag{3.41}$$

Let (T_n) be an increasing sequence converging to $+\infty$. There is a subsequence (x_{n_k},v_{n_k}) convergent on the interval $[0,T_1]$ to a solution $(\overline{x}_1,\overline{v}_1)$ to (3.41). Furthermore there is a subsequence $(x_{n_{k_l}},v_{n_{k_l}})$ convergent on the interval $[0,T_2]$ to a solution $(\overline{x}_2,\overline{v}_2)$ of (3.41) and $\overline{x}_1(t)=\overline{x}_2(t)$, $\overline{v}_1(t)=\overline{v}_2(t)$, for $t\in[0,T_1]$. Iterating the procedure we obtain a solution $(\overline{x},\overline{v})$ to (3.41) such that for every interval $[0,T]$ there is a subsequence of (x_n,v_n) convergent uniformly on $[0,T]$ to $(\overline{x},\overline{v})$. Thus for some measurable $\overline{u}(t)\in U$, $r(t)\in[0,C-L(\overline{x}(t),\overline{u}(t))]$, where $\overline{x}(t)=x(t;x_0,\overline{u})$ we have

$$\overline{v}'(t)=\overline{v}(t)-L(\overline{x}(t),\overline{u}(t))-r(t).$$

By Lemma 3.2.2, since $\overline{v}$ is bounded, $\overline{v}(t)=\int_0^\infty e^{-s}(L(\overline{x}(s+t),\overline{u}(s+t))+r(s+t))\,ds$ and therefore

$$\lim_{n\to\infty}v_n(0)=\overline{v}(0)\leq\int_0^\infty e^{-s}L(\overline{x}(s),\overline{u}(s))\,ds.$$

Hence $\overline{u}(\cdot)$ is an optimal control for the initial condition x_0. □

Proposition 3.2.5. *Assume that* (3.28)–(3.31) *hold true. If* $u \in A(x_0)$ *is an admissible control then the function* $t \to V(x_u(t))$ *is left continuous.*

Proof. Let $t < t_0$ and $y_t = x_u(t)$. By Proposition 3.2.4, there is an optimal control $\overline{u} \in A(y_t)$ such that $V(y_t) = \int_0^\infty e^{-s}L(x_{\overline{u}}(s), \overline{u}(s))\,ds$, where $x_{\overline{u}}(s) = x(s; y_t, \overline{u})$. We define

$$u_t(s) = \begin{cases} u(t+s) & \text{for } s \in [0, t_0 - t), \\ \overline{u}(s - (t_0 - t)) & \text{for } s > t_0 - t. \end{cases}$$

Let $x_t(s) = x(s; y_t, u_t)$. We have

$$V(y_t) \leq \int_0^\infty e^{-s}L(x_t(s), u_t(s))\,ds = \int_0^{t_0 - t} e^{-s}L(x_t(s), u_t(s))\,ds + e^{t - t_0}V(y_0).$$

Thus $\limsup_{t \to t_0 -} V(x_u(t)) \leq V(y_0)$. Combining it with the lower semicontinuity of V we obtain the desired statement. □

Proposition 3.2.6. *Assume that* (3.28)–(3.31) *hold true. Then the epigraph of the value function* $\mathcal{E}pi(V) \subset K \times \mathbb{R}$ *is a viability domain of* $F : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n \times \mathbb{R}$ *defined by* (3.40)

Proof. Let $u(\,\cdot\,) \in A(x_0)$ be an optimal control, i.e. $\int_0^\infty e^{-s}L(x_u(s), u(s))\,ds = V(x_0)$. Fix $t > 0$. Setting $u_1(s) = u(s + t)$ for $s \geq 0$ we obtain $u_1 \in A(x_u(t))$ and

$$\begin{aligned} &\int_0^\infty e^{-s}L(x_u(s), u(s))\,ds \\ &\qquad = \int_0^t e^{-s}L(x_u(s), u(s))ds + e^{-t}\int_0^\infty e^{-s}L(x_{u_1}(s), u_1(s))\,ds, \end{aligned}$$

where x_{u_1} is the output corresponding to the control u_1 starting from $x_u(t)$ at time 0. Hence,

$$\begin{aligned} V(x_u(t)) &\leq \int_0^\infty e^{-s}L(x_{u_1}(s), u_1(s))\,ds \\ &= e^t V(x_0) - e^t \int_0^t e^{-s}L(x_u(s), u(s))\,ds. \end{aligned} \tag{3.42}$$

Define $v(t) = e^t v_0 - e^t \int_0^t e^{-s}L(x_u(s), u(s))\,ds$, where $v_0 \geq V(x_0)$. It is easy to see that the function $(x_u(t), v(t))$ is a solution of the differential inclusion $(x', v') \in F(x, v)$ such that $(x(0), v(0)) = (x_0, v_0)$. By (3.42), we have

$$V(x_u(t)) \leq v(t).$$

Combining it with Viability Theorem 1.4.1, we obtain the desired conclusion. □

Proposition 3.2.7. *Assume that* (3.28)–(3.32) *hold true. Then the set* $\{(x, v) : x \in D$ *and* $v \geq V(x)\}$ *is a backward invariance domain of* F *defined by* (3.40).

Proof. Let $x_0 \in D$ and $v_0 \geq V(x_0)$. By (3.32) and Invariance Theorem 1.4.2, there exists $\varepsilon > 0$ such that for every measurable $u\colon [-\varepsilon, 0] \to U$ the solution $x_u\colon [-\varepsilon, 0] \to \mathbb{R}^n$ of the Cauchy problem

$$\begin{cases} x'(s) = f(x_u(s), u(s)), \\ x(0) = x_0, \end{cases}$$

satisfies $x_u(s) \in D$ for $s \in [-\varepsilon, 0]$. Let

$$v(s) = e^s v(0) - e^s \int_0^s e^{-\tau}(L(x(\tau), u(\tau)) + r(\tau))\, ds,$$

where $r(\,\cdot\,)$ is a measurable function such that $r(\tau) \in [0, C - L(x(\tau), u(\tau))]$. Fix $t \in [-\varepsilon, 0]$, $\overline{u} \in A(x_0)$ and denote $x_1 = x_u(t)$. Define

$$u_1(s) = \begin{cases} u(s+t) & \text{for } s \in [0, -t], \\ \overline{u}(s+t) & \text{for } s > -t. \end{cases}$$

We have $u_1 \in A(x_1)$ and

$$\begin{aligned} V(x_1) &\leq \int_0^\infty e^{-s} L(x_1(s), u_1(s))\, ds \\ &= \int_0^{-t} e^{-s} L(x_1(s), u_1(s))\, ds + e^t \int_0^\infty e^{-s} L(\overline{x}(s), \overline{u}(s))\, ds. \end{aligned}$$

Since $\overline{u}$ was arbitrary then

$$V(x_1) \leq \int_0^{-t} e^{-s} L(x_1(s), u_1(s)) ds + e^t V(x_0) \leq v(t).$$

By Invariance Theorem 1.4.2, we obtain the conclusion. □

Proposition 3.2.8. *Suppose that* (3.28)–(3.31) *hold true and the epigraph* $\mathcal{E}pi(W)$ *of a lower semicontinuous nonnegative function* $W\colon K \to \mathbb{R}$ *is a viability domain of* F *given by* (3.40). *Then*

$$W(x_0) \geq V(x_0) \quad \textit{for every } x_0 \in K.$$

Proof. By Viability Theorem 1.4.1, there is a solution $(x(\,\cdot\,), v(\,\cdot\,))$ to (3.41) such that $x(0) = x_0$, $v(0) = W(x_0)$ and $(x(t), v(t)) \in \mathcal{E}pi(W)$ for $t > 0$. Hence, $v(t) \geq W(x(t))$. There are measurable maps $u\colon [0, \infty) \to U$ and $r\colon [0, \infty) \to [0, C]$ such that

$$\begin{cases} x'(t) = f(x(t), u(t)), \\ v'(t) = v(t) - L(x(t), u(t)) - r(t). \end{cases}$$

Since the function v is nonnegative there is $D \geq 0$ such that $v(t) = De^t + \int_0^\infty e^{-s}L(x(s+t), u(s+t))\,ds$. Hence, $W(x_0) = D + \int_0^\infty e^{-s}L(x(s), u(s))\,ds \geq V(x_0)$, which completes the proof. □

Proposition 3.2.9. *Assume that* (3.28)–(3.33) *hold true. Then for every $y \in K$ there is a sequence $(y_n) \subset D$ convergent to y such that* $\lim_{n\to\infty} V(y_n) = V(y)$. *Moreover, the value function satisfies* (3.37) *with W replaced by V.*

Proof. By (3.33), there is $\overline{u} \in U$ such that $-f(y, \overline{u}) \notin P_K^M(y)$. By Proposition 1.4.3, there exists $\varepsilon > 0$ such that $\overline{x}(t) \in D$ for $t \in (-\varepsilon, 0)$, where $\overline{x}$ is the solution of the Cauchy problem

$$\begin{cases} x'(t) = f(x(t), \overline{u}), \\ x(0) = y. \end{cases}$$

Setting $y_n = \overline{x}(t_n)$, where $t_n \to 0^-$, we obtain $\lim_{n\to\infty} V(y_n) = V(y_0)$, by Proposition 3.2.5.

To prove the second part of the statement fix $x_0 \in M$ and $u \in U$ such that $f(x_0, u) \notin P_K^M(x_0)$. By Proposition 1.4.3, there exists $\delta > 0$ such that the solution of

$$\begin{cases} x'(t) = f(x(t), u), \\ x(0) = x_0, \end{cases}$$

satisfies $x(t) \in D$ for $t \in (-\delta, 0)$. Hence

$$V(x(t)) \leq \int_0^{|t|} e^{-s}L(x(t+s), u)\,ds + e^t V(x_0).$$

So, for every $p \in \partial_- V(x_0)$

$$V(x_0) + \langle -p, f(x_0, u)\rangle + L(x_0, u) \leq 0.$$

Hence the value function satisfies (3.37). □

Lemma 3.2.10. *If* (3.28)–(3.33) *hold true and $W\colon K \to \mathbb{R}$ is lower semicontinuous and satisfies* (3.37) *then for every $y \in M$*

$$\liminf_{y' \to y,\, y' \in D} W(y') = W(y).$$

Proof. From the lower semicontinuity of W it follows, that if $y \in D$ then the conclusion holds true.

Assume next that $y \notin D$. Then $\liminf_{y'\to y,\, y' \in D} W(y') \geq W(y)$. Let u be such that $f(y, u) \notin P_K^M(x)$. By Proposition 1.2.1, there exists $\varepsilon > 0$ such that for every $x \in K \cap B(y, \varepsilon)$

$$(x + (0, \varepsilon]B(-f(y, u), \varepsilon)) \cap K \subset K \setminus M = D.$$

Hence, taking ε smaller we get for every $x \in K \cap B(y, \varepsilon)$

$$(x + (0, \varepsilon]B(-f(x, u), \varepsilon)) \cap K \subset D.$$

This yields $f(x,u) \notin P_K^M(x)$. Thus $W(x) + \langle -p, f(x,u)\rangle + L(x,u) \le 0$ and we proved that for every $x \in K \cap B(y,\varepsilon)$

$$(-f(x,u), -W(x) - L(x,u)) \in \operatorname{co} T_{\mathcal{E}pi(W)}(x, W(x)).$$

(To apply the separation theorem and Rockafellar's result). Since

$$\liminf_{x\to y,\, z\to W(x),\, z\ge W(y)} \operatorname{co} T_{\mathcal{E}pi(W)}(x,z) \subset T_{\mathcal{E}pi(W)}(y, W(y)).$$

We get

$$(-f(y,u), -W(y) - L(y,u)) \subset T_{\mathcal{E}pi(W)}(y, W(y)).$$

Hence $D_\uparrow W(y)(-f(y,u)) \le W(y) + L(y,u)$. Consider $h_n \to 0^+$, $v_n \to f(y,u)$ such that $W(y - h_n v_n) \le W(y) - \varepsilon_n h_n$, where $\varepsilon_n \to 0^+$. Set $y_n = y - h_n v_n$. Then by (4.43) $y_n \in D$, $\limsup_{n\to\infty} W(y_n) \le W(y)$ and we get $\lim_{n\to\infty} W(y_n) = W(y)$. □

Proposition 3.2.11. *Assume that* (3.28)–(3.33) *hold true. Let* $W\colon K \to \mathbb{R}$ *be a bounded lower semicontinuous function such that*

$$\forall y \in M, \qquad \liminf_{y'\to y,\, y'\in D} W(y') = W(y).$$

If the set $\{(x,v) : x \in D$ *and* $v \ge V(x)\}$ *is a backward invariance domain of* F *(defined by* (3.40)*), then* $W(x_0) \le V(x_0)$ *for every* $x_0 \in K$.

Proof. Fix $x_0 \in K$, $u \in A(x_0)$. Let $x_u(s) = x(s; x_0, u)$.

Step 1. If $x_u(s) \in D$ for $s \in [t_1, t_0]$, then

$$W(x_u(s)) \le e^{s-t_0} W(x_u(t_0)) + \int_s^{t_0} e^{s-\tau} L(x_u(\tau), u(\tau))\, d\tau \tag{3.44}$$

for $s \in [t_1, t_0]$. Let t_2 be the infimum of $t \in [t_1, t_0]$ such that (3.44) holds true for all $s \in [t, t_0]$. Since W is lower semicontinuous we obtain that (3.44) holds true for $s = t_2$. Suppose that $t_1 < t_2$. By Theorem 1.4.2, there is t_3 ($t_1 \le t_3 < t_2$) such that the solution $v(\,\cdot\,)$ of the Cauchy problem

$$\begin{cases} v'(s) = v(s) - L(x_u(s), u(s)), \\ v(t_2) = W(x_u(t_2)), \end{cases}$$

satisfies $v(s) \ge W(x_u(s))$ for $s \in [t_3, t_2]$. Obviously, $v(s) = e^{s-t_2} v(t_2) + \int_s^{t_2} e^{s-\tau} L(x_u(\tau), u(\tau))\, d\tau$. It follows that (3.44) holds true for $s \in [t_3, t_0]$, which contradicts the definition of t_2.

Step 2. We claim that for every $t_0 > 0$ there is $T \in (0, t_0]$ such that (3.44) holds true for every $s \in [t_0 - T, t_0]$. If $x_u(t_0) \in D$ then there is $T > 0$ such that $x_u(s) \in D$ for every $s \in [t_0 - T, t_0]$ and we obtain our claim by Step 1.

Next consider the case $y_0 := x_u(t_0) \in M$. There is $\overline{u} \in U$ such that (3.33) holds true for $w = -f(y_0, \overline{u})$. By Lemma 1.2.1, there exists $R \in (0,1)$ such that $(y + (0, R)B(w, R)) \cap K \subset D$ for every $y \in K \cap B(y_0, R)$. It follows that if $y \in K \cap B(y_0, R)$, $\varepsilon < \mathbb{R}$, $x \in K$ and

$$|x - (y + \varepsilon w)| < \varepsilon R \tag{3.45}$$

then $x \in D$.

We choose a sequence $(y_n) \subset D$ convergent to y such that $\lim_{n\to\infty} W(y_n) = W(y)$. Let $M > 1$ be a bound of $f(x, u)$ and l be the Lipschitz constant of $f(\,\cdot\,, u)$ on the ball $B(y_0, M)$. We set $\varepsilon_n = 3|y_n - y_0|/R$. Moreover, we choose $T \in (0, R/M)$ such that

$$(M + 1)(e^{lT} - 1) < \frac{R}{3}.$$

Define $u_n\colon [t_0 - T, t_0] \to U$ by

$$u_n(s) = \begin{cases} \overline{u} & \text{for } s \in [t_0 - \varepsilon_n, t_0], \\ u(s + \varepsilon_n) & \text{for } s \in [t_0 - T, t_0 - \varepsilon_n). \end{cases}$$

Define $x_n(s) = x(s; t_0, y_n, u_n)$ for $s \in [t_0 - T, t_0]$. We shall show that for sufficiently large n, $x_n(s) \in D$ for $s \in [t_0 - T, t_0]$. By Proposition 1.4.3, there exists $\delta > 0$ such that $x(t; y, \overline{u}) \in D$ for $t \in (-\delta, 0)$ and $y \in K$, $|y - y_0| < \delta$. Thus for sufficiently large n and $s \in [t_0 - \varepsilon_n, t_0)$ we have $x_n(s) \in D$.

Now, we take $s \in [t_0 - T, t_0 - \varepsilon_n]$. Then

$$\begin{aligned}
|x_n(s) &- (x_u(s + \varepsilon_n) + \varepsilon_n w)| \\
&= \left| y_n + \int_{t_0}^{t_0 - \varepsilon_n} f(x_n(\tau), \overline{u})\, d\tau + \int_{t_0 - \varepsilon_n}^{s} f(x_n(\tau), u(\tau + \varepsilon_n))\, d\tau \right. \\
&\quad \left. - \left(y_0 + \int_{t_0}^{s + \varepsilon_n} f(x_u(\tau), u(\tau))\, d\tau + \varepsilon_n w \right) \right| \\
&\le |y_n - y_0| + \int_{t_0 - \varepsilon_n}^{t_0} |w + f(x_n(\tau), \overline{u})|\, d\tau \\
&\quad + \int_{s}^{t_0 - \varepsilon_n} |f(x_u(\tau + \varepsilon_n), u(\tau + \varepsilon_n)) - f(x_n(\tau), u(\tau + \varepsilon_n))|\, d\tau \\
&\le |y_n - y_0| + \left(\frac{1}{2} l M \varepsilon_n^2 + l |y_n - y_0| \varepsilon_n \right) \\
&\quad + \int_{s}^{t_0 - \varepsilon_n} (|y_n - y_0| + M\varepsilon_n) e^{l(\tau - s)} l\, d\tau \\
&\le \left(\frac{|y_n - y_0|}{\varepsilon_n} + \frac{1}{2} l M \varepsilon_n + l |y_n - y_0| + \left(\frac{|y_n - y_0|}{\varepsilon_n} + M \right)(e^{lT} - 1) \right) \varepsilon_n.
\end{aligned}$$

Hence, for sufficiently large n we obtain

$$|x_n(s) - (x_u(s + \varepsilon_n) + \varepsilon_n w)| \le R\varepsilon_n. \tag{3.46}$$

We set $t_1 = \inf\{t \in [t_0 - T, t_0 - \varepsilon_n] : \text{for all } s \in (t, t_0 - \varepsilon_n),\ x_n(s) \in D\}$. We claim that $t_1 = t_0 - T$. Since $x_n(t_0 - \varepsilon_n) \in D$, by (3.32) and Invariance Theorem 1.4.2, we have $t_1 < t_0 - \varepsilon_n$. Suppose to the contrary that $t_1 > t_0 - T$. Since $K = \mathrm{cl}(D)$, we obtain $x_n(t_1) \in K$. By (3.45), (3.46), we obtain $x_n(t_1) \in D$. Again by (3.32) and Invariance Theorem 1.4.2, there exist $t_2 < t_1$ such that $x_n(s) \in D$ for $s \in (t_2, t_1)$, which contradicts the definition of t_1.

By Step 1, (3.44) holds true for $s \in [t_0 - T, t_0]$ and sufficiently large n. Thus,

$$\begin{aligned} W(x_u(s)) &\le \liminf_{n\to\infty} W(x_n(s)) \\ &\le \liminf_{n\to\infty} e^{s-t_0} W(y_n) + \int_s^{t_0} e^{s-\tau} L(x_n(\tau), u_n(\tau))\, d\tau. \end{aligned}$$

Since

$$\begin{aligned} &\int_s^{t_0} e^{s-\tau}(L(x_n(\tau), u_n(\tau)) - L(x_u(\tau), u(\tau)))\, d\tau \\ &\quad = \int_{t_0-\varepsilon_n}^{t_0} e^{s-\tau} L(x_n(\tau), u_n(\tau)\, d\tau - \int_s^{s+\varepsilon_n} e^{s-\tau} L(x_u(\tau), u(\tau))\, d\tau \\ &\qquad + \int_{s+\varepsilon_n}^{t_0} e^{s-\tau}(e^{\varepsilon_n} L(x_n(\tau-\varepsilon_n), u_n(\tau-\varepsilon_n)) - L(x_u(\tau), u(\tau)))\, d\tau \end{aligned}$$

we deduce (3.44) for $x_u,\ u$.

Step 3. We show here that if $u \in A(x_0)$ and for every $t_0 \in (0, \infty)$ there is $T \in (0, t_0]$ such that (3.44) holds true for every $s \in [t_0 - T, t_0]$, then

$$W(x_u(0)) \le \int_0^\infty e^{-\tau} L(x_u(\tau), u(\tau))\, d\tau.$$

Fix $t_0 > 0$. We define $t_1 = \inf\{t \in [0, t_0) : \text{for all } s \in (t, t_0) \text{ such that (3.44) holds true}\}$. We choose $s_n \to t_1^+$ such that (3.44) holds true for s replaced by s_n and all n. By the lower semicontinuity of W, we obtain (3.44) for $s = t_1$. Suppose that $t_1 > 0$. Then there exists $T \in (0, t_1)$ such that for every $s \in [t_1 - T, t_1]$ we have

$$W(x_u(s)) \le e^{s-t_1} W(x_u(t_1)) + \int_s^{t_1} e^{-(\tau-s)} L(x_u(\tau), u(\tau))\, d\tau.$$

Hence,

$$\begin{aligned} W(x_u(s)) \le{}& e^{s-t_1} e^{t_1-t_0} W(x_u(t_0)) + e^{s-t_1} \int_{t_1}^{t_0} e^{-(\tau-t_1)} L(x_u(\tau), u(\tau))\, d\tau \\ &+ \int_s^{t_1} e^{-(\tau-s)} L(x_u(\tau), u(\tau))\, d\tau \end{aligned}$$

which contradicts the definition of t_1. Consequently $t_1 = 0$ and

$$W(x_u(0)) \le e^{-t_0} W(x_u(t_0)) + \int_0^{t_0} e^{-\tau} L(x_u(\tau), u(\tau))\, d\tau.$$

Since $\lim_{t\to\infty} e^{-t}W(x_u(t)) = 0$ we obtain

$$W(x_u(0)) \leq \int_0^\infty e^{-\tau} L(x_u(\tau), u(\tau))\, d\tau.$$

The admissible control $u \in A(x_0)$ being arbitrary, we finally obtain

$$W(x_0) \leq \inf_{u\in A(x_0)} \int_0^\infty e^{-\tau} L(x_u(\tau), u(\tau))\, d\tau,$$

which completes the proof. □

Proof of Theorem 3.2.1. By Propositions 3.2.3, 3.2.6, 3.2.7, we obtain the implication (a) ⇒ (b). The implication (b) ⇒ (a) is a direct conclusion from Propositions 3.2.8, 3.2.11 Lemma 3.2.10.

We prove next that (b) ⇒ (c). Let

$$H_F((x,v),(-p_x,-p_v)) = \sup_{u\in U,\, r\in[0,C-L(x,u)]} \langle (f(x,u), v - L(x,u) - r), (-p_x,-p_v)\rangle.$$

If $p_v < 0$ then

$$H_F((x,W(x)),(-p_x,-p_v)) = (-p_v)\left(W(x) + H\left(x, \frac{-p_x}{-p_v}\right)\right). \tag{3.47}$$

By Theorem 1.4.1 ((a) ⇒ (c)), (3.36) yields $H_F((x,W(x)),(-p_x,-p_v)) \geq 0$ for every $(p_x,p_v) \in [T_{\mathcal{E}pi(W)}(x,W(x))]^-$ and $x \in K$. If $n \in \partial_- W(x)$, then $(n,-1) \in [T_{\mathcal{E}pi(W)}(x,W(x))]^-$. From (3.47) we obtain that $W(x) + H(x,-n) = H_F((x,W(x)),(-n,1))$, for $n \in \partial_- W(x)$. Hence

$$\forall x \in K,\ \forall n \in \partial_- W(x) \quad W(x) + H(x,-n) \geq 0. \tag{3.48}$$

By Theorem 1.4.2 ((a) ⇒ (c)), (3.36) yields $H_F((x,W(x)),(-p_x,-p_v)) \leq 0$ for every $(p_x,p_v) \in [T_{\mathcal{E}pi(W)}(x,W(x))]^-$ and $x \in D$. By (3.47) we obtain

$$\forall x \in D,\ \forall n \in \partial_- W(x) \quad W(x) + H(x,-n) \leq 0. \tag{3.49}$$

It is obvious that (3.38), (3.39) are equivalent to (3.48), (3.49).

It remains to prove that (c) ⇒ (b). Let $(p_x,p_v) \in [T_{\mathcal{E}pi(W)}(x,W(x))]^-$. If $p_v < 0$ then $-p_x/p_v \in \partial_- W(x)$. Hence, (3.48) yields

$$W(x) + H\left(x, \frac{-p_x}{-p_v}\right) \geq 0.$$

By (3.47), it follows $H_F((x,W(x)),(-p_x,-p_v)) \geq 0$.

Next consider the case $p_v = 0$. By Lemma 1.3.4, there are sequences (x_n), (q_n), (p_n) such that $x_n \to x$, $x_n \in K$, $p_n \to p_x$, $q_n \to 0^-$, $(p_n,q_n) \in [T_{\mathcal{E}pi(W)}(x_n,W(x_n))]^-$. By (3.48), we have

$$W(x_n) + H\left(x_n, \frac{-p_n}{-q_n}\right) \geq 0.$$

Since W is bounded and (3.28)–(3.29) hold true, it follows

$$\begin{aligned}
H_F((x,W(x)),(-p_x,0)) &= \sup_{u\in U}\langle f(x,u),-p_x\rangle \\
&= \lim_{n\to\infty}\sup_{u\in U}(\langle f(x_n,u),-p_n\rangle-(-q_n)L(x_n,u)) \\
&= \lim_{n\to\infty}(-q_n)H\bigg(x_n,\frac{-p_n}{-q_n}\bigg) \\
&= \lim_{n\to\infty}(-q_n)\bigg(W(x_n)+H\bigg(x_n,\frac{-p_n}{-q_n}\bigg)\bigg)\geq 0.
\end{aligned}$$

By Viability Theorem 1.4.1 we obtain (3.35). In a similar way we prove that (3.49) implies (3.36). □

Example. We consider $D=\{(x,y):x<0 \text{ or } y<0\}$, $U=[0,1]$, $f(x,y,u) = (u,1-u)$, $L(x,y,u)=u$. One can easily check that D, $f(x,y,u)$ satisfy (3.33). The value function $V\colon K\to\mathbb{R}$ is given by

$$V(x,y)=\begin{cases}0 & \text{if } x\leq 0,\\ e^y & \text{if } x>0.\end{cases}$$

The above example shows that assumption (3.33) does not imply continuity of the value function. So, it is essentially different from the Soner condition

$$\forall x\in\partial D,\ \exists u\in U,\quad f(x,u)\cdot n(x)<-\beta$$

where D has a smooth boundary and $n(x)$ is the exterior normal to D and $\beta>0$. Soner condition was generalized to sets with nonsmooth boundary D by Ishii, Koike in [62]. The assumption (A3) in [62] can be formulated (in our notation) as follows

$$\forall x\in\partial D,\quad \mathrm{Int}(C_K(x))\cap\mathrm{co}(f(x,U))\neq\emptyset. \tag{3.50}$$

It was shown in [62] that (3.50) and (3.28)–(3.30) yield local Lipschitz continuity of the value function. So our assumption (3.33) is of essentially different nature.

3.3. Discontinuous Mayer problem

We apply results obtained for games (Theorem 5.2.2) to describe value function for the Mayer problem with fully discontinuous terminal cost. Next, we use the obtained description to characterize a value function of a control system with state constraints. Our characterization of the value function is provided in the framework of the concept of weak solutions given in Chapter 5.

Proposition 3.3.1. *Let $g\colon\mathbb{R}^n\mapsto\mathbb{R}$ be a bounded function. Assume that U is a compact metric space and $f\colon[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}^n$ satisfies* (5.2) *and* (5.3). *Then the value-function $W_g\colon(0,T]\times\mathbb{R}^n\to\mathbb{R}$ given by*

$$W_g(t,x)=\inf_{u\in\mathcal{U}(t)}g(x(T;t,x,u)),$$

where $\mathcal{U}(t)$ *denotes the set of measurable controls* $u\colon[t,T]\to U$*, is the unique generalized solution* (*in the meaning of Definition* 5.2.1) *to the Hamilton–Jacobi–Bellmann equation* (5.10) *where*

$$H(t,x,p) := \min_{u\in U}\langle f(t,x,u),p\rangle. \tag{3.51}$$

Proof. Fix $(t_0,x_0)\in(0,T]\times\mathbb{R}^n$. Let $\varepsilon>0$. There exists $u_\varepsilon\in\mathcal{U}(t_0)$ such that $g(x(T;t_0,x_0,u_\varepsilon)) < W_g(t_0,x_0)+\varepsilon$. We define $h\colon\mathbb{R}^n\to\mathbb{R}$ by

$$h(x) = \begin{cases} g(x) & \text{for } x = x(T;t_0,x_0,u_\varepsilon),\\ M & \text{for } x \neq x(T;t_0,x_0,u_\varepsilon),\end{cases}$$

where M is a bound of $\|g\|$. Obviousily, h is lower semicontinuous. By Theorem 5.2.2, the value W_h is a supersolution of (5.10). We have $W_h(t_0,x_0) < W_g(t_0,x_0)+\varepsilon$. Hence,

$$W_g(t_0,x_0) = \inf\{\psi(t_0,x_0) : \psi \text{ is a supersolution of (5.10)}, \ \psi(T,\cdot\,)\geq g(\,\cdot\,)\}.$$

We define $l\colon\mathbb{R}^n\to\mathbb{R}$ by

$$l(x) = \begin{cases} W_g(t_0,x_0) & \text{if } x\in\{x(T;t_0,x_0,u) : u\in\mathcal{U}(t_0)\},\\ -M & \text{if } x\notin\{x(T;t_0,x_0,u) : u\in\mathcal{U}(t_0)\}.\end{cases}$$

By (5.2), (5.3), the reachable set $\{x(T;t_0,x_0,u) : u\in\mathcal{U}(t_0)\}$ is closed. Thus, l is upper semicontinuous. Obviously, we have $W_g(t_0,x_0) = W_l(t_0,x_0)$. By Theorem 5.2.2 (in a version for upper semicontinuous terminal cost), we obtain W_l is a subsolution of (5.10). Hence,

$$W_g(t_0,x_0) = \sup\{\phi(t_0,x_0) : \phi \text{ is a subsolution of (5.10)}, \ \phi(T,\cdot\,)\leq g(\,\cdot\,)\}. \qquad \square$$

Remark. Proposition 3.3.1 is in fact the existence and uniqueness result for Hamilton–Jacobi equation (5.10) with Hamiltonian given by (3.51) and arbitrary terminal condition g. Uniqueness result in the case of lower semicontinuous g has been obtained in [14], [44] in the framework of different definition of solution. If g is lower semicontinuous then the solution in the meaning of Definition 5.2.1 as well as in the meaning of [14], [44] are equal to the value function W, so they coincides. We give an example of non semicontinuous g.

Example. Let $g\colon\mathbb{R}\to\mathbb{R}$ be the characteristic function of rationals. The dynamics $x' = f(t,x)$ of a system is given by a right hand side that depends neither on u nor on v and satisfies (5.2). In this case the value $V(t_0,x_0) = g(x(T;t_0,x_0))$ is discontinuous at every point. Despite of this, by Theorem 3.3.1, V is the unique solution (in the sense of Definition 5.2.1) of the corresponding problem (5.10). Let us remark, that the concepts of solution from [44] and [95] do not apply to the example.

Now, we apply Proposition 3.3.1 to the Mayer control problem with state constraints.

Let K be a closed subset of $\mathbb{R}^n$. We are interested in the characterization of the following value function $W_g^K\colon [0,T]\times K \to R$

$$
\begin{aligned}
(3.52)\quad & W_g^K(t_0,x_0)\\
&= \inf\left\{ g(x(T;\, t_0,x_0,u)) : \left\{ \begin{array}{l} u\in U(t_0),\\ x(t;t_0,x_0,u)\in K \quad \text{for } t\in[t_0,T). \end{array} \right. \right\}
\end{aligned}
$$

as the unique solution of the Hamilton–Jacobi equation. In the literature there are many attempts to solve this problem (see [92], [45]). The minimal requirement in order the function W_g^K is well defined by (3.52) is

$$
(3.53)\qquad \left\{ \begin{array}{l} \text{for any initial condition } (t_0,x_0)\in[0,T]\times K\\ \text{there exist a control } u\in U(t_0) \text{ such that}\\ \text{the solution } x(t;t_0,x_0,u) \text{ remains in set of constraints } K\\ \text{for every } t\in[t_0,T]. \end{array} \right.
$$

We provide a characterization of the value function W_g^K under assumption (3.53).

Theorem 3.3.2. *Let $K\subset\mathbb{R}^n$ be closed and $g\colon\mathbb{R}^n\mapsto\mathbb{R}$ be a function bounded by $M>0$. Assume that $f\colon[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}^n$ satisfies* (5.2), (5.3) *and that* (3.53) *holds true for f, K. Then*

$$W_g^K(t,x)=W(t,x,0)\quad \textit{for } x\in K$$

where $W\colon[0,T]\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is the unique solution of

$$
(3.54)\qquad \left\{ \begin{array}{l} \dfrac{\partial W}{\partial t}+\widetilde{H}\left(t,x,y,\dfrac{\partial W}{\partial x},\dfrac{\partial W}{\partial y}\right)=0,\\ W(T,x,y)=g(x)+(M+1)\chi_{(0,\infty)}(y), \end{array} \right.
$$

where $\widetilde{H}(t,x,y,p_x,p_y)=\min_{u\in U}\langle f(t,xu),p_x\rangle+d_K(x)p_y$ and $\chi_{(0,\infty)}$ denotes the characteristic function of the open interval $(0,\infty)$.

Proof. We adopt the classical method of adding an extra variable (usually used to reduce the Bolza problem to the Mayer one) and the technique of penalization function. We consider a new control problem

$$
\left\{ \begin{array}{l} x'(t)=f(t,x(t),u(t)),\\ y'(t)=d_K(x(t)), \end{array} \right.
$$

where $d_K(x)$ denotes the distance from x to K. It is obvious that (5.2), (5.3) hold true for the extended control system. By Theorem 3.3.1, we obtain that the value function

$$
\begin{aligned}
W(t_0,x_0,y_0) = {}& \inf_{u\in U(t_0)} g(x(T;t_0,x_0,u))\\
&+(M+1)\chi_{(0,\infty)}\left(y_0+\int_{t_0}^T d_K(x(t;t_0,x_0,u))\,dt\right)
\end{aligned}
$$

is the unique generalized solution of (3.54). On the other hand, just from the very definition, one can easily check that for every $x_0 \in K$ we have

$$W_g^K(t_0, x_0) = W(t_0, x_0, 0). \qquad \square$$

CHAPTER 4

TIME MEASURABLE DIFFERENTIAL GAMES

We study invariance of time-varying domains with respect to differential games with dynamics measurable in time. Invariance in the framework of differential games appears as a discriminating and leadership property. In the proof of the Discriminating Theorem we reduce the problem to the Viability Theorem thanks to the Nonexpansive Selections Theorem in ultrametric spaces. The geometric property describing discriminating and leadership domains is assumed to hold true for almost all "t". This leads to the concept of weak solution of the Hamilton–Jacobi–Isaacs equation that appeared already in Chapter 2, i.e. equation holds for almost all t.

4.1. Nonexpansive selections in ultrametric spaces

A metric ρ in a space M is an ultrametric if it satisfies strong triangle inequality

$$\rho(x,z) \le \max\left(\rho(x,y),\rho(y,z)\right).$$

We say that a subset K of an ultrametric space M is $(*)$-closed if for every sequence $\{y_n\} \subset K$ and every sequence $\{c_n\}$ $(c_n \ge c_{n+1} \ge 0)$ such that $\rho(y_n, y_{n+1}) \le c_n$, there is $\overline{y} \in K$ such that $\rho(\overline{y}, y_n) \le c_n$, for every n.

Remark. If D_1, D_2 are nonempty $(*)$-closed subsets of an ultrametric space M, then the Hausdorff distance $d_H(D_1, D_2) \le r$ if and only if for every $d_1 \in D_1$ there is $d_2 \in D_2$ such that $\rho(d_1, d_2) \le r$ and for every $d_2 \in D_2$ there is $d_1 \in D_1$ such that $\rho(d_1, d_2) \le r$.

We say that a set-valued map $A\colon N \rightsquigarrow M$ is a non-expansive set-valued map from an ultrametric space (N, ρ_N) into another ultrametric space (M, ρ_M) if, for all $(n_1, n_2) \in N \times N$, A satisfies:

(1) $\forall m_1 \in A(n_1),\ \exists m_2 \in A(n_2),\ \rho_M(m_1, m_2) \le \rho_N(n_1, n_2)$,
(2) $\forall m_2 \in A(n_2),\ \exists m_1 \in A(n_1),\ \rho_M(m_1, m_2) \le \rho_N(n_1, n_2)$.

Lemma 4.1.1 (Nonexpansive selection). *If $A: N \rightsquigarrow M$ is a non-expansive set-valued map from an ultrametric space (N, ρ_N) into an ultrametric space (M, ρ_M) with nonempty $(*)$-closed values, then there exists a non-expansive selection $\alpha: N \mapsto M$ of A.*

We procede the proof with some elementary properties of ultrametric spaces.

Proposition 4.1.2. *If $y_1, y_2, y_3 \in M$ and $\rho(y_1, y_2) < \rho(y_2, y_3)$, then*

$$\rho(y_1, y_3) = \rho(y_2, y_3).$$

Proof. We have $\rho(y_2, y_3) \leq \max(\rho(y_1, y_2), \rho(y_1, y_3))$. Thus

$$\rho(y_2, y_3) \leq \rho(y_1, y_3).$$

Moreover, $\rho(y_1, y_3) \leq \max(\rho(y_1, y_3), \rho(y_2, y_3)) = \rho(y_2, y_3)$. □

Let K be a nonempty subset of M. We denote by Q_K the family of nonempty subsets of K of the form $\{y \in K : \rho(y, y_0) \leq c\}$ where $y_0 \in K$ and $c \geq 0$.

Proposition 4.1.3. *If $D \in Q_K$ and $\overline{y} \in D$, then*

$$D = \{y \in K : \rho(y, \overline{y}) \leq \operatorname{diam} D\}.$$

Proof. Fix $y_0 \in K$, $c \geq 0$ and define $D = \{y \in K : \rho(y, y_0) \leq c\}$. Let $\overline{y} \in D$. Obviously, we have $D \subset \{y \in K : \rho(y, \overline{y}) \leq \operatorname{diam} D\}$. If $y \in K$ and $\rho(y, \overline{y}) \leq \operatorname{diam} D$, then $\rho(y, y_0) \leq \max(\rho(y, \overline{y}), \rho(\overline{y}, y_0)) \leq \max(\operatorname{diam} D, c) = c$.□

Proposition 4.1.4. *If $D \in Q_K$, $y_1 \in M$, $c \geq 0$ are such that $D_1 = \{y \in D : \rho(y, y_1) \leq c\}$ is a nonempty set, then $D_1 \in Q_K$.*

Proof. Fix $\overline{y} \in D_1$. By Proposition 4.1.3, we have $D = \{y \in K : \rho(y, \overline{y}) \leq \operatorname{diam} D\}$.

Case 1. If $\operatorname{diam} D \leq c$, then $D_1 = D$.

Indeed, for any $y \in D$, we have

$$\rho(y, y_1) \leq \max(\rho(y, \overline{y}), \rho(\overline{y}, y_1)) \leq \max(\operatorname{diam} D, c) = c.$$

Case 2. If $\operatorname{diam} D > c$, then $D_1 = \{y \in K : \rho(y, \overline{y}) \leq c\}$.

If $y \in D_1$, then $\rho(y, \overline{y}) \leq \max(\rho(y, y_1), \rho(y_1, \overline{y})) \leq c$. Thus $D_1 \subset \{y \in K : \rho(y, \overline{y}) \leq c\}$. If $y \in K$ and $\rho(y, \overline{y}) \leq c$, then $y \in D$ and

$$\rho(y, y_1) \leq \max(\rho(y, \overline{y}), \rho(\overline{y}, y_1)) \leq c. \qquad \square$$

Proposition 4.1.5. *If $D_1, D_2 \in Q_K$, $D_1 \subset D_2$ and $D_1 \neq D_2$, then* $\operatorname{diam} D_1$

Proof. Suppose that $\operatorname{diam} D_1 = \operatorname{diam} D_2 = d$ and $D_1 \subset D_2$.

If $y_1 \in D_1$ and $y_2 \in D_2$, then $D_1 = \{y \in K : \rho(y, y_1) \leq d\}$ and $D_2 = \{y \in K : \rho(y, y_2) \leq d\}$. We have $\rho(y_1, y_2) \leq d$. If $y \in D_2$, then $\rho(y, y_1) \leq \max(\rho(y, y_2), \rho(y_2, y_1)) \leq d$, which implies that $y \in D_1$. □

Proposition 4.1.6. *Suppose that K is a nonempty $(*)$-closed subset of M, and a family $\{D_\omega \in Q_K : \omega \in \Omega\}$ satisfies the following condition*

$$\forall \omega_1, \omega_2 \in \Omega, \quad D_{\omega_1} \subset D_{\omega_2} \text{ or } D_{\omega_2} \subset D_{\omega_1}.$$

Then:

(a) $\forall \omega_1, \omega_2 \in \Omega,\ (\operatorname{diam} D_{\omega_1} \leq \operatorname{diam} D_{\omega_2} \to D_{\omega_1} \subset D_{\omega_2})$.

(b) *If a sequence $\{\omega_n\} \subset \Omega$ satisfies the following conditions:*
- $\operatorname{diam} D_{\omega_{n+1}} \leq \operatorname{diam} D_{\omega_n}\ (:= d_n)$,
- $\lim_{n\to\infty} d_n = \inf_{\omega\in\Omega} \operatorname{diam} D_\omega\ (:= d)$,

then for every sequence $y_n \in D_{\omega_n}$ there is $\overline{y} \in K$ such that $\rho(\overline{y}, y_n) \leq d_n$ and

$$\bigcap_{\omega\in\Omega} D_\omega = \{y \in K : \rho(y, \overline{y}) \leq d\}. \tag{4.1}$$

(c) *The set $D = \bigcap_{\omega\in\Omega} D_\omega$ belongs to Q_K.*

Proof. Assertion (a) is an immediate consequence of Proposition 4.1.5.

By Proposition 4.1.3 $D_{\omega_n} = \{y \in K : \rho(y, y_n) \leq d_n\}$. According to assertion (a) $D_{\omega_{n+1}} \subset D_{\omega_n}$ for every n. Therefore $\rho(y_{n+1}, y_n) \leq d_n$. Since K is $(*)$-closed, there is $\overline{y} \in K$ such that $\rho(\overline{y}, y_n) \leq d_n$.

If $y \in D_{\omega_n}$, then $\rho(y, \overline{y}) \leq \max(\rho(y, y_n), \rho(y_n, \overline{y})) \leq d_n$.

Let us choose $y \in K$ such that $\rho(y, \overline{y}) \leq d$ and pick some $\omega \in \Omega$. There is ω_n such that $d_n \leq \operatorname{diam} D_\omega$. By the assertion (1), we have $D_{\omega_n} \subset D_\omega$. Moreover, $\rho(y, y_n) \leq \max(\rho(y, \overline{y}), \rho(\overline{y}, y_n)) \leq \max(d, d_n) = d_n$. Hence $y \in D_{\omega_n}$.

By (4.1), we obtain statement (c). □

Proof of Theorem 4.1.1. First, we define a partial order $(\mathcal{P}, \leq)$. The family $\mathcal{P}$ consists of all nonempty valued non-expansive maps $C\colon N \rightsquigarrow M$ such that $C(z) \in Q_{A(z)}$ for every $z \in N$. Since $A \in \mathcal{P}$, then the family $\mathcal{P}$ is nonempty. We say that $C_1 \leq C_2$ if $C_1(z) \subset C_2(z)$, for every $z \in N$.

Step 1. Let $\{C_\omega\}_{\omega\in\Omega} \subset \mathcal{P}$ be a chain. Define a set-valued map $C\colon N \rightsquigarrow M$ by $C(z) = \bigcap_{\omega\in\Omega} C_\omega(z)$. By Proposition 4.1.6(c), we have $C(z) \in Q_{A(z)}$, for every $z \in N$. Now, we show that C is a non-expansive map. Let us take $z_1, z_2 \in N$ and $\overline{y_1} \in C(z_1)$.

Case 1. $\rho(z_1, z_2) \geq \inf_{\omega\in\Omega} \operatorname{diam} C_\omega(z_2)$. We choose a sequence $\{\omega_n\} \subset \Omega$ such that $d_{n+1} \leq d_n\,(:= \operatorname{diam} C_{\omega_n}(z_2))$ and $\lim_{n\to\infty} d_n = \inf_{\omega\in\Omega} \operatorname{diam} C_\omega(z_2)$. Since C_{ω_n} is a non-expansive map, then there is $y_n \in C_{\omega_n}(z_2)$ such that $\rho(\overline{y_1}, y_n) \leq \rho(z_1, z_2)$. By Proposition 4.1.6(b), there is $\overline{y} \in C(z_2)$ such that $\rho(y_n, \overline{y}) \leq d_n$. Therefore $\rho(\overline{y_1}, \overline{y}) \leq \max(\rho(\overline{y_1}, y_n), \rho(y_n, \overline{y})) \leq \max(\rho(z_1, z_2), d_n)$.

Case 2. $\rho(z_1, z_2) < \inf_{\omega\in\Omega} \operatorname{diam} C_\omega(z_2)$.

Let us fix $\omega_0 \in \Omega$ and choose $y_0 \in C_{\omega_0}(z_2)$ such that $\rho(\overline{y_1}, y_0) \leq \rho(z_1, z_2)$. We claim that $y_0 \in C(z_2)$. We pick some $\omega \in \Omega$ and choose $y_\omega \in C_\omega(z_2)$ such that $\rho(\overline{y_1}, y_\omega) \le \rho(z_1, z_2)$. Thus $\rho(y_0, y_\omega) \le \max(\rho(y_0, y_1), \rho(y_1, y_\omega)) \le \rho(z_1, z_2)$. By Proposition 4.1.3, we have $C_\omega(z_2) = \{y \in A(z_2) : \rho(y, y_\omega) \le \operatorname{diam} C_\omega(z_2)\}$. Therefore $y_0 \in C_\omega(z_2)$.

Step 2. Suppose that $C \in \mathcal{P}$ and there is $z_0 \in N$ such that $\operatorname{diam} C(z_0) > 0$, i.e. C is not a single-valued map. We define a map $\widetilde{C}: N \rightsquigarrow M$ by

$$\widetilde{C}(z) = \begin{cases} C(z) & \text{if } \rho(z, z_0) \ge d, \\ \{y \in C(z) : \rho(y, y_0) \le \rho(z, z_0)\} & \text{if } \rho(z, z_0) < d, \end{cases}$$

where $d = \operatorname{diam} C(z_0)$ and y_0 is a fixed element of $C(z_0)$. Obviously, $\widetilde{C}(z_0) = \{y_0\} \ne C(z_0)$. Since C is a non-expansive map, then $\widetilde{C}(z) \ne \emptyset$ for every $z \in N$. By Proposition 4.1.4, we have $\widetilde{C}(z) \in Q_{A(z)}$ for every $z \in N$. Now, we show that $\widetilde{C}$ is a non-expansive map. Let us take $z_1, z_2 \in N$ and $y_1 \in \widetilde{C}(z_1)$.

Case 1. $\rho(z_1, z_0) \ge d$ and $\rho(z_2, z_0) < d$.

Let us take an arbitrary $y_2 \in \widetilde{C}(z_2)$. By Proposition 4.1.2, we have $\rho(z_1, z_2) = \rho(z_1, z_0)$. Since C is a non-expansive map, then there is $\overline{y_0} \in C(z_0)$ such that $\rho(y_1, \overline{y_0}) \le \rho(z_1, z_0)$. Therefore $\rho(y_1, y_2) \le \max(\rho(y_1, \overline{y_0}), \rho(\overline{y_0}, y_0), \rho(y_0, y_2)) \le \max(\rho(z_0, z_2), d) \le \rho(z_1, z_2)$.

Case 2. $\rho(z_1, z_0) < d$ and $\rho(z_2, z_0) < d$.

- $\rho(z_1, z_0) < \rho(z_2, z_0)$.

By Proposition 4.1.2, we have $\rho(z_1, z_2) = \rho(z_2, z_0)$. For any $y_2 \in \widetilde{C}(z_2)$ we have $\rho(y_1, y_2) \le \max(\rho(y_1, y_0), \rho(y_0, y_2)) \le \max(\rho(z_1, z_0), \rho(z_0, z_2)) = \rho(z_1, z_2)$.

- $\rho(z_1, z_0) > \rho(z_2, z_0)$.

By Proposition 4.1.2, we have $\rho(z_1, z_2) = \rho(z_1, z_0)$. Let y_2 be an arbitrary element of $\widetilde{C}(z_2)$. Therefore

$$\rho(y_1, y_2) \le \max(\rho(y_1, y_0), \rho(y_0, y_2)) \le \max(\rho(z_1, z_0), \rho(z_0, z_2)) = \rho(z_1, z_2).$$

- $\rho(z_1, z_0) = \rho(z_2, z_0)$.

Since C is a non-expansive map, then there is $y_2 \in C(z_2)$ such that $\rho(y_1, y_2) \le \rho(z_1, z_2)$. Observe that $\rho(z_1, z_2) \le \max(\rho(z_1, z_0), \rho(z_0, z_2)) = \rho(z_1, z_0)$. So $\rho(y_2, y_0) \le \max(\rho(y_2, y_1), \rho(y_1, y_0)) \le \max(\rho(z_2, z_1, \rho(z_1, z_0)) = \rho(z_1, z_0) = \rho(z_2, z_0)$. Therefore $y_2 \in \widetilde{C}(z_2)$.

By Steps 1 and 2 together with Kuratowski–Zorn's Lemma, we obtain the existence of a non-expansive (single-valued) selection $\alpha: N \mapsto M$ of the set-valued map $A: N \rightsquigarrow M$. $\square$

4.2. Discriminating domains

In the section we study a kind of viability problem for differential games. Let $P(t)$ be a time dependent set in $\mathbb{R}^n$ and let an initial condition (t_0, x_0) satisfy $x_0 \in P(t_0)$. The aim of the first player is to keep the trajectory of the game in the tube P, i.e. $x(t) \in P(t)$ for $t \in [t_0, T]$, despite of the behaviour of the second player. This can be formulated in a rigorous way as follows

$$\forall t_0 \in (0,T),\ \forall x_0 \in P(t_0),\ \exists \alpha \in \Gamma_{t_0},\ \forall z \in N_{t_0},\\ x(t;t_0,x_0,\alpha(z),z) \in P(t) \ \text{ for every } t \in [t_0,T]. \tag{4.2}$$

We provide a pointwise boundary condition which guarantees (4.2). The first results of this kind have been obtained by Cardaliaguet (see [24, Theorem 2.1]).

Definition 4.2.1 (Discriminating tube)**.** A tube $P\colon [0,T] \rightsquigarrow \mathbb{R}^n$ is a *discriminating tube* for $f\colon [0,T] \times \mathbb{R}^n \times Y \times Z \to \mathbb{R}^n$ if there exists a full measure set $C \subset [0,T]$ such that for every $t \in C$ and every $x \in P(t)$ we have

$$\forall (n_t, n_x) \in N^0_{\mathrm{Graph}(P)}(t,x),\ \forall z \in Z,\ \exists y \in Y,\\ \langle (n_t,n_x),(1,f(t,x,y,z))\rangle \le 0. \tag{4.3}$$

Theorem 4.2.2. *We assume that a tube $P\colon [0,T] \rightsquigarrow \mathbb{R}^n$ is left absolutely continuous and a right-hand side $f\colon [0,T] \times \mathbb{R}^n \times Y \times Z \to \mathbb{R}^n$ satisfies the following conditions:*

$$f(\,\cdot\,,x,y,z) \textit{ is measurable for every } x,y,z; \tag{4.4}$$

$$\exists l \in L^1(0,T),\ \forall x_1,\ x_2,\ \forall y \in Y,\ \forall z \in Z,\\ \|f(t,x_1,y,z) - f(t,x_2,y,z)\| \le l(t)\|x_1 - x_2\| \textit{ for a.a. } t \in [0,T]; \tag{4.5}$$

$$f(t,x,\,\cdot\,,\,\cdot\,) \textit{ is continuous for every } t,x; \tag{4.6}$$

$$\exists\ \mu \in L^1(0,T),\ \forall t,x,y,z, \quad \|f(t,x,y,z)\| \le \mu(t); \tag{4.7}$$

$$\forall (t,x,z) \in [0,T] \times \mathbb{R}^n \times Z, \quad \{f(t,x,y,z) : y \in Y\} \textit{ is convex.} \tag{4.8}$$

If P is a discriminating tube for f then for each $t_0 \in [0,T]$ and $x_0 \in P(t_0)$

$$\exists \alpha \in \Gamma_{t_0},\ \forall z(\,\cdot\,) \in N_{t_0},\ \forall t \in [t_0,T], \quad x(t;t_0,x_0,\alpha(z),z) \in P(t). \tag{4.9}$$

Conversely, if for each $t_0 \in [0,T]$ and $x_0 \in P(t_0)$

$$\forall \varepsilon > 0,\ \exists \alpha \in \Gamma_{t_0},\ \forall z(\,\cdot\,) \in N_{t_0},\ \forall t \in [t_0,T],\\ x(t;t_0,x_0,\alpha(z),z) \in P(t) + B(0,\varepsilon), \tag{4.10}$$

then P is a discriminating tube for f.

The proof of Theorem 4.2.2 makes use of a viability result for differential inclusions and a non-expansive selection theorem in ultrametric spaces.

Remark. Given $y_1, y_2 \in M_{t_0}$ we define

$$\rho(y_1,y_2) = T - \sup\{t \in [t_0,T] : y_1(s) = y_2(s) \text{ for a.a. } t \in [t_0,t]\}.$$

It is easy to see that (M_{t_0}, ρ) is an ultrametric space. Moreover, a strategy $\alpha\colon N_{t_0} \to M_{t_0}$ is nothing but a non-expansive map in the meaning of the ultrametric ρ.

Proof of Theorem 4.2.2. Fix $t_0 \in [0,T]$, $x_0 \in P(t_0)$ and $\widetilde{z}(\,\cdot\,) \in N_{t_0}$. We define a set-valued map $F_{\widetilde{z}(\,\cdot\,)}(t,x) = \{f(t,x,y,\widetilde{z}(t)) : y \in Y\}$. By the regularity of f: (4.4)–(4.8), the set-valued map $F_{\widetilde{z}(\,\cdot\,)}$ satisfies assumptions of Theorem 2.2.2. By the separation theorem and Theorem 4.2.2, we have for every $t \in C$ and $x \in P(t)$

$$\forall z \in Z,\ \exists y \in Y, \quad (1, f(t,x,y,z)) \in \overline{\mathrm{co}}(T_{\mathrm{Graph}(P)}(t,x)). \tag{4.11}$$

Thus $F_{\widetilde{z}(\,\cdot\,)}$ satisfies statement (a) in Theorem 2.2.2. Therefore there exists an absolutely continuous solution $\widetilde{x}\colon [t_0,T] \to \mathbb{R}^n$ of the differential inclusion $\widetilde{x}'(t) \in F_{\widetilde{z}(\,\cdot\,)}(t,\widetilde{x}(t))$ such that $\widetilde{x}(t_0) = x_0$ and $\widetilde{x}(t) \in P(t)$, for every $t \in [t_0,T]$. By Measurable Selection Theorem 8.2.10 in [7], there exists a measurable map $\widetilde{y}\colon [t_0,T] \to Y$ such that $x(t;t_0,x_0,\widetilde{y}(\,\cdot\,),\widetilde{z}(\,\cdot\,)) = \widetilde{x}(t)$ for $t \in [t_0,T]$.

We define a set-valued map $A\colon N_{t_0} \rightsquigarrow M_{t_0}$ by:

$$A(z(\,\cdot\,)) = \{y(\,\cdot\,) \in M_{t_0} : x(t;t_0,x_0,y(\,\cdot\,),z(\,\cdot\,)) \in P(t) \text{ for } t \in [t_0,T]\}.$$

We have shown that the values of the map A are nonempty. Now we verify that the map A satisfies the remaining assumptions of Lemma 4.1.1.

Let $z_1, z_2 \in N_{t_0}$ and $y_1 \in A(z_1)$. We set $t_1 = T - \rho(z_1,z_2)$ and $x_1 = x(t_1;t_0,x_0,y_1,z_1)$. We have $x_1 \in P(t_1)$. By (4.11) and Theorem 2.2.2, there exists a solution $\widehat{x}\colon [t_1,T] \to \mathbb{R}^n$ of a differential inclusion $\widehat{x}'(t) \in F_{z_2}(t,\widehat{x}(t))$ such that $\widehat{x}(t_1) = x_1$ and $\widehat{x}(t) \in P(t)$ for $t \in [t_1,T]$, where $F_{z_2}(t,x) = \{f(t,x,y,z_2(t)) : y \in Y\}$. By Theorem 8.2.10 in [7], there exists a measurable map $y_3\colon [t_1,T] \to Y$ such that $x(t;t_1,x_1,y_3,z_2) = \widehat{x}(t)$ for $t \in [t_1,T]$. Setting

$$y_2(t) = \begin{cases} y_1(t) & \text{for } t \in [t_0,t_1[, \\ y_3(t) & \text{for } t \in [t_1,T], \end{cases}$$

we get $y_2 \in A(z_2)$ such that $\rho(y_1,y_2) \le \rho(z_1,z_2)$, which means that the map A is non-expansive.

Now, we show that the set $A(z)$ is $(*)$-closed, for every $z \in N_{t_0}$. Let $0 \le \ldots \le c_{k+1} \le c_k \le \ldots \le c_1 \le T - t_0$, $c = \lim_{k\to\infty} c_k$ and $y_k \in A(z)$ satisfy $\rho(y_k,y_{k+1}) \le c_k$. We set $t_k = T - c_k$. Obviously, we have $x(t;t_0,x_0,y_k,z) = x(t;t_0,x_0,y_{k+1},z)$ for $t \in [t_0,t_k]$. We define a map $y_\infty\colon [t_0,T-c[\, \to Y$ by

$$y_\infty(t) = \begin{cases} y_1(t) & \text{for } t \in [t_0,t_1[, \\ y_k(t) & \text{for } t \in [t_{k-1},t_k[\text{ and } k = 2,3,\ldots \end{cases}$$

We set $x_\infty = \lim_{t\to(T-c)^-} x(t,t_0,x_0,y_\infty,z)$. It is easy to check that $x_\infty \in P(T-c)$. By (4.11) and Theorem 2.2.2, there exists a solution $\overline{x}\colon [T-c,T] \to \mathbb{R}^n$ of a differential inclusion $\overline{x}'(t) \in F_z(t,\overline{x}(t))$ such that $\overline{x}(T-c) = x_\infty$ and $\overline{x}(t) \in P(t)$

for $t \in [T-c, T]$. By Theorem 8.2.10 in [7], there exists a measurable map $\overline{y}\colon [T-c,T] \to Y$ such that $x(t; T-c, x_\infty, \overline{y}, z) = \overline{x}(t)$ for $t \in [T-c, T]$. Setting

$$y(t) = \begin{cases} y_\infty(t) & \text{for } t \in [t_0, T-c[, \\ \overline{y}(t) & \text{for } t \in [T-c, T], \end{cases}$$

we get $y \in A(z)$ such that $\rho(y_k, y) \leq c_k$, which means that the set $A(z)$ is $(*)$-closed.

Finally, by Lemma 4.1.1, there exists a non-expansive selection $\alpha\colon N_{t_0} \to M_{t_0}$ of A, which is the desired strategy.

For the converse, we set $F_z(t,x) = \{f(t,x,y,z) : y \in Y\}$. By Lemma 2.6 in [51], there is a full measure set $C \in [0,T]$ such that

$$\begin{aligned} &\forall (t_0, x_0, z) \in C \times \mathbb{R}^d \times Z,\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x(\,\cdot\,) \in \mathrm{Sol}_{F_z}(t_0, x_0), \\ &\qquad\qquad \forall 0 < |h| < \delta, \quad \frac{1}{h}(x(t_0+h) - x_0) \in F_z(t_0, x_0) + B(0, \varepsilon). \end{aligned}$$

Fix $t_0 \in C$, $x_0 \in P(t_0)$, $z_0 \in Z$. Applying (4.9) we obtain an $\alpha_n \in \Gamma_{t_0}$ such that $x_n(t) := x(t, t_0, x_0, \alpha_n(z), z) \in P(t) + B(0, 1/n)$, for $t \in [t_0, T]$, where $z(\,\cdot\,)$ is a constant control on $[t_0, T]$ equal to z_0. For fixed $h > 0$ let $x(t_0+h)$ be a condensing point of the sequence $(x_n(t_0+h))$. Obviously, we have $x(t_0+h) \in P(t_0+h)$ and for sufficiently small h

$$\frac{x(t_0+h) - x(t_0)}{h} \in F_{z_0}(t_0, x_0) + B(0, \varepsilon).$$

There is a sequence $h_n >$ tending to zero such that

$$v := \lim_{n\to\infty} \frac{x(t_0+h_n) - x(t_0)}{h_n} \in F_{z_0}(t_0, x_0).$$

We find $y_0 \in Y$ such that $v = f(t_0, x_0, y_0, z_0)$. We have $(1, v) \in T_{\mathrm{Graph}(P)}(t_0, x_0)$, which yields

$$\langle (n_t, n_x), (1, f(t_0, x_0, y_0, z_0)) \rangle \leq 0$$

for every $(n_t, n_x) \in N^0_{\mathrm{Graph}(P)}(t_0, x_0)$. $\square$

If we assume that f is also continuous with respect to the variable t then using exactly the same scheme of the proof (we use the viability theorem in the version of Theorem 1.4.1 instead of Theorem 2.2.2) we obtain the following

Proposition 4.2.3. *Assume that $f\colon [0,T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ is continuous, Lipschitz continuous with respect to x and $f(t,x,Y,z)$ is convex for every t, x, v. Suppose that the graph of the tube P is closed. If for every $t \in (0,T)$ and every $x \in P(t_0)$* (4.3) *holds true then* (4.2).

Proposition 4.2.3 is generalization of the Cardaliaguet result from [24].

4.3. Leadership domains

Definition 4.3.1 (Leadership tube)**.** The tube $P(\,\cdot\,)$ is a *leadership tube* for f if there exists a set C of full measure in $[0,T]$ such that for every $t \in C$ and $x \in P(t)$

$$\tag{4.12} \forall (n_t, n_x) \in N^0_{\mathrm{Graph}(P)}(t,x),\ \exists z \in Z,\ \forall y \in Y,\\ \langle (n_t, n_x), (1, f(t,x,y,z))\rangle \le 0.$$

Theorem 4.3.2. *We assume that a tube $P\colon [0,T] \rightsquigarrow \mathbb{R}^n$ is left absolutely continuous and that the right-hand side $f\colon [0,T] \times \mathbb{R}^n \times Y \times Z \to \mathbb{R}^n$ satisfies* (4.4)–(4.7)*. Then $P(\,\cdot\,)$ is a leadership tube for f if and only if for any $t_0 \in [0,T]$ and $x_0 \in P(t_0)$*

$$\tag{4.13} \forall \varepsilon > 0,\ \forall \alpha \in \Gamma_{t_0},\ \exists z(\,\cdot\,) \in N_{t_0},\ \forall t \in [t_0, T],\\ x(t; t_0, x_0, \alpha(z), z) \in P(t) + B(0,\varepsilon).$$

The proof is based on the following lemma.

Lemma 4.3.3. *Let f and $P(\,\cdot\,)$ be as in Theorem* 4.3.2*. The following assertions are equivalent:*

(a) *$P(\,\cdot\,)$ is a leadership tube for f.*
(b) *From any initial condition (t_0, z_0) belonging to* $\mathrm{Graph}(P)$*, for any measurable map $a\colon [0,T] \times Z \to Y$, there is at least one solution of the differential inclusion:*

$$\tag{4.14} \begin{cases} v'(t) \in \overline{\mathrm{co}} \bigcup\limits_z f(t, v(t), a(t,z), z) & a.e.\ in\ [t_0, T], \\ v(t_0) = v_0, \end{cases}$$

with $v(t) \in P(t)$ for all $t \in [t_0, T]$.

Proof. Assume that $P(\,\cdot\,)$ is a leadership tube. There exists a set C of full measure in $[t_0, T]$ such that for all $t \in C$, $x \in P(t)$, $(n_t, n_x) \in N^0_{\mathrm{Graph}(P)}(t,x)$,

$$\inf_z \sup_y \langle f(t,x,y,z), n_x\rangle + n_t \le 0.$$

For any measurable map $a\colon [t_0,T] \times Z \to Y$, the set-valued map F_a defined by

$$F_a(t,x) := \overline{\mathrm{co}} \bigcup_z f(t,x,a(t,z),z),$$

is measurable, integrably bounded by $\mu(\,\cdot\,)$, has convex compact values and, for almost every $t \in [t_0, T]$, $x \rightsquigarrow F_a(t,x)$ is $l(t)$-Lipschitz continuous.

Let us now prove that the tube $P(\cdot)$ is viable for F_a. Let $t \in C$, $x \in P(t)$, $(n_t, n_x) \in N^0_{\text{Graph}(P)}(t,x)$. Then

$$\begin{aligned}\inf_{w\in F_a(t,x)} \langle w, n_x\rangle + n_t &= \inf_z \langle f(t,x,a(t,z),z), n_x\rangle + n_t \\ &\le \inf_z \sup_y \langle f(t,x,y,z), n_x\rangle + n_t \le 0.\end{aligned}$$

So $P(\cdot)$ is a viability tube for F_a and Theorem 2.2.2 states that (b) holds true.

Conversely, assume that the tube $P(\cdot)$ enjoys property (b). Fix $n_x \in \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ and define:

$$\begin{aligned} Y_{n_x,x_0}(t,z) &:= \{\overline{y} \in Y : \langle f(t,x_0,\overline{y},z), n_x\rangle = \sup_y \langle f(t,x_0,y,z), n_x\rangle\}, \\ G_{n_x,x_0}(t,x) &:= \overline{\text{co}}\{f(t,x,y,z) : y \in Y_{n_x,x_0}(t,z) \text{ and } z \in Z\}.\end{aligned}$$

Let us prove in a first step that the tube $P(\cdot)$ is viable for the set-valued map G_{n_x,x_0} for any n_x and x_0. The set-valued map $Y_{n_x,x_0}(\cdot,\cdot)$ is measurable, so it enjoys a measurable selection $a_{n_x,x_0}(\cdot,\cdot)$. Note that

$$F_{a_{n_x,x_0}}(t,x) = \overline{\text{co}} \bigcup_z f(t,x,a_{n_x,x_0}(t,z),z) \subset G_{n_x,x_0}(t,x)$$

for almost every $t \in [t_0,T]$ and for all x. Thus, from (b), there is a solution of the differential inclusion for G_{n_x,x_0} which remain in the tube $P(\cdot)$.

Let us now point out that G_{n_x,x_0} is measurable and integrably bounded. Moreover, G_{n_x,x_0} is upper semi-continuous with respect to (n_x,x_0,x) and has convex compact values for almost every $t \in [t_0,T]$. Thus Lemma 2.6 of [51] yields the existence of a set C of full measure in $[t_0,T]$ such that: for all $(\tau,x_\tau,n_x,x_0) \in C \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, and all $\varepsilon > 0$, tehre exists $\delta > 0$ such that, for any solution $x(\cdot)$ to the differential inclusion for G_{n_x,x_0} starting at x_τ at time τ, one has:

$$\forall 0 < |h| < \delta, \quad \frac{1}{h}(x(\tau+h) - x_\tau) \in G_{n_x,x_0}(\tau,x_\tau) + \varepsilon B. \tag{4.15}$$

Let now $\tau \in C$, $x_\tau \in P(\tau)$ and $(n_t,n_x) \in N^0_{\text{Graph}(P(\cdot))}(\tau,x_\tau)$. We have already proved that there is a solution $x(\cdot)$ of the differential inclusion for G_{n_x,x_τ} starting from x_τ at time τ and which remains in the tube $P(\cdot)$ on $[\tau,T]$. From (4.15), for any $h \in]0,\delta[$, there is some $w_h \in G_{n_x,x_\tau}(\tau,x_\tau)$ such that

$$\frac{1}{h}(x(\tau+h) - x_\tau) \in w_h + \varepsilon B.$$

Since $G_{n_x,x_\tau}(\tau,x_\tau)$ is compact, w_h converges, up to a subsequence, to some $w \in G_{n_x,x_\tau}(\tau,x_\tau)$. Thus $(1,w)$ belongs to $T_{\text{Graph}(P(\cdot))}(\tau,x_\tau)$ and $\langle n_x,w\rangle + n_t \le 0$. From the very definition of $G_{n_x,x_\tau}(\tau,x_\tau)$, one has:

$$\begin{aligned} 0 \ge \langle n_x, w\rangle + n_t &\ge \inf_{v\in G_{n_x,x_\tau}(\tau,x_\tau)} \langle v, n_x\rangle + n_t \\ &= \inf_z \inf_{y \in Y_{n_x,x_\tau}(\tau,z)} \langle f(\tau,x_\tau,y,z), n_x\rangle + n_t = \inf_z \sup_y \langle f(\tau,x_\tau,y,z), n_x\rangle + n_t.\end{aligned}$$

So we have finally proved that, for any $\tau \in C$, for any $x_\tau \in P(\tau)$, for any $(n_t, n_x) \in N^0_{\mathrm{Graph}(P(\,\cdot\,))}(\tau, x_\tau)$,

$$\inf_y \sup_z \langle f(\tau, x_\tau, y, z), n_x\rangle + n_t \leq 0,$$

i.e. $P(\,\cdot\,)$ is a leadership tube. □

Proof of Theorem 4.3.2. Assume that $P(\,\cdot\,)$ enjoys the property described in Theorem 4.3.2 and let us prove that $P(\,\cdot\,)$ is a leadership tube. Let

$$a(\,\cdot\,,\,\cdot\,)\colon [0,T] \times Z \to Y$$

and define the non-anticipative strategy α in the following way:

$$\forall z(\,\cdot\,) \in N, \quad \alpha(z(\,\cdot\,))(t) := a(t, z(t)).$$

For any initial position (t_0, z_0) belonging to the graph of $P(\,\cdot\,)$, for any $\varepsilon > 0$, there is a control $z_\varepsilon(\,\cdot\,)$ such that the solution $x_\varepsilon(\,\cdot\,) := x(t_0, x_0, \alpha(z_\varepsilon(\,\cdot\,)), z_\varepsilon(\,\cdot\,))$ satisfies:

$$\forall t \in [t_0, T], \quad d_{P(t)}(x_\varepsilon(t)) \leq \varepsilon.$$

Note that the $x_\varepsilon(\,\cdot\,)$ are solutions of the differential inclusion (4.14). Moreover, the set of solutions of this differential inclusion being compact for the uniform convergence, a sub-sequence of the $x_\varepsilon(\,\cdot\,)$ converges to some solution $x(\,\cdot\,)$ of (4.14) satisfying $x(t) \in P(t)$ for any $t \in [t_0, T]$. Then Lemma 4.3.3 states that the tube $P(\,\cdot\,)$ is a leadership tube.

Conversely, assume now that $P(\,\cdot\,)$ is a leadership tube and fix any $\varepsilon > 0$. The idea of the proof consists in constructing the desired control $z(\,\cdot\,)$ step by step, on intervals $[n\tau, (n+1)\tau)$, where $\tau > 0$ is fixed and shall be chosen later as a function of ε.

For that purpose, we need the following estimation:

Lemma 4.3.4. *Let f and $P(\,\cdot\,)$ be as in Theorem 4.3.2, $t_0 \in [0,T)$ and $x_0 \notin P(t_0)$. Assume that $P(\,\cdot\,)$ is a leadership tube. For any non-anticipative strategy α, there is a control $z(\,\cdot\,)$ such that, if we set $x(\,\cdot\,) := x(t_0, x_0, \alpha(z(\,\cdot\,)), z(\,\cdot\,))$, for all $t \in [t_0, T]$,*

$$\begin{aligned} d^2_{P(t)}(x(t)) \leq \biggl(1 + 2\int_{t_0}^t l(s)\,ds\biggr) d^2_{P(t_0)}(x_0) + 4\biggl(\int_{t_0}^t \mu(s)\,ds\biggr)^2 \\ + 2d_{P(t_0)}(x_0) \int_{t_0}^t l(s) \int_{t_0}^s \mu(\sigma)\,d\sigma\,ds. \end{aligned}$$

Proof. The proof is based on Lemma 4.3.3. Let v_0 belong to the projection of x_0 onto $P(t_0)$. Set $\nu := x_0 - v_0$. Consider the following set-valued map:

$$(s,z) \rightsquigarrow \{\overline{y} \in Y : \langle f(s, v_0, \overline{y}, z), \nu\rangle = \max{}_y \langle f(s, v_0, \overline{y}, z), \nu\rangle\}.$$

This set-valued map is measurable and, so, enjoys a measurable selection $a(\,\cdot\,,\,\cdot\,)$. In the same way, the set-valued map

$$s \rightsquigarrow \{\overline{z} \in Z : \max{}_y \langle f(s, v_0, y, \overline{z}), \nu\rangle = \min_z \max{}_y \langle f(s, v_0, y, z), \nu\rangle\}$$

is measurable and enjoys a measurable selection $z(\,\cdot\,) \in N_{t_0}$.

Let us denote now $x(\,\cdot\,) := x(t_0, x_0, \alpha(z(\,\cdot\,)), z(\,\cdot\,))$ and let $v(\,\cdot\,)$ be a solution of

$$\begin{cases} v'(t) \in \overline{\mathrm{co}} \bigcup_z f(t, v(t), a(t, z), z) & \text{for a.e. } t \in [t_0, T], \\ v(t_0) = v_0, \end{cases}$$

which remains in the tube on $[t_0, T]$ (Lemma 4.3.3). Then

$$\begin{aligned} d^2_{P(t)}(x(t)) \le \|x(t) - v(t)\|^2 &= \|(x(t) - x_0) + (\nu) + (v_0 - v(t))\|^2 \\ &= \|x(t) - x_0\|^2 + \|\nu\|^2 + \|v_0 - v(t)\|^2 + 2\langle x(t) - x_0, \nu\rangle \\ &\quad + 2\langle x(t) - x_0, v_0 - v(t)\rangle + 2\langle \nu, v_0 - v(t)\rangle. \end{aligned}$$

Note that $\|x(t) - x_0\|^2$, $\|v_0 - v(t)\|^2$ and $\langle x(t) - x_0, v_0 - v(t)\rangle$ are bounded by $(\int_{t_0}^t \mu(s)\, ds)^2$. Note also that $\|\nu\|^2 = d^2_{P(t_0)}(x(t_0))$.

Let us now estimate $\langle x(t) - x_0, \nu\rangle$:

$$\begin{aligned} \langle x(t) - x_0, \nu\rangle &= \int_{t_0}^t \langle f(s, x(s), \alpha(z(\,\cdot\,))(s), z(s)), \nu\rangle\, ds \\ &\le \int_{t_0}^t \langle f(s, v_0, \alpha(z(\,\cdot\,))(s), z(s)), \nu\rangle\, ds + \|\nu\| \int_{t_0}^t l(s)\|x(s) - v_0\|\, ds. \end{aligned}$$

For almost every s,

$$\begin{aligned} &\langle f(s, v_0, \alpha(z(\,\cdot\,))(s), z(s)), \nu\rangle \\ &\qquad \le \langle f(s, v_0, a(s, z(s)), z(s)), \nu\rangle = \min_z \langle f(s, v_0, a(s, z), z), \nu\rangle \\ &\qquad = \min_{w \in \overline{\mathrm{co}} \bigcup_z f(s, v_0, a(s,z), z)} \langle w, \nu\rangle \le \langle v'(s), \nu\rangle + l(s)\|\nu\|\|v(s) - v_0\| \end{aligned}$$

from the very definition of $a(\,\cdot\,,\,\cdot\,)$ and of $z(\,\cdot\,)$ and because

$$x \rightsquigarrow \overline{\mathrm{co}} \bigcup_z f(s, v, a(s, z), z)$$

is $l(s)$-Lipschitz continuous for almost all s. So, we have finally:

$$\langle x(t) - x_0, \nu\rangle \le \langle v(t) - v(0), \nu\rangle + \|\nu\| \int_{t_0}^t l(s)(\|x(s) - v_0\| + \|v(s) - v_0\|)\, ds.$$

Since f is integrably bounded by $\mu(\,\cdot\,)$,

$$\|x(s) - v_0\| \le \int_{t_0}^s \mu(\sigma)\, d\sigma + \|\nu\| \quad \text{and} \quad \|v(s) - v_0\| \le \int_{t_0}^s \mu(\sigma)\, d\sigma,$$

so that

$$\langle x(t)-x_0,\nu\rangle+\langle \nu, v_0-v(t)\rangle \le \|\nu\| \int_{t_0}^t l(s)\bigg(\|\nu\|+2\int_{t_0}^s \mu(\sigma)\,d\sigma\bigg)\,ds.$$

In conclusion,

$$d^2_{P(t)}(x(t)) \le \|\nu\|^2+4\bigg(\int_{t_0}^t \mu(s)\,ds\bigg)^2+2\|\nu\|\int_{t_0}^t l(s)\bigg(\|\nu\|+2\int_{t_0}^s \mu(\sigma)\,d\sigma\bigg)\,ds. \ \square$$

Construction of $z(\,\cdot\,)$. We construct $z(\,\cdot\,)$ step by step, on intervals of the form $[n\tau,(n+1)\tau)$ where $\tau>0$ is fixed and shall be chosen bellow (τ depends mainly on ε).

Assume that we have already defined $z(\,\cdot\,)$ on $[0,n\tau]$. Then set $x_n := x(n\tau;t_0,x_0,\alpha(z(\,\cdot\,)),z(\,\cdot\,))$ (Note that x_n is well defined because α is non-anticipative).

- If x_n belongs to $P(n\tau)$, then choose any $z\in Z$ and set $z(\,\cdot\,):=z$ on $[n\tau,(n+1)\tau)$.
- Otherwise, let $z_1(\,\cdot\,)$ be the control defined in Lemma 4.3.4 for $(t_0,x_0):=(n\tau,x_n)$. Then we set $z(\,\cdot\,):=z_1(\,\cdot\,)$ on $[n\tau,(n+1)\tau)$.

Note that the distance between $x(t):=x(t;t_0,x_0,\alpha(z(\,\cdot\,)),z(\,\cdot\,))$ and $P(t)$ ($t\in[t_0,T]$) is maximal if $x_n\notin P(n\tau)$ for any $n>0$. In that case, this distance satisfies for all $t\in[n\tau,(n+1)\tau)$:

$$\begin{aligned} d^2_{P(t)}(x(t)) \le \bigg(1+2\int_{n\tau}^t l(s)\,ds\bigg)d^2_{P(n\tau)}(x_n)+4\bigg(\int_{n\tau}^t \mu(s)\,ds\bigg)^2 \\ +2d_{P(n\tau)}(x_n)\int_{n\tau}^t l(s)\int_{n\tau}^s \mu(\sigma)\,d\sigma\,ds \end{aligned}$$

from Lemma 4.3.4. In particular,

$$\forall t\in[n\tau,(n+1)\tau),\quad d^2_{P(t)}(x(t))\le d_{n+1}(\tau)$$

where $d_n(\tau)$ is the sequence defined by

$$d_0(\tau)=0,\quad d_{n+1}(\tau)=(1+\alpha_n(\tau))d_n(\tau)+\beta_n(\tau),$$

where $\alpha_n(\tau):=2\int_{n\tau}^{(n+1)\tau} l(s)\,ds$,

$$\beta:=\max\{4;2\sup_{z(\,\cdot\,)\in N_{t_0}}\ \sup_{t\in[t_0,T]} d_{P(t)}(x(t;t_0,x_0,\alpha(z(\,\cdot\,)),z(\,\cdot\,)))\}$$

(note that $\beta<+\infty$ because f is integrably bounded and $P(\,\cdot\,)$ is absolutely continuous) and

$$\beta_n(\tau):=\beta\bigg[\bigg(\int_{n\tau}^{(n+1)\tau}\mu(s)\,ds\bigg)^2+\int_{n\tau}^{(n+1)\tau} l(s)\int_{n\tau}^s \mu(\sigma)\,d\sigma\,ds\bigg].$$

To prove that the sequence constructed step-by-step satisfies the conclusion of Theorem 4.3.2, it is sufficient to apply the following lemma:

Lemma 4.3.5. *Let d_n be the sequence defined previously. For any $\varepsilon > 0$, there is $\tau_0 > 0$ such that, if $0 < \tau < \tau_0$, then*

$$\forall n \leq (T+\tau)/\tau, \quad d_n(\tau) \leq \varepsilon.$$

Proof. Fix $\varepsilon > 0$. To simplify the notations, we shall write d_i instead of $d_i(\tau)$, α_i instead of $\alpha_i(\tau)$, etc.

It is easy to prove by induction that

$$d_{n+1} = \sum_{i=0}^{n} \left(\prod_{j=i}^{n-1} (1+\alpha_j) \right) \beta_i$$

(where, by convention, $\prod_{j=n}^{n-1}(1+\alpha_j) = 1$). Note that

$$\left(\prod_{j=0}^{n} (1+\alpha_j) \right) = \exp\left[\sum_{j=0}^{n} \ln(1+\alpha_j) \right] \leq \exp\left[\sum_{j=0}^{n} \alpha_j \right] \leq \exp[2\|l(\,\cdot\,)\|_1]$$

from the very definition of α_i. So,

$$d_{n+1} \leq \exp[2\|l(\,\cdot\,)\|_1] \sum_{i=0}^{n} \beta_i.$$

Set $\varepsilon_0 := \varepsilon/(\beta e^{2\|l\|_1}\|l+\mu\|_1)$. Choose now τ small enough (say $\tau < \tau_0$) in such a way that $\int_{i\tau}^{(i+1)\tau} l(s)\,ds \leq \varepsilon_0$ and $\int_{i\tau}^{(i+1)\tau} \mu(s)\,ds \leq \varepsilon_0$ for any i such that $i\tau \leq T$. Then, for any $n \leq T/\tau$,

$$\sum_{i=0}^{n} \beta_i \leq \beta\varepsilon_0 \sum_{i=0}^{n} \int_{i\tau}^{(i+1)\tau} (\mu(s)+l(s))\,ds \leq \beta\varepsilon_0\|\mu+l\|_1$$

so that $d_{n+1} \leq \beta\varepsilon_0\|\mu+l\|_1 e^{2\|l\|_1} \leq \varepsilon$. □

Remark. If $Z = \{z_0\}$ then the differential game reduces to the control system with dynamics given by $\widehat{f}(t,x,y) = f(t,x,y,z_0)$. Assume moreover, that $\{f(t,x,y,z_0) : y \in Y\}$ is convex for every t and x. Then leadership tube condition (4.12) implies that

$$\forall y \in Y, \quad (1, f(t,x,y,z_0)) \in \overline{\mathrm{co}}(T_{\mathrm{Graph}(P)}(t,x))$$

and discriminating tube condition (4.3) implies that

$$\exists y \in Y, \quad (1, f(t,x,y,z_0)) \in \overline{\mathrm{co}}(T_{\mathrm{Graph}(P)}(t,x)).$$

4.4. Value function and Isaacs equations

In the section we mostly adopt the notations from Introduction. We shall study regularity of upper value U^+ and lower value U^- of differential game with dynamics f satisfying (4.4)–(4.7). Under additional assumptions about convexity of the sets $f(t,x,Y,z)$, $f(t,x,y,Z)$ we obtain the characterization of upper and lower value as a unique solutions of corresponding Hamilton–Jacobi–Isaacs' equations. As a consequence we obtain the existence of value under Isaacs' condition (4).

We shall formulate properties of value functions for upper value. The analogous results for lower value can be obtained by simple change of notations.

We start with dynamic programming property. The result come from [38]. It was formulated there for more regular f. Under our assumptions the same arguments can be used.

Theorem 4.4.1 ([38, Theorem 3.1])**.** *For each $0 \le t < t+h \le T$ and $x \in \mathbb{R}^n$*

$$U^+(t,x) = \sup_{\alpha\in\Gamma_t} \inf_{z\in N_t} U^+(t+h, x(t+h,t,x,\alpha(z),z)).$$

Next, we examine regularity of the value function. We recall that the modulus of continuity $m_{f,A}(\delta)$ of a function $f: X \to Y$ (X, Y are metric spaces) on a subset $A \subset X$ is given by

$$m_{f,A}(\delta) = \sup\{d(f(x_1), f(x_2)) : x_1, x_2 \in A,\ d(x_1,x_2) \le \delta\}$$

for $\delta > 0$. It is easy to check that f is uniformly continuous on A if and only if $\lim_{\delta\to 0^+} m_{f,A}(\delta) = 0$. Moreover, $m_{f,A}(\,\cdot\,)$ is nondecreasing and if $A \subset B \subset X$ then $m_{f,A}(\delta) \le m_{f,B}(\delta)$.

Proposition 4.4.2. *If f satisfies* (4.4)–(4.7) *and $g: \mathbb{R}^n \to \mathbb{R}$ is continuous then we have*

$$m_{U^+(t_0,\,\cdot\,),\ B(0,R)}(\delta) \le m_{g,\ B\left(0,R+\int_{t_0}^T \mu(s)\,ds\right)}\left(\delta \exp\left(\int_{t_0}^T l(s)\,ds\right)\right).$$

Proof. Fix $t_0 \in [0,T]$, α and z. By (4.5) and the Gronwall inequality, we have

$$\|x(T,t_0,x_1,\alpha(z),z) - x(T,t_0,x_2,\alpha(z),z)\| \le \exp\left(\int_{t_0}^T l(s)\,ds\right)\|x_1 - x_2\|.$$

By (4.7), we obtain

$$\|x(T,t_0,x_0,\alpha(z),z) - x_0\| \le \int_{t_0}^T \mu(s)\,ds.$$

Using the above estimations the proof is straightforward. $\square$

Corollary 4.4.3. *For every $t_0 \in [0,T]$ the function $U^+(t_0,\,\cdot\,)$ is continuous.*

We define the tubes $E, H\colon [0,T] \rightsquigarrow \mathbb{R}^n$ by $E(t) = \{(x,u) : u \geq U^+(t,x)\}$, $H(t) = \{(x,u) : u \leq U^+(t,x)\}$. We call E the epitube and H the hypotube generated by upper value $U+$. Obviously $\mathrm{Graph}(H) = \mathrm{Hyp}(U^+)$ and $\mathrm{Graph}(E) = \mathcal{E}pi(U^+)$.

Proposition 4.4.4. *If f satisfies* (4.7) *then the epitube E and the hypotube H generated by the upper value U^+ are left absolutely continuous, namely, for $t_1 < t_2$,*

$$E(t_1) \subset E(t_2) + \left(\int_{t_1}^{t_2} \mu(s)\right) B\,ds, \qquad H(t_1) \subset H(t_2) + \left(\int_{t_1}^{t_2} \mu(s)\right) B\,ds.$$

Proof. Fix $x \in \mathbb{R}^n$, $0 \leq t_1 < t_2 \leq T$. By Theorem 4.4.1,

$$U^+(t_1,x) = \sup_{\alpha\in\Gamma_{t_1}} \inf_{z\in N_{t_1}} U^+(t_2, x(t_2,t_1,x,\alpha(z),z)).$$

Take $\varepsilon > 0$. There is $\alpha_0 \in \Gamma_{t_1}$ such that

$$U^+(t_1,x) - \varepsilon < \inf_{z\in N_{t_1}} U(t_2, x(t_2,t_1,x,\alpha_0(z),z)) \leq U^+(t_1,x).$$

Next, there is $z_0 \in N_{t_1}$ such that

$$U^+(t_1,x) - \varepsilon < U^+(t_2, x(t_2,t_1,x,\alpha_0(z_0),z_0)) < U^+(t_1,x) + \varepsilon. \tag{4.16}$$

We set $x(\,\cdot\,) = x(\,\cdot\,,t_1,x,\alpha_0(z_0),z_0)$. If $u \geq U^+(t_1,x)$, then $U^+(t_2,x(t_2)) < u+\varepsilon$. Thus $(x(t_2), u+\varepsilon) \in E(t_2)$. Therefore

$$\mathrm{dist}((x,u),E(t_2)) \leq (\|x(t_2)-x\|^2 + \varepsilon^2)^{1/2}.$$

Hence

$$E(t_1) \subset E(t_2) + B\left(\int_{t_1}^{t_2} \mu(s)\,ds\right).$$

Now, let $(x,u) \in H(t_1)$. By (4.16), $u - \varepsilon < U^+(t_2,x(t_2))$. Thus $(x(t_2), u-\varepsilon) \in H(t_2)$. Therefore

$$\mathrm{dist}((x,u),H(t_2)) \leq (\|x(t_2)-x\|^2 + \varepsilon^2)^{1/2},$$

which completes the proof. □

Proposition 4.4.5. *If $U^+\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is an upper value then for each $t_0 \in [0,T]$ and $x_0 \in \mathbb{R}^n$*

$$\forall\varepsilon > 0,\ \exists\alpha \in \Gamma_{t_0},\ \forall z \in N_{t_0},\ \forall t \in [t_0,T], \quad U^+(t_0,x_0) \leq U^+(t, x(t;t_0,x_0,\alpha(z),z)) + \varepsilon, \tag{4.17}$$

$$\forall\varepsilon > 0,\ \forall\alpha \in \Gamma_{t_0},\ \exists z \in N_{t_0},\ \forall t \in [t_0,T], \quad U^+(t_0,x_0) \geq U^+(t, x(t;t_0,x_0,\alpha(z),z)) - \varepsilon. \tag{4.18}$$

Proof. Fix $t_0 \in [0,T]$, $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. First, we prove that if U^+ is the upper value then (4.17) holds true. By the definition of the value function, there exists an $\alpha_\varepsilon \in \Gamma_{t_0}$ such that

$$U^+(t_0,x_0) \le \inf_{z\in N_{t_0}} g(x(T;t_0,x_0,\alpha(z),z)) + \frac{\varepsilon}{2}.$$

We show that (4.17) holds true for α_ε. To the contrary assume that there are $z_0 \in N_{t_0}$ and $t_1 \in [0,T]$ such that

$$U^+(t_0,x_0) > U^+(t_1,x_1) + \varepsilon,$$

where $x_1 = x(t_1;t_0,x_0,\alpha_\varepsilon(z_0),z_0)$. Given $z_1 \in N_{t_1}$ we set

$$(z_0,z_1)(s) = \begin{cases} z_0(s) & \text{for } s \in [t_0,t_1), \\ z_1(s) & \text{for } s \in [t_1,T]. \end{cases}$$

Let $\alpha_1 \in \Gamma_{t_1}$ be given by

$$\alpha_1(z_1)(s) = \alpha_\varepsilon(z_0,z_1)(s) \tag{4.19}$$

for $s \in [t_1,T]$. Obviously, $U^+(t_1,x_1) \ge \inf_{z_1\in N_{t_1}} g(x(T;t_1,x_1,\alpha_1(z_1),z_1)$ and hence there is $z_1 \in N_{t_1}$ such that

$$\inf_{z\in N_{t_1}} g(x(T;t_1,x_1,\alpha_1(z),z) > g(x(T;t_1,x_1,\alpha_1(z_1),z_1)) - \frac{\varepsilon}{2}.$$

Setting $\widetilde{z} = (z_0,z_1)$ we have

$$x(T;t_0,x_0,\alpha_\varepsilon(\widetilde{z}),\widetilde{z}) = x(T;t_1,x_1,\alpha_1(z_1),z_1).$$

Thus

$$U^+(t_0,x_0) > U^+(t_1,x_1) + \varepsilon > g(x(T;t_0,x_0,\alpha_\varepsilon(\widetilde{z}),\widetilde{z}) - \frac{\varepsilon}{2} + \varepsilon,$$

which is the desired contradiction.

Fix $\alpha \in \Gamma_{t_0}$. We divide the proof of (4.18) into two steps.

Step 1. We fix a division $t_0 < \ldots < t_k = T$ of the interval $[t_0,T]$. By the dynamic programming property (Theorem 4.4.1), there is $z_0 \in N_{t_0}$ such that

$$U^+(t_0,x_0) > U^+(t_1,x_1) - \frac{\varepsilon}{2k}$$

where $x_1 = x(t_1;t_0,x_0,\alpha(z_0),z_0)$. Taking $\alpha_0 = \alpha$ in (4.19) we obtain an $\alpha_1 \in \Gamma_{t_1}$. By the dynamic programming property again, we obtain $z_1 \in N_{t_1}$ such that

$$U^+(t_1,x_1) > U^+(t_2,x_2) - \frac{\varepsilon}{2k}$$

where $x_1 = x(t_2;t_1,x_1,\alpha_1(z_1),z_1)$.

We proceed by induction getting a sequence $z_2 \in N_{t_2},\ldots,z_{k-1} \in N_{t_{k-1}}$. Setting $\widetilde{z}(s) = z_i(s)$, for $s \in [t_{i-1},t_i)$, we obtain

$$U^+(t_0,x_0) \ge U^+(t_i,x(t_i;t_0,x_0,\alpha(\widetilde{z}),\widetilde{z})) - \frac{i\varepsilon}{2k}.$$

Step 2. We set $R = \|x_0\| + 1$ and find $\delta > 0$ such that

$$m_{g,B\left(R+\int_{t_0}^{T}\mu(s)\,ds\right)}\left(\delta\int_{t_0}^{T} l(s)\,ds\right) < \frac{\varepsilon}{2}.$$

Next, we choose a division $t_0 < \ldots < t_k = T$ of the interval $[t_0, T]$ such that $\int_{t_{i-1}}^{t_i}\mu(s)\,ds < \delta/2$, for $i = 1, \ldots, k$. By Step 1, we find $\widetilde{z} \in N_{t_0}$ such that $U^+(t_0, x_0) > U^+(t_i, \widetilde{x}(t_i)) - \frac{\varepsilon}{2}$, where $\widetilde{x}(t) = x(t; t_0, x_0, \alpha(\widetilde{z}), \widetilde{z})$. Fix $t \in [t_{i-1}, t_i]$. By the dynamic programming property

$$U^+(t, \widetilde{x}(t)) \in \left[\inf\left\{U^+(t_i, y) : \|y - \widetilde{x}(t)\| \le \int_t^{t_i}\mu(s)\,ds\right\},\right.$$
$$\left.\sup\left\{U^+(t_i, y) : \|y - \widetilde{x}(t)\| \le \int_t^{t_i}\mu(s)\,ds\right\}\right] (:= J).$$

Since $\|\widetilde{x}(t_i - \widetilde{x}(t)\| \le \int_t^{t_i}\mu(s)\,ds$ then also $U^+(t_i, \widetilde{x}(t_i)) \in J$. Thus

$$\begin{aligned}&\|U^+(t, \widetilde{x}(t)) - U^+(t_i, \widetilde{x}(t_i)\|\\ &\le \sup\left\{\|U^+(t_1, y_1) - U^+(t_1, y_2)\| : y_1, y_2 \in B\left(\widetilde{x}(t_0), \int_t^{t_i}\mu(s)\,ds\right)\right\} \le \varepsilon/2,\end{aligned}$$

which completes the proof. □

Theorem. *Suppose that* $g\colon \mathbb{R}^n \to \mathbb{R}$ *is continuous and* $f\colon [0,T] \times \mathbb{R}^n \times Y \times Z \to \mathbb{R}^n$ *satisfies* (4.4)–(4.8). *Let the Hamiltonian* $H^+\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *and the value function* $U^+\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *be generated by* f, g. *Then a function* $W\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is equal to the uper value, i.e.* $W = U^+$, *if and only if* W *satisfies the following conditions:*

(a) $W(T, \cdot\,) = g(\,\cdot\,)$*;*
(b) $W(t, \cdot\,)$ *is a continuous function, for every* $t \in [0, T]$*;*
(c) *the epitube* E_W *and the hypotube* H_W *are left absolutely continuous, where* $E_W(t) = \{(x, w) \in \mathbb{R}^n \times \mathbb{R} : w \ge W(t, x)\}$ *and* $H_W(t) = \{(x, w) \in \mathbb{R}^n \times \mathbb{R} : w \le W(t, x)\}$*;*
(d) *there exists a full measure set* $C \subset [0, T]$ *such that for every* $t \in C$ *and* $x \in \mathbb{R}^n$

$$\forall (n_t, n_x, n_u) \in N^0_{\mathrm{Graph}(H_W)}(t, x, W(t, x)), \quad -n_t + H^+(t, x, -n_x) \ge 0, \tag{4.20}$$

$$\forall (n_t, n_x, n_u) \in N^0_{\mathrm{Graph}(E_W)}(t, x, W(t, x)), \qquad n_t + H^+(t, x, n_x) \le 0. \tag{4.21}$$

Remark. Note that, if $W = U^+$ is smooth, then equations (4.20) and (4.21) mean nothing but that W satisfies the Hamilton–Jacobi–Isaacs equation:

$$\frac{\partial W}{\partial t}(t, x) + H\left(t, x, \frac{\partial W}{\partial x}(t, x)\right) = 0.$$

Proof of Theorem 4.4.6. Suppose that $W = U^+$. Corollary 4.4.3 and Proposition 4.4.4 yield (b) and (c). Let $\widetilde{f}(t,x,u,y,z) = (f(t,x,y,z),0)$, $u \in R$. The function $(x(t),u(t)) = (x(t;t_0,x_0,\alpha(z),z),U^+(t_0,x_0))$ is the solution of the Cauchy problem

$$\begin{cases} (x'(t),u'(t)) = \widetilde{f}(t,x(t),u(t),\alpha(z),z), \\ (x(t_0),u(t_0)) = (x_0,U(t_0,x_0)). \end{cases}$$

From (4.17) it follows that (4.10) holds true for $P = H_W$ and $f = \widetilde{f}$. By Theorem 4.2.2, the hypotube H_W is a discriminating tube for $\widetilde{f}$. From this we conclude (4.20).

From (4.18) it follows that (4.13) holds true for $P = E_W$ and $f = \widetilde{f}$. By Theorem 4.3.2, the epitube E_W is a leadership tube for $\widetilde{f}$. From this we conclude (4.21).

Now, suppose that a function W satisfies (a)–(d). From (4.20) it follows that the hypotube H_W is a discriminating tube for $\widetilde{f}$. Fix t_0 and x_0. By Theorem 4.2.2, there is $\alpha \in \Gamma_{t_0}$ such that, for every $z \in N_{t_0}$,

$$(x(T;t_0,x_0,\alpha(z),z),W(t_0,x_0)) \in H_W(T).$$

Hence

$$\forall\alpha,\ \exists z, \quad W(t_0,x_0) \le W(T,x(T;t_0,x_0,\alpha(z),z)).$$

Thus

$$W(t_0,x_0) \le \sup_{\alpha}\inf_{z} g(x(T;t_0,x_0,\alpha(z),z)).$$

From (4.21) it follows that the epitube E_W is a leadership tube for $\widetilde{f}$. Fix t_0, x_0. By Theorem 4.3.2, for every $\varepsilon > 0$ and every $\alpha \in \Gamma_{t_0}$, there is $z \in N_{t_0}$

$$(x(T;t_0,x_0,\alpha(z),z),W(t_0,x_0)) \in E_W(T) + B(0,\varepsilon).$$

Since g is uniformly continuous on $B(x_0,\int_{t_0}^T \mu(s)ds)$, we have

$$W(t_0,x_0) \ge \sup_{\alpha}\inf_{z} g(x(T;t_0,x_0,\alpha(z),z)),$$

which completes the proof. □

Let us observe that Theorem 4.4.6 is not only the characterization of the upper (or lower) value. Since the statement of the theorem is "if and only if" so we obtained also uniqueness of weak solution of the corresponding Isaacs equation. By a weak solution we mean a function satisfying conditions (b)–(d) in Theorem 4.4.6. If the upper Hamiltonian H^+ is equal to the lower one H^- then condition (d) for the upper value and the lower value coincide. So we obtain the following:

Corollary 4.4.7. *Suppose that g, f satisfy assumptions of Theorem* 4.4.6 *and moreover,*

$$\forall (t,x,y) \in [0,T] \times \mathbb{R}^n \times Y, \quad \{f(t,x,y,z) : z \in Z\} \text{ is convex.}$$

If the Isaacs condition (4) *holds true for almost all t then for the game there exist a value, i.e.* $U^+ = U^-$.

CHAPTER 5

DIFFERENTIAL GAMES WITH DISCONTINUOUS TERMINAL COST

In this chapter we use the notation introduced in Chapter 4. We consider a terminal payoff function given by a function $g\colon \mathbb{R}^n \to \mathbb{R}$. We consider the case where g is lower semicontinuous. It causes the necessity of modifying definitions of upper and lower value functions.

$$\left\{ \begin{array}{l} V_g^+(t_0,x_0) := \sup_{\alpha\in\Gamma_{t_0}} \inf\{g(x) : x \in \mathrm{cl}(A_\alpha(t_0,x_0))\}, \\ V_g^-(t_0,x_0) := \inf_{\beta\in\Delta_{t_0}} \sup\{g(x) : x \in \mathrm{cl}(B_\beta(t_0,x_0))\}, \end{array} \right. \tag{5.1}$$

where cl means closure and $A_\alpha(t_0,x_0) = \{x(T;t_0,x_0,\alpha(z),z) : v \in N_{t_0}\}$, $B_\beta(t_0,x_0) = \{x(T;t_0,x_0,y,\beta(y)) : y \in M_{t_0}\}$ denote the reachable sets. Let us notice that when g is continuous, we can skip the closure in the definition (5.1) of value-functions. We provide an example with a discontinuous g showing that the two value functions V_g^+ and V_g^- are not equal when we do not take the closure in the definition (5.1).

We assume that $f\colon [0,T]\times\mathbb{R}^n\times Y\times Z \to \mathbb{R}^n$ satisfies

$$\left\{ \begin{array}{l} \bullet\ f(\,\cdot\,,\,\cdot\,,y,z) \text{ is Lipschitz continuous,} \\ \bullet\ f(t,x,\,\cdot\,,\,\cdot\,) \text{ is continuous,} \\ \bullet\ f \text{ has a linear growth, i.e.} \\ \quad \sup_{(t,u,v)} \|f(t,x,y,z)\| \le a(1+\|x\|) \\ \quad \text{for some given } a>0. \end{array} \right. \tag{5.2}$$

Throughout the chapter, we assume that

$$f(t,x,Y,z) \text{ is convex for every } t,\ x,\ z, \tag{5.3}$$

$$f(t,x,y,Z) \text{ is convex for every } t,\ x,\ y, \tag{5.4}$$

hold true.

5.1. Comparison result

For readers convenience we recall the Evans–Souganidis result [38, Theorem 4.1] stating that if the terminal cost is Lipschitz continuous then the upper value equals to the lower value.

Theorem 5.1.1. *If $f:[0,T]\times\mathbb{R}^n\times Y\times Z\to\mathbb{R}^n$ satisfies* (5.2), (4) *and $g:\mathbb{R}^n\to\mathbb{R}$ is Lipschitz continuous then the game has a value, i.e. $V_g^-=V_g^+$.*

Proposition 5.1.2. *If* (5.2), (4) *hold true and a terminal cost function g is locally bounded then*

$$V_g^+(t,x)\le V_g^-(t,x)\quad \textit{for every } (t,x)\in[0,T]\times\mathbb{R}^n.$$

The proof is a direct conclusion from the following Lemma.

Lemma 5.1.3. *Assume that* (5.2), (4) *hold true. Then*

$$\mathrm{cl}(A_\alpha(t_0,x_0))\cap\mathrm{cl}(B_\beta(t_0,x_0))\neq\emptyset$$

for each $\alpha\in\Gamma_{t_0}$, $\beta\in\Delta_{t_0}$.

Proof. Suppose, to the contrary, that there exist α_0, β_0 such that

$$\mathrm{cl}(A_{\alpha_0}(t_0,x_0))\cap\mathrm{cl}(B_{\beta_0}(t_0,x_0))=\emptyset.$$

Then, there exists a Lipschitz continuous function $h:\mathbb{R}^n\to[0,1]$ such that $h(x)=0$ for $x\in\mathrm{cl}(A_{\alpha_0})$ and $h(x)=1$ for $x\in\mathrm{cl}(B_{\beta_0})$. Hence,

$$V_h^-(t_0,x_0)=0<1=V_h^+(t_0,x_0).$$

This is a contradiction with Theorem 5.1.1. □

Now, we prove that any supersolution of Isaacs' Equation is greater then the corresponding lower value and that any subsolutions is smaller then the upper value.

Proposition 5.1.4. *Assume that* (5.2), (5.3) *hold true and suppose that $\psi:(0,T]\times\mathbb{R}^n\to\mathbb{R}$ is lower semicontinuous and is a supersolution of*

$$\psi_t+H^-(t,x,\psi_x)=0 \tag{5.5}$$

on $(0,T)\times\mathbb{R}^n$, when

$$H^-(t,x,p)=\max_{y\in Y}\min_{z\in Z}\langle f(t,x,y,z),p\rangle.$$

Then for every $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$ there exists a non-anticipative strategy $\beta\in\Delta_{t_0}$ such that, for every $u\in M_{t_0}$ and $t\in[t_0,T]$,

$$\psi(t_0,x_0)\ge\psi(t,x(t;t_0,x_0,u,\beta(u))). \tag{5.6}$$

Proof. By Proposition 1.6.1, for every $(t, x) \in (0, T) \times \mathbb{R}^n$ we have

$$\forall (n_t, n_x, n_u) \in N^0_{\mathcal{E}pi(\psi)}(t, x, \psi(t, x)) \quad n_t + \max_y \min_z \langle f(t, x, y, z), n_x \rangle \leq 0.$$

Thus, if (n_t, n_x, n_u) is a normal to the epigraph of ψ then

$$\forall y \; \exists z \quad \langle (1, f(t, x, y, z), 0), (n_t, n_x, n_u) \rangle \leq 0.$$

By Proposition 4.2.3 (players change their role), for every $(t_0, x_0) \in (0, T] \times \mathbb{R}^n$ there exists $\beta \in \Delta_{t_0}$ such that for every $y \in M_{t_0}$

$$(t, x(t; t_0, x_0, y, \beta(y)), \psi(t_0, x_0)) \in \mathcal{E}pi(\psi) \quad \text{for } t \in [t_0, T]. \qquad \square$$

Corollary 5.1.5. *Under the assumptions of Proposition* 5.1.4 *we obtain*

$$\psi(t, x) \geq V^-_g(t, x) \quad \textit{where } g(x) := \psi(T, x).$$

Proof. Since g is lower semicontinuous then for every subset $B \subset \mathbb{R}^n$ we have $\sup\{g(x) : x \in B\} = \sup\{g(x) : x \in \mathrm{cl}(B)\}$. Thus

$$V^-_g(t_0, x_0) = \inf_{\beta \in \Delta_{t_0}} \sup\{g(x) : x \in B_\beta(t_0, x_0)\}.$$

By Proposition 5.1.4 we obtain

$$\psi(t_0, x_0) \geq \inf_{\beta \in \Delta_{t_0}} \sup\{g(x) : x \in B_\beta(t_0, x_0)\},$$

which gives us the desired inequality. $\square$

Proposition 5.1.6. *Assume that* (5.2), (5.4) *hold true,* $\phi\colon (0, T] \times \mathbb{R}^n \to \mathbb{R}$ *is upper semicontinuous and is a subsolution of*

$$\phi_t + H^+(t, x, \phi_x) = 0 \quad \textit{on } (0, T) \times \mathbb{R}^n,$$

when

$$H^+(t, x, p) = \min_{z \in Z} \max_{y \in V} \langle f(t, x, y, z), p \rangle.$$

Then for every $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$ *there exists a non-anticipative strategy* $\alpha \in \Gamma_{t_0}$ *such that*

$$\phi(t_0, x_0) \leq \phi(t, x(t;\, t_0, x_0, \alpha(z), z))$$

for every $v \in N_{t_0}$ *and* $t \in [t_0, T]$.

The proof can be done using the same method as in the proof of Proposition 5.1.4.

Corollary 5.1.7. *Under the assumptions of Proposition* 5.1.6 *we obtain*

$$\phi(t,x) \le V_h^+(t,x) \quad \text{where } h(x) := \phi(T,x).$$

The proof is similar to the proof of Corollary 5.1.5.

If Isaacs' condition (4) holds true, then $H^- = H^+ (=: H)$ and previous results can be summarized in the following comparizon result (cf. Théorème d'unicité forte 4.10 in [10])

Proposition 5.1.8 (Comparison result)**.** *Assume that* (5.2)–(5.4)*,* (4) *hold true. Suppose that* $\psi: (0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is lower semicontinuous and is a supersolution of*

$$\psi_t + H(t,x,\psi_x) = 0 \tag{5.7}$$

on $(0,T) \times \mathbb{R}^n$ *and* $\phi: (0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is upper semicontinuous and is a subsolution of* (5.7) *on* $(0,T) \times \mathbb{R}^n$*. If* $\psi(T,x) \ge \phi(T,x)$*, for* $x \in \mathbb{R}^n$*, then* $\psi(t,x) \ge \phi(t,x)$*, for* $t \in (0,T]$ *and* $x \in \mathbb{R}^n$*.*

Proof. By Proposition 5.1.2, Corollaries 5.1.5, 5.1.7, we have

$$\phi(t,x) \le V_h^+(t,x) \le V_h^-(t,x) \le V_g^-(t,x) \le \psi(t,x)$$

where $h(x) = \phi(T,x)$ and $g(x) = \psi(T,x)$ for $x \in \mathbb{R}^n$. □

5.2. Existence of value

In the section we prove the existence of value and characterize it as a generalized solution of Isaacs' equation.

If the terminal cost g is discontinuous then so is the value-function. To describe the value function as a unique solution of the corresponding Hamilton–Jacobi equation we introduce the following definition.

Definition 5.2.1. Let $H: [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian. The function $(t,x) \mapsto u(t,x)$ is a *generalized solution* of the following Hamilton–Jacobi equation with terminal condition

$$\begin{cases} \dfrac{\partial u}{\partial t} + H\left(t,x,\dfrac{\partial u}{\partial x}\right) = 0, \\ u(T,x) = g(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{5.8}$$

if and only if

$$\begin{cases} \text{(a) } u \text{ is the supremum on the set of subsolutions } \phi \\ \qquad \text{such that } \phi(T,x) \le g(x), \text{ for all } x \in \mathbb{R}^n, \\ \text{(b) } u \text{ is the infimum on the set of supersolutions } \psi \\ \qquad \text{such that } \psi(T,x) \ge g(x), \text{ for all } x \in \mathbb{R}^n. \end{cases} \tag{5.9}$$

The above meaning of solution is similar to the envelope solution introduced in [12].

Theorem 5.2.2. *Assume that* (5.2)–(5.4) *and* (4) *hold true and* $g\colon \mathbb{R}^n \to \mathbb{R}$ *is a bounded from below lower semicontinuous function. Then the game has a value, i.e.*

$$V_g^+ = V_g^-(=: V).$$

The value function V *is the smallest supersolution of the Hamilton–Jacobi–Isaacs equation*

$$V_t + H(t, x, V_x) = 0 \tag{5.10}$$

satisfying $V(T,x) \geq g(x)$*, when* $H := H^+ = H^-$*. Moreover, the value function* V *is the unique generalized solution* (*Definition* 5.2.1) *of* (5.10) *satisfying* $V(T,x) = g(x)$.

We have stated the result in lower semicontinuous case. After typical reformulation it remains valid in upper semicontinuous case.

Proof of Theorem 5.2.2. We define a sequence $g_n\colon \mathbb{R}^n \to \mathbb{R}$ by

$$g_n(x) = \inf_{y\in\mathbb{R}^n} g(y) + n\|x - y\|.$$

The inf-convolutions g_n are Lipschitz continuous, $g_n(x) \leq g_{n+1}(x)$ and

$$\lim_n g_n(x) = g(x) \quad \text{for every } x \in \mathbb{R}^n.$$

Using Theorem 5.1.1, we have $V_{g_n}^+ = V_{g_n}^-(:= V_n)$ and V_n is a viscosity solution (i.e. super- and subsolution) to (5.10). Denote $W(t,x) = \lim_n V_n(t,x)$. By Lemma 1.6.2, W is a supersolution of (5.10). By Corollary 5.1.5, we obtain $W \geq V_g^-$. Since $V_g^+ \geq V_{g_n}^+$, we deduce $V_g^+ \geq W$. Hence $V_g^+ \geq V_g^-$. Combining it with Proposition 5.1.2, we obtain $V_g^+ = V_g^- = W$.

If $\psi\colon (0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a supersolution of (5.10) and $\psi(T,x) \geq g(x)$, then $\psi \geq V_g^-$. Thus V is the smallest supersolution of (5.10) satisfying $V(T,x) \geq g(x)$. Since V_n is a subsolution of (5.10), $V_n(T,x) \leq g(x)$ and $V = \lim_n V_n$, we obtain that V is a generalized solution of (5.10), $V(T,\,\cdot\,) = g(\,\cdot\,)$. $\square$

Remark. Due to general properties of monotone approximation V is also a solution of (5.10) in the Ishii sense. Namely, upper semicontinuous envelope of V coincides with the upper weak limit of V_n (cf. exercise in [10, p. 91]), which by Theorem 4.1 in [10] is a subsolution of (5.10).

The following example with a slight modification is taken from [9]. It served in [9] as a counter-example to uniqueness of discontinuous solution – in the Ishii sense – to a Hamilton–Jacobi's equation. Definition 5.2.1 is not equivalent to the notion of solution introduced by Ishii. In the example there exists a unique solution in the meaning of Definition 5.2.1 and there are several solutions in the Ishii sense [9].

Example. Let $U = V = [-1, 1]$. We define $f: (-\infty, 0] \times \mathbb{R} \times U \times V \to \mathbb{R}$ by

$$f(t, x, u, v) = \chi_{(x \le t)}(x - t)v + \chi_{(x \ge t)}(x - t)u.$$

It is easy to check that f satisfies (5.2)–(5.4) and the corresponding Hamiltonian is given by

$$H(t, x, p) = (x - t)|p|.$$

To define a terminal cost function $g: \mathbb{R} \to \mathbb{R}$, we fix $t_0 = x_0 < 0$. Let $b = x(0; t_0, x_0, u_1, v)$, $a = x(1; t_0, x_0, u_{-1}, v)$ where $u_1(t) = 1$, $u_{-1}(t) = -1$ for $t \in [t_0, 0]$, v is an arbitrary control. We define

$$g(x) = \begin{cases} 1 & \text{if } x \in (a, b), \\ -1 & \text{elsewhere.} \end{cases}$$

We set the terminal time T to be zero. By Theorem 5.2.2, the value V for this game exists and is the *unique* solution of the corresponding Hamilton–Jacobi equation:

$$\begin{cases} V_t + (x - t)|V_x| = 0, & \\ V(0, x) = g(x) & \text{for every } x \in \mathbb{R}. \end{cases}$$

Remark. The assumptions (5.3) and (5.4) concerning the convexity of the right-hand side are crucial for obtaining $V_g^+ \ge V_g^-$ because we used a viability approach which requires convexity. We recall that thanks to Proposition 5.1.2, inequality $V_g^+ \le V_g^-$ holds true.

5.3. On the definition of the values of the game

In the definition of upper and lower values (5.1) we have used the closure of reachable sets. They can be defined as well without closure

$$\begin{cases} U_g^+(t_0, x_0) := \sup_{\alpha \in \Gamma_{t_0}} \inf\{g(x) : x \in A_\alpha(t_0, x_0)\}, \\ U_g^-(t_0, x_0) := \inf_{\beta \in \Delta_{t_0}} \sup\{g(x) : x \in B_\beta(t_0, x_0)\}. \end{cases}$$

We shall exhibit an example where $U_g^- \ne U_g^+$.

Example. We provide an example of a differential game where $U_g^- > U_g^+$. For doing this we construct a pair of non-anticipative strategies (α, β) such that

$$A_\alpha \cap B_\beta = \emptyset.$$

(Let us notice that this implies that neither A_α nor B_β are closed by Proposition 5.1.2.) We consider the following differential game on $\mathbb{R}^2$

$$\begin{cases} x'(t) = u, \\ y'(t) = v, \end{cases}$$

where $U = V = [0, 1]$. We set $x_0 = 0$, $t_0 = 0$ and $T = 1$. We denote by x_u (y_v) the solution of the Cauchy problem $x'(t) = u(t)$, $x(0) = 0$ (resp. $y'(t) = v(t)$, $y(0) = 0$). We define the constant controls $u_0(t) = v_0(t) = 0$ and $u_1(t) = v_1(t) = 1$ for

$t \in [0,1]$. For measurable functions $w, z: [0,1] \to [0,1]$ we define an (ultrametric) distance

$$\rho(w,z) = 1 - \max\{t \in [0,1] : w(s) = z(s) \text{ for a.e. } s \in [0,t]\}.$$

Set $B = \{u \in M_0 : \rho(u,u_0) < 1\}$ and $S = \{u \in M_0 : \rho(u,u_0) = 1\}$. Define two non-anticipative strategies α, β as follows

$$\alpha(v) = \begin{cases} u_0 & \text{if } v \in B, \\ u_1 & \text{if } v \in S, \end{cases} \qquad \beta(u) = \begin{cases} v_1 & \text{if } u \in B, \\ v_0 & \text{if } u \in S. \end{cases}$$

If $u \in S$ then $x_u(1) > 0$. If $p \in (0,1]$ then there exists a control $u \in S$ such that $x_u(1) = p$.

If $u \in B$ then $x_u(1) < 1$. If $p \in [0,1)$ then there exists a control $u \in B$ such that $x_u(1) = p$.

We have

$$\begin{aligned} A_\alpha &= \{(x_{\alpha(v)}(1), y_v(1)) : v \in B\} \cup \{(x_{\alpha(v)}(1), y_v(1)) :\ v \in S\} \\ &= \{0\} \times [0,1) \cup \{1\} \times (0,1], \\ B_\beta &= \{(x_u(1), y_{\beta(u)}(1)) : u \in B\} \cup \{(x_u(1), y_{\beta(u)}(1)) : u \in S\} \\ &= [0,1) \times \{1\} \cup (0,1] \times \{0\}. \end{aligned}$$

Setting $g = \chi_{B_\beta}$ we obtain $U_g^+(0,0) = 0 < 1 = U_g^-(0,0)$.

We did not succeed to find an example where g is semicontinuous. Hence the question to know if $U_g^- = U_g^+$ (so, it would be equal to $V_g^- = V_g^+$) for semicontinuous g remains an open problem.

CHAPTER 6

OLEINIK–LAX FORMULAS AND MULTITIME HAMILTON–JACOBI SYSTEMS

We obtain explicit formulas for semicontinuous solutions of the Hamilton–Jacobi equation (1.27) associated with an Hamiltonian of the following form:

$$H(t, u, p) = \inf_{f \in F(t,u)} \langle f, p \rangle + \lambda(u).$$

Such explicit formulas have been obtained first by Hopf, Lax and Oleinik. In this section we obtain an explicit representation formula of the value function which generalizes some result obtained recently by Barron–Jensen-Liu [16] and Alvarez–Barron–Ishii [1]. Our approach extends results of these authors firstly to Hamiltonian depending on time and seconly, and mainly, to the case where the Lagrangian is nonconstant on its domain (in [16] and [1] the Lagrangian is constant and equal to zero on its domain).

Next, we study the multitime Hamilton–Jacobi systems using properties of commutation of semigroups of flows. To our knowledge, this question was firstly addressed in [73] with Hamiltonians which depend only on p (see also in [11] an extension to cases where H depends also on x).

We investigate an "overdetermined" system of multitime Hamilton Jacobi's equations.

$$\begin{cases} \dfrac{\partial W}{\partial t} + H_1\left(W, \dfrac{\partial W}{\partial x}\right) = 0, \\ \dfrac{\partial W}{\partial s} + H_2\left(W, \dfrac{\partial W}{\partial x}\right) = 0, \\ W(x, 0, 0) = g(x), \end{cases}$$

which solution is a function $W\colon (t, s, x) \in]-\infty, 0]^2 \times \mathbb{R}^N \mapsto \mathbb{R}$. Following Lions–Rochet [73], we reduce the question of solving the above system to a property of commutation of semigroups of flows. We provide a new result for this commutation property, the proof of which is based on commutation of reachable maps of differential inclusions. We apply this result to the existence and uniqueness of the above Hamilton–Jacobi system.

6.1. Oleinik–Lax's like formulas

We consider Lagrangians

$$L_F^\lambda(t,u,v)=\begin{cases}\lambda(u) & \text{if } v\in F(t,u),\\ +\infty & \text{elsewhere,}\end{cases}\tag{6.1}$$

where furthermore

$$\begin{cases}\bullet\ \lambda\colon\mathbb{R}\to[0,+\infty) \text{ is locally Lipschitz continuous and nonincreasing,}\\ \bullet\ F\colon[0,T]\times\mathbb{R}\rightsquigarrow\mathbb{R}^n \text{ is a locally Lipschitzg continuous map}\\ \quad\ \text{with nonempty convex compact values,}\\ \bullet\ F(t,u_1)\subset F(t,u_2) \text{ if } u_1\le u_2.\end{cases}\tag{6.2}$$

Obviously, the Lagrangian L_F^λ stands for a special case of the class L_F given by (1.20). In particular, Theorem 1.7.2 holds true for the Hamiltonian $H(t,u,p)$ corresponding to L_F^λ

$$H(t,u,p)=\inf_{f\in F(t,u)}\langle f,p\rangle+\lambda(u).\tag{6.3}$$

If $\lambda\equiv 0$ then $H(t,u,p)$ is positively homogeneous with respect to p.

The Cauchy problem

$$\begin{cases}u'(t)=-\lambda(u(t)),\\ u(T)=g_0,\end{cases}\tag{6.4}$$

have a left extendable up to 0 solution. Indeed, we have $u(t)\ge g_0$, from which follows $-\lambda(g_0)\le-\lambda(u(t))\le 0$ and $|u'(t)|$ is bounded. We define an operator $\Lambda\colon[0,T]\times\mathbb{R}\to\mathbb{R}$ by

$$\Lambda(t,g_0)=u(t)$$

where $u(\,\cdot\,)$ is the solution of (6.4). Having Λ we can define a kind of reachable map $R\colon[0,T]\times\mathbb{R}\rightsquigarrow\mathbb{R}^n$

$$R(t,g_0)=\int_t^T F(s,\Lambda(s,g_0))\,ds.$$

Sets $R(t,g_0)$ are nonempty convex compact and $R(t,g_1)\subset R(t,g_2)$ for $g_1\le g_2$. We shall also use a map $P\colon\mathbb{R}^n\times[0,T]\to\mathbb{R}\cup\{-\infty,+\infty\}$ given by

$$P(y,t)=\inf\{g_0 : y\in R(t,g_0)\}.$$

In the above formula and in the following ones we use the convention that $\inf\emptyset=+\infty$.

Theorem 6.1.1. *The unique semicontinuous solution of* (1.27), (*where the Hamiltonian is given by* (6.3) *and* λ, F *satisfy* (6.2)) *is represented by*

$$V(t,x) = \Lambda(t, \inf_{y\in\mathbb{R}^n} \max[g(y), P(y-x,t)]). \tag{6.5}$$

Proof. By Theorem 1.7.2, the semicontinuous solution of (1.27) is the corresponding value function. In the considered case the value function V given by the formula (1.24) with L replaced by L_F^λ can be represented as follows

$$V(t_0,x_0) = \inf\{u(t_0) : u'(t) = -\lambda(u(t)),\\ u(T) \geq g(x(T)),\ x'(t) \in F(t,u(t)),\ x(t_0) = x_0\}.$$

Observe that

$$\{u(t_0) : u'(t) = -\lambda(u(t)),\ u(T) \geq g(x(T)),\ x'(t) \in F(t,u(t)),\ x(t_0) = x_0\}\\ = \{\Lambda(t,g_0) : \exists y \in \mathbb{R}^n\ y \in R(t,g_0) \text{ and } g_0 \geq g(x_0+y)\}.$$

The rest of the proof is reduced to the following lemma. □

Lemma 6.1.2. *Assume that* $R\colon \mathbb{R} \to Y$ *is a set valued map such that*

$$R(u_1) \subset R(u_2) \quad \textit{for } u_1 \leq u_2$$

and Y *is a nonempty set. Let* $g\colon Y \rightsquigarrow \mathbb{R}\cup\{+\infty\}$ *be an arbitrary function and* $\Lambda\colon \mathbb{R}\to\mathbb{R}$ *be a nondecreasing function, continuous from the right. Then*

$$\inf\{\Lambda(u) : \exists y,\ y\in R(u) \textit{ and } u \geq g(y)\} = \Lambda(\inf_{y\in Y} \max((g(y), P(y))$$

where $P(y) = \inf\{u : y \in R(u)\}$.

Proof. Denote

$$\alpha = \inf\{\Lambda(u) : \exists y,\ y\in R(u) \text{ and } u \geq g(u)\},\\ \beta = \Lambda(\inf_{y\in Y} \max((g(y), P(y)).$$

First we show that $\alpha \leq \beta$. Let $\beta < b \in \mathbb{R}$. There exists $y \in Y$ such that $\Lambda(\max(g(y), P(y)) < b$. Since $\Lambda(P(y)) < b$ then there exists $u_1 \in \mathbb{R}$ such that $\Lambda(u_1) < b$ and $y \in R(u_1)$. We set $u = \max(u_1, g(y))$. By monotonicity of R we have $y \in R(u)$. Thus $\alpha \leq \Lambda(u) < b$.

Now, we show that $\beta \leq \alpha$. Let $\alpha < a \in \mathbb{R}$. There are $u \in \mathbb{R}$, $y \in R(u)$ such that $\Lambda(u) < a$ and $u \geq g(y)$. Thus $\max(g(y), P(y)) \leq u$ and $\beta \leq \Lambda(u) < a$. □

Corollary 6.1.3. *If* $\lambda \equiv 0$ *then the representation formula* (6.5) *simplifies to the following*

$$V(t_0,x_0) = \inf_{y\in\mathbb{R}^n} \max(g(y), P(y-x_0,t_0)\}$$

where

$$P(y,t_0)=\inf\left\{u\in\mathbb{R}:y\in\int_{t_0}^{T}F(t,u)\,dt\right\}.$$

If $\lambda\equiv 0$ *and* F *does not depend on the time* $(F(t,u)=F(u))$ *then the representation formula simplifies even more to the following*

$$V(t_0,x_0)=\inf_{y\in\mathbb{R}^n}\max\left(g(y),P\left(\frac{y-x_0}{T-t_0}\right)\right)$$

where $P(y)=\inf\{u\in\mathbb{R}:y\in F(u)\}$.

6.2. Commutation of flows and multitime Hamilton–Jacobi systems

We consider the problem of commutation of semigroups generated by two Hamilton–Jacobi's equations and the problem of existence of solution of multitime Hamilton–Jacobi systems (cf. [73]). We reduce the problem of commutation of semigroups to the problem of commutation of set-valued reachable sets generated by corresponding control systems. We shall use the previous approach in the case of time intervall $]-\infty,0]$ ($T=0$ is the terminal time).

The Hamiltonians $H(u,p)$ satisfy

Assumption A.

$$H(u,p)=\widetilde{H}(u,p)+\lambda(u)$$

where $\lambda\colon\mathbb{R}\to[0,+\infty)$ is nonincreasing and C^1 and $\widetilde{H}\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfies

- $\widetilde{H}(u,\,\cdot\,)$ is concave and positively homogenuous,
- $\widetilde{H}(\,\cdot\,,p)$ is nonincreasing and C^1.

Theorem 6.2.1. *Suppose that* H_1, H_2 *satisfy Assumption* A *and for all* u, p

$$\begin{cases}\dfrac{\partial\widetilde{H_1}}{\partial u}(u,p)\lambda_2(u)=\dfrac{\partial\widetilde{H_2}}{\partial u}(u,p)\lambda_1(u),\\ \lambda_1'(u)\lambda_2(u)=\lambda_2'(u)\lambda_1(u).\end{cases}\tag{6.6}$$

Then, for all $t_1,t_2<0$,

$$S_1(t_1)S_2(t_2)=S_2(t_2)S_1(t_1)$$

where (for $i=1,2$*)* $S_i(t)g=U_i(t,\,\cdot\,)$ *and* $U_i\colon(-\infty,0]\times\mathbb{R}^n\to R\cup\{+\infty\}$ *is the unique semicontinuous solution of*

$$\begin{cases}U_t+H_i(U,U_x)=0,\\ U(0,\,\cdot\,)=g(\,\cdot\,).\end{cases}\tag{6.7}$$

The semigroup S_i *acts on the space of extended, bounded from below, lower semicontinuous functions on* $\mathbb{R}^n$.

Proof. By Theorem 1.7.2, the solution of (6.7) is the value function of the corresponding generalized Bolza problem. The Lagrangian $L_i(u,v)$ corresponding to H_i is given by

$$L_i(u,v) = \begin{cases} \lambda_i(u) & \text{if } v \in F_i(u), \\ +\infty & \text{elsewhere,} \end{cases}$$

where

$$F_i(u) = \{v \in \mathbb{R}^n : \forall p \in \mathbb{R}^n,\ \langle p, v\rangle \geq \widetilde{H}_i(u,p)\}.$$

The value function $V_i\colon (-\infty, 0] \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ corresponding to the terminal value g at the terminal time $T = 0$ is given by

$$V_i(t_0, x_0) = \inf\{u(t_0) : x'(s) \in F_i(u(s))u'(s) = -\lambda_i(u(s))x(t_0) = x_0, \\ u(x(0)) \geq g(x(0))\}.$$

We define reachable maps

$$R_i((x_0,u_0),t_0) = \{(x(0),u(0)) : x'(s) \in F_i(u(s)), \\ x(t_0) = x_0,\ u'(s) = -\lambda_i(u(s)),\ u(t_0) = u_0\}.$$

We have $S_i(t)g(x_0) = \inf\{u_0 : \exists (y,v) \in R_i((x_0,u_0),t_0),\ v \geq g(y)\}$.

The problem of commutation of semigroups can be reduced to the problem of commutation of reachable maps thanks to the following, easy lemma. □

Lemma 6.2.2. *If $R_1(R_2((x,u),t_2),t_1) = R_2(R_1((x,u),t_1),t_2)$ then*

$$S_1(t_1)S_2(t_2) = S_2(t_2)S_1(t_1).$$

Let us notice that the converse is also true. Let $u(\,\cdot\,; t_0, u_0, i)$ denote the solution of

$$\begin{cases} u'(s) = -\lambda_i(u(s)), \\ u(t_0) = u_0. \end{cases}$$

We have

$$R_i((x_0,u_0),t_0) = \left(x_0 + \int_{t_0}^{0} F_i(u(s;\, t_0, u_0, i))ds, u(0; t_0, u_0, i)\right).$$

Fix $p \in \mathbb{R}^n$. We shall need the following easy claim.

Lemma 6.2.3. *If $G\colon [a,b] \mapsto \mathbb{R}^n$ is a bounded measurable map with convex compact values then*

$$\inf\left\{\langle p, q\rangle : q \in \int_a^b G(t)\,dt\right\} = \int_a^b \inf_{g \in G(t)} \langle p, g\rangle\,dt.$$

By Lemma 6.2.3,

$$\inf\{\langle p, r\rangle : r \in R_i((x_0,u_0),t_0)\} = \langle p, x_0\rangle + \int_{t_0}^{0} H_i(u(s;\, t_0, x_0, i), p)\,ds. \tag{6.8}$$

Let $t_1, t_2 < 0$ and $u_1 := u(0; t_1, u_0, 1)$, $u_2 := u(0; t_2, u_0, 2)$. We have:

$$\begin{aligned} R_{12} &:= R_1(R_2((x_0, u_0), t_2)t_1) \\ &= x_0 + \int_{t_2}^0 F_2(u(s;\, t_2, u_0, 2))\, ds + \int_{t_1}^0 F_1(u(s;\, t_1, u_2, 1))\, ds, \\ R_{21} &:= R_2(R_1((x_0, u_0), t_1)t_2) \\ &= x_0 + \int_{t_1}^0 F_1(u(s; t_1, u_0, 1))\, ds + \int_{t_2}^0 F_2(u(s; t_2, u_1, 2))\, ds. \end{aligned}$$

We claim that

$$\inf\{\langle p, r\rangle : r \in R_{12}\} = \inf\{\langle p, r\rangle : r \in R_{21}\}. \tag{6.9}$$

One can deduce this fact from the following result (given without proof).

Lemma 6.2.4. *If A, B are convex compact subsets of $\mathbb{R}^n$ then*

$$\inf\{\langle p, a + b\rangle : a \in A,\ b \in B\} = \inf\{\langle p, a\rangle : a \in A\} + \inf\{\langle p, b\rangle : b \in B\}.$$

By (6.8) and (6.9), we have

$$\begin{aligned} \inf\{\langle p, r\rangle : r \in R_{12}\} = \langle p, x_0\rangle &+ \int_{t_2}^0 H_2(u(s; t_2, u_0, 2), p)\, ds \\ &+ \int_{t_1}^0 H_1(u(s; t_1, u_2, 1), p)\, ds, \end{aligned} \tag{6.10}$$

$$\begin{aligned} \inf\{\langle p, r\rangle : r \in R_{21}\} = \langle p, x_0\rangle &+ \int_{t_1}^0 H_1(u(s; t_1, u_0, 1), p)\, ds \\ &+ \int_{t_2}^0 H_2(u(s; t_2, u_1, 2), p)\, ds. \end{aligned} \tag{6.11}$$

Now, we consider two planar ordinary differential equations ($i = 1, 2$)

$$\begin{cases} l'(t) = H_i(u(t), p), \\ u'(t) = \lambda_i(u(t)). \end{cases} \tag{6.12}$$

Denote the right hand sides by $f_i(l, u) = (H_i(u, p), \lambda_i(u))$. Since $Df_1\, f_2 - Df_2\, f_1 = 0$, by Corollary 1.11 in [66] the corresponding to (6.12) flows commute, which follows that (6.10) and (6.11) coincide.

Following Lions and Rochet [73], we apply commutation property to study the following system of PDE

$$\begin{cases} \dfrac{\partial W}{\partial t} + H_1\biggl(W, \dfrac{\partial W}{\partial x}\biggr) = 0 \quad \text{in } \mathbb{R}^N \times\,]{-\infty}, 0)^2, \\ \dfrac{\partial W}{\partial s} + H_2\biggl(W, \dfrac{\partial W}{\partial x}\biggr) = 0 \quad \text{in } \mathbb{R}^N \times\,]{-\infty}, 0)^2, \\ W(x, 0, 0) = g(x), \end{cases} \tag{6.13}$$

where $(t,s,x) \mapsto W(t,s,x)$ is the unknown solution and H_1 H_2 are some Hamiltonians.

We say that an extended function $W\colon (-\infty,0]^2 \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a multitime lower semicontinuous solution of the system (6.13) if for every fixed $s \in (-\infty,0]$ the function $W_s(t,x) = W(t,s,x)$ is a lower semicontinuous solution of

$$\frac{\partial W}{\partial t} + H_1\left(W, \frac{\partial W}{\partial x}\right) = 0,$$

i.e. for every $t < 0$, $x \in \mathbb{R}^n$ and every $(p_t, p_x) \in \partial_- W_s(t,x)$

$$p_t + H_1(W_s(t,x), p_x) = 0$$

and the function $W_t(s,x) = W(t,s,x)$ is a lower semicontinuous solution of

$$\frac{\partial W}{\partial t} + H_2\left(W, \frac{\partial W}{\partial x}\right) = 0,$$

i.e. for every $s < 0$, $x \in \mathbb{R}^n$ and every $(p_s, p_x) \in \partial_- W_t(s,x)$

$$p_s + H_2(W_t(s,x), p_x) = 0.$$

Corollary 6.2.5. *Suppose that assumptions of Theorem* 6.2.1 *holds true. If g is a bounded from below lower semicontinuous function then*

$$W(t,s,x) := S_1(t)S_2(s)g(x) = S_2(t)S_1(s)g(x)$$

is the unique multitime lower semicontinuous solution of (6.13).

Remark. If $(p_t, p_s, p_x) \in \partial_- W(t,s,x)$ then obviously $(p_t, p_x) \in \partial_- W_s(t,x)$ and $(p_s, p_x) \in \partial_- W_t(s,x)$. So, if $W(t,s,x)$ is a multitime lower semicontinuous solution of (6.13), then it is also a lower semicontinuous solution of (6.13).

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CONTENTS