Metamereology

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by
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REVISED AND EXTENDED EDITION

Translated from the Polish
by Matthew Carmody

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Foreword to the Polish edition [Pietruszczak, 2000]

Our aim in this book is not simply to provide an introduction to the topic of mereology but also to undertake a thorough analysis of it. Hence its name: *Metamereology* (in Polish: *Metamereologia*).

In Chapter I, entitled "Introduction to the problems of mereology", we introduce the philosophical problems connected with the concepts of *being a part of a whole* and of *being a set* (both a collective set and a distributive set). In Section 1 of the chapter, we discuss the basic properties of the relations of *being a part of* and of *being an ingrediens of* (i.e., being a part of a whole or being the whole itself). In Section 2, we concern ourselves with two meanings of the terms "set" and "element of a set". The following section, Section 3, is devoted to the logical foundations of set theory. In Section 4, we present Leśniewski's conception of classes (sets) and their elements. We compare his view to those of Cantor, Frege, and Whitehead and Russell. In order to better understand Leśniewski's theory, we introduce a sketch of his logic in Section 5. We compare Leśniewski's logic on the one hand with classical predicate calculus and on the other hand with so-called free logic. This comparison enables us to present an outline of Leśniewski's mereology as a theory of certain schemas based on the classical predicate logic (Section 6). Section 7, the final section of Chapter I, is an introduction to mereology understood as a theory of certain set-theoretic relational structures.

The rest of the book is divided into two parts and two appendices. Both parts are devoted to the theoretical foundations of mereology. Both appendices are fundamentally algebraic. Only in Section 1 of Appendix II do we recall the basic concepts of (elementary) first-order theories and their models. Such an analysis permits us to separate out established algebraic facts — facts independent of the mereological axioms. The reader familiar with lattice theory may wish to take no more than a glance at Appendix I in order simply to familiarise themselves with the terminology used in this book. Appendix II is devoted to the 'elementary side' of Boolean lattice theory. We will make use of it when analysing the 'elementary side' of mereology.

In Part A (chapters II–V), mereology is treated a theory of certain relational structures called *mereological structures*. We present our motivation for this approach in the introduction to this part on p. 70.

In Chapter II, we examine classical mereology. We start with the axioms chosen by Leśniewski and show their basic consequences. We finish the chapter with a representation theorem for mereological structures. In Chapter III, we introduce the connection between classical mereology and the theory of complete Boolean lattices. This connection allows us to show, for example, that the class of mereological structures is not elementarily axiomatisable. We also examine the connection between Leśniewski's axiomatization and the one used by Alfred Tarski. In Chapter IV, we give various equivalent axiomatisations of mereological structures using various primitive concepts. In Chapter V, the final chapter of Part A, we examine the dependence of various conditions that appear in the theory of mereological structures. This enables us to undertake an examination of the lattice of superclasses of the class of mereological structures itself.

Part B concerns certain elementary theories connected with mereology. In Chapter VI, we formulate a theory which we call *elementary mereology*. We examine the class of models of this theory, which is a proper superclass of the class of mereological structures. As a rule, elementary mereology is formulated with an infinite number of axioms. We prove that it is finitely axiomatisable. Although the class of mereological structures is narrower than the class of models of elementary mereology, it turns out that the theses of the elementary theory are all and only those formulae which are true in each mereological structure.

In Chapter VIII, we concern ourselves with a certain elementary theory in which it is possible to talk of collective sets composed both of individuals and of distributive sets. Besides the concepts of a *distributive set* and the *membership relation* (for distributive sets), the theory also employs three concepts: *collective set*, *being a collective part of*, and *being an individual*. We call it *the unitary theory of individuals and sets*.

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Foreword to the English edition

The English edition is a revised and extended version of my book [Pietruszczak, 2000]. Alongside corrections to a number of minor errors of content, it contains a greater amount of commentary and many of the key claims in the book have had their proofs filled out. Chapter VII in the Polish edition has become chapter VIII to allow for a new chapter VII. Chapter VIII itself has two additional sections (sections 4 and 5). Moreover, sections 1 and 13–18 in Appendix I are new.

The final part of the first section of chapter I has, however, been removed. It introduced and discussed a problem concerning the construction of a general theory of parts without assuming the transitivity of the relation *is a part of* (and also with assuming the transitivity of the relation *is an ingrediens of*). The reason for its removal is that its proper formulation and solution can be found in [Pietruszczak, 2012, 2014]. In addition, problems for various theories of parts without the assumption of transitivity and their solutions can also be found in Chapter IV of [Pietruszczak, 2013].

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I would also like to thank Dr Matthew Carmody for translating the original Polish version of this book into English.

Chapter I

An introduction to the problems of mereology

Mereology arose as a theory of collective sets. It was formulated by the Polish logician Stanisław Leśniewski.[1] Collective sets are certain wholes composed of parts. In general, the concept of a *collective set* can be defined with the help of the relation *is a part of* and mereology may therefore be considered as a theory of "the relation of part to the whole" (from the Greek: $\mu\varepsilon\rho o\varsigma$, *meros*, "part").

1. Parts and ingredienses

In everyday speech, the expression "part" is usually understood as having the sense of the expressions "fragment", "bit", and so forth. Thus understood, the relation of part to the whole has two properties:

(a) *no object is its own part*;

(b) *there are not two objects such that the first could be a part of the second and the second is a part of the first.*

Thanks to condition (a), we have no difficulty in interpreting the phrase "two objects" in condition (b). One can see that it concerns 'two different' objects.[2]

An example may be supplied in support of the properties (a) and (b): if I read all of a book, it would seem unnatural to say that I read

[1] Of course, Leśniewski did not invent the concept of a *collective set*. It is discussed, for example, by Whitehead and Russell in comments in *Principia Mathematica* [Whitehead and Russell, 1919] concerning the theory of classes developed in that work. Whitehead made use of such sets in his thoughts on the philosophy of space-time [cf., e.g., Whitehead, 1929].

[2] Without condition (a), the possibility of ambiguity arises, for linguistic custom permits us to talk of 'two objects, which turn out to be identical'. Yet this ambiguity does not, however, lead to complications. If we accept that the phrase "two objects" allows for such an understanding, under which there is the possibility that the two objects are identical, then condition (b) simply entails condition (a): *there is no object such that it may be a part of itself.*

a part of it; assuming the book has a number of chapters, then the first chapter is a part of the book, but not conversely. In the literature, the following kind of example is found: my left hand is not a part of my left hand; my left hand is a part of my left arm, and not conversely. It would appear that properties (a) and (b) are beyond question.

In the literature on mereology, the phrase "proper part" is often used instead of the expression "part" we have so far been using. The practice has become established of using "part" with a wider extension. It is that a part of a given object is that object itself or each of its parts in the everyday sense of that word. Each part of an object distinct from the object itself is called a *proper part*. With this new meaning, the expression "part" meets a condition contrary to — in the sense of traditional logic — condition (a). For it follows directly from the definition that *every object is its own* (*improper*) *part*. We obtain moreover that *no object is its own proper part*. If, however, one understands the phrase "two objects" in the sense of 'two different objects', then "part" understood this way meets condition (b).

Leśniewski did not ask us not to understand "part" with its everyday sense. Instead, he introduced the word "ingrediens" (in an older Polish form "ingredjens"). An *ingrediens* of a given object is the object itself or each of its parts, where "part" is understood with its everyday sense [cf. Leśniewski, 1928, p. 264, footnote 1 and definition I and hereafter p. 47]. We shall also be using Leśniewski's terminology. We shall therefore be using the word "part" (understood with its everyday sense) and "ingrediens" (as defined above).[3]

Leśniewski took the view that the relation *is a part of* has properties (a) and (b) and that it is transitive, i.e., that any part of a given object is also its part. In support of this property the following example was given: my left arm is a part of my body, which entails that my left hand is also a part of my body. Rescher [1955, p. 10] shows, however, that in the general case, the transitivity of the relation of a part to a whole is essentially problematic. He provides the following counterexample: a nucleus is a part of a cell, a cell is a part of an organ, but a nucleus is not a part of an organ. In fact, if we consider a part to be a direct functional constituent of a whole, a nucleus is not a part of an organism.

[3] By accepting the convention broadening the sense of the word "part", we may sometimes be led to certain misunderstandings. The strange-sounding term "ingrediens" is in this case an 'ally', as it reminds us that it is an 'artificial' concept.

Yet Simons [1987, p. 107–108] shows that the concept of a *part* with transitivity corresponds to spatio-temporal inclusion and in that sense it is true that a nucleus is a part of an organ. Simons states that the fact that the word "part" has an additional meaning does not undermine the mereological concept of a *part*, because it is not being claimed that the mereological concept includes all the meanings of the word "part" but rather those that are fundamental and of greatest importance. He says that the transitivity of the relation *is a part of* causes no special difficulties when we refer to spatio-temporal relations, including those between events.

In order to avoid difficulities of interpretation arising from the use of everyday language, in this book we shall make use of a formal language. In this language "y", "z", "u", "v", and "w" (with or without indices) are individual variables ranging over arbitrary objects. The symbols "$\neg$", "$\wedge$", "$\vee$", "$\Rightarrow$", and "$\Leftrightarrow$" are correspondingly the truth-connectives: negation, conjunction, inclusive disjunction, material implication (conditional), and material biconditional.[4] The symbols "$\forall$" and "$\exists$" are the universal and existential quantifiers, respectively, which shall be binding individual variables.[5] In this formal language, conditions (a) and (b) are written as follows:

$$\neg\exists_x\; x \text{ is a part of } x, \quad \text{or} \quad \forall_x\; \neg\, x \text{ is a part of } x\,,^{6} \tag{irr$_\mathrm{P}$}$$

$$\begin{gathered}\neg\exists_{x,y}(x \neq y \wedge x \text{ is a part of } y \wedge y \text{ is a part of } x), \quad \text{or}\\ \forall_{x,y}\; \neg(x \neq y \wedge x \text{ is a part of } y \wedge y \text{ is a part of } x).^{7}\end{gathered} \tag{antis$_\mathrm{P}$}$$

We observe that the conjunction of (irr$_\mathrm{P}$) and (antis$_\mathrm{P}$) is logically equivalent to the sentence below:

$$\begin{gathered}\neg\exists_{x,y}(x \text{ is a part of } y \wedge y \text{ is a part of } x), \quad \text{or}\\ \forall_{x,y}\neg(x \text{ is a part of } y \wedge y \text{ is a part of } x).\end{gathered} \tag{as$_\mathrm{P}$}$$

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[4] We may express these symbols in turn with the phrases: "it is not the case that", "and", "or", "if … then … ", "if and only if" (hereafter abbreviated to "iff").

[5] We may express these symbols with the phrases "every object … is such that" and "some object … is such that".

[6] Two forms are given, one employing the universal quantifier and one employing the existential quantifier. Via Morgan's law for quantifiers formulae $\ulcorner\neg\exists_x\varphi(x)\urcorner$ and $\ulcorner\forall_x\neg\varphi(x)\urcorner$ are logically equivalent.

[7] In these formalised sentences the phrase "two objects" is to be interpreted so as not to allow that they be 'identical objects'.

Remark 1.1. One may observe that these are formalisations of sentence (b) in which we do allow the phrase "two objects" to be interpreted so as to allow that those objects be identical (cf. footnote 2).

Sentences ($\mathrm{irr}_{\mathrm{P}}$) and ($\mathrm{antis}_{\mathrm{P}}$) state respectively that the relation of part to a whole is reflexive and antisymmetric. The sentence (as_{P}) states that the relation of part to a whole is asymmetric. It is a known result that a relation is asymmetric iff it is a reflexive and antisymmetric (cf. Lemma 2.2(iii) in Appendix I). □

We recall that the use of different variables does not mean that we are referring to different objects. In order to understand better the how variables 'operate', we may say by way of paraphrase that they refer to the picking-out of objects and not directly to the objects themselves.[8] Let us suppose, that x is an object picked out the first time and y an object picked out the second time. If we pick out different objects at both times (i.e., $x \neq y$), then condition ($\mathrm{antis}_{\mathrm{P}}$) precludes the possibility that both x is a part of y and y is a part of x. If we pick out the same object at both times (i.e., $x = y$), then condition ($\mathrm{irr}_{\mathrm{P}}$) precludes the possibility that x is a part of x. Conditions ($\mathrm{irr}_{\mathrm{P}}$) and ($\mathrm{antis}_{\mathrm{P}}$) therefore entail condition (as_{P}). Conversely, under the preceding assumption concerning x and y, condition (as_{P}) rules out the possibility that both x is a part of y and y is a part of x. If we therefore pick out the same object twice, we obtain condition ($\mathrm{irr}_{\mathrm{P}}$). If, however, we pick out two different objects, we obtain condition ($\mathrm{antis}_{\mathrm{P}}$).

Obviously, conditions ($\mathrm{antis}_{\mathrm{P}}$) and ($\mathrm{as}_{\mathrm{P}}$) may be formulated so as to be logically equivalent:

$$\begin{aligned}&\forall_{x,y}(x \text{ is a part of } y \wedge x \neq y \implies \neg\, y \text{ is a part of } x), \quad \text{or}\\&\forall_{x,y}(x \text{ is a part of } y \wedge y \text{ is a part of } x \implies x = y),\end{aligned} \tag{$\mathrm{antis}'_{\mathrm{P}}$}$$

$$\forall_{x,y}(x \text{ is a part of } y \implies \neg\; y \text{ is a part of } x). \tag{$\mathrm{as}'_{\mathrm{P}}$}$$

Employing the paraphrase 'pickings-out of objects', we may read off two versions of sentence ($\mathrm{antis}'_{\mathrm{P}}$): if x is the object picked out the first time and y the object picked out the second time, then (i) if x is a part

[8] We shall not paraphrase things so as to employ the concept of a valuation of the variables since we want to use variables to talk about objects and not about the variables themselves. Furthermore, an expression of the sort "a given object is the value of the variable 'x'" would not help us much. The formal phrase "object p is the value of the variable 'x'" clarifies nothing regarding the use of individual variables, as it itself features the variable "p".

of y and x is distinct from y, then y is not a part of x; (ii) if x is a part of y and y is a part of x, then we have picked out the same object both times.

Remark 1.2. Since (irr_P) says that if x is a part of y, then $x \neq y$, the second assumption in the antecedent of (i) therefore seems inessential. Irreflexivity and antisymmetry automatically yield asymmetry, such as (as'_P). Analogously, the antecedent in (ii) seems contradictory. Via (ii), it follows that $x = y$ whereas via (irr_P), it follows that $x \neq y$. □

The formula (as'_P) may, however, be read: if x is the object picked out the first time and y the object picked out the second time and x is a part of y, then y is not a part of x. In this formulation, the logical equivalence of the sentence (as'_P) with the conjunction of the sentences (irr_P) and ($\mathrm{antis}'_\mathrm{P}$) is easily displayed.

Let U be any non-empty set of objects.[9] Let $\sqsubset$ be the binary relation *is a part of* holding between objects from a set U, i.e., we put:

$$\sqsubset \;:=\; \{\, \langle x, y\rangle \in U \times U : x \text{ is part of } y \,\}.$$

Instead of "$\langle x, y\rangle \in \sqsubset$" and "$\langle x, y\rangle \notin \sqsubset$" we will write for short: "$x \sqsubset y$" and "$x \not\sqsubset y$", respectively. We extend this policy to other combinations of variables ranging over objects from U. Condition (as_P) (resp. (as'_P)) states that the relation $\sqsubset$ is asymmetric in the set U, i.e., we have:

$$\begin{array}{ll} \forall_{x,y\in U}\, \neg(x \sqsubset y \wedge y \sqsubset x), & \text{or} \\ \forall_{x,y\in U}(x \sqsubset y \implies y \not\sqsubset x). & \end{array} \tag{$\mathrm{as}_\sqsubset$}$$

It logically follows from ($\mathrm{as}_\sqsubset$) that the relation $\sqsubset$ is irreflexive in U,[10] i.e., we obtain:

$$\forall_{x\in U}\; x \not\sqsubset x\,. \tag{$\mathrm{irr}_\sqsubset$}$$

In accordance with the definition of the relation *is an ingrediens of* accepted by Leśniewski (cf. p. 16), we have, for arbitrary $x, y \in U$:

$$x \text{ is an ingrediens of } y \;:\Longleftrightarrow\; x = y \vee x \sqsubset y\,. \tag{df-ingr}$$

[9] In this section, the term "set" is being used exclusively in the distributive sense (compare the next few sections, in which we shall talk about collective sets and the difference between the two types).

On the issue of terminology and notation, see Appendix I. In accordance with the convention established there, in the later parts of this book, $\mathcal{P}(X)$ is the set of all subsets of a freely chosen set X. Furthermore, let $\mathcal{P}_+(X) := \mathcal{P}(X) \setminus \{\emptyset\}$, i.e., $\mathcal{P}_+(X)$ is the set of all non-empty subsets of a set X.

[10] Compare the sentence (irr_P). As we have just observed, the asymmetry of the relation $\sqsubset$ is equivalent to its irreflexivity and antisymmetry.

Let $\sqsubseteq$ be the binary relation *is an ingrediens of* holding between objects from the set U, i.e.:

$$\sqsubseteq := \{ \langle x, y\rangle \in U \times U : x \text{ is an ingrediens of } y \}.$$

We shall pursue the same policy of abbreviation with $\sqsubseteq$ as we are for $\sqsubset$. We have, therefore, for arbitrary $x, y \in U$:

$$x \sqsubseteq y \;:\Longleftrightarrow x = y \vee x \sqsubset y\,. \tag{df $\sqsubseteq$}$$

From Lemma 2.3(i,iii) in Appendix I, the relation $\sqsubseteq$ is reflexive and antisymmetric,[11] i.e., we have:

$$\forall_{x\in U}\; x \sqsubseteq x\,, \tag{r$_\sqsubseteq$}$$

$$\forall_{x,y\in U}(x \sqsubseteq y \wedge y \sqsubseteq x \implies x = y). \tag{antis$_\sqsubseteq$}$$

Moreover, it follows from Lemma 2.3(ii,iii) in Appendix I that:

$$\forall_{x,y\in U}(x \sqsubset y \iff x \sqsubseteq y \wedge x \neq y) \tag{$\sqsubset = \sqsubseteq \setminus$id}$$

$$\forall_{x,y\in U}(x \sqsubset y \iff x \sqsubseteq y \wedge y \not\sqsubseteq x). \tag{$\sqsubset = \sqsubseteq \setminus \sqsupseteq$}$$

We will not be assuming that the relation $\sqsubset$ is transitive in any set U of objects. In mereology, however, only such structures $\langle M, \sqsubset\rangle$ interest us, where M is a non-empty set and $\sqsubset$ is also a transitive relation in M, i.e., we have:

$$\forall_{x,y,z\in M}(x \sqsubset y \wedge y \sqsubset z \implies x \sqsubset z). \tag{t$_\sqsubset$}$$

In such structures, by (df $\sqsubseteq$) and (t$_\sqsubset$), we obtain that also the relation $\sqsubseteq$ is transitive, i.e., we have:

$$\forall_{x,y,z\in M}(x \sqsubseteq y \wedge y \sqsubseteq z \implies x \sqsubseteq z). \tag{t$_\sqsubseteq$}$$

Remark 1.3. At this point in the Polish edition of this book [Pietruszczak, 2000], there featured a discussion of a problem concerning the construction of a general theory of parts without the assumed transitivity of the relation *is a part of* (and also for the assumed transitivity of the relation *is an ingrediens of*). We will not be presenting the problem in this work, as its proper formulation and solution can be found in [Pietruszczak, 2012, 2014]. Furthermore, problems for various theories of parts without the assumption of transitivity and their solutions can be found in Chapter IV of [Pietruszczak, 2013] (at present, this work is only available in Polish but an English version is in the pipeline). □

[11] Concerning the other meaning of term "part", which allows for so-called improper parts.

2. Two meanings of the terms "set" ("class") and "element of a set" ("element of a class")

When we talk about sets (resp. classes) and their elements, certain misunderstandings can arise because of the multiplicity of meanings the terms "set" and "element of a set" (resp. "member of set") possess. Consider following passage from [Borkowski, 1977]:[12]

> The terms "set" and "element of a set" are used with two meanings. Understood with the first of these meanings, the term "set" signifies objects composed of parts, collections and conglomerations of a different kind. The elements of such type of set are to be understood as arbitrary parts of that set, where the term "part" is understood in its everyday sense, with which, for example, the leg of a table is a part of the table. A pile of stones is in this sense a set of those stones. The elements of that set are both individual stones along with the various parts of those stones, and thus, for example, the molecules or atoms of which those stones are composed. With this meaning, the set of given stones is identical to, for example, the set of all the atoms from which they are composed. Elements of a set so understood, such as the set of all tables, would be not only the individual tables but, for example, the legs of those tables or other of their parts. We shall say that we are using here the term "set" in its *collective* sense, as we are using it with that sense. A theory of sets and the relation *is a part of* understood in line with the above has been constructed by S. Leśniewski, who called it *mereology*.
>
> We use the terms "set" and "element of a set", with the second meaning in the following example: when talking about the et of European countries, we consider as elements of that set particular European countries, such as Poland, France and Italy, and we do not consider as elements the parts of those countries. With this meaning, the Tatra mountains or the Małopolska Upland are not elements of the set of European countries even though they are parts of certain European countries. We also use these terms with this meaning for example when, talking about the set of Polish towns, we consider as elements of that set towns such as Wrocław and Warsaw whilst not considering as elements of that set particular streets or squares or other parts of those cities. The terms "set" and "element of a set" have long been used with this meaning in logic, when speaking of extensions of names

[12] We will be drawing on some rather long excerpts from various sources in this initial chapter both to introduce the reader to key concepts and to assure them that we are not inventing new meanings for these concepts but employing established ones.

> or concepts as certain sets of objects. In contrast to the first meaning, it is not possible to identify the concept of an element with the common concept of a part. [Borkowski, 1977, p. 146]

The second meaning of the term "set" has come to be called the *distributive* or *set-theoretic meaning*.

Let us add further a section from the final paragraph of a book by Słupecki and Borkowski [1984] that makes some philosophical on sets.

> [...] the word "set" has two clearly distinct meanings in everyday speech, of which one is call the collective meaning and the second the distributive. With the collective meaning — a set of a certain objects is a whole composed of those objects in the same way that a chain is composed of links and a pile of a sand of grains of sand. With this meaning, a set of concrete, sensually perceptible objects is also a concrete and perceptually-available object. Using the term "set" with this meaning, we understand "x is an element of the set A" as having the same sense as the expression "x is a part of the set A" (with the word "part" having that meaning such that the leg of a table is a part of the table). A set theory understood in this way was built by S. Leśniewski under the name *mereology*. Using the term "set" with its distributive meaning, we consider the sentence "Mars is an element of the set of planets in our Solar System" as equivalent to the sentence "Mars is a planet in our Solar System". The difference in meaning is attested to by the fact that certain true sentences where "set" is understood with its first meaning are false when it is understood with its second meaning. For example, where the meaning is collective, it is true that a tenth part of Mars is an element of the set of planets in our Solar System, because it is a part of the whole arrangement; whereas that sentence is false if the meaning is distributive, because no tenth part of Mars is a planet in our Solar system. [Słupecki and Borkowski, 1984, p. 279]

It is evident from the above texts that the terms "collective set" and "distributive set" have different meanings. It would seem indeed that the single common characteristic is that, in both cases, it is possible to say that "a set of certain objects is a whole composed of those objects" [Murawski, 1995, p. 164]. To put it another way, there may be a similar "way of creating sets" for both concepts. As Hao Wang writes:

> There are two familiar and natural ways of construing sets.[13] On the one: hand, given a multiplicity of objects, some or all of these objects

[13] Both conceptions of the creation of sets described here obviously concern distributive sets.

> can be conceived together as forming a set; the process can be iterated indefinitely. This way may be called "the extensional conception of set." On the other hand, a set may be seen as the extension of a concept or a property in the sense that it consists of all and only the objects which have the property. This way may be called "the intensional conception of set." We tend to use both conceptions and expect no conflict between them. Yet in practice it makes a difference whether one takes the one or the other conception as basic.
>
> Roughly speaking, Frege begins with the intensional conception and Cantor begins with the extensional conception. [Wang, 1994, p. 267]

Since collective sets and distributive sets (understood according to the "extensional conception") are created in a similar way, as 'collections into one whole' of certain objects, the difference therefore consists in this: the basis for how they are collected. Above all, it must be observed that the word "collection" is understood differently in each case.

In the case of collective sets, the "collection" ("grouping") of certain objects may be compared to the 'gluing together' of those objects. As Słupecki and Borkowski have written: if all 'collections' of elements are concrete[14], then the collective set thereby obtained is a concrete object.[15] If, however, one or other of the 'grouped' objects is abstract, then that same collective set must be recognised as an abstract object as well.[16]

In the passages above from [Borkowski, 1977] and [Słupecki and Borkowski, 1984], it was said that, understood with its everyday sense, the concept of *being an element of* a given collective set is supposed just to amount to the concept of *being a part of* (it). In Leśniewski's mereology there is, however, a slightly different situation. Each part of a given collective set is an (collective) element of it. The set itself is, however, also collective element of itself. In mereology therefore the concept of *being an element of* a given collective set overlaps with the concept of *being an ingrediens of* that set (see below (5.7) on p. 51).

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[14] By concrete object we understand here that which exists in space-time. An object which is not concrete we call abstract.

[15] In practice, we talk as a rule of collective sets composed exclusively of concrete objects.

[16] Leśniewski himself illustrated his theory of collective sets composed of geometrical segments [see Leśniewski, 1927, pp. 186–187 and hereafter p. 45]. It is, however, hard to regard such geometrical segments as concrete objects. (Leśniewski might have thought differently, though.)

For collective sets composed of abstract objects, see, e.g., [Pietruszczak, 1996, 1997] and hereafter Chapter VIII.

Used with their distributive senses, the terms "set" and "class" are often treated as synonyms. In certain versions of modern set theory a distinction is made between them. In such theories, *each set has to be a class, but not conversely.* Sets are a special kind of class: they are those classes which are elements of other classes. Classes not belonging to any class are called *proper classes*. Classes composed exclusively of concrete objects are sets. In the passages below, therefore, if there is talk of distributive classes composed of concrete objects, then such classes are also sets.

In the case of distributive classes (sets), the collection — i.e., collecting of objects, regardless of their type — must be understood always in an abstract sense and not a spatio-temporal one. Quine writes:

> The reassuring phrase 'mere aggregates' must be received warily as a description of classes. Aggregates, perhaps; but not in the sense of composite concrete objects or heaps. Continental United States is an extensive physical body (of arbitrary depth) having the several states as parts; at the same time it is a physical body having the several counties as parts. It is the same concrete object, regardless of the conceptual dissections imposed; the heap of states and the heap of counties are identical. The class of states, however, cannot be identified with the class of counties; for there is much that we want to affirm of the one class and deny of the other. We want to say e.g. that the one class has exactly 48 members, while the other has 3075. We want to say that Delaware is a member of the first class and not of the second, and that Nantucket is a member of the second class and not of the first. These classes, unlike the single concrete heaps which their members compose, must be accepted as two entities of a non-spatial and abstract kind.
>
> [Quine, 1981, p. 120]

With their distributive meaning, the terms "class" ("set") and "element" satisfy the principle given below in the form of a schema:

$$\forall_x (x \text{ is an element of the class (set) of } \mathsf{P}\text{s} \iff x \text{ is a } \mathsf{P}). \tag{2.1}$$

Remark 2.1. In this schema, as in others, the letter "P" is not a variable in the language of classical logic. It is just a so-called *schematic letter*. It stands for (in the sense of 'appearing instead of') an arbitrary general name (term). Crudely put, the letter "P" (as with the letter "S") indicates an empty space which we may fill by putting in an arbitrary general name.

Moreover, in (2.1) the letter "x" (as used hereinafter the letters "y", "z", "u", etc.) is an individual variable in the language of classical logic. If it is not bound by a quantifier, one may replace it with an arbitrary name having exactly one referent (for details see point 3 of Section 5). □

Remark 2.2. In this work, the expressions "name" and "term" we will use interchangeably. A *referent* (or *designatum*) of a name is an arbitrary object which it signifies. Names that do not have any referents are called *empty*.

We will use the traditional division: *general names – singular names*. In short, general names are suitable for building one-place predicates of the form "is a P". Singular names include, for example: proper names, singular definite descriptions ("the highest mountain of the world", "the youngest daughter of the author of this book"), and demonstratives ("that dog").[17] A singular name is intended to refer to exactly one object, but does not have to. There are empty names for both general ("crocodile living on the moon", "son of the author of this book") and singular names (e.g., "Princess Snow White", "the youngest son of the author of this book").

Leśniewski employed only one syntactic category of names. He thought that from a logical point of view, the only way basis for dividing names was the number of their referents. He distinguished the following three categories:

- empty names,
- names each of which has exactly one referent,
- names each of which has at least two referents.

In order to employ the above tripartite division, we shall introduce two new terms for the latter two categories. A name which has exactly one referent we will call *monoreferential*. A name which has at least two referents we will call *polyreferential*. It should be clear that in the category of monoreferential names we find singular names and general names. Are there polyreferential singular names? Do we only find general names in the category of polyreferential names? This is not a question we will look further into in the current work.

Leśniewski's category of *empty names* is also our category. His category of *singular names* ("kategoria *nazw jednostkowych*" in Polish) is our category of *monoreferential names* and his category of *general names* ("kategoria *nazw ogólnych*" in Polish) is our category of *polyreferential*

[17] For singular names see, e.g., [Stirton, 2000].

names. It is because "singular name" and "general name" have different meanings in English that we have decided to introduce the terms "monoreferential name" and "polyreferential name". A singular name is not what we understand by a monoreferential name both because there can be monoreferential general names and empty singular names. □

Delaware is a state of the USA but it is not a county whereas with Nantucket it is the reverse. Therefore, by (2.1), the class of states of the USA is different from the class of counties because "Delaware is a member of the first class and not of the second, and that Nantucket is a member of the second class and not of the first" [Quine, 1981, p. 120]. This non-identity attests to the fact that the aforementioned distributive classes may not be identified with any spatio-temporal object. Essentially, the USA is the only spatio-temporal object with which those classes may be identified. Under that identity we would have the equality rather than the inequality of those classes (transitivity and the commutativity of identity). In other words, what Quine is saying in the previous passage is that if the class of the states of the USA occupied some 'place' in space, then it would the very same place that is occupied by the USA. The same would be true of the class of counties in the USA. We should therefore identify these distributive classes, contrary to condition (2.1).

The analyses presented in both the passage by Quine and the paragraph about still do not prove that they are distributive classes. The analyses show only that if there are distributive classes, they are abstract objects.[18]

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[18] In this paragraph the word "exists" has consciously not been used in place of the two expressions "is" and "are". We do not want to take any position on the question of whether being is the same as existence. (Which is also why in footnote 5, the existential quantifier is to be read as "some object ... is such that".)

Some philosophers do not regard the meanings of the terms "exist" and "be" as identical. They believe that all that exists also is, but not conversely (which means that there are objects which do not exist). For example, "Pegasus" signifies a (fictional) object which does not exist. A number of these philosophers may identify existence with *being a concrete* (*physical*) *object*.

Other philosophers identify the meanings of "exist" and "be". For example, Quine replies to the question "What is there?" with "Everything" [Quine, 1953, p. 1]. Preserving the use the terminology of the preceeding paragraph for the purposes of comparison — all that *is* also *exists* and vice versa. Quine stresses that "However, this is merely to say that there is what there is. There remains room for disagreement over cases; and so the issue has stayed alive down the centuries" [Quine, 1953, p. 1].

Leśniewski would definitely have replied to the question "What is there?" in a similar way (in his theory of what he calls *ontology*: something is an object iff it

We may paraphrase the preceding considerations: it is possible 'to collect abstractly' the states of the USA whilst not collecting their counties and vice versa. Quine provides us with another example in support of the theory of the abstractness of distributive classes (sets):

> The fact that classes are universals, or abstract entities, is sometimes obscured by speaking of classes as mere aggregates or collections, thus likening a class of stones, say, to a heap of stones. The heap is indeed a concrete object, as concrete as the stones that make it up; but the class of stones in the heap cannot properly be identified ,with the heap. For, if it could, then by the same token another class could be identified with the same heap, namely, the class of molecules of stones in the heap. But actually these classes have to be kept distinct; for we want to say that the one has just, say, a hundred members, while the other has trillions. Classes, therefore, are abstract entities; we may call them aggregates or collections if we like, but they are universals. That is, if there *are* classes. [Quine, 1953, pp. 114–115]

As in the previous quoted passes, Quine is saying that if a class of stones occupied some 'place' in space, then it would be a pile of stones. A similar thing would be said of the molecules in the stones. It is possible to 'abstractly take' the stones 'without moving' their molecules or vice versa.

Let us remind ourselves that in "the intensional conception of [distributive] set", "a set may be seen as the extension of a concept or a property in the sense that it consists of all and only the objects which have the property" [Wang, 1994, p. 267]. Thus, not as a property, but as its extension. A further excerpt from [Quine, 1981] will help clarify what is meant:

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exists). Leśniewski would consider himself differing from Quine in his views on existence (being) "in particular cases". Taking into account Leśniewski's oft-broadcast nominalist views [cf. his works or e.g. Küng, 1977a,b, 1981] one may say that for him only concrete (physical) objects exist (are). Leśniewski thus rejects the existence of objects such as distributive classes. Ironically, he "senses" that they have "the scent of mythical paradigms from a rich gallery of 'invented' objects" [Leśniewski, 1927, p. 204]. Arguments given by Whitehead and Russell in [1919] — similar to Quine's arguments in the citation above from [Quine, 1981] — inclined Leśniewski to say that since distributive classes do not exist "in the world", there are therefore no such things. [Leśniewski, 1927, pp. 204–205; an similar passage from [Leśniewski, 1927] is to be found on p. 43]. To put it another way: for Leśniewski, the term "distributive class", as with the term "Pegasus", signifies nothing.

Reading Quine, one is inclined to say that, for him, although Pegasus does not exist, there do exist, however, mathematical beings such as distributive sets.

> Once classes are freed thus of any deceptive hint of tangibility, there is little reason to distinguish them from properties. It matters little whether we read '$x \in y$' as 'x is a member of the class y' or 'x has the property y'. If there is any difference between classes and properties, it is merely this: classes are the same when their members are the same, whereas it is not universally conceded that properties are the same when possessed by the same objects. The class of all marine mammals living in 1940 is the same as the class of all whales and porpoises living in 1940, whereas the property of being a marine mammal alive in 1940 might be regarded as differing from the property of being a whale or porpoise alive in 1940. But classes may be thought of as properties if the latter notion is so qualified that properties become identical when their instances are identical. Classes may be thought of as properties in abstraction from any differences which are not reflected in differences of instances. For mathematics certainly, and perhaps for discourse generally, there is no need of countenancing properties in any other sense.
> [Quine, 1981, pp. 120–121]

Quine later relaxed his views on the identification of properties possessed by the same objects, i.e., coextensive properties (features):

> If it makes clear sense to speak of properties, it should make clear sense to speak of sameness and difference of properties; but it does not. If a thing has this property and not that, then certainly this property and that are different properties. But what if everything that has this property has that one as well, and vice versa? Should we then say that they are the same property? If so, well and good; no problem. But people do not take that line. I am told that every creature with a heart has kidneys, and vice versa; but who will say that the property of having a heart is the same as that of having kidneys?
>
> In short, coextensiveness of properties is not seen as sufficient for their identity. What then is? If an answer is given, it is apt to be that they are identical if they do not just happen to be coextensive, but are necessarily coextensive. But NECESSITY, *q.v.*, is too hazy a notion to rest with.
>
> [...] why not clean up our act by just declaring coextensive properties identical? Only because it would be a disturbing breach of usage, as seen in the case of the heart and kidneys. To ease that shock, we change the word: we speak no longer of properties, but of *classes*.
> [Quine, 1987, pp. 22–23]

Properties do not therefore have to be identical when they are coextensive. Such is the case with the following three pairs of properties: (i) the property of *having a heart* and the property of *having kidneys*; (ii) the property of *being the first even number* and the property of *being a positive even number less than 3*; (iii) the property of *being a crocodile living on the moon* and the property of *being a house on Mars*. Coextensive properties correspond to one distributive class, which is their common extension. The properties in (i) determine one distributive set which can be signified by the two expressions "the set of creatures with a heart" and "the set of creatures with kidneys". The properties in (ii) hold of one and the same object — the number 2. They therefore determine the same one-element set, names for which might be "the set of first even numbers", "the set of positive even numbers less than 3", and "$\{2\}$". The pair of properties in (iii) do not hold of any object. They determine a set which one may refer to using the names "the set of crocodiles on the moon", "the set of houses on Mars", "the empty set", and "$\emptyset$".

Remark 2.3. It is clear that when synonymous terms appear in place of the letters "S" and "P", the property of *being* S is identical with the property of *being* P [cf. Stanosz, 1971, p. 521]. The same may, however, be true when terms S and P are not synonymous. As Barbara Stanosz observes:

> [...] predicates expressing the same property [...] may differ in respect of meaning. This is the case, for example, with the predicates "has the colour of a ripe lemon" and "has a bright yellow colour", which, although not synonymous, express the same feature; similarly-behaved are the predicates "has the shape of the Egyptian pyramids" and "has the shape of a square-based pyramid" and many others.
>
> [Stanosz, 1971, p. 520]

Stanosz [1971] carefully investigated the identity-conditions of features. More precisely, she gives two versions of a definition of how two predicates determine the same feature. □

Many interesting remarks on the matters sketched above may be found the chapters of Quine's [1987] entitled "Classes versus Properties" and "Classes versus Sets". Let us give one further fragment from [Słupecki and Borkowski, 1984] to finish this section:

> It can be said without difficulty that, in the set theory, the term "set" is used in a distributive sense rather than a collective one; however, the relation of element to set is not here understood as the relation of part to whole. The latter relation has different basic properties; for example, it is transitive, whereas the law of transitivity is not valid for "$\in$". The following fact attests to this: the formulae $\emptyset \in \{\emptyset\}$,

> $\{\emptyset\} \in \{\{\emptyset\}\}$ are true, but the formula $\emptyset \in \{\{\emptyset\}\}$ (equivalent to $\emptyset = \{\emptyset\}$) is false. As B. Russell observes, if we were to treat sets as heaps or conglomerates (a therefore as sets in the collective sense), then "it impossible to understand how there can be such a class as the null-class, which has no members at all and cannot be regarded as an aggregate; we should also find it very hard to understand how it comes about that a class which has only one member is not identical with that one member"[19] [Słupecki and Borkowski, 1984, pp. 279–280]

3. Distributive classes (sets)

In this section we shall understand the terms "class", "set", and "element" ("member") exclusively with their distributive sense.

It might seem that condition (2.1) defines the concepts of *being a class* (resp. of *being a set*) and of *being an element*. This is not, however, the case. In its first version, condition (2.1) allows us to eliminate the terms "class" (resp. "set") and "element" only when they occur in predicates of the form "is an element of the class of Ps" (resp. "is an element of the set of Ps"). More precisely, it allows us to eliminate that predicate in favour of "is a P". Condition (2.1) does not, in the general case, suffice for the complete elimination of the term "class" (resp. "set"). There are sentences of the form "the class of Ps is ..." from whose subject we may not eliminate the term "the class of Ps". We must therefore have it guaranteed that this term has in general some referent.

A different situation is to be found in the theory which Quine calls "the virtual theory of classes, or the theory of virtual classes" [cf. Quine, 1970, pp. 70–72 and Quine, 1969, p. 16]. In this theory, we are allowed to use 'non-decomposable' predicates of the type "is an element of the class of Ps", which may be eliminated in favour of predicates of the type "is a P"; i.e., that is, with the help of principle (2.1). We are further allowed to use sentences of the form "the class of Ss is included in the class of Ps" as a 'conventional' abbreviation for "$\forall_x(x$ is an element of

[19] The footnote gave the reference as Russell's *Introduction to Mathematical Philosophy*, p. 268. In [Słupecki and Borkowski, 1984], a passage from the Polish version of the book [Russell, 2010] was used. Here is the passage from the English original (pp. 146–147): "I do not mean to assert, or to deny, that there are such entities as 'heaps.' As a mathematical logician, I am not called upon to have an opinion on this point. All that I am maintaining is that, if there are such things as heaps, we cannot identify them with the classes composed of their constituents" [Russell, 2010, p. 147].

the class of Ss $\Rightarrow x$ is an element of the class of Ps)" and to reduce this — using (2.1) — to "$\forall_x(x$ is an $\mathsf{S} \Rightarrow x$ is a $\mathsf{P})$".[20] Similarly, in the virtual theory of sets a sentence of the form "the class of Ss is identical with the class of Ps" is only a 'conventional' abbreviation of "$\forall_x(x$ is an element of the class of Ss $\Leftrightarrow x$ is an element of the class of Ps)", which we may in turn reduce to "$\forall_x(x$ is an $\mathsf{S} \Leftrightarrow x$ is a $\mathsf{P})$".[21] This virtual theory is for that reason a theory of virtual classes, because classes themselves do not belong to the range of its variables.

If we go outside of the virtual theory of classes, it is easy to see that condition (2.1) is a restricted principle. It works only in those cases in which a term of the form "class of Ps" (resp. "set of Ps") is a monoreferential name. The assumption seems natural that, for an arbitrary term standing in place of "P" at most one object is signified by terms of the form "class of Ps" and "set of Ps".[22] The problem is

[20] We are using a formal language, but not everyday speech, because the latter can cause us interpretational difficulties in certain cases. For example, a sentence of the form "$\forall_x(x$ is an $\mathsf{S} \Rightarrow x$ is a $\mathsf{P})$" expresses the informal sentential schema "Every S is a P". If the term standing in place of "S" is empty, then we have a problem with interpreting this informal sentence: is it true or false — or perhaps neither true nor false? There is no such problem with the formal language, because if in place of "S" stands an empty term, the schema "x is an S" is simply not satisfied by any object, and thus the formal sentence is true. Hence, the empty class (that is, the class of S when there are no S; otherwise $\emptyset$) is included in any arbitrary class.

[21] In the virtual theory of classes we can also consider predicates of the form "is an element of the class of Ss" in which in place of "S" can feature the expression "elements of the class of Ps", i.e., a predicate of the form "is an element of the class of elements of the class of Ps". Using principle (2.1), this second predicate may be reduced to a predicate "is of the elements of the class of Ps" and in turn to the predicate "is a P".

This thought suggests the possibility of allowing iterations of the expressions "element", "class" and other Boolean operations, permitting us to construct a Boolean algebra of virtual classes. We must, however, first introduce an inductive definition of the class terms. The details shall not be presented here: the reader is instead referred to [Quine, 1969, 1970]. We will note only that for all such class expressions we must generalise the 'conventional' abbreviations for the expressions of inclusiveness and identity. In particular cases, predicates with the names of Boolean operations will look as follows: "is an element of the complemented class of Ps" and "is an element of the product (resp. sum) of the class of Ss with the class of Ps". We can correspondingly reduce these to the predicates "is not a P" and "is an S and (or) a P".

[22] We are clearly concerned with the context in which the terms "class of Ps" and "set of Ps" are to mean the same as the terms "class of all Ps" and "set of all Ps", respectively. Sometimes these terms are used in the same way as "numerical set", with the sense "set composed of some numbers" (the expression "some" allows for the possibility of all). In such a case a term of the form "class composed of some Ps"

therefore whether terms of that form have in each case a referent. If in a given case a term of the form "class of Ps" signifies nothing, then it is hard to understand the term "element of the class of Ps".

Let us stress that we are not concerned here with the possibility of a term of the form "element of the class of Ps" being empty. That is the case when and only when the term standing in place of 'P" is empty. In such a case, however, the term "class of Ps" has a referent (the empty class) and hence there is no problem of interpretation.

To make formulation still further more convenient, we shall abbreviate the non-relational predicates (one-place predicates) "is a class", "is a set", and "is a proper class" using in turn the symbols "Cl", "Set", and "pCl". The relational (a two-place) predicate "is an element of" we shall abbreviate with "$\in$", as is standard.[23]

In Zermelo's system of set theory only the one term, "set", appears 'officially'.[24] It follows from the axioms of that system that every set is an element of some set.[25]

As we have already recalled above, in some versions of contemporary set theory stemming from von Neumann and Bernays, a distinction is introduced between the terms "class" and "set". In these theories, sets

(resp. "set composed of some Ps" may be empty, or monoreferential, or polyreferential. If — in a given context — the term "class of Ps" were to have two referents, then — in accordance with (2.1) — these different classes would not be distinguished by their elements. And this clashes with the idea of the concept of a class of Ps as 'a collection into a single whole' of all Ps.

[23] We are not assuming here any formal grammar for the language we are using. We believe that, from the point of view of natural language, it would be an artificial solution to regard the predicates given above as primitive expressions of the language, with whose help atomic sentential formulae would be constructed (using additionally variables and individual names). Such a solution is proper just when we design a formal language (e.g., a first-order language). Speaking informally, we talk of an "element of a set" or say "every set is a class but not every class is a set". The primitive terms here are the non-relative (in this context) terms "class" and "set" and the relative term "element".

[24] There are ways of understanding Zermelo's system in which one may speak of proper classes, but as objects of a 'different kind' than sets. More precisely, proper classes are not the values of variables in this system [see, e.g., Jech, 1971, § 1, and Shoenfield, 1977, § 7]. Instead of objects of a 'different kind' one may construct a Quinean "virtual theory of classes" on top of Zermelo's system, as a theory in which proper classes (and not sets) are 'virtual objects'.

[25] We also have a thesis saying that an arbitrary object is an element of some set, whose single element is that object. Formally: $\forall_x \exists_y (\mathrm{Set}\, y \wedge x \in y \wedge \forall_z (z \in y \Leftrightarrow z = x))$.

are classes which are the elements of other classes; and proper classes are those classes which are not sets. We shall therefore accept the following formal definitions:

$$\mathrm{Set}\, x \;:\Longleftrightarrow\; \mathrm{Cl}\, x \wedge \exists_y (\mathrm{Cl}\, y \wedge x \in y), \tag{df Set}$$

$$\mathrm{pCl}\, x \;:\Longleftrightarrow\; \mathrm{Cl}\, x \wedge \neg\, \mathrm{Set}\, x\,. \tag{df pCl}$$

We shall take it that only classes possess elements.[26] Formally:

$$\forall_{x,y} (x \in y \implies \mathrm{Cl}\, y). \tag{3.1}$$

Thus, the formula "$x \in y \wedge \mathrm{Cl}\, y$" is equivalent to the formula "$x \in y$".

It is known that principle (2.1) leads to contradiction under the assumption that in each case expressions of the form "class of Ps" (resp. "set of Ps") have referents. We can reconstruct Russell's paradox (antinomy) by putting in place of "P", for example, the expression "object which is not an element of itself". In fact, assume that the object R is the referent of the expression "class of objects which are not elements of themselves". Then $R \in R$ iff (by (2.1)) R is an object which is not an element of itself iff R is an object and $R \notin R$. Therefore, since R is an object, we obtain the contradiction: $R \in R$ iff $R \notin R$ (or: $R \in R$ and $R \notin R$). We may therefore draw the conclusion, that the expression"class of objects that are not elements of themselves" has no referent. We get an analogous paradox for the second use of principle (2.1) by using the expression "set of objects which are elements of themselves". Moreover, we obtain similar paradoxes when we replace "P" in the first version of principle (2.1) with the expression "class that is not an element of itself" (i.e., for $R :=$ the class of classes that are not elements of themselves") and in the second version of that principle with "set that is not an element of itself" (i.e., for $R :=$ the set of sets that are not elements of themselves"; details below p. 34).

From the perspective of the "extensional conception", the assumption that there might be classes (sets) which are their own elements appears unnatural.[27] Classes and sets which are not their own elements are

[26] In theories of classes built for the needs of 'pure' mathematics, it is assumed that there are only classes. They must therefore be 'pure classes', such as the empty class and classes whose elements are exclusively classes. In these theories the statement "$\forall_x\ \mathrm{Cl}\, x$" would be a thesis, and so the predicate "Cl" and principle (3.1) would be superfluous.

[27] For example, taking (2.1) into consideration, a class (set) of Ps which is itself an P. See: the class (set) of towns is not a town, the class (set) of nouns is not a noun, etc.

called *normal*. There is also not much support, from the perspective of the "intensional conception" for non-normal classes (sets). We will later show that the terms "class of objects", "class of classes" and "set of sets" are objectless and the expression "class of sets" signifies no set but at most a class.[28]

We observe that the use of a 'non-decomposable' predicate of the form "is a class of Ps" does not lead to contradiction. The case is similar with the predicate "is a set of Ps". We may therefore instead of (2.1) accept the following two correct definitions in which (2.1) is 'embedded':

$$x \text{ is a class of } \mathsf{Ps} :\Longleftrightarrow \mathrm{Cl}\, x \wedge \forall_y (y \in x \Leftrightarrow y \text{ is a } \mathsf{P}), \tag{df cl of Ps}$$
$$x \text{ is a set of } \mathsf{Ps} :\Longleftrightarrow \mathrm{Set}\, x \wedge \forall_y (y \in x \Leftrightarrow y \text{ is a a } \mathsf{P}). \tag{df set of Ps}$$

Since every set is a class, we obtain from these definitions the following:

$$\forall_x (x \text{ is a set of } \mathsf{Ps} \iff \mathrm{Set}\, x \wedge x \text{ is a class of } \mathsf{Ps}). \tag{3.2}$$

Directly from (df cl of Ps) and (df Set) (or (df set of Ps)) we obtain:

(a) There is no class of objects that are not their own elements.
(b) There is no class of normal classes.
(c) There is no set of objects that are not their own elements.
(d) There is no set of normal sets.

For (a) and (c) we repeat the reasoning used previously with the analysis of Russell's paradox (see pp. 33–33). For (b): Assume that x is the referent of the term "class of normal classes". Therefore, in the light of (df cl of Ps), $\mathrm{Cl}\, x$ and: $x \in x$ iff x is a class which is not its own element iff $\mathrm{Cl}\, x$ and $x \notin x$. It follows from this equivalence that $\neg \mathrm{Cl}\, x$. We therefore have a contradiction: $\mathrm{Cl}\, x$ and $\neg \mathrm{Cl}\, x$. For (d): We use (b) and (df Set).

It follows directly from (df cl of Ps) that if there is there is at least one proper class, then there is no class of classes and so there is no class of objects. Formally:

$$\exists_x \ \mathrm{pCl}\, x \implies \neg \exists_x \ x \text{ is a class of classes.} \tag{3.3}$$
$$\exists_x \ \mathrm{pCl}\, x \implies \neg \exists_x \ x \text{ is a class of objects.} \tag{3.4}$$

Assume x is a proper class and that y is a class of class (resp. objects). In the light of (df cl of Ps), we have $x \in y$ (in the second case, since x is

[28] On condition that we distinguish in general the meaning of the terms "class" and "set".

a class, it therefore an object), which contradicts the claim that x is a proper class.

It also results directly from (df cl of Ps) that if something is a class of normal sets, then it is not a set; i.e., that it is a proper class. Formally:

$$\forall_x (x \text{ is a class of normal sets} \implies \mathrm{pCl}\, x). \tag{3.5}$$

Assume that x is a class of normal sets. In the light of (df cl of Ps) we have $\mathrm{Cl}\, x$ and: $x \in x$ iff x is a set that is not its own element iff $\mathrm{Set}\, x$ and $x \notin x$. It follows from the equivalence that $\neg \mathrm{Set}\, x$. Therefore $\mathrm{Cl}\, x$ and $\neg \mathrm{Set}\, x$; that is $\mathrm{pCl}\, x$.

Obviously, we may not affirm without making additional assumptions, that the expression "class of normal sets" is non-empty.

A further direct consequence of (df cl of Ps) says that if two terms (S and P) designate the same objects and one of them (e.g., S) determines a class, then the second term (P) determines the same class:[29]

$$\begin{aligned} &\forall_y (y \text{ is an } \mathsf{S} \Leftrightarrow y \text{ is a } \mathsf{P}) \implies \\ &\qquad \forall_x (x \text{ is a class of } \mathsf{S}\text{s} \Rightarrow x \text{ is a class of } \mathsf{P}\text{s}). \end{aligned} \tag{3.6}$$

Also following directly from (df cl of Ps) is a result that is in a certain sense the opposite of condition (3.6). It says that if two terms determine the same class, then they refer to the same objects:

$$\begin{aligned} &\exists_x (x \text{ is a class of } \mathsf{S}\text{s} \wedge x \text{ is a class of } \mathsf{P}\text{s}) \implies \\ &\qquad \forall_y (y \text{ is an } \mathsf{S} \Leftrightarrow y \text{ is a } \mathsf{P}). \end{aligned} \tag{3.7}$$

Result (3.7) allows us to affirm that something which is a class determined by one term is not a class determined by a second term if those terms differ by even one referent.[30]

Conditions (3.6) and (3.7) say that in the case where terms of the form "class of Ss" and "class of Ps" are monoreferential names, we have the following equivalence: the class of Ss = the class of Ps iff $\forall_y (y$ is an $\mathsf{S} \Leftrightarrow y$ is a $\mathsf{P})$. That that each of these terms has at most one referent give us the so-called axiom of extensionality (3.8). That these terms are not empty (in certain cases) gives us axiom (3.11).

[29] A formalisation is given only for the term "class". In the condition below and further in condition (3.7), the term "class" may, however, be replaced with the term "set".

[30] Observe that we still have not made use of axiom of extensionality (3.8).

For all classes we assume the so-called axiom of extensionality which states that there are no distinct classes possessing exactly the same elements:

$$\forall_{x,y}\big(\mathrm{Cl}\,x \wedge \mathrm{Cl}\,y \wedge \forall_z(z \in x \Leftrightarrow z \in y) \implies x = y\big). \tag{3.8}$$

It follows from (df cl of Ps) and from the axiom of extensionality that what we assumed on p. 31, namely that the term "class of Ps" (resp. "set of Ps") never has more than one referent:

$$\forall_{x,y}(x \text{ is a class of } \mathsf{Ps} \wedge y \text{ is a class of } \mathsf{Ps} \implies x = y). \tag{3.9}$$

If x and y are classes of Ps then, on the light of (df cl of Ps), $\mathrm{Cl}\,x$ and $\mathrm{Cl}\,y$ and $\forall_z(z \in x \Leftrightarrow z \in y)$. Hence $x = y$, by (df Set).

From (3.6) and (3.9) follows a generalisation of the latter: if two terms signify the same objects, then they determine the exact same class:

$$\begin{aligned}&\forall_z(z \text{ is an } \mathsf{S} \Leftrightarrow z \text{ is a } \mathsf{P}) \implies\\ &\forall_{x,y}\big((x \text{ is a class of } \mathsf{Ss} \wedge y \text{ is a class of } \mathsf{Ps}\big) \Rightarrow x = y).\end{aligned} \tag{3.10}$$

If $\forall_z(z \text{ is an } \mathsf{S} \Leftrightarrow z \text{ is a } \mathsf{P})$ and y is a class of Ps, then y is also a class of Ss, by (3.6). So $x = y$, by (3.9).

It follows from the above that if the term "class" were to interest us only the context of "class of Ps", then (3.9) would suffice as the axiom of extensionality. In fact, (3.10) follows from (3.9) and (df cl of Ps). We thus obtain from (3.10) and (df cl of Ps): if x is a class of Ss, y is a class of Ps, and x and y have the same elements, then $x = y$.

We should accept a principle which generates classes but protects us against contradictions. Let us establish a terminological short-hand before we proceed to its formulation, namely that instead of the predicate "is an element of some class" we will just use "is an element".[31] In

[31] One may accept this also as an abbreviation of the predicate "is an element of something". In the light of (3.1), only classes have elements.

This reflects our everyday way of talking. For example, the predicate "is a father" may be used as an abbreviation of "is a father of someone" or "is a father of some person". Speaking thus not only makes our utterances shorter but frees us from an interpretational problem with plurals. The problem is: what form is the predicate "is an element of some class" to have in the plural? It seems that neither "are elements of some class" nor "are elements of some classes" express what we are after. The former may suggest that we are talking of objects belonging to one common class. The latter may suggest that each object about which we are speaking is to belong to more than one class. By using the abbreviation, we can simply have "are elements" as the plural. This is to mean "they are objects of which each is an element" and only here shall we use the long form of this abbreviation for "is an element".

the framework of Morse's system of the theory of classes[32], we accept a principle of the following form (for the second formulation we use (df cl of Ps)):[33]

$$\exists_x \big(\mathrm{Cl}\, x \wedge \forall_z (z \in x \iff (z \text{ is a } \mathsf{P} \wedge \exists_y\, z \in y))\big), \quad \text{or} \qquad (3.11)$$

There is a class of Ps which are elements.

Using principles (3.8) and (3.11), we obtain that an arbitrary term of the form "class of Ps being elements" is a monoreferential name. If all Ps are elements then that class is also the unique designation of the term "class of Ps". In fact, in this case, the term in place of "P" and the term obtained from "P which is an element" signify exactly the same objects, so we use (3.6). Therefore we have:

$$\forall_z (z \text{ is a } \mathsf{P} \Rightarrow \exists_y\, z \in y) \implies \exists_x\, x \text{ is a class of } \mathsf{P}\text{s}, \quad \text{or} \qquad (3.12)$$

Every P is an element $\implies$ there is a class of Ps.

Note that from (3.12) we obtain:

$$\forall_x \exists_y \big(\mathrm{Cl}\, y \wedge \forall_z (z \in y \Leftrightarrow z \text{ is a } \mathsf{P} \wedge z \in x)\big) \quad \text{or} \qquad (3.13)$$

$\forall_x$ there is a class of Ps belonging to x.

If x is not a class then we can put $y = \emptyset$ in the light of (3.1).[34] If x is a class then we substitute in (3.12) the schema "P belonging to x" in place of "P". Then the antecedent of (3.12) is true. So something is a class of Ps belonging to x.

In virtue of (3.12), we have:

(e) There is a class of elements.
(f) There is a class of sets.
(g) There is a class of normal sets.

We are not going to assume that some objects are not classes nor that if in general there are non-classes[35], then every non-class is an element.

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[32] On the subject of Morse's system, see, e.g., Chapter VII, where this system is formulated in a first-order language.

[33] In the system NBG (von Neumann-Bernays-Gödel), the situation is more complicated. In brief, we assume in that system that all quantificational expressions occurring in the expression in place of "P" must refer to elements.

[34] Of course, this solution can also be adopted if either $x = \emptyset$ or there is no P.

[35] Cf. footnote 26 and [Mendelson, 1964, p. 160].

Non-classes which are elements we call *individuals* or *urelements* (from the German prefix: ur-, 'primordial'). As we are not assuming that there are non-classes, we cannot therefore assume that there is some individual.[36] In virtue of (3.12), we have:

(h) There is a class of individuals.

Moreover, by (df cl of Ps), (3.9), and either (3.11) or (3.12), we have:

(i) there is a class of Ps which are not Ps,
(j) there is a empty set.
(k) there is a non-empty set.

For (i): We exploit the fact that the term "P which is not a P" is contradictory, so also empty, and thus signifies exactly the same objects as the term "P which is not a P but which is an element". By (3.11), the second term postulates some class. Via the axiom of extensionality, there is only one such class. Moreover, by (3.10), both terms postulate the same class. For (j): In the light of (df cl of Ps), this class has no element. We call x the *empty class* and symbolise it as "$\emptyset$". One of the postulates of Morse's theory states that:

$$\emptyset \text{ is a set.}$$

For (k): The term "the object identical with $\emptyset$" signifies only $\emptyset$. Thus, (3.12) guarantees us that something is a class of objects identical with $\emptyset$. In virtue of (df cl of Ps) the postulated class has exactly one member. It is $\emptyset$. We represent this class by "$\{\emptyset\}$". We obtain in the light of a certain axiom in Morse's theory the result that $\{\emptyset\}$ is a set. We therefore have our first non-empty set. By (3.11), for any sets x and y there is a class consisting of x and y (so-called a *pair*). In fact, in place of "P" we put the general term "object which is identical with x or y" (since x and y are sets, so they are elements). We represent this class by "$\{x, y\}$"; if $x = y$ then we use "$\{x\}$". In Morse's theory we have that for any sets x and y the pair $\{x, y\}$ is a set.

We say that a class x *is included in* a class y (we write: $x \subseteq y$) iff $\forall_z(z \in x \Rightarrow z \in y)$.

The terms "class of elements", "class of sets", "class of normal sets", and "class of individuals" are monoreferential names. For theses term let us establish, respectively, the following abbreviating symbols: "V", "S",

[36] In [Nowaczyk, 1985, p. 53] all non-classes are individuals and there are at least two of them.

"NS", and "I". We see that $\emptyset \in \mathsf{NS}$, $\mathsf{NS} \subseteq \mathsf{S}$, and $\mathsf{S} \subseteq \mathsf{V}$. It is not possible to state any other dependencies between NS, S, and V without further assumptions. It is also not possible to claim that I is a set nor that $\mathrm{I} \neq \emptyset$.

From (3.5) it follows that:

(l) NS is a proper class.

From this and, respectively (3.3) or (3.4), it follows that:

(m) There is no class of classes.
(n) There is no class of objects.

We observe that by using (3.13), (a) and (b) we obtain (m) and (n) without using (3.3), (3.4), and (3.5). In fact, assume for a contradiction that y is a class of objects (resp. classes). Then, via (3.13), something would be a class of objects (resp. classes) which are not their own elements, but elements of y. It follows from this that something would be a class of objects (resp. classes) which are not their own elements because being an element of y reduces to being an object (resp. class). This contradicts statement (a) (resp. (b)).

We will show further that in the Morse's system, the non-empty terms "class of elements" and "class of sets" (compare (e) and (f)) also do not signify sets. In order to prove them, however, we must accept a further additional axiom.

A possibility would be the so-called "axiom of subsets", which is often formulated without appeal to principle (3.11), by accepting directly that all common members of some set and some class create a set. Formally:

$$\forall_{x,y}(\mathrm{Set}\, x \wedge \mathrm{Cl}\, y \implies \exists_z(\mathrm{Set}\, z \wedge \forall_u(u \in z \Leftrightarrow u \in x \wedge u \in y))). \quad (3.14)$$

Using the axiom of extensionality (3.8) and (3.14) we obtain:

$$\forall_{x,y,z}(\mathrm{Set}\, x \wedge \mathrm{Cl}\, y \wedge \mathrm{Cl}\, z \wedge \forall_u(u \in z \Leftrightarrow u \in x \wedge u \in y) \implies \mathrm{Set}\, z) \quad (3.15)$$

From (3.15) and (df cl of Ps) we have:

$$\forall_{x,y,z}((\mathrm{Set}\, x \wedge \mathrm{Cl}\, y \wedge z \text{ is a class of common members of } x \text{ and } y) \implies \mathrm{Set}\, z).$$

Hence, by (3.13) and the axiom of extensionality, we obtain:

$$\forall_{x,y}(\mathrm{Set}\, x \wedge \mathrm{Cl}\, y \implies \text{the class of common elements } x \text{ and } y \text{ is a set}). \quad (3.16)$$

This formulation suggests that it suffices to accept for the theory of classes the principle of "subsets" whose form is (3.16).

It follows from the axiom of subsets that any class which is included in some set is also a set. In fact, this class is the class of its members and members of this set. Formally we have:

$$\forall_{x,y}(\mathrm{Set}\,x \wedge \mathrm{Cl}\,y \wedge y \subseteq x \implies \mathrm{Set}\,y). \tag{3.17}$$

From this it follows that we can strengthen principle (3.13):

$$\begin{aligned}&\forall_x(\mathrm{Set}\,x \implies \exists_y(\mathrm{Set}\,y \wedge \forall_z(z \in y \Leftrightarrow z \text{ is a } \mathsf{P} \wedge z \in x))) \quad \text{or}\\ &\forall_x(\mathrm{Set}\,x \implies \text{the class of } \mathsf{P}\text{s belonging to } x \text{ is a set}).\end{aligned} \tag{3.18}$$

Let x be a set. Then the class of Ps belonging to x is included in x. We can get this directly from (3.14). In fact, let y be the class of Ps which are elements; i.e., for any z: $z \in y$ iff z is a P and $\exists_u\, z \in u$. Hence: $z \in y$ and $z \in x$ iff z is a P and $z \in x$ and $\exists_u\, z \in u$ iff z is a P and $z \in x$.

From (3.17) and (l) it follows that:

(o) V is a proper class.
(p) S is a proper class.

Since NS $\subseteq$ V, NS $\subseteq$ S NS is not a set, so also V and S are not sets.

Note that facts (o) and (p) are equivalent, respectively, to:

(q) There is no set of elements.
(r) There is no set of sets.

For any x we have: x is a set of elements iff Set x and $\forall_z(z \in x \Leftrightarrow z$ is an element) iff Set x and $\forall_z(z \in x \Leftrightarrow z$ is an element) and Cl x iff x is a class of elements. We can prove this this in a similar way for a set of sets.

Note that the principle (3.18) holds in Zermelo's system as well. Moreover, in this system statements (c) and (d) follow from (df set of Ps) itself. From this and (3.18) we obtain in the system the corresponding statements (q) and (r). That is, we do not have here a sets of set nor — a fortiori — a set of elements. In fact, assume that x is a set of sets. We put the term "normal set" in place of "P" in (3.18). It follows from the accepted definitions that for any y: $y \in x$ and y is a normal set iff y is a normal set. From this and (3.18) it follows that there is a set of normal sets. We have therefore obtained a contradiction with (d). We therefore have (r). From this, reasoning in an analogous fashion as before, we get (q).

In Morse's theory, we accept (amongst others) the axiom of foundation from which it follows that each class is normal:

$$\forall_x(\mathrm{Cl}\, x \implies x \notin x). \tag{3.19}$$

In the light of (3.1), only non-empty classes have elements. From this — via (3.19) — it follows that no object is its own element:

$$\forall_x\, x \notin x. \tag{3.20}$$

In Morse's theory therefore, the terms "object which is not its own element", "normal class", and "normal set" signify, respectively, exactly the same objects as the terms "object", "class", and "set". From this with (a)–(d)[37], (l), and (3.6) we obtain respectively (m)–(r) without making use of the principle of subsets.

4. Leśniewski's conception of classes (sets) and their elements

In this section we shall show that Leśniewski accepted a collective conception of classes (sets) and their elements and that in general did not allow the existence of distributive classes (sets).[38] In other words, and taking a neutral stance on the meaning of the word "exist", he treated the terms "distributive set" and "distributive class" as empty.

We will reproduce below some long passages from [Leśniewski, 1927] in order to present Leśniewski's views accurately. To this end, let us say straightaway that, for Leśniewski, the non-relative terms "set" and "class" would be equivalent in meaning (obviously, they would both be understood in the collective sense). The same goes for the non-relational predicates (one-argument predicates) "is a set" and "is a class". In brief, for Leśniewski — as we will show in Section 6 — each set would be a class and vice versa. We are speaking subjunctively, as it were, because such

[37] It is easy to see that by having (3.20) one may obtain (a)–(d) without incurring Russell's paradox. And so for (a), using (3.20), we have: if x is a class of objects which are not their own elements, then $\forall_y\, y \in x$. Hence $x \in x$ which contradicts (3.20). We obtain contradictions for (b)–(d) in an analogous way.

[38] Cf. footnote 18, in which Leśniewski's philosophical views are sketched. Leśniewski thus considered that collective classes (sets) exist and that nothing exists which may be called a distributive set or distributive class [see the passage from Leśniewski, 1927 on p. 43].

non-relative terms appear only in commentaries on Leśniewski and not in theses of his theory.

With Leśniewski, we find only counterparts of "class of Ps" and "set of Ps". The term "class of Ps" — if we use the non-relative term "class" ("set") and the metaphor of footnote 22 — means the same as "class (set) of all Ps" [cf. Quine, 1953, p. 186], and the term "set of Ps" means the same as the term "class (set) composed of some Ps" (the word "some" allows for the possibility of all). The first of these names may be for Leśniewski either empty or monoreferential, and the second one may be either empty, or monoreferential, or polyreferential. It is not possible to say in the theory that the class of Ps is the set of all Ps, because the term "set of all Ps" cannot be formed in the language of the theory.

In accordance with Leśniewski's theory: if x is a class of Ps, then x is a set of Ps; moreover: if x is a set of Ps, then x is a class of Ps which are elements of x or — simply — x is a class of elements of x. Every class is therefore a set and every set a class, if we accept that a class (resp. set) is a class (resp. set) of something.

Leśniewski stated that Cantor's theory of sets relates to collective sets — just as with his mereology. The follows passages show this:[39]

> My conception is, in this respect, on the one hand (as far as I have managed to observe) entirely consistent with the way the expressions "class" and "set" are used in the common, everyday language of people who have never held neither any "theory of classes" nor any "theory of multitudes". On the other hand, it is based on a strong academic tradition, running more or less continuously through countless past and present scholars, and in particular through George Cantor.
>
> [Leśniewski, 1927, p. 190]

In Leśniewski's opinion therefore, mereology deserves the title of "The foundation of mathematics" in the same way as in Cantor's theory, since both theories are concerned with the same sets (classes). Leśniewski's main work, in which he presented his mereology, he thus called "On the foundations of mathematics" [Leśniewski, 1927, 1928, 1929, 1930, 1931]. An earlier work pertaining to mereology carried the title "The foundations of the general theory of multitudes" [Leśniewski, 1916].

[39] The passages from Leśniewski's papers have been translated from the original and not taken from the English edition of this work [Leśniewski, 1991b].

It is today beyond discussion that Cantor was analysing distributive sets in his theory. Leśniewski might have been misled by the 'definitions' of set given by Cantor:

> By a set is meant a gathering into one whole objects which are quite distinct in our intuition or our thought.
>
> [Cantor, 1932, p. 282]; see [Burbaki, 1994, p. 25]
>
> By the concept of 'set' (*Menge*) we are to understand each collection into one whole M of specified, clearly distinct objects m of our inspection or our thought (which are called elements of M).
>
> [Murawski, 1995, p. 68]; cf. also [Murawski, 1984, p. 78]

The theory of types created by Russell and Whitehead and the theory of classes as the extensions of concepts created by Frege were both for Leśniewski objectless:

> I don't know what Russell and Whitehead understand in the commentaries on their system by class. The fact that, on their position, "class" is supposed to be the same as "extension" does not help me in the slightest, as I don't know what these authors mean by extension. I don't therefore know either, when they consider the matter of the existence or non-existence of objects as such whether their thoughts on the puzzle of existence and non-existence address those objects which are classes. [...] Not understanding the relevant terminology of Whitehead and Russell, I am not in particular aware where and to what degree their doubts as to the existence of objects, which are classes in their understanding of that term[40], may bear on particular positions I take in the theory of classes sketched earlier. In "*Principia Mathematica*", I did not find a single paragraph which I felt there was even the weakest presumption of calling into question the existence of classes as I understand them. Sensing in the "classes" of Whitehead and Russell, in a

[40] The authors of the *Principia Mathematica* do in fact introduce as a problem the question of the existence of distributive sets. This is why, as Quine [1953, p. 122–123] writes: "Russell ([...], *Principia*) had a no-class theory. Notations purporting to refer to classes were so defined, in context, that all such references would disappear on expansion. This result was hailed by some, notably Hans Hahn, as freeing mathematics from Platonism, as reconciling mathematics with an exclusively concrete ontology. But this interpretation is wrong. Russell's method eliminates classes, but only by appeal to another realm of equally abstract or universal entities — so-called propositional functions. The phrase 'propositional function' is used ambiguously in *Principia Mathematica*; sometimes it means an open sentence and sometimes it means an attribute. Russell's no-class theory uses propositional functions in this second sense as values of bound variables; so nothing can be claimed for the theory beyond a reduction of certain universals to others, classes to attributes." See also [Quine, 1981, p. 121, footnote 1] and [Quine, 1970, p. 68].

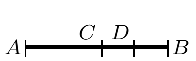
> similar fashion as with the "extensions of concepts" of Frege, the scent of mythical paradigms from a rich gallery of "invented" objects, I cannot for my part divest myself of the inclination to sympathise "on credit" with the doubts of the authors on the matter of whether objects that are such "classes" exist in the world. — On the matter of the relation of my conception of class to the views represented in the commentaries of Whitehead and Russell on their system, a certain light may be here thrown by the views of Russell on "heaps". Russell writes in one of his works: "We cannot take classes in the *pure* extensional way as simply heaps or conglomerations. If we were to attempt to do that, we should find it impossible to understand how there can be such a class as the null-class, which has no members at all and cannot be regarded as a "heap"; we should also find it very hard to understand how it comes about that a class which has only one member is not identical with that one member. I do not mean to assert, or to deny, that there are such entities as "heaps". As a mathematical logician, I am not called upon to have an opinion on this point. All that I am maintaining is that, if there are such things as heaps, we cannot identity them with the classes composed of their constituents" [The passage Leśniewski refers to is to be found in Russell's *Introduction to Mathematical Philosophy*, p. 183]. If I understand the cited paragraph correctly, then the fact that a certain object P is a "heap" of some as, composed of all as, would still not be for Russell a sufficient basis on which to affirm that the object P is a "class" of objects a. Russell's terminology would remain most clearly in complete discord with my terminology; in accordance with his use of the expressions "class" and "set", and the use of the expression "heap" in our common, everyday language [...], I could always say of a "heap" of some as, that it is a set of objects a [of as], but of a "heap" of objects a [of as] composed of all as, that it is the class of objects a [of as]. [...] The difficulty is in understanding in what consists the difference a "heap" of objects a [of as] and "class" of objects a [of as] from Russell's point of view, if both such things existed and if each of them were *composed* of all as and it is a difficulty which I do not know how to overcome. [Leśniewski, 1927, pp. 204–205]

In the text above, one of the statements Leśniewski makes may be paraphrased in the following way: I can always say of any 'heap' of some Ps that it is a collective set of Ps, whereas of a 'heap' of Ps composed of all Ps, that it is a collective class of Ps.[41] Leśniewski is right about this point. He could not, however, understand "on what rests the difference

[41] Compare the comments on p. 41 on Leśniewski's understanding of the counterparts of the schema "class of Ps" and "set of Ps".

between" a 'heap' of Ps (that is, a collective class of Ps) and a distributive class of Ps, "if both such things existed and if each of them were composed of all" Ps.

Leśniewski was 'inspired' to create his own conception of classes (sets) and their elements by Russell's antinomy.[42] In order to solve the problem of 'the class of classes not being their own elements', Leśniewski made a certain assumption, which is introduced in the text below and from which it follows that his conception distinguishes itself considerably from Cantor's.

> Wishing "to conceive of something" and not knowing at the same time how to find any reasonable fault in any of the aforementioned assumptions on which the earlier "antinomy" rests, nor also in the reasoning leading to contradiction on the basis of those assumptions, I began to muse on examples of situations in which in practice I consider or do not consider such and such objects as classes or sets of such and such objects [...] and to submit for critical analysis my faith in the particular assumptions of the "antimony" in hand from that point of view (the puzzle of "empty classes" was not the theme of my considerations on that occasion because I treated the conception of "empty classes" from my first moment of contact with it as a "mythical" conception, taking without any hesitation the position that:
>
> (1) if any object is a class of objects a, then some object is an a.)
>
> I came via that to the conviction that:
>
> (2) it still happens, that such and such an object is a class of such and such objects and at the same time a class of completely different objects (as, for example, the segment AB in Figure 1 is the class of segments being the segment AC or the segment CB, and at the same time the class of segments being the segment AD or the segment DB) and that
>
> 
>
>
> Fig. 1
>
> (3) if one and only one object is P, then P is a class of objects of P (as, for example, the segment AB from Figure 1 is the class of segments of AB from Figure 1).
>
> Trying to grasp in what way I also really use expressions of the type "P is subordinated to a class K", which [...] I used *promiscue* with corresponding expressions of the form "P is an element of a class K', I established a definition according to which I stated that:

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[42] Various versions of Russell's antimony (paradox) were presented on p. 33. We will look at Leśniewski's solution on pages 46 and 66.

(4) P is subordinated to a class K if and only if for some meaning of a expression "a" the following conditions are fulfilled: α) K is a class of objects a [of as], β) P is an a. [Leśniewski, 1927, pp. 185–187]

Leśniewski's solution to Russell's antinomy under the assumptions was already 'straightforward'. From (2) and (4) it followed that every class is its own element (more exactly: each object is a class and is its own element), i.e., that there are no normal classes and all classes are non-normal classes. Since the term "normal class" is empty, we obtain from (1) that "class of normal classes" is also empty, because there are no empty classes (cf. p. 66).

We can already see in this solution two fundamental differences between Leśniewski's theory and Cantor's theory. First, in the second of these theories something is an empty class.[43] Second, according to the "extensional conception" accepted by Cantor, it is completely justified to accept a principle that says that every class is normal.[44] Thus one may regard the term "non-normal class" as empty.

Leśniewski states in his commentary on definition (4) that "[it] harmonises completely with the common way the expression 'element" is used in practice by 'theoreticians of multitudes'" [Leśniewski, 1927, p. 187]. It may be accepted in a certain sense that this is essentially so — on condition that we successfully 'deal with' the matter of how to understand the clause "for some meaning of 'a'" occurring in (4). The problem lies with the fact that that clause in condition (α) is not understood by Leśniewski in the same way as by those "theoreticians of multitudes", i.e., in the same way as it appears in (df cl of Ps). This last is connected with principle (2.1), which Leśniewski simply did not accept.[45] Leśniewski would therefore consider that a given object is a

[43] In the theory of distributive classes, the term "class of normal classes" is also objectless (cf. (b) on p. 34), but this does not argue for the conclusion that there is no empty class.

[44] See [Wang, 1994, p. 267]. Cf. also the axiom of foundation (3.19).

[45] Leśniewski's definition (4) has nothing in common with principle (2.1). In Leśniewski's terminology principle (2.1) would have the form: an object P is an element of a class of as iff P is an a. It follows from (2) that Leśniewski accepts just the right-to-left part of (2.1). That is, he identifies: segment AB; the set of segments AC and CB; the set of segments AD and DB. Therefore for him the segment AC is an element of the class of segments AD and DB, because we may take it that "a" in (4) means just the same as "segment which is one of the segments of AC and CB."

Leśniewski [1927, p. 189] directly writes about the rejection of (2.1) on the basis of his conception of classes and their elements.

class of as "for some of a expression 'a' ", however, for "theoreticians of multitudes", that object would not be a class of as.

Let us now turn our attention to one further disagreement between Leśniewski's theory and the theory deriving from Cantor. *Ad* 2: Under Cantor's conception of sets, the segment AB from Figure 1 is a (distributive) set of certain points and not a (distributive) set of segments. The two-element set composed of the segments AC and CB is disjoint from the two-element set composed of the segments AD and DB. In no case is it possible to identify these distributive sets with the segment AB. That segment is simply the set-theoretic sum of segments AC and CB and the sum of segments AD and DB. These segments — as distributive sets composed of certain points — are subsets of the segment AB. *Ad* 3: In Cantor's theory, the one-element set composed of the segment AB is not the segment AB. *Ad* 4: It follows from definition (4) and the definition of being a class of as, it follows that being a (collective) element reduces to being an ingrediens.[46]

The primary concept in Leśniewski's theory is that of being part. As we have already noted in Section 1, Leśniewski assumed that the relation *is a part of* is symmetric and transitive. Let us give his formulation of these assumptions given in [Leśniewski, 1928], as we shall need them in the next section for the analysis of Leśniewski's logical system:

> *Axiom I.* If P is a part of an object Q, then Q is not a part of the object P.
>
> *Axiom II.* If P is a part of an object Q and Q is a part of an object R, then P is a part of the object R. [Leśniewski, 1928, pp. 263–264]

Leśniewski next establishes two definitions: *is an ingrediens of* and *is a class of*:

> *Definition I.* P is an ingrediens of an object Q if and only if P is the same object as Q or is a part of the object Q.
>
> *Definition II*, P is a class of objects a [of as] if and only if the following conditions are met:
>
> α) P is an object;
>
> β) every a is an ingrediens of P;
>
> γ) for any Q — , if Q is an ingrediens of the object P then some ingrediens of the object Q is an ingrediens of some a.
>
> [Leśniewski, 1928, p. 264]

[46] See thesis XVI and XVII in [Leśniewski, 1928, p. 272] and further (5.7). It is not possible at this point to give a proof of this fact, because we still have not given a definition of being a class of as.

This raises the following question: *What role does condition* (α) *play in Definition II?* We shall attempt to answer this in the following section.

Using these concepts, Leśniewski introduces two axioms:

> *Axiom III.* If P is a class of objects a [of as] and Q is a class of objects a [of as], then P is Q.
>
> *Axiom IV.* If some object is an a, then some object is a class of objects a [of as]. [Leśniewski, 1928, p. 266]

Leśniewski defines two further concepts we have been concerned with in this section as follows:

> *Definition III.* P is a set of objects a [of as] if and only if the following conditions are met:
> α) P is an object;
> β) for any Q — , if Q is an ingrediens of an object P, then some ingrediens of the object Q is an ingrediens of some a being an ingrediens of the object P.
>
> *Definition IV.* P is an element of an object Q if and only if for some a — (Q is a class of objects a [of as] and P is an a).
> [Leśniewski, 1928, pp. 270 and 272]

Definitions II and III have the same condition (α). Conditions (α)–(γ) in Definition II entail condition (β) in Definition III [Leśniewski, 1928, Theorem VI]. The class of as is therefore one of the sets of as [Leśniewski, 1928, Theorem XIII; cf. (6.3)]. Definition IV is a slightly different formulation of definition (4) given on p. 46 in the quote from [Leśniewski, 1927]. We also have problems with the interpretation — in the context in which it is used — of the quantified expression "for some a".

5. Leśniewski's logic

1. In order to reply to the question of why condition (α) appears in definitions II and III, we must get to know more about the system of logic used by Leśniewski. We must therefore learn about the way in which variables, quantifiers and the connectives "is" and "is not" are used by him.

Leśniewski's mereology was formulated in a specific way, distancing it from the standard formulations. The theory was 'built on top of' another system of Leśniewski's which he called "ontology".

Leśniewski considered that it was not the task of logic to look into whether a given name is empty, or monoreferential, or polyreferential (see Remark 2.2). He believed that the laws of logic should be valid for all names. This also explains why he wanted it to be possible for his systems to be used (ontology and mereology) with arbitrary names replacing free variables. He did not want the substitution of empty names to lead to contradiction, as happens in the case of their use in classical logic.[47] To this end, Leśniewski fixed a specific interpretation of the copula "is" and — this being connected with it — a specific interpretation of the expression "is not". He formulated these interpretations only in his ontology — in the theory which appeared after his theory of mereology. We need not give the formalisation of his ontology here. It suffices to state only what Leśniewski assumed explicitly in [Leśniewski, 1927].

Let "n" be the index of the category of names and "s" be the index of the category of sentences. For Leśniewski, the copula "is" and the expression "is not" are sentence-forming functors taking two names as arguments, i.e., they have the following index [see Ajdukiewicz, 1934] :

$$\frac{\mathrm{s}}{\mathrm{n}\quad\mathrm{n}}\,.$$

If the letters "S" and "P" represent arbitrary names, then — for Leśniewski — expressions of schemas "S is a P" and "S is not a P" are sentences in a logical sense and have true-values (i.e., they are true or false). A sentence whose schema is "S is a P" (resp. "S is not a P") *is true* iff the name represented by "S" is monoreferential and its only referent is (resp. is not) a referent of the name represented by "P"; otherwise, such a statement *is false*.

Note that schemata "S is an object" and "S is an S" are equivalent and not tautological. In fact, both sentences of such forms are true iff the name represented by "S" is monoreferential.

Moreover, the schema "S is not a P" is equivalent to "S is an object and $\neg$ S is a P". Hence the schema "S is not an object" is contradictory (it is equivalent to "S is not an object and $\neg$ S is an object', which has the form "$p \wedge \neg p$"'). So the schema "$\neg$ S is not an object" is tautological (it is equivalent to "S is an object or $\neg$ S is an object", which has the form "$p \vee \neg p$").

Although Leśniewski used only one syntactic category of names (see Remark 2.2), we see in the axioms and definitions two types of name-

[47] Let we stress straightaway that his system was not a system of free logic in which empty names have replaced free variables. We will return to this matter in points 3 and 4.

variable: upper-case letters "P", "Q", "R"; and the lower-case letter "a". This has, however, does not — speaking formally — matter. It is a 'purely visual' procedure. Upper-case letters — used as variables — are supposed to 'suggest' that a given atomic formula changes into a true sentence only with the substitution of monoreferential names for those variables.

We may write the fact, therefore, that *is a part of* is irreflexive and *is an ingrediens of* is reflexive within Leśniewski's system thus [cf., e.g., Leśniewski, 1928, theorems I and II]:

(5.1) If P is an object, the P is not a part of the object P.
(5.2) If P is an object, then P is an ingrediens of the object P.

We have used an upper-case letter as a variable, because the antecedent turns into a true sentence only with the substitution of a monoreferential name for it. On the other hand, if we substitute an arbitrary monoreferential name for the variable, then the antecedents and the consequents will be true. These the consequents expressing, respectively, the irreflexivity of the concept of *being a part of* and the reflexivity of the concept of *being an ingrediens of.* We further observe that, with the substitution of an empty term or a polyreferential term for the variable, both pairs of antecedents and consequences will be false (see the interpretation of the expressions "is" and "is not"), and therefore that without the antecedent the theorems (only the consequences) would not be tautological.

Analogous comments relate to the conclusion below which follows from Definition II and (5) [cf. Leśniewski, 1928, Theorem VII]:

(5.3) If P is an object, then P is the class of ingrediens of the object P.

For Leśniewski, the expressions "part of", "ingrediens of" , and "element of" were name-forming functors taking names as arguments and therefore had the index:

$$\frac{\mathrm{n}}{\mathrm{n}}\,.$$

The following 'interpretational' theorem holds for mereology (based on ontology).

Theorem 5.1. (i) *We obtain from the name-formula "part of Q" a non-empty term only if we substitute a monoreferential name for the variable "Q".*[48]

[48] The theorem is not reversible because mereology does not rule out objects without parts.

(ii) *We obtain from the name-formula "ingrediens of Q"* (*resp. "element of Q"*) *a non-empty term if and only if we substitute a monoreferential term for the variable "Q".*

Before giving a proof of these theorems, let us first note that they do not appear in Leśniewski's work. Point (i) and the left-to-right part from (ii) written in the 'intra-system' notation would have the forms:

(5.4) If P is a part of Q, then P is an object and Q is an object.

(5.5) If P is an ingrediens of Q, then P is an object and Q is an object.

(5.6) If P is an element of Q, then P is an object and Q is an object.

These statements do not appear explicitly in Leśniewski's work either. It is possible to find traces of (5.4) and (5.5) in the proof of Theorem X [Leśniewski, 1928, pp. 269–270], where the premise "P is an ingrediens of Q" reduces (with the help of Definition I) to the form "P is the same object as Q or P is a part of Q", and from this the conclusion "Q is an object" may be drawn — underlining that this results from Axiom I. This justification clearly pertains to the conclusion drawn from the second disjunct of the disjunction: if P is a part of Q, then — in virtue of Axiom I — we have that Q is not a part of P, and hence that Q is an object. Statement (5.6) follows from (5.5) and from the left-to-right part in the statement below, which Leśniewski formulated in theorems XVI and XVII in [Leśniewski, 1928, p. 272]:

(5.7) P is an element of Q iff P is an ingrediens of Q.

Assume that that P is an element of Q. Then — in virtue of (4) or Definition IV — for some meaning of "a", we have that Q is a class of as and P is an a. It follows from Definition II that P is an ingrediens of Q. Conversely, let P be an ingrediens of Q. Then P is an object and Q is an object. (see (5.5)). Let us assume that "a" means the same as "ingrediens of Q". It follows from Definition II that Q is a class of ingredienses of Q [cf. Leśniewski, 1928, theorems VII and X]. Hence, in the light of the assumption and Definition IV, we obtain that P is an element of Q.

Since the non-emptiness of the predicate "part of Q" is essential for the truth of a sentence of the form "P is a part of Q", in virtue of Theorem 5.1, we use an upper-case letter as the variable. It is similarly the case for the predicates "ingrediens of Q" and "element of Q".

Proof of Theorem 5.1. (i) Suppose that we have obtained a non-empty term for some substitution of the name-formula "part of Q". In order to prove our statement, we will use its 'intra-system' version, i.e., (5.4) and certain theorems in Leśniewski's ontology. Namely, from (5.4) we obtain a different 'intra-system' theorem: "if there is an object P such that P is a part of Q, then Q is an object". Hence we have "if some object is a part of Q, then Q is an object". The antecedent of the previous statement expresses exactly the assumption we had made (i.e., "part of Q" is a non-empty term). We therefore obtain a true antecedent by carrying out the substitution. And this is so only when we substitute a non-empty term for "Q".

(ii) We prove the left-to-right part in the same way as (i), but using (5.5) (resp. (5.6)). The right-to-left part results from (5.2) (resp. both (5.2) and (5.7)). □

To recapitulate: in Leśniewski's theory, the name-formula "part of P" gives us:

(i) an empty term iff for "P" we substitute either an empty name, or polyreferential, or a monoreferential name whose sole referent has no part.
(ii) a term having at last referents iff for "P" we substitute a monoreferential term whose sole referent has at last two parts.[49]

The expressions "ingrediens of P" and "element of P" give us:

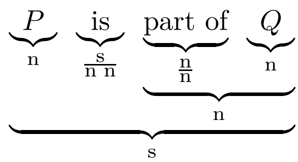
(i) an empty term iff for "P" we substitute either an empty term or a term having at last two referents.
(ii) a monoreferential term iff for "P" we substitute a monoreferential term whose sole referent has no part[50].
(iii) a polyreferential term iff for "P" we substitute a monoreferential term whose sole referent has at lest two parts.

In Leśniewski's theory, the expressions "is a part of" and "is not a part of" are not in general functors. This is borne out by, for example,

[49] In Leśniewski's mereology no object has exactly one part. This is why the name-formula "part of P" will not give us a monoreferential term. This is easy to prove using definitions I and II and axioms I and III. It follows from these definitions that if Q is an object, then Q is a class of Q [see Leśniewski, 1928, Theorem VIII]. Assume therefore that an object Q is the only part of an object P. It then follows from definitions I and II, that P is a class of Q [cf. Leśniewski, 1928, Theorem IX]. Hence P is Q, by Axiom III. This contradicts the claim that Q is a part of P (cf. (5.3) which results from Axioms I).

[50] Then the only ingrediens of the object P is P itself.

the category diagram below of the formula "*P* is a part of *Q*", where the syntactic categories are those employed by Leśniewski:[51]



We can therefore write the formula "*P* is a part of *Q*" symbolically — using Leśniewski's categories — as "$P\ \varepsilon\ \mathrm{pr}\, Q$". In this formula "ε" symbolises the copula "is" from Leśniewski's ontology and "pr" symbolises the name-forming functor "part of" interpreted in the way presented below.

2. We can now reply to the question: *why does condition* (α) *appear in definitions II and III?* If we replace an arbitrary empty name for the variable "*P*" in those definitions then — in line with the interpretation of the copula "is" — their definienda and condition (α) are transformed into false sentences. Thus, their definienses will also be false, i.e., from the equivalence we obtain a true sentence.

We note, however, that for a substitution of an empty name for "*P*", conditions (γ) in Definition II and (β) in Definition III are transformed into true sentences (the implications in them have unsatisfied antecedents). It is therefore not possible to eliminate condition (α) from Definition III.

We observe in turn that, using the so-called weak understanding of "Every … is …", condition (β) in Definition II is transformed into a true sentence if we substitute an empty name for the variable "*a*". Under the weak interpretation of the expression "Every … is …"[52] it is therefore not possible to eliminate condition (α) from Definition II, because, for a substitution of empty names for the variables "*P*" and "*a*", its definiendum would be false and conditions (β) and (γ) would be true.

Leśniewski, however — as he himself stresses in footnote 2 on p. 264 in [Leśniewski, 1928] — used the expression "Every … is …" in its

[51] We obtain an analogous diagram if we replace "is" by "is not" and/or the expression "part of *Q*" by the expressions "ingrediens of *Q*" and "element of *Q*".

[52] Under the weak interpretation of the expression "Every … is …" a true sentence will always be generated if an empty name appears in the subject.

so-called strong sense.[53] Then if (β) in Definition II is true, then the name substituted for "a" is non-empty. Hence something there is an ingrediens of P; so — in virtue of (5.5) — P is an object. To recapitulate, under the strong interpretation of the expression "Every ... is ..." in Definition II, condition (α) follows from condition (β). In view of this, Leśniewski replaced Definition II with the equivalent Definition F, changing condition (β) for the following weaker condition, which was a universally-quantified sentence "Every a is an ingrediens of P" in the weaker interpretation: "for any X — if X is an a, then X is an ingrediens of P". "This condition in conjunction with condition (γ) does not entail condition (α)" [Leśniewski, 1930, p. 78, footnote 1].[54]

3. One can see from the conclusions of the examples cited above that the formalised rules of proof Leśniewski's logical calculus do not differ from the rules of the classical predicate calculus (CPC). For example, as one does also does in CPC, Leśniewski uses a rule of generalisation which takes from an arbitrary formula $\varphi(\zeta)$ with a free variable ζ to yield the formula $\forall_\zeta\ \varphi(\zeta)$. We may write this symbolically as follows:

$$\frac{\varphi(\zeta)}{\forall_\zeta\ \varphi(\zeta)}$$

In both CPC and in Leśniewski's calculus we have the following as admissible rules of proof, which always take us from theses to theses:

$$\frac{\forall_\zeta\ \varphi(\zeta)}{\varphi(\zeta)} \qquad \frac{\varphi(\zeta)}{\exists_\zeta\ \varphi(\zeta)} \qquad \frac{\forall_\zeta\ \varphi(\zeta)}{\exists_\zeta\ \varphi(\zeta)} \tag{rd1}$$

[53] Under the strong interpretation of the expression "Every ... is ..." a false sentence will always be generated if an empty name appears in the subject. We will generate a true sentence only if a non-empty name appears in the subject. Leśniewski writes: "I use and have used sentences of the type 'Every a is b' as equivalents of appropriate sentences of the type 'Some object is a and for any X —, if X is an a, then X is a b', but not as equivalents of sentences of the type 'For any X —, if X is an a, then X is a b' " [Leśniewski, 1928, p. 264, footnote 2].

[54] A similar derivation of the above was presented by Leśniewski himself in the proof of Theorem LX in [Leśniewski, 1929]. He made the following comment on this: "From Theorem LX we can see that conditions (β) and (γ) in Definition II entail condition (α) on the basis of my "general theory of multitudes" [Leśniewski, 1929, p. 64, footnote 1] (he writes similarly in [Leśniewski, 1930, p. 78, footnote 1]). Leśniewski evidently meant that in Definition II the same conditions (β) and (γ) suffice under the strong interpretation of universally-quantified sentences.

If we were to use the term "object" in CPC — 'embedded' in CPC's quantifiers — in the form of a one-place predicate "is an object", then the following statement would be a thesis of CPC: "$\forall_x$ x is an object"[55] which — taking into consideration the reading of the universal quantifier given in footnote 8 — says the same as the following sentence of ordinary English: "every object is an object". In view of the third rule in (rd1), we would obtain as a theorem in CPC: "$\exists_x$ x is an object", i.e., "some object is an object".[56]

In Leśniewski's system, the term "object" is not 'embedded' in the quantifiers. The sentence "$\forall_P$ P is an object" is not a thesis of his system. In fact, if it were, then — by the use of the first rule of (rd1) — we would obtain as a thesis the formula "P is an object" which turns into a false sentence with the substitution for "P" of an arbitrary term which either is empty or polyreferential.[57] In a similar way — by applying the third rule of (rd1) — we would obtain the sentence "$\exists_P$ P is an object", which is not a thesis of Leśniewski's system.[58]

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[55] From this perspective, the predicate "is an object" is superfluous in CPC.

[56] Shown here is the so-called "ontological assumption" of CPC of the non-emptiness of the universe of objects. If someone chooses to understand "to exist" as meaning the same as "is an object", then this "ontological assumption" will be an "existential assumption".

[57] Recall that in Leśniewski's system we are allowed to substitute an arbitrary name for the variable "P" and — in accordance with the interpretation of the copula "is" in this system — the formula "P is an object" produces a true sentence iff for "P" we substitute a monoreferential term.

[58] Some scholars comment on this saying that Leśniewski never accepted an "ontological assumption" as part of his system which reduces to an "existential assumption" in his ontology (cf. footnote 56). In fact, Leśniewski nowhere neither assumes nor states that *there are* (*exist*) some objects. This does not, however, mean that Leśniewski did not assume a non-empty universe for his analysis. For there are theses in his system which begin with existential quantifiers, such as: "$\exists_a \forall_P$(if P is an a then P is an a)" which we get from the thesis "$\forall_a \forall_P$(if P is an a then P is an a), by applying the third rule of (rd1). It is, however, true that since the sentence "$\exists_P$ P is an object" in not a thesis in his system, there are therefore no theses of the form "$\exists_P$(P is an object and $\varphi(P)$)".

We therefore see that it is not possible to interpret Leśniewski's quantifiers in a way which would be conveyed by the use of the phrases "every object ... is such that ..." and "some object ... is such that ...", respectively. In view of this, some scholars think that Leśniewski was using a substitutional interpretation of the quantifiers [cf., e.g., Küng, 1970]. This view is not shared by Guido Küng [Küng, 1977a,b, 1981]. He believe that "Leśniewski's quantifiers are neither referential (objectual) nor substitutional" and that "in reality there is [...] a third possible way of quantification.

Let us add that the sentence "$\exists_P$ P is an object" is equivalent in Leśniewski's system to the sentence "some object is an object", i.e., that the second is not a thesis of his system. Leśniewski writes:

> The thesis stating that some object is an object is not obtainable on the basis [...] of the system of "ontology" ([...] I use sentences of the type "some a is a b" as equivalents of appropriate sentences of the form "for some X — (X is an a and X is a b)" in which the expression "for some X" plays the role of a "quantifier" "$\exists X$", — the sentence "some object is an object" is for me equivalent to the sentence "for some X — (X is an object and X is an object)" and thus to the sentence "for some X — X is an object"). [Leśniewski, 1928, footnote 4, p. 265]

For Leśniewski, the sentence "$\forall_P$ P is an object" does not correspond to the sentence "every object is an object". The second — under the strong interpretation which Leśniewski uses — means the same as the conjunction: "some object is an object and for any X — if X is an object, then X is an object".[59] Since the first conjunct is not a thesis in Leśniewski's system, the whole conjunction is therefore not a thesis.[60]

The difference between CPC and Leśniewski's logic can be expressed only when we use in both systems admissible rules relating to the sub-

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[...] Since for Leśniewski general names not only signify objects but also possess extensions, the domain of quantification should be composed of these extensions [...]. In spite of quantification, extensions remain extensions which names possess, but will not be named objects; the domain of objects does not become widened. In Leśniewski's logic, the domain of objects and the domain of values of variables are not identical — just as in the particular case of Russell's logic. [...] If extensions remain just what they are and do not become named objects (are not objectified) then Leśniewski's logic still remains nominalistic; but if extensions are quantified over, then it comes close to platonism. It is something of a sui generis nominalism which in fact is equivalent to a platonistic Russellian theory of types. [...] via quantifiers [we appeal to] the extensions of names [...]" [Küng, 1977a, p. 96–97]. This approach might solve many formal problems connected with the interpretation of the quantifiers (or: the domain of the values of variables) we find with Leśniewski. Judging from the material cited in Section 4, however, it would for him come too "close to platonism". Besides this, what should we do with a formula of the form "P is an object"? Are extensions supposed to be values of the variable "P", where these extensions "are not objectified". We cannot further touch on the question of the proper interpretation of Leśniewski's quantifiers.

[59] Compare the interpretation used by Leśniewski of sentences of the form "every a is a b", which we presented in footnote 53. We substituted the term "object" for both variables "a" and "b".

[60] The second conjunct is clearly a thesis in Leśniewski's system. This conjunct would correspond, however, to the universally-quantified sentence under the weak interpretation.

stitution of name constants alongside the rules of (rd1). Let a formula $\varphi(\zeta/\tau)$ arise from a formula $\varphi(\zeta)$ by a substitution of a term τ for each free occurrence of the variable ζ in $\varphi(\zeta)$. Both systems had admissible rules of proof with the following identical forms:

$$\frac{\forall_\zeta\ \varphi(\zeta)}{\varphi(\zeta/\tau)} \qquad \frac{\varphi(\zeta)}{\varphi(\zeta/\tau)} \qquad \frac{\varphi(\zeta/\tau)}{\exists_\zeta\ \varphi(\zeta)} \tag{rd2}$$

In both systems we accept, however, a different "range of applicability" of these rules. In CPC we limit their application to monoreferential terms, i.e., τ must be such a term. In Leśniewski's logic we have no fixed limit, i.e., τ can be an arbitrary term.[61]

4. Similarly in the case of Leśniewski's logic, the admissible rules of proof of free logic can be used with empty terms. This should not be seen as saying, however, that his logic is a free logic. The rules of (rd2) in Leśniewski's logic are simply not admissible in a free logic — at least in its standard version, which Lambert presents in his essay "The nature of free logic" [Lambert, 1991]. For example, to obtain an admissible rule in free logic, it is necessary to strengthen the first rule of (rd2) by the addition of the premise $\ulcorner\exists_x\ x = \tau\urcorner$ to obtain:

$$\frac{\forall_\zeta\ \varphi(\zeta) \qquad \exists_x\ x = \tau}{\varphi(\zeta/\tau)}$$

Since the interpretation of the quantifiers in free logic is referential (objectual), this additional premise simply states that some object is the referent of the term τ.[62]

6. Collective classes (sets)

In this section we will translate Leśniewski's axioms I–IV and definitions I–IV into the schematic language of classical logic. We will therefore be

[61] Formally, this difference manifests itself in the conditions which the definitions of the name-constants in theories based on CPC must fulfil. Before introducing a new name-constant in such a theory, one must prove that it signifies exactly one object. In Leśniewski's system no limits on the definition of name-constants is made.

[62] In both Leśniewski's logic and in CPC this additional assumption is superfluous. In the former, the rule is just admissible without this assumption. In CPC the assumption is tautological, because we have a restriction after all: τ must be a monoreferential term.

using the variables "x", "y", "z" etc. It is possible to substitute only monoreferential terms — in accordance with the rules of (rd2).

Axioms I and II relate to the relation *is a part of*. The first states that it is asymmetric. In classical logic we express it with the help of the formula (as_{P}) or — using the name "$\sqsubset$" of the relation *is a part of* — with the help of the formula ($\mathrm{as}_{\sqsubset}$). The second axiom of Leśniewski's mereology states that the relation *is a part of* is transitive. We express it with the help of the formula ($\mathrm{t}_{\sqsubset}$). The formula (df-ingr), or symbolically: (df $\sqsubseteq$), gives Definition I of the relation *is an ingrediens of*. Using only this concept Leśniewski defined the concepts *collective class of* P*s* and *collective set of* P*s* (see definitions II and III), where "P" is a schematic letter representing arbitrary names.

We shall continue to use the nous "class" and "set" both in a collective and in a distributive sense. In the first case we will use the expressions "$\mathrm{class_c}$" and "$\mathrm{set_c}$", respectively. In the second, "$\mathrm{class_d}$" and "$\mathrm{set_d}$", respectively. We shall then put Leśniewski's definitions in the following forms:

x is a $\mathrm{class_c}$ of Ps iff every P is an ingrediens of x and every ingrediens of x has a common ingrediens with some P, (6.1)

x is a $\mathrm{set_c}$ of Ps iff every ingrediens of x has a common ingrediens with some P which is an ingrediens of x. (6.2)

Recall that we are understanding sentences that are instances of the schema "Every S is a P" with the following sense: $\forall_x(x$ is an $\mathsf{S} \Rightarrow x$ is a $\mathsf{P})$". We shall therefore give the definitions above the following symbolic forms:

$$x \text{ is a class}_{\mathrm{c}} \text{ of } \mathsf{Ps} \;:\Longleftrightarrow \forall_y\big(y \text{ is a } \mathsf{P} \Rightarrow y \sqsubseteq x\big) \wedge \forall_z\big(z \sqsubseteq x \Rightarrow \exists_{y,u}(y \text{ is a } \mathsf{P} \wedge u \sqsubseteq y \wedge u \sqsubseteq z)\big), \tag{6.1$'$}$$

$$x \text{ is a set}_{\mathrm{c}} \text{ of } \mathsf{Ps} \;:\Longleftrightarrow \forall_z\big(z \sqsubseteq x \Rightarrow \exists_{y,u}(y \text{ is a } \mathsf{P} \wedge y \sqsubseteq x \wedge u \sqsubseteq y \wedge u \sqsubseteq z)\big). \tag{6.2$'$}$$

It follows directly from the definitions above that each class is a set:

$$\forall_x(x \text{ is a class}_{\mathrm{c}} \text{ of } \mathsf{Ps} \implies x \text{ is a set}_{\mathrm{c}} \text{ of } \mathsf{Ps}). \tag{6.3}$$

Let x be a $\mathrm{class_c}$ of Ps. Then each P is an ingrediens of x and: if $y \sqsubseteq x$ then y has a common ingrediens with some P which is an ingrediens of x. Therefore x is a $\mathrm{set_c}$ of Ps.

Conversely as well, each set_c is a class_c. By substituting in (6.1) for "P" an expression of the form "P which is an ingrediens of x", we obtain the tautological first conjunct of definiens of (6.1). Thus:

$$\forall_x(x \text{ is a set}_c \text{ of } \mathsf{P}\text{s} \iff x \text{ is a class}_c \text{ of } \mathsf{P}\text{s being ingredienses of } x). \tag{6.4}$$

It follows directly from (6.1) and (df $\sqsubseteq$) that each object is a class_c of something (and so, in the light of (6.3), it is also a set_c of this something):

$$\forall_x\; x \text{ is a class}_c \text{ of ingredienses of } x, \tag{6.5}$$

$$\forall_x\; x \text{ is a class}_c \text{ of objects identical with } x.\text{[63]} \tag{6.6}$$

For (6.5): we substitute for "P" in (6.1) the expression "ingrediens of x". Then the first conjunct in the definiens is tautological: each ingrediens of x is an ingrediens of x. The second conjunct is also true: an arbitrary ingrediens of x has a common ingrediens with some ingrediens of x, because it has a common ingrediens with itself. In virtue of (6.1) therefore, x is a class_c of ingredienses of x. For (6.6): substitute in (6.1) for "P" a general term "object identical with x" which for any x denotes x and only x. Then the first conjunct in the definiens is true, because $x \sqsubseteq x$. Moreover, if $y \sqsubseteq x$ then y has a common ingrediens with x, since $y \sqsubseteq y$.

It follows from the above considerations that the non-relative predicates "is a class_c" and "is a set_c" would be superfluous for mereology, because each object turns out to be a class_c and set_c. In fact, these non-relative predicates would have to fulfil the following principles:

$$\forall_x(x \text{ is a class}_c \text{ of } \mathsf{P}\text{s} \Longrightarrow x \text{ is a class}_c),$$

$$\forall_x(x \text{ is a set}_c \text{ of } \mathsf{P}\text{s} \Longrightarrow x \text{ is a set}_c).$$

Hence, using (6.5) (or (6.6)) and (6.3) we obtain:

$$\forall_x\; x \text{ is a class}_c,$$

$$\forall_x\; x \text{ is a set}_c.$$

We see therefore that the non-relative expressions "class_c" and "set_c" are just as universal as the term "object" (everything is an object): everything is a class_c and everything is a set_c. Thus, the only interesting roles which these expressions can play are the roles of name-forming functors

[63] Clearly, the object identical with x is x and only x. We have used this form in view the non-natural plural form "xs". (Leśniewski wrote "objects of P"; see point (3) in the passage on p. 45 take from [Leśniewski, 1927].)

taking names as argument, which serve to build general names whose schemata are “class$_c$ of Ps” and “set$_c$ of Ps”.

We stress that up to this point we have been using only Definitions I–III themselves. In addition, not making use of any specific axiom of mereology, it follows from definitions (6.1), (6.2), and (df $\sqsubseteq$) that if the letter “P” represents an empty name then the names represented by the schemata “class$_c$ of Ss” and “set$_c$ of Ps” are also empty. This accords with our informal understanding of the expression “aggregate” — ‘there can be no assemblage of that of which there is none’, i.e., ‘there is no aggregate of Ps if there are not any Ps’:

$$\begin{gathered}\text{There is a set}_c\text{ of Ps} \implies \text{there is a P}, \quad \text{or}\\ \text{There is no P} \implies \text{there is no set}_c\text{ of Ps.}\end{gathered} \tag{6.7}$$

Suppose that there is no P. Moreover, assume for a contradiction that some x is a set$_c$ of Ps. Then, by (6.2), each ingrediens of x has some common ingrediens with some P. Because, by (df $\sqsubseteq$), we have $x \sqsubseteq x$, so x has a common ingrediens with some P. But there are no P. So we have obtained a contradiction.

Directly from (6.3) and (6.7) we obtain:[64]

$$\begin{gathered}\text{There is a class}_c\text{ of Ps} \implies \text{there is a P}, \quad \text{or}\\ \text{There is no P} \implies \text{tere is no class}_c\text{ of Ps.}\end{gathered} \tag{6.8}$$

Of course, this also can be obtained by applying (6.1) and (df $\sqsubseteq$).

It also follows from definitions (6.1) and (df $\sqsubseteq$) that

$$\forall_x(x \text{ has some part} \iff x \text{ is a class}_c \text{ of parts of } x). \tag{6.9}$$

Assume that x has some part. Substitute for the definition “P” in (6.1) the expression “part of x”. Then, by (df $\sqsubseteq$), the first conjunct in the definiens is true: each part of x is an ingrediens of x. The second conjunct is also true. If $y \sqsubseteq x$ then either $x = y$ or $y \sqsubset x$. In the first case — in the light of the assumption — for some z we have: $z \sqsubset x$; so also $z \sqsubseteq x$. Since also $z \sqsubseteq z$, so y has a common ingrediens with z. In the second case y has a common ingrediens with y, because $y \sqsubseteq y$. Hence in both cases an arbitrary ingrediens of x has some common ingrediens with some part of x. Therefore, via (6.1), x is a class$_c$ of parts of x. The converse implication results from (6.8).

[64] Compare point (1) in the passage from [Leśniewski, 1927] reprinted on p. 45.

The final results drawn directly from (6.1) and (6.2), respectively, will be the counterparts of condition (3.6) which we formulated for distributive classes and sets. They state that if two terms signify the same objects and one of them determines some class$_c$ (resp. set$_c$), then the other of them determines the same class$_c$ (resp. set$_c$):

$$\forall_y(y \text{ is an } \mathsf{S} \Leftrightarrow y \text{ is a } \mathsf{P}) \implies \forall_x(x \text{ is a class}_c \text{ of } \mathsf{Ss} \Rightarrow x \text{ is a class}_c \text{ of } \mathsf{Ps}). \tag{6.10}$$

$$\forall_y(y \text{ is an } \mathsf{S} \Leftrightarrow y \text{ is a } \mathsf{P}) \implies \forall_x(x \text{ is a set}_c \text{ of } \mathsf{Ss} \Rightarrow x \text{ is a set}_c \text{ of } \mathsf{Ps}). \tag{6.11}$$

Condition (3.7) was for distributive classes and sets. Its counterpart for collective classes and sets is not in general true. Two terms can determine the same collective class in spite of the fact that they do not refer to the same objects. What is more, it may be that something is a class$_c$ of Ss and a class$_c$ of Ps (resp. a set$_c$ of Ss and a set$_c$ of Ps), even though no S is a P. Leśniewski presents such a case in point (3) in the passage take from [Leśniewski, 1927] given on p. 45. Similar cases are to be found in the examples in the quoted texts from [Borkowski, 1977; Quine, 1981, 1953] on pages 21, 27, and 24, respectively. And so — in agreement with [Borkowski, 1977; Quine, 1953] — a given pile of stones is a class$_c$ of stones comprising that pile. It is also the class$_c$ of atoms of which the stones in that pile are composed and it is also the class$_c$ of molecules of which the stones in that pile are composed.[65] Analogously, the USA is the class$_c$ of states in the USA and the class$_c$ of counties in the USA.

Axiom III states that terms of the form "class$_c$ of Ps" never signify more than one object. We may express this as the following formula:

$$\forall_{x,y}(x \text{ is a class}_c \text{ of } \mathsf{Ps} \wedge y \text{ is a class}_c \text{ of } \mathsf{Ps} \implies x = y). \tag{6.12}$$

Hence if "class$_c$ of Ps" represents a non-empty term then it is also monoreferential and so we may use the term "the class$_c$ of Ps". Therefore, from (6.5), (6.6), and (6.9) we get:

$$x = \text{the class}_c \text{ of ingredienses of } x, \tag{6.13}$$

$$x = \text{the class}_c \text{ of objects identical with } x, \tag{6.14}$$

$$x \text{ has some part} \implies x = \text{the class}_c \text{ of parts of } x. \tag{6.15}$$

[65] Similarly, in accordance with (6.2): a given piles of stones is a set$_c$ of stones, a set$_c$ of atoms, and also a set$_c$ of molecules.

Moreover, it follows from (6.6) (or l(6.14)), (6.3), and (6.12), that if the name represented by "P" signifies exactly one object, then this object is also the unique referent of the names "class$_c$ of Ps" and "set$_c$ of Ps". In each case, we are not dealing with any 'grouping' of elements into a 'whole'; the 'whole' is that unique P. Thus:

$$\text{there is exactly one } \mathsf{P} \implies \text{the class}_c \text{ of } \mathsf{Ps} = \text{the } \mathsf{P} = \text{the set}_c \text{ of } \mathsf{Ps}. \tag{6.16}$$

If x is the unique P, then x is a class$_c$ of Ps and is a set$_c$ of Ps. The rest follows from (6.12).

Axiom IV is written as the converse implication of implication (6.8):

$$\text{There is a } \mathsf{P} \implies \text{there is a class}_c \text{ of } \mathsf{Ps}. \tag{6.17}$$

With regard to (6.8), it is obviously the maximally strong assumption one may make in mereology without falling into contradiction. We shall show in Chapter II that this theory is in fact consistent. In Section 5 of Chapter II we will discuss the doubts raised by Axiom (6.17).

Both assumptions (6.12) and (6.17) guarantee that:

$$\text{There is a } \mathsf{P} \implies \text{there is exactly one class}_c \text{ of } \mathsf{Ps}. \tag{6.18}$$

Thus, if the letter "P" represents a non-empty term then the schema "class$_c$ of Ps" represents a monoreferential term and so we may substitute "the class$_c$ of Ps" for the free variables "x", "y", "z", etc. One of the consequences of (6.1) therefore is:

$$\text{There is a } \mathsf{P} \implies \text{each } \mathsf{P} \text{ is an ingrediens of the classc of } \mathsf{Ps}.^{66} \tag{6.19}$$

It suffices to substitute "the class$_c$ of Ps" for "x" in (6.1) and, by (6.18), detach the schema "the class$_c$ of Ps is a class$_c$ of Ps".

We note finally that thanks to (6.18) we may write condition (6.10) as follows:

$$\forall_y (y \text{ is an } \mathsf{S} \Leftrightarrow y \text{ is a } \mathsf{P}) \wedge \text{there is an } \mathsf{S} \implies \text{the class}_c \text{ of } \mathsf{Ss} = \text{the class}_c \text{ of } \mathsf{Ps}. \tag{6.20}$$

We know that the above implication is not reversible.

[66] Note that if we replace "ingrediens" by "part", then we do not generally obtain a true condition. Essentially, the same class$_c$ of Ps can be of one the Ps, but it is not a part of it. Compare the example given on p. 66 and result (6.16).

We have already mentioned tha ambiguity of the expression "element of a class" (resp. "element of a set"). It might seem that this ambiguity is entirely a matter of the connection this expression has with the equally ambiguous the term "class" (resp. "set") and that the expression "element of a collective class" (resp. "element of a collective set") has just one meaning. We will show that this is not, however, the case and that this is caused by the universality of the term "collective class" (resp. "collective set"). Take two arbitrary objects x and y. In virtue of (6.13), the sentence "y is an element of the class_{c} of ingredienses of x" has the same logical value as the sentence "y is an element of x". The concept of *an object* is, however, universal and also includes distributive classes (sets). Therefore, it might turn out that an arbitrarily chosen object x is a distributive set. In this case, the sentence "y is an element of x" would not be unambiguous. We do not know whether we are talking about being an element in the distributive sense or being an element in the collective sense. This second concept — as we have already mentioned and will later show — boils down to the concept of *being an ingrediens of*. Since every object is its own ingrediens, every object is therefore is own element in the collective sense; so also all distributive classes are their own elements in a collective sense. On the other hand, only distributive classes have elements in the distributive sense and only non-normal distributive sets are their own elements in a distributive sense. We shall therefore use the term "element" with a subscripted "c" or "d", depending on whether we are using that term in a collective or distributive sense.

Let us turn to the analysis of the term "$\text{element}_{\text{c}}$". In order to formulate Leśniewski's Definition IV given, it is not enough to use the schemata of object-language expressions. We need to use a metalinguistic formulation so as to preserve an appropriate level of precision. Let us establish that the letter "n" is a metalinguistic variable (metavariable) representing names.[67] The general name of the class_{c} of all designations of a given name n will then be ⌜the class_{c} of ns⌝.[68] In its metalinguistic

[67] We substitute the name of a given name (e.g., a quotational name of a given name) in place of the metalinguistic variable "n".

[68] We have used 'corner quotes', Quine's so-called quasi-quotation marks, and have thus created quasi-quotational names. Such a name refers to the name (it is a name of a name) which is a concatenation of the noun "class" (used in its collective sense) with a suitably (grammatically) adjusted name n. If, for example n = "dog", then ⌜the class_{c} of ns⌝ = "the class_{c} of dogs". (In place of "n" we have inserted the same expression, but not its quotational name. We have ⌜n⌝ = n, for any name n.)

formulation, condition (6.18) states that for any name n we have:

$$n \text{ is non-empty} \implies \ulcorner\text{class}_\text{c} \text{ of } n\text{s}\urcorner \text{ is a monoreferential term.} \quad (6.18')$$

In its metalinguistic formulation, Definition IV appears thus for any name n and all objects x and y:

$$x \text{ is an element}_\text{c} \text{ of } y \;:\Longleftrightarrow \exists_n (x \text{ is a referent of } n \wedge y \text{ is the referent of } \ulcorner\text{the class}_\text{c} \text{ of } n\text{s}\urcorner). \quad (6.21)$$

We shall provide below a logico-philosophical commentary on definition (6.21).

With regard to the left-to-right direction in (6.21): Assume that x is an element$_\text{c}$ of y. In keeping with our informal understanding of the term "element", y is a class$_\text{c}$ of some objects and x is one of those objects.[69] Thus, for a certain name n we obtain that x is a referent of the name n and the name ⌜the class$_\text{c}$ of ns⌝ signifies y.[70]

The right-to-left direction of (6.21) has its source in the analytic statement: every P is an element$_\text{c}$ of the class$_\text{c}$ of Ps.[71] Clearly, we are

On the subject of quasi-quotation, see, e.g., [Quine, 1981, pp. 39–42], [Quine, 1953, pp. 151–154], [Mostowski, 1948, pp. 321–313].

If we were to use the usual quotational name "class$_\text{c}$ of ns" — instead of the quasi-quotational name — then this would signify an expression which is the concatenation of "class of" with the letter "n" [cf., e.g. Tarski, 1933, pp. 8–10].

[69] Speaking figuratively: y is a 'gluing together' of those objects to which x belongs. Or: for the class$_\text{c}$ y we can carry out a sort of 'conceptual dismemberment' under which x falls.

[70] To formulate this in a way that preserves precision, we have had to move to the metalinguistic level. It is not possible to write this formulation precisely using a sentential schema at the object-language level. The formulation would have to have the following form: if x is an element$_\text{c}$ of y, then the letter "P" may represent such name, where x is a P and y = the class$_\text{c}$ of Ps. It is possible to express this using second-order logic:

$$x \text{ is an element}_\text{c} \text{ of } y \implies \exists_X (x \text{ is an } X \wedge y = \text{the class}_\text{c} \text{ of } X\text{s}),$$

but then problems arise with the interpretation of the quantifier binding the variable "X".

[71] We observe that it is not possible to reverse this statement, i.e., it may be that not every element$_\text{c}$ of a class$_\text{c}$ of Ps will be P. This follows from the fact that the 'gluing together' of other elements$_\text{c}$ than Ps may yield that same 'whole', i.e., the class$_\text{c}$ of Ps. We have given an example on p. 61. Each state of the USA is an element$_\text{c}$ of the class$_\text{c}$ of states of the USA; each county in the USA is an element$_\text{c}$ of the class$_\text{c}$ of counties in the USA. But the class$_\text{c}$ of states in the USA = USA = the class$_\text{c}$ of counties in the USA; and therefore some element$_\text{c}$ of the class$_\text{c}$ of states in the USA is not a state of the USA.

'quietly' assuming here that in general there is a class$_{\mathrm{c}}$ of Ps, that is that there is at least one P. Under this assumption, using (6.18), the given analytic sentence may be rendered as follows:

$$\begin{aligned} &\text{There is a } \mathsf{P} \implies \\ &\qquad \forall_x (x \text{ is a } \mathsf{P} \Rightarrow x \text{ is an element}_{\mathrm{c}} \text{ of the class}_{\mathrm{c}} \text{ of } \mathsf{P}\text{s}). \end{aligned} \tag{6.22}$$

By moving to the metalinguistic level, we can dispense with the assumption "there is a P". The formulation will be correct for any name n:

$$\begin{aligned} \forall_{x,y,\mathsf{n}}\big((x \text{ is a referent of } \mathsf{n} \wedge y \text{ is the referent of } \ulcorner\text{the class}_{\mathrm{c}} \text{ of } \mathsf{n}\text{s}\urcorner) \\ \implies x \text{ is an element}_{\mathrm{c}} \text{ of } y\big). \end{aligned}$$

Using the rule regarding quantifiers, we get:

$$\begin{aligned} \forall_{x,y}\big(\exists_{\mathsf{n}}(x \text{ is a referent of } \mathsf{n} \wedge y \text{ is the referent of } \ulcorner\text{the class}_{\mathrm{c}} \text{ of } \mathsf{n}\text{s}\urcorner) \\ \implies x \text{ is an element}_{\mathrm{c}} \text{ of } y\big). \end{aligned}$$ [72]

These conditions suffice for us to show that in the case of collective sets, being an element$_{\mathrm{c}}$ of a given set reduces to being its ingrediens. In other words, the following biconditional holds:

$$\forall_{x,y}(x \text{ is an element}_{\mathrm{c}} \text{ of } y \iff x \text{ is an ingrediens of } y). \tag{6.23}$$

For the left-to-right direction: If x is an element$_{\mathrm{c}}$ of y, then — in virtue of (6.21) — for a certain name n_0, we have that x is an referent of n_0 and that the name $\ulcorner$the class$_{\mathrm{c}}$ of ns$\urcorner$ signifies y. Substituting in (6.19) the name n_0 in place of the letter "P" we have that each referent of n_0 is an ingrediens of the unique referent of the name $\ulcorner$the class$_{\mathrm{c}}$ of ns$\urcorner$. Therefore $x \sqsubseteq y$. For the right-to-left direction: Assume that $x \sqsubseteq y$. Then x is an referent of "ingrediens of y". By (6.13), we have: $y =$ the class$_{\mathrm{c}}$ of ingredienses of y, i.e., y is the referent of the name "the class$_{\mathrm{c}}$ of ingredienses of y". So the right-hand side of definition (6.21) is true, i.e., x is an element$_{\mathrm{c}}$ of y.[73]

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[72] By applying analogous rules for the use of second-order quantifiers we obtain from the consequence of (6.22) the converse implication to that in footnote 70. Thus, we may formulate the definition of being an element$_{\mathrm{c}}$ of a class$_{\mathrm{c}}$ in a second-order language:

$$x \text{ is an element}_{\mathrm{c}} \text{ of } y \;:\Longleftrightarrow \exists_X(x \text{ is an } X \wedge y = \text{the class}_{\mathrm{c}} \text{ of } X\text{s}).$$

[73] It is also possible to prove the right-to-left direction of (6.23) by substituting the name "ingrediens of y" in place of the letter "P" in the schema (6.22).

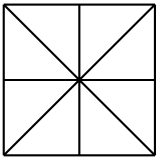

Figure 2.

Condition (6.23) shows that any class_c (sets_c) has some element_c. Hence there is no empty collective class (set). It also follows from condition (6.23) that all classes_c (resp. sets_c) are non-normal, i.e., that they are their own elements_c. Hence we obtain a solution to Russell's antinomy in the 'field' of mereology, the one which we have mentioned already on p. 46: since the terms "normal class_c" and "normal set_c" are empty, we get from (6.7) that the terms "set_c of normal classes_c" and "set_c of normal sets_c" are empty. Moreover, by (6.8), the terms "class_c of normal classes_c" and "class_c of normal sets_c" are also empty. There is therefore 'nothing to say' — there is 'no problem'.

Also by (6.23), we have the obvious statement:

$$\forall_x (x \text{ has exactly one element}_\text{c} \iff x \text{ has no parts}). \tag{6.24}$$

Thus, in the case where there exists exactly one P, it is not possible to say in a general way of P that it is a one-element class_c (set_c) and this one elements is the set itself.[74] It is clearly true that an arbitrary class_c (set_c) which has only one element_c is identical to this unique element_c. We must not, however, confuse this statement with (6.16).

It may even happen that the class_c of Ss = the class_c of Ps when there exists exactly one S and at least two Ps. Then the class_c of Ss has at least two elements_c (and if the unique S is not an P, then the class_c of Ss has at least three elements_c). For example, let the letter "S" represent the name "square with a side of two centimetres in Figure 2" and the letter "P" represent the name "square in Figure 2". The first of these names signifies exactly one object and the second has five referents.

We clearly have an identity: the class_c of Ss = the class_c of Ps. Thus, the class_c of Ss has at least five elements_c. These are not, however, its

[74] Some do sometimes write in this way, though: see, e.g., the quote from Russell's book printed here on page 30. Recall that condition (6.16) just says: if there exists exactly one P, then the P = the class_c of Ps = the set_c of Ps.

only elements$_c$. In fact, in keeping with the conception of collective sets, the following equality also holds: the class$_c$ of Ss = the class$_c$ of triangles in Figure 2 = the class$_c$ of rectangles in Figure 2, etc.

7. Mereology with set theory

In the previous section, we were able to avoid both an analysis employing object-language schemata and metalinguistic schemata. It sufficed to use distributive sets to pursue our investigations into collective classes (sets).[75]

Let S be an arbitrary set$_d$. Substitute in definition (6.1) the expression "element$_d$ of S" in place of the schematic letter "P". Then the schematic phrase "is a P" changes into the phrase "is an element$_d$ of S" and (6.1) takes the form:

x is a class$_c$ of elements$_d$ of S iff every element$_d$ of S is an ingrediens of x and every ingrediens of x has some common ingrediens with some elements$_d$ of S. (7.1)

By replacing the statement "y is an element$_d$ of S" with "$y \in S$" we obtain the following symbolic form corresponding to the schema (6.1′):

$$x \text{ is a class}_c \text{ of elements}_d \text{ of } S :\Longleftrightarrow \forall_z(z \in S \Rightarrow z \sqsubseteq x) \wedge \forall_y\big(y \sqsubseteq x \Rightarrow \exists_{z,u}(z \in S \wedge u \sqsubseteq z \wedge u \sqsubseteq y)\big). \tag{7.1′}$$

It follows from this definition that if x is a class$_c$ of elements$_d$ of S, then $S \neq \emptyset$, i.e., S has in general some element$_d$.

In this convention results (6.13) and (6.14) have the following form:

$$x = \text{the class}_c \text{ of elements}_d \text{ of the set}_d \ \{y : y \sqsubseteq x\}, \tag{7.2}$$

$$x = \text{the class}_c \text{ of elements}_d \text{ of the set}_d \ \{x\}. \tag{7.3}$$

We may similarly write down the other schematic formulae that appeared in the previous section (including the counterparts of axioms III and IV of the mereology).

[75] This approach would have been 'completely foreign' to Leśniewski, who did not recognise the existence of distributive classes (sets).

Limiting the scope of our analysis to a non-empty set$_{\mathrm{d}}$ M, if $x \in M$ and $S \in \mathcal{P}_{+}(M)$, then the fact that x is a class$_{\mathrm{c}}$ of elements$_{\mathrm{d}}$ of S can be reduced to a certain relation holding between x and S. This relation is the subset of the Cartesian product $M \times \mathcal{P}_{+}(M)$. It is precisely this relation which we will be concerned with in the following chapters of this book [see also, e.g., Gruszczyński and Pietruszczak, 2010].

By using the language of set theory, we may dispense with the metalinguistic formulations of definition (6.21). Instead of names we may talk of sets$_{\mathrm{d}}$ and instead of the statements "x is a referent of a name *n*" and "y is a referent of "⌜the class$_{\mathrm{c}}$ of *n*s⌝" we can respectively use "$x \in S$" and "y is the class$_{\mathrm{c}}$ of elements$_{\mathrm{d}}$ of S". Definition (6.21) therefore takes on the form:

$$x \text{ is an element}_{\mathrm{c}} \text{ of } y \;:\Longleftrightarrow \exists_S (x \in S \wedge \\ y \text{ is the class}_{\mathrm{c}} \text{ of elements}_{\mathrm{d}} \text{ of } S). \tag{7.4}$$

From (7.4) we may derive all the results which we obtained in the previous section where we made use of (6.21). Since being an element$_{\mathrm{c}}$ comes down in mereology to being an ingrediens, we shall therefore not be using the former concept.

Part A

MEREOLOGICAL STRUCTURES

In this part of the book we shall be treating mereology as a theory of certain relational structures called *mereological structures*.

Following Leśniewski [1928], mereology — pursued as a theory of certain relational structures — concerns ordered pairs of the form $\langle M, \sqsubset \rangle$ in which $\sqsubset$ is the relation *is a part of* in a non-empty set M.[a] For, as Alfred Tarski notes, "it should be emphasized that mereology, as it was conceived by its author, is not to be regarded as a formal theory where primitive notions may admit many different interpretations" [Tarski, 1956c, p. 334, footnote 1 form p. 333]. Following Leśniewski we accept that the set M is strictly partially ordered by the relation $\sqsubset$, which meets further additional conditions.[b] The strict partial order will be established by axioms (L1) and (L2), and these additional conditions by axioms (L3) and (L4) (cf. pp. 82 and 87, respectively).

If we assume only conditions (L1)–(L4) regarding the relation *is a part of* in a non-empty set M, then — using set-theoretic language — it is not possible to state that a binary relation R in M which satisfies the aforementioned conditions is ultimately the 'true' relation *is a part of*.[c] We shall therefore call any relational structure $\langle M, R \rangle$ a *mereological structure* if a binary relation R satisfies conditions (L1)–(L4). The relation R itself will be signified by the symbol "$\sqsubset$" and called the relation *is a part of*.

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[a] Henceforth, the terms "set" and "class" will be used in a distributive sense, unless otherwise stated. On the matter of terminology and notation, see Section I.1 and sections 2 and 3 of Appendix I.

Clearly, the set-theoretic approach to mereology is alien to Leśniewski's philosophical views. We know that Leśniewski did not recognise the existence of distributive sets. The structure $\langle M, \sqsubset \rangle$ of set theory and both its constituents did not exist for him. In his mereology, Leśniewski used the logical term "object" (universal name) and the name-generating functor "part". These terms did not have extensions for him because he did not recognise their existence either.

[b] That is, the relation $\sqsubset$ is transitive and irreflexive, and so also asymmetric. The set-theoretic predicate of set-membership will feature in the formulation of the additional conditions for the relation $\sqsubset$ along with a variable ranging over the power set of the universe of structures (cf. Section I.7). The mereological structures will not be elementarily axiomatisable.

[c] Similarly, in Leśniewski's logical language, his conditions do not determine a unique meaning for the term "part".

Chapter II

Classical mereology

1. First axioms

Let M be a non-empty set and $\sqsubset$ a binary relation in M. Later we will assume that the relation $\sqsubset$ meets the condition which Leśniewski imposed on the relation *is a part of*. The first axioms state that the relation $\sqsubset$ is a strict partial order in the set M; more exactly, that the relation $\sqsubset$ is asymmetric, transitive, and irreflexive in M, i.e., the relation $\sqsubset$ satisfies in M conditions ($\mathrm{as}_{\sqsubset}$), ($\mathrm{t}_{\sqsubset}$), and ($\mathrm{irr}_{\sqsubset}$), respectively, from pp. 19–20. As is well known, it is sufficient to assume that the relation $\sqsubset$ is transitive and satisfies one of the remaining two conditions.[1] Following Leśniewski, we choose conditions ($\mathrm{as}_{\sqsubset}$) and ($\mathrm{t}_{\sqsubset}$) as the first two axioms. So we put:

$$\forall_{x,y\in M}(x \sqsubset y \implies y \not\sqsubset x), \tag{L1}$$

$$\forall_{x,y,z\in M}(x \sqsubset y \wedge y \sqsubset z \implies x \sqsubset z). \tag{L2}$$

Let **L12** be the class of structures of the form $\langle M, \sqsubset\rangle$ satisfying (L1) and (L2). In other words, **L12** is the class **SPOS** of all strictly partially ordered sets. We shall further consider a certain class **MS** of *mereological structures* which will be a proper subclass of **L12**. We will define the class **MS** by introducing two further, non-elementary axioms (L3) and (L4) for it (see pp. 82 and 87).[2]

We will be examining the 'elementary aspects' of the class **MS** with the help of the elementary language $\mathrm{L}_{\sqsubset}$ with identity and one specific constant which is the binary predicate "$\sqsubset$". This predicate we may read as "is a part of". The language $\mathrm{L}_{\sqsubset}$ is created by the rules given in Section 1 of Appendix II.

[1] See, e.g., Lemma 2.2 from Section 2 of Appendix I and Section I.1.

[2] Axiom (L3) may be replaced with another condition that will be elementary.

In accordance with the convention adopted in Section 1 of Appendix II, we accept that the predicate "$\sqsubset$" is interpreted in $\mathfrak{M} = \langle M, \sqsubset \rangle$ as the relation $\sqsubset$. It is possible to treat the structure $\mathfrak{M}$ as a set-theoretic interpretation of the language $L_{\sqsubset}$ and to call it an $L_{\sqsubset}$-*structure*. For an arbitrary valuation of variables $V \colon \mathrm{Var} \to M$ for atomic formulae we have: $\mathfrak{M} \models \ulcorner x_i \sqsubset x_j \urcorner [V]$ iff $V(x_i) \sqsubset V(x_j)$; and $\mathfrak{M} \models \ulcorner x_i = x_j \urcorner [V]$ iff $V(x_i) = V(x_j)$.

Remark 1.1. The introduction of the language $L_{\sqsubset}$ allows us:

1) to observe that different parts of our analysis, which we are carrying out in the confines of set theory, can also be carried out in various elementary theories built in the language $L_{\sqsubset}$, i.e., without using the concepts of *set* and of *relation*;

2) to build in the language $L_{\sqsubset}$ a certain first-order theory which may be called *elementary mereology* and which elementarily axiomatises a certain class of structures **qMS** closely connected with Leśniewski's mereology (see Chapter VI). □

If a structure $\mathfrak{M} = \langle M, \sqsubset \rangle$ satisfies conditions (L1) and (L2) then the following two sentences from $L_{\sqsubset}$ are true in $\mathfrak{M}$:

$$\forall_x \forall_y (x \sqsubset y \to \neg\, y \sqsubset x) \tag{λ1}$$

$$\forall_x \forall_y \forall_z (x \sqsubset y \land y \sqsubset z \to x \sqsubset z) \tag{λ2}$$

Of course, the class **L12** is equal to the class of $L_{\sqsubset}$-structures composed of all models of the set $\{(\lambda1), (\lambda2)\}$. Thus, the class **L12** is finitely elementarily axiomatisable.[3]

2. Auxiliary definitions

In this section we shall describe three binary relations and one set, which will all be elementarily definable (or short: e-definable) in $\mathfrak{M} = \langle M, \sqsubset \rangle$.[4] We begin with the binary relation $\sqsubseteq$ which we shall describe with the aid of the formula (df $\sqsubseteq$) given on p. 20.[5] In this case, if $\sqsubset$ is the 'true'

[3] We shall later prove that the class **MS**, which we mentioned in the introduction to this part of the book and in one of previous paragraphs, is not elementarily axiomatisable.

[4] For elementary definability see, e.g., Section 2 in Appendix II.

[5] The $L_{\sqsubset}$-formula used for e-definition of $\sqsubseteq$ in $\mathfrak{M}$ is "$x \sqsubset y \lor x = y$". Clearly, the relation $\sqsubset$ itself is e-definable with the help of the formula "$x \sqsubset y$", and the sets $\emptyset$

relation *is a part of*, then the relation $\sqsubseteq$ is the relation *is an ingrediens of*, which we discussed on p. 19. We shall in future always call this the relation $\sqsubseteq$.

The relation $\sqsubseteq$ partially orders the set M; more exactly, the relation $\sqsubset$ is irreflexive, antisymmetric and transitive in M, i.e., the relation $\sqsubseteq$ satisfies in M conditions ($\mathrm{r}_{\sqsubseteq}$), ($\mathrm{antis}_{\sqsubseteq}$), and ($\mathrm{t}_{\sqsubseteq}$), respectively, from p. 20. Moreover, the relation $\sqsubseteq$ has other properties given in the formulae in ($\sqsubset=\sqsubseteq\setminus\mathrm{id}$) and ($\sqsubset=\sqsubseteq\setminus\sqsupseteq$) from p. 20, and below for all $x, y, z \in M$:[6]

$$x \sqsubset y \wedge y \sqsubseteq z \implies x \sqsubset z\,, \tag{2.1}$$
$$x \sqsubseteq y \wedge y \sqsubset z \implies x \sqsubset z\,. \tag{2.2}$$

In order to abbreviate our formulations, let us introduce two auxiliary functions $\mathbb{P}$ and $\mathbb{I}$ from M into $\mathcal{P}(M)$ which assign each $x \in M$ the set of parts of x and the set of ingredienses of x, respectively:

$$\mathbb{P}(x) := \{y \in M : y \sqsubset x\},$$
$$\mathbb{I}(x) := \{y \in M : y \sqsubseteq x\}.$$

For any $x \in M$ the sets $\mathbb{P}(x)$ and $\mathbb{I}(x)$ are e-definable in $\mathfrak{M}$ with the parameter x.[7] By the above definitions we obtain:

$$\mathbb{I}(x) = \mathbb{P}(x) \cup \{x\}. \tag{2.3}$$

Directly from ($\mathrm{irr}_{\sqsubset}$) and ($\mathrm{as}_{\sqsubset}$), for all $x, y \in M$ we have, respectively:

$$x \notin \mathbb{P}(x),$$
$$\mathbb{P}(x) \subsetneq \mathbb{I}(x),$$
$$x \in \mathbb{P}(y) \implies y \notin \mathbb{P}(x).$$

Moreover, from ($\mathrm{irr}_{\sqsubset}$) and ($\mathrm{t}_{\sqsubset}$), for all $x, y \in M$ we obtain:

$$x \sqsubset y \implies \mathbb{P}(x) \subsetneq \mathbb{P}(y), \tag{$\mathrm{mono}_{\mathbb{P}}$}$$
$$x \sqsubset y \implies \mathbb{I}(x) \subsetneq \mathbb{I}(y). \tag{$\mathrm{mono}_{\mathbb{I}}$}$$

---

and M are e-definable in $\mathfrak{M}$ with the help of "`x ≠ x`" and "`x = x`", respectively. If $\mathfrak{M} \in$ **L12** then these sets are also e-definable in $\mathfrak{M}$ without identity, with the help of "`x ⊏ x`" and "`¬ x ⊏ x`", respectively.

[6] See, e.g., the facts given in sections 2–4 of Appendix I.

[7] For $\mathbb{P}(x)$ and $\mathbb{I}(x)$, the $\mathrm{L}_{\sqsubset}$ formulae used for e-definability with the parameter x are, respectively, the same formulae which were used for the e-definability of the relations $\sqsubset$ and $\sqsubseteq$.

Let $x \sqsubset y$. Then for (mono$_{\mathbb{P}}$), by (t$_{\sqsubset}$), we have $\mathbb{P}(x) \subseteq \mathbb{P}(y)$. Moreover, by (irr$_{\sqsubset}$), $x \notin \mathbb{P}(x)$ and $x \in \mathbb{P}(y)$; so $\mathbb{P}(x) \subsetneq \mathbb{P}(y)$. For (mono$_{\mathbb{I}}$): $\mathbb{I}(x) := \mathbb{P}(x) \cup \{x\} \subseteq \mathbb{P}(y) \subsetneq \mathbb{I}(y)$.

Note that the pair of conditions (r$_{\sqsubseteq}$) and (t$_{\sqsubseteq}$) is equivalent to the following condition (see Lemma 2.1(ii) in Appendix I):

$$\begin{aligned} x \sqsubseteq y &\iff \forall_{z \in M}(z \sqsubseteq x \Rightarrow z \sqsubseteq y), \quad \text{or} \\ x \sqsubseteq y &\iff \mathbb{I}(x) \subseteq \mathbb{I}(y). \end{aligned} \tag{r\&t$_{\sqsubseteq}$}$$

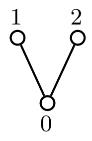
Finally, by (r$_{\sqsubseteq}$) and (antis$_{\sqsubseteq}$), we get the "principle of extensionality" with respect to $\mathbb{I}$ (or $\sqsubseteq$), i.e., for all $x, y \in M$ we have:

$$\begin{aligned} &\forall_{z \in M}(z \sqsubseteq x \Leftrightarrow z \sqsubseteq y) \implies x = y\,, \quad \text{or} \\ &\mathbb{I}(x) = \mathbb{I}(y) \implies x = y. \end{aligned} \tag{ext$_{\sqsubseteq}$}$$

Remark 2.1. Of course, we can not get an analogous "principle of extensionality" with respect to $\mathbb{P}$ that would take the following form: "$\mathbb{P}(x) = \mathbb{P}(y) \Rightarrow x = y$". Classical mereology allows for the existence objects without any part; so-called mereological atoms (see below). So for two (different) atoms x and y we have $\mathbb{P}(x) = \emptyset = \mathbb{P}(y)$, but $x \neq y$.

Formally, we put the structure $\langle \{1, 2\}, \sqsubset \rangle$ with the empty relation $\sqsubset$. It is easy to see that all axioms (L1)–(L4) are true in this structure. But $\mathbb{P}(1) = \emptyset = \mathbb{P}(2)$ and $1 \neq 2$.

So an analogous "principle of extensionality" with respect to $\mathbb{P}$ should have the following form: "$\mathbb{P}(x) = \mathbb{P}(y) \neq \emptyset \Rightarrow x = y$". But this will not get in the class **L12**. In fact, we put the structure $\langle \{0, 1, 2\}, \sqsubset \rangle$, where $\sqsubset := \{\langle 0, 1\rangle, \langle 0, 2\rangle\}$. We can illustrate this by the graph below:



This structure belongs to **L12**, but $\mathbb{P}(1) = \{0\} = \mathbb{P}(2)$ and $1 \neq 2$.

The "principle of extensionality" with respect to $\mathbb{P}$ will be obtained after the addition of axiom (L3) (see (ext$_{\sqsubset}$) on p. 82). □

Classical mereology allows for the existence of so-called mereological atoms. We say that a given object is is a *mereological atom* iff it has no part. Let $\mathfrak{at}$ be the set of *mereological atoms* in the structure $\mathfrak{M}$,[8] i.e.,

[8] The $L_{\sqsubset}$-formula used for e-definability of the set $\mathfrak{at}$ is "$\neg \exists_y\ y \sqsubset x$". Atoms in mereological structures will be discussed in detail in Section VI.6.

for any $x \in M$ we have:

$$x \in \mathbb{at} \;:\Longleftrightarrow\; \neg\exists_{z \in M}\, z \sqsubset x \;\Longleftrightarrow\; \mathbb{P}(x) = \emptyset\,. \tag{df $\mathbb{at}$}$$

Applying definitions and ($\mathrm{r}_{\sqsubseteq}$), for each $x \in M$ we obtain:

$$x \in \mathbb{at} \;\Longleftrightarrow\; \mathbb{I}(x) = \{x\} \;\Longleftrightarrow\; \forall_{z \in M}(z \sqsubseteq x \;\Leftrightarrow\; z = x). \tag{2.4}$$

Remark 2.2. Let us immediately highlight that each possible axiomatisation of mereological structures of the form $\langle M, \sqsubset\rangle$, in which $\sqsubset$ is supposed to be the relation *is a part of*, has to ensure that — apart from the case in which M is a singleton — there will not be a least element in M ('zero') with respect to $\sqsubset$. In other words, if the universe M has at least two members then there is no its member which is an ingrediens of all members of M, i.e., the following sentence will be true: $\neg\exists_{x \in M}\forall_{y \in M}\, x \sqsubseteq y$ (see ($\nexists 0$) on p. 85). This is supposed to rule out the existence of so-called 'empty objects' or 'zeros'. So if M is not a singleton then also no mereological atoms will be zeros. The case in which the set M is one-membered is of no interest on account of its triviality. □

Note that for any $x \in M$ we obtain:

$$x \in \mathbb{at} \;\Longleftrightarrow\; \forall_{y \in M}\big(\exists_{z \in M}(z \sqsubseteq x \wedge z \sqsubseteq y) \Rightarrow x \sqsubseteq y\big). \tag{2.5}$$

If $x \in \mathbb{at}$ and for some z we have $z \sqsubseteq x$ and $z \sqsubseteq y$. Hence, by (df $\mathbb{at}$), we have $z = x$ and therefore $x \sqsubseteq y$. Conversely, if $x \notin \mathbb{at}$ then for some y we have $y \sqsubset x$. Hence, by (df $\sqsubseteq$) and ($\mathrm{r}_{\sqsubseteq}$), we have $y \sqsubseteq x$ and $y \sqsubseteq y$, but by ($\mathrm{as}_{\sqsubset}$) and ($\mathrm{irr}_{\sqsubset}$), we obtain $x \not\sqsubseteq y$.

The second e-definable relation in $\mathfrak{M}$ is the binary relation $\bigcirc$ *overlaps*.[9] Two objects overlap iff they have a common ingrediens; i.e., for all $x, y \in M$ we have:

$$x \bigcirc y \;:\Longleftrightarrow\; \exists_{z \in M}(z \sqsubseteq x \wedge z \sqsubseteq y).\text{[10]} \tag{df $\bigcirc$}$$

---

[9] We take our intuitions about the meaning of this term from the case where $\sqsubset$ is the 'true' relation *is a part of* and $\sqsubseteq$ is the 'true' relation *is an ingrediens of*. The term "overlapping" comes from Leonard and Goodman [1940, p. 47] and is also used by Simons [1987], who also uses the same symbol as we do here. In [Leonard and Goodman, 1940] this relation is signified by '**o**'.

[10] Note that $\bigcirc := \sqsupseteq \circ \sqsubseteq$, where $\sqsupseteq$ is the converse relation to the relation $\sqsubseteq$ and $\circ$ is the relative product of two binary relations (see Appendix I). The $\mathrm{L}_{\sqsubset}$-formula used for the e-definition is "`∃z((z ⊏ x ∨ z = x) ∧ (z ⊏ y ∨ z = y))`".

Remark 2.3. Let us immediately highlight that each possible axiomatisation of mereological structures of the form $\langle M, \sqsubset \rangle$, in which $\sqsubset$ is supposed to be the relation *is a part of*, has to ensure that — apart from the case in which M is a singleton — there will exist in M elements which do not overlap (see ($\exists\wr$) on p. 85). □

It follows directly from the definition that the relation *overlaps* is symmetric and from ($\mathrm{r}_\sqsubseteq$) that it is reflexive and includes the relation *is an ingrediens of*, i.e., for arbitrary $x, y \in M$ we have:

$$x \bigcirc y \iff y \bigcirc x\,, \tag{$\mathrm{s}_\circ$}$$

$$x \bigcirc x\,, \tag{$\mathrm{r}_\circ$}$$

$$x \sqsubseteq y \implies x \bigcirc y\,. \tag{$\sqsubseteq\subseteq\circ$}$$

Moreover, by ($\mathrm{t}_\sqsubseteq$), for arbitrary $x, y, z \in M$ we obtain:

$$x \sqsubseteq y \wedge z \bigcirc x \implies z \bigcirc y\,. \tag{$\mathrm{mono}_\circ$}$$

Connected with the relation $\circ$ is the mapping $\mathbb{O}\colon M \to \mathcal{P}_+(M)$ which assigns each $x \in M$ the set $\mathbb{O}(x)$ of objects overlapping x.

$$\mathbb{O}(x) := \{y \in M : y \bigcirc x\}.$$

For any $x \in M$ the set $\mathbb{O}(x)$ is e-definable in $\mathfrak{M}$ with the parameter x. For arbitrary $x, y \in M$ we have:

$$y \in \mathbb{O}(x) \iff x \in \mathbb{O}(y),$$

$$x \in \mathbb{O}(x),$$

$$\mathbb{I}(x) \subseteq \mathbb{O}(x), \tag{$\mathbb{I} \subseteq \mathbb{O}$}$$

$$x \sqsubseteq y \implies \mathbb{O}(x) \subseteq \mathbb{O}(y), \tag{$\mathrm{mono}_\mathbb{O}$}$$

$$\mathbb{I}(x) \subseteq \mathbb{O}(y) \iff \mathbb{O}(x) \subseteq \mathbb{O}(y). \tag{2.6}$$

For (2.6): Assume that $\mathbb{I}(x) \subseteq \mathbb{O}(y)$ and $z \in \mathbb{O}(x)$. Then for some u we have $u \sqsubseteq z$ and $u \sqsubseteq x$. Hence, in virtue of the assumption, $u \in \mathbb{O}(y)$. Therefore for some v we have $v \sqsubseteq u$ and $v \sqsubseteq y$. So, by ($\mathrm{t}_\sqsubseteq$), we have $v \sqsubseteq z$ and $v \sqsubseteq y$, i.e., $z \in \mathbb{O}(y)$. The converse implication follows from ($\mathbb{I} \subseteq \mathbb{O}$).

The third e-definable relation is the binary relation $\wr$ *is exterior to*. One object is exterior to another iff they have no common ingrediens,[11]

[11] We take our intuitions again about the meaning of this term from the case where $\sqsubset$ is the 'true' relation *is a part of* of and $\sqsubseteq$ is the 'true' relation *is an ingre-*

i.e., for arbitrary $x, y \in M$ we have:

$$x \wr y \ :\Longleftrightarrow \neg\exists_{z\in M}(z \sqsubseteq x \wedge z \sqsubseteq y).^{12} \tag{df$\wr$}$$

It follows directly from the definitions that the relation $\wr$ *is exterior to* is the set-theoretical complement of the relation $\circ$ *overlaps* and $\wr$ is symmetric. It follows from ($\mathrm{r}_\circ$) that $\wr$ is irreflexive and from ($\mathrm{mono}_\circ$) we obtain the analogous property for $\wr$. So for all $x, y \in M$ we have:

$$x \wr y \iff \neg\, x \circ y\,, \tag{$\wr{=}{-}\circ$}$$

$$x \wr y \iff y \wr x\,, \tag{$\mathrm{s}_\wr$}$$

$$\neg\, x \wr x\,, \tag{$\mathrm{irr}_\wr$}$$

$$x \sqsubseteq y \wedge z \wr y \implies z \wr x\,. \tag{$\mathrm{mono}_\wr$}$$

3. The mereological sum of elements of a set

Let "P" be a schematic letter representing some general name of certain members of the set M. The definition of the collective class of Ps given by Leśniewski [1928] may be represented in the form of a sentential schema (I.6.1). Leśniewski established that if a name represented by "P" is non-empty, then exactly one object is a $\mathrm{class}_\mathrm{c}$ of Ps. (cf. (I.6.18)).

The range of the aforementioned general name is the distributive set of Ps, i.e., the set $\{y \in M : y \text{ is a } \mathsf{P}\}$. Modelling ourselves on Tarski [1956b], instead of saying that x is a $\mathrm{class}_\mathrm{c}$ of Ps we shall say that x is a *mereological sum* of all elements of the set $\{y \in M : y \text{ is a } \mathsf{P}\}$, i.e., the mereological sum of all Ps. As we said in Section I.7, this way of speaking allows us to dispense with schematic letters in our analysis. Instead of speaking of the mereological sum of all Ps, we can speak of

diens of. The term comes from Leśniewski (in Polish, the term is "jest zewnętrzny względem"). The relation $\wr$ is understood identically in [Leonard and Goodman, 1940, p. 46] and [Simons, 1987, p. 28], where it is named "discreteness" and "disjointness", respectively. The symbol being used with here is modelled on the symbol used in Simons [1987]. In Leonard and Goodman [1940] the symbol "⅂" is used. The term "separation" is perhaps inappropriate here, because it is possible to give an interpretation of mereology in so-called "pointless (point-free) geometry" in which two objects are exterior to one another but 'touch each other' [see, e.g., Gruszczyński and Pietruszczak, 2009]. In Remark 2.3, it was already noted that — excepting the case where M is a singleton — there will exist in M elements which are exterior to themselves (see ($\exists\wr$))

[12] The $\mathrm{L}_\sqsubseteq$-formula used in the definition for $\wr$ is the negation of the formula which we used for $\circ$.

the mereological sum of all elements of a given subset of the set M. For arbitrary $x \in M$ and $S \in \mathcal{P}(M)$, Leśniewski's definitional schema (I.6.1) is transformed into definition (I.7.1) which we may write using the term "mereological sum" in the following way:

> x is a *mereological sum* of all members of a set S iff
> every members of S is an ingrediens of x
> and every ingrediens of x has some common
> ingrediens with some members of S.

As we have already mentioned in Section I.7, the fact that x is a mereological sum of all elements of a set S will be expressed with the aid of the relation Sum holding between x and the set S. It will be included in the Cartesian product $M \times \mathcal{P}(M)$. Having established this, we may write the above definition of the relation Sum *is a mereological sum of all elements of a given set* as (cf. (I.7.1′)):

$$x \mathrel{\mathsf{Sum}} S \;:\Longleftrightarrow\; \forall_{z\in S}\, z \sqsubseteq x \wedge \forall_{y\in M}\big(y \sqsubseteq x \;\Rightarrow\; \exists_{z\in S}\exists_{u\in M}(u \sqsubseteq y \wedge u \sqsubseteq z)\big). \tag{df Sum}$$

By making use of the relation $\bigcirc$, which was defined by (df $\bigcirc$), we can write definition (df Sum) as:

$$x \mathrel{\mathsf{Sum}} S \iff \forall_{z\in S}\, z \sqsubseteq x \wedge \forall_{y\in M}(y \sqsubseteq x \;\Rightarrow \exists_{z\in S}\, y \bigcirc z). \tag{df′ Sum}$$

Moreover, by making use of the functions $\mathbb{I}$ and $\mathbb{O}$, we can write definition (df Sum) as:

$$x \mathrel{\mathsf{Sum}} S \iff S \subseteq \mathbb{I}(x) \wedge \forall_{y\in\mathbb{I}(x)}\exists_{z\in S}\, \mathbb{I}(y) \cap \mathbb{I}(z) \neq \emptyset\,. \tag{df″ Sum}$$

$$x \mathrel{\mathsf{Sum}} S \iff S \subseteq \mathbb{I}(x) \subseteq \bigcup \mathbb{O}[S]\,. \tag{df‴ Sum}$$

Remark 3.1. $\mathbb{O}[S]$ is the family of sets which is the image of a set S determined by the function $\mathbb{O}$ and $\bigcup \mathbb{O}[S]$ is the set-theoretical sum of the family $\mathbb{O}[S]$, i.e.:

$$\mathbb{O}[S] := \{Y \in \mathcal{P}_+(M) : \exists_{z\in S}\, Y = \mathbb{O}(z)\} = \{\mathbb{O}(z) : z \in S\}.$$

$$\bigcup \mathbb{O}[S] := \{y \in M : \exists_{Y\in\mathbb{O}[S]}\, y \in Y\}.$$

Thus, we have:

$$\bigcup \mathbb{O}[S] = \bigcup\{\mathbb{O}(z) : z \in S\} = \{y \in M : \exists_{z\in S}\, y \bigcirc z)\}. \quad \square$$

Finally, by making use of the relation $\wr$, which was defined by (df$\wr$), we can write (df Sum) in the form that Tarski [1956b] adopted:

$$x \mathsf{Sum} S \iff \forall_{z\in S}\, z \sqsubseteq x \wedge \neg\exists_{y\in M}(y \sqsubseteq x \wedge \forall_{z\in S}\, y \wr z). \quad (\text{df T}\,\mathsf{Sum})$$

Before continuing our analysis of the relation Sum we shall prove the following lemma.

LEMMA 3.1. *For arbitrary $x \in M$ and $S \in \mathcal{P}(M)$ we have: $x \mathsf{Sum} S$ iff the following three conditions hold*:

(a) $S \neq \emptyset$,
(b) $\forall_{z\in S}\, z \sqsubseteq x$,
(c) $\forall_{y\in M}(y \sqsubset x \Rightarrow \exists_{z\in S}\, y \bigcirc z)$.

PROOF. '⇒' Let $x \mathsf{Sum} S$. Then $\forall_{y\in M}(y \sqsubseteq x \Rightarrow \exists_{z\in S}\, z \bigcirc y)$. Hence $S \neq \emptyset$, since $x \sqsubseteq x$, by ($\mathrm{r}_{\sqsubseteq}$). Moreover, we obtain (c), because $y \sqsubset x$ implies $y \sqsubseteq x$.

'⇐' Let $y \sqsubseteq x$. If $y \sqsubset x$ then $\exists_{z\in S}\, z \bigcirc y$, by (c). Suppose therefore that $y = x$. By (a), for some z_0 we have $z_0 \in S$. Moreover, by (b), $z_0 \sqsubseteq x$. Hence, by ($\mathrm{r}_{\sqsubseteq}$), we have $z_0 \bigcirc x$. So we obtain: $\exists_{z\in S}\, z \bigcirc y$. Hence, by (ii), we obtain $x \mathsf{Sum} S$, in the light of (df′ Sum). □

Making use of functions $\mathbb{P}$, $\mathbb{I}$, and $\mathbb{O}$, we may write Lemma 3.1 as:

$$x \mathsf{Sum} S \iff \emptyset \neq S \subseteq \mathbb{I}(x) \wedge \mathbb{P}(x) \subseteq \bigcup \mathbb{O}[S]\,. \quad (3.1)$$

Hence no member of M is in the relation Sum with the set $\emptyset$:[13]

$$\neg\exists_{x\in M}\, x \mathsf{Sum} \emptyset\,. \quad (3.2)$$

For all $x \in M$ and $S \in \mathcal{P}_+(M)$, we say that x is the *greatest element* of S in $\mathfrak{M}$ iff $x \in S$ and for each $z \in S$ we have $z \sqsubseteq x$. There can at most be one greatest element of S in $\mathfrak{M}$. Briefly, if x and y are greatest elements of S, then $x \sqsubseteq y$ and $y \sqsubseteq x$. Therefore, $x = y$, by ($\mathrm{antis}_{\sqsubseteq}$). If $S = X$ then we will shortly say that x is a *greatest element of* $\mathfrak{M}$.

Applying ($\sqsubseteq\subseteq\circ$) we get:

LEMMA 3.2. *If x is the greatest element of S in $\mathfrak{M}$, then $x \mathsf{Sum} S$.*

[13] There is no mereological sum of members of the empty set (as there are no members of $\emptyset$). Cf. (I.6.8) and Section I.7.

Since, in virtue of ($\mathrm{r}_{\sqsubseteq}$), x is the greatest element in $\{x\}$ and $\mathbb{I}(x)$, so from ($\sqsubseteq\subseteq\circ$) or directly from Lemma 3.2 we have:[14]

$$x \operatorname{Sum} \{x\}\,, \tag{3.3}$$
$$x \operatorname{Sum} \mathbb{I}(x)\,. \tag{3.4}$$

Remark 3.2. With the identification of the relation Sum with the relation *is a collective set of certain objects*, condition (3.2) says that there is no collective empty set. Is this collective set really a 'collection into one whole' of some objects given this identification, however? As can be seen from (3.3) the definition does not keep the plural form "some" because it allows the 'summing' of one object. Lemma 3.2 says that it allows the 'summing' a group of objects in which these objects are ingredienses of one of them. We will give an alternative proposal for the formal definition of this collective set in sections 2 and 4 of Chapter IV. □

From Lemma 3.1 and ($\mathrm{r}_\circ$) it follows that (cf. (I.6.9) and (I.6.15)):

$$\forall_{x\in M}(\mathbb{P}(x) \neq \emptyset \iff x \operatorname{Sum} \mathbb{P}(x)). \tag{3.5}$$

In the light of (2.3) and (3.1) for all $S \in \mathcal{P}(M)$ and $x \in M$ we have:

$$x \operatorname{Sum} S \wedge \mathbb{P}(x) = \emptyset \implies S = \{x\}\,. \tag{3.6}$$

Moreover, the following holds for atoms:

$$x \operatorname{Sum} S \wedge y \in \mathfrak{at} \implies (y \sqsubseteq x \Leftrightarrow \exists_{z\in S}\, y \sqsubseteq z)\,. \tag{3.7}$$

If $x \operatorname{Sum} S$, $y \in \mathfrak{at}$, and $y \sqsubseteq x$, then there is a $z \in S$ such that $y \circ z$. Hence, in the light of (2.5), we have $y \sqsubseteq z$. Moreover, if $x \operatorname{Sum} S$ and for some $z \in S$ we have $y \sqsubseteq z$, then $y \sqsubseteq x$, because $S \subseteq \mathbb{I}(x)$ and the relation $\sqsubseteq$ is transitive.

For finite sets the relation Sum has an interesting property, namely that for arbitrary x, y_1, …, $y_n \in M$ ($n \geqslant 1$) the following holds:

$$x \operatorname{Sum} \{y_1, \ldots, y_n\} \iff x \operatorname{Sum} \{z \in M : \exists_{0\leqslant i\leqslant n}\, z \sqsubseteq y_i\}. \tag{3.8}$$

Firstly, in the light of ($\mathrm{mono}_{\mathbb{I}}$): $\forall_{i\leqslant n} y_i \sqsubseteq x$ iff $\forall_{i\leqslant n}\forall_z(z \sqsubseteq y_i \Rightarrow z \sqsubseteq x)$ iff $\forall_z(\exists_{i\leqslant n}\, z \sqsubseteq y_i \Rightarrow z \sqsubseteq x)$. Secondly, if we have $\forall_z(z \sqsubseteq x \Rightarrow \exists_{i\leqslant n}\, y_i \circ z)$, then also $\forall_z(z \sqsubseteq x \Rightarrow \exists_{i\leqslant n}\exists_u(u \sqsubseteq y_i \wedge u \sqsubseteq z))$. Hence, by ($\sqsubseteq\subseteq\circ$), we

[14] Cf. (I.6.5), (I.6.6), (I.6.13), (I.6.16), (I.7.2) and (I.7.3).

have $\forall_z(z \sqsubseteq x \Rightarrow \exists_u(\exists_{i\leqslant n} u \sqsubseteq y_i \wedge u \bigcirc z))$. Conversely, if $\forall_z(z \sqsubseteq x \Rightarrow \exists_u(\exists_{i\leqslant n}\, u \sqsubseteq y_i \wedge u \bigcirc z))$ then $\forall_z(z \sqsubseteq x \Rightarrow \exists_{i\leqslant n}\, y_i \bigcirc z)$, by (mono$_\circ$).

From the transitivity of the relation $\sqsubseteq$ and our definitions, so only by axiom (L2), we obtain:

$$\begin{aligned} x \,\mathbb{S}\mathrm{um}\, S &\implies \forall_y(y \bigcirc x \Leftrightarrow \exists_{z\in S}\, y \bigcirc z)\,, \qquad \text{or} \\ x \,\mathbb{S}\mathrm{um}\, S &\implies \mathbb{O}(x) = \bigcup \mathbb{O}[S]\,.^{15} \end{aligned} \tag{3.9}$$

Let $x \,\mathbb{S}\mathrm{um}\, S$. For '$\Rightarrow$': Suppose that $y \bigcirc x$, i.e., for some $u \in M$ we have $u \sqsubseteq y$ and $u \sqsubseteq x$. Then, by (df $\mathbb{S}\mathrm{um}$), there are $z \in S$ and $v \in M$ such that $v \sqsubseteq z$ and $v \sqsubseteq u$. Hence, by (t$_\sqsubseteq$), also $v \sqsubseteq y$. So $y \bigcirc z$. For '$\Leftarrow$': Suppose that for some $z \in S$ we have $z \bigcirc y$, i.e., there is $u \in M$ such that $u \sqsubseteq z$ and $u \sqsubseteq y$. Then, by (df $\mathbb{S}\mathrm{um}$), $z \sqsubseteq x$. Hence, by (t$_\sqsubseteq$), also $u \sqsubseteq x$. So $x \bigcirc y$.

Remark 3.3. Since the relation $\mathbb{S}\mathrm{um}$ is included in the Cartesian product $M \times \mathcal{P}_+(M)$, it therefore has type $(*, (*))$ in the hierarchy of types over the set M.[16] If we assume that for each $S \in \mathcal{P}_+(M)$ there exists exactly one $x \in M$ such that $x \,\mathbb{S}\mathrm{um}\, S$ (axioms (L3) and (L4) say just this, as we shall see), then we can create from the relation $\mathbb{S}\mathrm{um}$ a certain operation $\bigsqcup\colon \mathcal{P}_+(M) \to M$ of mereological sum of members of a given set. This operation assigns to the set S the sum $\bigsqcup S$ of its members, which is a member of the set M; $\bigsqcup S := (\iota\, x)\, x \,\mathbb{S}\mathrm{um}\, S$, i.e., $\bigsqcup S$ is the only x such that $x \,\mathbb{S}\mathrm{um}\, S$ (see Remark 5.4 and Section 7).

In particular, for the set $\{y, z\} \in \mathcal{P}_+(M)$ we have $\bigsqcup\{y, z\} =: y \sqcup z$, and thus we get the operation of mereological sum of two elements from $M \times M$ into M.

Such a use of the term "sum" in mereology does not clash with the use of that term in set theory, where it is used in the context "sum of all elements of a given family of sets". The relation *is a sum of* (a family of sets) is included in the Cartesian product $\mathcal{P}(M) \times \mathcal{P}(\mathcal{P}(M))$, i.e., it has the type $((*), ((*)))$ in the hierarchy of types over the set M. Associated with that relation is the operation $\bigcup\colon \mathcal{P}(\mathcal{P}(M)) \to \mathcal{P}(M)$ of sum of members of a given family of sets. This operation converts an arbitrary family of sets $\mathcal{S}$ from $\mathcal{P}(\mathcal{P}(M))$ in the set $\bigcup \mathcal{S}$ belonging to $\mathcal{P}(M)$; $\bigcup \mathcal{S} := \{x \in M : \exists_{S\in\mathcal{S}}\, x \in S\}$.

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[15] Theorem IV.3.1 will show that implication (3.9) may be reversed in mereological structures.

[16] On the hierarchy of types over a base set, see e.g. [Mostowski, 1948, p. 325].

In particular, for the family $\{X, Y\} \in \mathcal{P}_+(\mathcal{P}(M))$ we have $\bigcup\{X, Y\} = X \cup Y$, and therefore we have the operation of sum of two sets from $\mathcal{P}(M) \times \mathcal{P}(M)$ into $\mathcal{P}(M)$. In this case therefore there is no clash with the mereological use of the term "sum". □

We note finally in this section that thanks to ($\mathrm{t}_{\sqsubseteq}$), the relation $\mathbb{S}\mathrm{um}$ is also transitive, i.e., for arbitrary $x \in M$ and $S, Y_z \in \mathcal{P}_+(M)$, for $z \in S$ we have:

$$x \,\mathbb{S}\mathrm{um}\, S \wedge \forall_{z\in S}\, z \,\mathbb{S}\mathrm{um}\, Y_z \implies x \,\mathbb{S}\mathrm{um} \textstyle\bigcup_{z\in S} Y_z\,.\text{[17]} \tag{3.10}$$

Let $x \,\mathbb{S}\mathrm{um}\, S$ and $\forall_{z\in S}\, z \,\mathbb{S}\mathrm{um}\, Y_z$. Suppose that $y \in \bigcup_{z\in S} Y_z$, i.e., for some $z \in S$ we have $y \in Y_z$. Then — in virtue of the assumptions made — we have $y \sqsubseteq z$ and $z \sqsubseteq x$. So also $y \sqsubseteq x$, by ($\mathrm{t}_{\sqsubseteq}$). Moreover, suppose that $y \sqsubseteq x$. Then, by (df $\mathbb{S}\mathrm{um}$), there are $z \in S$ and $u \in M$ such that $u \sqsubseteq z$ and $u \sqsubseteq y$. Hence, in virtue of the second assumption, for some $v \in Y_z$ and $w \in M$ we have $w \sqsubseteq v$ and $w \sqsubseteq u$. Hence, by ($\mathrm{t}_{\sqsubseteq}$), we have $w \sqsubseteq y$. So $v \bigcirc y$. Of course, $v \in \bigcup_{z\in S} Y_z$. Thus, $x \,\mathbb{S}\mathrm{um} \bigcup_{z\in S} Y_z$.

4. The uniqueness of $\mathbb{S}\mathrm{um}$

The following axiom admitted by Leśniewski states that if a set has a mereological sum then it is unique:

$$\forall_{S\in\mathcal{P}(M)}\forall_{x,y\in M}(x \,\mathbb{S}\mathrm{um}\, S \wedge y \,\mathbb{S}\mathrm{um}\, S \implies x = y). \tag{L3}$$

Remark 4.1. Axiom (L3) should not raise any interpretational worries. Moreover — as has just been mentioned in footnote 2 — Theorem 4.4 says that this axiom may be replaced with another elementary. We will discuss the independence of axioms and the 'strength' of particular formulae in Chapter V. □

From (3.5) and (L3) we obtain the "principle of extensionality" with respect to $\mathbb{P}$ (or $\sqsubset$), i.e.:

$$\forall_{x,y\in M}(\emptyset \neq \mathbb{P}(x) = \mathbb{P}(y) \implies x = y). \tag{ext$_\sqsubset$}$$

If $\mathbb{P}(x) \neq \emptyset \neq \mathbb{P}(y)$, then $x \,\mathbb{S}\mathrm{um}\, \mathbb{P}(x)$ and $y \,\mathbb{S}\mathrm{um}\, \mathbb{P}(y)$, by (3.5). So if also $\mathbb{P}(x) = \mathbb{P}(y)$, then $y \,\mathbb{S}\mathrm{um}\, \mathbb{P}(x)$. Therefore $x = y$, by (L3).[18]

[17] Axioms (L1) and (L2) themselves and the conditions $\forall_{z\in S}\, z \,\mathbb{S}\mathrm{um}\, Y_z$ and $x \,\mathbb{S}\mathrm{um} \bigcup_{z\in S} Y_z$ do not entail that $x \,\mathbb{S}\mathrm{um}\, S$ (cf. (6.5)).

[18] As we showed in Remark 2.1, the assumption "$\mathbb{P}(x) \neq \emptyset$" is important.

Axiom (L3) directly entails the condition which states that $\{x\}$ is the only singleton whose sum is x (cf. (3.3)), i.e.,

$$\forall_{x,y\in M}(x \mathbin{\mathsf{Sum}} \{y\} \implies x = y). \tag{$\mathrm{S}_{\mathsf{Sum}}$}$$

Let $x \mathbin{\mathsf{Sum}} \{y\}$. In virtue of (3.3), we have $y \mathbin{\mathsf{Sum}} \{y\}$. Therefore $x = y$, by (L3).[19]

Simons [1987] accepts in place of (L3) a condition which he calls the *Weak Supplementation Principle* which follows from (L1) and (L3) (resp. ($\mathrm{irr}_{\sqsubset}$) and (L3); see Lemma 4.1(v) below) and which says that if x is a part of y then y has another part which is exterior to x:[20]

$$\begin{aligned} &\forall_{x,y\in M}(x \sqsubset y \implies \exists_{z\in M}(z \sqsubset y \wedge z \wr x)), \quad \text{or} \\ &\forall_{x,y\in M}(\mathbb{P}(y) \subseteq \mathbb{O}(x) \implies x \not\sqsubset y), \end{aligned} \tag{WSP}$$

what it intuitively says is that if x is a part of y, then we can find some z being a part of y and external to x. By contraposition, in the light of ($\wr$=−$\circ$), both versions of (WSP) are equivalent.

Now we consider structures of the form $\langle M, \sqsubset\rangle$ with a primitive relation $\sqsubset$ included in $M \times M$, where relations $\sqsubseteq$, $\circ$, and Sum are defined by (df $\sqsubseteq$), (df $\circ$), and (df Sum). The lemma below shows that (WSP) results from (L1)–(L3) and, moreover, (WSP) and (L2) ental (L1).[21]

Lemma 4.1. (i) ($\mathrm{irr}_{\sqsubset}$) *follows from* (WSP)*.*
(ii) (L1) *follows from* (L2) *and* (WSP)*.*
(iii) (WSP) *is equivalent to the set* $\{(\mathrm{irr}_{\sqsubset}), (\mathrm{S}_{\mathsf{Sum}})\}$*.*
(iv) *Three sets* $\{(\mathrm{L2}), (\mathrm{WSP})\}$*,* $\{(\mathrm{L1}), (\mathrm{L2}), (\mathrm{S}_{\mathsf{Sum}})\}$ *and* $\{(\mathrm{irr}_{\sqsubset}), (\mathrm{L2}), (\mathrm{S}_{\mathsf{Sum}})\}$ *are equivalent.*
(v) ($\mathrm{irr}_{\sqsubset}$) *and* (L3) *entail* (WSP)*. So also* (L1) *and* (L3) *entail* (WSP)*.*

Proof. *Ad* (i): Directly from definitions we obtain (2.3) and ($\mathbb{I} \subseteq \mathbb{O}$), i.e., $\mathbb{P}(x) \subseteq \mathbb{I}(x) \subseteq \mathbb{O}(x)$, for any $x \in M$. Hence $x \not\sqsubset x$, by (WSP).[22]

[19] Tarski [1937] accepts condition ($\mathrm{S}_{\mathsf{Sum}}$) in place of (L3) (cf. theorems IV.1.2 and IV.5.10).

[20] For this principle see also Lemma IV.1.1, Theorem IV.1.2 and Section IV.8.

[21] We are discussing this because Simons [1987], and others following him, e.g., Chisholm [1993], Libardi [1990], Smith [1995], have amongst their axioms (L1), (L2), (WSP).

[22] The elementary proof: Assume for a contradiction that for some $x \in M$ we have $x \sqsubset x$. Then, by virtue of (WSP), for some z we have: $z \sqsubset x$ and $z \wr x$. From the first of these, in the light of (df $\sqsubseteq$) and (df $\circ$), we have $z \circ x$. Therefore, by virtue of ($\wr$=−$\circ$), we have a contradiction.

Ad (ii): By (i), (WSP) entails ($\mathrm{irr}_{\sqsubset}$). But (L2) and ($\mathrm{irr}_{\sqsubset}$) entail (L1).

Ad (iii): '⇒' By (i), (WSP) entails ($\mathrm{irr}_{\sqsubset}$). So let $x \,\mathbb{S}\mathrm{um}\, \{y\}$ and assume for a contradiction that $x \neq y$. Then, by (df $\mathbb{S}$um) and (df $\sqsubseteq$), we have $y \sqsubset x$. Hence, in virtue of (WSP), for some $z \in M$ we have $z \sqsubset x$ and $z \wr y$. Therefore we get a contradiction, because $z \sqsubset x$ and $x \,\mathbb{S}\mathrm{um}\, \{y\}$ entails $z \circ y$.

'⇐' Let $x \sqsubset y$ and assume for a contradiction that $\neg\exists_{z\in M}(z \sqsubset y \wedge z \wr x)$, i.e., $\mathbb{P}(y) \subseteq \mathbb{O}(x)$. Then $y \,\mathbb{S}\mathrm{um}\, \{x\}$. Hence, by ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$), we have $x = y$. But this contradicts our assumption in virtue of ($\mathrm{irr}_{\sqsubset}$).

Ad (iv): '⇒': By (ii) (or (i)) and the '⇒'-part of (iii). '⇐': By the '⇐'-part of (iii). Moreover, (L1) and (L2) entail ($\mathrm{irr}_{\sqsubset}$). So .

Ad (v): We prove that (L3) entails ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$). So we use the '⇐'-part of (iii). □

Moreover, in transitive structures that meet (WSP), Lemma 3.2 can be strengthened. The following lemma will be used in Section 4 of Chapter IV:

LEMMA 4.2. *Let $\mathfrak{M} = \langle M, \sqsubset\rangle$ be transitive structure satisfying* (WSP). *If x is the greatest element of a set S in $\mathfrak{M}$, then x is the only mereological sum of S.*

PROOF. Let $x \in S$ be the greatest element in S. Then (a) $\forall_{z\in S}\, z \sqsubseteq x$. Furthermore, by virtue of Lemma 3.2, we have $x \,\mathbb{S}\mathrm{um}\, S$. Assume that (b) $y \,\mathbb{S}\mathrm{um}\, S$. Hence $x \sqsubseteq y$. If $x \neq y$ then $x \sqsubset y$. Therefore, by virtue of (WSP), for some u we have: (c) $u \sqsubset y$ and (d) $u \wr x$. From (a), (d) and ($\mathrm{t}_{\sqsubset}$) we obtain $\forall_{z\in S}\, z \wr u$, which contradicts (b), (c), and (df T $\mathbb{S}$um). Therefore $x = y$. □

Conditions (WSP) and ($\mathrm{irr}_{\wr}$) entail the following:

$$\forall_{x,y\in M}\big(x \sqsubset y \implies \exists_{z\in M}(z \sqsubset y \wedge z \neq x)\big), \tag{4.1}$$

which says that no element in M has exactly one part:

$$\forall_{x\in M}\ \mathrm{Card}\,\mathbb{P}(x) \neq 1\,. \tag{4.2}$$

The above fact is also easily obtained from ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$) and ($\mathrm{irr}_{\sqsubset}$). Assume for a contradiction that $\mathbb{P}(x) = \{y\}$, for some $x, y \in M$. Hence $y \sqsubset x$. Furthermore, $x \,\mathbb{S}\mathrm{um}\, \{y\}$, because $x \,\mathbb{S}\mathrm{um}\, \mathbb{P}(x)$, by (3.5). Hence, in virtue of ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$), we have $x = y$, which contradicts ($\mathrm{irr}_{\sqsubset}$).

It also follows from (WSP) that if M is not a singleton, then M has two members exterior to one another. Moreover, thanks to ($\mathrm{irr}_{\wr}$), the reverse implication holds. Thus:

$$\mathrm{Card}\, M > 1 \iff \exists_{x,y\in M}\, x \wr y\,. \tag{$\exists\wr$}$$

Let $\mathrm{Card}\, M > 1$ and assume for a contradiction that all members of M overlap one another. Then for some $x_1, x_2 \in M$ we have $x_1 \neq x_2$ and for some y_0 we have $y_0 \sqsubseteq x_1$ and $y_0 \sqsubseteq x_2$. Hence $y_0 \sqsubset x_1$ or $y_0 \sqsubset x_2$. In both cases, in virtue of (WSP), there exists $z \in M$ such that $z \wr y_0$: a contradiction.

It follows from ($\exists\wr$) that unless M is not a singleton, then there is no member in it which is a part of all the remaining elements, i.e., in M there is no object which is an ingrediens of all members of M. In other words, if M has at least two elements, the in M there is no least element. Thus:

$$\exists_{x\in M}\forall_{y\in M}\, x \sqsubseteq y \iff \mathrm{Card}\, M = 1\,. \tag{$\nexists 0$}$$

For the simpler implication: if in M there exists a least element then $\forall_{x,y}\, x \bigcirc y$. Hence, in virtue of ($\exists\wr$), we have $\mathrm{Card}\, M = 1$. For the converse implication: if M is a singleton then—in virtue of ($\mathrm{r}_{\sqsubseteq}$)—its only element is clearly the least element.

We prove below that in strict partial orders axiom (L3) is equivalent to the "principle of extensionality" with respect to the relation $\bigcirc$ (see Theorem 4.4). To this end, the following lemma will be useful, where we do not assume that the relation $\sqsubseteq$ is defined by (df $\sqsubseteq$). So we consider structures of the form $\langle M, \sqsubseteq\rangle$ with a primitive relation $\sqsubseteq$ included in $M \times M$. We assume, however, that two relations $\bigcirc$ and $\mathbb{S}\mathrm{um}$ are defined by (df $\bigcirc$) and (df $\mathbb{S}\mathrm{um}$) using $\sqsubseteq$. Similarly, we define the functions $\mathbb{I}$ and $\mathbb{O}$ as on pages 73 and 76, using $\sqsubseteq$ and $\bigcirc$, respectively.

Lemma 4.3. *From* ($\mathrm{r}_{\sqsubseteq}$) *and* ($\mathrm{t}_{\sqsubseteq}$) *it follows that for arbitrary* $x, y \in M$:

$$\mathbb{I}(x) \subseteq \mathbb{O}(y) \implies x\ \mathbb{S}\mathrm{um}\ \mathbb{I}(x) \cap \mathbb{I}(y).$$

Proof. Let $\mathbb{I}(x) \subseteq \mathbb{O}(y)$. From this and ($\mathrm{r}_{\sqsubseteq}$) it follows that $x \bigcirc y$. Hence $\emptyset \neq \mathbb{I}(x) \cap \mathbb{I}(y) \subseteq \mathbb{I}(x)$. Furthermore, take an arbitrary $u \sqsubseteq x$. In virtue of the assumption, we have $u \bigcirc y$. Therefore, there is v such that $v \sqsubseteq u$ and $v \sqsubseteq y$. From this and ($\mathrm{t}_{\sqsubseteq}$) it follows that $v \in \mathbb{I}(x) \cap \mathbb{I}(y)$. In virtue of ($\sqsubseteq\subseteq\bigcirc$) we have $v \bigcirc u$. Thus, $x\ \mathbb{S}\mathrm{um}\ \mathbb{I}(x) \cap \mathbb{I}(y)$. □

Of course, Lemma 4.3 also holds when we consider structures of the form $\langle M, \sqsubset \rangle$ with a primitive relation $\sqsubset$ included in $M \times M$, where relations $\sqsubseteq$, $\bigcirc$, and $\mathbb{S}\mathrm{um}$ are defined by (df $\sqsubseteq$), (df $\bigcirc$), and (df $\mathbb{S}\mathrm{um}$).

THEOREM 4.4. *Let $\mathfrak{M} = \langle M, \sqsubset \rangle$ be a transitive structure. Then axiom* (L3) *is equivalent to the "principle of extensionality" with respect to the relation $\bigcirc$, which has the following form*:

$$\begin{aligned} &\forall_{x,y\in M}\big(\forall_{z\in M}(z \bigcirc x \;\Leftrightarrow\; z \bigcirc y) \implies x = y\big), \quad \textit{or} \\ &\forall_{x,y\in M}\big(\mathbb{O}(x) = \mathbb{O}(y) \implies x = y\big). \end{aligned} \tag{ext$_\circ$}$$

PROOF. Let $\mathfrak{M} = \langle M, \sqsubset \rangle$ satisfy axiom (L2).

Assume that (L3) holds and that $\mathbb{O}(x) = \mathbb{O}(y)$. Then $\mathbb{I}(x) \subseteq \mathbb{O}(y)$ and $\mathbb{I}(y) \subseteq \mathbb{O}(x)$, by ($\mathbb{I} \subseteq \mathbb{O}$). Hence $x \;\mathbb{S}\mathrm{um}\; \mathbb{I}(x) \cap \mathbb{I}(y)$ and $y \;\mathbb{S}\mathrm{um}\; \mathbb{I}(x) \cap \mathbb{I}(y)$, by Lemma 4.3. Thus, applying (L3), we get $x = y$.

Now assume that (ext$_\circ$) holds and that $x \;\mathbb{S}\mathrm{um}\; S$ and $y \;\mathbb{S}\mathrm{um}\; S$. Then, in virtue of (3.9), we have $\mathbb{O}(x) = \bigcup \mathbb{O}[S] = \mathbb{O}(y)$. Hence, applying (ext$_\circ$), we get $x = y$. □

So condition (ext$_\circ$) is a thesis of our theory. Hence, by ($\wr{=}{-}\bigcirc$), we obtain an analogous principle with respect to $\wr$:

$$\forall_{x,y\in M}\big(\forall_{z\in M}(z \wr x \;\Leftrightarrow\; z \wr y) \implies x = y\big). \tag{ext$_\wr$}$$

Let **L123** be the class of structures of the form $\langle M, \sqsubset \rangle$ satisfying conditions (L1), (L2), and (L3). By Theorem 4.4 we obtain that the class **L123** is finitely elementarily axiomatisable. In order to more easily formulate this fact let us broaden the language $L_{\sqsubset}$ to include two-argument predicates: "$\sqsubseteq$", "O", and "$\wr$". The two predicates are supposed to correspond to the relations $\sqsubseteq$, $\bigcirc$ and $\wr$, respectively; so in any structure $\langle M, \sqsubset \rangle$ we will interpret them by the corresponding relations. Therefore, "$\sqsubseteq$", "O" and "$\wr$" we will read as "is an ingrediens", "overlaps with" and "is exterior to", respectively.

Let us broaden the language $L_{\sqsubset}$ to the elementary language $L_{\sqsubset}^{\sqsubseteq\mathsf{O}}$, which arises in the same way except for the use of the three predicates (and "="). In $L_{\sqsubset}^{\sqsubseteq\mathsf{O}}$ we define the predicates "$\sqsubseteq$", "O" "$\wr$" with the help of the elementary sentences below which correspond to conditions (df $\sqsubseteq$), (df $\bigcirc$) and (df $\wr$), respectively:

$$\forall_x \forall_y (x \sqsubseteq y \equiv (x \sqsubset y \vee x = y)) \tag{$\delta \sqsubseteq$}$$

$$\forall_x \forall_y (x \,\mathsf{O}\, y \equiv \exists_z (z \sqsubseteq x \wedge z \sqsubseteq y)) \tag{$\delta \mathsf{O}$}$$

$$\forall_x \forall_y (x \wr y \equiv \neg \exists_z (z \sqsubseteq x \wedge z \sqsubseteq y)) \tag{$\delta \wr$}$$

In the light of Theorem 4.4 we see that the class **123** is equal to the class of $\mathrm{L}_{\sqsubset}^{\sqsubseteq\circ}$-structures composed of all models of the sentences (λ1), (λ2), ($\delta\sqsubseteq$), ($\delta\circ$), ($\delta\wr$), and the following one corresponding to ($\mathrm{ext}_\circ$)

$$\forall_x \forall_y (\forall_z (z \circ x \equiv z \circ y) \to x = y) \tag{$\varepsilon_\circ$}$$

Thus, we have shown that the class **L123** is finitely elementarily axiomatisable.

Of course, we can build a suitable elementary theory in the language $\mathrm{L}_{\sqsubset}$ by eliminating definitions ($\delta\sqsubseteq$), ($\delta\circ$) and ($\delta\wr$), and by substituting, e.g., $\ulcorner\exists_z((z \sqsubset x_i \vee z = x_i) \wedge (z \sqsubset x_j \vee z = x_j))\urcorner$ in place of $\ulcorner x_i \circ x_j\urcorner$.

5. Mereological structures

We find ourselves with the following problem: *for which subsets of the universe M are mereological sums of their members supposed to exist?* A possible solution would be to accept the strongest axiom of existence for mereological sums which definition (df $\mathbb{Sum}$) itself allows. We know from Lemma 3.1 that this is an axiom ensuring the existence of a sum of all members of an arbitrary non-empty set included in M:

$$\forall_{S\in\mathcal{P}_+(M)}\exists_{x\in M}\ x \ \mathbb{Sum}\ S\,. \tag{L4}$$

A structure $\langle M, \sqsubset\rangle$ in which axioms (L1)–(L4) are true will be called a *mereological structure*. As we mentioned in Section 1, let **MS** be the class of all mereological structures. The class **MS** is non-empty. For example, for $M := \{a\}$ with the empty relation $\sqsubset$ we obtain a trivial mereological structure. Axioms (L1) and (L2) are satisfied in this structure. Furthermore, $\sqsubseteq\, = \mathrm{id}_{\{a\}} = \circ$, i.e., $a \ \mathbb{Sum}\ \{a\}$ and therefore (L3) and (L4) are satisfied.

Remark 5.1. For any set S, we used the binary relation $\mathrm{id}_S := \{\langle x, y\rangle : x, y \in S \wedge x = y\} = \{\langle x, x\rangle : x \in S\}$ of *identity* on S. □

Remark 5.2. (i) The solution provided by axiom (L4) very often raises doubts. We are not going to go through them here because in the majority of cases they boil down to a problem of the following sort: the moon and the heart of the author of this book ("hab" for short) are material objects. Therefore if something is the mereological sum of the elements of this set $\{\mathrm{moon}, \mathrm{hab}\}$, it would also be a material entity[23].

[23] We do not need a further abstract object because we already have one — the distributive set $\{\mathrm{moon}, \mathrm{hab}\}$.

Is there, however, such a material entity? For some, the falsity of axiom (L4) provides a negative answer to this question. In our opinion that negative response says just that the set M, in which the moon and hab would appear as members, cannot be the bearer of the mereological structure. The 'strength' of axiom (L4) causes there to be a restriction on the applications of the theory of mereological structures. Perhaps these applications come down to certain varieties of geometry; e.g., for so-called free-point geometry [see Gruszczyński and Pietruszczak, 2008, 2009].

(ii) A different and "extreme" solution to the problem would be to dodge the question and not accept any sum existence axiom.[24] We will give a 'tautological' answer: apart from the cases resulting from (df $\mathbb{Sum}$) (cf. (3.3)–(3.5)) the existence of a mereological sum is dependent on the existence of M itself (some of its subsets can have a sum of their members and some can lack such a sum). □

From (L3) and (L4) the following sentence clearly follows:

$$\forall_{S\in\mathcal{P}_+(M)}\exists_{x\in M}\big(x \;\mathbb{Sum}\; S \wedge \forall_{y\in M}(y \;\mathbb{Sum}\; S \;\Rightarrow\; x = y)\big). \qquad \text{(L3-L4)}$$

Remark 5.3. Of course, (L4) follows from (L3-L4). Moreover, since the relation $\sqsubseteq$ is reflexive, then we may derive from ($\mathrm{t}_{\sqsubseteq}$) and (df $\mathbb{Sum}$) result (3.1), which says that there is no mereological sum for $\emptyset$. From this and (L3-L4) follows (L3) as well. In fact, suppose that $x \;\mathbb{Sum}\; S$ and $y \;\mathbb{Sum}\; S$. Then $S \neq \emptyset$, by (3.1). Hence, in the light of (L3-L4), for some x_0 we have $x_0 \;\mathbb{Sum}\; S$ and $x = x_0 = y$.

Thus, the conjunction of the sentences (L3) and (L4) in the theory of class **MS** is therefore equivalent to the sentence (L3-L4). □

Remark 5.4. Throughout the book we use the description operator "ι" forming the expression "$(\iota\, x)\ \varphi(x)$" which is the individual constant ⌜the only object x such that $\varphi(x)$⌝, where a formula $\varphi(x)$ has "x" as a free variable. To use it, first we have to prove that there exists exactly one object x such that $\varphi(x)$, i.e., the formula $\varphi(x)$ must fulfill the following two conditions:

$$\exists_x\, \varphi(x)\,,$$
$$\forall_{x,y}\big(\varphi(x) \wedge \varphi(y) \Rightarrow x = y\big).$$

[24] We are accepting in its place stronger axioms for the relation $\sqsubset$, for example, the "Strong Supplementation Principle" (SSP) given later. A wide-ranging list of acceptable sum existence axioms is given in chapters II and III of the book [Pietruszczak, 2013].

or, equivalently, the following one:

$$\exists_x \big(\varphi(x) \wedge \forall_y (\varphi(y) \Rightarrow x = y)\big).$$

We will express this formally thus: write: $\exists!_x\, \varphi(x)$, □

By (L3) and (L4) (resp. (L3-L4)), we obtain:

$$\exists!_x\; x \;\mathsf{Sum}\; M\,,$$

since $M \neq \emptyset$ (in the above remark we put $\varphi(x) :=$ "$x \;\mathsf{Sum}\; M$"). Thus, we can introduce the following individual constant:

$$\mathbb{1} := (\iota\, x)\; x \;\mathsf{Sum}\; M, \tag{df $\mathbb{1}$}$$

i.e., $\mathbb{1}$ is the only object $x \in M$ such that $x \;\mathsf{Sum}\; M$ (see Remark 5.4). That is, in a mereological structure $\mathfrak{M}$, the object $\mathbb{1}$ is the mereological sum of all members of M. It follows from definition (df Sum) that:

$$\begin{aligned} &\mathbb{1} \text{ is the greatest element in } \mathfrak{M}, \qquad \text{or} \\ &\forall_{z \in M}\; z \sqsubseteq \mathbb{1}\,. \end{aligned} \tag{5.1}$$

Therefore, we will call $\mathbb{1}$ the *unity* in the structure $\mathfrak{M}$.

6. Some important properties of mereological structures

We will show that in mereological structures a condition of polarisation holds, i.e., a condition of separation which Simons [1987, p. 29] calls the *Strong Supplementation Principle*:

$$\begin{aligned} &\forall_{x,y \in M} \big(x \not\sqsubseteq y \implies \exists_{z \in M} (z \sqsubseteq x \wedge z \wr y)\big), \quad \text{or} \\ &\forall_{x,y \in M} \big(\mathbb{I}(x) \subseteq \mathbb{O}(y) \implies x \sqsubseteq y\big). \end{aligned} \tag{SSP}$$

What this intuitively says is that if x is not an ingrediens of y, then we can find some z being an ingrediens of x and external to y. In the light of ($\wr$=$-\!\circ$) both versions of (SSP) are equivalent.

Lemma IV.1.3 on p. 135 says that (SSP) does not follow from axioms (L1)–(L3) themselves and in Lemma 6.4 it is stated that (L3) follows from (L2) and (SSP). Now we prove:

Theorem 6.1. *Condition* (SSP) *holds in all mereological structures.*

Proof. Let $\mathfrak{M} \in \mathsf{MS}$ and assume for a contradiction that (SSP) is false in $\mathfrak{M}$, i.e., that for some $x_0, y_0 \in M$ we have (a) $x_0 \not\sqsubseteq y_0$ and (b) $\mathbb{I}(x_0) \subseteq \mathbb{O}(y_0)$.

Since $\mathbb{I}(y_0) \cup \{x_0\} \neq \emptyset$, therefore—in virtue of (L4)—for some $z_0 \in M$ we have z_0 Sum $(\mathbb{I}(y_0) \cup \{x_0\})$. Thus, from(df′ Sum) we have (c) $\forall_{y \in \mathbb{I}(y_0)} y \sqsubseteq z_0$, (d) $x_0 \sqsubseteq z_0$ and (e) $\forall_z (z \sqsubseteq z_0 \Rightarrow \exists_u ((u \sqsubseteq y_0 \vee u = x_0) \wedge u \bigcirc z))$, i.e., $\forall_z (z \sqsubseteq z_0 \Rightarrow (x_0 \bigcirc z \vee \exists_{u \in \mathbb{I}(y_0)} u \bigcirc z))$. We will show that from (b) it follows that condition $x_0 \bigcirc z$ entails $\exists_{u \in \mathbb{I}(y_0)}\, u \bigcirc z$.

Let $x_0 \bigcirc z$. Then, by (df $\bigcirc$), for some x_1 we have $x_1 \sqsubseteq x_0$ and $x_1 \sqsubseteq z$. In virtue of (b) we have $x_1 \bigcirc y_0$. Hence, by (df $\bigcirc$), for some x_2 we have $x_2 \sqsubseteq x_1$ and $x_2 \sqsubseteq y_0$. In the light of ($\mathrm{t}_\sqsubseteq$) we have $x_2 \sqsubseteq z$. From this and ($\sqsubseteq \subseteq \circ$) we obtain $x_2 \bigcirc z$. Thus, $x_2 \sqsubseteq y_0$ and $x_2 \bigcirc z$, i.e., we get (f): $\forall_z (z \sqsubseteq z_0 \Rightarrow \exists_{u \in \mathbb{I}(y_0)}\, u \bigcirc z)$.

It follows from (c) and (f) that z_0 Sum $\mathbb{I}(y_0)$. In virtue of (3.4), we have y_0 Sum $\mathbb{I}(y_0)$. Thus, in virtue of (L3), we have $y_0 = z_0$. And this contradicts (a) and (d) because from them it follows that $y_0 \neq z_0$. □

From (SSP), ($\mathrm{r}_\sqsubseteq$), and ($\mathrm{t}_\sqsubseteq$) we obtain that for arbitrary $x, y \in M$:

$$\begin{aligned} x \sqsubseteq y &\iff \forall_{z \in M} (z \sqsubseteq x \Rightarrow\; z \bigcirc y), \quad \text{or} \\ x \sqsubseteq y &\iff \mathbb{I}(x) \subseteq \mathbb{O}(y). \end{aligned} \tag{6.1}$$

Hence, by (2.6), we can get the stronger condition from ($\mathrm{mono}_\circ$):

$$\begin{aligned} x \sqsubseteq y &\iff \forall_{z \in M} (z \bigcirc x \Rightarrow\; z \bigcirc y), \quad \text{or} \\ x \sqsubseteq y &\iff \mathbb{O}(x) \subseteq \mathbb{O}(y)\,. \end{aligned} \tag{$\mathrm{df}_\circ \sqsubseteq$}$$

Hence, by ($\wr = - \bigcirc$), we can get:

$$x \sqsubseteq y \iff \forall_{z \in M} (z \wr y \Rightarrow z \wr x). \tag{$\mathrm{df}_\wr \sqsubseteq$}$$

Before pursuing our analysis of the sentence (SSP), we shall prove a lemma and a corollary with the same assumptions as used in the proof of Lemma 4.3. Thus, we do not assume that the relation $\sqsubseteq$ is defined by (df $\sqsubseteq$). So we consider structures of the form $\langle M, \sqsubseteq \rangle$ with a primitive relation $\sqsubseteq$ included in $M \times M$. We assume, however, that two relations $\bigcirc$ and Sum are defined by (df $\bigcirc$) and (df Sum) using $\sqsubseteq$.

Lemma 6.2. (i) *From* ($\mathrm{t}_\sqsubseteq$) *and* (SSP) *follows what we shall call the first principle of monotonicity*:

$$\forall_{S \in \mathcal{P}(M)} \forall_{x,y \in M} (\mathbb{I}(x) \subseteq \bigcup \mathbb{O}[S] \wedge S \subseteq \mathbb{I}(y) \implies x \sqsubseteq y). \tag{M1}$$

(ii) *From* (M1) *it follows the sentence we shall call the second principle of monotonicity*:

$$\forall_{S_1,S_2\in\mathcal{P}(M)}\forall_{x,y\in M}(x \mathrel{\mathsf{Sum}} S_1 \wedge y \mathrel{\mathsf{Sum}} S_2 \wedge S_1 \subseteq S_2 \implies x \sqsubseteq y). \tag{M2}$$

(iii) [Gruszczyński and Pietruszczak, 2014, p. 129] (M2) *entails* ($\mathrm{r}_\sqsubseteq$).

(iv) (M1) *entails* ($\mathrm{r}_\sqsubseteq$).

(v) ($\mathrm{t}_\sqsubseteq$) *and* (SSP) *entail* ($\mathrm{r}_\sqsubseteq$).

(vi) ($\mathrm{t}_\sqsubseteq$) *and* (M2) *entail* (M1).

(vii) (M1) *entails* (SSP).

(viii) ($\mathrm{antis}_\sqsubseteq$) *and* (M2) *entail* (L3).

(ix) ($\mathrm{t}_\sqsubseteq$), ($\mathrm{antis}_\sqsubseteq$), *and* (SSP) *entail* (L3).

PROOF. *Ad* (i): Let $\mathbb{I}(x) \subseteq \bigcup \mathbb{O}[S]$, i.e., $\mathbb{I}(x) \subseteq \bigcup_{z\in S}\mathbb{O}(z)$, and $S \subseteq \mathbb{I}(y)$. Then $\mathbb{I}(x) \subseteq \bigcup_{z\in\mathbb{I}(y)}\mathbb{O}(z)$. We have $\forall_{z\in\mathbb{I}(y)}\mathbb{O}(z) \subseteq \mathbb{O}(y)$, in virtue of condition ($\mathrm{mono}_\circ$) which follows from ($\mathrm{t}_\sqsubseteq$). Thus, $\mathbb{I}(x) \subseteq \bigcup_{z\in\mathbb{I}(y)}\mathbb{O}(z) \subseteq \mathbb{O}(y)$. Hence $x \sqsubseteq y$, by (SSP).

Ad (ii): Assume that $x \mathrel{\mathsf{Sum}} S_1$, $y \mathrel{\mathsf{Sum}} S_2$, and $S_1 \subseteq S_2$. Then $\mathbb{I}(x) \subseteq \bigcup\mathbb{O}[S_1] \subseteq \bigcup\mathbb{O}[S_2]$ and $S_2 \subseteq \mathbb{I}(y)$. Hence $x \sqsubseteq y$, in virtue of (M1).

Ad (iii): (M2) entails (a): $\forall_{x\in M}(x \not\sqsubseteq x \Rightarrow \neg\exists_{S\in\mathcal{P}(M)}\ x \mathrel{\mathsf{Sum}} S)$. Hence we obtain (b): $\forall_{x\in M}(x \not\sqsubseteq x \Rightarrow \exists_{y\in M}(y \sqsubseteq x \wedge y \neq x \wedge y \not\sqsubseteq y)$. Assume for a contradiction that $x \not\sqsubseteq x$ and for any $y \in M$ such that $y \sqsubseteq x$ and $y \neq x$ we have $y \sqsubseteq y$. We put $\mathrm{P}(x) := \{z \in M : z \sqsubseteq x \wedge z \neq x\}$. Then (d): $x \mathrel{\mathsf{Sum}} \mathrm{P}(x)$. In fact, firstly $\forall_{z\in\mathrm{P}(x)}\ z \sqsubseteq x$, by definition. Secondly, suppose that $y \sqsubseteq x$. Then $y \neq x$, since $x \not\sqsubseteq x$. So in (df Sum) we can take $z = y = u$. But (d) contradicts (a).

Moreover, we obtain (c): $\forall_{x\in M}(x \not\sqsubseteq x \Rightarrow x \mathrel{\mathsf{Sum}} \mathrm{P}(x))$. Let $x \not\sqsubseteq x$. Then $\forall_{z\in\mathrm{P}(x)}\ z \sqsubseteq x$. Now suppose that $y \sqsubseteq x$. Then $y \neq x$, since $x \not\sqsubseteq x$. Moreover, if $y \sqsubseteq y$, then in (df Sum) we can take $z = y = u$. If $y \not\sqsubseteq y$, then by (b) there is $u \in M$ such that $u \sqsubseteq y$. So in (df Sum) we can take $z = y$.

By (a) and (c) we have that ($\mathrm{r}_\sqsubseteq$) holds.

Ad (iv): By (ii) and (iii).

Ad (v): By (i) and (iv).

Ad (vi): (M2) entails ($\mathrm{r}_\sqsubseteq$), by (iii). Let $\mathbb{I}(x) \subseteq \mathbb{O}[S]$ and $S \subseteq \mathbb{I}(y)$. From this and ($\mathrm{t}_\sqsubseteq$) we obtain $\mathbb{I}(x) \subseteq \mathbb{O}(y)$, as we have shown in (i). Hence — using ($\mathrm{r}_\sqsubseteq$), ($\mathrm{t}_\sqsubseteq$) and Lemma 4.3 — we have $x \mathrel{\mathsf{Sum}} \mathbb{I}(x) \cap \mathbb{I}(y)$. Furthermore, from ($\mathrm{r}_\sqsubseteq$), we have $y \mathrel{\mathsf{Sum}} \mathbb{I}(y)$. Therefore, in virtue of (M2), we get $x \sqsubseteq y$.

Ad (vii): (M1) entails ($r_\sqsubseteq$), by (iv). Let $\mathbb{I}(x) \subseteq \mathbb{O}(y)$. Then $\mathbb{I}(x) \subseteq \bigcup \mathbb{O}[\{y\}]$ and $\{y\} \subseteq \mathbb{I}(y)$, by ($r_\sqsubseteq$). Thus, $x \sqsubseteq y$, in virtue of (M1).

Ad (viii): Let $x \mathrel{\mathsf{Sum}} S$ and $y \mathrel{\mathsf{Sum}} S$. Then, in virtue of (M2), we get $x \sqsubseteq y$ and $y \sqsubseteq x$. Thus, $x = y$, by ($\text{antis}_\sqsubseteq$).

Ad (ix): By (i) and (ii), from ($t_\sqsubseteq$) and (SSP) we have (M1) and (M2). From this and ($\text{antis}_\sqsubseteq$) we obtain (L3), by (viii). □

Corollary 6.3. *It follows from* ($t_\sqsubseteq$) *that*:

(i) *Conditions* (SSP), (M1), *and* (M2) *are equivalent.*
(ii) *Condition* (SSP) *is equivalent to the conjunction of* ($r_\sqsubseteq$) *and the sentence we shall call the third principle of monotonicity*:

$$\forall_{S \in \mathcal{P}(M)} \forall_{x,y \in M} \big(x \mathrel{\mathsf{Sum}} S \wedge S \subseteq \mathbb{I}(y) \implies x \sqsubseteq y\big). \tag{M3}$$

Proof. *Ad* (i): By Lemma 6.2(i,ii,vi,vii).

Ad (ii): Suppose that ($t_\sqsubseteq$) holds. '⇒' Assume that (SSP) holds. Then also ($r_\sqsubseteq$) and (M1) hold, by Lemma 6.2(v) and (i), respectively. Let $x \mathrel{\mathsf{Sum}} S$ and $S \subseteq \mathbb{I}(y)$. Then $\mathbb{I}(x) \subseteq \bigcup \mathbb{O}[S]$. Hence $x \sqsubseteq y$, by (M1). '⇐' Suppose that ($r_\sqsubseteq$) and (M3) hold. Then, by Lemma 4.3, if $\mathbb{I}(x) \subseteq \mathbb{O}(y)$ then $x \mathrel{\mathsf{Sum}} \mathbb{I}(x) \cap \mathbb{I}(y)$. Since $\mathbb{I}(x) \cap \mathbb{I}(y) \subseteq \mathbb{I}(y)$, so $x \sqsubseteq y$, by (M3). Therefore, (SSP) holds. □

Of course, Lemma 6.2 and Corollary 6.3 also feature when we consider structures of the form $\langle M, \sqsubset \rangle$ with the primitive relation $\sqsubset$ included in $M \times M$, where relations $\sqsubseteq$, $\circ$, and Sum are defined by (df $\sqsubseteq$), (df $\circ$), and (df Sum).[25] Let us note that with such an assumption, (M1)–(M3) follow from (L2) and (SSP), and, moreover, we obtain the following:

Lemma 6.4. (L2) *and* (SSP) *entail* (L3).

Proof. By Lemma 6.2(ix), since ($t_\sqsubset$) and ($\text{antis}_\sqsubset$) follow from (L2). □

Now we obtain two results which — written in the language of Leśniewski's original system — were first proved by Tarski [see Leśniewski, 1930, p. 87, theses (*b*) and (*c*)]. The first of Tarski's results says that for all $S \in \mathcal{P}(M)$ and $x \in M$:

$$x \mathrel{\mathsf{Sum}} S \iff S \neq \emptyset \wedge S \subseteq \mathbb{I}(x) \wedge \forall_{y \in M}(S \subseteq \mathbb{I}(y) \Rightarrow x \sqsubseteq y). \tag{6.2}$$

[25] Then, however, points (iii)–(v) of this lemma are devoid of purpose, because condition ($r_\sqsubseteq$) it is obtained directly from (df $\sqsubseteq$).

'⇒' Assume that $x \mathbin{\mathsf{Sum}} S$. First we use (3.2); so $S \neq \emptyset$. Second, by (SSP) and Corollary 6.3(ii), we can use (M3). '⇐' Suppose that (a): $S \neq \emptyset$; and (b): $S \subseteq \mathbb{I}(x)$ and $\forall_{y\in M}(S \subseteq \mathbb{I}(y) \Rightarrow x \sqsubseteq y)$. From (a) and (L4) we obtain that there is $z \in M$ such that $z \mathbin{\mathsf{Sum}} S$. Hence $S \subseteq \mathbb{I}(z)$ and $\forall_{y\in M}(S \subseteq \mathbb{I}(y) \Rightarrow z \sqsubseteq y)$, by (M3). From this and (b) we have $x \sqsubseteq z$ and $z \sqsubseteq x$. Therefore, by (antis$_\sqsubseteq$), $x = z$; so $x \mathbin{\mathsf{Sum}} S$.

The second of Tarski's results says that for all $S \in \mathcal{P}(M)$ and $x \in M$:

$$x \mathbin{\mathsf{Sum}} S \iff S \neq \emptyset \wedge \forall_{y\in M}(x \sqsubseteq y \Leftrightarrow S \subseteq \mathbb{I}(y)). \tag{6.3}$$

'⇒' Assume that $x \mathbin{\mathsf{Sum}} S$. So also $S \subseteq \mathbb{I}(x)$. First we use (3.2); so $S \neq \emptyset$. Second, if $x \sqsubseteq y$ then $S \subseteq \mathbb{I}(x) \subseteq \mathbb{I}(y)$, by (t$_\sqsubseteq$). Conversely, if $S \subseteq \mathbb{I}(y)$ then $x \sqsubseteq y$, by (M3). '⇐' Assume that (a): $S \neq \emptyset$; and (b): $\forall_{y\in M}(x \sqsubseteq y \Leftrightarrow S \subseteq \mathbb{I}(y))$. From (a) and (L4) we obtain that there is a $z \in M$ such that $z \mathbin{\mathsf{Sum}} S$. Hence (c): $S \subseteq \mathbb{I}(z)$ and (d): $\forall_{y\in M}(S \subseteq \mathbb{I}(y) \Rightarrow z \sqsubseteq y)$, by (M3). Now from (c) and (b) we obtain $x \sqsubseteq z$. Moreover, since $x \sqsubseteq x$, so $S \subseteq \mathbb{I}(x)$, by (b). So $z \sqsubseteq x$, by (d). Therefore, by (antis$_\sqsubseteq$), $x = z$; so $x \mathbin{\mathsf{Sum}} S$.

Kuratowski was the first to prove the following equivalence for Leśniewski's original system [see Leśniewski, 1930, p. 87, Thesis (a)]:

$$x \mathbin{\mathsf{Sum}} S \iff S \neq \emptyset \wedge S \subseteq \mathbb{I}(x) \wedge \forall_{y\in M}(S \subseteq \mathbb{I}(y) \wedge y \sqsubseteq x \Rightarrow y = x). \tag{6.4}$$

'⇒' Let $x \mathbin{\mathsf{Sum}} S$. Then, by (6.2), we have: (a) $S \neq \emptyset$; (b) $S \subseteq \mathbb{I}(x)$ and (c) $\forall_{y\in M}(S \subseteq \mathbb{I}(y) \Rightarrow x \sqsubseteq y)$. Hence if $S \subseteq \mathbb{I}(y)$ and $y \sqsubseteq x$, then $x \sqsubseteq y$, by (c), and so $y = x$, in virtue of (antis$_\sqsubseteq$). '⇐' Assume that (a): $S \neq \emptyset$; (b): $S \subseteq \mathbb{I}(x)$; and (c): $\forall_{y\in M}(S \subseteq \mathbb{I}(y) \wedge y \sqsubseteq x \Rightarrow y = x)$. From (a) and (L4) we obtain that there is $z \in M$ such that $z \mathbin{\mathsf{Sum}} S$, that is, (d): $S \subseteq \mathbb{I}(z) \subseteq \bigcup \mathbb{O}[S]$. From this, (b) and (M1) we have $z \sqsubseteq x$. Hence, by (c) and (d), we have $z = x$; so $x \mathbin{\mathsf{Sum}} S$.

We may use condition (M2) to strengthen condition (3.10), i.e., for arbitrary $x \in M$ and $S, Y_z \in \mathcal{P}(M)$ we have:

$$\forall_{z\in S}\, z \mathbin{\mathsf{Sum}} Y_z \implies \big(x \mathbin{\mathsf{Sum}} S \Leftrightarrow x \mathbin{\mathsf{Sum}} \textstyle\bigcup_{z\in S} Y_z\big). \tag{6.5}$$

Let $\forall_{z\in S}\, z \mathbin{\mathsf{Sum}} Y_z$. Then the implication follows from (3.10). Assume therefore, that $x \mathbin{\mathsf{Sum}} \bigcup_{z\in S} Y_z$. First, we show that $\forall_{z\in S}\, z \sqsubseteq x$. Take an arbitrary $z \in S$. Then $Y_z \subseteq \mathbb{I}(x)$, in virtue of the second assumption. Since $x \mathbin{\mathsf{Sum}} \mathbb{I}(x)$, so $z \sqsubseteq x$, in virtue of (M2) and the first assumption. Second, let $y \sqsubseteq x$. Then, in virtue of the second assumption, for some $z \in S$, $v \in Y_z$, and $u \in M$ we have: $u \sqsubseteq v$ and $u \sqsubseteq y$. In virtue of the first assumption we have $v \sqsubseteq z$. Hence — via ($\mathrm{t}_{\sqsubseteq}$) — we have $u \sqsubseteq z$. So $z \bigcirc y$. Thus, $x \operatorname{Sum} S$.

It follows from (SSP) that if x has some part and every part of x is a part of y, then x is an ingrediens of y (cf. ($\mathrm{mono}_{\mathbb{P}}$)). This statement Simons [1987] calls the Proper Parts Principle.[26] We have therefore for arbitrary $x, y \in M$:

$$\emptyset \neq \mathbb{P}(x) \subseteq \mathbb{P}(y) \implies x \sqsubseteq y\,. \tag{PPP}$$

Let $z_0 \sqsubset x$ and $\mathbb{P}(x) \subseteq \mathbb{P}(y)$. Thus, also $z_0 \sqsubset y$. So also $x \bigcirc y$. Furthermore, assume for a contradiction that $x \not\sqsubseteq y$. Then, by (SSP), for some z_1 we have: $z_1 \sqsubseteq x$ and $z_1 \wr y$. We have by this $z_1 \neq x$, because $z_1 = x$ would entail that $x \wr y$. Therefore $z_1 \sqsubset x$. Hence, in virtue of our assumption, we have $z_1 \sqsubset y$. Yet this contradicts $z_1 \wr y$.[27]

It is obvious that from (PPP) it follows that for arbitrary $x, y \in M$:

$$\emptyset \neq \mathbb{P}(x) \subsetneq \mathbb{P}(y) \implies x \sqsubset y\,. \tag{PPP$'$}$$

Remark 6.1. From (PPP$'$) and ($\mathrm{ext}_{\sqsubset}$) we obtain (PPP). In fact, if $\emptyset \neq \mathbb{P}(x) \subsetneq \mathbb{P}(y)$ then $x \sqsubset y$, by (PPP$'$). If $\emptyset \neq \mathbb{P}(x) = \mathbb{P}(y)$ then $x = y$, in virtue of ($\mathrm{ext}_{\sqsubset}$). In both cases therefore, we have $x \sqsubseteq y$. □

7. The operation of mereological sum

Thanks to axioms (L3) and (L4) (resp. (L3-L4)) in any mereological structure $\mathfrak{M} = \langle M, \sqsubset \rangle$, for any non-empty subset S of M, there exists exactly one object $x \in M$ such that $x \operatorname{Sum} S$ (in Remark 5.4 we put $\varphi(x) :=$ "$x \operatorname{Sum} S$"). Thus, we can define ON the family $\mathcal{P}_+(M)$ the following unary operation $\bigsqcup \colon \mathcal{P}_+(M) \to M$ of sum of all members of a given non-empty set:

$$\bigsqcup S := (\iota\, x)\; x \operatorname{Sum} S\,, \tag{df $\bigsqcup$}$$

i.e., $\bigsqcup S$ is the only object $x \in M$ such that $x \operatorname{Sum} S$ (see Remark 5.4). That is, in a mereological structure $\mathfrak{M}$, the object $\bigsqcup S$ is the mereological sum of all members of S.

[26] Simons [1987] uses "proper part" with the meaning we are attaching in this work to "part".

[27] We obtain another proof of the fact (PPP) from (M2) and (3.5). Yet another proof is given by Simons [1987, p. 29]. One can see from the proof given above that the assumption about the existence of some part of x was necessary for us to be able to say that $x \bigcirc y$. The assumption is clearly essential in (PPP).

Thanks to (df $\mathbb{1}$), we obtain:

$$\mathbb{1} = \bigsqcup M\,. \tag{df$'$ $\mathbb{1}$}$$

For a given $S \in \mathcal{P}_+(M)$ we have the following directly from (3.9):

$$\mathbb{O}(\bigsqcup S) = \bigcup \mathbb{O}[S]\,. \tag{7.1}$$

From this it follows that for arbitrary $x \in M$ and $S \in \mathcal{P}_+(M)$: x is a part of the mereological sum of all members of S iff each part of x overlaps with some members of S.

$$\begin{aligned} &x \sqsubseteq \bigsqcup S \iff \forall_{y\in M}(y \sqsubseteq x \Rightarrow \exists_{z\in S}\, y \bigcirc z), \quad \text{or} \\ &x \sqsubseteq \bigsqcup S \iff \mathbb{I}(x) \subseteq \mathbb{O}[\bigsqcup S]\,. \end{aligned} \tag{7.2}$$

Let $x \sqsubseteq \bigsqcup S$. Then, by virtue of ($\mathrm{t}_{\sqsubseteq}$), (df$'''$ $\mathbb{S}\mathrm{um}$), and (7.1), we have $\mathbb{I}(x) \subseteq \mathbb{I}(\bigsqcup S) \subseteq \bigcup \mathbb{O}[S] = \mathbb{O}(\bigsqcup S)$. Conversely, if $\mathbb{I}(x) \subseteq \bigcup \mathbb{O}[S] = \mathbb{O}(\bigsqcup S)$ then $x \sqsubseteq \bigsqcup S$, in virtue of (SSP).

Since we have $\{x, y\} \in \mathcal{P}_+(M)$, for any $x, y \in M$, we can therefore generate the binary operation $\sqcup\colon M \times M \to M$ of *mereological sum of two elements*:

$$x \sqcup y := \bigsqcup\{x, y\}. \tag{df $\sqcup$}$$

Note that, in virtue of (3.8) and (df $\bigsqcup$), the operation $\sqcup$ may also be defined by the identity below:

$$x \sqcup y = \bigsqcup\{z \in M : z \sqsubseteq x \vee z \sqsubseteq y\}\,. \tag{7.3}$$

It follows from (3.3) that the operation $\sqcup$ is idempotent and in the light of the definition itself, it is also commutative, i.e., for arbitrary $x, y \in M$ we have:

$$x = x \sqcup x\,, \tag{7.4}$$

$$x \sqcup y = y \sqcup x\,. \tag{7.5}$$

It follows from definitions (df $\sqcup$) and (df $\mathbb{S}\mathrm{um}$) that for any $x, y \in M$:

$$x \sqsubseteq x \sqcup y\,. \tag{7.6}$$

Note that by putting $S := \{x, y\}$ in (7.1), we get for any $x, y, z \in M$:

$$z \bigcirc x \sqcup y \iff z \bigcirc x \vee z \bigcirc y\,, \tag{7.7}$$

$$z \wr x \sqcup y \iff z \wr x \wedge z \wr y\,. \tag{7.8}$$

For (7.7): $z \bigcirc x \sqcup y$ iff $z \in \mathbb{O}[\bigsqcup\{x, y\}]$ iff $z \in \bigcup \mathbb{O}[\{x, y\}]$ iff $z \bigcirc x$ or $z \bigcirc y$. For (7.8): We use (7.7) and ($\wr{=}{-}\bigcirc$).

By putting $S := \{x, y\}$ in (7.2), by (7.7), we get for any $x, y, z \in M$:

$$z \sqsubseteq x \sqcup y \iff \forall_{u \in M}(u \sqsubseteq z \Rightarrow (u \bigcirc x \vee u \bigcirc y)). \tag{7.9}$$

Let us observe that for arbitrary $x, y, z \in M$ we have:

$$x \sqcup y \sqsubseteq z \iff x \sqsubseteq z \wedge y \sqsubseteq z\,. \tag{7.10}$$

The '$\Rightarrow$'-part we obtain by (7.6), (7.5), and ($\mathrm{t}_{\sqsubseteq}$). For '$\Leftarrow$': Let $x \sqsubseteq z$ and $y \sqsubseteq z$. Assume for a contradiction that $x \sqcup y \not\sqsubseteq z$. Then, by (SSP), there is $u \in M$ such that $u \sqsubseteq x \sqcup y$ and $u \wr z$. In the light of ($\mathrm{r}_{\sqsubseteq}$) and (7.9), we have: $u \bigcirc x$ or $u \bigcirc y$. Hence, in virtue of our assumption and ($\mathrm{mono}_{\circ}$), we obtain a contradiction in both cases, namely: $u \bigcirc z$.

In the light of conditions (7.7), (7.10), and (df $\mathbb{S}\mathrm{um}$), we obtain the associativity of the operation $\sqcup$, i.e., for any $x, y, z \in M$ we have:

$$(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z). \tag{7.11}$$

We have: $u = (x \sqcup y) \sqcup z$ iff $u \ \mathbb{S}\mathrm{um}\ \{x \sqcup y, z\}$ iff $x \sqcup y \sqsubseteq u$, $z \sqsubseteq u$ and $\forall_v(v \sqsubseteq u \Rightarrow (x \sqcup y \bigcirc v \vee z \bigcirc v))$ iff $x \sqsubseteq u$, $y \sqsubseteq u$, $z \sqsubseteq u$ and $\forall_v(v \sqsubseteq u \Rightarrow (x \bigcirc v \vee y \bigcirc v \vee z \bigcirc v))$ iff $u = x \sqcup (y \sqcup z)$.

Now, by (7.9) and (SSP), we obtain

$$z \sqsubseteq x \sqcup y \wedge z \wr x \implies z \sqsubseteq y\,. \tag{7.12}$$

Let $z \sqsubseteq x \sqcup y$ and $z \not\sqsubseteq y$. Then, by (SSP), for some $u \in M$ we have $u \sqsubseteq z$ and $u \wr y$. Hence, in virtue of (7.9), we have $u \bigcirc x$. Hence $z \bigcirc x$, by ($\mathrm{mono}_{\circ}$) and ($\mathrm{s}_{\circ}$).

To finish this section, by making use of (7.12) and (SSP), for arbitrary $x, y, z, u \in M$ we have:

$$u \sqsubseteq x \sqcup y \wedge u \sqsubseteq x \sqcup z \wedge y \wr z \implies u \sqsubseteq x. \tag{7.13}$$

Let $u \sqsubseteq x \sqcup y$, $u \sqsubseteq x \sqcup z$, and $u \not\sqsubseteq x$. Then. by (SSP), there is $w \in W$ such that $w \sqsubseteq u$ and $w \wr x$. In virtue of ($\mathrm{t}_{\sqsubseteq}$) we have $w \sqsubseteq x \sqcup y$ and $w \sqsubseteq x \sqcup z$. Hence, by (7.12), we have $w \sqsubseteq y$ and $w \sqsubseteq z$; so $y \bigcirc z$.

8. The relation $\mathbb{S}\mathrm{um}$ versus the relation of supremum

Let $\mathfrak{M} = \langle M, \sqsubset \rangle$ be any mereological structure, $x \in M$, and $S \in \mathcal{P}(M)$. We say that x is an *upper bound* of S in $\mathfrak{M}$ iff all members of S are ingredienses of x, i.e., $S \subseteq \mathbb{I}(x)$. Moreover, since the relation $\sqsubseteq$ partially orders the set M, we can therefore define in $M \times \mathcal{P}(M)$ the relation *is a supremum of* with respect to $\sqsubseteq$, in accordance with (df $\mathsf{sup}_{\leq}$) from Section 4 in Appendix I. Thus, we say that x is a *supremum* of S in $\mathfrak{M}$ (we write: $x \mathbin{\mathsf{sup}_{\sqsubseteq}} S$) iff x is the least upper bound of S in $\mathfrak{M}$. This may be put symbolically as follows for all $x \in X$ and $S \in \mathcal{P}(X)$:

$$x \mathbin{\mathsf{sup}_{\sqsubseteq}} S \;:\Longleftrightarrow\; S \subseteq \mathbb{I}(x) \wedge \forall_{y \in M}(S \subseteq \mathbb{I}(y) \;\Rightarrow\; x \sqsubseteq y). \qquad (\mathrm{df}\ \mathsf{sup}_{\sqsubseteq})$$

The immediate consequences of (df $\mathsf{sup}_{\sqsubseteq}$) are stated in the following lemma.

Lemma 8.1. (i) *To show that* $\mathsf{sup}_{\sqsubseteq}$ *is monotonic, we need only definition* (df $\mathsf{sup}_{\sqsubseteq}$)*; so we obtain*:

$$\forall_{S_1,S_2 \in \mathcal{P}(M)} \forall_{x,y \in M}(x \mathbin{\mathsf{sup}_{\sqsubseteq}} S_1 \wedge y \mathbin{\mathsf{sup}_{\sqsubseteq}} S_2 \wedge S_1 \subseteq S_2) \implies x \sqsubseteq y. \qquad (\mathrm{M}_{\mathsf{sup}})$$

(ii) *From* ($\mathrm{r}_{\sqsubseteq}$) *it follows that*:

$$\forall_{x \in M}\ x \mathbin{\mathsf{sup}_{\sqsubseteq}} \{x\}\,, \qquad (8.1)$$

$$\forall_{x \in M}\ x \mathbin{\mathsf{sup}_{\sqsubseteq}} \mathbb{I}(x)\,. \qquad (8.2)$$

(iii) *From* ($\mathrm{antis}_{\sqsubseteq}$) *it follows that if a set has a supremum then it is unique, i.e.*:

$$\forall_{S \in \mathcal{P}(M)} \forall_{x,y \in M}(x \mathbin{\mathsf{sup}_{\sqsubseteq}} S \wedge y \mathbin{\mathsf{sup}_{\sqsubseteq}} S \implies x = y). \qquad (\mathrm{U}_{\mathsf{sup}})$$

(iv) *From* ($\mathrm{r}_{\sqsubseteq}$) *and* ($\mathrm{antis}_{\sqsubseteq}$) *it follows that*:

$$\forall_{x,y \in M}(y \mathbin{\mathsf{sup}_{\sqsubseteq}} \{x\} \implies x = y). \qquad (\mathrm{S}_{\mathsf{sup}})$$

To begin, we will examine the relationships that hold between the relations Sum and sup when none of axioms (L1)–(L4) is assumed. In other words, we will make the same assumption as with lemmas 4.3 and 6.2, and Corollary 6.3. That is, we consider structures of the form $\langle M, \sqsubseteq\rangle$ with a primitive relation $\sqsubseteq$ included in $M \times M$ (so we do not assume that $\sqsubseteq$ is defined by (df $\sqsubseteq$))). We assume, however, that $\circ$ and $\mathbb{S}\mathrm{um}$ are defined by (df $\circ$) and (df $\mathbb{S}\mathrm{um}$) using $\sqsubseteq$.

Lemma 8.2. *It follows from* ($\mathrm{r}_{\sqsubseteq}$) *and* ($\mathrm{t}_{\sqsubseteq}$) *that*:

$\mathbb{S}\mathrm{um} \subseteq \mathsf{sup}_{\sqsubseteq}$ *iff* (SSP) *holds iff* (M1) *holds iff* (M2) *holds.*

Proof. Suppose that ($\mathrm{r}_{\sqsubseteq}$) and ($\mathrm{t}_{\sqsubseteq}$) hold. First, by Corollary 6.3(i), conditions (SSP), (M1), and (M2) are equivalent.

Second, by Corollary 6.3(ii), we obtain that if (SSP) holds then also (M3) holds. So if x $\mathbb{S}\mathrm{um}$ S, then $S \subseteq \mathbb{I}(x)$ and $\forall_{y\in M}(S \subseteq \mathbb{I}(y) \Rightarrow x \sqsubseteq y)$. Hence $x \ \mathsf{sup}_{\sqsubseteq} \ S$, by (df $\mathsf{sup}_{\sqsubseteq}$). So we obtain that $\mathbb{S}\mathrm{um} \subseteq \mathsf{sup}_{\sqsubseteq}$.

Thirdly, suppose that $\mathbb{S}\mathrm{um} \subseteq \mathsf{sup}_{\sqsubseteq}$. By Lemma 4.3, if $\mathbb{I}(x) \subseteq \mathbb{O}(y)$, then x $\mathbb{S}\mathrm{um}$ $\mathbb{I}(x) \cap \mathbb{I}(y)$. Hence also $x \ \mathsf{sup}_{\sqsubseteq} \ \mathbb{I}(x) \cap \mathbb{I}(y)$. By (8.2), we have $y \ \mathsf{sup}_{\sqsubseteq} \ \mathbb{I}(y)$. Therefore, $x \sqsubseteq y$, by Lemma 8.1(i). □

Let us note that directly by (6.2) and (df $\mathsf{sup}_{\sqsubseteq}$), in all mereological structures *being a mereological sum of all members of* a given non-empty set coincides with *being a supremum of* all members of that set, i.e.:

$$\forall_{S\in\mathcal{P}(M)}\forall_{x\in M}(x \ \mathbb{S}\mathrm{um} \ S \iff S \neq \emptyset \wedge x \ \mathsf{sup}_{\sqsubseteq} \ S). \qquad (\mathbb{S}\mathrm{um}\text{-}\mathsf{sup}_{\sqsubseteq})$$

Hence we obtain:

$$\forall_{S\in\mathcal{P}_+(M)} \bigsqcup S = \sup\nolimits_{\sqsubseteq} S\,, \tag{8.3}$$

where $\sup_{\sqsubseteq} S := (\iota\, x)\ x \ \mathsf{sup}_{\sqsubseteq} \ S$ (for any $S \in \mathcal{P}_+(M)$). In fact, thanks to (L3) and (L4), for any $S \in \mathcal{P}_+(M)$ is defined $\bigsqcup S$. In virtue of ($\mathbb{S}\mathrm{um}$-$\mathsf{sup}_{\sqsubseteq}$) and ($\mathrm{U}_{\mathsf{sup}}$), $\bigsqcup S$ is the only member of M such that $\bigsqcup S \ \mathsf{sup}_{\sqsubseteq} \ S$.

Remark 8.1. Condition (6.2), from which we received ($\mathbb{S}\mathrm{um}$-$\mathsf{sup}_{\sqsubseteq}$), directly corresponds to (df $\mathsf{sup}_{\sqsubseteq}$) (and to (df $\mathsf{sup}_{\leq}$) from Appendix I). Similarly, condition (6.3) corresponds to condition (4.5) from Appendix I. Let us note, however, that the latter condition was only obtained from the reflexivity and transitivity of the relation $\leq$, but for conditions (6.2) and (6.2) we needed axiom (L4). This just shows what role this axiom plays in obtaining ($\mathbb{S}\mathrm{um}$-$\mathsf{sup}_{\sqsubseteq}$). □

We will show that the relations $\mathbb{S}\mathrm{um}$ and $\mathsf{sup}_{\sqsubseteq}$ are distinct iff the set M is a singleton, i.e.:

$$\mathbb{S}\mathrm{um} = \mathsf{sup}_{\sqsubseteq} \iff \mathrm{Card}\, M > 1\,. \tag{8.4}$$

'⇒' Let $\mathbb{S}\mathrm{um} = \mathsf{sup}_{\sqsubseteq}$. If x $\mathbb{S}\mathrm{um}$ S then $x \ \mathsf{sup}_{\sqsubseteq} \ S$, in virtue of (df $\mathsf{sup}_{\sqsubseteq}$) and (6.2). If x $\mathbb{S}\mathrm{um}$ S then $S \neq \emptyset$, because — in virtue of ($\nexists 0$) — there is no least element ('zero') in $\mathfrak{M}$. Hence x $\mathbb{S}\mathrm{um}$ S, in virtue of (df $\mathsf{sup}_{\sqsubseteq}$) and (6.2). '⇐' let $\mathrm{Card}\, M = 1$ and suppose that $M = \{a\}$. Then $a \ \mathsf{sup}_{\sqsubseteq} \ \emptyset$ and $\neg\, a$ $\mathbb{S}\mathrm{um}$ $\emptyset$, since there is no mereological sum of $\emptyset$, in virtue of (3.2). Therefore $\mathbb{S}\mathrm{um} \neq \mathsf{sup}_{\sqsubseteq}$.

Remark 8.2. Theses ($\mathbb{S}\mathrm{um}$-$\mathsf{sup}_{\sqsubseteq}$), (8.3), and (8.4) do not show that, in axioms (L3) and (L4), we may use the relation $\mathsf{sup}_{\sqsubseteq}$ instead of the relation $\mathbb{S}\mathrm{um}$. With this new form, (L3) (= ($\mathrm{U}_{\mathsf{sup}}$)) would be dependent on (L1) and (L2), and instead of (L4) we would have $(*)$: $\forall_{S\in\mathcal{P}_+(M)}\exists_{x\in M} x \mathsf{sup}_{\sqsubseteq} S$. The collection of conditions (L1), (L2) and $(*)$ is essentially weaker than the collection (L1)–(L4). In fact, this first collection would be an axiomatisation of complete lattices with zero removed and the second collection — as we will show in the next chapter — is an axiomatisation of complete Boolean lattices with zero removed. Alongside (L1), (L2), and $(*)$ one needs to assume ($\mathbb{S}\mathrm{um}$-$\mathsf{sup}_{\sqsubseteq}$) in order to obtain a collection equivalent to the collection (L1)–(L4). □

9. The operation of mereological product

Let $\mathfrak{M} = \langle M, \sqsubset\rangle$ by any mereological structure. By virtue of (L4) and (3.2), the equivalence below is satisfied for an arbitrary $S \in \mathcal{P}(M)$:

$$\bigcap \mathbb{I}[S] \neq \emptyset \iff \exists_{x\in M}\ x \,\mathbb{S}\mathrm{um} \bigcap \mathbb{I}[S]\,. \tag{9.1}$$

Remark 9.1. $\mathbb{I}[S]$ is the family of sets which is the image of a set S determined by the function $\mathbb{I}$ and $\bigcap \mathbb{I}[S]$ is the set-theoretical product of the family $\mathbb{I}[S]$, i.e.:

$$\mathbb{I}[S] := \{Y \in \mathcal{P}_+(M) : \exists_{z\in S}\, Y = \mathbb{I}(z)\} = \{\mathbb{I}(z) : z \in S\},$$
$$\bigcap \mathbb{I}[S] := \{y \in M : \forall_{Y\in\mathbb{I}[S]}\, y \in Y\}.$$

Thus, we have:

$$\bigcap \mathbb{I}[S] = \bigcap\{\mathbb{I}(z) : z \in S\} = \{y \in M : \forall_{z\in S}\, y \sqsubseteq z\}.$$

Note that, in the light of ($\mathrm{r}_{\sqsubseteq}$), for any $S \in \mathcal{P}(M)$ we have: $\mathbb{I}(S) = \emptyset$ iff $S = \emptyset$. Hence:

$$\bigcap \mathbb{I}[\emptyset] = \{y \in M : \forall_{Y\in\emptyset}\, y \in Y\} = \bigcap \emptyset = \{y \in M : \forall_{z\in\emptyset}\, y \sqsubseteq z)\} = M. \quad \square$$

By applying (L3) and (L4) therefore, we can define IN the set $\mathcal{P}(M)$ a PARTIAL unary operation $\bigsqcap \colon \mathcal{P}(M) \to M$ by the following condition:

$$\bigcap \mathbb{I}[S] \neq \emptyset \implies \bigsqcap S := \bigsqcup \bigcap \mathbb{I}[S]. \tag{df$\bigsqcap$}$$

It follows from condition (9.1) that the domain of the operation $\bigsqcap$ is the family $\{S \in \mathcal{P}(M) : \bigcap \mathbb{I}[S] \neq \emptyset\}$. If $\bigsqcap S$ exists, then we will call it the *mereological product of all members of* the set S. We know from ($\nexists 0$) that as long as the set M is not a singleton, then there is no element in

M which is an ingrediens of all elements of M. Leaving aside the case of a 'trivial structure', therefore, the product of all members of the set M does not exist.

Furthermore, since $\bigcap \mathbb{I}[\emptyset] = M \neq \emptyset$, by (df $\bigsqcap$) and (df′ $\mathbb{1}$), we obtain:

$$\bigsqcap\emptyset = \bigsqcup M = \mathbb{1}\,.$$

For all $x \in M$ and $S \in \mathcal{P}(M)$, we say that x is an *lower bound* of S in $\mathfrak{M}$ iff x is an ingrediens of all members of S, i.e., $\forall_{z\in S}\, x \sqsubseteq z$. Moreover, since the relation $\sqsubseteq$ partially orders the set M, we can therefore define in $M \times \mathcal{P}(M)$ the relation *is an infimum of* with respect to $\sqsubseteq$, in accordance with (df $\mathsf{inf}_{\leq}$) from Section 4 in Appendix I. Thus, we say that x is a *infimum* of S in $\mathfrak{M}$ (we writhe: $x\ \mathsf{inf}_{\sqsubseteq}\ S$) iff x is the *greatest lower bound* This may be put symbolically as follows for all $x \in X$ and $S \in \mathcal{P}(X)$:

$$x\ \mathsf{inf}_{\sqsubseteq}\ S \ :\Longleftrightarrow \forall_{z\in S}\, x \sqsubseteq z \wedge \forall_{y\in M}(\forall_{z\in S}\, y \sqsubseteq z \ \Rightarrow\ y \sqsubseteq x). \qquad (\text{df}\ \mathsf{inf}_{\sqsubseteq})$$

The immediate consequences of (df $\mathsf{inf}_{\sqsubseteq}$) are stated in the following lemma.

LEMMA 9.1. (i) *To show that* $\mathsf{inf}_{\sqsubseteq}$ *is monotonic, we need only definition* (df $\mathsf{inf}_{\sqsubseteq}$)*; so we obtain*:

$$\forall_{S_1,S_2\in\mathcal{P}(M)}\forall_{x,y\in M}(x\ \mathsf{inf}_{\sqsubseteq}\ S_1 \wedge y\ \mathsf{inf}_{\sqsubseteq}\ S_2 \wedge S_1 \subseteq S_2) \implies y \sqsubseteq x. \qquad (\mathrm{M}_{\mathsf{inf}})$$

(ii) *From* ($\mathrm{r}_{\sqsubseteq}$) *it follows that*:

$$\forall_{x\in M}\ x\ \mathsf{inf}_{\sqsubseteq}\ \{x\}\,. \qquad (9.2)$$

(iii) *From* ($\mathrm{antis}_{\sqsubseteq}$) *it follows that if a set has an infimum then it is unique, i.e.:*

$$\forall_{S\in\mathcal{P}(M)}\forall_{x,y\in M}\big(x\ \mathsf{inf}_{\sqsubseteq}\ S \wedge y\ \mathsf{inf}_{\sqsubseteq}\ S \implies x = y\big). \qquad (\mathrm{U}_{\mathsf{inf}})$$

(iv) *The relations* $\mathsf{inf}_{\sqsubseteq}$ *and* $\mathsf{sup}_{\sqsubseteq}$ *are interdefinable, i.e., for arbitrary* $S \in \mathcal{P}(M)$ *and* $x \in M$ *the following conditions hold* (cf. (4.10) and (4.11) in Appendix I):

$$x\ \mathsf{inf}_{\sqsubseteq}\ S \Longleftrightarrow x\ \mathsf{sup}_{\sqsubseteq}\ \{y \in M : \forall_{z\in S}\, y \sqsubseteq z\} \Longleftrightarrow x\ \mathsf{sup}_{\sqsubseteq} \bigcap \mathbb{I}[S], \qquad (9.3)$$

$$x\ \mathsf{sup}_{\sqsubseteq}\ S \iff x\ \mathsf{inf}_{\sqsubseteq}\ \{y \in M : S \subseteq \mathbb{I}(y)\}. \qquad (9.4)$$

So we have: $x\ \mathsf{inf}_{\sqsubseteq}\ \emptyset$ *iff* $x\ \mathsf{sup}_{\sqsubseteq}\ M$ *iff* $x = \mathbb{1}$.

By (df $\bigsqcap$), (8.3), and (9.3), for any $S \in \mathcal{P}(M)$ we have:

$$\bigcap \mathbb{I}[S] \neq \emptyset \implies \bigsqcap S = \inf_{\sqsubseteq} S\,, \tag{9.5}$$

where $\inf_{\sqsubseteq} S := (\iota\, x)\; x \; \mathsf{inf}_{\sqsubseteq}\; S$. In fact, for an arbitrary $S \in \mathcal{P}(M)$ such that $\bigcap \mathbb{I}[S] \neq \emptyset$ we have $\bigsqcap S := \bigsqcup \bigcap \mathbb{I}[S] = \sup_{\sqsubseteq} \bigcap \mathbb{I}[S] = \inf_{\sqsubseteq} S$.

In an analogous way for the operation $\sqcup$ we may generate a PARTIAL binary operation $\sqcap\colon M \times M \to M$ of *mereological product of two elements* (cf. also (9.5)):

$$\begin{aligned} x \bigcirc y \implies x \sqcap y &:= \bigsqcap\{x, y\} \\ &= \inf_{\sqsubseteq}\{x, y\}. \end{aligned} \tag{df $\sqcap$}$$

It follows from (9.1) that the domain of the operation $\sqcap$ is the set $\{\langle x, y\rangle \in M \times M : x \bigcirc y\}$. Apart from the case in which M is a singleton, the operation $\sqcap$ is partial, because in virtue of condition ($\exists\wr$), if $\operatorname{Card} M > 1$ then there exist $x, y \in M$ such that $x \wr y$ which do not belong to the domain of this operation.

It follows from the definition alone that the operation is $\sqcap$ commutative and from (3.4) we obtain that $\sqcap$ is idempotent, i.e., for all $x, y \in M$:

$$x \bigcirc y \implies x \sqcap y = y \sqcap x\,, \tag{9.6}$$

$$x = x \sqcap x\,. \tag{9.7}$$

Moreover, by (df $\sqcap$) and (df $\bigsqcap$), for all $x, y \in M$ such that $x \bigcirc y$ the product $x \sqcap y$ is an ingrediens both x and y, and it is the mereological sum (supremum) of the common ingrediens of x and y, i.e.:

$$x \bigcirc y \implies x \sqcap y \sqsubseteq x \wedge x \sqcap y \sqsubseteq y\,, \tag{9.8}$$

$$x \bigcirc y \implies x \sqcap y = \bigsqcup\{z \in M : z \sqsubseteq x \wedge z \sqsubseteq y\}. \tag{9.9}$$

If $x \bigcirc y$ then the product x and y exists. So we use (df $\sqcap$). Moreover, $x \sqcap y = \bigsqcup \bigcap \mathbb{I}[\{x, y\}] = \bigsqcup\{z \in M : z \sqsubseteq x \wedge z \sqsubseteq y\}$.[28]

Thus, for all $x, y \in M$ we obtain:

$$x \bigcirc y \implies (z \sqsubseteq x \sqcap y \iff z \sqsubseteq x \wedge z \sqsubseteq y). \tag{9.10}$$

Let $x \bigcirc y$. Then the implication follows from (9.8) and ($\mathrm{t}_{\sqsubseteq}$). The converse implication follows from (9.9).

[28] Cf. also (df $\sqcap$) and (4.6) from Appendix I.

The associativity of the operation $\sqcap$ follows from (df $\sqcap$) and (9.10), i.e., for all $x, y, z \in M$ we obtain:[29]

$$\exists_{u\in M}(u \sqsubseteq x \wedge u \sqsubseteq y \wedge u \sqsubseteq z) \implies (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z). \quad (9.11)$$

Suppose that $\exists_{u\in M}(u \sqsubseteq x \wedge u \sqsubseteq y \wedge u \sqsubseteq z)$. Then exist both $x \sqcap y$, $x \sqcap z$, and $y \sqcap z$. Moreover, $u = (x \sqcap y) \sqcap z$ iff $u \operatorname{inf}_{\sqsubseteq} \{x \sqcap y, z\}$ iff $u \sqsubseteq x \sqcap y$ and $u \sqsubseteq z$ and $\forall_{v\in M}(v \sqsubseteq x \sqcap y \wedge v \sqsubseteq z \Rightarrow v \sqsubseteq x)$ iff $u \sqsubseteq x$ and $u \sqsubseteq y$ and $u \sqsubseteq z$ and $\forall_{v\in M}(v \sqsubseteq x \wedge v \sqsubseteq y \wedge v \sqsubseteq z \Rightarrow v \sqsubseteq x)$ iff $u = x \sqcap (y \sqcap z)$.

Note that, by ($\mathrm{t}_{\sqsubseteq}$), ($\mathrm{antis}_{\sqsubseteq}$), (5.1), (9.6), (9.7), (9.8) and (9.10), for arbitrary $x, y \in M$ we have:

$$x \bigcirc y \implies (x \sqcap y = \mathbb{1} \Leftrightarrow x = \mathbb{1} = y), \quad (9.12)$$

$$x \sqcap \mathbb{1} = \mathbb{1} \sqcap x = x. \quad (9.13)$$

From (9.10), ($\mathrm{t}_{\sqsubseteq}$), and ($\mathrm{r}_{\sqsubseteq}$), we have, for arbitrary $x, y, u, v \in M$:

$$x \bigcirc y \wedge x \sqsubseteq u \wedge y \sqsubseteq v \implies u \bigcirc v \wedge x \sqcap y \sqsubseteq u \sqcap v. \quad (9.14)$$

Suppose that $x \bigcirc y$, $x \sqsubseteq u$, and $y \sqsubseteq v$. Then there is a z such that $z \sqsubseteq x \sqsubseteq u$ and $z \sqsubseteq y \sqsubseteq v$. Hence $u \bigcirc v$, by ($\mathrm{t}_{\sqsubseteq}$). Let us pick an arbitrary w such that $w \sqsubseteq x \sqcap y$. By virtue of (9.10), we have $w \sqsubseteq x$ and $w \sqsubseteq y$. Therefore, also $w \sqsubseteq u$ and $w \sqsubseteq v$, by our assumption. Hence, by similarly applying (9.10), we have $w \sqsubseteq u \sqcap v$. From the arbitrariness of the choice of w, we have $\mathbb{I}(x \sqcap y) \subseteq \mathbb{I}(u \sqcap v)$. Thus, $x \sqcap y \sqsubseteq u \sqcap v$., by virtue of ($\mathrm{mono}_{\mathbb{I}}$).

To close this section, let us note that, for all $x, y, z \in M$ we have:

$$z \sqsubseteq x \wedge z \bigcirc y \implies x \bigcirc y \wedge z \bigcirc x \sqcap y. \quad (9.15)$$

If $z \sqsubseteq x$ and $z \bigcirc y$, then for some u we have: $u \sqsubseteq z \sqsubseteq x$ and $u \sqsubseteq y$. Hence, from ($\mathrm{t}_{\sqsubseteq}$) and (9.10), we have $x \bigcirc y$ and $u \sqsubseteq x \sqcap y$. So $z \bigcirc x \sqcap y$.

10. Distributivity

Let $\mathfrak{M} = \langle M, \sqsubset \rangle$ be an arbitrary mereological structure. In view of this, since — leaving aside the trivial case — $\mathfrak{M}$ is not a lattice, the conditions of

[29] Note that, with the exception of the cases of trivial structures, structures of the form $\langle M, \sqcap \rangle$ are not lattices.

distributivity are going to have a somewhat complicated form.[30] Before formulating the condition for the distributivity of the operation $\sqcup$ with respect to the operation $\sqcap$, let us observe that since we have $x \sqsubseteq x \sqcup y$ and $x \sqsubseteq x \sqcup z$, for arbitrary $x, y, z \in M$, we therefore have:

$$(x \sqcup y) \bigcirc (x \sqcup z).$$

Thus, we will now prove that for arbitrary $x, y, z \in M$ we obtain:

$$(x \sqcup y) \sqcap (x \sqcup z) = \begin{cases} x \sqcup (y \sqcap z) & \text{if } y \bigcirc z, \\ x & \text{if } y \wr z. \end{cases} \qquad (\Delta_1)$$

We need to consider two cases. (A) $y \wr z$: We have $x \sqsubseteq x \sqcup y$ and $x \sqsubseteq x \sqcup z$. Hence $x \sqsubseteq (x \sqcup y) \sqcap (x \sqcup z)$, by virtue of (9.10). Conversely, pick an arbitrary u such that $u \sqsubseteq (x \sqcup y) \sqcap (x \sqcup z)$. Then $u \sqsubseteq x \sqcup y$ and $u \sqsubseteq x \sqcup z$, by virtue of (9.10). Hence $u \sqsubseteq x$, by virtue of (7.13). From the arbitrariness of u we have $(x \sqcup y) \sqcap (x \sqcup z) \sqsubseteq x$. (B) $y \bigcirc z$: We have $y \sqsubseteq x \sqcup y$ and $z \sqsubseteq x \sqcup z$. Hence $y \sqcap z \sqsubseteq (x \sqcup y) \sqcap (x \sqcup z)$, by (9.14). From this and (A) we have $x \sqcup (y \sqcap z) \sqsubseteq (x \sqcup y) \sqcap (x \sqcup z)$, by (7.10). Assume for a contradiction that $(x \sqcup y) \sqcap (x \sqcup z) \not\sqsubseteq x \sqcup (y \sqcap z)$. Hence, by virtue of (SSP), there is a u such that $u \sqsubseteq (x \sqcup y) \sqcap (x \sqcup z)$ and $u \wr x \sqcup (y \sqcap z)$. From the first of these facts we have $u \sqsubseteq x \sqcup y$ and $u \sqsubseteq x \sqcup z$. From the second, by virtue of (7.8), we get $u \wr x$ and $u \wr y \sqcap z$. Therefore—applying (7.12)—we have $u \sqsubseteq y$ and $u \sqsubseteq z$. Hence we have $u \sqsubseteq y \sqcap z$ which contradicts $u \wr y \sqcap z$.

In formulating the condition of distributivity of the operation $\sqcap$ with respect to the operation $\sqcup$, we will use fact (7.7). We will prove that, for arbitrary $x, y, z \in M$ we obtain:

$$x \bigcirc y \vee x \bigcirc z \implies$$
$$x \sqcap (y \sqcup z) = \begin{cases} (x \sqcap y) \sqcup (x \sqcap z) & \text{if } x \bigcirc y \text{ and } x \bigcirc z, \\ x \sqcap y & \text{if } x \bigcirc y \text{ and } x \wr z, \\ x \sqcap z & \text{if } x \wr y \text{ and } x \bigcirc z. \end{cases} \qquad (\Delta_2)$$

We need to consider three cases. (A) $x \bigcirc y$ and $x \wr z$: we have $x \sqcap y \sqsubseteq x$ and $x \sqcap y \sqsubseteq y \sqsubseteq y \sqcup z$. Hence $x \sqcap y \sqsubseteq x \sqcap (y \sqcup z)$. Conversely, assume for

[30] One must not confuse these conditions with the conditions for the property of so-called conditional distributivity in lattices with a unity. We have ‘full’ distributivity in a ‘partial’ lattice.

a contradiction that $x \sqcap (y \sqcup z) \not\sqsubseteq x \sqcap y$. Then, by virtue of (SSP), there is a u such that $u \sqsubseteq x \sqcap (y \sqcup z)$ and $u \wr x \sqcap y$. Thus, $u \sqsubseteq x$, $u \sqsubseteq y \sqcup z$, and we have $u \wr y$, by virtue of (9.15). Hence, by virtue of (7.12), we get $u \sqsubseteq z$. Therefore $x \bigcirc z$, which contradicts $x \wr z$. (B) $x \wr y$ and $x \bigcirc z$: We have $x \sqcap z \sqsubseteq x$ and $x \sqcap z \sqsubseteq z \sqsubseteq y \sqcup z$. Hence $x \sqcap z \sqsubseteq x \sqcap (y \sqcup z)$. Conversely, as in (A). (C) $x \bigcirc y$ and $x \bigcirc z$: From (A) and (B), by virtue of (7.10), we have $(x \sqcap y) \sqcup (x \sqcap z) \sqsubseteq x \sqcap (y \sqcup z)$. Conversely, assume for a contradiction that $x \sqcap (y \sqcup z) \not\sqsubseteq (x \sqcap y) \sqcup (x \sqcap z)$. Then, by virtue of (SSP), there is a u such that $u \sqsubseteq x \sqcap (y \sqcup z)$ and $u \wr (x \sqcap y) \sqcup (x \sqcap z)$. Therefore $u \sqsubseteq x$, $u \sqsubseteq y \sqcup z$, $u \wr x \sqcap y$ and $u \wr x \sqcap z$. By virtue of (9.15), we have $u \wr y$ and $u \wr z$. Hence, by virtue of (7.8), we get $u \wr y \sqcup z$ which contradicts $u \sqsubseteq y \sqcup z$.

11. The mereological complement operation

In any mereological structure $\mathfrak{M}$ for each $x \in M$ we have:

$$x \neq \mathbb{1} \iff \exists_{y \in M}\, y \wr x\,. \tag{11.1}$$

If $x \neq \mathbb{1}$ then $x \sqsubset \mathbb{1}$, by (5.1). Hence, by virtue of (WSP), for some y we have $y \wr x$. Conversely, for each $y \in M$ we have $y \bigcirc \mathbb{1}$, by virtue of (5.1) and ($\sqsubseteq\subseteq\bigcirc$).

In the light of (11.1), if $x \neq \mathbb{1}$ then $\{y \in M : y \wr x\} \neq \emptyset$. Hence $\bigsqcup\{y \in M : y \wr x\}$ is defined, by virtue of (L3) and (L4). Moreover:

$$x \neq \mathbb{1} \implies \bigsqcup\{y \in M : y \wr x\} \neq \mathbb{1}\,. \tag{11.2}$$

Assume for a contradiction that $x \neq \mathbb{1}$ and $\bigsqcup\{y \in M : y \wr x\} = \mathbb{1}$. Then, in virtue of (5.1), ($\sqsubseteq\subseteq\bigcirc$), (7.1), we have $x \in \mathbb{O}(\mathbb{1}) = \bigcup \mathbb{O}[\{y \in M : y \wr x\}] = \{z \in M : \exists_{y \in M}(y \wr x \wedge z \bigcirc y)\}$. Hence for some $y \in M$, we obtain the contradiction: $x \bigcirc y$ and $y \wr x$.

It follows from the above considerations that if $M \neq \{\mathbb{1}\}$ then we may define ON the set $M \setminus \{\mathbb{1}\}$ the unary operation ${}^{\mathsf{c}} \colon M \setminus \{\mathbb{1}\} \to M \setminus \{\mathbb{1}\}$:

$$\begin{aligned} x^{\mathsf{c}} &:= \bigsqcup\{y \in M : y \wr x\} \\ &= \sup_{\sqsubseteq}\{y \in M : y \wr x\}\,. \end{aligned} \tag{df ${}^{\mathsf{c}}$}$$

The object x^{c} will be called the *mereological complement of* x. This object is not only the mereological sum (supremum) of the indicated set, but also belongs to this set. Namely, for any $x \in M$ we have:

$$x \neq \mathbb{1} \implies x \wr x^{\mathsf{c}}. \tag{11.3}$$

Assume for a contradiction that $x \neq \mathbb{1}$ and $x \bigcirc x^{\mathsf{C}}$. Then for some $y \in M$ we have (a) $y \sqsubseteq x$ and (b) $y \sqsubseteq x^{\mathsf{C}} := \bigsqcup\{z \in M : z \wr x\}$. By (b), (7.2) and ($\mathrm{r}_{\sqsubseteq}$), for some $z \in M$ we obtain (c) $z \wr x$ and (d) $z \bigcirc y$. Thus, by (a), (d) and ($\mathrm{mono}_{\circ}$), we have $z \bigcirc x$, which contradicts (c).

Moreover, the operation of mereological complement has the following properties for arbitrary $x, y \in M$:

$$x \neq \mathbb{1} \implies y \wr x \Leftrightarrow y \sqsubseteq x^{\mathsf{C}}, \tag{11.4}$$

$$x \neq \mathbb{1} \neq y \implies x \sqsubseteq y \Leftrightarrow y^{\mathsf{C}} \sqsubseteq x^{\mathsf{C}}, \tag{11.5}$$

$$x \neq \mathbb{1} \implies x^{\mathsf{CC}} = x\,, \tag{11.6}$$

$$x \neq \mathbb{1} \neq y \wedge x^{\mathsf{C}} \bigcirc y^{\mathsf{C}} \implies x \sqcup y \neq \mathbb{1} \neq x^{\mathsf{C}} \sqcap y^{\mathsf{C}}. \tag{11.7}$$

For (11.4): Let $x \neq \mathbb{1}$. If $y \wr x$, then $y \sqsubseteq x^{\mathsf{C}} := \bigsqcup\{z \in M : z \wr x\}$. Conversely, if $y \sqsubseteq x^{\mathsf{C}}$, then $\mathbb{O}(y) \subseteq \mathbb{O}(x^{\mathsf{C}})$, by virtue of ($\mathrm{mono}_{\circ}$). Thus, $x \notin \mathbb{O}(y)$, since $x \notin \mathbb{O}(x^{\mathsf{C}})$, by (11.3). For (11.5): Assume that $x \neq \mathbb{1} \neq y$. Then: $x \sqsubseteq y$ iff (by ($\mathrm{df}_{\circ}\sqsubseteq$)) $\mathbb{O}(x) \subseteq \mathbb{O}(y)$ iff $\{z : z \wr y\} \subseteq \{z : z \wr x\}$ iff (by (11.4)) $\mathbb{I}(y^{\mathsf{C}}) \subseteq \mathbb{I}(x^{\mathsf{C}})$ iff (by ($\mathrm{mono}_{\mathbb{I}}$)) $y^{\mathsf{C}} \sqsubseteq x^{\mathsf{C}}$. For (11.6): By virtue of (11.2)–(11.4), for any $x \neq \mathbb{1}$ we have $x \sqsubseteq x^{\mathsf{CC}}$. Applying this to x^{C} we get $x^{\mathsf{C}} \sqsubseteq x^{\mathsf{CCC}}$. Hence $x^{\mathsf{CC}} \sqsubseteq x$, by virtue of (11.5). Therefore $x = x^{\mathsf{CC}}$, by virtue of ($\mathrm{antis}_{\sqsubseteq}$). For (11.7): Let $x \neq \mathbb{1} \neq y$ and $x^{\mathsf{C}} \bigcirc y^{\mathsf{C}}$. Then there is a z such that $z \sqsubseteq x^{\mathsf{C}}$ and $z \sqsubseteq y^{\mathsf{C}}$. By (11.4) we have $z \wr x$ and $z \wr y$. Hence $\neg\, \mathbb{1}\ \mathrm{Sum}\ \{x, y\}$, by (5.1). Moreover, assume for a contradiction that $x^{\mathsf{C}} \sqcap y^{\mathsf{C}} = \mathbb{1}$. Then, (9.12), we have $x^{\mathsf{C}} = \mathbb{1} = y^{\mathsf{C}}$, which contradicts (11.2).

Note that we have:

$$\forall_{x \in M}(x \neq \mathbb{1} \implies x \sqcup x^{\mathsf{C}} = \mathbb{1}). \tag{11.8}$$

Assume for a contradiction that $x \neq \mathbb{1}$ and for some $y \in M$ we have: $y \wr x$ and $y \wr x^{\mathsf{C}}$. Then $y \sqsubseteq x^{\mathsf{C}}$ and $y \sqsubseteq x$, by (11.4) and (11.6). Hence, in the light of (11.3), we get the contradiction: $x \bigcirc x^{\mathsf{C}}$. We therefore have for each $y \in M$: $y \bigcirc x$ or $y \bigcirc x^{\mathsf{C}}$. Hence $\mathbb{1}\ \mathrm{Sum}\ \{x, x^{\mathsf{C}}\}$.

Now we prove that:

$$\forall_{x,y \in M}\big(x \sqcup y = \mathbb{1} \wedge x \wr y \implies x \neq \mathbb{1} \neq y \wedge y = x^{\mathsf{C}} \wedge x = y^{\mathsf{C}}\big). \tag{11.9}$$

Let $x \sqcup y = \mathbb{1}$ and $x \wr y$. Then, by (11.1), we have $x \neq \mathbb{1} \neq y$. We will show that $y\ \mathrm{Sum}\ \{z \in M : z \wr x\}$, i.e., $y = x^{\mathsf{C}}$. First, assume for a contradiction that that for some z_0 we have $z_0 \wr x$ and $z_0 \not\sqsubseteq y$. Then, by (SSP), for some u_0 we have $u_0 \sqsubseteq z_0$ and $u_0 \wr y$. Hence, by ($\mathrm{mono}_{\wr}$), we have $u_0 \wr x$. From this and (7.8) we obtain $u_0 \wr x \sqcup y = \mathbb{1}$. But this contradicts (11.1). Second, assume for a contradiction that $u \sqsubseteq y$ but for any z such that $z \wr x$ we have $z \wr u$. Then, by ($\mathrm{df}_{\wr} \sqsubseteq$), we have $u \sqsubseteq x$. Yet this contradicts $x \wr y$. Thus, we obtain $y = x^{\mathsf{c}}$. Hence also $x = y^{\mathsf{c}}$, by (11.6).

From (11.3), (11.8) and (11.9) we obtain:

$$\forall_{x \in M \setminus \{\mathbb{1}\}} \forall_{y \in M} \big(x \sqcup y = \mathbb{1} \wedge x \wr y \iff y = x^{\mathsf{c}}\big). \tag{11.10}$$

We observe that, by (5.1), (11.4) and (11.6), for all $x, y \in M$ we have:

$$x \not\sqsubseteq y \iff y \neq \mathbb{1} \wedge x \circ y^{\mathsf{c}}.$$

So for any $x, y \in M$ such that $x \not\sqsubseteq y$ we can put:

$$x \smallsetminus y := x \sqcap y^{\mathsf{c}}\,.$$

The object $x \smallsetminus y$ can be treated as the *mereological difference* of x and y, or the *relative mereological complement of* y *with respect to* x.

Note that for arbitrary $x, y \in M$ we have:

$$x \not\sqsubseteq y \implies x \smallsetminus y \sqsubseteq x \wedge x \smallsetminus y \sqsubseteq y^{\mathsf{c}}, \tag{11.11}$$

$$x \not\sqsubseteq y \implies x \smallsetminus y \wr y\,. \tag{11.12}$$

If $x \not\sqsubseteq y$, then $y \neq \mathbb{1}$, so we use (9.8). Moreover, by (11.4): $x \smallsetminus y \sqsubseteq y^{\mathsf{c}}$ iff $x \smallsetminus y \wr y$.

In the light of the above conditions we see the difference $x \smallsetminus y$ belongs to the set $\{z \in M : z \sqsubseteq x \wedge z \wr y\}$, and moreover, in virtue of (9.9) and (11.4), it is equal to the mereological sum (supremum) of this set, i.e.:

$$\begin{aligned} x \not\sqsubseteq y \implies x \smallsetminus y &= \textstyle\bigsqcup\{z \in M : z \sqsubseteq x \wedge z \wr y\} \\ &= \sup_{\sqsubseteq}\{z \in M : z \sqsubseteq x \wedge z \wr y\}\,. \end{aligned} \tag{11.13}$$

By making use of the above result (11.11)–(11.13), we may strengthen condition (SSP) to the following:

$$\begin{aligned} \forall_{x,y \in M} \big(x \not\sqsubseteq y \implies{}& \exists_{z \in M} (z \sqsubseteq x \wedge z \wr y \wedge \\ & \forall_{u \in M} (u \sqsubseteq x \wedge u \wr y \Rightarrow u \sqsubseteq z))\big), \end{aligned} \tag{SSP+}$$

which we will call the *Super Strong Supplementation Principle*. What it intuitively says is that if x is not an ingrediens of y, then we can not only find some z being an ingrediens of x and external to y, but we can also find an element of the structure in question satisfying the aforementioned property and being the greatest such object in the structure. In the light of (11.4) and (11.11)–(11.13), we see that the postulated element z is the difference $x \smallsetminus y$.

To finish this section, we shall prove the conditional De Morgan's laws. First, in virtue of (9.12) and (9.13), for all $x, y \in M$ we can prove:

$$x \bigcirc y \wedge x \sqcap y \neq \mathbb{1} \implies (x \sqcap y)^{\complement} = \begin{cases} x^{\complement} & \text{if } x \neq \mathbb{1} \text{ and } y = \mathbb{1} \\ y^{\complement} & \text{if } x = \mathbb{1} \text{ and } y \neq \mathbb{1} \\ x^{\complement} \sqcup y^{\complement} & \text{if } x \neq \mathbb{1} \neq y \end{cases} \qquad (\mathrm{DM}_1)$$

If $x \bigcirc y$ and $x \sqcap y \neq \mathbb{1}$, then $x \sqcap y$ and $(x \sqcap y)^{\complement}$ are defined, and—in virtue of (9.12)—one of the three cases given above holds. In the first two cases, via (9.13), we have $(x \sqcap \mathbb{1})^{\complement} = x^{\complement}$ and $(\mathbb{1} \sqcap y)^{\complement} = y^{\complement}$, respectively. In the third case we will show that $(x \sqcap y)^{\complement}$ Sum $\{x^{\complement}, y^{\complement}\}$. To this end, we note that $x^{\complement} \sqsubseteq (x \sqcap y)^{\complement}$ and $y^{\complement} \sqsubseteq (x \sqcap y)^{\complement}$, by (9.8) and (11.5). Furthermore, for an arbitrary z, if $z \sqsubseteq (x \sqcap y)^{\complement}$, then $z \bigcirc x^{\complement}$ or $z \bigcirc y^{\complement}$. In essence, in the converse case $z \wr x^{\complement}$ and $z \wr y^{\complement}$, and thus—by virtue of (11.4) and (11.6)—we would have $z \sqsubseteq x$ and $z \sqsubseteq y$. And hence we would obtain $z \sqsubseteq x \sqcap y$. By virtue of (11.3), this would contradict our assumption.

Second, in the light of (11.7), for all $x, y \in M$ we can prove:

$$x \neq \mathbb{1} \neq y \wedge x^{\complement} \bigcirc y^{\complement} \implies (x \sqcup y)^{\complement} = x^{\complement} \sqcap y^{\complement}. \qquad (\mathrm{DM}_1)$$

If $x \neq \mathbb{1} \neq y$ and $x^{\complement} \bigcirc y^{\complement}$, then $x^{\complement} \sqcap y^{\complement}$ is defined. Moreover, by virtue of (11.7), we have $x \sqcup y \neq \mathbb{1} \neq x^{\complement} \sqcap y^{\complement}$. Therefore, in the light of the third case in (DM_1) and (11.6), we have $(x^{\complement} \sqcap y^{\complement})^{\complement} = x^{\complement\complement} \sqcup y^{\complement\complement} = x \sqcup y$. Hence $(x \sqcup y)^{\complement} = (x^{\complement} \sqcap y^{\complement})^{\complement\complement} = x^{\complement} \sqcap y^{\complement}$.

12. Filters in mereological structures

Let $\mathfrak{M} = \langle M, \sqsubset \rangle$ be any mereological structure. We will call a non-empty subset F of the set M a *filter* in $\mathfrak{M}$ iff it satisfies the following two conditions:

(f_1) if $x, y \in F$, then $x \bigcirc y$ and $x \sqcap y \in F$;
(f_2) if $x \in F$ and $x \sqsubseteq y$, then $y \in F$.

Let $\mathrm{F}_{\mathfrak{M}}$ be the family of all filters in $\mathfrak{M}$.

THEOREM 12.1. *For each $x \in M$ there is $F \in \mathrm{F}_{\mathfrak{M}}$ such that $x \in F$.*

PROOF. We show that the set $F_x := \{y : x \sqsubseteq y\}$ is a filter such that $x \in F_x$. First, by ($\mathrm{r}_{\sqsubseteq}$), we have $x \in F_x$. Second, by (9.10), if $y, z \in F_x$, then $x \sqsubseteq y$ and $x \sqsubseteq z$, i.e., $y \circ z$ and $y \sqcap z$. Thirdly, by ($\mathrm{t}_{\sqsubseteq}$), if $y \in F_x$ and $y \sqsubseteq z$, then $z \in F_x$. □

Thus, $\mathrm{F}_{\mathfrak{M}} \neq \emptyset$ and, via (f_1) and ($\exists\wr$), we obtain:

$$M \in \mathrm{F}_{\mathfrak{M}} \iff \mathrm{Card}\, M = 1. \tag{12.1}$$

We observe that by virtue of (5.1), (f_1), (f_2) and (11.3), we have for all $F \in \mathrm{F}_{\mathfrak{M}}$ and $x \in M$:

$$\mathbb{1} \in F\,, \tag{12.2}$$

$$x \neq \mathbb{1} \implies x \notin F \vee x^{\complement} \notin F\,. \tag{12.3}$$

LEMMA 12.2. *If $F \in \mathrm{F}_{\mathfrak{M}}$, $x \neq \mathbb{1}$ and $x^{\complement} \notin F$, then $F \subseteq \mathbb{O}(x)$.*

PROOF. Suppose that $F \in \mathrm{F}_{\mathfrak{M}}$, $x \neq \mathbb{1}$, $x^{\complement} \notin F$, and $y \in F$. If $y \wr x$, then $y \sqsubseteq x^{\complement}$, by (11.4). Hence $x^{\complement} \in F$, which contradicts our assumption. □

We say that a family $\mathcal{F} \subseteq \mathrm{F}_{\mathfrak{M}}$ is a *chain of filters* in $\mathfrak{M}$ iff $\mathcal{F} \neq \emptyset$ and for arbitrary $F_1, F_2 \in \mathfrak{F}$ either $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$.

LEMMA 12.3. *If $\mathcal{F} \subseteq \mathrm{F}_{\mathfrak{M}}$ is a chain of filters in $\mathfrak{M}$, then $\bigcup \mathcal{F} \in \mathrm{F}_{\mathfrak{M}}$.*

PROOF. Since $\mathcal{F} \neq \emptyset$, so $\bigcup \mathcal{F} \neq \emptyset$. If $x, y \in \bigcup \mathcal{F}$, then there are $F_1, F_2 \in \mathcal{F}$ such that $x \in F_1$ and $y \in F_2$. Since $\mathcal{F}$ is a chain, then either $x, y \in F_1$ or $x, y \in F_2$. In both cases we have $x \circ y$ and $x \sqcap y \in \bigcup \mathcal{F}$, by ($\mathrm{f}_1$) for F_1 or F_2. Finally, if $x \in \bigcup \mathcal{F}$ and $x \sqsubseteq y$, then $x \in F$, for some $F \in \mathrm{F}_{\mathfrak{M}}$. Hence $y \in \bigcup \mathcal{F}$, by ($\mathrm{f}_2$) for F. □

We say that a filter F is an *ultrafilter* in $\mathfrak{M}$ iff F is a maximal filter in $\mathfrak{M}$ with respect to set-theoretical inclusion, i.e., there does not exist a $G \in \mathrm{F}_{\mathfrak{M}}$ such that $F \subsetneq G$. Let $\mathrm{Ult}_{\mathfrak{M}}$ be the family of all ultrafilters of $\mathfrak{M}$. In the standard way, by applying the Kuratowski-Zorn lemma and Lemma 12.3, we can prove:

LEMMA 12.4. *Every filter in $\mathfrak{M}$ is included in some ultrafilter in $\mathfrak{M}$.*

Hence $\mathrm{Ult}_{\mathfrak{M}} \neq \emptyset$, because $\mathrm{F}_{\mathfrak{M}} \neq \emptyset$. Moreover,

THEOREM 12.5. *For each $x \in M$ there exists an $F \in \mathrm{Ult}_{\mathfrak{M}}$ such that $x \in F$.*

Proof. For any $x \in M$, in the light of Theorem 12.1, for some $F_x \in \mathrm{F}_{\mathfrak{M}}$ we have $x \in F_x$. In the light of Lemma 12.4, there exists a maximal filter F such that $F_x \subseteq F$. □

We call the filter $F \in \mathrm{F}_{\mathfrak{M}}$ *primary* iff it satisfies a third condition:

(f3) if $x \sqcup y \in F$, then either $x \in F$ or $y \in F$.

Let $\mathrm{PF}_{\mathfrak{M}}$ be the family of all primary filters in $\mathfrak{M}$.

Lemma 12.6. *For any* $F \in \mathrm{F}_{\mathfrak{M}}$: $F \in \mathrm{PF}_{\mathfrak{M}}$ *iff for each* $x \in M \setminus \{\mathbb{1}\}$ *either* $x \in F$ *or* $x^{\complement} \in F$.

Proof. '⇒' Let $F \in \mathrm{PF}_{\mathfrak{M}}$ and $x \neq \mathbb{1}$. Then, by (11.8), we have $x \sqcup x^{\complement} = \mathbb{1}$. Since $\mathbb{1} \in F$, either $x \in F$ or $x^{\complement} \in F$, by (f_3).

'⇐' Let F be a filter, which meets the given condition. Pick arbitrary $x, y \in M$ such that $x \sqcup y \in F$. If $x = \mathbb{1}$ or $y = \mathbb{1}$, then $x \in F$ or $y \in F$, by (12.2). Suppose therefore that $x \neq \mathbb{1} \neq y$. Assume for a contradiction that $x \notin F$ and $y \notin F$. Then, in the light of the assumed condition, we have $x^{\complement} \in F$ and $y^{\complement} \in F$. Therefore, by virtue of (f_1), we have $x^{\complement} \bigcirc y^{\complement}$ and $x^{\complement} \sqcap y^{\complement} \in F$. Hence, by virtue of (11.7), we have $(x \sqcup y)^{\complement} \neq \mathbb{1}$. Moreover, by virtue of (DM_1), we have $(x \sqcup y)^{\complement} \in F$. And this gives us a contradiction, via (12.3). □

Theorem 12.7. $\mathrm{Ult}_{\mathfrak{M}} = \mathrm{PF}_{\mathfrak{M}}$.

Proof. "⊆" Let $F \in \mathrm{Ult}_{\mathfrak{M}}$. We will show that F satisfies the following condition: for each $x \in M \setminus \{\mathbb{1}\}$ either $x \in F$ or $x^{\complement} \in F$. Hence, by virtue of Lemma 12.6, we have: $F \in \mathrm{PF}_{\mathfrak{M}}$.

Suppose that $x \in M \setminus \{\mathbb{1}\}$ and $x^{\complement} \notin F$. We put $G := \{z \in M : \exists_{y \in F}(y \bigcirc x \wedge y \sqcap x \sqsubseteq z)\}$. Firstly, let us note that $x \in G$, since $\mathbb{1} \in F$ and $\mathbb{1} \sqcap x = x$. Secondly, let us note that $F = F \cap \mathbb{O}(x)$, by Lemma 12.2. Thirdly, let us note that $F = F \cap \mathbb{O}(x) \subseteq G$. In fact, if $y \in F \cap \mathbb{O}(x)$, then $y \bigcirc x$ and $y \sqcap x \sqsubseteq y$; so $y \in G$. Finally, we will show that G is a filter. Condition (f_2) we obtain by ($\mathrm{t}_{\sqsubseteq}$). To prove (f_1), assume that $z_1, z_2 \in G$, i.e., for some $y_1, y_2 \in F$, we have $y_1 \bigcirc x$, $y_2 \bigcirc x$, $y_1 \sqcap x \sqsubseteq z_1$, and $y_2 \sqcap x \sqsubseteq z_2$. Then $y_1 \bigcirc y_2$ and $y_1 \sqcap y_2 \in F$, by virtue of (f_1) for F. Hence $y_1 \sqcap y_2 \in \mathbb{O}(x)$, via Lemma 12.2. So $y_1 \sqcap y_2 \sqcap x$ is defined. Furthermore, $y_1 \sqcap y_2 \sqcap x \sqsubseteq y_1 \sqcap x$ and $y_1 \sqcap y_2 \sqcap x \sqsubseteq y_2 \sqcap x$, by (9.6), (9.8) and (9.11). So $y_1 \sqcap y_2 \sqcap x \sqsubseteq z_1$ and $y_1 \sqcap y_2 \sqcap x \sqsubseteq z_2$, by ($\mathrm{t}_{\sqsubseteq}$). Therefore $y_1 \sqcap y_2 \sqcap x \sqsubseteq z_1 \sqcap z_2$, in virtue of (9.10). Hence $z_1 \sqcap z_2 \in G$.

Since F is a maximal filter, then $x \in F = G$.

"$\supseteq$" Assume for a contradiction that $F \in \mathrm{PF}_{\mathfrak{M}}$ and $F \notin \mathrm{Ult}_{\mathfrak{M}}$. Then there exists a $G \in \mathrm{F}_{\mathfrak{M}}$ such that $F \subsetneq G$. Hence for some $x \in M$ we have $x \in G$ and $x \notin F$. Therefore $x \neq \mathbb{1}$ and $x^{\complement} \in F \subsetneq G$, by Lemma 12.6. We thus obtain a contradiction via (12.3). □

13. A representation theorem for mereological structures

Let $\mathfrak{M} = \langle M, \sqsubset\rangle$ be a mereological structure. The *Stone map* of $\mathfrak{M}$ is the function $\mathrm{s}\colon M \to \mathcal{P}_{+}(\mathrm{Ult}_{\mathfrak{M}})$ defined by:

$$\mathrm{s}(x) := \{F \in \mathrm{Ult}_{\mathfrak{M}} : x \in F\}.$$

Clearly, $\mathrm{s}[M]$ is the image of the set M determined by the function s, i.e., $\mathrm{s}[M] := \{\mathrm{s}(x) : x \in M\}$. By virtue of Theorem 12.5 we have $\mathrm{s}(x) \neq \emptyset$, for each $x \in M$. Hence we obtain:

Lemma 13.1. $\emptyset \notin \mathrm{s}[M]$.

Now we prove that the Stone map is a monomorphism from $\mathfrak{M}$ into $\langle \mathcal{P}_{+}(\mathrm{Ult}_{\mathfrak{M}}), \subsetneq\rangle$.

Lemma 13.2. (i) *The Stone map* s *is a monomorphism from* $\langle M, \sqsubseteq\rangle$ *into* $\langle \mathcal{P}_{+}(\mathrm{Ult}_{\mathfrak{M}}), \subseteq\rangle$, *i.e., for arbitrary* $x, y \in M$:

$$x \sqsubseteq y \iff \mathrm{s}(x) \subseteq \mathrm{s}(y),$$
$$x = y \iff \mathrm{s}(x) = \mathrm{s}(y).$$

(ii) *The Stone map* s *is a monomorphism from* $\langle M, \sqsubset\rangle$ *into* $\langle \mathcal{P}_{+}(\mathrm{Ult}_{\mathfrak{M}}), \subsetneq\rangle$, *i.e., for any* $x, y \in M$:

$$x \sqsubset y \iff \mathrm{s}(x) \subsetneq \mathrm{s}(y).$$

Proof. *Ad* (i): Let $x \sqsubseteq y$ and $F \in \mathrm{s}(x)$. Then $x \in F$ and so $y \in F$, by (f_2). Hence $\mathrm{s}(x) \subseteq \mathrm{s}(y)$.

Conversely, let $x \not\sqsubseteq y$. Then, by virtue of (SSP), there exists a $z \in M$ such that $z \sqsubseteq x$ and $z \wr y$. In the light of Lemma 13.1 and the previously proved implication, we have $\emptyset \neq \mathrm{s}(z) \subseteq \mathrm{s}(x)$. Therefore, there exists an $F_0 \in \mathrm{Ult}_{\mathfrak{M}}$ such that $z, x \in F_0$. Since $z \wr y$, then $y \notin F_0$, by virtue of (f_1). Therefore $F_0 \in \mathrm{s}(x)$ and $F_0 \notin \mathrm{s}(y)$. Hence $\mathrm{s}(x) \nsubseteq \mathrm{s}(y)$.

Moreover, let $\mathrm{s}(x) = \mathrm{s}(y)$, i.e., $\mathrm{s}(x) \subseteq \mathrm{s}(y)$ and $\mathrm{s}(y) \subseteq \mathrm{s}(x)$. Then $x \sqsubseteq y$ and $y \sqsubseteq x$. Hence $x = y$, by ($\mathrm{antis}_{\sqsubseteq}$).

Ad (ii): Direct conclusion from (i). □

We obtain directly from the above lemma the following representation theorem for mereological structures:

THEOREM 13.3. *For any mereological structure* $\mathfrak{M} = \langle M, \sqsubset \rangle$:

(i) *The Stone map* s *is an isomorphism of the structure* $\langle M, \sqsubseteq \rangle$ *on the structure* $\langle \mathrm{s}[M], \subseteq \rangle$.

(ii) *The Stone map* s *is an isomorphism of* $\mathfrak{M}$ *on* $\langle \mathrm{s}[M], \subsetneq \rangle$.

Now note that the Stone map s for arbitrary $x, y \in M$ has the following properties:

$$\mathrm{s}(\mathbb{1}) = \mathrm{Ult}_{\mathfrak{M}} \,. \tag{13.1}$$

$$\mathrm{s}(x \sqcup y) = \mathrm{s}(x) \cup \mathrm{s}(y) \,, \tag{13.2}$$

$$x \circ y \implies \mathrm{s}(x \sqcap y) = \mathrm{s}(x) \cap \mathrm{s}(y) \,, \tag{13.3}$$

$$x \neq \mathbb{1} \implies \mathrm{s}(x^{\complement}) = \mathrm{Ult}_{\mathfrak{M}} \setminus \mathrm{s}(x) \,. \tag{13.4}$$

For (13.1): $\mathrm{s}(\mathbb{1}) = \mathrm{Ult}_{\mathfrak{M}}$, by (12.2). For (13.2): If $F \in \mathrm{s}(x \sqcup y)$ then either $x \in F$ or $y \in F$, by (f_3), since $\mathrm{Ult}_{\mathfrak{M}} = \mathrm{PF}_{\mathfrak{M}}$. Conversely, if $F \in \mathrm{s}(x)$ or $F \in \mathrm{s}(y)$, then $x \sqcup y \in F$, by (f_2), since $x \sqsubseteq x \sqcup y$, $y \sqsubseteq x \sqcup y$, and either $x \in F$ or $y \in F$. For (13.3): Let $x \circ y$. If $F \in \mathrm{s}(x \sqcap y)$, then $x \in F$ and $y \in F$, by (f_2), since $x \sqcap y \sqsubseteq x$ and $x \sqcap y \sqsubseteq y$. Conversely, if $F \in \mathrm{s}(x)$ and $F \in \mathrm{s}(y)$, then $x \sqcap y \in F$, by (f_1). For (13.4): Let $x \neq \mathbb{1}$. If $F \in \mathrm{s}(x^{\complement})$, then $x^{\complement} \in F$ and so $x \notin F$, by (12.3). Conversely, if $F \in \mathrm{Ult}_{\mathfrak{M}}$ and $F \notin \mathrm{s}(x)$, then $x \notin F$. Hence $x^{\complement} \in F$ and $F \in \mathrm{s}(x^{\complement})$, in the light of Lemma 12.6 and Theorem 12.7.

Finally, note that:

THEOREM 13.4. *For any mereological structure* $\mathfrak{M} = \langle M, \sqsubset \rangle$:

(i) *The family* $\mathrm{s}[M] \cup \{\emptyset\}$ *is an algebra (a field) of sets over* $\mathrm{Ult}_{\mathfrak{M}}$.

(ii) $\langle \mathrm{s}[M] \cup \{\emptyset\}, \subseteq, \emptyset, \mathrm{Ult}_{\mathfrak{M}} \rangle$ *is a complete Boolean lattice.*[31]

PROOF. *Ad* (i): Firstly, $\mathrm{Ult}_{\mathfrak{M}} \in \mathrm{s}[M]$, by (13.1). Secondly, let $\mathcal{X}, \mathcal{Y} \in \mathrm{s}[M] \cup \{\emptyset\}$. If $\mathcal{X} = \emptyset$ or $\mathcal{Y} = \emptyset$, then $\mathcal{X} \cup \mathcal{Y} \in \mathrm{s}[M] \cup \{\emptyset\}$. Assume therefore that $\mathcal{X}, \mathcal{Y} \in \mathrm{s}[M]$. Then for some $x, y \in M$ we have $\mathcal{X} = \mathrm{s}(x)$ and $\mathcal{Y} = \mathrm{s}(y)$. Hence $\mathcal{X} \cup \mathcal{Y} = \mathrm{s}(x \sqcup y) \in \mathrm{s}[M]$, by (13.2). So $\mathcal{X} \cup \mathcal{Y} \in \mathrm{s}[M] \cup \{\emptyset\}$. Thirdly, let $\mathcal{X} \in \mathrm{s}[M] \cup \{\emptyset\}$. If $\mathcal{X} = \emptyset$ then $\mathrm{Ult}_{\mathfrak{M}} \setminus \mathcal{X} = \mathrm{Ult}_{\mathfrak{M}}$. If $\mathcal{X} = \mathrm{Ult}_{\mathfrak{M}}$ then $\mathrm{Ult}_{\mathfrak{M}} \setminus \mathcal{X} = \emptyset$. In the remaining cases, for some $x \neq \mathbb{1}$ we have $\mathcal{X} = \mathrm{s}(x)$. Hence $\mathrm{Ult}_{\mathfrak{M}} \setminus \mathcal{X} = \mathrm{s}(x^{\complement}) \in \mathrm{s}[M]$, by (13.4). So $\mathrm{Ult}_{\mathfrak{M}} \setminus \mathcal{X} \in \mathrm{s}[M] \cup \{\emptyset\}$.

[31] The field $\mathrm{s}[M] \cup \{\emptyset\}$ is not necessarily complete, i.e., there may be a family $\mathcal{F}$ such that $\mathcal{F} \subseteq \mathrm{s}[M] \cup \{\emptyset\}$ and $\bigcup \mathcal{F} \notin \mathrm{s}[M]$.

Ad (ii): Since the family $\mathrm{s}[M] \cup \{\emptyset\}$ is a field, the structure $\langle \mathrm{s}[M] \cup \{\emptyset\}, \subseteq\rangle$ is a Boolean lattice. It remains to be shown that for any family $\mathcal{F} \in \mathcal{P}(\mathrm{s}[M] \cup \{\emptyset\})$ there exists an $\mathcal{X} \in \mathrm{s}[M] \cup \{\emptyset\}$ such that $\mathcal{X} \ \mathsf{sup}_{\subseteq} \ \mathcal{F}$. If $\mathcal{F} = \emptyset$ or $\mathcal{F} = \{\emptyset\}$, then $\emptyset \ \mathsf{sup}_{\subseteq} \ \mathcal{F}$. Therefore let $\emptyset \neq \mathcal{F} \neq \{\emptyset\}$. Let us put $S := \{x \in M : \mathrm{s}(x) \in \mathcal{F}\}$. In the light of the assumption, we have $S \neq \emptyset$. There exists therefore an $x \in M$ such that $x = \bigsqcup S = \sup_{\sqsubseteq} S$. Hence, by virtue of Lemma 13.2(i), we have $\mathrm{s}(x) \ \mathsf{sup}_{\subseteq} \ \mathrm{s}[S]$. We observe that if $\emptyset \notin \mathcal{F}$, then $\mathrm{s}[S] = \mathcal{F}$. Otherwise $\mathcal{F} = \mathrm{s}[S] \cup \{\emptyset\}$. Therefore $\mathrm{s}(x) \ \mathsf{sup}_{\subseteq} \ \mathcal{F}$. □

Theorems 13.3 and 13.4 signal the presence of a certain connection between mereological structures and complete Boolean lattices. Let us add to the universe of a mereological structure an element $\mathit{0}$ less than all others — the zero element. Then the Stone map expanded by the condition $\mathrm{s}(\mathit{0}) := \emptyset$ is an isomorphism of this new structure on the structure $\langle \mathrm{s}[M] \cup \{\emptyset\}, \subsetneq\rangle$. The structure $\langle M \cup \{\mathit{0}\}, \sqsubseteq\rangle$ with the expanded relation $\sqsubseteq$ satisfying the condition $\mathit{0} \in \mathbb{I}(M \cup \{\mathit{0}\})$ will therefore be a complete Boolean lattice. This theme will be a particular concern of the next chapter.

Chapter III

Mereology and complete Boolean lattices

1. Mereological structures and complete Boolean lattices

In this section we will prove two theorems which introduce a connection between mereological structures and complete Boolean lattices[1]. Tarski [1935] was the first to draw attention to this connection in a footnote in an article on the subject of Boolean algebra. In the English version of this paper [Tarski, 1956c], the footnote is to be found as footnote 1 on pp. 333–334. In the first paragraph of this footnote, Tarski asks how one might axiomatise mereological structures other than in a way equivalent to the way he does in his book. In the second paragraph, he sketches proofs of certain theorems which we shall later reconstruct. As the footnote gives the connection between mereological structures and complete Boolean lattices in a synoptic way, we reprint it below:

> The formulation of Posts. $\mathfrak{B}_4$ and $\mathfrak{B}_4^*$ [[2]] as well as some fragments of the proofs of Ths. 1 and 2 have been influenced by the researches of S. Leśniewski. The extended system of Boolean algebra [[3]] is closely related to the deductive theory developed by S. Leśniewski and called by him *mereology*. The foundations of mereology have been briefly discussed in article II [i.e., in Tarski, 1956b], where bibliographical references to the relevant works of Leśniewski will also be found. The relation of the part to the whole [[4]], which can be regarded as the only primitive notion of mereology, is the correlate of Boolean-algebraic in-

[1] Complete Boolean lattices are discussed in Section 11 of Appendix I.

[2] Postulate $\mathfrak{B}_4^*$ is expressed in this book as condition ($\bigstar$) in Theorem 12.1 listed in Appendix I. By changing the expression "exists exactly one x" in ($\bigstar$) for "exists at least one x", we obtain the formulation corresponding to postulate $\mathfrak{B}_4$.

[3] This is what Tarski calls a system axiomatising a class of complete Boolean lattices. Theorems 1 and 2, which are given above, say the same thing as Theorem 12.1 in Appendix I.

[4] Tarski used the term "part" with a wider meaning, that is, the same way that Leśniewski uses the term "ingrediens"

clusion. The postulate system $[\mathfrak{B}_2, \mathfrak{B}_4^*]$ has been obtained by a slight modification of the postulate system for mereology (Posts. I and II) suggested in II; regarding the relation of the latter system to the original postulate system of Leśniewski see II, p. 25, footnote 2. [5]

The formal difference between mereology and the extended system of Boolean algebra reduces to one point: the axioms of mereology imply (under the assumption of the existence of at least two different individuals) that there is no individual corresponding to the Boolean-algebraic zero, i.e. an individual which is a part of every other individual. If a set B of elements (together with the relation of inclusion) constitutes a model of the extended system of Boolean algebra, then, by removing the zero element from B, we obtain a model for mereology; if, conversely, a set C is a model for mereology, then, by adding a new element to C and by postulating that this element is in the relation of inclusion to every element of G, we obtain a model for the extended system of Boolean algebra. Apart from these formal differences and similarities, it should be emphasized that mereology, as it was conceived by its author, is not to be regarded as a formal theory where primitive notions may admit many different interpretations. Regarding Ths. 1 and 2 see Tarski, A. (78).

[Tarski, 1956c, pp. 333–334, footnote 1]

Theorem 1.1. *Let $\mathfrak{B} = \langle B, \leq\ \mathsf{0}, \mathsf{1}\rangle$ be non-trivial complete Boolean lattice. We put $M := B \setminus \{\mathsf{0}\}$ and $\sqsubset\ := \ \leq|_M \setminus \mathrm{id}_M$. Then:*

(i) *$M \neq \emptyset$ and the relation $\sqsubset$ strictly partially orders the set M.*

(ii) $\sqsubseteq\ =\ \leq|_M$.

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[5] In speaking of “the original postulates of Leśniewski’s system”, Tarski does not have in mind the system with the primitive relation $\sqsubset$ and axioms (L1)–(L4), but the system with the primitive relation $\sqsubseteq$, which was studied by Leśniewski [1930].

In view of the fact that, in axioms (L3) and (L4), the non-primitive concept *being an ingrediens* occurs, Leśniewski [1930] wanted to reconstruct his mereology such that this concept would be a primitive concept. In Theorem m in [Leśniewski, 1930, p. 85] he proved that the reflexivity of this concept (our $(\mathrm{t}_\sqsubseteq)$) follows from its transitivity and antisymmetry along with axioms III and IV (our$(\mathrm{t}_\sqsubseteq)$, $(\mathrm{antis}_\sqsubseteq)$, (L3), and (L4)). Thus Leśniewski [1930, p. 82] accepted four corresponding axioms. Tarski noted that the axiom expressing the antisymmetry of the relation *is an ingrediens of* follows from the others. He writes about this in [Tarski, 1956b], p. 25 in the second footnote. Thus only the axiom expressing the transitivity of the relation remains, along with axioms (L3) and (L4), which Tarski joined together into one axiom. Postulates I and II are introduced in [Tarski, 1956b, p. 25]. Postulate I $= \mathfrak{B}_2$ and says that the relation *is an ingrediens of* is transitive. Postulate II is logically equivalent to the conjunction of axioms (L3) and (L4). This postulate corresponds to postulate $\mathfrak{B}_4^*$ ($\mathfrak{B}_4$ would correspond to postulate (L4)).

The relation of Postulates I and II to the system (L1)–(L4) will be discussed in Section 4.

(iii) *For all* $x, y \in M$ *we have:* $x \bigcirc y$ *iff* $x \cdot y \in M$.
(iv) *For each* $S \in \mathcal{P}_+(M)$ *we have* $\sup_\leq S \ \mathbb{S}\mathrm{um}\ S$.
(v) $1 \ \mathbb{S}\mathrm{um}\ M$.
(vi) $\mathfrak{M} := \langle M, \sqsubset \rangle$ *is a mereological structure.*
(vii) *For each* $S \in \mathcal{P}_+(M)$ *we have* $\sup_\sqsubseteq S = \sup_\leq S = \bigsqcup S$.
(viii) *Let* $+$, $\cdot$, *and* $-$ *be the operations of* $\mathfrak{B}$ (i.e., $\langle B, +, \cdot, -, 0, 1 \rangle$ is the complete Boolean algebra obtained from $\mathfrak{B}$; see pp. 265–270 and Remark 9.1 from Appendix I). *Then for all* $x, y \in M$ *we have* $x \sqcup y = x + y$; *for all* $x, y \in M$ *such that* $x \bigcirc y$ *we have* $x \sqcap y = x \cdot y$; *for any* $x \in M \setminus \{1\}$ *we have* $x^\complement = -x$.

Proof. *Ad* (i): Since $0 \neq 1$, then $1 \in M$. Because $\leq$ partially orders the set B, then $\leq|_M$ partially orders the set M. Hence, by virtue of results Lemma 2.4(iv,v) from Appendix I, the relation $\sqsubset$ is asymmetric and transitive.

Ad (ii): Since the relation $\leq|_M$ is reflexive, then — in the light of Lemma 2.4(iii) from Appendix I — we have $\sqsubseteq \ := \ \sqsubset \cup \mathrm{id}_M := (\leq|_M \setminus \mathrm{id}_M) \cup \mathrm{id}_M = \ \leq|_M$.

Ad (iii): By virtue of (ii) and conditions (df $\bigcirc$) and (6.14) from Appendix I, for all $x, y \in M$: $x \cdot y \neq 0$ iff $\exists_{u \in B \setminus \{0\}} (u \leq x \wedge u \leq y)$ iff $\exists_{u \in M} (u \sqsubseteq x \wedge u \sqsubseteq y)$ iff $x \bigcirc y$.

Ad (iv): Let $S \in \mathcal{P}_+(M)$. Then $\sup_\leq S \neq 0$, i.e., $\sup_\leq S \in M$. By virtue of (ii) and from the definition of a supremum, we have (a): $\forall_{z \in S}\, z \sqsubseteq \sup_\leq S$. We show that the object $\sup_\leq S$ also meets for S the second condition of (df $\mathbb{S}\mathrm{um}$). We take an arbitrary $y \in M$ such that $y \sqsubseteq \sup_\leq S$. So, by (ii), also $y \leq \sup_\leq S$. Assume for a contradiction that $\forall_{z \in S}\, y \wr z$. Then, by (iii), we have: $\forall_{z \in S}\, y \cdot z = 0$. Hence, by (7.5) from Appendix I, we have a contradiction: $y = 0$. Therefore $\exists_{z \in S}\, y \bigcirc z$.

Ad (v): $1 := \sup_\leq B = \sup_\leq (B \setminus \{0\}) = \sup_\leq M$. By virtue of (iv), therefore, we have $1 \ \mathbb{S}\mathrm{um}\ M$.

Ad (vi): By virtue of (i) and (iv), in $\mathfrak{M}$ axioms (L1), (L2), and (L4) hold. Since $\mathfrak{B}$ is a Boolean lattice, then condition (6.17) from Appendix I holds for it, which we may write as follows:

$$\forall_{x,y \in M} \big(x \not\sqsubseteq y \iff \exists_{z \in M} (z \sqsubseteq x \wedge z \wr y) \big).$$

Thus in the structure $\langle M, \sqsubset \rangle$ condition (SSP) holds. Hence, in the light of Lemma II.6.2, we get (L3).

Ad (vii): Let $S \in \mathcal{P}_+(M)$. Then $\sup_\leq S \in M$. Furthermore, by virtue of (ii), $\sqsubseteq \ = \ \leq|_M$. Therefore $\sup_\sqsubseteq S = \sup_\leq S$. The rest follows from (iv) and (v).

Ad (viii): By virtue of (iii), (vii), (df $\sqcup$), (df $+$), (df $\sqcap$), (df $\cdot$), (df $^{\complement}$), and Lemma 8.2 from Appendix I. □

Remark 1.1. We also have another proof of Theorem 1.1(vi) which is based on some of Tarski's results. Pick an arbitrary $S \in \mathcal{P}_+(M)$ and an $x \in M$ such that $x \;\mathsf{Sum}\; S$. Then (A) $\forall_{z\in S}\, z \sqsubseteq x$ and (B) $\forall_{y\in M}(y \sqsubseteq x \Rightarrow \exists_{z\in S}\, y \bigcirc z)$. From (A), by virtue of (ii), we have $\forall_{z\in S}\, z \leq x$. From (B), (ii) and (iii) we get: $\forall_{y\in B}(y \leq x \wedge y \neq \mathrm{o} \Rightarrow \exists_{z\in S}\, y \cdot z \neq \mathrm{o})$. Hence $\forall_{y\in B}(y \leq x \wedge \forall_{z\in S}\, y \cdot z = \mathrm{o} \Rightarrow y = \mathrm{o})$. We see therefore that, for the set S, x satisfies conditions (a) and (b) in ($\star$) in Theorem 11.2 from Appendix I. By virtue of (7.5) from Appendix I, $\sup_{\leq} S$ also has this property. By virtue of Theorem 11.2 from Appendix I, only one member of B has this property. Therefore $x = \sup_{\leq} S$. Analogously, $y \;\mathsf{Sum}\; S$ entails $y = \sup_{\leq} S$, i.e., $x = y$. Thus (L3) is true in $\mathfrak{M}$. □

Theorem 1.1 may therefore be expressed in the following way[6]: *We obtain a mereological structure from an arbitrary non-trivial complete Boolean lattice which has had its zero removed and which includes a strict inclusion.* We will prove below a theorem from which it follows that we obtain a non-trivial complete Boolean lattice from every mereological structure to which a 'zero' has been added.

THEOREM 1.2. *Let $\mathfrak{M} = \langle M, \sqsubset\rangle$ be a mereological structure and let 0 be an arbitrary object which does not belong to M. We put $M^0 := M \cup \{0\}$ and let us define in the set M^0 a binary relation $\sqsubseteq^o := \sqsubseteq \cup (\{0\} \times M^0)$, i.e., for all $x, y \in M^0$ we put: $x \sqsubseteq^o y \;:\Longleftrightarrow\; x \sqsubseteq y \vee x = 0$. Then:*

(i) $\sqsubseteq \,=\, \sqsubseteq^o|_M$ *and* $\forall_{x\in M^0}\; 0 \sqsubseteq^o x$.
(ii) $\mathfrak{M}^o := \langle M^0, \sqsubseteq^o, 0, \mathbb{1}\rangle$ *is a non-trivial complete Boolean lattice in which 0 is the zero and $\mathbb{1}$ is the unity.*
(iii) *For each $S \in \mathcal{P}(M^0)$: $0 = \sup_{\sqsubseteq^o} S$ iff either $S = \emptyset$ or $S = \{0\}$.*
(iv) *For each $S \in \mathcal{P}_+(M)$ we have:*

$$\bigsqcup S = \sup_{\sqsubseteq} S = \sup_{\sqsubseteq^o} S = \sup_{\sqsubseteq^o}(S \cup \{0\}) \in M\,.$$

(v) *Let $+$, $\cdot$, and $-$ be the operations of the Boolean lattice $\mathfrak{M}^o$* (i.e., $\langle M^0, +, \cdot, -, 0, \mathbb{1}\rangle$ is the complete Boolean algebra obtained from $\mathfrak{M}^o$; see pp. 265–270 and Remark 9.1 from Appendix I). *Then for*

[6] In the passage reprinted earlier, Tarski was writing about mereological structures with the primitive relation *ing* and hence left out a reflexive Boolean inclusion.

all $x, y \in M^{0}$ *we have:*

$$x + y = \begin{cases} x \sqcup y & \text{if } x, y \in M \\ x & \text{if } y = 0 \\ y & \text{if } x = 0 \end{cases} \qquad -x = \begin{cases} x^{\complement} & \text{if } x \in M \setminus \{\mathbb{1}\} \\ 0 & \text{if } x = \mathbb{1} \\ \mathbb{1} & \text{if } x = 0 \end{cases}$$

$$x \cdot y = \begin{cases} x \sqcap y & \text{if } x, y \in M \text{ and } x \bigcirc y \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for all $x, y \in M^{0}$: $x \sqsubseteq^{0} y$ *iff* $x + y = y$ *iff* $x \cdot y = x$.

(vi) *In the set* M^{0} *we define three operations* $\sqcup^{o}$, $\sqcap^{o}$, *and* ${}^{\complement^{o}}$ *by*

$$x \sqcup^{o} y := \begin{cases} x \sqcup y & \text{if } x, y \in M \\ x & \text{if } y = 0 \\ y & \text{if } x = 0 \end{cases} \qquad x^{\complement^{o}} := \begin{cases} x^{\complement} & \text{if } x \in M \setminus \{\mathbb{1}\} \\ 0 & \text{if } x = \mathbb{1} \\ \mathbb{1} & \text{if } x = 0 \end{cases}$$

$$x \sqcap^{o} y := \begin{cases} x \sqcap y & \text{if } x, y \in M \text{ and } x \bigcirc y \\ 0 & \text{otherwise.} \end{cases}$$

then $\langle M^{0}, \sqcup^{o}, \sqcap^{o}, {}^{\complement^{o}}, \mathrm{o}, \mathbb{1}\rangle$ *is a non-trivial complete Boolean algebra, the same that we obtain from the Boolean lattice* $\mathfrak{M}^{o}$. *Moreover, for all* $x, y \in M^{0}$ *we have:* $x \sqsubseteq^{0} y$ *iff* $x \sqcup^{o} y = y$ *iff* $x \sqcap^{o} y = x$.

Proof. *Ad* (i): We have $\sqsubseteq \,\subseteq\, \sqsubseteq^{o} \cap (M \times M)$, by the assumption and by virtue of the definition of the relation $\sqsubseteq^{o}$. Conversely, if $x, y \in M$ and $x \sqsubseteq^{o} y$, then $x \sqsubseteq y$, since $x \neq 0$. We also have directly from the definition of the relation $\sqsubseteq^{o}$: if $x = 0$ then $x \sqsubseteq^{o} y$, for each $y \in M^{0}$.

Ad (ii): We have directly from the definition of $\sqsubseteq^{o}$ that:

$$\forall_{x,y \in M^{0}} (x \sqsubseteq y \iff 0 \neq x \sqsubseteq^{o} y), \tag{A}$$

$$\forall_{x \in M^{0}} (x = 0 \iff \forall_{y \in M^{0}} x \sqsubseteq^{o} y). \tag{B}$$

As we know, the relation $\sqsubseteq \,:=\, \sqsubset \cup \mathrm{id}_M$ is reflexive, transitive and antisymmetric. We will now prove that the relation $\sqsubseteq^{o}$ also has these properties.

Since $\sqsubseteq$ is reflexive, it therefore includes the relation id_M. So $\mathrm{id}_{M^{0}} = \mathrm{id}_M \cup \mathrm{id}_{\{0\}} \subseteq \,\sqsubseteq \cup (\{0\} \times M^{0}) =: \,\sqsubseteq^{o}$. Hence $\sqsubseteq^{o}$ is reflexive.

Assume that $x \sqsubseteq^{o} y$ and $y \sqsubseteq^{o} x$. Then, by simple transformations, we have either $x \sqsubseteq y \sqsubseteq x$ or $x \sqsubseteq y = 0$ or $y \sqsubseteq x = 0$ or $x = 0 = y$. The second and third cases are false and $x = y$ follows from the first, since the relation $\sqsubseteq$ is antisymmetric. Thus the relation $\sqsubseteq^{o}$ is also antisymmetric.

Assume that $x \sqsubseteq^o y$ and $y \sqsubseteq^o z$. Then, by simple transformations, we have either $x \sqsubseteq y \sqsubseteq z$ or $x \sqsubseteq y = \mathit{0}$ or $x = \mathit{0} \wedge y \sqsubseteq z$ or $x = \mathit{0} = y$. The second case is false and $x \sqsubseteq z$ follows from the first, since the relation $\sqsubseteq$ is transitive. Therefore in this case also $x \sqsubseteq^o z$. In the third and fourth cases we have $x \sqsubseteq^o z$, by (B). Thus the relation $\sqsubseteq^o$ is transitive.

We will now prove that the pair $\langle M^{\mathit{0}}, \sqsubseteq^o\rangle$ is a lattice, i.e., that for arbitrary $x, y \in M^{\mathit{0}}$ there are both an infimum and an supremum for the set $\{x, y\}$ with respect to the relation $\sqsubseteq^o$. In order to do this, we will make use of an the existence of the operations $\sqcup, \sqcap \colon M \times M \to M$, which, on the basis of the results given in sections II.7 and II.9 (cf. (II.8.3) and (II.9.5)) for all $x, y \in M$ satisfy the following conditions: $x \sqcup y = \sup_{\sqsubseteq}\{x, y\}$; if $x \bigcirc y$ then $x \sqcap y = \inf_{\sqsubseteq}\{x, y\}$. It thereby follows that for arbitrary $x, y \in M^{\mathit{0}}$ we have:

$$\sup_{\sqsubseteq^o}\{x, y\} = \begin{cases} x & \text{if } y = \mathit{0} \\ y & \text{if } x = \mathit{0} \\ x \sqcup y & \text{if } x, y \in M \end{cases}$$

$$\inf_{\sqsubseteq^o}\{x, y\} = \begin{cases} \mathit{0} & \text{if } x = \mathit{0} \text{ or } y = \mathit{0} \\ \mathit{0} & \text{if } x \wr y \\ x \sqcap y & \text{if } x \bigcirc y \end{cases}$$

It follows from (B) that $\mathit{0}$ is the zero in the lattice $\langle M^{\mathit{0}} \sqsubseteq^o\rangle$.

Since $\forall_{x \in M} x \sqsubseteq \mathbb{1}$ and $\mathit{0} \sqsubseteq^o \mathbb{1}$, then $\forall_{x \in M^{\mathit{0}}}\, x \sqsubseteq^o \mathbb{1}$. From this and the antisymmetry of the relation $\sqsubseteq^o$ it follows that $\mathbb{1}$ is the unity in $\langle M^{\mathit{0}}, \sqsubseteq^o\rangle$.

As with every lattice, let introduce the operations $+, \cdot \colon M \times M \to M$ by the conditions: $x + y := \sup_{\sqsubseteq^o}\{x, y\}$ and $x \cdot y := \inf_{\sqsubseteq^o}\{x, y\}$. Obviously, these operations are commutative.

Now, by (II.11.3) and (II.11.8), for any $x \in M \setminus \{\mathbb{1}\}$ we have:

$$x \sqcup x^{\complement} = \mathbb{1} \wedge x \wr x^{\complement}.$$

We have $\mathbb{1} + \mathit{0} := \sup_{\sqsubseteq^o}\{\mathbb{1}, \mathit{0}\} = \mathbb{1}$ and $\mathbb{1} \cdot \mathit{0} := \inf_{\sqsubseteq^o}\{\mathbb{1}, \mathit{0}\} = \mathit{0}$. Moreover, for any $x \in M \setminus \{\mathbb{1}\}$ there is $y \in M$ such that $\mathbb{1} = x \sqcup y = \sup_{\sqsubseteq^o}\{x, y\} =: x + y$ and $x \wr y$, i.e., $x \cdot y := \inf_{\sqsubseteq^o}\{x, y\} = \mathit{0}$. Therefore $\mathfrak{M}^o$ satisfies the condition (cf. p. 270), since this lattice is distributive:

$$\forall_{x \in M^{\mathit{0}}} \exists!_{y \in M^{\mathit{0}}} (x + y = \mathbb{1} \wedge x \cdot y = \mathit{0}). \tag{c!}$$

Thus $\mathfrak{M}^o$ is a complemented lattice and for the operation $- \colon M^{\mathit{0}} \to M^{\mathit{0}}$ we have: $-\mathit{0} = \mathbb{1}$, $-\mathbb{1} = \mathit{0}$, and $x^{\complement} = -x$, for any $x \in M \setminus \{\mathbb{1}\}$.

We will now show that the lattice $\mathfrak{M}^o$ is distributive. Pick arbitrary x, y, $z \in M^0$. If $x = 0$ then $(x+y)\cdot(x+z) = y\cdot z = 0 + (y\cdot z) = x + (y\cdot z)$. Assume therefore, that $x \in M$. If $y = 0 \neq z$ then $y \cdot z = 0$ and $(x + y)\cdot(x + z) = x \sqcap (x \sqcup z) = x = x + 0 = x + (y \cdot z)$. Similarly, when $z = 0 \neq y$. Finally, let $x, y, z \in M$. Then, in the light of (Δ_1), we have: $(x + y)\cdot(x + z) = (x \sqcup y)\cdot(x \sqcup z) = (x \sqcup y) \sqcap (x \sqcup z) = \begin{cases} x \sqcup (y \sqcap z) & \text{if } y \bigcirc z \\ x = x + 0 & \text{if } y \wr z \end{cases} = x + (y \cdot z)$, because if $y \wr z$ then $y \cdot z = 0$. For lattices, it suffices to check just one condition of distributivity.

Thus $\mathfrak{M}^o$ is a Boolean lattice, as a complemented distributive lattice. It remains only to be shown that it is a complete lattice.

Let $S \in \mathcal{P}(M^0)$. From the definitions of the relations $\sqsubseteq^o$ and $\mathsf{sup}_{\sqsubseteq^o}$ themselves, it follows that $0 = \sup_{\sqsubseteq^o} \emptyset = \sup_{\sqsubseteq^o}\{0\}$. We may therefore assume that $\emptyset \neq S \neq \{0\}$, i.e., $S = S_+ \cup \{0\}$, where $S_+ := S \setminus \{0\}$ and $S_+ \neq \emptyset$. For S_+ there exists an $x \in M$ such that $x = \bigsqcup S_+ = \sup_{\sqsubseteq} S_+$, by virtue of (II.8.3). From the equality $\sqsubseteq\ =\ \leq|_M$ we get $\sup_{\sqsubseteq} S_+ = \sup_{\sqsubseteq^o} S_+$. By virtue of the definitions of the relations $\sqsubseteq^o$ and $\mathsf{sup}_{\sqsubseteq^o}$ themselves we get: $x\ \mathsf{sup}_{\sqsubseteq^o}\ S_+$ iff $x\ \mathsf{sup}_{\sqsubseteq^o}\ S$. Therefore $\sup_{\sqsubseteq^o} S = \sup_{\sqsubseteq} S_+$.

Ad (iii): If $0 = \sup_{\sqsubseteq^o} S$ then $\forall_{x\in S}\, x \sqsubseteq^o 0$. Hence either $S = \emptyset$ either $S = \{0\}$. From the definition of the relations $\sqsubseteq^o$ and $\mathsf{sup}_{\sqsubseteq^o}$ itself, it follows that $0 = \sup_{\sqsubseteq^o} \emptyset = \sup_{\sqsubseteq^o}\{0\}$.

Ad (iv): For $S \in \mathcal{P}_+(M)$ there exists $x \in M$ such that $x = \bigsqcup S = \sup_{\sqsubseteq} S$, by virtue of (II.8.3). The next equality holds in virtue of $\sqsubseteq\ =\ \sqsubseteq^o|_M$. The final equality holds by virtue of the definition of $\sup_{\sqsubseteq^o}$.

Ad (v) *and* (v): This is proven in point (ii). □

A certain fragment of the above proof can be based on some of Tarski's results (Theorem 11.2 from Appendix I).

Proof of Theorem 1.2(ii) *using Tarski's theorem.* This second version begins at the point where we showed that the triple $\langle M^0, \sqsubseteq^o, 0\rangle$ is a lattice with zero. We will prove that the relation $\sqsubseteq^o$ satisfies condition ($\star$) from Theorem 11.2 in Appendix I, i.e., that for any $S \in \mathcal{P}(M^0)$ we have exactly one $x \in M^0$ such that: (a) $\forall_{z\in S}\, z \sqsubseteq^o x$ and (b) $\forall_{y\in M^0}(y \sqsubseteq^o x \wedge \forall_{z\in S}\, y\cdot z = 0 \Rightarrow y = 0)$. To begin with, for arbitrary $x \in M^0$ and $S \in \mathcal{P}(M^0)$ we shall transform condition (b): $\forall_{y\in M^0}(y \sqsubseteq^o x \wedge y \neq 0 \Rightarrow \exists_{z\in S}\, y\cdot z \neq 0)$ iff (b′) $\forall_{y\in M}(y \sqsubseteq x \Rightarrow \exists_{z\in S\setminus\{0\}}\, y \bigcirc z)$.

We now consider two cases of the set S.

1. Either $S = \emptyset$ or $S = \{0\}$: Clearly 0 satisfies conditions (a) and (b′). If, however, $x \neq 0$, then $x \in M$ and $x \sqsubseteq x$, and hence from (b′) we get a contradiction. Thus x does not therefore satisfy condition (b′).

2. $S \cap M \neq \emptyset$: Clearly, $S \setminus \{0\} \in \mathcal{P}_+(M)$. By virtue of conditions (A) and (B), for an arbitrary $x \in M^0$ we have: $\forall_{z \in S}\, z \sqsubseteq^o x$ iff $x \in M$ and $\forall_{z \in S \setminus \{0\}}\, z \sqsubseteq x$. From this and the equivalence of conditions (b) and (b′), it follows that x and S satisfy conditions (a) and (b) from ($\star$) iff $x \ \mathsf{Sum}\ (S \setminus \{0\})$. By virtue of (L3) and (L4), there exists exactly one $x \in M$ such that $x \ \mathsf{Sum}\ (S \setminus \{0\})$.

We have therefore shown that for an arbitrary $S \in \mathcal{P}(M^0)$ there exists exactly one $x \in M^0$ which meets conditions (a) and (b). Hence, by virtue of Theorem 11.2 from Appendix I, the structure $\langle M^0, \sqsubseteq^o \rangle$ is a complete Boolean lattice in which 0 is the zero. □

Note that using Theorem II.13.3 we can simplify the proof of point (ii) of Theorem 1.2. However, in the proof of Theorem II.13.3 we used Lemma II.12.4, which follows from the Kuratowski-Zorn lemma (equivalent to the the axiom of choice). We will prove Theorem 1.2(ii) below with using the axiom of choice.

Proof of Theorem 1.2(ii) *with the axiom of choice.* Let us expand the Stone map $\mathrm{s}\colon M \to \mathcal{P}_+(\mathrm{Ult}_{\mathfrak{M}})$ of $\mathfrak{M}$ defined in Section II.13 to the mapping $\mathrm{s}^o\colon M^0 \to \mathcal{P}_+(\mathrm{Ult}_{\mathfrak{M}})$

$$\mathrm{s}^o(x) := \begin{cases} \emptyset & \text{if } x = 0, \\ \mathrm{s}(x) & \text{if } x \in M. \end{cases}$$

Clearly, $\mathrm{s}^o[M^0] = \mathrm{s}[M] \cup \{\emptyset\}$. In the light of Theorem II.13.3, the mapping s^o is an isomorphism of the structure $\mathfrak{M}^o$ on the complete Boolean lattice $\langle \mathrm{s}^o[M^0], \subseteq, \emptyset, \mathrm{Ult}_{\mathfrak{M}} \rangle$. Therefore $\mathfrak{M}^o$ is also a complete Boolean lattice. Since $\emptyset$ and $\mathrm{Ult}_{\mathfrak{M}}$ are respectively the zero and the unity in $\langle \mathrm{s}^o(M^0), \subseteq \rangle$, and furthermore $\mathrm{s}^o(0) = \emptyset$ and $\mathrm{s}^o(\mathbb{1}) = \mathrm{Ult}_{\mathfrak{M}}$, then 0 and $\mathbb{1}$ are respectively the zero and the unity in $\mathfrak{M}^o$ (the former also follows from point (i)). □

It will be useful to have some knowledge of the construction of finite mereological structures for the analyses to be carried out in chapters IV and V of this work. To this end, we shall make use of theorems 1.1 and 1.2, from which follows a pair of statements which state, speaking loosely, that *every mereological structure arises from some non-trivial* *complete Boolean lattice whose zero has been removed* and, conversely, *every non-trivial complete Boolean lattice arises from some mereological structure to which a zero has been added.* In light of theorems 1.1 and 1.2 we obtain the following theorem.

THEOREM 1.3 (Pietruszczak, 2013). *For any non-empty set M and for any binary relation $\sqsubseteq$ in R the following conditions are equivalent*:

(a) $\langle M, \sqsubseteq\rangle$ *is a mereological structure, were* $\mathbb{1}$ *is the unity.*
(b) *For some* (*equivalently*: *any*) $0 \notin M$, *for* $M^0 := M \cup \{0\}$ *and for* $\sqsubseteq^o := \sqsubseteq \cup (\{0\} \times M^0)$ *the structure* $\langle M^0, \sqsubseteq^o, 0, \mathbb{1}\rangle$ *is a non-trivial complete Boolean lattice.*
(c) *For some non-trivial complete Boolean lattice* $\langle B, \leq, \mathrm{o}, \mathrm{1}\rangle$ *we have* $M = B \setminus \{\mathrm{o}\}$, $\sqsubseteq\, = \,\leq|_M$, *and* $\mathbb{1} = \mathrm{1}$.
(d) *For some non-trivial complete Boolean algebra* $\langle A, +, *, -, 0, 1\rangle$ *we have* $M = A \setminus \{0\}$, $\mathbb{1} = 1$, *and* $\sqsubseteq\, = \,\leq|_M$, *where* $\leq$ *is defined by*: $x \leq y \;:\Longleftrightarrow x + y = y$.

PROOF. '(a)⇒(b)' By Theorem 1.2.

'(b)⇒(c)' We put $B := M^0$, $\leq\, := \,\sqsubseteq^o$, $\mathrm{o} := 0$, and $\mathrm{1} := \mathbb{1}$. Then $M = B \setminus \{\mathrm{o}\}$ and $\sqsubseteq\, = \,\leq|_M$.

'(b)⇒(d)' In a non-trivial complete Boolean lattice $\langle M^0, \sqsubseteq^o, 0, \mathbb{1}\rangle$ by means of (df +), (df ·) and (df −) we define three operations $+$, $\cdot$ and $-$, respectively. So $\langle M^0, +, \cdot, -, 0, \mathbb{1}\rangle$ is a complete Boolean algebra and, by Theorem 1.2, the relation $\leq$ is equal to $\sqsubseteq^o$. So $\sqsubseteq\, = \,\leq|_M$.

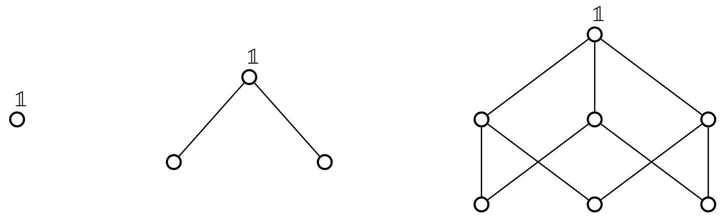
'(c)⇒(a)' By Theorem 1.1.

'(iv)⇒(i)' By the relationship between Boolean lattices and Boolean algebras (cf. Remark 9.1 from Appendix I) and Theorem 1.1. □

Every finite boolean lattice is full and its universe has 2^n members for some natural number $n \in \mathbb{N}$ (see p. 274). The number of elements of a finite mereological structure amounts to $2n - 1$, for a natural number $n > 0$. Graphs are given on p. 273 of Boolean lattices for $n = 0$, 1, 2, 3. To these non-trivial lattices in the graphs correspond the mereological structures given by the graphs in Model 1 (and conversely).[7]

[7] In these graphs "∘" means that a given element is not in the relation $\sqsubset$ with itself (the relation $\sqsubset$ is irreflexive). The possibility of moving upwards along lines leading from x to y signifies that $x \sqsubset y$ (the relation $\sqsubset$ is transitive and asymmetric). In the opposite case $x \not\sqsubset y$.

Graphs with the relation $\sqsubseteq$ differ from those graphs with the relation $\sqsubset$ only in that in place of each"∘" is "•" which means that a given element is in the relation $\sqsubseteq$ with itself ($\sqsubseteq$ is reflexive).



Model 1. Examples of finite mereological structures

2. The class MS is not elementarily axiomatisable

The title of this section is the content of Theorem 2.4 which we will prove below. This result is connected with the result that the class **CBL** of complete Boolean lattices is not elementarily axiomatisable (see Section 5 in Appendix II). The proof of Theorem 2.4 will rest on theorems 1.1 and 1.2 and those results given in Appendix II. Moreover, we shall also be making use of Lemma 2.1, which is easy to prove.

For mereological structures we use the elementary language $\mathrm{L}_{\sqsubset}$ with the identity predicate "=" and only one binary predicate "⊏" (see p. 71). Of course, all mereological structures are $\mathrm{L}_{\sqsubset}$-structures (see p. 72 and Remark 1.1). We assign to an arbitrary $\mathrm{L}_{\sqsubset}$-structure $\mathfrak{A} = \langle A, \sqsubset\rangle$ an arbitrarily established $0 \notin A$ along with the structure $\mathfrak{A}^o = \langle M^0, \sqsubseteq^o\rangle$ defined in Theorem 1.2. We connect the structure $\mathfrak{A}^o$ with an elementary language $\mathrm{L}^0_{\leq}$ with the identity "=" and two specific constants: the binary predicate "≤" and the individual constant "0".[8] These constants are interpreted in $\mathfrak{A}^o$ with the help of the relation $\sqsubseteq^o$ and the element 0, respectively.

LEMMA 2.1. *Let σ be an arbitrary $\mathrm{L}_{\sqsubset}$-sentence. We transform it into a $\mathrm{L}^0_{\leq}$-sentence σ^o with the help of the following transformation: in place of the subformula* $\ulcorner x_i \sqsubset x_j \urcorner$ *we substitute the $\mathrm{L}_{\leq}$-formula* $\ulcorner x_i \leq x_j \wedge x_i \neq x_j \urcorner$*; and we exchange an arbitrary quantifier binding x_i for a quantifier limited by condition* $\ulcorner x_i \neq 0 \urcorner$.[9] *Then:* $\mathfrak{A} \models \sigma$ *iff* $\mathfrak{A}^o \models \sigma^o$.

Let $\mathbb{N}$ be the set of all natural numbers, $\mathcal{P}(\mathbb{N})$ be the family of all subsets of $\mathbb{N}$, and $\mathcal{FC}(\mathbb{N})$ be the family of all finite and all co-finite subsets of $\mathbb{N}$, i.e., those subsets of $\mathbb{N}$ whose complements are finite (see

[8] $\mathrm{L}^0_{\leq}$ is created according to the rules given in Section 1 in Appendix II

[9] Formally: after exchanging the predicate "⊏", instead of $\forall_{x_i}\psi$ and $\exists_{x_j}\chi$ we take $\ulcorner \forall_{x_i}(x_i \neq 0 \rightarrow \psi)\urcorner$ and $\ulcorner \exists_{x_j}(x_i \neq 0 \wedge \chi)\urcorner$, respectively.

Example 1.2 in Appendix I). In Appendix I we analysed the Boolean lattices $\mathfrak{P}_{\mathbb{N}} := \langle \mathcal{P}(\mathbb{N}), \subseteq, \emptyset, \mathbb{N}\rangle$ and $\mathfrak{FC}_{\mathbb{N}} := \langle \mathcal{FC}(\mathbb{N}), \subseteq, \emptyset, \mathbb{N}\rangle$. The first one is complete and the second one is not complete (cf. examples 11.1 and 11.4 in Appendix I, respectively).

Now we introduce the following two $\mathrm{L}_{\sqsubset}$-structures:

$$\mathfrak{P}_{\mathbb{N}}^{+} := \langle \mathcal{P}_{+}(\mathbb{N}), \subsetneq\rangle,$$
$$\mathfrak{FC}_{\mathbb{N}}^{+} := \langle \mathcal{FC}_{+}(\mathbb{N}), \subsetneq\rangle,$$

where $\mathcal{P}_{+}(\mathbb{N}) := \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$ and $\mathcal{FC}_{+}(\mathbb{N}) := \mathcal{FC}(\mathbb{N}) \setminus \{\emptyset\}$. We notice that:

- $\mathfrak{P}_{\mathbb{N}}^{+}$ and $\mathfrak{FC}_{\mathbb{N}}^{+}$ are obtained from the Boolean lattices $\mathfrak{P}_{\mathbb{N}}$ and $\mathfrak{FC}_{\mathbb{N}}$, respectively, as a result of the operation described in Theorem 1.1.
- $\mathfrak{P}_{\mathbb{N}}$ and $\mathfrak{FC}_{\mathbb{N}}$ are obtained from $\mathfrak{P}_{\mathbb{N}}^{+}$ and $\mathfrak{FC}_{\mathbb{N}}^{+}$, respectively, as a result of the operation described in Theorem 1.2, i.e., we have $(\mathfrak{P}_{\mathbb{N}}^{+})^{o} = \mathfrak{P}_{\mathbb{N}}$ and $(\mathfrak{FC}_{\mathbb{N}}^{+})^{o} = \mathfrak{FC}_{\mathbb{N}}$.

Thus, by theorems 1.1 and 1.2, since the Boolean lattice $\mathfrak{P}_{\mathbb{N}}$ is complete and the Boolean lattice $\mathfrak{FC}_{\mathbb{N}}$ is not complete, so we obtain:

LEMMA 2.2. *$\mathfrak{P}_{\mathbb{N}}^{+}$ is a mereological structure, but $\mathfrak{FC}_{\mathbb{N}}^{+}$ is not a mereological structure.*

Moreover, we can also show that:

LEMMA 2.3. *$\mathrm{L}_{\sqsubset}$-structures $\mathfrak{P}_{\mathbb{N}}^{+}$ and $\mathfrak{FC}_{\mathbb{N}}^{+}$ are elementarily equivalent, i.e., we have* $\mathrm{Th}(\mathfrak{P}_{\mathbb{N}}^{+}) = \mathrm{Th}(\mathfrak{FC}_{\mathbb{N}}^{+})$.

PROOF. Since we have $(\mathfrak{P}_{\mathbb{N}}^{+})^{o} = \mathfrak{P}_{\mathbb{N}}$ and $(\mathfrak{FC}_{\mathbb{N}}^{+})^{o} = \mathfrak{FC}_{\mathbb{N}}$, then we can use Lemma 2.1. Moreover, $\mathfrak{P}_{\mathbb{N}}$ and $\mathfrak{FC}_{\mathbb{N}}$ are elementarily equivalent (see Lemma 5.1 in Appendix II). Therefore, for any $\mathrm{L}_{\sqsubset}$-sentence σ we have:

$$\begin{aligned} \sigma \in \mathrm{Th}(\mathfrak{P}_{\mathbb{N}}^{+}) \ &\text{iff}\ \ \sigma^{o} \in \mathrm{Th}(\mathfrak{P}_{\mathbb{N}}) && \text{(by Lemma 2.1)}\\ &\text{iff}\ \ \sigma^{o} \in \mathrm{Th}(\mathfrak{FC}_{\mathbb{N}}) && \text{(since } \mathrm{Th}(\mathfrak{P}_{\mathbb{N}}) = \mathrm{Th}(\mathfrak{FC}_{\mathbb{N}}))\\ &\text{iff}\ \ \sigma \in \mathrm{Th}(\mathfrak{FC}_{\mathbb{N}}^{+}) && \text{(by Lemma 2.1).} \end{aligned}$$ □

Remark 2.1. The proof of Lemma 2.3 can be obtained in a different way. Namely, Tsai [2013] proved that the structures $\mathfrak{P}_{\mathbb{N}}^{+}$ and $\mathfrak{FC}_{\mathbb{N}}^{+}$ are models of some complete elementary theory. So they are elementarily equivalent (see Proposition 1.3 in Appendix II). □

By Proposition 1.1 from Appendix II, every elementarily axiomatisable class is closed under elementary equivalence. Thus, by Lemma 2.3, we obtain:

Theorem 2.4. *The class* **MS** *is not elementarily axiomatisable.*

Proof. By Lemma 2.3, $\mathfrak{P}^{+}_{\mathbb{N}}$ and $\mathfrak{FC}^{+}_{\mathbb{N}}$ are elementarily equivalent, but the former is a mereological structure whereas the latter is not. Hence the class **MS** is not closed under elementary equivalence. Thus, this class is not elementarily axiomatisable, by Proposition 1.1 in Appendix II. □

3. Tarski's system

In this section we shall introduce the system of axioms for classical mereology established by Tarski [1929, 1956b]. We have already mentioned this system in Section 1.[10] In the next section we shall compare this system with the system of axioms (L1)–(L4).

The primitive notion in Tarski's system is the relation $\sqsubseteq$ *is an ingrediens of*. With the help of the relation $\sqsubseteq$ Tarski defines three auxiliary relations. The first is the relation $\sqsubset$ — *is a* (proper) *part of* — which he defines by using the formula ($\sqsubset=\sqsubseteq\setminus\mathrm{id}$). The second is the relation $\wr$ — *is exterior to* — which Tarski defines by using the formula (df $\wr$).[11] By employing the relations $\sqsubseteq$ and $\wr$ Tarski defines with the help of formula (df T $\mathbb{S}\mathrm{um}$) a third relation $\mathbb{S}\mathrm{um}$ — *is a mereological sum of* — i.e., he has it that x is a sum of all members of a set S iff each member of S is an ingrediens of x and no ingrediens of x is exterior to all members of S [cf. Tarski, 1956b, Definition III]. By adding (df $\circ$) we obtain ($\wr=-\circ$) and from this it follows that the definition of the relation $\mathbb{S}\mathrm{um}$ used by Tarski is equivalent to (df $\mathbb{S}\mathrm{um}$).

The mereological structures used in [Tarski, 1956b] are therefore of the form $\langle M, \sqsubseteq\rangle$, where the relation $\sqsubseteq$ satisfies two postulates. The first (Postulate I) says that the relation $\sqsubseteq$ is transitive, i.e., that condition ($\mathrm{t}_{\sqsubseteq}$) holds. The second (Postulate II) states that in the structure $\langle M, \sqsubseteq\rangle$ condition (L3-L4) holds.

Remark 3.1. Compare the passage from [Tarski, 1956c] and footnote 5. As Theorem 3.2 (proven below) shows — this being what Tarski was talking about in that passage — it follows from ($\mathrm{t}_{\sqsubseteq}$) and (L3-L4) that the relation $\sqsubseteq$ is also reflexive and antisymmetric. □

[10] Cf. the passage from [Tarski, 1956c] on p. 113 and footnote 5.

[11] That is, x is a (proper) part of y iff $x \sqsubseteq y$ and $x \neq y$; and x is exterior to y iff x and y do not have a common ingrediens [cf. Tarski, 1956b, definitions I and II].

LEMMA 3.1. *Let R be any reflexive relation in R. By putting in place of the symbol "$\sqsubseteq$" the letter "R" from definition* (df $\mathbb{S}\mathrm{um}$) *we generate the following definition of the relation* is the sum with respect to R of members of a given subset of M. *For all $x \in M$ and $S \in \mathcal{P}(M)$ we put*:

$$x \,\mathsf{sum}_R\, S \;:\Longleftrightarrow \forall_{z\in S}\, z\,R\,x \,\wedge\, \forall_{y\in M}\big(y\,R\,x \Rightarrow \exists_{z\in S}\exists_{u\in M}(u\,R\,y \wedge u\,R\,z)\big). \qquad (\mathrm{df}\ \mathsf{sum}_R)$$

In the same way, from conditions (L3), (L4), *and* (L3-L4) *we obtain the following conditions*:

$$\forall_{S\in\mathcal{P}(M)}\forall_{x,y\in M}(x\,\mathsf{sum}_R\,S \wedge y\,\mathsf{sum}_R\,S \implies x = y), \qquad (\mathrm{L}3_R)$$

$$\forall_{S\in\mathcal{P}_+(M)}\exists_{x\in M}\, x\,\mathsf{sum}_R\,S\,, \qquad (\mathrm{L}4_R)$$

$$\forall_{S\in\mathcal{P}_+(M)}\exists_{x\in M}\big(x\,\mathsf{sum}_R\,S \wedge \forall_{y\in M}(y\,\mathsf{sum}_R\,S \Rightarrow x = y)\big). \qquad (\mathrm{L}3\text{-}\mathrm{L}4_R)$$

In the class of all reflexive relational structures of the form $\langle M, R\rangle$, the conjunction of ($\mathrm{L}3_R$) *and* ($\mathrm{L}4_R$) *is equivalent to* ($\mathrm{L}3\text{-}\mathrm{L}4_R$).

PROOF. Condition ($\mathrm{L}3\text{-}\mathrm{L}4_R$) logically follows from sentences ($\mathrm{L}3_R$) and ($\mathrm{L}4_R$). Condition ($\mathrm{L}4_R$) logically follows from ($\mathrm{L}3\text{-}\mathrm{L}4_R$), too. Moreover, from ($\mathrm{r}_R$) and (df sum_R) we obtain that there is no $x \in M$ such that $x\,\mathsf{sum}_R\,\emptyset$ (cf. the proof of (II.3.1)). From this and from ($\mathrm{L}3\text{-}\mathrm{L}4_R$) follows ($\mathrm{L}3_R$) as well. Indeed, suppose that $x\,\mathsf{sum}_R\,S$ and $y\,\mathsf{sum}_R\,S$. Then $S \neq \emptyset$. Therefore, in the light of ($\mathrm{L}3\text{-}\mathrm{L}4_R$), for some $x_0 \in M$ we have: $x_0\,\mathsf{sum}_R\,S$ and $x = x_0 = y$. □

Remark 3.2. (i) Since the reflexivity of $\sqsubseteq$ is a consequence of definition (df $\sqsubseteq$), so — by virtue of Lemma 3.1 — the conjunction of (L3) and (L4) is equivalent to (L3-L4).

(ii) As mentioned in Remark 3.1, condition ($\mathrm{r}_\sqsubseteq$) is also a consequence of the pair conditions ($\mathrm{t}_\sqsubseteq$) and (L3-L4). Thus, by virtue of Lemma 3.1, also in Tarski's theory the conjunction of (L3) and (L4) is equivalent to (L3-L4). □

We will connect Tarski's system with a certain class of structures **TS** with one binary relation. We will introduce a different meaning for that relation than "$\sqsubseteq$", in order not to prove the various established facts about the relation $\sqsubseteq$, e.g., that it is reflexive and antisymmetric. We will therefore replace the symbol "$\sqsubseteq$" with the symbol "$\blacktriangleleft$", as in Lemma 3.1 we replaced "$\sqsubseteq$" with the letter "R". Thus, if we put $R := \blacktriangleleft$ then from (df sum_R), ($\mathrm{L}3_R$), ($\mathrm{L}4_R$) and ($\mathrm{L}3\text{-}\mathrm{L}4_R$) we obtain (df $\mathsf{sum}_\blacktriangleleft$), ($\mathrm{L}3_\blacktriangleleft$), ($\mathrm{L}4_\blacktriangleleft$) and ($\mathrm{L}3\text{-}\mathrm{L}4_\blacktriangleleft$), respectively.

The class **TS** is composed of those and only those structures of the form $\langle M, \blacktriangleleft\rangle$ in which the relation $\blacktriangleleft$ is transitive and (L3-L4$_\blacktriangleleft$) holds.

Remark 3.3. (i) If $\blacktriangleleft\ =\ \sqsubseteq$, then $\mathsf{sum}_\sqsubseteq\ =\ \mathbb{S}\mathrm{um}$, and (df $\mathsf{sum}_\blacktriangleleft$) and (L3-L4$_\blacktriangleleft$) reduce to (df $\mathbb{S}\mathrm{um}$) and (L3-L4), respectively.

(ii) From (i) and from Remark II.5.3 it follows that, if $\langle M, \sqsubset\rangle \in \mathbf{MS}$ and the relation $\sqsubseteq$ is defined by (df $\sqsubseteq$), then $\langle M, \sqsubseteq\rangle \in \mathbf{TS}$. □

Applying Theorem 12.1 from Appendix I for the class **TS**, we may prove the theorem which Tarski was talking about in the passage earlier on pp. 113–114.

THEOREM 3.2. *Let $\mathfrak{T} = \langle M, \blacktriangleleft\rangle$ be structure of the class* **TS** *and $0 \notin M$. We put $M^0 := M \cup \{0\}$ and we define in M^0 a binary relation $\blacktriangleleft^o := \blacktriangleleft \cup (\{0\} \times M^0)$, i.e., for arbitrary $x, y \in M^0$: $x \blacktriangleleft^o y :\Longleftrightarrow x \blacktriangleleft y \vee x = 0$. Then*:

(i) $\blacktriangleleft\ =\ \blacktriangleleft^o|_M$ *and* $\forall_{x\in M^0}\ 0 \blacktriangleleft^o x$.
(ii) $\mathfrak{T}^o := \langle M^0, \blacktriangleleft^o\rangle$ *is a non-trivial complete Boolean lattice in which 0 is its zero and the object $(\iota\, x)\ x\ \mathsf{sum}_\blacktriangleleft\ M$ is its unity.*
(iii) *The relation $\blacktriangleleft$ is reflexive and antisymmetric.*
(iv) *For any $S \in \mathcal{P}(M^0)$: $0 = \sup_{\blacktriangleleft^o} S$ iff either $S = \emptyset$ or $S = \{0\}$.*
(v) *For any $S \in \mathcal{P}_+(M)$*

$$(\iota\, x)x\ \mathsf{sum}_\blacktriangleleft\ S = \sup_\blacktriangleleft\ S = \sup_{\blacktriangleleft^o} S = \sup_{\blacktriangleleft^o}(S \cup \{0\}) \in M\,.$$

PROOF. Directly from the definition of the relation $\blacktriangleleft^o$ we have:

$$\forall_{x,y\in M^0}(x \blacktriangleleft y \iff 0 \neq x \blacktriangleleft^o y), \tag{A}$$

$$\forall_{x\in M^0}(x = 0 \iff \forall_{y\in M^0} x \blacktriangleleft^o y). \tag{B}$$

It follows from (B) that 0 is the zero in $\mathfrak{T}^o$.

In a manner analogous to that by which we demonstrated of the transitivity of the relation $\sqsubseteq^o$ in the proof of Theorem 1.2, we show:

$$\blacktriangleleft^o \text{ is transitive in } M^0. \tag{C}$$

We will show below that:

$$\forall_{x\in M}\exists_{y\in M}\ y \blacktriangleleft x\,. \tag{D}$$

Indeed, it follows from (L3-L4$_\blacktriangleleft$) that for any $x \in M$ there is a $z \in M$ such that $z\ \mathsf{sum}_\blacktriangleleft\ \{x\}$. Therefore, by virtue of (df $\mathsf{sum}_\blacktriangleleft$), we have $x \blacktriangleleft z$. Hence for some $y \in M$ we have $y \blacktriangleleft x$, also by virtue of (df $\mathsf{sum}_\blacktriangleleft$).

Ad (i): By (A) and (B).

Ad (ii): We will show that the transitive relation $◀^o$ satisfies condition (★) from Theorem 12.1 in Appendix I, i.e., that for an arbitrary $S \in \mathcal{P}(M^0)$ in M^0 we have exactly one x such that:

(a) $\forall_{z \in S}\, z ◀^o x$ and
(b) $\forall_{y \in M^0}\big(y ◀^o x \wedge \forall_{z \in S}\forall_{u \in M^0}(u ◀^o y \wedge u ◀^o z \Rightarrow \forall_{v \in M^0} u ◀^o v) \implies \forall_{v \in M^0} y ◀^o v\big)$.

To begin with, for arbitrary $x \in M^0$ and $S \in \mathcal{P}(M^0)$ we shall transform condition (b) in an equivalent way:

$$x \text{ and } S \text{ satisfy (b) from } (★)$$
$$\text{iff}$$
$$\forall_{y \in M^0}(y ◀^o x \wedge \neg\forall_{v \in M^0}\, y ◀^o v \implies \exists_{z \in S}\exists_{u \in M^0}(u ◀^o y \wedge u ◀^o z \wedge \neg\forall_{v \in M^0} u ◀^o v))$$
$$\text{iff}$$
$$\forall_{y \in M^0}(y ◀^o x \wedge y \neq 0 \implies \exists_{z \in S}\exists_{u \in M^0}(u ◀^o y \wedge u ◀^o z \wedge u \neq 0))$$
$$\text{iff}$$
$$\forall_{y \in M}(y ◀ x \implies \exists_{z \in S}\exists_{u \in M}(u ◀^o y \wedge u ◀^o z))$$
$$\text{iff}$$
$$(\mathrm{b}') \quad \forall_{y \in M}(y ◀ x \implies \exists_{z \in S \setminus \{0\}}\exists_{u \in M}(u ◀^o y \wedge u ◀^o z))$$

Let us consider two cases for the set S.

Either $S = \emptyset$ or $S = \{0\}$: Clearly, 0 satisfies conditions (a) and (b′) (in an empty way). If $x \neq 0$ then $x \in M$, and so — by virtue of (D) — for some $y \in M$ we have $y ◀ x$. From this and (b′) we get a contradiction. Therefore $x \neq 0$ does not satisfy condition (b′).

$S \cap M \neq \emptyset$: Clearly, $S \setminus \{0\} \in \mathcal{P}_+(M)$. By virtue of (A) and (B) for any $x \in M^0$ we have: $\forall_{z \in S}\, z ◀^o x$ iff $x \in M$ and $\forall_{z \in S \setminus \{0\}}\, z ◀ x$. From this and and the equivalence of conditions (b) and (b′) it follows that x and S satisfy conditions (a) and (b) iff $x \ \mathsf{Sum}\ (S \setminus \{0\})$. By virtue of (L3-L4$_◀$), there exists exactly one $x \in M$ such that $x \ \mathsf{Sum}\ (S \setminus \{0\})$.

We have therefore shown that for an arbitrary $S \in \mathcal{P}(M^0)$ there exists exactly one $x \in M^0$ which meets conditions (a) and (b). Hence, in the light of Theorem 12.1 from Appendix I:

- the relation $◀^o$ is reflexive and antisymmetric;
- the structure $\mathfrak{T}^o := \langle M^0, ◀^o\rangle$ is a complete Boolean lattice.

Thus, 0 is the zero in $\mathfrak{T}^o$. Moreover, since $\forall_{x \in M}\, x ◀ (\iota y)\, y \ \mathsf{sum}_◀\ M$ and $0 ◀^o (\iota y)\, y \ \mathsf{sum}_◀\ M$, so $\forall_{x \in M^0}\, x ◀^o (\iota y)\, y \ \mathsf{sum}_◀\ M$. It thereby follows from the antisymmetry of the relation $\blacktriangleleft^o$ that $(\iota\, y)\; y \;\mathsf{sum}_{\blacktriangleleft}\; M$ is the unity of $\mathfrak{T}^o$.

Ad (iii): Since $\blacktriangleleft\; = \;\blacktriangleleft^o|_M$, then the relation $\blacktriangleleft$ is reflexive and antisymmetric.

Ad (iv): If $0 = \sup_{\blacktriangleleft^o} S$ then $\forall_{x\in S}\, x \blacktriangleleft^o 0$. Hence either $S = \emptyset$ or $S = \{0\}$. Conversely, if $S = \emptyset$, then — by virtue of (ii) — we have: $0 = \sup_{\blacktriangleleft^o} \emptyset$. In the second case, this follows from the definition of the function $\sup_{\blacktriangleleft^o}$.

Ad (v): The third equality is true for an arbitrary subset of the set M^0, by virtue of the definition of the function $\sup_{\blacktriangleleft^o}$. Let $S \in \mathcal{P}_+(M)$. Then the second equality holds, since $\blacktriangleleft\; = \;\blacktriangleleft^o|_M$. Clearly, $\sup_{\blacktriangleleft} S \in M$.

We will now prove the equality $\sup_{\blacktriangleleft} S = (\iota\, x)\; x \;\mathsf{sum}_{\blacktriangleleft}\; S$. From the definition of a supremum, the first condition in (df $\mathsf{sum}_{\blacktriangleleft}$) holds: $\forall_{z\in S}\, z \blacktriangleleft \sup_{\sqsubseteq} S$. We now verify the second condition. Since we have shown already that $\langle M^0, \blacktriangleleft^o\rangle$ is a complete Boolean lattice and $\blacktriangleleft\; = \;\blacktriangleleft^o|_M$, then we may employ the condition analogous to (iii) from Theorem 1.1. We therefore pick an arbitrary $y \in M$ such that $y \blacktriangleleft \sup_{\blacktriangleleft} S$. Assume for a contradiction that $\forall_{z\in S}\exists_{u\in M}(u \blacktriangleleft y \wedge u \blacktriangleleft z)$. Then, by virtue of (iii) from Theorem 1.1, we have $y \blacktriangleleft^o \sup_{\blacktriangleleft^o} S$ and $\forall_{z\in S}\, y\cdot z = 0$. By virtue of condition (7.5) from Appendix I, our assumption yields a contradiction: $y = 0$. □

Since any structure from **MS** has an irreflexive relation and any structure from **TS** has a reflexive relation, then we obtain:

Corollary 3.3. $\mathbf{MS} \cap \mathbf{TS} = \emptyset$.

For structures from the class **TS** a variant of Theorem 1.1 also holds, which Tarski writes about in the passage on p. 113. The following theorem has similar to the proof of Theorem 1.1.

Theorem 3.4. *Let $\mathfrak{B} = \langle B, \leq, \mathrm{o}, 1\rangle$ a non-trivial complete Boolean lattice. We put $M := B \setminus \{\mathrm{o}\}$ and $\blacktriangleleft\; := \;\leq|_M$. Then:*

(i) *$M \neq \emptyset$ and the relation $\blacktriangleleft$ partially orders the set M.*
(ii) *For all $x, y \in M$ we have: $x \wr y$ iff $x \cdot y = \mathrm{o}$.*
(iii) *For any $S \in \mathcal{P}_+(M)$ we have $\sup_{\leq} S \;\mathsf{sum}_{\blacktriangleleft}\; S$.*
(iv) *$1 \;\mathsf{sum}_{\blacktriangleleft}\; M$.*
(v) *$\langle M, \blacktriangleleft\rangle$ belongs to **TS**;*
(vi) *For any $S \in \mathcal{P}_+(M)$ we have $\sup_{\blacktriangleleft} S = \sup_{\leq} S = (\iota\, x)\; x \;\mathsf{sum}_{\blacktriangleleft}\; S$.*

We can prove that the class **TS** is not elementarily axiomatisable. We need only establish that all structures from **TS** are $\mathrm{L}_{\sqsubseteq}$-structures, i.e., that the predicate "$\sqsubseteq$" in them is interpreted with the help of the relation $\blacktriangleleft$. We use counterparts of lemmas 2.1, 2.2 and 2.3 for $\mathrm{L}_{\sqsubseteq}$-structures (we analyse suitable versions of structures $\mathfrak{P}^{+}_{\mathbb{N}}$ and then prove the result in a manner analogous to that used for Theorem 2.4, using theorems 3.4 and 3.2 respectively in place of theorems 1.1 and 1.2.

4. A comparison of the classes **MS** and **TS**

We recall that we connected the class **MS** with an elementary language $\mathrm{L}_{\sqsubset}$ with identity described in Section 1 of Chapter II. We will connect the class **TS** with an elementary language $\mathrm{L}_{\sqsubseteq}$ with the identity "=" and one specific constant, which is the binary predicate "$\sqsubseteq$". This predicate may be read as "is an ingrediens of". The language $\mathrm{L}_{\sqsubseteq}$ is generated according to the rules given in Section 1 of Appendix II.

Let $\mathfrak{T} = \langle N, \blacktriangleleft \rangle$ belongs to **TS**. The predicate "$\sqsubseteq$" is interpreted in $\mathfrak{T}$ as the relation $\blacktriangleleft$. It is possible to treat the structure $\mathfrak{T}$ as a set-theoretic interpretation of the language $\mathrm{L}_{\sqsubseteq}$ and call it an $\mathrm{L}_{\sqsubseteq}$-*structure*.

We will prove that the classes **MS** and **TS** are elementarily definitionally equivalent in the sense of Szmielew [1983]:

Theorem 4.1. *The function* $\Delta\colon \mathbf{MS} \to \mathbf{TS}$ *is defined for an arbitrary* $\mathfrak{M} = \langle M, \sqsubset \rangle$ *by condition* $\Delta(\mathfrak{M}) := \langle M, \sqsubset \cup \mathrm{id}_M \rangle$ *is a bijection and the relation* $\sqsubset \cup \mathrm{id}_M$ *is is e-definable in* $\mathfrak{M}$*, and the relation* $\sqsubset$ *is e-definable in* $\Delta(\mathfrak{M})$ (see p. 292).

Furthermore, the converse bijection $\Delta^{-1}\colon \mathbf{TS} \to \mathbf{MS}$ *is defined for an arbitrary* $\mathfrak{T} = \langle N, \blacktriangleleft \rangle$ *by condition* $\Delta^{-1}(\mathfrak{T}) := \langle N, \blacktriangleleft \setminus \mathrm{id}_N \rangle$*, and the relation* $\blacktriangleleft \setminus \mathrm{id}_N$ *is e-definable in* $\mathfrak{N}$ *and the relation* $\blacktriangleleft$ *is is e-definable in* $\Delta^{-1}(\mathfrak{T})$.

These properties of the function Δ follow from the lemmas below:

Lemma 4.2. *Let* $\mathfrak{M} \in \mathbf{MS}$*. Then the relation* $\sqsubseteq$ *defined by condition* (df $\sqsubseteq$) *is e-definable in* $\mathfrak{M}$ *and the structure* $\langle M, \sqsubseteq \rangle$ *belongs to* **TS**.

Proof. Firstly, $\sqsubseteq = \{\langle x, y \rangle \in M \times M : \mathfrak{M} \models (\mathtt{x} \sqsubset \mathtt{y} \lor \mathtt{x} = \mathtt{y})[x/\mathtt{x}, y/\mathtt{y}]$. The rest follows from conditions ($\mathrm{t}_{\sqsubseteq}$), (L3), and (L4). □

Lemma 4.3. *Let* $\mathfrak{T} = \langle N, \blacktriangleleft \rangle$ *belong to* **TS**. *Then the relation* $\blacktriangleleft \setminus \mathrm{id}_N$ *is e-definable in* $\mathfrak{T}$, $(\blacktriangleleft \setminus \mathrm{id}_N) \cup \mathrm{id}_N = \blacktriangleleft$, *and the structure* $\langle N, \blacktriangleleft \setminus \mathrm{id}_M \rangle$ *belongs to* **MS**.

Proof. Firstly, $\blacktriangleleft \setminus \mathrm{id}_N = \{\langle x, y\rangle \in N \times N : \mathfrak{T} \models (\mathtt{x} \sqsubseteq \mathtt{y} \wedge \mathtt{x} \neq \mathtt{y})[x/\mathtt{x}, y/\mathtt{y}]$. Furthermore, since $\blacktriangleleft$ partially orders the set N, then the relation $\blacktriangleleft \setminus \mathrm{id}_N$ strictly partially orders the set N, i.e., in the structure $\langle N, \blacktriangleleft \setminus \mathrm{id}_N\rangle$ condition (L1) and (L2) hold. We now observe that $(\blacktriangleleft \setminus \mathrm{id}_N) \cup \mathrm{id}_N = \blacktriangleleft$, since the relation $\blacktriangleleft$ is reflexive. Hence conditions (L3) and (L4) hold in $\langle N, \blacktriangleleft \setminus \mathrm{id}_N\rangle$, because in $\mathfrak{T}$ condition (L3-L4$_\blacktriangleleft$) holds (cf. Remark 3.2). Thus, $\langle N, \blacktriangleleft \setminus \mathrm{id}_N\rangle$ belongs to **MS**. □

Proof of Theorem 4.1. In the light of Lemma 4.2, for an arbitrary structure $\mathfrak{M} = \langle M, \sqsubset \rangle$ from **MS** we have $\Delta(\mathfrak{M}) \in$ **TS**.

We will show that the function Δ is a bijection, i.e., it is an one-to-one function from the class **MS** onto the class **TS**.

Let $\Delta(\mathfrak{M}_1) = \Delta(\mathfrak{M}_2)$ holds for $\mathfrak{M}_1 = \langle M_1, \sqsubset_1 \rangle$ and $\mathfrak{M}_2 = \langle M_2, \sqsubset_2 \rangle$ from **MS**. Then, by virtue of the definition of the function Δ, we have $M_1 = M_2$ and $\sqsubset_1 \cup \mathrm{id}_{M_1} = \sqsubset_2 \cup \mathrm{id}_{M_2}$. Hence $\sqsubset_1 = \sqsubset_2$, since — by (L1) — we have $\sqsubset_1 \cap \mathrm{id}_{M_1} = \emptyset = \sqsubset_2 \cap \mathrm{id}_{M_2}$. Thus, $\mathfrak{M}_1 = \mathfrak{M}_2$, i.e., Δ is injective.

We pick an arbitrary $\mathfrak{T} = \langle N, \blacktriangleleft \rangle$ from **TS**. In the light of Lemma 4.3, the structure $\langle N, \sqsubset \rangle$, in which $\sqsubset := \blacktriangleleft \setminus \mathrm{id}_N$, belongs to **MS** and $\mathfrak{T} = \Delta(\langle N, \sqsubset \rangle)$. The function Δ is therefore surjective.

By virtue of Lemma 4.2, the relation $\sqsubset \cup \mathrm{id}_M$ is e-definable in $\mathfrak{M}$. By virtue of ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$), we have $\sqsubset = \sqsubseteq \setminus \mathrm{id}_M$. Therefore $\sqsubset$ is e-definable in $\Delta(\mathfrak{M})$, as in the proof of Lemma 4.3. □

From Theorem 4.1 follows the conclusion below:

Corollary 4.4. *Let* $\mathfrak{M} = \langle M, \sqsubset \rangle$ *belong to* **MS**, $\mathfrak{T} = \langle M, \blacktriangleleft \rangle$ *belong to* **TS**, *and* $\mathfrak{T} = \Delta(\mathfrak{M})$, *i.e.,* $\blacktriangleleft = \sqsubseteq$. *Then* $\sqsubset = \blacktriangleleft \setminus \mathrm{id}_M$ *and the structures* $\mathfrak{M}$ *and* $\mathfrak{T}$ *are elementarily definitionally equivalent, i.e., the relation* $\sqsubset$ *is e-definable in* $\mathfrak{T}$, *and the relation* $\blacktriangleleft$ *is e-definable in* $\mathfrak{M}$.

Also from theorems 4.1 and 2.4 it follows that the class **TS** — as the class **MS** — is not elementarily axiomatisable.

As we have already said on p. 124, the primitive notion in Tarski's system is the relation *is an ingrediens of*. The following conclusion shows that by considering mereology in the style of Tarski, we may come back from the 'old' meaning of "$\sqsubseteq$" and with it to axioms ($\mathrm{t}_\sqsubseteq$) and (L3-L4).

COROLLARY 4.5. *Let M be non-empty set and $\sqsubset$ and $\sqsubseteq$ be binary relations in M. Then the following two conditions are equivalent*:

- *The structure $\langle M, \sqsubset\rangle$ belongs to* **MS** *and the relation $\sqsubseteq$ satisfies condition* (df $\sqsubseteq$).
- *The structure $\langle M, \sqsubseteq\rangle$ belongs to* **TS** *and the relation $\sqsubset$ satisfies condition* ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$).

5. The class MS*

Theorems 1.2 and 3.2 may be generalised to a wider class of relational structures than the sum of the classes **MS** and **TS**.

Let M be a non-empty set and $\lhd$ be a transitive binary relation in M. Moreover, we put $\unlhd := \lhd \cup \mathrm{id}_M$. So the relation $\unlhd$ is reflexive and transitive.

We put $R := \unlhd$ in conditions (df sum_R), ($\mathrm{L3}_R$), ($\mathrm{L4}_R$), and ($\mathrm{L3}$-$\mathrm{L4}_R$). Then we obtain (df $\mathsf{sum}_\unlhd$), ($\mathrm{L3}_\unlhd$), ($\mathrm{L4}_\unlhd$) and ($\mathrm{L3}$-$\mathrm{L4}_\unlhd$), respectively.

Let **MS*** be the class of relational structures of the form $\langle M, \lhd\rangle$ in which $\lhd$ is transitive and for which ($\mathrm{L3}$-$\mathrm{L4}_\unlhd$) holds.[12]

We have the following result:

PROPOSITION 5.1. *Let $\mathfrak{M} = \langle M, \lhd\rangle \in$* **MS***. *Then*:

(i) $\mathfrak{M} \in$ **MS** *iff the relation $\lhd$ is irreflexive.*
(ii) $\mathfrak{M} \in$ **TS** *iff the relation $\lhd$ is reflexive iff $\lhd = \unlhd$.*

PROOF. *Ad* (i): By Lemma 3.1, since the relation $\unlhd$ is reflexive.

Ad (ii): Firstly, the relation $\lhd$ is reflexive iff $\lhd = \lhd \cup \mathrm{id}_M =: \unlhd$. Secondly, if $\mathfrak{M} \in$ **TS** then the relation $\lhd$ is obviously reflexive. Conversely, if the relation $\lhd$ is reflexive, then $\lhd = \unlhd$ and therefore we have written the relation $\mathsf{sum}_\unlhd$ and the condition ($\mathrm{L3}$-$\mathrm{L4}_\unlhd$) in terms of relation $\lhd$, i.e., $\mathfrak{M} \in$ **TS**. □

PROPOSITION 5.2. (i) *For all $\langle M, \sqsubset\rangle \in$* **MS**, *$\langle M, \lhd\rangle \in$* **MS***, *if $\sqsubset \subseteq \lhd \subseteq \sqsubseteq$, then $\sqsubseteq = \unlhd$, so also $\mathbb{S}\mathrm{um} = \mathsf{sum}_\unlhd$.*
(ii) *For all $\langle M, \blacktriangleleft\rangle \in$* **TS**, *$\langle M, \lhd\rangle \in$* **MS***, *if $\blacktriangleleft \setminus \mathrm{id}_M \subseteq \lhd \subseteq \blacktriangleleft$, then $\unlhd = \blacktriangleleft$, so also $\mathsf{sum}_\unlhd = \mathsf{sum}_\blacktriangleleft$.*
(iii) **MS** $\cup$ **TS** $\subsetneq$ **MS***.

[12] To put it crudely: we take Tarski's Postulate I for the relation *is a part of* ($\lhd$) and Postulate II for the relation *is an ingrediens of* ($\unlhd$).

Proof. *Ad* (i): If $\sqsubset \subseteq \lhd \subseteq \sqsubseteq$, then $\sqsubseteq = \sqsubset \cup \mathrm{id}_M \subseteq \lhd \cup \mathrm{id}_M = \unlhd \subseteq \sqsubseteq \cup \mathrm{id}_M = \sqsubseteq$.

Ad (ii): If $\blacktriangleleft \setminus \mathrm{id}_M \subseteq \lhd \subseteq \blacktriangleleft$, then $\blacktriangleleft = (\blacktriangleleft \setminus \mathrm{id}_M) \cup \mathrm{id}_M \subseteq \lhd \cup \mathrm{id}_M = \unlhd \subseteq \blacktriangleleft \cup \mathrm{id}_M = \blacktriangleleft$.

Ad (iii): By Lemma 3.1, we have $\mathbf{MS} \cup \mathbf{TS} \subseteq \mathbf{MS}^*$. Now for any non-trivial structure $\langle M, \sqsubset \rangle$ from **MS** and $x \in M$ we put $\lhd := \sqsubset \cup \{\langle x, x\rangle\}$. Then $\sqsubset \subsetneq \lhd \subsetneq \sqsubseteq$, and moreover $\lhd$ is transitive and neither reflexive nor irreflexive. By (i), we have $\unlhd = \sqsubseteq$, so also $\mathsf{sum}_{\unlhd} = \mathbb{S}\mathrm{um}$. Thus, $\langle M, \lhd \rangle \in \mathbf{MS}^*$, but $\langle M, \lhd \rangle$ belongs neither **MS** nor **TS**. □

Remark 5.1. Take any structure $\langle M, \sqsubset \rangle$ from **MS** and 'make reflexive' only some of members of the universe. In this way, create a relation $\lhd$. Then $\sqsubset \subseteq \lhd \subseteq \sqsubseteq$. So $\sqsubseteq = \unlhd$ and $\mathbb{S}\mathrm{um} = \mathsf{sum}_{\unlhd}$. Thus, $\langle M, \lhd \rangle$ belongs to $\mathbf{MS}^*$, but does not belong to $\mathbf{MS} \cup \mathbf{TS}$. □

Theorem 5.3. *Let $\langle M, \lhd \rangle$ be a structure from* $\mathbf{MS}^*$ *and let $0 \notin M$. We put $M^0 := M \cup \{0\}$ and define in M^0 a binary relation $\unlhd^o := \unlhd \cup (\{0\} \times M^0)$, i.e., for all $x, y \in M^0$: $x \unlhd^o y :\Longleftrightarrow x \unlhd y \vee x = 0$. Then conditions* (i)–(v) *from Theorem 3.2 hold for the structure $\mathfrak{M}^o = \langle M^0, \unlhd^o \rangle$, if we replace "$\blacktriangleleft$" and "$\blacktriangleleft^o$" in it with "$\unlhd$" and "$\unlhd^o$", respectively.*

Proof. Under the assumptions made, conditions (A)–(D) hold from the proof of Theorem 3.2 formulated for the relation $\unlhd$, in which (D) is obvious, because the relation $\unlhd$ is reflexive by definition. We therefore have (i) and (ii).

Ad (iii): We show the antisymmetry of the relation $\unlhd$ just as we did in (iii) in the proof of Theorem 3.2. Hence, since $\lhd \subseteq \unlhd$, then $\lhd$ is also antisymmetric.

Ad (iv) and (v): We prove these in the same way as we did in the proof of Theorem 3.2. □

By using Theorem 5.3, one may prove that the class $\mathbf{MS}^*$ is not elementarily axiomatisable. One needs only to establish, that all structures from $\mathbf{MS}^*$ are $\mathrm{L}_{\sqsubset}$-structures, i.e., that the predicate "$\sqsubset$" in them is interpreted with the help of the relation $\lhd$. We continue the proof in the same way as we did for Theorem 2.4, using counterparts of lemmas 2.1, 2.2 and 2.3, and Theorem 1.1 and Theorem 5.3 (in place of Theorem 1.2).

6. A formal comparison of the classes MS, TS, and MS* with the class of complete Boolean lattices

We assign to an arbitrary lattice $\mathfrak{B}$ belonging to the class **CBL** of complete Boolean lattices a structure $\mathfrak{B}^+_\sqsubset \in$ **MS** which arises from $\mathfrak{B}$ as a result of the operation carried out in Theorem 1.1. Analogously, we assign to an arbitrary lattice $\mathfrak{B} \in$ **CBL** a structure $\mathfrak{B}^+_\blacktriangleleft \in$ **TS** which arises from $\mathfrak{B}$ as a result of the operation carried out in Theorem 3.4. Finally, we assign to an arbitrary structure $\mathfrak{M} \in$ **MS*** an arbitrarily chosen $\mathit{0} \notin M$ and the complete Boolean lattice $\mathfrak{M}^o = \langle M^{\mathit{0}}, \trianglelefteq^o \rangle$ described in Theorem 5.3. Under these assignments, for each $\mathfrak{M} \in$ **MS*** we have:

$$\mathfrak{M} \in \mathbf{MS} \iff \mathfrak{M} = (\mathfrak{M}^o)^+_\sqsubset, \tag{6.1}$$

$$\mathfrak{M} \in \mathbf{TS} \iff \mathfrak{M} = (\mathfrak{M}^o)^+_\blacktriangleleft. \tag{6.2}$$

Note that if $\mathfrak{M} = \langle M, \triangleleft \rangle$ belongs to **MS***, then — by virtue of Theorem 5.3 — the zero in M^{0} is $\mathit{0}$. Hence the set M is the universe of the structures $(\mathfrak{M}^o)^+_\sqsubset$ and $(\mathfrak{M}^o)^+_\blacktriangleleft$. Further for (6.1): If $\mathfrak{M} \in$ **MS** ($\triangleleft = \sqsubset$), then $\trianglelefteq^o = \sqsubseteq \cup (\{\mathit{0}\} \times M^{\mathit{0}})$. Therefore $\trianglelefteq^o|_M = \sqsubseteq = \sqsubset \cup \mathrm{id}_M$, i.e., $\sqsubset = \trianglelefteq^o|_M \setminus \mathrm{id}_M$, by virtue of ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$). Hence $\mathfrak{M} = (\mathfrak{M}^o)^+_\sqsubset$. Conversely, let $\mathfrak{M} = (\mathfrak{M}^o)^+_\sqsubset$, i.e., $\triangleleft = \trianglelefteq^o|_M \setminus \mathrm{id}_M = (\triangleleft \cup \mathrm{id}_M) \setminus \mathrm{id}_M$. Hence, by virtue of lemmas 2.2 and 2.3 from Appendix I, the relation $\triangleleft$ is irreflexive and antisymmetric. Since $\mathfrak{M} \in$ **MS***, then $\mathfrak{M} \in$ **MS**. For (6.2): if $\mathfrak{M} \in$ **TS** ($\triangleleft = \blacktriangleleft$), then $\trianglelefteq^o := \blacktriangleleft \cup (\{\mathit{0}\} \times M^{\mathit{0}})$, i.e., $\blacktriangleleft = \trianglelefteq^o|_M$. Hence $\mathfrak{M} = (\mathfrak{M}^o)^+_\blacktriangleleft$. Conversely, if $\mathfrak{M} = (\mathfrak{M}^o)^+_\blacktriangleleft$, then $\triangleleft = \trianglelefteq^o|_M = \triangleleft \cup \mathrm{id}_M = \trianglelefteq$, that is, the relation $\triangleleft$ is reflexive. Since $\mathfrak{M} \in$ **MS***, then $\mathfrak{M} \in$ **TS**.

By applying (6.1) and (6.2) we get the equalities:

$$\mathbf{MS} = \{\mathfrak{B}^+_\sqsubset : \mathfrak{B} \in \mathbf{CBL}\},$$

$$\mathbf{TS} = \{\mathfrak{B}^+_\blacktriangleleft : \mathfrak{B} \in \mathbf{CBL}\}.$$

Suppose that $\mathfrak{M} \in$ **MS**. Then, in the light of Theorem 1.2, we have $\mathfrak{M}^o \in$ **CBL**. Moreover, by virtue of (6.1), we have $\mathfrak{M} = (\mathfrak{M}^o)^+_\sqsubset$. Hence for some $\mathfrak{B} \in$ **CBL** we have $\mathfrak{M} = \mathfrak{B}^+_\sqsubset$. Conversely, if $\mathfrak{B} \in$ **CBL**, then — by virtue of Theorem 1.1 — we have $\mathfrak{B}^+_\sqsubset \in$ **MS**. Analogously for **TS**.

The following equalities clearly hold:

$$\mathbf{CBL} = \{\mathfrak{M}^o : \mathfrak{M} \in \mathbf{MS}\} = \{\mathfrak{M}^o : \mathfrak{M} \in \mathbf{MS}^*\} = \{\mathfrak{M}^o : \mathfrak{M} \in \mathbf{TS}\}.$$

If $\mathfrak{B} \in$ **CBL**, then $\mathfrak{B}^+_\sqsubset \in$ **MS** (resp. $\mathfrak{B}^+_\blacktriangleleft \in$ **TS**). By carrying out the operation $(\mathfrak{B}^+_\sqsubset)^o$ (resp. $(\mathfrak{B}^+_\blacktriangleleft)^o$), using the zero of the lattice $\mathfrak{B}$, we get back the lattice $\mathfrak{B}$. Conversely, if $\mathfrak{M} \in$ **MS***, then $\mathfrak{M}^o \in$ **CBL**.

Chapter IV

Equivalent axiomatisations of mereological structures

In the next five sections we shall be examining structures of the form $\mathfrak{M} = \langle M, \sqsubset \rangle$ with the primitive relation $\sqsubset$. By saying that the structure $\mathfrak{M}$ satisfies given conditions, the thought is that those conditions are satisfied by the relation $\sqsubset$ and potentially the relations $\sqsubseteq$, $\bigcirc$, $\wr$ and $\mathbb{S}\mathrm{um}$, which are defined with the help of the definitions (df $\sqsubseteq$), (df $\bigcirc$), (df $\wr$) and (df $\mathbb{S}\mathrm{um}$), respectively.

1. Other axiomatisations with the primitive relation $\sqsubset$

As we know from Lemma II.6.4, condition (L3) follows from conditions (L2) and (SSP). Proposition 1.3, to be given in this chapter, shows that condition (SSP) does not follow from conditions (L1)–(L3). Furthermore, as we have shown in Section II.4, condition ($S_{\mathbb{S}\mathrm{um}}$) follows from (L3) and by adding the irreflexivity of the relation $\sqsubset$ we get (WSP).

In Chapter V we shall show that (L3) does not follow from (L1), (L2) and (WSP) (resp. ($S_{\mathbb{S}\mathrm{um}}$)). The situation is different if we add condition (L4).

Lemma 1.1. *Condition* (L3) *follows from* (L2), (WSP) *and* (L4).

Proof. The transitivity of the relation $\sqsubseteq$ follows from (L2). Assume that (A) $x \;\mathbb{S}\mathrm{um}\; S$ and (B) $y \;\mathbb{S}\mathrm{um}\; S$. From this and (WSP) it follows that (C) $x \bigcirc y$ and (D) $y \not\sqsubset x$.

For (C): We have $\emptyset \neq S \subseteq \mathbb{I}(x) \cap \mathbb{I}(y)$. Therefore for some $z \in S$ we have $z \sqsubseteq x$ and $z \sqsubseteq y$. So $x \bigcirc y$.

For (D): Assume for a contradiction that $y \sqsubset x$. Then, by (WSP), for some u we have (a) $u \sqsubset x$ and (b) $u \wr y$. By virtue of (a) and (A) there is a z such that $z \bigcirc u$. Therefore for some w we have (c) $w \sqsubseteq z$ and (d) $w \sqsubseteq u$. On the other hand, (B) entails (e) $z \sqsubseteq y$. From (C) and (e) we have (f) $w \sqsubseteq y$. Therefore (d) and (f) yield $u \bigcirc y$, which contradicts (b).

Now assume for a contradiction that $y \sqsubset x$. Then, by (D), we obtain (D′) $y \not\sqsubseteq x$. Therefore, by virtue of (L4), for some s, we have (α) $s \operatorname{Sum} \{x, y\}$. Hence $y \sqsubseteq s$ and $x \sqsubseteq s$. From this and (D′) we have $x \neq s$, i.e., (β) $x \sqsubset s$. From this and (WSP) for some u we have (γ) $u \sqsubset s$ and (δ) $u \wr x$. From (α) and (γ): either $x \bigcirc u$ or $y \bigcirc u$. From this and (δ) it follows that (ε) $y \bigcirc u$. There exists therefore a w such that (ζ) $w \sqsubseteq y$ and (η) $w \sqsubseteq u$. From (B) and (ζ) for some $z \in S$ we have $z \bigcirc w$. And hence there exists a v such that (θ) $v \sqsubseteq z$ and (ι) $v \sqsubseteq w$. From (A) we have $z \sqsubseteq x$. From this and (θ) we have $v \sqsubseteq x$. From (ι) and (η) we have $v \sqsubseteq u$. Hence $x \bigcirc u$, which contradicts (δ). □

We have the following equivalent axiomatisations of the class **MS**:

THEOREM 1.2. *The nine groups of conditions below are equivalent*:

1° (L1)–(L4);
2° (L1), (L2), *and* (L3-L4);
3° (L1), (L2), ($\mathrm{ext}_\circ$), *and* (L4);
4° (L1), (L2), (SSP), *and* (L4);
5° (L1), (L2), (M1), *and* (L4);
6° (L1), (L2), (M2), *and* (L4);
7° (L1), (L2), (M3), *and* (L4);
8° (L1), (L2), ($\mathrm{S}_{\mathrm{Sum}}$), *and* (L4);
9° (L2), (WSP), *and* (L4).

PROOF. ‘1° ⇔ 2°’ See Remark II.5.3.

‘1° ⇔ 3°’ By Theorem II.4.4, condition ($\mathrm{ext}_\circ$) is equivalent to (L3) in all strict partial orders.

‘1° ⇔ 4°’ Theorem II.6.1 says that (SSP) follows from (L1)–(L4). By virtue of Lemma II.6.4, condition (L3) follows from (L2) and (SSP).

‘4° ⇔ 5° ⇔ 6° ⇔ 7°’ By Corollary II.6.3, conditions (SSP), (M1), (M2), and (M3) are equivalent in all transitive structures.

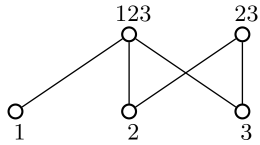
‘1° ⇒ 8°’ On p. 83 we showed that ($\mathrm{S}_{\mathrm{Sum}}$) follows from (L3).

‘8° ⇔ 9°’ By virtue of Lemma II.4.1(iv).

‘9° ⇒ 1°’ Condition (L1) follows from (WSP) and (L2), by virtue of Lemma II.4.1ii. The rest follows by virtue of Lemma 1.1. □

At the end of this section we will prove the lemma we will use in the next section and in Chapter V.

PROPOSITION 1.3. *Condition* (SSP) *does not follow from* (L1)–(L3).



Model 2. Conditions (L1)–(L3) hold, but (SSP) does not hold

Proof. In model 2, in which $\mathrm{M}_1 = \{1, 2, 3, 23, 123\}$ and the relation $\sqsubseteq$ is defined by the graph[1], conditions (L1)–(L3) hold, but condition (SSP) does not hold. In fact, we have $\mathbb{I}(23) \subseteq \mathbb{O}(123)$ and $23 \not\sqsubseteq 123$. □

2. A different definition of a collective set. The fusion of members of a set

In Chapter X of his "Foundations of Mathematics" [1931], Leśniewski adopts a different understanding of the concept of *collective class of certain objects*. In [1940], H. S. Leonard and N. Goodman capture this concept in the language of set theory as the relation *is a fusion of* all members of a given distributive set. In [Leonard and Goodman, 1940] this relation is signified by "$\mathbb{Fu}$".[2]

In [1931, Chapter X] Leśniewski adopts as the primitive relation the binary relation $\wr$ *is exterior with respect to*. This relation, included in $M \times M$, is also the primary concept of the system introduced in [Leonard and Goodman, 1940].[3] The definition ($\mathfrak{C}$) of the expression "P is a class of objects a [of as]" (cf. p. 47) from [Leśniewski, 1931, p. 142] may be written in the terminology of [Leonard and Goodman, 1940] in the manner given below, as the definition of the relation $\mathbb{Fu}$ included in $M \times \mathcal{P}(M)$:

$$x \mathbin{\mathbb{Fu}} S \;:\Longleftrightarrow \forall_{y \in M}(y \wr x \;\Leftrightarrow \forall_{z \in S}\, z \wr y). \tag{df $\mathbb{Fu}$}$$

In [Leśniewski, 1931, Chapter X] there are two axioms signified by '($\mathfrak{A}$)' and '($\mathfrak{B}$)'. Since they are to be written in the terminology of Leonard and Goodman [1940], we will swap these symbols for lower-case

[1] The matter of how to interpret these graphs was discussed in footnote 7 of Chapter III on p. 121.

[2] Leonard and Goodman [1940] the relation $\mathbb{Fu}$ also called the relation *is a sum-individual of*.

[3] [Leonard and Goodman, 1940] appeal to [Leśniewski, 1931] a number of times.

Gothic letters. Axiom ($\mathfrak{A}$) pertains to the relation $\wr$:[4]

$$\forall_{x,y\in M}\big(x \wr y \iff \forall_{z\in M}\exists_{u\in M}((u \wr x \vee u \wr y) \wedge \neg\, u \wr z)\big). \qquad (\mathfrak{a})$$

The symmetry and irreflexivity of the relation $\wr$ follow directly from ($\mathfrak{a}$). That is, we obtain ($\mathrm{s}_\wr$) and ($\mathrm{irr}_\wr$). In fact, $x \wr x$ entails a contradiction: $\exists_u(u \wr x \wedge \neg\, u \wr x)$.

Axiom ($\mathfrak{B}$) was written by Leśniewski with the help of a so-called functor variable[5]. In the terminology of [Leonard and Goodman, 1940], this axiom states that for an arbitrary non-empty subset of the set M there exists exactly one element in M, which is a fusion of all members of the subset:

$$\forall_{S\in\mathcal{P}_+(M)}\exists_{x\in M}\big(x \,\mathbb{F}\mathrm{u}\, S \wedge \forall_{y\in M}(y \,\mathbb{F}\mathrm{u}\, S \Rightarrow x = y)\big). \qquad (\mathfrak{b})$$

Of course, from ($\mathfrak{b}$) we have:

$$\forall_{S\in\mathcal{P}_+(M)}\exists_{x\in M}\; x \,\mathbb{F}\mathrm{u}\, S\,. \qquad (\exists\mathbb{F}\mathrm{u})$$

In [Leśniewski, 1931, Chapter 10 X] the relations $\sqsubseteq$ and $\sqsubset$ are respectively defined with the help of formulae ($\mathfrak{D}$) and ($\mathfrak{E}$). The first of these is written in the language of the set theory as follows:

$$x \sqsubseteq y \;:\Longleftrightarrow \exists_{S\in\mathcal{P}(M)}(y \,\mathbb{F}\mathrm{u}\, S \wedge x \in S), \qquad (\mathfrak{d})$$

and the second corresponds to formula ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$), which says that a part a part of an object is any its ingrediens which is distinct from the object (and vice versa). The relation $\sqsubset$ is therefore irreflexive.

From (df $\mathbb{F}\mathrm{u}$) and ($\mathrm{irr}_\wr$) it follows that

$$\neg\exists_{x\in M}\; x \,\mathbb{F}\mathrm{u}\, \emptyset\,. \qquad (2.1)$$

In fact, by (df $\mathbb{F}\mathrm{u}$), from $x \,\mathbb{F}\mathrm{u}\, \emptyset$ we have $\forall_y\, y \wr x$, and hence $x \wr x$, which contradicts ($\mathrm{irr}_\wr$).

Of course, from ($\mathfrak{b}$) and (2.1) we obtain that if a set has a fusion then it is unique:

$$\forall_{S\in\mathcal{P}(M)}\forall_{x,y\in M}(x \,\mathbb{F}\mathrm{u}\, S \wedge y \,\mathbb{F}\mathrm{u}\, S \implies x = y). \qquad (\mathrm{U}_{\mathbb{F}\mathrm{u}})$$

In fact, suppose that $x \,\mathbb{F}\mathrm{u}\, S$ and $y \,\mathbb{F}\mathrm{u}\, S$. Then $S \neq \emptyset$, by (2.1). Hence, in the light of ($\mathfrak{b}$), for some x_0 we have $x_0 \,\mathbb{F}\mathrm{u}\, S$ and $x = x_0 = y$.

[4] We will recognise its 'meaning', when we prove formula ($\mathrm{df}_\wr \sqsubseteq$) in the system currently under consideration.

[5] A tool derived from Leśniewski's "protothetic". Leśniewski built his mereology on top of this and his ontology.

For $S = \{x\}$, thanks to ($\mathrm{s}_{\wr}$), the left-hand side of (df $\mathbb{Fu}$) is transformed into a tautology. Hence for each $x \in M$ we have:

$$x \ \mathbb{Fu}\ \{x\}. \tag{2.2}$$

From this and ($\mathfrak{d}$) we have the reflexivity of the relation $\sqsubseteq$, i.e., ($\mathrm{r}_{\sqsubseteq}$).

Observe that from (df $\mathbb{Fu}$), ($\mathfrak{d}$) and ($\mathrm{s}_{\wr}$) for any $x, y \in M$ we have:

$$y \wr x \iff \forall_{z \in M}(z \sqsubseteq x \ \Rightarrow\ z \wr y). \tag{2.3}$$

In fact, let $y \wr x$ and $z \sqsubseteq x$. Then for some S_0 we have $x \ \mathbb{Fu}\ S_0$ and $z \in S_0$. Hence $z \wr y$. Conversely, let $\forall_{z \in M}(z \sqsubseteq x \Rightarrow z \wr y)$. Since $x \sqsubseteq x$, then $x \wr y$.

Directly from (2.3) and (df $\mathbb{Fu}$) for each $x \in M$ we have:

$$x \ \mathbb{Fu}\ \mathbb{I}(x)\,. \tag{2.4}$$

By applying (2.3) we can also prove the transitivity and antisymmetry of the relation $\sqsubseteq$ in Leśniewski's system along with the fact that formula ($\mathrm{df}_{\wr} \sqsubseteq$) holds.

For ($\mathrm{t}_{\sqsubseteq}$): Assume that $x \sqsubseteq y$ and $y \sqsubseteq z$. Then, by virtue of ($\mathfrak{d}$), for some sets S_1 and S_2 we have (a) $y \ \mathbb{Fu}\ S_1$, (b) $x \in S_1$, (c) $z \ \mathbb{Fu}\ S_2$ and $y \in S_2$. It suffices to show that (@) $z \ \mathbb{Fu}\ (S_1 \cup S_2)$. Then, by virtue of ($\mathfrak{d}$) and (b), we have $x \sqsubseteq z$. For the proof of (@) we show that the appropriate *definiens* in (df $\mathbb{Fu}$) is satisfied. Pick an arbitrary $u \in M$. Assume that (d) $u \wr z$. Since $y \sqsubseteq z$, then we have (e) $y \wr u$, by virtue of the '$\Rightarrow$'-part in (2.3). Pick an arbitrary $w \in S_1 \cup S_2$. Then, by virtue of (a), (c), (df $\mathbb{Fu}$) and either (e) or (d), we have $w \wr u$. Conversely, assume that for each $w \in S_1 \cup S_2$ we have $w \wr u$, that is, that also for each $w \in S_2$ we have $w \wr u$. From this and (c) we have $u \wr z$, by virtue of (df $\mathbb{Fu}$).

For ($\mathrm{antis}_{\sqsubseteq}$): Assume that $x \sqsubseteq y$ and $y \sqsubseteq x$. We observe that by virtue of (df $\mathbb{Fu}$) we have: $y \ \mathbb{Fu}\ \{x\}$ iff $\forall_z(z \wr y \Leftrightarrow x \wr z)$. The right-hand side of this equivalence is satisfied, by virtue of our assumptions, (2.3), and ($\mathrm{s}_{\wr}$). Therefore $y \ \mathbb{Fu}\ \{x\}$. Moreover, we have $x \ \mathbb{Fu}\ \{x\}$. Thus, by virtue of ($\mathrm{U}_{\mathbb{Fu}}$), we have $x = y$.

Since the relation $\sqsubseteq$ is reflexive, transitive and antisymmetric, the relation $\sqsubset$ is asymmetric and transitive, i.e., conditions (L1) and (L2) hold. Furthermore, since the relations $\sqsubset$ and $\sqsubseteq$ satisfy condition ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$), then there holds between them the connection expressed by formula (df $\sqsubseteq$).[6]

[6] Cf. Lemma 2.4 in Appendix I.

For ($\mathrm{df}_{\wr}\sqsubseteq$): Assume that $x \sqsubseteq y$ and $z \wr y$. Then, by virtue of (2.3) and ($\mathrm{s}_{\wr}$), we have $z \wr x$. Conversely, let $\forall_z(z \wr y \Rightarrow z \wr x)$. Then $\forall_z(z \wr y \Leftrightarrow (z \wr x \wedge z \wr y))$. Hence, by the exclusive application of (df $\mathbb{F}\mathrm{u}$), we have $y \,\mathbb{F}\mathrm{u}\, \{x, y\}$. Now, applying ($\mathfrak{d}$), we get: $x \sqsubseteq y$.

Having proved formula ($\mathrm{df}_{\wr}\sqsubseteq$), axiom ($\mathfrak{a}$) can be reduced to formula (df$\wr$).[7] The following lemma shows this.

Lemma 2.1. *If the relations $\wr$ and $\sqsubseteq$ satisfy condition* ($\mathrm{df}_{\wr}\sqsubseteq$)*, then conditions* ($\mathfrak{a}$) *and* (df$\wr$) *are equivalent.*

Proof. We subject ($\mathfrak{a}$) to 'value-preserving' changes, in which we apply — besides the logical rules — just condition ($\mathrm{df}_{\wr}\sqsubseteq$):

$$\begin{aligned} x \wr y &\iff \neg\exists_{z\in M}\forall_{u\in M}\big((u \wr x \vee u \wr y) \Rightarrow u \wr z\big), \\ &\iff \neg\exists_{z\in M}\big(\forall_{u\in M}(u \wr x \Rightarrow u \wr z) \wedge \forall_{u\in M}(u \wr y \Rightarrow u \wr z)\big), \\ &\iff \neg\exists_{z\in M}(z \sqsubseteq x \wedge z \sqsubseteq y). \end{aligned}$$ □

Following Leśniewski [1931, Chapter X], it may be proved that the system above with the primitive relation $\wr$ is definitionally equivalent to the system with the primitive relation $\sqsubset$ and axioms (L1)–(L4). This means that the following two results hold: theorems 2.2 and 3.2 (cf. also theorems 6.4 and 6.5):

Theorem 2.2. *Let M be a non-empty set and $\wr$ be a binary relation in M. Moreover, in structure $\langle M, \wr\rangle$ we define relations $\mathbb{F}\mathrm{u}$, $\sqsubseteq$, $\sqsubset$, and $\mathbb{S}\mathrm{um}$ by the definitions* (df $\mathbb{F}\mathrm{u}$), ($\mathfrak{d}$), ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$)*, and* (df $\mathbb{S}\mathrm{um}$)*, respectively. Suppose that $\langle M, \wr\rangle$ satisfies conditions* ($\mathfrak{a}$) *and* ($\mathfrak{b}$)*. Then $\mathbb{F}\mathrm{u} = \mathbb{S}\mathrm{um}$ and formulae* (L1)–(L4), (df $\sqsubseteq$)*, and* (df$\wr$) *hold in $\langle M, \wr\rangle$.*

Proof. We proved that formulae (L1), (L2), (df $\sqsubseteq$), (df$\wr$) ($\mathrm{t}_{\sqsubseteq}$), ($\mathrm{df}_{\wr}\sqsubseteq$), (2.3), and (df$\wr$) hold in $\langle M, \wr\rangle$. Now we prove $\mathbb{F}\mathrm{u} = \mathbb{S}\mathrm{um}$.

Suppose that $x \,\mathbb{F}\mathrm{u}\, S$. Then, by (df $\mathbb{F}\mathrm{u}$), for any $y \in M$ we have: $y \wr x$ iff $\forall_{z\in S}\, y \wr z$. First, assume for a contradiction that for some $z_0 \in S$ we have $z_0 \not\sqsubseteq x$. Then, by (2.3), for some u_0 we have: $u_0 \wr x$ and for some

[7] Formula (df$\wr$) is clearly not the definition of the relation $\wr$ in the system under consideration (in this system, the relation $\wr$ is a primitive concept). Condition (df$\wr$) shows only the connection between relations $\wr$ and $\sqsubseteq$ (the second defined by condition ($\mathfrak{d}$)). It is just this connection which axiom ($\mathfrak{a}$) determines.

Leśniewski did not consider the relation $\circ$. He simply writes that some ingrediens of one object is an ingrediens of a second. If this connection is expressed by means of the relation $\circ$, then the definition will be adopted for it. We will thereby also obtain condition ($\wr = -\circ$).

w_0 we have $w_0 \sqsubseteq z_0$ and $w_0 \sqsubseteq u_0$. Since for any $z \in S$ we have $u_0 \wr z$, so we obtain a contradiction. Thus, we have shown that $\forall_{z\in S}\, z \sqsubseteq x$. Now pick any $y \in M$ such that $y \sqsubseteq x$. Assume for a contradiction that for any $z \in S$ there is no $u \in M$ such that $u \sqsubseteq y$ and $u \sqsubseteq z$, i.e., $y \wr z$. Then, we obtain a contradiction: $y \wr x$. Thus, we have shown that $x \mathbb{Sum}\, S$.

Suppose that $x \mathbb{Sum}\, S$. Then, by (df $\mathbb{Sum}$), we have $\forall_{z\in S}\, z \sqsubseteq x$ and for any $y \in M$ such that $y \sqsubseteq x$ for some $z \in S$ and $u \in M$ we have: $u \sqsubseteq y$ and $u \sqsubseteq z$. Pick any $y \in M$ such that $y \wr x$ and any $z \in S$. Then $z \sqsubseteq x$ and so $z \wr y$, by (2.3). Conversely, assume for a contradiction that $\forall_{z\in S}\, y \wr z$ and for some $v_0 \in M$ we have $v_0 \sqsubseteq x$ and $v_0 \sqsubseteq y$. Then for some $z_0 \in S$ and $u_0 \in M$ we have: $u_0 \sqsubseteq v_0$ and $u_0 \sqsubseteq z_0$. Moreover, by ($\mathrm{t}_{\sqsubseteq}$), we obtain a contradiction: $u_0 \sqsubseteq y$ and $u_0 \sqsubseteq z_0$. Thus, we have shown that $x \mathbb{Fu}\, S$.

Since $\mathbb{Fu} = \mathbb{Sum}$, then from ($\mathrm{U}_{\mathbb{Fu}}$) and ($\mathfrak{b}$) we obtain (L3) and (L4), respectively. □

Since the system of Leśniewski presented here is close the system of Leonard and Goodman in [1940], we shall therefore present the latter in the following remark.

Remark 2.1. Leonard and Goodman [1940] take as primitive the relation $\wr$ *is exterior to*. The relation *is a fusion of* they define by condition (df $\mathbb{Fu}$). The first axiom of their system states the existence of a fusion for an arbitrary non-empty subset of the universe [I.1 in Leonard and Goodman, 1940], i.e., this axiom is condition ($\exists\mathbb{Fu}$).

In [Leonard and Goodman, 1940], the relation $\sqsubseteq$ *is an ingrediens of* is defined by ($\mathrm{df}_{\wr} \sqsubseteq$).[8] Therefore, by virtue of the definition itself, the relation $\sqsubseteq$ in Goodman and Leonard's system is transitive and reflexive [theses I.3 and I.31 in Leonard and Goodman, 1940; our ($\mathrm{t}_{\sqsubseteq}$) and ($\mathrm{r}_{\sqsubseteq}$)]. Its antisymmetry (our ($\mathrm{antis}_{\sqsubseteq}$)) is established in Leonard and Goodman [1940] by axiom I.12. The relation $\sqsubset$ *is a part of* is defined by condition ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$). In Leonard and Goodman's system, therefore, the relation $\sqsubset$ is irreflexive, asymmetric and transitive [theses I.325, I.326 and I.328 in Leonard and Goodman, 1940]. The relation $\bigcirc$ *overlapping* is defined in [Leonard and Goodman, 1940] with the help of condition (df $\bigcirc$), i.e., it is symmetric and reflexive. The final axiom in [Leonard and Goodman, 1940] is the equivalence: $x \bigcirc y \Leftrightarrow \neg\, x \wr y$ [I.13 in Leonard and Goodman,

[8] This explains in a certain sense the 'strength' of Leonard and Goodman's theory, because formula ($\mathrm{df}_{\wr} \sqsubseteq$) along with the other axioms of the theory have the 'power' of formula (SSP) (see below).

1940; the counterpart to our ($\wr = -\bigcirc$)]. Thus, the relation $\wr$ is symmetric and irreflexive.

In Leonard and Goodman's system, axioms ($\mathfrak{a}$) and ($\mathfrak{b}$) adopted by Leśniewski become theorems.[9] In Corollary 6.3 we will show that their system is definitionally equivalent to Leśniewski's system in [Leśniewski, 1931, Chapter X] with the relation $\wr$ as primitive. Furthermore, in theorems 6.4 and 6.5 we will prove that in both these systems the definitions (df $\mathbb{F}\mathrm{u}$) and (df $\mathbb{S}\mathrm{um}$) define the same relation and are definitionally equivalent to a system with $\sqsubset$ as the primary relation and axioms (L1)–(L4).

Leonard and Goodman, besides the relation $\mathbb{F}\mathrm{u}$, consider also a second relation included in $M \times \mathcal{P}(M)$. This is the relation *is a product* (*product-individual*) *of* all elements of a given distributive set. This relation is signified by the symbol "$\mathbb{N}\mathrm{u}$" (*nucleus*) and its definition is as follows:

$$x \,\mathbb{N}\mathrm{u}\, S \iff \forall_{y \in M}(y \sqsubseteq x \Leftrightarrow \forall_{z \in S}\, y \sqsubseteq z). \tag{df $\mathbb{N}\mathrm{u}$}$$

It follows from the axioms and definitions we have adopted that the relation $\sqsubseteq$ partially orders the set M and the right-hand side of (df $\mathbb{N}\mathrm{u}$) is simply the right-hand side of condition (4.15) in Appendix I written for the relation $\sqsubseteq$. Thus:

$$\mathbb{N}\mathrm{u} = \mathsf{inf}_{\sqsubseteq}. \tag{2.5}$$

For theses I.56–I.58 in [Leonard and Goodman, 1940], for an arbitrary $S \in \mathcal{P}(M)$ we have respectively (cf. Corollary 3.4):

$$\bigcap \mathbb{I}(S) \neq \emptyset \implies \exists_{x \in M}\, x \,\mathbb{N}\mathrm{u}\, S\,, \tag{$\exists\mathbb{N}\mathrm{u}$}$$

$$\forall_{x,y \in M}(x \,\mathbb{N}\mathrm{u}\, S \wedge y \,\mathbb{N}\mathrm{u}\, S \Rightarrow x = y), \tag{$\mathrm{U}_{\mathbb{N}\mathrm{u}}$}$$

$$\bigcap \mathbb{I}(S) \neq \emptyset \implies \exists_{x \in M}(x \,\mathbb{N}\mathrm{u}\, S \wedge \forall_{y \in M}(y \,\mathbb{N}\mathrm{u}\, S \Rightarrow x = y)). \tag{$\exists!\mathbb{N}\mathrm{u}$}$$

3. Fusion in place of sum

Pick an arbitrary structure $\mathfrak{M} = \langle M, \sqsubset \rangle$ in which the relations $\sqsubseteq$, $\bigcirc$, $\wr$, $\mathbb{S}\mathrm{um}$, and $\mathbb{F}\mathrm{u}$ are defined by (df $\sqsubseteq$), (df $\bigcirc$), (df $\wr$), (df $\mathbb{S}\mathrm{um}$) and (df $\mathbb{F}\mathrm{u}$), respectively. By virtue of (df $\sqsubseteq$) we have ($\mathrm{r}_{\sqsubseteq}$). By virtue of (df $\bigcirc$) and (df $\wr$) we have ($\mathrm{r}_{\circ}$), ($\mathrm{s}_{\circ}$), ($\mathrm{irr}_{\wr}$), ($\mathrm{s}_{\wr}$), and ($\wr = -\bigcirc$). Therefore (df $\mathbb{F}\mathrm{u}$) may be reduced to the following form:

$$\begin{aligned} &x \,\mathbb{F}\mathrm{u}\, S \iff \forall_{y \in M}(y \bigcirc x \Leftrightarrow \exists_{z \in S}\, z \bigcirc y), \qquad \text{or} \\ &x \,\mathbb{F}\mathrm{u}\, S \iff \mathbb{O}(x) = \bigcup \mathbb{O}(S). \end{aligned} \tag{df$'$ $\mathbb{F}\mathrm{u}$}$$

---

[9] For ($\mathfrak{a}$) this follows from definitions I.01 and I.02 along with axiom I.13; for ($\mathfrak{b}$) this is just Thesis I.53.

Since for each $x \in M$ we have $\mathbb{O}(x) \neq \emptyset = \bigcup \mathbb{O}(\emptyset)$, then we have (2.1). Furthermore, since for $x \in M$ we have $\mathbb{O}(x) = \mathbb{O}(\{x\})$, then (2.2) holds. If the relation $\sqsubseteq$ is reflexive and transitive, then $\mathbb{O}(x) = \bigcup \mathbb{O}(\mathbb{I}(x))$ and we also have (2.4).[10]

THEOREM 3.1. *Suppose that a relation $\sqsubset$ is transitive in a non-empty set M, i.e., $\sqsubset$ satisfies condition* (L2)*. Then*:

(i) $\mathbb{S}\mathrm{um} \subseteq \mathbb{F}\mathrm{u}$.
(ii) *Condition* (SSP) *is equivalent to the inclusion* $\mathbb{F}\mathrm{u} \subseteq \mathbb{S}\mathrm{um}$.[11]

PROOF. Suppose that a relation $\sqsubset$ is transitive in a non-empty set M.

Ad (i): The inclusion $\mathbb{S}\mathrm{um} \subseteq \mathbb{F}\mathrm{u}$ follows from the transitivity of $\sqsubseteq$ and our definitions, so only by (L2) and our definitions; see the proof of (II.3.9).

Ad (ii): Assume (SSP). Let $x\ \mathbb{F}\mathrm{u}\ S$, i.e., let $(*)$: $\mathbb{O}(x) = \bigcup \mathbb{O}(S)$. From (L2) and (df $\sqsubseteq$) it follows that ($\mathrm{r}_{\sqsubseteq}$) and ($\mathrm{t}_{\sqsubseteq}$). From this and (SSP) we have ($\mathrm{df}_{\circ} \sqsubseteq$). From $(*)$ we obtain: $\forall_{y \in M}(\exists_{z \in S}\, z \bigcirc y \Rightarrow y \bigcirc x)$, which is equivalent to $\forall_{z \in S} \forall_{y \in M}(y \bigcirc z \Rightarrow y \bigcirc x)$. From this and ($\mathrm{df}_{\circ} \sqsubseteq$) we have $(**)$: $\forall_{z \in S}\, z \sqsubseteq x$. Furthermore, from $(*)$ we obtain $\forall_{y \in M}(y \bigcirc x \Rightarrow \exists_{z \in S}\, z \bigcirc y)$. From this and ($\sqsubseteq\subseteq\circ$) we have: $\forall_{y \in M}(y \sqsubseteq x \Rightarrow \exists_{z \in S}\, z \bigcirc y)$. From this and $(**)$, by applying (II.df′ $\mathbb{S}\mathrm{um}$), we get: $x\ \mathbb{S}\mathrm{um}\ S$. Thus, $\mathbb{F}\mathrm{u} \subseteq \mathbb{S}\mathrm{um}$.[12]

Conversely, let $\mathbb{F}\mathrm{u} \subseteq \mathbb{S}\mathrm{um}$. For the proof of (SSP) we assume the inclusion $\mathbb{I}(x) \subseteq \mathbb{O}(y)$. Then, by virtue of (II.2.6), we have $\mathbb{O}(x) \subseteq \mathbb{O}(y)$, i.e., $\mathbb{O}(y) = \mathbb{O}(x) \cup \mathbb{O}(y)$. Hence $y\ \mathbb{F}\mathrm{u}\ \{x, y\}$. Therefore $y\ \mathbb{S}\mathrm{um}\ \{x, y\}$ as well. And from this it follows that $x \sqsubseteq y$. □

As a conclusion, we obtain the following theorem:

THEOREM 3.2. *In all mereological structures*:

(i) $\mathbb{S}\mathrm{um} = \mathbb{F}\mathrm{u}$.
(ii) *Formulae* ($\mathfrak{a}$), ($\mathfrak{b}$), *and* ($\mathfrak{d}$) *hold.*[13]

---

[10] About the relation $\mathbb{F}\mathrm{u}$ see also [Gruszczyński and Pietruszczak, 2010, sect. 5].

[11] This is the reverse of implication (II.3.9) mentioned in footnote II.15.

[12] The equality $\mathbb{S}\mathrm{um} = \mathbb{F}\mathrm{u}$ also holds in Leonard and Goodman's system presented in Remark 2.1. In essence, in this system the relation $\sqsubset$ satisfies conditions (L1) and (L2). Furthermore, conditions ($\sqsubseteq\subseteq\circ$), ($\mathrm{df}_{\wr} \sqsubseteq$), ($\mathrm{df}\,\wr$), and ($\mathrm{df}_{\circ} \sqsubseteq$) hold, these being used in the proof (in [Leonard and Goodman, 1940] the first is thesis I.332 and the second follows from axiom I.12 and the definition of the relation $\sqsubseteq$).

[13] Formula ($\mathfrak{b}$) is not the definition of the relation $\sqsubseteq$ in the system of the class **MS** of mereological structures (in this system, the relation $\sqsubseteq$ is defined by (df $\sqsubseteq$)).

Proof. *Ad* (i): By virtue of Theorem II.6.1, in mereological structures, condition (SSP) holds. We may therefore apply Theorem 3.1.

Ad (ii): By (i), $\mathbb{Fu} = \mathbb{Sum}$. So from (L3-L4) we have ($\mathfrak{b}$). Moreover, let $y \sqsubseteq x$. Since $y \mathbb{Sum} \mathbb{I}(y)$, then for $S := \mathbb{I}(y)$ we have: $y \mathbb{Sum} S$ and $x \in S$. Conversely, if $y \mathbb{Sum} S$ and $x \in S$, then $x \sqsubseteq y$, by (df $\mathbb{Sum}$). Thus, we have shown that ($\mathfrak{d}$) holds in all mereological structures.

Finally, by Lemma 2.1, since (df$_\wr$ $\sqsubseteq$) and (df$\wr$) hold in all mereological structures, then ($\mathfrak{a}$) also holds. □

Corollary 3.3. *Let $\langle M, \sqsubset\rangle$ satisfy conditions* (L2) *and* (SSP)*. Then conditions* (L4) *and* ($\exists\mathbb{Fu}$) *are equivalent, and* (L3-L4) *and also conditions* ($\mathfrak{b}$) *are equivalent.*

Proof. If (L2) and (SSP) hold in $\langle M, \sqsubset\rangle$, then $\mathbb{Sum} = \mathbb{Fu}$, by virtue of Theorem 3.1(i). Thus, ($\exists\mathbb{Fu}$) and ($\mathfrak{b}$) arise from (L4) and (L3-L4), respectively, when "$\mathbb{Sum}$" is exchanged for "$\mathbb{Fu}$", and vice versa. □

Corollary 3.4. *Conditions* ($\exists\mathbb{Nu}$), ($\mathrm{U}_{\mathbb{Nu}}$) *and* ($\exists!\mathbb{Nu}$)

(i) *follow from conditions* (L2), (SSP), *and* ($\exists\mathbb{Fu}$);
(ii) *hold in all mereological structures.*

Proof. *Ad* (i): Firstly, from (2.5) and (4.10) in Appendix I we have: $x \mathbb{Nu} S$ iff $x \mathsf{inf}_{\sqsubseteq} S$ iff $x \mathsf{sup}_{\sqsubseteq} \bigcap \mathbb{I}(S)$. If conditions (L2) and (SSP) hold, then $\mathbb{Fu} = \mathbb{Sum}$, and so conditions ($\exists\mathbb{Fu}$) and (L4) say the same thing. Hence ($\exists\mathbb{Fu}$) and (SSP) entail ($\mathbb{Sum}$-$\mathsf{sup}_{\sqsubseteq}$). Hence if $\bigcap \mathbb{I}(S) \neq \emptyset$ then: $x \mathsf{sup}_{\sqsubseteq} \bigcap \mathbb{I}(S)$ iff $x \mathbb{Sum} \bigcap \mathbb{I}(S)$ iff $x \mathbb{Fu} \bigcap \mathbb{I}(S)$, i.e.: $x \mathbb{Nu} S$ iff $x \mathbb{Fu} \bigcap \mathbb{I}(S)$. Thus, we obtain ($\exists\mathbb{Nu}$) from ($\exists\mathbb{Fu}$).

Condition ($\mathrm{U}_{\mathbb{Nu}}$) follows from (2.5) and from the fact that the relation $\mathsf{inf}_{\sqsubseteq}$ is a function of the second argument. Condition ($\exists!\mathbb{Nu}$) follows from ($\exists\mathbb{Nu}$) and ($\mathrm{U}_{\mathbb{Nu}}$).

Ad (ii): Because conditions (L2), (SSP), and ($\exists\mathbb{Fu}$) hold in all mereological structures. □

Directly from (df $\mathbb{Nu}$) and (df $\sqsubseteq$) for an arbitrary $S \in \mathcal{P}(M)$ follows the converse implication to ($\exists\mathbb{Nu}$):

$$\exists_{x \in M}\, x \mathbb{Nu} S \implies \bigcap \mathbb{I}(S) \neq \emptyset\,. \tag{3.1}$$

Condition $x \mathbb{Nu} S$ entails: $\exists_{y \in M}\, y \sqsubseteq x$ iff $\exists_{y \in M} \forall_{z \in S}\, y \sqsubseteq z$ iff $\bigcap \mathbb{I}(S) \neq \emptyset$. Since $x \sqsubseteq x$, then the left-hand side is satisfied.

Proposition 3.5. *The inclusion $\mathbb{Fu} \subseteq \mathbb{Sum}$ cannot be derived from conditions* (L1)–(L3).

PROOF. This follows from Proposition 1.3 and Theorem 3.1(ii). It is also visible from model 1, in which conditions (L1)–(L3) hold, but $\mathbb{F}\mathrm{u} \not\subseteq \mathbb{S}\mathrm{um}$. In fact, we since $\mathbb{O}(123) = \mathrm{M}_1 = \bigcup \mathbb{O}(\mathrm{M}_1)$, then $123 \; \mathbb{F}\mathrm{u} \; \mathrm{M}_1$. But it is not the case that $123 \; \mathbb{S}\mathrm{um} \; \mathrm{M}_1$. □

LEMMA 3.6. *For any transitive structure* $\mathfrak{M} = \langle M, \sqsubset \rangle$, *conditions* (L3) *and* ($\mathrm{U}_{\mathbb{F}\mathrm{u}}$) *are equivalent.*

PROOF. Let $\mathfrak{M} = \langle M, \sqsubset \rangle$ satisfy condition (L2).

Assume that (L3) holds and that $x \; \mathbb{F}\mathrm{u} \; S$ and $y \; \mathbb{F}\mathrm{u} \; S$. Then, by virtue of (df′ $\mathbb{F}\mathrm{u}$), we have $\mathbb{O}(x) = \bigcup \mathbb{O}(S) = \mathbb{O}(y)$. Hence $x = y$, by virtue of condition ($\mathrm{ext}_\circ$) which — by virtue of Theorem II.4.4 — is equivalent in class **L12** to condition (L3).

Assume that ($\mathrm{U}_{\mathbb{F}\mathrm{u}}$) holds and that $x \; \mathbb{S}\mathrm{um} \; S$ and $y \; \mathbb{S}\mathrm{um} \; S$. Then — by virtue of Theorem 3.1(i) — we have $x \; \mathbb{F}\mathrm{u}$ and $y \; \mathbb{F}\mathrm{u} \; S$. Hence $x = y$. □

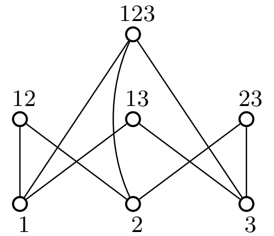
Corollary 3.3 and Lemma 3.6 show that by applying conditions ($\exists\mathbb{F}\mathrm{u}$) and ($\mathrm{U}_{\mathbb{F}\mathrm{u}}$) we may get two further equivalent axiomatisations of mereological structures. In the group 1° in Theorem 1.2, formula (L3) may be exchanged in an equivalent way for formula ($\mathrm{U}_{\mathbb{F}\mathrm{u}}$). Moreover, in the group 4°–7° formula (L4) may be exchanged for formula ($\exists\mathbb{F}\mathrm{u}$). We shall further prove that it is not possible to carry out this second change in the groups 1°, 3°, 8° and 9°. Finally, with regards to groups 1°–9°, we will show that it is not possible to exchange the last two formulae in them for the formulae ($\mathrm{U}_{\mathbb{F}\mathrm{u}}$) and ($\exists\mathbb{F}\mathrm{u}$).

THEOREM 3.7. *The nine groups from Theorem 1.2 are also equivalent to the five groups of conditions below*:

10° (L1), (L2), ($\mathrm{U}_{\mathbb{F}\mathrm{u}}$), *and* (L4);
11° (L1), (L2), (SSP), *and* ($\exists\mathbb{F}\mathrm{u}$);
12° (L1), (L2), (M1), *and* ($\exists\mathbb{F}\mathrm{u}$);
13° (L1), (L2), (M2), *and* ($\exists\mathbb{F}\mathrm{u}$).
14° (L1), (L2), (M3), *and* ($\exists\mathbb{F}\mathrm{u}$).

PROOF. '1° ⇔ 10°' By Lemma 3.6, conditions (L3) and ($\mathrm{U}_{\mathbb{F}\mathrm{u}}$) are equivalent in all transitive structures.

'4° ⇔ 11° ⇔ 12° ⇔ 13° ⇔ 14°' By Corollary 3.3, conditions (L4) and ($\exists\mathbb{F}\mathrm{u}$) are equivalent in all structures satisfying conditions (L2) and (SSP). Moreover, by Corollary II.6.3, conditions (SSP), (M1), (M2), and (M3) are equivalent in all transitive structures. □



Model 3. Conditions (L1)–(L3), $(\exists\mathbb{Fu})$ hold, but (L4) does not hold

Directly by virtue of Lemma 3.6 and Theorem II.4.4, respectively, we obtain:

PROPOSITION 3.8. *The three sets* {(L2), (L3)}, {(L2), $(\mathrm{U}_{\mathbb{Fu}})$}, *and* {(L2), $(\mathrm{ext}_{\circ})$} *are equivalent. So the three sets* {(L1), (L2), (L3), $(\exists\mathbb{Fu})$}, {(L1), (L2), $(\mathrm{ext}_{\circ})$, $(\exists\mathbb{Fu})$}, *and* {(L1), (L2), $(\mathrm{U}_{\mathbb{Fu}})$, $(\exists\mathbb{Fu})$} *are equivalent, too.*

We shall prove below that

PROPOSITION 3.9. (i) *By putting condition* $(\exists\mathbb{Fu})$ *in place of condition* (L4) *in the equivalent groups* 1°, 3°, *and* 10° *we obtain groups which are too weak for the axiomatisation of mereological structures.*

(ii) *So, by putting conditions* $(\mathrm{U}_{\mathbb{Fu}})$ *and* $(\exists\mathbb{Fu})$ *in place of conditions* (L3) *and* (L4) *in the group* 1° *we obtain a group which is too weak for the axiomatisation of mereological structures.*

(iii) *So, by putting condition* $(\mathfrak{b})$ *in place of condition* (L3-L4) *in the group* 2° *we obtain a group which is too weak for the axiomatisation of mereological structures.*

PROOF. *Ad* (i): First, by virtue of theorems 3.2, 1.2, and 3.7, formula $(\exists\mathbb{Fu})$ follows from each of equivalent groups 1°, 3°, and 10°.

Second, we prove that the equivalent sets {(L1), (L2), (L3), $(\exists\mathbb{Fu})$}, {(L1), (L2), $(\mathrm{ext}_{\circ})$, $(\exists\mathbb{Fu})$}, and {(L1), (L2), $(\mathrm{U}_{\mathbb{Fu}})$, $(\exists\mathbb{Fu})$} from Proposition 3.8 are essentially weaker than each of equivalent groups 1°, 3°, and 10°. We show that (L4) does not follow from the set {(L1), (L2), (L3), $(\exists\mathbb{Fu})$}. In fact, (L1), (L2), $(\mathrm{ext}_{\circ})$, and $(\exists\mathbb{Fu})$ hold in model 3, but (L4) does not hold. On p. 122 we presented a single (with respect to isomorphism) seven-element structure from **MS**. Thus, model 3 is not a model of mereology. So (L4) does not hold in this model.

Let $\mathrm{M}_2 := \{1, 2, 3, 12, 13, 23, 123\}$ be the universe of model 3. In model 3 conditions (L1) and (L2) hold. Condition ($\mathrm{ext}_\circ$) also holds. In fact, $\mathbb{O}(1) = \{1, 12, 13, 123\}$, $\mathbb{O}(2) = \{2, 12, 23, 123\}$, $\mathbb{O}(3) = \{3, 13, 23, 123\}$, $\mathbb{O}(12) = \mathrm{M}_2 \setminus \{3\}$, $\mathbb{O}(13) = \mathrm{M}_2 \setminus \{2\}$, $\mathbb{O}(23) = \mathrm{M}_2 \setminus \{1\}$, $\mathbb{O}(123) = \mathrm{M}_2$. Moreover, the equalities given above show that for any at least a two-element subset S of M_2, we obtain:

$$\begin{aligned} 12 \;\mathbb{Fu}\; S &\iff 3 \notin \bigcup \mathbb{O}(S),\\ 13 \;\mathbb{Fu}\; S &\iff 2 \notin \bigcup \mathbb{O}(S),\\ 23 \;\mathbb{Fu}\; S &\iff 1 \notin \bigcup \mathbb{O}(S),\\ 123 \;\mathbb{Fu}\; S &\iff \bigcup \mathbb{O}(S) = \mathrm{M}_2. \end{aligned}$$

From this and (2.2) it follows that sentence ($\exists\mathbb{Fu}$) holds in model 3.

Note that it is obvious that (L4) is false in model 3. In this model, there is no unity and therefore: $\neg\exists_{x\in\mathrm{M}_2}\, x \;\mathbb{Sum}\; \mathrm{M}_2$. Moreover, any two-element subset of {12,13, 23} does not also have a sum.

Ad (ii): Directly by (i).

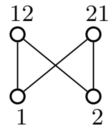
Ad (iii): By (i) Proposition 3.8, conditions ($\mathrm{U}_{\mathbb{Fu}}$) and ($\exists\mathbb{Fu}$) hold in model 3. Therefore, condition ($\mathfrak{b}$) holds, too. But (L3-L4) does not hold in model 3, since (L4) follows from (L3-L4). □

It is clear that it follows from the above proposition, that putting condition ($\exists\mathbb{Fu}$) in place of condition (L4) in the groups 8° and 9° in Theorem 1.2, we get groups of conditions which are also too weak for the axiomatisation of mereological structures. We will show below that, having made the above change, we obtain even weaker sets than those from Proposition 3.8, i.e., those sets obtained after the changes made in Proposition 3.9(i). However, we first notice that by Lemma II.4.1(iv) we obtain the following fact.

Proposition 3.10. *The sets* {(L1), (L2), ($\mathrm{S}_{\mathbb{Sum}}$)} *and* {(L2), (WSP)} *are equivalent. So the sets* {(L1), (L2), ($\mathrm{S}_{\mathbb{Sum}}$), ($\exists\mathbb{Fu}$)} *and* {(L2), (WSP), ($\exists\mathbb{Fu}$)} *are equivalent too.*

Proposition 3.11. *The two sets* {(L1), (L2), ($\mathrm{S}_{\mathbb{Sum}}$), ($\exists\mathbb{Fu}$)} *and* {(L2), (WSP), ($\exists\mathbb{Fu}$)} *are essentially weaker than each of the three equivalent sets* {(L1), (L2), (L3), ($\exists\mathbb{Fu}$)}, {(L1), (L2), ($\mathrm{ext}_\circ$), ($\exists\mathbb{Fu}$)}, *and* {(L1), (L2), ($\mathrm{U}_{\mathbb{Fu}}$), ($\exists\mathbb{Fu}$)}.

Proof. First, formulae (L1)–(L3) entail ($\mathrm{S}_{\mathbb{Sum}}$) and (WSP) (see p. 83 and Lemma II.4.1(v)).



Model 4. Conditions (L1), (L2), (WSP), ($S_{\mathbb{Sum}}$), ($\exists\mathbb{Fu}$) hold, but (L3), ($\mathrm{ext}_{\circ}$), ($U_{\mathbb{Fu}}$), ($\exists\sqcap$), ($\mathrm{ext}_{\sqsubset}$) do not hold

Second, we prove the two sets $\{$(L1), (L2), ($S_{\mathbb{Sum}}$), ($\exists\mathbb{Fu}$)$\}$ and $\{$(L2), (WSP), ($\exists\mathbb{Fu}$)$\}$ are essentially weaker than each of the three equivalent sets $\{$(L1), (L2), (L3), ($\exists\mathbb{Fu}$)$\}$, $\{$(L1), (L2), ($\mathrm{ext}_{\circ}$), ($\exists\mathbb{Fu}$)$\}$, and $\{$(L1), (L2), ($U_{\mathbb{Fu}}$), ($\exists\mathbb{Fu}$)$\}$. We show that (L3) does not follow from the set $\{$(L1), (L2), (WSP), ($S_{\mathbb{Sum}}$), ($\exists\mathbb{Fu}$)$\}$. In fact, (L1), (L2), (WSP), ($S_{\mathbb{Sum}}$) and ($\exists\mathbb{Fu}$) hold in model 4, but (L3) does not hold.

Let $\mathrm{M}_3 := \{1, 2, 12, 21\}$ be the universe of model 4. In the model conditions (L1), (L2), (WSP) hold. Condition ($\exists\mathbb{Fu}$) also holds. In fact, by virtue of (2.2), for each $x \in \mathrm{M}_3$, we have $x \mathbb{Fu} \{x\}$. Moreover, for any at least two-element subset S of M_3, we have $\bigcup \mathbb{O}(S) = \mathrm{M}_3 = \mathbb{O}(12) = \mathbb{O}(21)$, i.e., $12 \mathbb{Fu} S$ (and $21 \mathbb{Fu} S$). Thus, every non-empty subset of M_3 has a fusion.

Condition (L3) does not hold in model 4, because $12 \mathbb{Sum} \{1, 2\}$ and $21 \mathbb{Sum} \{1, 2\}$, but $12 \neq 21$.[14] □

Remark 3.1. (i) Certain 'misunderstandings' attend the groups of conditions (L2), (WSP) and ($\exists\mathbb{Fu}$) which relate to Simons' "Classical Extensional Mereology" from his [1987, pp. 37–41]. Simons adopts elementary counterparts to these conditions as axioms in his theory. The elementary counterpart of ($\exists\mathbb{Fu}$) is a certain sentential schema which is obtained from ($\exists\mathbb{Fu}$) by 'unravelling' the definition ($\mathrm{df}'\ \mathbb{Fu}$) and replacing the set-theoretic formula "$z \in S$" with the sentential schema "Fz" (see condition (GSP) on p. 166. One may, however, substitute various formulae for the schema "Fz", such as "'$z = x \vee z = y$", etc. In [Simons, 1987], an elementary counterpart of condition (L1) is also (superfluously) adopted, which follows from (L2) and (WSP) (cf. Lemma II.4.1(ii)).

Simons is of the view that the group of conditions he accepts axiomatises Leśniewski's mereology in an elementary fashion. He mistakenly believes that from (L1), (L2), (WSP) and (GSP) follows a third axiom

[14] Let us also notice that it follows from the comment made on p. 122 that there are no four-element mereological structures. So model 4 is not a model of mereology.

of Breitkopf's system in [1978] (see condition (8.2) on p. 167), which is an elementary counterpart of condition ($\exists\mathbb{N}\Cup$). We shall return to this in Section 8.

(ii) Observe that model 4 is also a model of Simons' "Classical Extensional Mereology". An arbitrary non-empty 'whole' of elements in M_3 is finite and therefore is 'described' by the elementary formula "$z = x_1 \vee \cdots \vee z = x_n$" (where $1 \leqslant n \leqslant 4$ and $x_i \in \{1, 2, 21, 21\}$), which we may substitute in (GSP) for "Fz". □

The situation is entirely different if we add the condition below[15] to the equivalents sets $\{$(L1), (L2), ($\mathrm{S}_{\mathsf{Sum}}$), ($\exists\mathbb{F}\Cup$)$\}$ and $\{$(L2), (WSP), ($\exists\mathbb{F}\Cup$)$\}$.

$$\forall_{x,y\in M}\big(x \bigcirc y \implies \exists_{u\in M}\forall_{z\in M}(z \sqsubseteq u \iff z \sqsubseteq x \wedge z \sqsubseteq y)\big). \qquad (\exists\sqcap)$$

PROPOSITION 3.12. (i) *Condition* ($\exists\sqcap$) *follows from* ($\exists\mathbb{N}\Cup$).
(ii) *Condition* ($\exists\sqcap$) *holds in all mereological structures.*

PROOF. *Ad* (i): For all $x, y \in M$ such that $x \bigcirc y$ we have $\bigcap \mathbb{I}(\{x, y\} \neq \emptyset$. Therefore, by virtue of ($\exists\mathbb{N}\Cup$), there is a u such that $u \ \mathbb{N}\Cup\ \{x, y\}$, i.e., for any $z \in M$: $z \sqsubseteq u$ iff $\forall_{w\in\{x,y\}}\, z \sqsubseteq w$ iff $z \sqsubseteq x$ and $z \sqsubseteq y$.[16]

Ad (ii): By (i) and Corollary 3.4. □

By using ($\mathrm{r}_{\sqsubseteq}$) and (df $\bigcirc$), condition ($\exists\sqcap$) may be strengthened to the following equivalence:

$$\forall_{x,y\in M}\big(x \bigcirc y \iff \exists_{u\in M}\forall_{z\in M}(z \sqsubseteq u \iff z \sqsubseteq x \wedge z \sqsubseteq y)\big).$$

In Theorem 3.15 we will show that by adding ($\exists\sqcap$) to the sets (L1), (L2), ($\mathrm{S}_{\mathsf{Sum}}$), ($\exists\mathbb{F}\Cup$)$\}$ and $\{$(L2), (WSP), ($\exists\mathbb{F}\Cup$)$\}$ we get sets of axioms of mereology. For this we will need:

LEMMA 3.13. (i) (SSP) *follows from* (L2), (WSP), *and* ($\exists\sqcap$).
(ii) (SSP) *follows from* (L1), (L2), ($\mathrm{S}_{\mathsf{Sum}}$) *and* ($\exists\sqcap$).

PROOF. *Ad* (i): Let $x \not\sqsubseteq y$. Then if $x \wr y$, then the consequent of (SSP) holds, since $x \sqsubseteq x$. Assume therefore that $x \bigcirc y$. Then, by virtue of ($\exists\sqcap$), for some v it is the case that (b) for each z: $z \sqsubseteq v$ iff both $z \sqsubseteq x$

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[15] This is axiom SA6 of "Minimal Extensional Mereology" considered by Simons in [1987, p. 31]. The formulae SA1–SA3 are other specific axioms of this system (see [Simons, 1987, p. 31]), i.e., which are counterparts to our conditions (L1), (L2). and (WSP) (the first of these is inessential in this group).

[16] Another proof: Condition ($\exists\sqcap$) holds in the class **MS**, because the formula (II.9.10) holds. So, in mereological structures, the element postulated in ($\exists\sqcap$) is simply the product $x \sqcap y$ (cf. also (2.5)).

and $z \sqsubseteq y$. Therefore — since $v \sqsubseteq v$ — we have (c) $v \sqsubseteq x$ and (d) $v \sqsubseteq y$. From (d) and (a) we have $v \neq x$, i.e., $v \sqsubset x$, by virtue of (c). Hence, by virtue of (WSP), for some u we have: (e) $u \sqsubset x$ and (f) $u \wr v$. We will show that (g) $u \wr y$. Assume for a contradiction that $u \circ y$. Then for some w we have: (h) $w \sqsubseteq u$ and (i) $w \sqsubseteq y$. From (h) and (e) we have $w \sqsubseteq x$, by virtue of (L2). From this and (i) and (b) we have $w \sqsubseteq v$. From this and (h) we obtain $u \circ v$, which contradicts (f). Thus, (e) and (g) give us the thesis.[17]

Ad (ii): By virtue of Lemma II.4.1(iv), condition (WSP) follows from (L1), (L2), and (S_{Sum}). We therefore apply (i). □

LEMMA 3.14. (i) *Conditions* ($r_{\sqsubseteq}$), ($t_{\sqsubseteq}$) *and the following*

$$\forall_{S \in \mathcal{P}(M)} \forall_{u \in M} \big(u \mathbb{Fu}\, S \iff S \neq \emptyset \wedge u \,\mathsf{sup}_{\sqsubseteq}\, S\big). \tag{$*$}$$

entail

$$\forall_{S \in \mathcal{P}(M)} \forall_{u \in M} \big(u \mathbb{Nu}\, S \iff u \mathbb{Fu} \textstyle\bigcap \mathbb{I}(S)\big). \tag{$**$}$$

(ii) *Condition* ($**$) *entails*

$$\forall_{x,y,u \in M} \big(v \mathbb{Nu} \{x, y\} \iff u \mathbb{Fu} \textstyle\bigcap \mathbb{I}(\{x, y\})\big). \tag{$***$}$$

(iii) *Conditions* ($\exists\mathbb{Fu}$) *and* ($***$) *entail* ($\exists\sqcap$).

PROOF. *Ad* (i): If $\bigcap \mathbb{I}(S) = \emptyset$ then it follows directly from the definitions (df $\mathbb{Nu}$) and (df′ $\mathbb{Fu}$) that both sides of the equivalence ($*$) are false. Assume therefore that $\bigcap \mathbb{I}(S) \neq \emptyset$. Since the relation $\sqsubseteq$ is reflexive and transitive, we can apply (4.15) of Appendix I. Furthermore, by applying (4.10) of Appendix I we obtain: $u \mathbb{Nu}\, S$ iff $u \,\mathsf{inf}_{\sqsubseteq}\, S$ iff $u \,\mathsf{sup}_{\sqsubseteq} \bigcap \mathbb{I}(S)$ iff $u \mathbb{Fu} \bigcap \mathbb{I}(S)$.

Ad (ii): We apply ($**$) for $S = \{x, y\}$.

Ad (iii): Let $x \circ y$. Then $\bigcap \mathbb{I}(\{x, y\}) = \{z \in M : z \sqsubseteq x \wedge z \sqsubseteq y\} \neq \emptyset$. Therefore, by virtue of ($\exists\mathbb{Fu}$), for some u we have $u \mathbb{Fu} \{z \in M : z \sqsubseteq x \wedge z \sqsubseteq y\}$. Hence, by virtue of ($***$), we have $u \mathbb{Nu} \{x, y\}$, which, by virtue of (df $\mathbb{Nu}$), gives, for each $z \in M$: $z \sqsubseteq u$ iff both $z \sqsubseteq x$ and $z \sqsubseteq x$. □

THEOREM 3.15. *The groups in theorems 1.2 and 3.7 are also equivalent to the five groups of conditions below*:

15° (L1), (L2), (S_{Sum}), ($\exists\sqcap$), *and* ($\exists\mathbb{Fu}$);
16° (L2), (WSP), ($\exists\sqcap$), *and* ($\exists\mathbb{Fu}$);

[17] A similar proof of this result is given by Simons in [1987, p. 31]. In Simons' proof, "$x \cdot y$" occurs in place of "v".

17° (L2), (WSP), ($\exists\mathbb{F}\mathrm{u}$) *and* ($\ast\ast\ast$);
18° (L2), (WSP), ($\exists\mathbb{F}\mathrm{u}$) *and* ($\ast\ast$);
19° (L2), (WSP), ($\exists\mathbb{F}\mathrm{u}$) *and* ($\ast$).

Proof. '15° ⇔ 16°' By virtue of Lemma II.4.1(iv).

'1° ⇒ 19°' By virtue of Theorem 3.2, in mereological structures we have $\mathbb{S}\mathrm{um} = \mathbb{F}\mathrm{u}$. We obtain therefore ($\exists\mathbb{F}\mathrm{u}$) from (L4), and ($\ast$) from ($\mathbb{S}\mathrm{um}$-$\mathsf{sup}_{\sqsubseteq}$).

'19° ⇒ 18° ⇒ 17° ⇒ 16°' By virtue of Lemma 3.14.

'15° ⇒ 11°' By virtue of Lemma 3.13(ii). □

4. Aggregate of the elements of a set

In line with the intuitions presented in Chapter I, a collective set is a 'joining together in one whole' of some objects, out of which it is to be composed. Adopting the concept of *being a mereological sum* as an explication of the concept of *being a collective set* does, however, generate certain 'unnatural' consequences. These are that the definition of sum allows it to be the case that:

- the collective set itself may be amongst the objects out of which it is composed;
- the collective set may be the only object 'composed into a whole' (we therefore lose the desired 'plural reading' of the phrase "some objects").

We shall introduce a new relation which should capture the intuitions associated with the concept of *being an aggregate.*[18] It should satisfy the condition: *if an object x is an aggregate of objects $y_1, \ldots, y_n$, then no $y_1, \ldots, y_n$ is identical with x.* It follows from this that: $n > 1$ *and* $y_1 \sqsubset x, \ldots, y_n \sqsubset x$.[19]

[18] The term "aggregate" replaces Goodman's "fusion". The term "aggregate" for a collective set is used by Quine [1953] and Russell [2010], amongst others. See the passages from [Quine, 1953] and [Słupecki and Borkowski, 1984] on pages 24 and 30.

[19] For example: The European Union arose through the joining together of certain European nations and not through the joining together of the European Union and certain nations. Can a fusion (as a type of aggregate) arise out of one company (in the sense of a business)? Can a fusion of companies arise in which one of them already contains others?

Let us accept below that M is an arbitrary non-empty set, that $\sqsubset$ is an arbitrary binary relation in M, and that the relations $\sqsubseteq$, $\bigcirc$, $\wr$ and $\mathbb{S}\mathrm{um}$ are defined by means of (df $\sqsubseteq$), (df $\bigcirc$), (df $\wr$) and (df $\mathbb{S}\mathrm{um}$), respectively.

The fact that x is an aggregate of all elements of a set S will be expressed with the help of the relation $\mathbb{A}\mathrm{gr}$ holding between x and S, i.e. (a relation) included in $M \times \mathcal{P}(M)$. We accept the following definition for the relation $\mathbb{A}\mathrm{gr}$:

$$x \;\mathbb{A}\mathrm{gr}\; S \;:\Longleftrightarrow S \neq \emptyset \wedge \forall_{z \in S}\, z \sqsubset x \wedge \forall_{y \in M}(y \sqsubset x \Rightarrow \exists_{z \in S}\, z \bigcirc y). \tag{df $\mathbb{A}\mathrm{gr}$}$$

Thus, x is an aggregate of all elements of a set S iff (a) the set S has in general some elements (cf. Proposition 4.2), (b) each element of S is a PART of x and (c) each part of x overlaps some element of S.[20] By exploiting the functions $\mathbb{P}$ and $\mathbb{O}$, the definition above may be written for arbitrary $x \in M$ and $S \in \mathcal{P}(M)$ as:

$$x \;\mathbb{A}\mathrm{gr}\; S \iff \emptyset \neq S \subseteq \mathbb{P}(x) \subseteq \bigcup \mathbb{O}(S). \tag{df$'$ $\mathbb{A}\mathrm{gr}$}$$

Therefore, directly from (df$'$ $\mathbb{A}\mathrm{gr}$), (II.3.1) and ($\mathrm{r}_{\sqsubseteq}$) we obtain:

$$\mathbb{A}\mathrm{gr} \subseteq \mathbb{S}\mathrm{um}\,. \tag{4.1}$$

Moreover, observe that the definition of an aggregate arose by swapping condition (ii) in Lemma II.3.1 for the stronger condition (b). Thus, for arbitrary $x \in M$ and $S \in M$ (in an arbitrary structure $\mathfrak{M}$) we obtain:

$$x \;\mathbb{S}\mathrm{um}\; S \wedge x \notin S \implies x \;\mathbb{A}\mathrm{gr}\; S, \tag{4.2}$$

$$x \;\mathbb{S}\mathrm{um}\; S \wedge S \text{ has no greatest element} \implies x \;\mathbb{A}\mathrm{gr}\; S. \tag{4.3}$$

In fact, if $x \;\mathbb{S}\mathrm{um}\; S$ and S has no greatest element, then $x \notin S$.

PROPOSITION 4.1. *If $\mathfrak{M} = \langle M, \sqsubset \rangle$ is an irreflexive structure then*

$$\forall_{S \in \mathcal{P}(M)} \forall_{x \in M}(x \;\mathbb{A}\mathrm{gr}\; S \iff x \;\mathbb{S}\mathrm{um}\; S \wedge x \notin S).$$

PROOF. '$\Rightarrow$' If $x \;\mathbb{A}\mathrm{gr}\; S$, then $x \;\mathbb{S}\mathrm{um}\; S$, by (4.1). If it were the case that $x \in S$, then — by virtue of the assumption and ($\mathrm{irr}_{\sqsubset}$) — there would exist a $z \in S$ such that $z \not\sqsubset x$, since $x \not\sqsubset x$. '$\Leftarrow$' By (4.2). □

[20] About the relation $\mathbb{A}\mathrm{gr}$ see also [Gruszczyński and Pietruszczak, 2010, sections 6 and 7].

Proposition 4.2. *If $\langle M, \sqsubset\rangle$ is a transitive structure satisfying* (WSP)*, then for arbitrary $x \in M$ and $S \in \mathcal{P}(M)$ we have:*

$$\forall_{S\in\mathcal{P}(M)}\forall_{x\in M}(x \ \mathbb{A}\mathrm{gr}\ S \iff x \ \mathsf{Sum}\ S \wedge S \text{ has no greatest element}).$$

Thus, if $x \ \mathbb{A}\mathrm{gr}\ S$ then $\operatorname{Card} S > 1$.

Proof. '⇒' By Lemma II.4.1.(i), from (WSP) we obtain ($\mathrm{irr}_{\sqsubset}$). Thus, by Proposition 4.1, if $x \ \mathbb{A}\mathrm{gr}\ S$ then $x \ \mathsf{Sum}\ S$ and $x \notin S$. Assume for a contradiction that S has a greatest element and that it is z_0. So $z_0 \sqsubset x$. Therefore, by virtue of (WSP), for some y_0 we have $y_0 \sqsubset x$ and $y_0 \wr z_0$. Therefore, by ($\mathrm{t}_{\sqsubset}$), we have $\forall_{z\in S}\, z \wr y_0$, which contradicts $y_0 \sqsubset x$ with $x \ \mathbb{A}\mathrm{gr}\ S$, by (df $\mathbb{A}\mathrm{gr}$). '⇐' By (4.3). □

Proposition 4.3. *If $\mathfrak{M} = \langle M, \sqsubset\rangle$ is an irreflexive structure then:*

$$\forall_{x\in M}\big(\mathbb{P}(x) \neq \emptyset \iff x \ \mathbb{A}\mathrm{gr}\ \mathbb{P}(x)\big).$$

Proof. '⇒' If $\mathbb{P}(x) \neq \emptyset$, then $x \ \mathsf{Sum}\ \mathbb{P}(x)$, by (II.3.5). Since by ($\mathrm{irr}_{\sqsubset}$) we have $x \notin \mathbb{P}(x)$, then also $x \ \mathbb{A}\mathrm{gr}\ \mathbb{P}(x)$, by (4.2). '⇐' By (df $\mathbb{A}\mathrm{gr}$). □

Proposition 4.4. *Let $\mathfrak{M} = \langle M, \sqsubset\rangle$ be any structure.*

(i) *If in $\mathfrak{M}$ condition* (L3) *holds, then $\mathfrak{M}$ satisfies the following condition of the uniqueness of the relation $\mathbb{A}\mathrm{gr}$:*

$$\forall_{S\in\mathcal{P}(M)}\forall_{x,y\in M}(x \ \mathbb{A}\mathrm{gr}\ S \wedge y \ \mathbb{A}\mathrm{gr}\ S \implies x = y) \qquad (\mathrm{U}_{\mathbb{A}\mathrm{gr}})$$

(ii) *If in $\mathfrak{M}$ conditions* (L2), (WSP) *and* ($\mathrm{U}_{\mathbb{A}\mathrm{gr}}$) *hold, then $\mathfrak{M}$ satisfies condition* (L3).

Proof. *Ad* (i): Assume that $\mathfrak{M}$ satisfies (L3). If $x \ \mathbb{A}\mathrm{gr}\ S$ and $y \ \mathbb{A}\mathrm{gr}\ S$, then also $x \ \mathsf{Sum}\ S$ and $y \ \mathsf{Sum}\ S$, by (4.1). So $x = y$, by (L3).

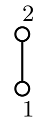
Ad (ii): Suppose that $\mathfrak{M}$ satisfies (L2), (WSP) and ($\mathrm{U}_{\mathbb{A}\mathrm{gr}}$). Let $x \ \mathsf{Sum}\ S$ and $y \ \mathsf{Sum}\ S$. Then if S has a greatest element, then $x = y$, by virtue of Lemma II.4.2. If, however, S has no greatest element, then $x \ \mathbb{A}\mathrm{gr}\ S$ and $y \ \mathbb{A}\mathrm{gr}\ S$, by (4.3). So $x = y$, by ($\mathrm{U}_{\mathbb{A}\mathrm{gr}}$). □

Proposition 4.5. (i) *Condition* ($\mathrm{U}_{\mathbb{A}\mathrm{gr}}$) *does not follow from* (L1), (L2) ($\mathrm{S}_{\mathsf{Sum}}$) *and* (WSP).

(ii) *Condition* (WSP) *does not follow from* (L1), (L2) *and* ($\mathrm{U}_{\mathbb{A}\mathrm{gr}}$)*, and hence* (L3) *does not either.*

Proof. *Ad* (i): In model 4 we have $12 \ \mathbb{A}\mathrm{gr}\ \{1,2\}$ and $21 \ \mathbb{A}\mathrm{gr}\ \{1,2\}$.

Ad (ii): In model 5 conditions (L1) and (L2) hold. Furthermore, we only have $2 \ \mathbb{A}\mathrm{gr}\ \{1\}$, and therefore ($\mathrm{U}_{\mathbb{A}\mathrm{gr}}$) holds. Condition (WSP) and



Model 5. Conditions (L1), (L2), ($\mathrm{U}_{\mathbb{Agr}}$) and (PPP) hold, but (WSP), ($\mathrm{S}_{\mathbb{Sum}}$), ($\nexists 0$), (L3) do not hold

($\mathrm{S}_{\mathbb{Sum}}$) do not hold, since: $1 \sqsubset 2$ and $\mathbb{P}(2) = \{1\} \subseteq \{1,2\} = \mathbb{O}(1)$; and $1 \mathbin{\mathbb{Sum}} \{1\}$ and $2 \mathbin{\mathbb{Sum}} \{1\}$. □

Proposition 4.6. *If in an irreflexive structure $\langle M, \sqsubset\rangle$ the following condition of 'aggregate existence' holds*:

$$\forall_{S \in \mathcal{P}_+(M)}(S \textit{ has no greatest element} \implies \exists_{x \in M}\, x \mathbin{\mathbb{Agr}} S) \qquad (\exists\mathbb{Agr})$$

then in $\mathfrak{M}$ there exists a greatest element.

Proof. Assume for a contradiction that $\mathfrak{M}$ satisfies ($\mathrm{irr}_{\sqsubset}$) and ($\exists\mathbb{Agr}$), and M has no greatest element. Then, by virtue of ($\exists\mathbb{Agr}$), there exists an x such that $x \mathbin{\mathbb{Agr}} M$. From this we obtain $\forall_{y \in M}\, y \sqsubset x$, and so $x \sqsubset x$, which contradicts ($\mathrm{irr}_{\sqsubset}$). □

Proposition 4.7. *In an arbitrary structure $\langle M, \sqsubset\rangle$, conditions (L4) and ($\exists\mathbb{Agr}$) are equivalent.*

Proof. Assume that (L4) holds and S is an non-empty set which has no greatest element. Then there exists an $x \in M$ such that $x \mathbin{\mathbb{Sum}} S$. Hence, by (4.3), we have $x \mathbin{\mathbb{Agr}} S$.

Assume that ($\exists\mathbb{Agr}$) holds and pick an arbitrary $S \in \mathcal{P}_+(M)$. If x is the greatest element in S, then $x \mathbin{\mathbb{Sum}} S$, by virtue of Lemma II.3.2. If S has no greatest element, then — by virtue of ($\exists\mathbb{Agr}$) — for some x we have $x \mathbin{\mathbb{Agr}} S$. Hence $x \mathbin{\mathbb{Sum}} S$, by (4.1). □

With propositions 4.4 and 4.7 established, we may now obtain the following theorem which concerns further conditions relation to the axiomatization of mereological structures.

Theorem 4.8. *The groups of conditions in theorems 1.2, 3.7 and 3.15 are also equivalent to the groups of conditions which arise from groups 1°, 3°–10° by the replacement of (L4) for ($\exists\mathbb{Agr}$). In addition, in 1° we can replace condition (L3) by two conditions (WSP) and ($\mathrm{U}_{\mathbb{Agr}}$).*

Mereological structures may therefore be axiomatised by using the relation $\mathbb{A}\mathrm{gr}$ in place of the relation $\mathbb{S}\mathrm{um}$.

5. Axiomatisations with the primary relation $\sqsubseteq$. Either (WSP), (S) or (SSP) instead of (L3)

In this section we shall take, just as Tarski did, the relation $\sqsubseteq$ *is an ingrediens of* to be primary. By saying that the structure $\mathfrak{T} = \langle M, \sqsubseteq\rangle$ satisfies certain conditions, we mean to say that that the relation $\sqsubseteq$ satisfies them and potentially the relations $\sqsubset$, $\bigcirc$, $\wr$, $\mathbb{S}\mathrm{um}$ and $\mathbb{F}\mathrm{u}$ too, which are to be defined respectively via the definitions ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$), (df $\bigcirc$), (df $\wr$), (df $\mathbb{S}\mathrm{um}$) and (df $\mathbb{F}\mathrm{u}$). By virtue of ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$), we have ($\mathrm{irr}_{\sqsubset}$), i.e., $\sqsubset$ is irreflexive. By virtue of (df $\bigcirc$) and (df $\wr$), we have , ($\mathrm{s}_{\bigcirc}$), ($\mathrm{s}_{\wr}$) and ($\wr = -\bigcirc$), i.e.. $\bigcirc$ and $\wr$ are symmetric and $\wr$ is the set-theoretical complement of $\bigcirc$. Thus (df $\mathbb{F}\mathrm{u}$) and (df$'$ $\mathbb{F}\mathrm{u}$) are equivalent. Finally, if $\sqsubseteq$ is reflexive, i.e., we have ($\mathrm{r}_{\sqsubseteq}$), then $\bigcirc$ is also reflexive ($\mathrm{r}_{\bigcirc}$) and $\wr$ is irreflexive ($\mathrm{irr}_{\wr}$).

We note first of all, that the following results hold (see also lemmas 4.3, 6.2 and 6.4 from Chapter II):

LEMMA 5.1. *For an arbitrary structure* $\mathfrak{T} = \langle M, \sqsubseteq\rangle$:

(i) *If* $\mathfrak{T}$ *satisfies* (WSP), *then it also satisfies* ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$).
(ii) *If* $\mathfrak{T}$ *satisfies* ($\mathrm{r}_{\sqsubseteq}$) *and* ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$), *then it also satisfies* (WSP).
(iii) *If* $\mathfrak{T}$ *satisfies* ($\mathrm{r}_{\sqsubseteq}$) *and* (L3), *then it also satisfies* ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$).
(iv) *If* $\mathfrak{T}$ *satisfies* ($\mathrm{r}_{\sqsubseteq}$) *and* (L3), *then it also satisfies* (WSP).
(v) *If* $\mathfrak{T}$ *satisfies* ($\mathrm{r}_{\sqsubseteq}$), ($\mathrm{antis}_{\sqsubseteq}$) *and* (M2), *then it also satisfies* ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$) *and* (WSP).

PROOF. *Ad* (i): Assume for a contradiction that $y \,\mathbb{S}\mathrm{um}\, \{x\}$ and $x \neq y$. Then, by (df $\mathbb{S}\mathrm{um}$), we have $x \sqsubseteq y$; and so $x \sqsubset y$, since $\sqsubset \,:=\, \sqsubseteq \setminus \mathrm{id}_M$. Hence, in virtue of (WSP), for some $z \in M$ we have $z \sqsubset y$ and $z \wr x$. Therefore we get a contradiction, since $z \sqsubset y$ and $y \,\mathbb{S}\mathrm{um}\, \{x\}$ entail $z \bigcirc x$.

Ad (ii): Let $x \sqsubset y$, i.e., $x \sqsubseteq y$ and $x \neq y$. Note that, by ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$) and ($\mathrm{r}_{\sqsubseteq}$), for any $z \in M$ we have: $z \sqsubseteq y$ iff $z \sqsubset y$ or $z = y$. Assume for a contradiction that $\neg\exists_{z\in M}(z \sqsubset y \wedge z \wr x)$, i.e., $\mathbb{P}(y) \subseteq \mathbb{O}(x)$. Then, by ($\mathrm{r}_{\sqsubseteq}$), we have $x \sqsubseteq x$; and so $y \bigcirc x$. Thus, $y \,\mathbb{S}\mathrm{um}\, \{x\}$. Hence, by ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$), we have a contradiction: $x = y$.

Ad (iii): By ($\mathrm{r}_{\sqsubseteq}$) we have $x \,\mathbb{S}\mathrm{um}\, \{x\}$. Therefore, if $y \,\mathbb{S}\mathrm{um}\, \{x\}$, then $x = y$, by (L3).

Ad (iv): From (ii) and (iii).

Ad (v): By Lemma II.6.2(viii), conditions (antis$_{\sqsubseteq}$) and (M2) entail (L3). So we use (iii) and (iv), respectively. □

LEMMA 5.2. *For an arbitrary structure* $\mathfrak{T} = \langle M, \sqsubseteq \rangle$:

(i) *If* $\mathfrak{T}$ *satisfies conditions* (S$_{\mathbb{S}\mathrm{um}}$) *and* (L4), *then it also satisfies* (r$_{\sqsubseteq}$).
(ii) *If* $\mathfrak{T}$ *satisfies* (r$_{\sqsubseteq}$), (t$_{\sqsubseteq}$) *and* (S$_{\mathbb{S}\mathrm{um}}$), *then it also satisfies* (antis$_{\sqsubseteq}$).
(iii) *If* $\mathfrak{T}$ *satisfies* (r$_{\sqsubseteq}$), (t$_{\sqsubseteq}$) *and* (WSP), *then it also satisfies* (antis$_{\sqsubseteq}$).
(iv) *If* $\mathfrak{T}$ *satisfies* (r$_{\sqsubseteq}$), (t$_{\sqsubseteq}$) *and* (L3), *then it also satisfies* (antis$_{\sqsubseteq}$).

PROOF. *Ad* (i): Pick an arbitrary $x \in M$. By virtue of (L4) there exists a $y \in M$ such that $y \ \mathbb{S}\mathrm{um}\ \{x\}$. Hence $x = y$ and $x \sqsubseteq y$, by virtue of (S$_{\mathbb{S}\mathrm{um}}$) and (df $\mathbb{S}\mathrm{um}$), respectively. Therefore, $x \sqsubseteq x$.

Ad (ii): Assume that (a) $x \sqsubseteq y$ and (b) $y \sqsubseteq x$. We show that (c) $y \ \mathbb{S}\mathrm{um}\ \{x\}$. In fact, for any $z \in M$ such that $z \sqsubseteq y$ we have $z \sqsubseteq x$, by virtue of (b) and (t$_{\sqsubseteq}$). Hence $z \bigcirc x$, because $z \sqsubseteq z$, by (r$_{\sqsubseteq}$). From this and (a) we have $y \ \mathbb{S}\mathrm{um}\ \{x\}$. Thus, $x = y$, by virtue of (S$_{\mathbb{S}\mathrm{um}}$).

Ad (iii): By virtue of Lemma 5.1(i), condition (S$_{\mathbb{S}\mathrm{um}}$) follows from (WSP). So we use (ii).

Ad (iv): By virtue of Lemma 5.1(iii), condition (S$_{\mathbb{S}\mathrm{um}}$) follows from (r$_{\sqsubseteq}$) and (L3). So we use (ii). □

By the above lemma and Lemma 1.1 we also obtain:

LEMMA 5.3. (i) *Condition* (L3) *follows from* (t$_{\sqsubseteq}$), (WSP) *and* (L4).
(ii) *Condition* (L3) *follows from* (t$_{\sqsubseteq}$), (S$_{\mathbb{S}\mathrm{um}}$) *and* (L4).

PROOF. *Ad* (i): By virtue of Lemma 5.1(i), condition (S$_{\mathbb{S}\mathrm{um}}$) follows from (WSP). By virtue of Lemma 5.2(i), condition (r$_{\sqsubseteq}$) follows from (S$_{\mathbb{S}\mathrm{um}}$) and (L4). From (r$_{\sqsubseteq}$), (t$_{\sqsubseteq}$) and (WSP) we obtain (antis$_{\sqsubseteq}$), by Lemma 5.2(iii). From (antis$_{\sqsubseteq}$) and (t$_{\sqsubseteq}$) — by virtue of Lemma 2.4 from Appendix I — the relation $\sqsubset$ satisfies (L2). From (L2), (WSP) and (L4) we have (L3), by virtue of Lemma 1.1.

Ad (ii): By virtue of Lemma 5.1(ii), condition (WSP) follows from (S$_{\mathbb{S}\mathrm{um}}$) and (r$_{\sqsubseteq}$). So we use (i). □

As Lemma 3.13(i) we obtain:

LEMMA 5.4. *Condition* (SSP) *follows from* (t$_{\sqsubseteq}$), (WSP) *and* ($\exists\sqcap$).

LEMMA 5.5. *Ifi* $\mathfrak{T}$ *satisfies conditions* (t$_{\sqsubseteq}$) *and* (L3-L4), *then it also satisfies conditions* (r$_{\sqsubseteq}$), (antis$_{\sqsubseteq}$), (L3), (L4) *and* (SSP).

Proof. *Ad* ($\mathrm{r}_{\sqsubseteq}$), ($\mathrm{antis}_{\sqsubseteq}$), *and* (L4): The structure $\mathfrak{T}$ meeting the conditions ($\mathrm{t}_{\sqsubseteq}$) and (L3-L4) belongs to **TS**. Hence, by virtue of Theorem III.3.2(iii) the relation $\sqsubseteq$ is also reflexive and anti-symmetric, i.e., it meets conditions ($\mathrm{r}_{\sqsubseteq}$) and ($\mathrm{antis}_{\sqsubseteq}$). Furthermore, (L4) follows from (L3-L4).

Ad (L3): In Remark II.5.3 we showed that (L3) follows from ($\mathrm{r}_{\sqsubseteq}$) and (L3-L4).

Ad (SSP): In the proof of Theorem II.6.1 we derived condition (SSP) by using only ($\mathrm{t}_{\sqsubseteq}$), (L3), (L4), (df $\mathbb{S}\mathrm{um}$), ($\sqsubseteq\subseteq\circ$), and (II.3.4). The two final conditions follow directly from ($\mathrm{r}_{\sqsubseteq}$).

In the proof of Theorem III.3.2 we used Theorem 12.1 from Appendix I, which is not proved in this book. We therefore also 'elementarily' derive conditions ($\mathrm{r}_{\sqsubseteq}$) and ($\mathrm{antis}_{\sqsubseteq}$) from conditions ($\mathrm{t}_{\sqsubseteq}$) and (L3-L4).

Ad ($\mathrm{r}_{\sqsubseteq}$): Pick an arbitrary $x \in M$. We have from (L4) that there is a y such that $y \ \mathbb{S}\mathrm{um}\ \{x\}$. From (df $\mathbb{S}\mathrm{um}$) we obtain (a) $x \sqsubseteq y$ and (b) $\forall_u(u \sqsubseteq y \Rightarrow u \circ x)$. From this it follows that $x \circ x$. By applying (df $\circ$), we get $\exists_v v \sqsubseteq x$, i.e., $\mathbb{I}(x) \neq \emptyset$. Thus, by virtue of (L4), there exists a z such that $z \ \mathbb{S}\mathrm{um}\ \mathbb{I}(x)$. Now from (df $\mathbb{S}\mathrm{um}$) we have $\forall_u(u \sqsubseteq x \Rightarrow u \sqsubseteq z)$ and $\forall_u(u \sqsubseteq z \Rightarrow \exists_v(v \sqsubseteq x \wedge v \circ u))$. (c) $\forall_v(v \sqsubseteq x \Rightarrow \exists_v(v \sqsubseteq x \wedge v \circ u))$. This suffices — applying (df $\mathbb{S}\mathrm{um}$) — for us to state that $x \ \mathbb{S}\mathrm{um}\ \mathbb{I}(x)$.

We shall show that $y \ \mathbb{S}\mathrm{um}\ \mathbb{I}(x)$ as well. Hence, by virtue of (L3-L4), we will get $x = y$ which — by virtue of (a) — will give us $x \sqsubseteq x$.

Firstly, by virtue of (a) and ($\mathrm{t}_{\sqsubseteq}$), we obtain: $\forall_u(u \sqsubseteq x \Rightarrow u \sqsubseteq y)$. Secondly, by virtue of (b) and (df $\circ$), we have $\forall_u(u \sqsubseteq y \Rightarrow \exists_w(w \sqsubseteq x \wedge w \sqsubseteq u))$. From this and (c) we obtain $\forall_u(u \sqsubseteq y \Rightarrow \exists_{v,w}(v \sqsubseteq x \wedge v \circ w \wedge w \sqsubseteq u))$. Thus, $\forall_u(u \sqsubseteq y \Rightarrow \exists_{v,w,u'}(v \sqsubseteq x \wedge u' \sqsubseteq v \wedge u' \sqsubseteq w \wedge w \sqsubseteq u))$. Hence, by virtue of ($\mathrm{t}_{\sqsubseteq}$) and (df $\circ$), we have $\forall_u(u \sqsubseteq y \Rightarrow \exists_v(v \sqsubseteq x \wedge v \circ u))$. From both of the above results along with (df $\mathbb{S}\mathrm{um}$) it follows that $y \ \mathbb{S}\mathrm{um}\ \mathbb{I}(x)$.

Ad ($\mathrm{antis}_{\sqsubseteq}$): Both ($\mathrm{r}_{\sqsubseteq}$) and (L3) follow from ($\mathrm{t}_{\sqsubseteq}$) and (L3-L4). Hence, by Lemma 5.1(iii), we have ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$). Thus, by Lemma 5.2(ii), condition ($\mathrm{antis}_{\sqsubseteq}$) holds. □

Lemma 5.6. *If $\mathfrak{T}$ satisfies conditions* ($\mathrm{antis}_{\sqsubseteq}$), ($\mathrm{t}_{\sqsubseteq}$) *and* (SSP), *then it also satisfies* (L3) *and* ($\mathrm{U}_{\mathbb{F}\mathrm{u}}$).

Proof. Lemma II.6.2 says that: (M1) follows from z ($\mathrm{t}_{\sqsubseteq}$) and (SSP); that (M2) follows from (M1); and that (L3) follows from (M2) and (antis$_\sqsubseteq$). From (t$_\sqsubseteq$) and (SSP) we obtain $\mathbb{F}\mathrm{u} = \mathbb{S}\mathrm{um}$, by Theorem 3.1. So from (L3) we have (U$_{\mathbb{F}\mathrm{u}}$). □

LEMMA 5.7. *If $\mathfrak{T}$ satisfies* (t$_\sqsubseteq$) *and* (SSP), *then it also satisfies* (r$_\sqsubseteq$).

PROOF. Assume for a contradiction that for some $x \in M$ we have $x \not\sqsubseteq x$. Then, by virtue of (SSP), for some $y \in M$ we have (a) $y \sqsubseteq x$ and (b) $y \wr x$. From (b) — by applying (df$\wr$) — we have (c): $\forall_z (z \sqsubseteq y \Rightarrow z \not\sqsubseteq x)$. From (a), (c) and (t$_\sqsubseteq$) we have $\neg\exists_z\, z \sqsubseteq y$. Hence $y \not\sqsubseteq y$. By applying (SSP) therefore, we get a contradiction: $\exists_z\, z \sqsubseteq y$. □

Notice that the inclusion $\mathbb{S}\mathrm{um} \subseteq \mathbb{F}\mathrm{u}$ follows from the transitivity of $\sqsubseteq$ and our definitions; see the proof of (II.3.9). So we obtain:

LEMMA 5.8. *Condition* (L3) *follows from conditions* (t$_\sqsubseteq$) *and* (U$_{\mathbb{F}\mathrm{u}}$).

PROPOSITION 5.9. *Condition* (antis$_\sqsubseteq$) *does not follow from conditions* (t$_\sqsubseteq$), (SSP) *and* (L4).

PROOF. Let us take the structure $\mathfrak{T}_0 = \langle\{1,2\}, \sqsubseteq_0\rangle$ in which $\sqsubseteq_0$ is a full relation, i.e., $\sqsubseteq_0 := \{1,2\} \times \{1,2\}$. Thus $\sqsubseteq_0$ is transitive and satisfies (SSP), but it is not antisymmetric. In $\mathfrak{T}_0$ the relation $\bigcirc$ is full and therefore condition (L4) holds (but (L3) does not hold). □

We have the follow equivalent axiomatisations of the class **TS**:[21]

THEOREM 5.10. *The groups of conditions below are equivalent*:

1° (t$_\sqsubseteq$) *and* (L3-L4);
2° (t$_\sqsubseteq$), (L3) *and* (L4);
3° (t$_\sqsubseteq$), (antis$_\sqsubseteq$), (SSP) *and* (L4);
4° (t$_\sqsubseteq$), (antis$_\sqsubseteq$), (SSP) *and* ($\exists\mathbb{F}\mathrm{u}$);
5° (t$_\sqsubseteq$), (U$_{\mathbb{F}\mathrm{u}}$) *and* (L4);
6° (t$_\sqsubseteq$), (antis$_\sqsubseteq$), (S$_{\mathbb{S}\mathrm{um}}$), ($\exists\sqcap$) *and* (L4);
7° (t$_\sqsubseteq$), (antis$_\sqsubseteq$), (S$_{\mathbb{S}\mathrm{um}}$), ($\exists\sqcap$) *and* ($\exists\mathbb{F}\mathrm{u}$);
8° (t$_\sqsubseteq$), (antis$_\sqsubseteq$), (WSP), ($\exists\sqcap$) *and* (L4);
9° (t$_\sqsubseteq$), (antis$_\sqsubseteq$), (WSP), ($\exists\sqcap$) *and* ($\exists\mathbb{F}\mathrm{u}$);
10° (t$_\sqsubseteq$), (WSP) *and* (L4);
11° (t$_\sqsubseteq$), (S$_{\mathbb{S}\mathrm{um}}$) *and* (L4).

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[21] The following theorem and Proposition 5.9 show that if the relation $\sqsubseteq$ is not antisymmetric, then, in the structure $\langle M, \sqsubseteq\rangle$, the 'strong' condition (SSP) is weaker than (S$_{\mathbb{S}\mathrm{um}}$) and the 'weak' condition (WSP).

PROOF. '1° ⇔ 2°' Firstly, (L3-L4) follows from (L3) and (L4). Secondly, we use Lemma 5.5.

'2° ⇔ 3°' By virtue of lemmas 5.5 and 5.6, respectively

'3° ⇔ 4°' By virtue of Lemma 5.8 and Theorem 3.1, in both groups we have $\mathbb{F}\mathrm{u} = \mathbb{S}\mathrm{um}$.

'3° ⇒ 5°' By Lemma 5.6, from ($\mathrm{antis}_{\sqsubseteq}$), ($\mathrm{t}_{\sqsubseteq}$) and (SSP) we have ($\mathrm{U}_{\mathbb{F}\mathrm{u}}$).

'5° ⇒ 2°' By Lemma 5.8, condition (L3) follows from ($\mathrm{t}_{\sqsubseteq}$) and ($\mathrm{U}_{\mathbb{F}\mathrm{u}}$).

'6° ⇔ 8°' and '7° ⇔ 9°' Firstly, from ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$) and (L4) we have ($\mathrm{r}_{\sqsubseteq}$), by Lemma 5.2(i). Moreover, from ($\mathrm{r}_{\sqsubseteq}$) and ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$) we obtain (WSP), by Lemma 5.1(ii). Secondly, from (WSP) we have ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$), by Lemma 5.1(i).

'8° ⇔ 9°' By Lemma 5.4, in both groups condition (SSP) follows from ($\mathrm{t}_{\sqsubseteq}$), (WSP), and ($\exists\sqcap$). Hence, by Lemma 5.8 and Theorem 3.1, in both groups we have $\mathbb{F}\mathrm{u} = \mathbb{S}\mathrm{um}$.

'8° ⇒ 3°' By virtue of Lemma 5.4, condition (SSP) follows from ($\mathrm{t}_{\sqsubseteq}$), (WSP) and ($\exists\sqcap$).

'1° ⇒ 6°' By Corollary III.4.5, Proposition 3.12(i), and Lemma 5.5.

'3° ⇒ 10°' By virtue of Lemma 5.6, (L3) follows from ($\mathrm{antis}_{\sqsubseteq}$), ($\mathrm{t}_{\sqsubseteq}$) and (SSP). Since the relation $\sqsubset\ :=\ \sqsubseteq\setminus\mathrm{id}_M$ is irreflexive, we have (WSP), by virtue of Lemma II.4.1(iv) and (L3).

'10° ⇒ 11°' By virtue of Lemma 5.1(i).

'11° ⇒ 10°' By Lemma 5.2(i), condition ($\mathrm{r}_{\sqsubseteq}$) follows from ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$) and (L4). So we use Lemma 5.1(ii).

'10° ⇒ 2°' By virtue of Lemma 5.3(i). □

Theorems 1.2, 3.7, 3.15, 4.8 and 5.10 may be used to extend Corollary III.4.5, which presents the connection between the axiomatisations of the classes **MS** and **TS**. If the relations $\sqsubset$, $\sqsubseteq$, $\circ$, $\wr$, $\mathbb{S}\mathrm{um}$, $\mathbb{F}\mathrm{u}$ and $\mathbb{A}\mathrm{gr}$ satisfy the conditions associated with the groups of those theorems, then $\langle M, \sqsubset\rangle \in$ **MS** and $\langle M, \sqsubseteq\rangle \in$ **TS**.

6. Axiomatisations with the primitive relation $\wr$

Let us return to Leonard and Goodman's system as well as Leśniewski's system discussed in Section 2. We will begin with the following auxiliary results.

LEMMA 6.1. *For arbitrary binary relations $\wr$ and $\sqsubseteq$ in a non-empty set M, the following groups of conditions are equivalent:*

(a) $(\mathrm{t}_{\sqsubseteq})$, (SSP) *and* $(\mathrm{df}\wr)$;
(b) $(\mathrm{df}_{\wr}\sqsubseteq)$ *and* $(\mathrm{df}\wr)$;
(c) $(\mathrm{df}_{\wr}\sqsubseteq)$ *and* $(\mathfrak{a})$.

Proof. '(a) ⇒ (b)' By virtue of Lemma 5.7, $(\mathrm{r}_{\sqsubseteq})$ follows from (SSP) and $(\mathrm{t}_{\sqsubseteq})$.

If $x \sqsubseteq y$ and $\exists_u(u \sqsubseteq z \wedge u \sqsubseteq x)$, then by virtue of $(\mathrm{t}_{\sqsubseteq})$, we obtain $\exists_u(u \sqsubseteq z \wedge u \sqsubseteq y)$. Thus conditions $(\mathrm{t}_{\sqsubseteq})$ and $(\mathrm{df}\wr)$ entail the implication '⇒' in w $(\mathrm{df}_{\wr}\sqsubseteq)$.

For the proof of the implication '⇐' in $(\mathrm{df}_{\wr}\sqsubseteq)$ assume that (i): $\forall_z(z \wr y \Rightarrow z \wr x)$. From (i), $(\mathrm{r}_{\sqsubseteq})$ and $(\mathrm{df}\wr)$ we have (ii): $\forall_z(z \sqsubseteq x \Rightarrow \neg\, z \wr y)$. In fact, if $z \sqsubseteq x$, then $\neg\, z \wr x$. From this and (i) we have $\neg\, z \wr y$. From (ii) and (SSP) we have $x \sqsubseteq y$.

'(b) ⇒ (a)' From $(\mathrm{df}_{\wr}\sqsubseteq)$ we get $(\mathrm{t}_{\sqsubseteq})$. Assume that (i) $\forall_z(z \sqsubseteq x \Rightarrow \neg\, z \wr y)$, From (i) and $(\mathrm{t}_{\sqsubseteq})$ we have (ii) $\forall_z(z \wr y \Rightarrow z \wr x)$. Basically, let $\neg\, z \wr x$. Then, by virtue of $(\mathrm{df}\wr)$, for some u we have $u \sqsubseteq z$ and $u \sqsubseteq x$. From this and (i): $\neg\, u \wr y$. For a certain v we have therefore $v \sqsubseteq u$ and $v \sqsubseteq y$. By virtue of $(\mathrm{t}_{\sqsubseteq})$ that is, $v \sqsubseteq z$ and $v \sqsubseteq y$, i.e., $\neg\, z \wr y$. From (ii) and $(\mathrm{df}_{\wr}\sqsubseteq)$ we have $x \sqsubseteq y$, which proves (SSP).

'(b) ⇔ (c)' By virtue of Lemma 2.1. □

Lemma 6.2. *Let M be an arbitrary non-empty set, $\wr$ and $\sqsubseteq$ be binary relations in M, and $\mathbb{Fu} \subseteq M \times \mathcal{P}(M)$ be a relation defined by* $(\mathrm{df}\,\mathbb{Fu})$. *Then for $\wr$, $\sqsubseteq$ and $\mathbb{Fu}$ the groups of conditions below are equivalent*:

1° $(\mathrm{t}_{\sqsubseteq})$, (SSP), $(\mathrm{df}\wr)$, $(\mathrm{antis}_{\sqsubseteq})$ *andi* $(\exists\mathbb{Fu})$;[22]
2° $(\mathrm{df}_{\wr}\sqsubseteq)$, $(\mathfrak{a})$, $(\mathrm{antis}_{\sqsubseteq})$ *and* $(\exists\mathbb{Fu})$;
3° $(\mathrm{df}_{\wr}\sqsubseteq)$, $(\mathrm{df}\wr)$, $(\mathrm{antis}_{\sqsubseteq})$ *and* $(\exists\mathbb{Fu})$;
4° $(\mathrm{df}_{\wr}\sqsubseteq)$, $(\mathrm{df}\wr)$, $(\mathfrak{b})$ *and* $(\mathfrak{d})$;
5° $(\mathfrak{a})$, $(\mathfrak{b})$ *and* $(\mathfrak{d})$.

Proof. '1° ⇔ 2° ⇔ 3°' By virtue of Lemma 6.1.

'3° ⇒ 4°' From $(\mathrm{df}_{\wr}\sqsubseteq)$ we have $(\mathrm{r}_{\sqsubseteq})$ and $(\mathrm{t}_{\sqsubseteq})$, and from $(\mathrm{df}\wr)$ we have $(\mathrm{irr}_{\wr})$ and $(\mathrm{s}_{\wr})$.

Ad $(\mathfrak{b})$: Choose an arbitrary $S \in \mathcal{P}_+(M)$. By virtue of $(\exists\mathbb{Fu})$ there exists an x such that $x \,\mathbb{Fu}\, S$. Assume that $y \,\mathbb{Fu}\, S$. Then — by virtue of $(\mathrm{df}\,\mathbb{Fu})$ — we have: $\forall_u(u \wr x \Leftrightarrow u \wr y)$. Hence, by virtue of $(\mathrm{df}_{\wr}\sqsubseteq)$, we have: $x \sqsubseteq y$ and $y \sqsubseteq x$. Therefore, by virtue of $(\mathrm{antis}_{\sqsubseteq})$, we have $x = y$.

[22] Since $(\mathrm{r}_{\sqsubseteq})$ results from $(\mathrm{t}_{\sqsubseteq})$ and (SSP), the group $(\mathrm{t}_{\sqsubseteq})$, $(\mathrm{antis}_{\sqsubseteq})$ and (SSP) defines the class of polarised partial orders of the form $\langle M, \sqsubseteq\rangle$ with the relation $\wr$.

Ad ($\mathfrak{d}$): Condition (2.3) follows from ($\mathrm{df}_{\wr}\sqsubseteq$) and ($\mathrm{r}_{\sqsubseteq}$). Hence — by applying (df $\mathbb{Fu}$) — we have y $\mathbb{Fu}$ $\mathbb{I}(y)$. Therefore if $x \sqsubseteq y$, then for a certain S we have y $\mathbb{Fu}$ S and $x \in S$. Conversely, for a certain S let y $\mathbb{Fu}$ S and $x \in S$. Hence — applying (Def Fu)(df $\mathbb{Fu}$) — we have $\forall_u(u \wr y \Rightarrow u \wr x)$. Therefore $x \sqsubseteq y$, by virtue of ($\mathrm{df}_{\wr}\sqsubseteq$).

'4° ⇒ 5°' From (df $\wr$) and ($\mathrm{df}_{\wr}\sqsubseteq$) we obtain (a), by Lemma 2.1.

'5° ⇒ 2°' On p. 138 we derived conditions ($\mathrm{antis}_{\sqsubseteq}$) and ($\mathrm{df}_{\wr}\sqsubseteq$) from 5°. Condition ($\exists\mathbb{Fu}$) follows directly from condition ($\mathfrak{b}$).[23] □

Group 3° with the additional condition (df $\mathbb{Fu}$) is definitionally equivalent to the group of axioms and the definition of Leonard and Goodman's system as introduced in Remark 2.1. To see this, one needs merely to define two binary relations $\sqsubset$ and $\circ$ by condition ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$) and the identity $\circ = -\wr$. That condition is also a definition in Leonard and Goodman's system, and the identity is an axiom. Furthermore, under this identity, (df $\wr$) and (df $\circ$) are equivalent.

Group 5° with the added condition (df $\mathbb{Fu}$) is a restriction of the group of axioms and definitions of the system from [Leśniewski, 1931, ch. X]. We obtain Leśniewski's full system by adding condition ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$) which defines the relation $\sqsubset$.

From the above comments and from Lemma 6.2 we obtain the following conclusion:

Proposition 6.3. *Leonard and Goodman's system in* [1940] *and Leśniewski's system in* [1931, ch. X] *are definitionally equivalent.*

Proof. It suffices to describe the relation $\circ$ (in Leśniewski's system) either either by the identity $\circ = -\wr$ or by condition (df $\circ$), because condition (df $\wr$) holds in his system. □

One may accept that in groups 3° and 4° — as in Leonard and Goodman's system — the primitive relation is $\wr$ and condition ($\mathrm{df}_{\wr}\sqsubseteq$) defines the relation $\sqsubseteq$. It is equally 'natural' to accept that the primitive relation is $\sqsubseteq$ and the relation $\wr$ is defined by (df $\wr$).

We will prove below that the systems mentioned in Proposition 6.3 are equivalent to the systems of axioms and definitions defining structures from the class **MS** (resp. **TS**), namely that the following theorems hold.

[23] We have used the 'power' of condition ($\mathfrak{b}$), this being greater than the power of condition ($\exists\mathbb{Fu}$), to derive condition ($\mathrm{antis}_{\sqsubseteq}$). Clearly, for the derivation of ($\mathrm{antis}_{\sqsubseteq}$) condition ($\mathrm{U}_{\mathbb{Fu}}$) itself sufficed, this resulting from ($\mathfrak{b}$) and (2.1).

Theorem 6.4. *Let M be an arbitrary non-empty set, $\wr$ and $\sqsubseteq$ be binary relations in M satisfying some group of conditions from 1°–5°, and $\mathbb{Fu}, \mathbb{Sum} \subseteq M \times \mathcal{P}(M)$ be relations defined by conditions (df $\mathbb{Fu}$) and (df $\mathbb{Sum}$), respectively. Then*:

(i) $\mathbb{Fu} = \mathbb{Sum}$.

(ii) *The structure $\langle M, \sqsubseteq\rangle$ belongs to* **TS**.

(iii) *If the binary relation $\sqsubset$ is defined by condition ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$), then the structure $\langle M, \sqsubset\rangle$ belongs to* **MS**.

Proof. To begin with, we shall note a few results which we shall make use of later. From ($\mathrm{df}_{\wr} \sqsubseteq$) we have ($\mathrm{r}_{\sqsubseteq}$) and ($\mathrm{t}_{\sqsubseteq}$) directly. Hence the relation $\sqsubseteq$ partially orders M. From this and the identity $\sqsubset := \sqsubseteq \setminus \mathrm{id}_M$ it follows that the relation $\sqsubset$ satisfies conditions (L1) and (L2).

Thus, if we will prove that (i) holds, then from condition ($\mathfrak{b}$) we obtain condition(L3-L4), i.e., (ii) holds. Furthermore, (L4) follows from (L3-L4), and from ($\mathrm{r}_{\sqsubseteq}$) and (L3-L4) we have (L3). Therefore (iii) holds.

The identity $\mathbb{Fu} = \mathbb{Sum}$ remains to be proved. In order not to repeat earlier proofs in a different form, however, let us introduce the auxiliary relation $\bigcirc$ defined by the identity $\bigcirc := -\wr$.

Since the relations $\sqsubseteq$ and $\wr$ satisfy conditions ($\mathrm{df}_{\wr} \sqsubseteq$) and (df $\wr$), the relations $\sqsubseteq$ and $\bigcirc$ therefore satisfy conditions ($\mathrm{df}_{\bigcirc} \sqsubseteq$) and (df $\bigcirc$). It follows from the second condition and ($\mathrm{r}_{\sqsubseteq}$) that $\sqsubseteq$ is included in $\bigcirc$. Furthermore, the relations $\mathbb{Fu}$ and $\mathbb{Sum}$ satisfy conditions (df′ $\mathbb{Fu}$) and (II.df′ $\mathbb{Sum}$), respectively.

On p. 81 we proved the inclusion $\mathbb{Sum} \subseteq \mathbb{Fu}$ by using just (df $\mathbb{Sum}$), (df′ $\mathbb{Fu}$), (df $\bigcirc$) and ($\mathrm{t}_{\sqsubseteq}$). On p. 142 we proved the inclusion $\mathbb{Fu} \subseteq \mathbb{Sum}$ by using only (df $\mathbb{Sum}$), (df′ $\mathbb{Fu}$), (df $\bigcirc$), ($\mathrm{df}_{\bigcirc} \sqsubseteq$) and ($\mathrm{r}_{\sqsubseteq}$). □

Theorem 6.5. *Let M be an arbitrary non-empty set, $\sqsubset$, $\sqsubseteq$ and $\wr$ be binary relations in M, and $\mathbb{Fu}, \mathbb{Sum} \subseteq M \times \mathcal{P}(M)$ be relations defined by conditions (df $\mathbb{Fu}$) and (df $\mathbb{Sum}$), respectively. Then in both of the cases below, when*

(a) *the relation $\wr$ satisfies condition (df $\wr$) and the structure $\langle M, \sqsubseteq\rangle$ belongs to* **TS**,

(b) *the relations $\sqsubseteq$ and $\wr$ satisfy conditions (df $\sqsubseteq$) and (df $\wr$), and the structure $\langle M, \sqsubset\rangle$ belongs to* **MS**,

then the relations $\sqsubseteq$ and $\wr$ also satisfy the conditions in groups 1°–5°.

Proof. Suppose that (a) holds. Then for the relation $\sqsubset := \sqsubseteq \setminus \mathrm{id}_M$, the structure $\langle M, \sqsubset\rangle$ belongs to **MS** and condition (df $\sqsubseteq$) is satisfied.

Suppose that (b) holds. Then we proved earlier that all the conditions in 3° follow from conditions (L1)–(L4) and definitions (df $\sqsubseteq$), (df $\wr$), (df $\mathbb{S}\mathrm{um}$) and (df $\mathbb{F}\mathrm{u}$). □

By introducing a suitable elementary language with a predicate associated with the relation $\wr$ and by using theorems 6.4 and 6.5, we can prove theorems which affirm the elementary definitional equivalence of the classes **MS** and **TS** with the class of structures of the form $\langle M, \wr\rangle$, in which the relation $\wr$ and two defined relations $\mathbb{F}\mathrm{u}$ and $\sqsubseteq$ together satisfy some system from 1°–5° and (df $\mathbb{F}\mathrm{u}$).[24]

7. Axiomatisations with the primitive relation $\bigcirc$

The follow lemmas will come in useful for presenting the next definitionally equivalent axiomatisation of mereological structures:

LEMMA 7.1. *Assume that the binary relations $\bigcirc$ and $\sqsubseteq$ in a non-empty set M satisfy condition* ($\mathrm{df}_\circ \sqsubseteq$). *Then:*

(i) *Conditions* ($\mathrm{antis}_\sqsubseteq$) *and* ($\mathrm{ext}_\circ$) *are equivalent.*

(ii) *Condition* (df $\bigcirc$) *is equivalent to the following*[25]

$$x \bigcirc y \iff \exists_{z\in M}\forall_{u\in M}\big(u \bigcirc z \Rightarrow (u \bigcirc x \wedge u \bigcirc y)\big). \tag{G_1}$$

PROOF. *Ad* (i): Condition ($\mathrm{ext}_\circ$) follows in an obvious way from ($\mathrm{df}_\circ \sqsubseteq$) and ($\mathrm{antis}_\sqsubseteq$). Furthermore, if $x \sqsubseteq y$ and $y \sqsubseteq x$, then — by virtue of ($\mathrm{df}_\circ \sqsubseteq$) — we have: $\forall_z(z \bigcirc x \Leftrightarrow z \bigcirc y)$. Therefore ($\mathrm{ext}_\circ$) gives $x = y$.

Ad (ii): We subject (G_1) to 'equivalence' transformations, in which we apply — besides the logical rules — only condition ($\mathrm{df}_\circ \sqsubseteq$):

$$x \bigcirc y \iff \exists_{z\in M}\big(\forall_{u\in M}(u \bigcirc z \Rightarrow u \bigcirc x) \wedge \forall_{u\in M}(u \bigcirc z \Rightarrow u \bigcirc y)\big),$$

$$x \bigcirc y \iff \exists_{z\in M}\big(z \sqsubseteq x \wedge z \sqsubseteq y\big).$$

The final formula is (df $\bigcirc$). □

[24] Compare p. 129. The formula elementarily defining the relation $\wr$ in terms of the relation $\sqsubseteq$ (resp. $\sqsubset$) would arise from the elementary notation of condition (df $\wr$) (resp. the 'compilation' of conditions (df $\wr$) and (df $\sqsubseteq$)) and, in the other direction, of the condition ($\mathrm{df}_\wr \sqsubseteq$) (resp. ($\sqsubset = \sqsubseteq \setminus \mathrm{id}$) and ($\mathrm{df}_\wr \sqsubseteq$)).

If we accept that the relation $\sqsubseteq$ is primitive and $\wr$ defined then — with the groups of conditions we have accepted — the class of structures of the form $\langle M, \sqsubseteq\rangle$ is simply the class **TS**.

[25] Condition (G_1) is the first of the axioms of the calculus of individuals from [Goodman, 1951] (formula 2.41 on p. 44). If the identity $\wr = -\bigcirc$ holds, then it is easy to show that (G_1) is equivalent to ($\mathfrak{a}$) (cf. lemmas 2.1 and 6.1).

LEMMA 7.2. *If relations* $\bigcirc$ *and* $\sqsubseteq$ *satisfy conditions* ($t_\sqsubseteq$) *and* (SSP), *then* $\sqsubseteq$ *is reflexive.*

PROOF. Assume for a contradiction that for a certain $x \in M$, we have $x \not\sqsubseteq x$. By virtue of (SSP), there exists a y such that $y \sqsubseteq x$ and $\neg\, y \bigcirc x$. By applying (df $\bigcirc$) we further get a contradiction, just as in the proof of Lemma 5.7. □

LEMMA 7.3. *For arbitrary binary relations* $\bigcirc$ *and* $\sqsubseteq$ *in a non-empty set* M, *the following systems of conditions are equivalent*:

(a) ($t_\sqsubseteq$), (SSP) *and* (df $\bigcirc$);
(b) (df$_\circ$ $\sqsubseteq$) *and* (df $\bigcirc$);
(c) (df$_\circ$ $\sqsubseteq$) *and* (G_1).

PROOF. '(a) ⇔ (b)' By virtue of lemma 7.2, ($r_\sqsubseteq$) follows from (SSP) and ($t_\sqsubseteq$). We showed on p. 90 that (df$_\circ$ $\sqsubseteq$) follows from (SSP), ($r_\sqsubseteq$) and ($t_\sqsubseteq$).

From (df$_\circ$ $\sqsubseteq$) we obtain the reflexivity and transitivity of the relation $\sqsubseteq$. Condition (II.2.6) follows from ($r_\sqsubseteq$), ($t_\sqsubseteq$) and (df $\bigcirc$), i.e., for all $x, y \in M$ we have $\mathbb{I}(x) \subseteq \mathbb{O}(y)$ iff $\mathbb{O}(x) \subseteq \mathbb{O}(y)$. From this and (df$_\circ$ $\sqsubseteq$) for all $x, y \in M$ we get: if $\mathbb{I}(x) \subseteq \mathbb{O}(y)$ then $x \sqsubseteq y$, i.e., (SSP) holds.

'(b) ⇔ (c)' By virtue of Lemma 7.1(ii). □

From lemmas 7.1(i) and 7.3 follows:

PROPOSITION 7.4. *If we add one of conditions* (antis$_\sqsubseteq$) *or* (ext$_\circ$) *to the groups* (a)–(c) *from Lemma* 7.3, *then we will obtain six equivalent groups of conditions.*

LEMMA 7.5. *Let* M *be an arbitrary non-empty set,* $\bigcirc$ *and* $\sqsubseteq$ *be binary relations in* M *satisfying one of the six expanded systems from Proposition* 7.4 *and* $\mathbb{Fu}, \mathbb{Sum} \subseteq M \times \mathcal{P}(M)$ *be relations defined by conditions* (df′ $\mathbb{Fu}$) *and* (df $\mathbb{Sum}$), *respectively. Then*:

(i) $\mathbb{Fu} = \mathbb{Sum}$.
(ii) *Conditions* (L4) *and* ($\exists\mathbb{Fu}$) *are equivalent.*

PROOF. On p. 81, by using just (df $\mathbb{Sum}$), (df′ $\mathbb{Fu}$), (df $\bigcirc$) and ($t_\sqsubseteq$), we proved the inclusion $\mathbb{Sum} \subseteq \mathbb{Fu}$. On p. 142, by using just (df $\mathbb{Sum}$), (df′ $\mathbb{Fu}$), (df $\bigcirc$), (df$_\circ$ $\sqsubseteq$) and ($r_\sqsubseteq$), we proved the inclusion $\mathbb{Fu} \subseteq \mathbb{Sum}$. We therefore have: $\mathbb{Sum} = \mathbb{Fu}$, i.e., conditions (L4) and ($\exists\mathbb{Fu}$) state the same thing. □

The theorem below may be used in deriving the following various equivalent axiomatisations of mereological structures.

THEOREM 7.6. *Let M be any non-empty set, $\bigcirc$ and $\sqsubseteq$ be binary relations in M, and $\mathbb{F}\mathrm{u}, \mathbb{S}\mathrm{um} \subseteq M \times \mathcal{P}(M)$ be defined by (df′ $\mathbb{F}\mathrm{u}$) and (df $\mathbb{S}\mathrm{um}$). Moreover, suppose that $\bigcirc$, $\sqsubseteq$, $\mathbb{F}\mathrm{u}$, $\mathbb{S}\mathrm{um}$ meet one of the six groups from Proposition 7.4 plus one of the equivalent sentences ($\exists\mathbb{F}\mathrm{u}$) or (L4). Then:*

(i) *If the binary relation $\sqsubset$ is defined by the condition ($\sqsubset=\sqsubseteq\setminus\mathrm{id}$), then the structure $\langle M, \sqsubset\rangle$ belongs to* **MS**.

(ii) *The structure $\langle M, \sqsubseteq\rangle$ belongs to* **TS**.

PROOF. *Ad* (i): If x $\mathbb{S}\mathrm{um}$ S and y $\mathbb{S}\mathrm{um}$ S then, by Lemma 7.5(i), x $\mathbb{F}\mathrm{u}$ S and y $\mathbb{F}\mathrm{u}$ S as well. Therefore $\mathbb{O}(x) = \bigcup \mathbb{O}(S) = \mathbb{O}(y)$, by virtue of (df′ $\mathbb{F}\mathrm{u}$). Hence $x = y$, by virtue of ($\mathrm{ext}_\circ$). We have therefore proved (L3).

By virtue of our assumptions, we have: ($\mathrm{r}_\sqsubseteq$), ($\mathrm{t}_\sqsubseteq$), ($\mathrm{antis}_\sqsubseteq$). The relation $\sqsubset$, defined by the equality$\sqsubset := \sqsubseteq \setminus \mathrm{id}_M$, therefore meets conditions (L1) and (L2).

Ad (ii): We have ($\mathrm{t}_\sqsubseteq$), and (L3-L4) follows from (L3) and (L4). □

Eberle [1967] and Smith [1993] have considered mereology as a first-order theory with identity with a primitive two-place predicate "$<$" associated with the relation $\sqsubseteq$ (corresponding to our "$\sqsubseteq$"). The predicate associated with the relation $\bigcirc$ they defined with the help of a formula whose counterpart here is (df $\bigcirc$). They established that the relations $\sqsubseteq$ and $\bigcirc$ satisfy conditions ($\mathrm{antis}_\sqsubseteq$) and ($\mathrm{df}_\circ \sqsubseteq$). Furthermore, they adopted an infinite number of elementary axioms falling under one schema. We obtain these axioms from ($\exists\mathbb{F}\mathrm{u}$) by 'unravelling' the definition (df′ $\mathbb{F}\mathrm{u}$) and replacing the formula "$x \in S$" with an arbitrary formula $\varphi(x)$ of their theory, which has at least one free variable "x". It may therefore be accepted that Eberle and Smith were using a system composed out of ($\mathrm{df}_\circ \sqsubseteq$), (df $\bigcirc$), ($\mathrm{antis}_\sqsubseteq$) and ($\exists\mathbb{F}\mathrm{u}$), whilst bypassing the fact that they explored mereology as a first-order theory.[26]

In his calculus of individuals Goodman [1951] also considers mereology as a first-order theory with a primitive two-place predicate "o" associated with the relation $\bigcirc$. With the help of this predicate, other two-place predicates are defined: "$\ll$", "$<$", "$\urcorner$" and "$=$". The first three predicates correspond to our relations $\sqsubset$, $\sqsubseteq$, and $\wr$, respectively. They are defined by elementary counterparts of conditions ($\sqsubset=\sqsubseteq\setminus\sqsupseteq$), ($\mathrm{df}_\circ \sqsubseteq$)

[26] We shall talk about the difference between both approaches in Chapter VI. Under the 'elementary' approach — from a structure-theoretical point of view — we will assume only the existence of fusions for non-empty sets $\{x \in M : \varphi(x)\}$, i.e., elementarily definable sets (they can be elementary definability with parameters when other free variables occur in the formula $\varphi(x)$; see p. 292 in Appendix II).

and ($\wr{=}{-}\bigcirc$). Furthermore, Goodman adopts a near-undefined predicate in his theory, which is marked schematically by "$\ldots x$".[27] The predicate "=" is defined via an elementary counterpart of condition ($\mathrm{ext}_\circ$) and is supposed to fill the role of identity, i.e., the axiom-schema "$x = y \wedge \ldots x \Rightarrow \ldots y$" is implicitly accepted.[28] The axioms are moreover certain elementary formulae, of which the first corresponds to condition (G_1) and the others guarantee that in an arbitrary model $\langle M, \bigcirc\rangle$ of the theory for arbitrary $x, y \in M$, there exists a sum $x + y$ such that $x + y \ \mathbb{Fu}\ \{x, y\}$; if $x \bigcirc y$, then there exists a product xy such that $xy \ \mathbb{Nu}\ \{x, y\}$; there exists a complement $-x$ such that $\mathbb{I}(-x) = \{z \in M : z \wr x\}$, if the indicated set is non-empty, because the relation $\sqsubseteq$ is reflexive, i.e, $-x \in \mathbb{I}(-x)$.

Breitkopf [1978] broadens Goodman's calculus of individuals [1951]. Breitkopf adopts the same definitions as Goodman. The first axiom in [Breitkopf, 1978] — as in [Goodman, 1951] — is an elementary counterpart of (G_1). The second axiom is an elementary sentential schema arising from ($\exists\mathbb{Fu}$), which we mentioned in footnote 3.1 on p. 147. Bypassing the fact, therefore, that an elementary theory was explored in [Breitkopf, 1978], Breitkopf adopted system: ($\mathrm{df}_\circ \sqsubseteq$), ($\mathrm{G}_1$), ($\mathrm{ext}_\circ$) and ($\exists\mathbb{Fu}$).[29]

We may also prove an counterpart of Theorem 6.5 for the six equivalent groups of Proposition 7.4 by swapping in them every occurrence of the symbol "$\wr$" and condition (df $\mathbb{Fu}$) for the symbol "$\bigcirc$" and condition (df′ $\mathbb{Fu}$), respectively. The final conclusion of this counterpart should have the form: *the relations $\sqsubseteq$ and $\bigcirc$ also satisfy the conditions given in the extended groups of Proposition 7.4.*

[27] Which corresponds to the sentential schema "Fx".

[28] This schema is mentioned by Libardi [1990, p. 134] as the third axiom of Goodman's theory. This schema is essential. As a matter of fact, it is possible to prove the reflexivity, symmetry and transitivity of the predicate "=" from the axioms and definitions he adopts, and that it behaves like identity with respect to the predicates "o", "$\ll$", "<" and "$\rceil$". We will not, however, prove the general schema "$x = y \wedge Fx \Rightarrow Fy$".

[29] Breitkopf also accepted a third axiom, which is an elementary counterpart of our condition ($\exists\mathbb{Nu}$) (see condition (8.2) on p. 167) . It is a certain sentential schema obtained from ($\exists\mathbb{Nu}$) by 'unravelling' definition (df $\mathbb{Nu}$) and replacing the set-theoretic formula "$z \in S$" with the sentential schema "Fz". As Simons [1987, (fn. 22, p. 36] correctly observes this third axiom is dependent on the first two (cf. Corollary 3.4 and Theorem 7.6). Simons' proof of this is unsuccessful, however, and because of this draws the false conclusion, that the conditions ($\exists\sqcap$) is not necessary in the group of conditions: (L2), (WSP), ($\exists\sqcap$) and ($\exists\mathbb{Fu}$) (the group 16° in Theorem 3.15). We shall return to this matter in the next section.

We shall end by observing that, for structures of the form $\langle M, \bigcirc \rangle$, which satisfy the conditions given in Theorem 7.6, analogous comments to those made after Theorem 6.5 may be made.

8. Simons' Classical Extensional Mereology

Classical Extensional Mereology, Simons' theory from [1987, p. 37], is formulated in a schematic first-order language with certain specific symbols added to it. The first of these is a two-place predicate "$\ll$" (corresponding to our predicate "$\sqsubset$") associated with the relation $\sqsubset$. Other two-argument predicates "$<$" (corresponding to our predicate "$\sqsubseteq$"), "$\bigcirc$" and "$\wr$" are associated with the relations $\sqsubseteq$, $\bigcirc$ and $\wr$, respectively, and are defined with the help of certain elementary formulae corresponding to our (df $\sqsubseteq$), (df $\bigcirc$) and ($\wr{=}{-}\bigcirc$). In presenting the theory from [Simons, 1987], we shall use our terminology because the formulae expressed in terms of it may be easily translated into the elementary formulae as expressed by Simons.

The axioms of Simons' Classical Extensional Mereology are: (L1), (L2), (WSP)[30] and the equivalent of the condition signified in [Simons, 1987, pp. 36–37] by "SA24" and "GSP":[31]

$$\exists_x Fx \implies \exists_y \forall_z \big(z \bigcirc y \Leftrightarrow \exists_x (Fx \wedge x \bigcirc z)\big). \tag{GSP}$$

Considering the condition (df′ $\mathbb{F}\mathbb{u}$) (equivalent to (df $\mathbb{F}\mathbb{u}$), based on ($\wr{=}{-}\bigcirc$)), we see that the schema (GSP) states the same as in our terminology the following schema:

$$\exists_{x \in M} Fx \implies \exists_{y \in M}\, y \,\mathbb{F}\mathbb{u}\, \{x \in M : Fx\}\,, \tag{8.1}$$

i.e., condition ($\exists\mathbb{F}\mathbb{u}$) applied to the set $S := \{x \in M : Fx\}$.

It may be taken that Simons intended to axiomatise 'ordinary' classical mereology. This is attested to by the results below and — in the context of the equivalence of systems 1°–19° from theorems 1.2, 3.7 and 3.15 — the passage from his [Simons, 1987] which is reprinted below.

The group of axioms described on p 165 (and in footnote 29) accepted by Breitkopf [1978] axiomatises classical mereology (in an elementary

[30] We have already noted (in footnote 15) that axiom (L1) is inessential here.

[31] In (GSP), so as to standardise things, we have swapped some symbols used in [Simons, 1987] for the symbols we are using here. And so we have used brackets instead of the 'corners' playing the role of brackets in [Simons, 1987].

form). Simons [1987, p. 36, fn. 22] observes that the third axiom in [Breitkopf, 1978] is not independent of the first two. As was already mentioned in footnote 29, Brietkopf's third axiom, i.e., the schema (1.3) in Breitkopf [1978] corresponds to the condition ($\exists\mathbb{N}\Cup$) from p. 141 which was given the following elementary form:

$$\exists_z\forall_x(Fx \Rightarrow z \sqsubseteq x) \implies \exists_y\forall_z\big(z \sqsubseteq y \Leftrightarrow \forall_x(Fx \Rightarrow z \sqsubseteq x)\big). \tag{8.2}$$

That is, it corresponds to condition ($\exists\mathbb{N}\Cup$) applied to $S = \{x \in M : Fx\}$. By substituting the formula "$x = u \vee x = v$" for the schema "Fx", we obtain $S = \{u, v\}$. Thus, as on p. 148, we derive condition ($\exists\sqcap$) from (8.2), the former being condition (1.19) in [Breitkopf, 1978].[32]

Simons considers that (8.2), i.e., ($\exists\sqcap$) as well, is not independent of (L1), (L2), (WSP) and (GSP). Since he has earlier derived condition(SSP) from the axioms of his "Minimal Extensional Mereology" — i.e., from (L1), (L2), (WSP) and ($\exists\sqcap$) — by applying "the rule of cut", he comes to the conclusion that he can also derive condition (SSP) from the axioms of this Classical Extensional Mereology.[33]

Condition ($\exists\sqcap$) is, however, independent of (L1), (L2), (WSP) and (GSP) and the definitions established in [Simons, 1987]. As we have already noted in footnote 14, model 4 is a model of Simons' theory, but condition ($\exists\sqcap$) does not hold in this model. For in that structure we have $12 \bigcirc 21$, but there does not exist a $y \in M_3$ such that for any $z \in M_3$ we have: $z \sqsubseteq y$ iff $z \sqsubseteq 12$ and $z \sqsubseteq 21$.

We observe also that since we have $12 \not\sqsubseteq 21$ and $\mathbb{I}(12) \subseteq \mathbb{O}(21)$ in model 4, condition (SSP) is therefore not true in it. Hence condition (SSP) does not follow from the axioms and definitions accepted by Simons. Thus the formula SCT12 should not be amongst the list of theorems he gives, as it is — written in an elementary way — the counterpart of our formula (II.6.1). In a similar respect — as we have shown in the proof of Proposition 3.11 — formulae SCT15 and SCT16 should

[32] Breitkopf [1978] does not in general use the third axiom to introduce condition (1.19), but takes a somewhat roundabout route which uses just the first two axioms (G_1) and (GSP). From the first and from definition ($\mathrm{df}_\circ \sqsubseteq$) he gets ($\mathrm{df} \bigcirc$). He later substitutes the formula "$x \sqsubseteq u \wedge x \sqsubseteq v$" in (GSP) for the schema "Fx", in other words, he applies ($\exists\mathbb{F}\Cup$) to $S = \{x : x \sqsubseteq u \wedge x \sqsubseteq v\}$. Besides this, he makes use of condition ($\mathrm{df}_\circ \sqsubseteq$) many times. In short, Breitkopf shows that condition ($\exists\sqcap$) follows from: ($\mathrm{r}_\sqsubseteq$), ($\mathrm{t}_\sqsubseteq$), ($\mathrm{df} \bigcirc$), (SSP) and (GSP) (resp. ($\mathrm{df}' \mathbb{F}\Cup$) and ($\exists\mathbb{F}\Cup$)).

[33] From (SSP) and (L1), (L2) and (GSP), in fact, we get 'true' classical elementary mereology. Compare the group 11° in Theorem 3.7.

not find themselves included in his list, since they are the elementary counterparts of our formulae (df$_\circ \sqsubseteq$) and (ext$_\circ$). Simons [1987] does not prove that the aforementioned formulae are theses of his system. He just cites the list of theorems given by Breitkopf [1978]. Simons' axioms and definitions, however, create an essentially weaker system than the system created by Breitkopf's axioms and definitions. As we proved above, by extending the axiomatisation of his minimal extensional mereology by the inclusion of (GSP), Simons unnecessarily rejected axiom ($\exists\sqcap$). We will try to explain below what drove Simons to this.

Simons [1987, pp. 35 and 37] introduced definition SD9 by means of a singular description, in which the definiendum "$\sigma x(Fx)$" is supposed to be a 'general sum' (fusion) of all F-ers, formally corresponding — in our terminology — to the sum $\bigsqcup\{x \in M : Fx\}$. Definition SD9 is as follows:

$$\sigma x(Fx) \approx (\iota\, y)\, \forall_z \big(z \bigcirc y \Leftrightarrow \exists_x (Fx \wedge x \bigcirc z)\big), \tag{SD9}$$

where the symbol "$\approx$" is used with the following sense: a formula of the form "$a \approx b$" says that a and b signify the same object or signify nothing.[34]

In the language we are using, definition (SD9) may be written as a definition of the partial operator Σ of the sum (fusion) of all elements of a given subset of the universe, i.e., an operator, which need not be defined on $\mathcal{P}_+(M)$. The definition of the operator Σ therefore looks like this:

$$\Sigma(S) := (\iota\, y)\, \forall_{z\in M} \big(z \bigcirc y \Leftrightarrow \exists_{x\in S}\, x \bigcirc z\big). \tag{df Σ}$$

The formulation "$\sigma x(Fx)$" used by Simons would correspond to "$\Sigma\{x : Fx\}$" in our language. By applying the definitions in [Simons, 1987] and (df$'$ $\mathbb{F}\mathrm{u}$) and (df Σ) we see that $\Sigma(S) = (\iota\, y) y \mathbin{\mathbb{F}\mathrm{u}} S$. Hence, for the domain $\operatorname{dom}\Sigma$ of the operator Σ we have $\operatorname{dom}\Sigma = \{S \in \mathcal{P}(M) : \exists!_{y\in M}\, y \mathbin{\mathbb{F}\mathrm{u}} S\}$. Moreover, it follows from axioms ($\exists\mathbb{F}\mathrm{u}$) that $\operatorname{dom}\Sigma \subseteq \mathcal{P}_+(M)$.

[34] On p. 22 in [Simons, 1987] there is a table in which the formula "$a \approx b$" is explained as "Truth-conditions 'a' and 'b' designate the same individual or are both empty". Notice that on both sides of the symbol the names are empty not only when there is no such y for the predicate F which would satisfy the the condition "$\forall_z(z \bigcirc y \Leftrightarrow \exists_x(Fx \wedge x \bigcirc z))$". Both names are also empty when there is a y but not just one y. Axiom (GSP) ensures that if something is F, then we have a y which satisfies the aforementioned formula. Yet — as Proposition 3.11 entails — all the axioms accepted by Simons do not ensure that this y is unique. This is not the case in mereological structures, where an expression of the type "$\bigsqcup\{x \in M : Fx\}$" is a monoreferential term if and only if some element of the set M is F

The operator Σ is associated with the relation σ included in $M \times \operatorname{dom}\Sigma$ and defined by the following condition:

$$\sigma := \{\langle y, S\rangle \in M \times \operatorname{dom}\Sigma \;:\; y = \Sigma(S)\}. \tag{df σ}$$

Clearly, $\sigma \subseteq \mathbb{Fu}$, but in model 4 we have $\sigma \subsetneq \mathbb{Fu}$. In fact, $12 \;\mathbb{Fu}\; \{1,2\}$, but $\{1,2\} \notin \operatorname{dom}\Sigma$, because $21 \;\mathbb{Fu}\; \{1,2\}$ as well, i.e., $\neg\, 12\, \sigma\, \{1,2\}$.[35]

Alongside the definition of a "general sum" on p. 35 Simons introduces the definition SD10, in which the definiendum "$\pi\, x(Fx)$" is supposed to be the "general product" of all F-ers, which corresponds to the product $\bigsqcap\{x \in M : Fx\}$. Definition SD10 is as follows:

$$\pi x(Fx) \approx (\iota\, y)\, \forall_z \big(y < z \Leftrightarrow \forall_x (Fx \;\Rightarrow\; x < z)\big), \tag{SD10}$$

Simons adds that he adopts SD10 from Breitkopf. In our language, definition SD10 is none other than the definition of the partial operator $\inf_{\sqsubseteq}$ of the lower bound with respect to relation $\sqsubseteq$, which satisfies the following condition: $\inf_{\sqsubseteq}(S) := (\iota\, y)\; y \;\mathsf{inf}_{\sqsubseteq}\; S$. One just needs to consider condition (4.15) from Appendix I, that the relation $\sqsubseteq$ is reflexive and transitive and that and the relation $\inf_{\sqsubseteq}$ satisfies condition ($\mathrm{U}_{\mathsf{inf}}$). It follows from this last result and from (2.5) that, for arbitrary $y \in M$ and S belonging to the domain of the operator $\inf_{\sqsubseteq}$ we have:

$$y = \inf_{\sqsubseteq}(S) \iff y \;\mathsf{inf}_{\sqsubseteq}\; S \iff y \;\mathbb{Nu}\; S\,. \tag{8.3}$$

Having accepted axiom (GSP) and definition (SD10), Simons writes the following, however:

> There is no need for a special axiom for products, since if something is a part [36] of all F-ers, them the product may be defined as the sum of all such common parts. [37] In fact, in this case it may equally well

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[35] We observe that the group of axioms (L1), (L2), (WSP) and ($\exists\mathbb{Fu}$) expanded by the equality $\sigma = \mathbb{Fu}$ is equivalent to the group: (L1), (L2), ($\mathrm{U}_{\mathbb{Fu}}$) and ($\exists\mathbb{Fu}$) (and also — by virtue of Lemma 3.6 — to the group: (L1), (L2), (L3) and ($\exists\mathbb{Fu}$)). In fact, the pair ($\exists\mathbb{Fu}$) and $\sigma = \mathbb{Fu}$ is equivalent to the pair ($\exists\mathbb{Fu}$) and $\operatorname{dom}\Sigma = \mathcal{P}_{+}(M)$ (i.e., $y = \Sigma S$ iff $y \;\mathbb{Fu}\; S$, by virtue of (df Σ)), and this is equivalent to the pair ($\exists\mathbb{Fu}$) and ($\mathrm{U}_{\mathbb{Fu}}$). Furthermore, (WSP) follows from (L1), (L2) and (L3).

Thus, even if we were additionally to accept that $\sigma = \mathbb{Fu}$, we would not obtain an axiomatisation for mereology, because the formulae in that extended group are true in model 3.

[36] Remember that Simons' "part" is our "ingrediens".

[37] Footnote 22 added here reads: "Thus the third axiom of BREITKOPF 1978 is not independent."

> be defined as the least upper bound of all such common parts. The proof that the two definitions of product are equivalent uses the Strong Supplementation Principle and the existence of *binary* products to derive a contradiction from the assumption that the least upper bound of all common parts is *not* itself a common part. Since it therefore *is* a common part, it is then straightforward to show that this least upper bound is also the sum of common parts. [Simons, 1987, p. 36]

Taking into account the previously-indicated connection between (SD10) and the definition of the lower bound, we see that the first sentence in the passage above states that:

$$\bigcap \mathbb{l}(S) \neq \emptyset \implies \inf_{\sqsubseteq}(S) = \Sigma(\bigcap \mathbb{l}(S)), \tag{8.4}$$

or — with reference to (8.3) — asserts the truth of condition $(**)$. Simons later justifies (8.4) by saying that the lower bound is definable by the upper bound (i.e., $\inf_{\sqsubseteq}(S) = \sup_{\sqsubseteq}(\bigcap \mathbb{l}(S))$) and that from (SSP) and $(\exists\sqcap)$ we can derive:

$$\bigcap \mathbb{l}(S) \neq \emptyset \implies \sup_{\sqsubseteq}(\bigcap \mathbb{l}(S)) = \Sigma(\bigcap \mathbb{l}(S)). \tag{8.5}$$

Since it is easy to derive condition $(\exists\mathbb{N}\Cup)$ from (8.3), (8.4) and $(\exists\mathbb{F}\Cup)$,[38] we may also derive condition $(\exists\sqcap)$.[39] Simons has 'overlooked' the fact, however, that without $(\exists\sqcap)$ he cannot derive (8.5), which is the justification for condition (8.4). Quite simply, as Theorem 3.15 shows, in the context of formulae (L2), (WSP) and $(\exists\mathbb{F}\Cup)$, the formulae $(\exists\sqcap)$ and $(**)$ (resp. (8.4)) are equivalent.

In accordance with his position as presented in the first sentence of the passage above, Simons introduces the following definition in presenting his Classical Extensional Mereology on p. 37.

$$\pi x(Fx) \approx \sigma x\big(\forall_y(Fy \Rightarrow x \sqsubseteq y)\big), \tag{$\lightning$}$$

giving it once again the name "SD10".[40] Definition ($\lightning$) may be transformed into our terminology into the definition of the partial operator

[38] That is why in [Simons, 1987, s. 36] there was the need for footnote 22 (see footnote 37).

[39] Cf. Lemma 3.14.

[40] Considering the passage cited as a whole, it follows that Simons considers that definition ($\lightning$) is equivalent in his theory with definition SD10 (previously taken from Breitfkopf).

$\mathsf{\Pi}$ of the product of all elements of a given subset of the universe. We therefore define it by the identity $\mathsf{\Pi}(S) := \mathsf{\Sigma}(\bigcap \mathbb{I}(S))$. It follows from (8.5) that $\mathsf{\Pi} = \inf_{\sqsubseteq}$ or that:

$$\bigcap \mathbb{I}(S) \neq \emptyset \implies \mathsf{\Pi}(S) = (\iota\, x)\ x \ \mathbb{N}\mathbb{u}\ S\,. \tag{8.6}$$

But in Simons' theory condition (8.5) is not satisfied.

To end, let us add that formula SCT61 should also not be on Simons' list, this being the elementary counterpart of our formula (8.6). Nor should formulae SCT65 and SCT66 appear there, these being elementary counterparts of the formulae below:

$$S \neq \emptyset \implies \mathsf{\Sigma}(S) = (\iota\, x)\ x \ \mathsf{sup}_{\sqsubseteq}\ S\,,$$
$$S \neq \emptyset \implies \mathsf{\Sigma}(S) = (\iota\, x)\ x \ \mathbb{S}\mathbb{u}\mathbb{m}\ S\,.$$

Simons [1987, p. 40, fn. 24] states that, in his theory, the following are equivalent: his definition of a sum, the definition of a sum used by Tarski, and the definition of an upper bound. This is not true.

To sum up, we see that Simons has implicitly adopted condition (∗∗), which corresponds to both (8.4) and (8.5). Theorem 3.15 shows that Simons could have equally well adopted — in the context of (8.5) — he more general condition (∗) instead of (∗∗), or — the context of (8.4) — the more particular condition (∗∗∗). He could also have simply not rejected sentence (∃⊓).

Chapter V

The lattice of certain superclasses of the class **MS** — independence of conditions

Throughout this chapter, we shall be considering $L_{\sqsubset}$-structures of the form $\langle M, \sqsubset\rangle$. As in Chapter II — irrespective of the properties we established for the relation $\sqsubset$ — we define in the structure $\langle M, \sqsubset\rangle$ auxiliary relations $\sqsubseteq$, $\bigcirc$, $\wr$ and $\mathbb{S}\mathrm{um}$ by applying (df $\sqsubseteq$), (df $\bigcirc$), (df $\wr$) and (df $\mathbb{S}\mathrm{um}$), respectively. It follows from the definitions themselves, that the relations $\sqsubseteq$ and $\bigcirc$ are reflexive, that the relation $\wr$ is irreflexive, that $\bigcirc$ and $\wr$ are symmetric, that $\sqsubseteq$ is included in $\bigcirc$, and that the equality $\wr = -\bigcirc$ holds.

Theorem III.2.4 says that the class **MS** is not elementarily axiomatisable. All the remaining superclasses of the class **MS** considered in this chapter will be finitely axiomatisable (see Theorem 8.1)

1. The lattice of certain classes between L123 and L12

First we use the following three conditions:

$$\forall_{x,y\in M}(x \sqsubset y \implies y \not\sqsubset x), \tag{L1}$$

$$\forall_{x,y,z\in M}(x \sqsubset y \wedge y \sqsubset z \implies x \sqsubset z), \tag{L2}$$

$$\forall_{S\in\mathcal{P}(M)}\forall_{x,y\in M}(x \;\mathbb{S}\mathrm{um}\; S \wedge y \;\mathbb{S}\mathrm{um}\; S \implies x = y), \tag{L3}$$

(L1) and (L2) say that the relation $\sqsubset$ is asymmetric and transitive, respectively, and (L3) says that if a set has a mereological sum then it is unique. On p. 71 we described **L12** as the class of all those structures in which the relation $\sqsubset$ satisfies conditions (L1) and (L2). Thus, **L12** is the class **SPOS** of all strictly partially ordered sets in which the relation $\sqsubset$ is also irreflexive, i.e.:

$$\forall_{x\in M}\; x \not\sqsubset x\,. \tag{$\mathrm{irr}_{\sqsubset}$}$$

We define **L123** as the class of all structures of **L12** in which (L3) holds. From Proposition IV.4.5(ii) we have:

PROPOSITION 1.1. **L123** $\subsetneq$ **L12**.

In this section we will examine the lattice of certain classes of structures $\boldsymbol{K}$ such that $\mathbf{L123} \subsetneq \boldsymbol{K} \subsetneq \mathbf{L12}$. The addition of the name of a condition C to the name of a class $\boldsymbol{K}$ of structures generates the name of the class of those structures in which C holds. Furthermore, the connection between these names of conditions creates the name of the class of structures in which the conditions hold.

In this section we will use the following conditions to designate classes of structures:

$$\exists_{x \in M} \forall_{y \in M}\, x \sqsubseteq y \iff \operatorname{Card} M = 1\,, \tag{$\nexists 0$}$$

$$\operatorname{Card} M > 1 \iff \exists_{x,y \in M}\, x \wr y\,, \tag{$\exists \wr$}$$

$$\forall_{x,y \in M} (\emptyset \neq \mathbb{P}(x) = \mathbb{P}(y) \implies x = y), \tag{$\mathrm{ext}_{\sqsubset}$}$$

$$\forall_{x,y \in M} \big(\mathbb{O}(x) = \mathbb{O}(y) \implies x = y\big), \tag{$\mathrm{ext}_{\circ}$}$$

$$\forall_{x,y \in M} (x \mathrel{\mathsf{Sum}} \{y\} \implies x = y), \tag{$\mathrm{S}_{\mathsf{Sum}}$}$$

$$\forall_{x,y \in M} \big(x \sqsubset y \implies \exists_{z \in M} (z \sqsubset y \wedge z \wr x)\big), \tag{WSP}$$

$$\forall_{x,y \in M} \big(\emptyset \neq \mathbb{P}(x) \subseteq \mathbb{P}(y) \implies x \sqsubseteq y\big), \tag{PPP}$$

$$\forall_{x,y \in M} \big(\emptyset \neq \mathbb{P}(x) \subsetneq \mathbb{P}(y) \implies x \sqsubset y\big). \tag{PPP$'$}$$

Of course we have for all classes $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ such structures we have:

$$\boldsymbol{K}_1 \subseteq \boldsymbol{K}_2 \quad \text{iff} \quad \begin{array}{l}\text{from the set of conditions defining } \boldsymbol{K}_1 \\ \text{follow all conditions defining } \boldsymbol{K}_2.\end{array}$$

For example, we have:

1. $\mathbf{L12} = (\mathrm{L2}) + (\mathrm{irr}_{\sqsubset})$
2. $\mathbf{L12} + (\mathrm{WSP}) = (\mathrm{L2}) + (\mathrm{WSP}) = \mathbf{L12} + (\mathrm{S}_{\mathsf{Sum}})$ by Lemma II.4.1
3. $\mathbf{L123} := \mathbf{L12} + (\mathrm{L3}) = \mathbf{L12} + (\mathrm{ext}_{\circ})$ by Theorem II.4.4
4. $\mathbf{L12} + (\exists \wr) \subseteq \mathbf{L12} + (\nexists 0)$ see p. 85
5. $(\mathrm{L2}) + (\mathrm{WSP}) \subseteq (\mathrm{L2}) + (\exists \wr)$ see p. 85
6. $\mathbf{L123} \subseteq (\mathrm{L2}) + (\mathrm{WSP}) + (\mathrm{ext}_{\sqsubset})$ see p. 83
7. $\mathbf{L12} + (\mathrm{PPP}) = \mathbf{L12} + (\mathrm{PPP}') + (\mathrm{ext}_{\sqsubset})$ see Remark II.6.1

We will prove that the classes featured in Diagram 1 (which are composed out of structures we shall be examining in this chapter) create a lattice in which the relation of inclusion $\subseteq$ is a partial order. The dependence $\boldsymbol{K}_1 \to \boldsymbol{K}_2$ in Diagram 1 indicates that $\boldsymbol{K}_1 \subsetneq \boldsymbol{K}_2$. The fact that the appropriate inclusions hold in diagram 1 follows either from the definitions of the classes that occur there or from the above points 2–6. We will now show that these inclusions are proper and that no others hold.

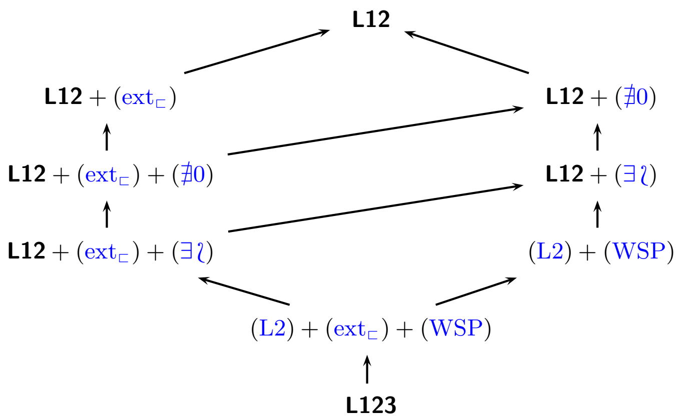
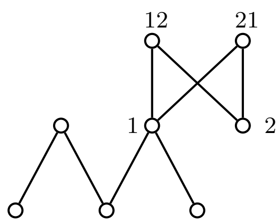


Diagram 1. The lattice of certain classes between **L123** and **L12**



Model 6. (L1), (L2), (WSP) and (ext$_\sqsubset$) hold, but (L3) does not hold

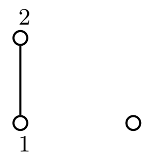
Proposition 1.2. *Condition* (L3) *does not follow from the set* {(L1), (L2), (WSP), (ext$_\sqsubset$)}. *Thus, we obtain*:

- **L123** $\subsetneq$ (L2) + (WSP) + (ext$_\sqsubset$).

Proof. By expanding model 4 we obtain model 6 of the formulae (L1), (L2), (WSP) and (ext$_\sqsubset$), in which (L3) is not true, since we have both 12 $\mathbb{S}\mathrm{um}$ $\{1, 2\}$ and 21 $\mathbb{S}\mathrm{um}$ $\{1, 2\}$.[1] □

Proposition 1.3. *Condition* (ext$_\sqsubset$) *does not follow from the set* {(L1), (L2), (WSP)}, *so also it does not follow both from* {(L1), (L2), (∃≀)} *and* {(L1), (L2), (∄0)}. *Thus, we obtain*:

[1] Note that model 4 suffices to show that (L3) does not follow from (L1), (L2) and (WSP). Furthermore, model 5 suffices to show that (L3) does not follow from (L1), (L2) and (ext$_\sqsubset$).



Model 7. (L1), (L2), $(\mathrm{ext}_{\sqsubset})$, $(\exists\wr)$, (PPP) hold, but (WSP) does not hold

- **L12** + (WSP) + $(\mathrm{ext}_{\sqsubset})$ $\subsetneq$ (L2) + (WSP),
- **L12** + $(\mathrm{ext}_{\sqsubset})$ + $(\exists\wr)$ $\subsetneq$ **L12** + $(\exists\wr)$,
- **L12** + $(\mathrm{ext}_{\sqsubset})$ + $(\nexists 0)$ $\subsetneq$ **L12** + $(\nexists 0)$,
- **L12** + $(\mathrm{ext}_{\sqsubset})$ $\subsetneq$ **L12**,
- **L12** + (WSP) $\not\subseteq$ **L12** + $(\mathrm{ext}_{\sqsubset})$ + $(\exists\wr)$,
- **L12** + $(\exists\wr)$ $\not\subseteq$ **L12** + $(\mathrm{ext}_{\sqsubset})$ + $(\nexists 0)$,
- **L12** + $(\nexists 0)$ $\not\subseteq$ **L12** + $(\mathrm{ext}_{\sqsubset})$.

PROOF. In model 4, (L1), (L2) and (WSP) hold, but $(\mathrm{ext}_{\sqsubset})$ does not hold, because $\mathbb{P}(12) = \{1, 2\} = \mathbb{P}(21)$. □

PROPOSITION 1.4. *Condition* (WSP) *does not follow from the set* {(L1), (L2),(PPP),$(\exists\wr)$}, *so also it does not follow from the set* {(L1), (L2), $(\mathrm{ext}_{\sqsubset})$, $(\exists\wr)$}. *Thus, we obtain*:

- **L12** + $(\mathrm{ext}_{\sqsubset})$ + (WSP) $\subsetneq$ **L12** + $(\mathrm{ext}_{\sqsubset})$ + $(\exists\wr)$,
- **L12** + (WSP) $\subsetneq$ **L12** + $(\exists\wr)$,
- **L12** + $(\mathrm{ext}_{\sqsubset})$ + $(\exists\wr)$ $\not\subseteq$ **L12** + (WSP).

PROOF. In model 7 of conditions (L1), (L2), $(\mathrm{ext}_{\sqsubset})$ and $(\exists\wr)$ condition (WSP) does not hold, because $1 \sqsubset 2$ and $\mathbb{P}(2) \subseteq \mathbb{O}(1)$.[2] □

PROPOSITION 1.5. *Condition* $(\exists\wr)$ *does not follow from the set* {(L1), (L2), (PPP), $(\nexists 0)$}, *so also it does not follow from the set* {(L1), (L2), $(\mathrm{ext}_{\sqsubset})$, $(\nexists 0)$}. *Thus, we obtain*:

- **L12** + $(\mathrm{ext}_{\sqsubset})$ + $(\exists\wr)$ $\subsetneq$ **L12** + $(\mathrm{ext}_{\sqsubset})$ + $(\nexists 0)$,
- **L12** + $(\exists\wr)$ $\subsetneq$ **L12** + $(\nexists 0)$,
- **L12** + $(\mathrm{ext}_{\sqsubset})$ + $(\nexists 0)$ $\not\subseteq$ **L12** + $(\exists\wr)$.

PROOF. Let $\mathfrak{M} = \langle M, \sqsubset \rangle$ be a structure in which M is the set of negative integers and $\sqsubset$ is the relation $<$. Then $\sqsubseteq$ is the relation $\leqslant$, $\bigcirc$ is the full relation, and $\wr$ is the empty one. Thus, conditions (L1), (L2), $(\nexists 0)$ and (PPP) hold, but $(\exists\wr)$ do not hold. □

[2] Model 5 suffices to show that (WSP) does not follow from {(L1), (L2), (PPP)}.

PROPOSITION 1.6. *Condition* ($\nexists 0$) *does not follow from the set* {(L1), (L2), (PPP)}, *so also it does not follow from the set* {(L1), (L2), ($\mathrm{ext}_\sqsubset$)}. *Thus, we obtain*:

- $\mathbf{L12} + (\mathrm{ext}_\sqsubset) + (\nexists 0) \subsetneq \mathbf{L12} + (\mathrm{ext}_\sqsubset)$,
- $\mathbf{L12} + (\nexists 0) \subsetneq \mathbf{L12}$,
- $\mathbf{L12} + (\mathrm{ext}_\sqsubset) \nsubseteq \mathbf{L12} + (\nexists 0)$.

PROOF. In model 5 of conditions (L1), (L2) and (PPP) (we have $\mathbb{P}(1) = \emptyset \neq \mathbb{P}(2)$) condition ($\nexists 0$) does not hold. □

Moreover notice that conditions ($\mathrm{irr}_\sqsubset$) (and thus a fortiori condition (L1)) and (WSP) do not follow from the set composed of conditions: (L2), (L3), (L4), ($\mathrm{ext}_\sqsubset$), ($\exists\wr$), ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$), ($\nexists 0$), ($\mathrm{ext}_\circ$), (PPP), (PPP′), (SSP), (M1), (M2), where

$$\forall_{S \in \mathcal{P}_+(M)} \exists_{x \in M}\; x \;\mathbb{S}\mathrm{um}\; S\,, \tag{L4}$$

$$\forall_{x,y \in M} \big(x \not\sqsubseteq y \implies \exists_{z \in M} (z \sqsubseteq x \wedge z \wr y)\big), \tag{SSP}$$

$$\forall_{S \in \mathcal{P}(M)} \forall_{x,y \in M} \big(\mathbb{I}(x) \subseteq \bigcup \mathbb{O}[S] \wedge S \subseteq \mathbb{I}(y) \implies x \sqsubseteq y\big), \tag{M1}$$

$$\forall_{S_1, S_2 \in \mathcal{P}(M)} \forall_{x,y \in M} \big(x \;\mathbb{S}\mathrm{um}\; S_1 \wedge y \;\mathbb{S}\mathrm{um}\; S_2 \wedge S_1 \subseteq S_2 \implies x \sqsubseteq y\big). \tag{M2}$$

We take the structure with the one-element universe $\{0\}$ and such that $0 \sqsubset 0$. Thus, the structure does not fulfil conditions ($\mathrm{irr}_\sqsubset$) and (WSP). It is clear that conditions (L2), ($\mathrm{ext}_\sqsubset$), (PPP), (PPP′) and ($\mathrm{S}_{\mathbb{S}\mathrm{um}}$) are true in it. By virtue of the definitions we have $0 \sqsubseteq 0$ and $0 \circ 0$. Thus, the equivalences ($\exists\wr$) and ($\nexists 0$) hold, because both sides of the former are false and both sides of the latter are true. Furthermore, $0 \;\mathbb{S}\mathrm{um}\; \{0\}$.

2. The lattice of certain classes between L12+(SSP) and L12

To the class $\mathbf{L12}$ + (SSP) belong those and only those structures of the form $\langle M, \sqsubset\rangle$ which are polarised strict partial orders. In the class **L12**, condition (SSP) is equivalent to each of the conditions (M1) and (M2) (cf. Lemma II.6.2). Hence we have:

8. $\mathbf{L12} + (\mathrm{SSP}) = \mathbf{L12} + (\mathrm{M1}) = \mathbf{L12} + (\mathrm{M2})$.

Furthermore, by virtue of Theorem IV.3.1, the identity $\mathbb{S}\mathrm{um} = \mathbb{F}\mathrm{u}$ holds for the relation $\mathbb{F}\mathrm{u}$ as defined by condition (df $\mathbb{F}\mathrm{u}$).

The theory of the class $\mathbf{L12}$ + (SSP) is the strongest one which does not postulate the existence of mereological sums [see Pietruszczak, 2013, pp. 67–68]. So we might call it either *Neutral Existential Mereology* or *Non-Existential Mereology*.

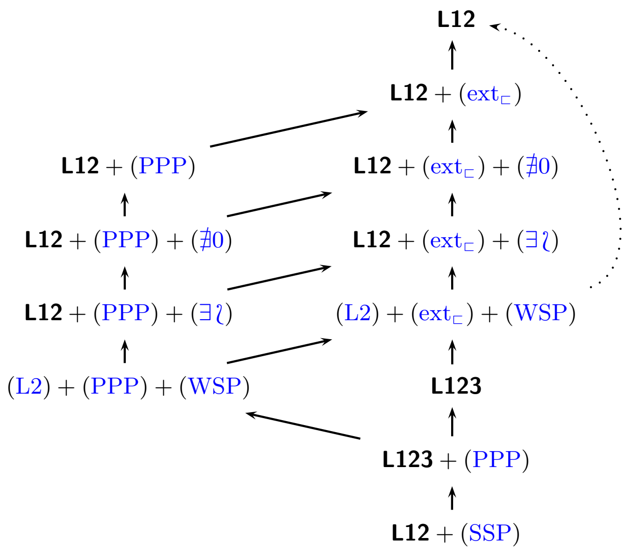


Diagram 2. The lattice of certain classes between **L12** + (SSP) and **L12**

On the strength of identity 7 we also have the following equalities:

9. (L2) + (PPP) + (WSP) = (L2) + (PPP′) + ($\mathrm{ext}_{\sqsubset}$) + (WSP),
10. **L123** + (PPP) = **L123** + (PPP′) + ($\mathrm{ext}_{\sqsubset}$).

We will now expand the lattice from diagram 1 to diagram 2 with classes that are created using conditions (PPP) and (SSP).

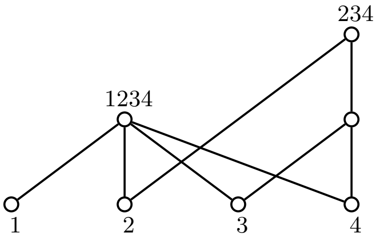
The fact that all corresponding inclusions hold in diagram 2 follows from the definitions of the classes occurring there or from results proved in II.4. By using propositions 1.4–1.6 and the results proven below we will show that these inclusions are proper and that no others hold.

Proposition 2.1. *Condition* (SSP) *does not follow from the set* {(L1), (L2), (L3), (PPP)}. *Thus, we obtain*:

- **L12** + (SSP) $\subsetneq$ **L123** + (PPP).

Proof. In model 8 of formulae (L1)–(L3) and (PPP), the sentence (SSP) is false, because $\mathbb{I}(234) \subseteq \mathbb{O}(1234)$ and $234 \not\sqsubseteq 1234$.[3] □

[3] It is also easy to observe that in model 8, conditions (M1) and (M2) are false and that the inclusion $\mathbb{Fu} \subseteq \mathbb{Sum}$ does not hold. Essentially, we have $234 \not\sqsubseteq 1234$ but $\mathbb{I}(234) \subseteq \mathbb{O}(\{1234\})$; $1234 \,\mathbb{Sum}\, \{1,2,3,4\}$ and $234 \,\mathbb{Sum}\, \{2,3,4\}$; $\mathbb{O}(1234) = M = \bigcup \mathbb{O}(M)$ and $\neg\, 1234 \,\mathbb{Sum}\, M$.



Model 8. Conditions (L1)–(L3) and (PPP) hold, but (SSP) does not hold

PROPOSITION 2.2. *Condition* (PPP′) (*and so also* (PPP)) *does not follow from the set* {(L1), (L2), (L3)}, *so also it does not follow both from* {(L1), (L2), ($\mathrm{ext}_\sqsubset$), ($\exists\wr$)}, {(L1), (L2), ($\mathrm{ext}_\sqsubset$), (WSP)} *and* {(L1), (L2), ($\mathrm{ext}_\sqsubset$), ($\nexists 0$)}. *Thus, we obtain*:

- **L123** + (PPP) $\subsetneq$ **L123**,
- **L12** + (PPP) + ($\exists\wr$) $\subsetneq$ **L12** + ($\mathrm{ext}_\sqsubset$) + ($\exists\wr$),
- **L12** + (PPP) + (WSP) $\subsetneq$ **L12** + ($\mathrm{ext}_\sqsubset$) + (WSP),
- **L12** + (PPP) + ($\nexists 0$) $\subsetneq$ **L12** + ($\mathrm{ext}_\sqsubset$) + ($\nexists 0$),
- **L12** + ($\mathrm{ext}_\sqsubset$) $\subsetneq$ **L12** + (PPP),
- **L123** $\nsubseteq$ **L12** + (PPP) + (WSP),
- **L12** + ($\mathrm{ext}_\sqsubset$) + (WSP) $\nsubseteq$ **L12** + (PPP) + ($\exists\wr$),
- **L12** + ($\mathrm{ext}_\sqsubset$) + ($\exists\wr$) $\nsubseteq$ **L12** + (PPP) + ($\nexists 0$),
- **L12** + ($\mathrm{ext}_\sqsubset$) + ($\nexists 0$) $\nsubseteq$ **L12** + (PPP).

PROOF. In model 2 of conditions (L1)–(L3), conditions (PPP′) and (PPP) do not hold, because $\emptyset \neq \mathbb{P}(23) \subsetneq \mathbb{P}(123)$ and $23 \nsubseteq 123$. □

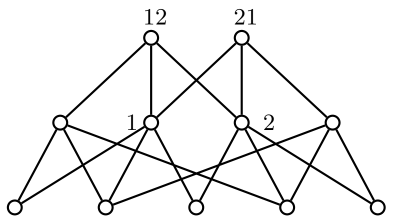
PROPOSITION 2.3. *Condition* (L3) *does not follow from the set* {(L2), (PPP), (WSP)}, *so also it does not follow from the set* {(L2), ($\mathrm{ext}_\sqsubset$), (WSP)}. *Thus, we obtain*:

- **L123** $\subsetneq$ **L12** + ($\mathrm{ext}_\sqsubset$) + (WSP),
- **L12** + (PPP) + (WSP) $\nsubseteq$ **L123**.

PROOF. In model 9 of conditions (L1), (L2), (PPP) and (WSP), condition (L3) does not hold, because $12 \;\mathsf{Sum}\; \{1,2\}$ and $21 \;\mathsf{Sum}\; \{1,2\}$. □

Moreover, by propositions 1.4–1.6 we obtain:

- **L12** + (PPP) + (WSP) $\subsetneq$ **L12** + (PPP) + ($\exists\wr$),
- **L12** + ($\mathrm{ext}_\sqsubset$) + (WSP) $\subsetneq$ **L12** + (PPP) + ($\exists\wr$),
- **L12** + (PPP) + ($\exists\wr$) $\subsetneq$ **L12** + (PPP) + ($\nexists 0$),



Model 9. (L1), (L2), (PPP) and (WSP) hold, but (L3) does not hold

- $\mathbf{L12} + (\text{PPP}) + (\nexists 0) \nsubseteq \mathbf{L12} + (\text{ext}_{\sqsubset}) + (\exists \wr)$,
- $\mathbf{L12} + (\text{PPP}) + (\nexists 0) \subsetneq \mathbf{L12} + (\text{PPP})$,
- $\mathbf{L12} + (\text{PPP}) \nsubseteq \mathbf{L12} + (\text{ext}_{\sqsubset}) + (\nexists 0)$.

3. The relation $\mathbb{S}\mathrm{um}$ versus the relation of supremum in classes between $\mathbf{L12} + (\text{SSP})$ and $\mathbf{L12}$

If $\langle M, \sqsubset\rangle$ belongs to $\mathbf{L12}$, then the relation $\sqsubseteq$ partially orders the set M. The relation of the least upper bound $\mathsf{sup}_{\sqsubseteq}$ defined by the condition (df $\mathsf{sup}_{\sqsubseteq}$) therefore has well-known properties (see Lemma II.8.1).

1. By virtue of Lemma II.8.2, in $\mathbf{L12}$ condition (SSP) is also equivalent to the following:

$$\mathbb{S}\mathrm{um} \subseteq \mathsf{sup}_{\sqsubseteq} . \tag{†}$$

Thus, to the class $\mathbf{L12} + (\text{SSP})$ belong those and only those strict partial orders in which the inclusion $\mathbb{S}\mathrm{um} \subseteq \mathsf{sup}_{\sqsubseteq}$ holds. Thus, we have:

11. $\mathbf{L12} + (\text{SSP}) = \mathbf{L12} + (\dagger)$.

Proposition 3.1. *The following condition*

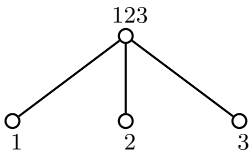
$$\forall_{S\in\mathcal{P}_+(M)}\forall_{x\in M}(x \;\mathsf{sup}_{\sqsubseteq}\; S \implies x \;\mathbb{S}\mathrm{um}\; S). \tag{‡}$$

does not follow from the set $\{(\text{L1}), (\text{L2}), (\text{SSP})\}$. *Thus, we obtain*:

- $\mathbf{L12} + (\text{SSP}) + (\ddagger) \subsetneq \mathbf{L12} + (\text{SSP})$.

Proof. In model 10 of conditions (L1), (L2) and (SSP), the set $\{1, 2\}$ has no mereological sum, but we have $123 \;\mathsf{sup}_{\sqsubseteq}\; \{1, 2\}$. □

2. From conditions (L1)–(L3) and (PPP) themselves follows no general connection between the relation $\mathbb{S}\mathrm{um}$ and $\mathsf{sup}_{\sqsubseteq}$, besides the fact that



Model 10. Conditions (L1), (L2), (SSP), (WSP) and ($\exists\sqcap$) hold, but condition (‡) does not hold

these relations coincide on singletons. In fact, in model 8 of the formulae (L1)–(L3) and (PPP) we have: $234 \ \mathbb{S}\mathrm{um}\ \{2,3,4\}$ but $\neg\exists_x x \ \mathsf{sup}_{\sqsubseteq}\ \{2,3,4\}$; and $1234 \ \mathsf{sup}_{\sqsubseteq}\ \{1,2\}$ but $\neg\exists_x\ x \ \mathbb{S}\mathrm{um}\ \{1,2\}$. Therefore both $\mathbb{S}\mathrm{um} \nsubseteq \mathsf{sup}_{\sqsubseteq}$ and $\mathsf{sup}_{\sqsubseteq} \nsubseteq \mathbb{S}\mathrm{um}$.[4]

3. We will prove that in all structures of the class (L2) + (WSP), if the mereological sum and the least upper bound of a given set exist, then they are equal to one another. In fact, we obtain:

Lemma 3.2. *In the class* (L2) + (WSP) *the following condition holds*:

$$\forall_{S\in\mathcal{P}(M)}\forall_{x,y\in M}(x \ \mathsf{sup}_{\sqsubseteq}\ S \wedge y \ \mathbb{S}\mathrm{um}\ S \implies x = y). \qquad (\diamond)$$

Proof. Let (a) $x \ \mathsf{sup}_{\sqsubseteq}\ S$ and (b) $y \ \mathbb{S}\mathrm{um}\ S$. From (b) we have $S \subseteq \mathbb{I}(y)$. Hence, by virtue of (a) and (df $\mathsf{sup}_{\sqsubseteq}$), we have $x \sqsubseteq y$. Assume for a contradiction that $x \neq y$. Therefore $x \sqsubset y$. Thus, by virtue of (WSP), for some $v \in M$ we have (c) $v \sqsubset y$ and (d) $v \wr x$. From (b) and (c), by virtue of (df $\mathbb{S}\mathrm{um}$), there exist $\in zS$ and $u \in M$ such that $u \sqsubseteq v$ and $u \sqsubseteq z$. Since $z \sqsubseteq x$ by virtue of (a), then from (L2) we have $u \sqsubseteq x$. Therefore $v \bigcirc x$, which contradicts condition (d).[5] □

Moreover, we also have:

Lemma 3.3 (Pietruszczak, 2013). *In* **L12** + ($\diamond$) *condition* (WSP) *holds.*

Proof. Assume for a contradiction that in **L12** + ($\diamond$) condition (WSP) does not hold. Then for some $x, y \in M$: (a) $x \sqsubset y$ and (b) $\mathbb{P}(y) \subseteq \mathbb{O}(x)$. By (a) and ($t_{\sqsubseteq}$) we have (c) $\mathbb{I}(x) = \mathbb{I}(x) \cap \mathbb{I}(y)$. Moreover, by (a), (b)

[4] If conditions (L1)–(L3) were the only ones to interest us, then model 2 would suffice by itself. For in that model we have: $23 \ \mathbb{S}\mathrm{um}\ \{2,3\}$ but $\neg\exists_x\ x \ \mathsf{sup}_{\sqsubseteq}\ \{2,3\}$; and $123 \ \mathsf{sup}_{\sqsubseteq}\ \{1,2\}$ but $\neg\exists_x\ x \ \mathbb{S}\mathrm{um}\ \{1,2\}$.

[5] For structures of **L12** + (SSP) we have an easier proof of condition ($\diamond$). Assume that $x \ \mathsf{sup}_{\sqsubseteq}\ S$ and $y \ \mathbb{S}\mathrm{um}\ S$. Since $\mathbb{S}\mathrm{um} \subseteq \mathsf{sup}_{\sqsubseteq}$, then $y \ \mathsf{sup}_{\sqsubseteq}\ S$. Thus $x = y$, by ($\mathrm{U}_{\mathsf{sup}}$).

and ($r_\sqsubseteq$), also $\mathbb{I}(y) \subseteq \mathbb{O}(x)$. Hence, in virtue of Lemma II.4.3, we have $y \;\mathbb{S}\mathrm{um}\; \mathbb{I}(x) \cap \mathbb{I}(y)$. So, by (c), also $y \;\mathbb{S}\mathrm{um}\; \mathbb{I}(x)$. However, $x \;\mathsf{sup}_\sqsubseteq\; \mathbb{I}(x)$, by ($r_\sqsubseteq$). Thus, by ($\diamond$), we have $x = y$, and by (a) we obtain a contradiction: $x \sqsubset x$.[6] □

Thus, the two above lemmas give:

12. (L2) + (WSP) = **L12** + ($\diamond$).

4. Mereological strictly partially ordered sets

Any strictly partially ordered set satisfying the following condition:

$$\forall_{S \in \mathcal{P}(M)} \forall_{x \in M} (x \;\mathbb{S}\mathrm{um}\; S \iff S \neq \emptyset \wedge x \;\mathsf{sup}_\sqsubseteq\; S). \qquad (\mathbb{S}\mathrm{um}\text{-}\mathsf{sup}_\sqsubseteq)$$

will be called a *mereological strictly partially ordered set*. Let **ML12** be the class of all these structures, i.e., we put:

$$\mathbf{ML12} := \mathbf{L12} + (\mathbb{S}\mathrm{um}\text{-}\mathsf{sup}_\sqsubseteq).$$

By definition, in all mereological strictly partially ordered sets conditions (†) and (‡) hold. Hence — by virtue of Lemma II.8.2 — in all structures of **ML12**, condition (SSP) holds, i.e., **ML12** ⊆ **L12** + (SSP). In other worlds, all mereological strictly partially ordered sets are polarised and we obtain:

13. **ML12** = **L12** + (†) + (‡) = **ML12** + (SSP) + (‡).

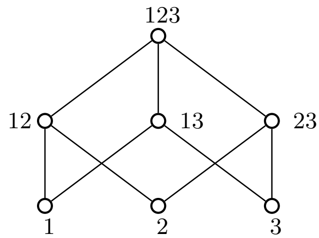
Thus, by Proposition 3.1 we obtain:

- **ML12** $\subsetneq$ **L12** + (SSP).

Remark 4.1. Condition (‡) does not explicitly postulate the existence of a mereological sum that it implicitly postulates one.

Consider the structure from the model 10. Since 123 is the least upper bound for the pairs $\{1, 2\}$, $\{2, 3\}$ and $\{1, 3\}$, then the presence in the theory of condition (‡) would enforce the existence of mereological sums for these pairs. Therefore, enriching the theory with this condition, we force the 'closure' of model 10 by sums: $1 \sqcup 2$, $2 \sqcup 3$ and $1 \sqcup 3$. We obtain model 11. Thus, we see that the addition of condition (‡) as an axiom will mean that two objects that have the least upper bound automatically must also have a mereological sum. □

[6] Because, by ($r_\sqsubseteq$), we also have that $x \;\mathbb{S}\mathrm{um}\; \mathbb{I}(x)$, so the above proof can be converted into the proof that in the class **L123** condition (WSP) holds.



Model 11. From model 10 to the structure satisfying condition (‡)

5. Simons' Minimal Extensional Mereology

Let **MEM** be the class of models of Simons' Minimal Extensional Mereology (cf. footnote 15 on p. 148), i.e., **MEM** := (L2) + (WSP) + ($\exists\sqcap$), where

$$\forall_{x,y\in M}\big(x \bigcirc y \implies \exists_{u\in M}\forall_{z\in M}(z \sqsubseteq u \iff z \sqsubseteq x \wedge z \sqsubseteq y)\big). \qquad (\exists\sqcap)$$

Just as in mereological structures, the element postulated in ($\exists\sqcap$) is simply the product $x \sqcap y$. In fact, using the relation $\mathbb{N}\mathbb{u}$ defined by (df $\mathbb{N}\mathbb{u}$) we see that condition ($\exists\sqcap$) is equivalent to the following:

$$\forall_{x,y\in M}\big(x \bigcirc y \implies \exists_{u\in M}\ u\ \mathbb{N}\mathbb{u}\ \{x,y\}\big).$$

But in the class **L12** we have $\mathbb{N}\mathbb{u} = \mathsf{inf}_{\sqsubseteq}$ (see p. 141). So in the class, condition ($\exists\sqcap$) is equivalent to the following:

$$\forall_{x,y\in M}\big(x \bigcirc y \implies \exists_{u\in M}\ u\ \mathsf{inf}_{\sqsubseteq}\ \{x,y\}\big). \qquad (5.1)$$

Thus, by ($\mathrm{U}_{\mathsf{inf}}$), we have:

$$\forall_{x,y\in M}\big(x \bigcirc y \implies \exists!_{u\in M}\ u\ \mathsf{inf}_{\sqsubseteq}\ \{x,y\}\big).$$

Thus, in the class **MEM** — in an analogous way for the operation $\sqcap$ in mereological structures — we may generate a PARTIAL binary operation $\sqcap\colon M \times M \to M$:

$$x \bigcirc y \implies x \sqcap y := \inf_{\sqsubseteq}\{x,y\}.$$

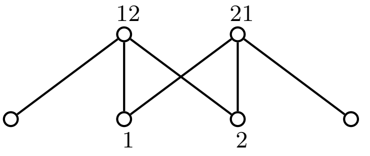
This operation has properties (II.9.6)–(II.9.11) and (II.9.14)–(II.9.15).

Notice that, by points 2 and 12, we have:

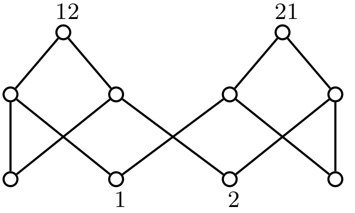
14. **MEM** = **L12**+(WSP)+($\exists\sqcap$) = **L12**+($\mathrm{S}_{\mathsf{Sum}}$)+($\exists\sqcap$) = **L12**+($\diamond$)+($\exists\sqcap$).

Moreover, by Lemma IV.3.13, we have:

15. **MEM** $\subseteq$ **L12** + (SSP).



Model 12. Conditions (L1), (L2) and (SSP) hold, but (∃⊓) does not hold



Model 13. (L1), (L2), (SSP) and (‡) hold, but (∃⊓) and (SSP+) do not hold

Proposition 5.1. *Condition* (∃⊓) *does not follows from the set* {(L1), (L2), (SSP)}. *Thus, we obtain*:

- $\mathsf{MEM} \subsetneq \mathsf{L12} + (\mathrm{SSP})$.

Proof. In model 12 of conditions (L1), (L2) and (SSP) we have $12 \circ 21$ and there is no element whose ingredienses would be the elements 1 and 2 and only those elements. Thus elements 12 and 21 have no product. □

However, we can strengthen the last proposition:

Proposition 5.2. *Condition* (∃⊓) *does not follows from the set* {(L1), (L2), (SSP), (‡)}. *Thus, we obtain*:

- $\mathsf{MEM} \not\subseteq \mathsf{ML12}$,
- $\mathsf{MEM} \cap \mathsf{ML12} \subsetneq \mathsf{MEM}$.

Proof. In model 13 of conditions (L1), (L2), (SSP) and (‡) we have $12 \circ 21$ and there is no element whose ingredienses would be the elements 1 and 2 and only those elements. The elements 12 and 21 do not therefore have a product. □

Notice that, by lemmas II.4.1(v) and II.6.4, conditions (L2) and (SSP) entail (WSP). Moreover, by Lemma IV.3.13, conditions (L2), (WSP) and (∃⊓) entail (SSP). Therefore we obtain:

$$\begin{aligned}\mathsf{MEM} \cap \mathsf{ML12} &= \mathsf{L12} + (\exists\sqcap) + (\mathrm{SSP}) + (\ddagger)\\ &= (\mathrm{L2}) + (\mathrm{WSP}) + (\exists\sqcap) + (\ddagger).\end{aligned}$$

Now we prove:

PROPOSITION 5.3. *Condition* (‡) *does not follow from the set* {(L1), (L2), (WSP), (∃⊓)}. *Thus, we obtain*:

- **ML12** $\not\subseteq$ **MEM**,
- **MEM** $\cap$ **ML12** $\subsetneq$ **ML12**.

PROOF. In model 10 of conditions (L1), (L2), (WSP) and (∃⊓), the set $\{1, 2\}$ has no mereological sum, but we have $123 \operatorname{sup}_{\sqsubseteq} \{1, 2\}$. □

6. The class L12+(SSP+)

Now we will examine the class of structures from **L12** that satisfy the *Super Strong Supplementation Principle*, i.e., the following condition:

$$\forall_{x,y \in M}(x \not\sqsubseteq y \implies \exists_{z \in M}(z \sqsubseteq x \wedge z \wr y \wedge \forall_{u \in M}(u \sqsubseteq x \wedge u \wr y \Rightarrow u \sqsubseteq z))). \tag{SSP+}$$

Condition (SSP+) intuitively says is that if x is not an ingrediens of y, then we can not only find some z being an ingrediens of x and external to y, but we can also find an element of the structure in question possessing the aforementioned property and being the greatest such object in the structure. Thus, if $x \not\sqsubseteq y$ then there is a greatest element in the following set:

$$D^x_y := \{u \in M : u \sqsubseteq x \wedge u \wr y\}.$$

By Lemma II.3.2, in class **L12** the the greatest element in a given set is also the mereological sum of the set. Thus, for all $x, y \in M$ we have:

- $D^x_y \neq \emptyset$ iff $x \not\sqsubseteq y$.
- If $x \not\sqsubseteq y$, then there is a greatest element in the set D^x_y.
- If $x \not\sqsubseteq y$, then there is exactly one $z \in D^x_y$ such that $z \operatorname{sup}_{\sqsubseteq} D^x_y$.
- If $x \not\sqsubseteq y$, then there is exactly one $z \in D^x_y$ such that $z \operatorname{Sum} D^x_y$.

Thus, in the class **L12** + (SSP+) we may generate a PARTIAL binary operation $\smallsetminus : M \times M \to M$:

$$x \not\sqsubseteq y \implies x \smallsetminus y := \text{the greatest element in } D^x_y = \sup_{\sqsubseteq} D^x_y = \bigsqcup D^x_y.$$

By definition, if $x \not\sqsubseteq y$, then $x \smallsetminus y \sqsubseteq x$ and $x \smallsetminus y \wr y$ and: $x = x - y$ iff $x \wr y$. Moreover, if $x \circ y$ and $x \not\sqsubseteq y$, then $x \not\sqsubseteq x \smallsetminus y$; and so there is an element $x \smallsetminus (x \smallsetminus y)$.

LEMMA 6.1 (Pietruszczak, 2013). *In the class* **L12**+(SSP+) *the following condition holds*:

$$\forall_{x,y\in M}\big(x \bigcirc y \wedge x \not\sqsubseteq y \implies x \smallsetminus (x \smallsetminus y) \;\mathsf{inf}_{\sqsubseteq}\; \{x,y\}\big).$$

Moreover, if $y \sqsubseteq x$ *and* $x \not\sqsubseteq y$, *then* $y = x \smallsetminus (x \smallsetminus y)$.

Of course, if $x \sqsubseteq y$ then $x \;\mathsf{inf}_{\sqsubseteq}\; \{x, y\}$. Thus, we have:

PROPOSITION 6.2 (Pietruszczak, 2013). *In the calss* **L12** + (SSP+) *conditions* (5.1) *and* ($\exists\sqcap$) *hold.*

Thus, in the class **L12** + (SSP+) — as in **MEM** — we may generate a PARTIAL binary operation $\sqcap\colon M \times M \to M$:

$$x \bigcirc y \implies x \sqcap y := \inf_{\sqsubseteq}\{x, y\}.$$

This operation has the following property for all $x, y \in M$:

$$x \bigcirc y \implies x \sqcap y := \begin{cases} x & \text{if } x \sqsubseteq y \\ y & \text{if } y \sqsubseteq x \\ x \smallsetminus (x \smallsetminus y) & \text{if } x \not\sqsubseteq y \text{ and } y \not\sqsubseteq x \end{cases}$$

Of course, from (SSP+) we obtain (SSP); and so also (†).

PROPOSITION 6.3. *In the class* **L12** + (SSP+) *both conditions* (‡) *and* ($\mathbb{S}\mathrm{um}$-$\mathsf{sup}_{\sqsubseteq}$) *hold.*

PROOF. For (‡): Let us assume that (a) $x \;\mathsf{sup}_{\sqsubseteq}\; S$, (b) $S \neq \emptyset$ and indirectly that (c) $\neg\, x \;\mathbb{S}\mathrm{um}\; S$. From (a) and (c) it follows that for some $u_0 \in M$ we have: (d) $u_0 \sqsubseteq x$ and (e) $\forall_{z\in S}\, z \wr u_0$.

Now assume for a contradiction that $u_0 = x$. Then, by virtue of (a), (b) and (e), for some $z_0 \in S$ we have $z_0 \sqsubseteq x$ and and $z_0 \wr x$, which contradicts ($\mathrm{r}_{\sqsubseteq}$). So $u_0 \neq x$. Thus, (f) $x \not\sqsubseteq u_0$, by (d) and ($\mathrm{antis}_{\sqsubseteq}$).

From (f), by virtue of (SSP+), for some $y_0 \in M$ we have (g) $y_0 \sqsubseteq x$, (h) $y_0 \wr u_0$ and (i) for an arbitrary u: $u \sqsubseteq x$ and $u \wr u_0$ entails $u \sqsubseteq y_0$. From (a) and (e) it follows that for an arbitrary $z \in S$ we have: $z \sqsubseteq x$ and $z \wr u_0$. Hence, by (i), we have $\forall_{z\in S}\, z \sqsubseteq y_0$. So $x \sqsubseteq y_0$, by virtue of (a). Hence, by (g) and ($\mathrm{antis}_{\sqsubseteq}$), we have $x = y_0$, which contradicts (d) $\wedge$ (h).

For ($\mathbb{S}\mathrm{um}$-$\mathsf{sup}_{\sqsubseteq}$): Condition (SSP) gives (†) and two conditions (†) and (‡) give ($\mathbb{S}\mathrm{um}$-$\mathsf{sup}_{\sqsubseteq}$). □

By propositions 5.2, 6.2 and 6.3 we obtain:

PROPOSITION 6.4. $\mathbf{L12} + (\text{SSP+}) \subsetneq \mathbf{ML12}$.

PROOF. Firstly, from Proposition 6.3 we have $\mathbf{L12} + (\text{SSP+}) \subseteq \mathbf{ML12}$.

Secondly, by propositions 5.2 and 6.2, condition (SSP+) does not follow from the set $\{(\text{L1}), (\text{L2}), (\text{SSP}), (\ddagger)\}$. □

By propositions 5.2, 6.2 and 6.4 we obtain:

PROPOSITION 6.5. $\mathbf{L12} + (\text{SSP+}) \subsetneq \mathbf{MEM}$.

PROOF. Firstly, by virtue of Lemma II.6.4, condition (L3) follows from $\{(\text{L2}), (\text{SSP})\}$. Moreover, by Lemma II.4.1(v), conditions (L1) and (L3) entail (WSP). Hence $\mathbf{L12} + (\text{SSP+}) \subseteq \mathbf{MEM}$, by Proposition 6.2.

Secondly, by propositions 5.2 and 6.4, if $\mathbf{L12} + (\text{SSP+}) = \mathbf{MEM}$ then we obtain a contradiction: $\mathbf{MEM} \nsubseteq \mathbf{ML12}$ and $\mathbf{MEM} \subseteq \mathbf{ML12}$. □

In [Pietruszczak, 2013, Fact 6.11, p. 102] we prove:

PROPOSITION 6.6. *Condition* (SSP+) *does not follows from the set* $\{(\text{L1}), (\text{L2}), (\exists\sqcap), (\text{SSP}), (\ddagger)\}$.

Hence, by propositions 6.4 and 6.5, we obtain:

PROPOSITION 6.7. $\mathbf{L12} + (\text{SSP+}) \subsetneq \mathbf{MEM} \cap \mathbf{ML12}$.

7. Grzegorczyk's mereological structures

Let **GMS** be the class of $\mathrm{L}_{\sqsubseteq}$-structures which are models of Grzegorczyk's theory in [1955], i.e., these and only these $\mathrm{L}_{\sqsubseteq}$-structures that satisfy the following conditions: (L1), (L2), (SSP+) and

$$\forall_{x,y\in M}\exists_{z\in M}\; z \,\mathsf{sup}_{\sqsubseteq}\, \{x,y\}. \qquad (\exists_{\text{pair}}\mathsf{sup})$$

We remember that (SSP+) entails (SSP). Elements of **GMS** we call *Grzegorczyk mereological structures* or *mereological fields.*[7]

Remark 7.1. (i) Grzegorczyk [1955] accepted the additional axiom (5.1). Since the relation $\sqsubseteq$ in the class **GMS** partially orders the universe, then $\mathbb{N}\mathbb{u} = \mathsf{inf}_{\sqsubseteq}$, i.e., (5.1) is equivalent to $(\exists\sqcap)$. We know, however (see Lemma 6.2), that condition (5.1) follows from $\{(\text{L1}), (\text{L2}), (\text{SSP+})\}$.

[7] The class **GMS** has been thoroughly researched in [Pietruszczak, 2013]

(ii) In place of axioms (L1) and (L2) Grzegorczyk assumes the reflexivity, antisymmetry and transitivity of the relation $\sqsubseteq$, i.e., conditions ($\mathrm{r}_{\sqsubseteq}$), ($\mathrm{antis}_{\sqsubseteq}$) and ($\mathrm{t}_{\sqsubseteq}$). Although the relation $\sqsubseteq$ is defined in [Grzegorczyk, 1955] (in the original, the relation is named *ingr*) with the help of condition (df $\sqsubseteq$), it may be taken that this is only an 'intuitive' explanation of this relation. If this were not the case, then, in an obvious way, the axiom ($\mathrm{r}_{\sqsubseteq}$) which follows from the definition (df $\sqsubseteq$) itself would be inessential. In any case, this axiom is indeed inessential in [Grzegorczyk, 1955], as we know that ($\mathrm{r}_{\sqsubseteq}$) follows from $\{(\mathrm{t}_{\sqsubseteq}), (\mathrm{SSP})\}$. □

It follows from the results given in Chapter II (sections 8 and 11) that $\mathbf{MS} \subseteq \mathbf{GMS}$. Furthermore we have:

THEOREM 7.1. (i) *All finite structures from* **Grz** *belong to* **MS**.
(ii) $\mathbf{MS} \subsetneq \mathbf{GMS}$.

PROOF. *Ad* (i): Via ($\exists_{\mathrm{pair}}\mathsf{sup}$), for an arbitrary non-empty finite set $S \subseteq M$ there exists an $x \in M$ such that $x \ \mathsf{sup}_{\sqsubseteq}\ S$. By virtue of Proposition 6.3, we have $x \ \mathbb{Sum}\ S$. Thus, in finite structures in **GMS** condition (L4) holds. Furthermore, (L3) follows from $\{(\mathrm{L2}), (\mathrm{SSP})\}$, by virtue of Lemma II.6.4.

Ad (ii): Let X be an arbitrary infinite set and let $\mathrm{F}_+(X)$ be the family of all finite and non-empty subsets of the set X. Then $\langle \mathrm{F}_+(X), \subsetneq \rangle$ belongs to the class **GMS**, but does not belong to the set **MS**.

The first result follows from the observation, that for all $A, B \in \mathrm{F}_+(X)$: $A \sqsubseteq B$ iff $A \subseteq B$; $A \circ B$ iff $A \cap B \neq \emptyset$; and $A \wr B$ iff $A \cap B = \emptyset$. Furthermore, $\mathsf{sup}_{\sqsubseteq}\ \{A, B\} = A \cup B \in \mathrm{F}_+(X)$; and if $A \cap B \neq \emptyset$, then $\mathsf{inf}_{\sqsubseteq}\ \{A, B\} = A \cap B \in \mathrm{F}_+(X)$. Finally, if $A \nsubseteq B$, then $A \setminus B \in \mathrm{F}_+(X)$ and this is the set postulated in (SSP+).

The structure $\langle \mathrm{F}_+(X), \subsetneq \rangle$ does not belong to **MS**, as $\langle \mathrm{F}_+(X) \cup \{\emptyset\}, \subseteq \rangle$ is not a Boolean lattice. □

The results proven above may be depicted by complementing diagram 2 with diagram 3.

8. Finite elementary axiomatisability of superclasses of MS

THEOREM 8.1. *All superclasses of the class* **MS** *in diagrams 1–3 are finitely elementarily axiomatisable.*

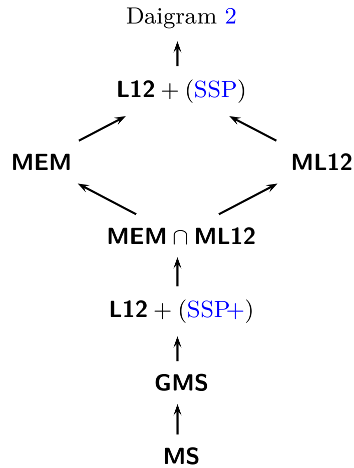


Diagram 3. The lattice of certain superclasses of the class **MS**

PROOF. If $\boldsymbol{K} \neq \mathbf{L123}$, then we can take the formulae of the language $L_{\sqsubset}$ as axioms, which arise in an obvious manner from the conditions describing the class $\boldsymbol{K}$. We make use of the $L_{\sqsubset}$-formulae e-defining the relations $\sqsubset$, $\sqsubseteq$, $\bigcirc$ and $\wr$ and the sets that are the values of the operators $\mathbb{P}$, $\mathbb{I}$ and $\mathbb{O}$. Clearly, conditions $\operatorname{Card} M > 1$ and $\operatorname{Card} M = 1$ can be translated in to the $L_{\sqsubset}$-sentences $\ulcorner \exists_x \exists_y \neg x = y \urcorner$ and $\ulcorner \forall_x \forall_y \; x = y \urcorner$, respectively. For the class **L123** we use the identity 3 and proceed as above (see p. 87). $\square$

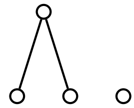
9. Elements isolated in superclasses of the class MS

We shall give the name "isolated element" to an arbitrary element in the structure $\mathfrak{M} = \langle M \sqsubset \rangle$ which has no part and which is not a part of any element. We will therefore define the e-definable set:

$$\mathbb{is} := \{x \in M : \forall_{y \in M}(y \not\sqsubset x \wedge x \not\sqsubset y)\}\,.$$

We were not concerned earlier with isolated elements because, quite simply, there are not any in non-trivial mereological structures. This is also the case for all structures of the class **GMS**:

$$\mathfrak{M} \in \mathbf{GMS} \implies (\mathbb{is} \neq \emptyset \Leftrightarrow \operatorname{Card} M = 1 \Leftrightarrow \mathbb{is} = M).$$



Model 14. Conditions (L1), (L1) and (SSP+) hold

In fact, if $\operatorname{Card} M = 1$, then — by virtue of ($\mathrm{irr}_{\sqsubset}$) — we have: $\mathbb{is} = M \neq \emptyset$. Assume for a contradiction that $\operatorname{Card} M > 1$ and $x \in \mathbb{is}$. Then there exists a y such that (a) $y \neq x$ and (b) $y \not\sqsubset x$. By virtue of ($\exists_{\mathrm{pair}}\mathsf{sup}$), there exists a z such that $z \ \mathbb{Sum}\ \{x, y\}$. Thus, (c) $x \sqsubseteq z$ and (d) $y \sqsubseteq z$. By virtue of (a), (b) and (d) we have $z \neq x$. From this and (c) we have $x \sqsubset z$ which contradicts our assumption. Thus $\mathbb{is} \neq \emptyset$ entails $\operatorname{Card} M = 1$. Finally, if $\mathbb{is} = M$ then $\mathbb{is} \neq \emptyset$.

For all of the proper superclasses of the class **GMS** considered in this chapter, it is not possible to say anything general about isolated elements. For example, model 14 belongs to the class **L12** + (SSP+). In this model there occurs only one isolated element.

Let C be one of conditions (L1)–(L3), ($\mathrm{ext}_{\sqsubset}$), ($\nexists 0$), ($\exists\,\wr$), ($\mathrm{S}_{\mathbb{Sum}}$), (WSP), (PPP), (SSP), ($\exists\sqcap$), ($\mathbb{Sum}$-$\mathsf{sup}_{\sqsubseteq}$) and (SSP+). Assume that the structure $\mathfrak{M}' = \langle M', \sqsubset' \rangle$ arise from $\mathfrak{M}$ by the addition of a non-empty set of isolated elements. It is then easy to see that $\mathfrak{M}'$ meets a condition C iff $\mathfrak{M}$ meets C. If $\boldsymbol{K}$ is therefore one of the proper superclasses of the **GMS** considered here, then

$$\mathfrak{M}' \in \boldsymbol{K} \iff \mathfrak{M} \in \boldsymbol{K}\,.$$

Part B

ELEMENTARY THEORIES CONNECTED WITH MEREOLOGY

In this part of the book we will be concerned with various first-order theories connected with mereology.

In Chapter VI, we will formulate the theory **M**, which is a first-order (elementary) theory with identity constructed in the language $\mathrm{L}_{\sqsubset}$ and which we will call *elementary mereology*. The class of the models of **M** will be the class of $\mathrm{L}_{\sqsubset}$-structures **qMS** which we mentioned on p. 72. This class is narrower than the class **MEM** and wider than the class **MS**.

We will formulate **M** with an infinite number of axioms, this being how it is usually done. We will, however, also give a finite axiomatisation for **M**, which is to say that the class **qMS** will turn out to be finitely elementarily axiomatisable (see p. 290 in Appendix II). We will obtain these results by making use of the relationship between the class **qMS** and the class **ecBL** of e-complete Boolean lattices, this being a relation analogous to the one that holds between the classes **MS** of mereological structures and **CBL** of complete Boolean lattices.

By drawing on the above relationships and the identity Th(**ecBL**) = Th(**CBL**) we will show the identity Th(**qMS**) = Th(**MS**), i.e., that a given $\mathrm{L}_{\sqsubset}$-sentence is true in all structures of the class **qMS** iff it is true in all structures of the class **MS** (see Section 1 of Appendix II). From this and from Gödel's completeness theorem it follows that a given $\mathrm{L}_{\sqsubset}$-formula is a thesis of the theory **M** iff it is true in all structures of the class **MS**.

In Chapter VII, we will present a first-order theory **MDC** in which we can speak of a mereological sum composed of distributive classes. Besides the concepts of *being a distributive class* and *being a member of a distributive set*, it will involve the concepts of *being a mereological sum of* and *being a mereological part of*. We will interpret Morse's [1965] first-order class theory in this theory. We will show that our theory **MDC** has a model, if Morse's theory has one.

In the final chapter of the book we will present unitary theories of individuals and sets. This is possible thanks to a mereological broadening of ZF set theory. We will show how in that theories it is possible to define the concept of an *individual* on the basis of the following primitive concepts: *being a part of*, *being a distributive set*, and *being a member of a distributive set*. We will show that our theories have models, if ZF set theory has one.

Chapter VI

The elementary aspect of mereology

1. Elementary mereology

Let us build in the language $\mathrm{L}_{\sqsubset}$ a theory $\mathbf{M}$ (first-order with the identity predicate "$=$") which will be determined by the set of extralogical axioms $\mathrm{Ax}^{\mathbf{M}}$ (that is, $\mathbf{M} := \mathrm{Cn}(\mathrm{Ax}^{\mathbf{M}})$) and which we will name *elementary mereology*. The set $\mathrm{Ax}^{\mathbf{M}}$ will be composed of an infinite number of $\mathrm{L}_{\sqsubset}$-formulae. This is a standard way of presenting $\mathbf{M}$. We will later prove that it also has a finite axiomatisation (see Section 9).

In order to more easily formulate the theory $\mathbf{M}$, we will extend the language $\mathrm{L}_{\sqsubset}$ by three two-argument predicates: "$\sqsubseteq$", "$\circ$" and "$\wr$". These three predicates, however, will be definable in the theory $\mathbf{M}$ by the definitions used in Chapter II. That is, we put:

$$\forall_x \forall_y (x \sqsubseteq y \equiv (x \sqsubset y \vee x = y)) \tag{$\delta\sqsubseteq$}$$

$$\forall_x \forall_y (x \circ y \equiv \exists_z (z \sqsubseteq z \wedge z \sqsubseteq y)) \tag{$\delta\circ$}$$

$$\forall_x \forall_y (x \wr y \equiv \neg \exists_z (z \sqsubseteq z \wedge z \sqsubseteq y)) \tag{$\delta\wr$}$$

Thus, as in Chapter II, these predicates we can read as "is an ingrediens of", "overlaps with" and "is exterior to", respectively. Let us extend the language $\mathrm{L}_{\sqsubset}$ to the elementary language $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$, which arises in the same way except for the use of the predicates: "$\sqsubset$", "$\sqsubseteq$", "$\circ$" and "$\wr$". Set-theoretic interpretations of the language $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$ are provided by $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-structures of the form $\langle M, \sqsubset, \sqsubseteq, \circ, \wr \rangle$ in which the predicates "$\sqsubset$" and "$\sqsubseteq$" we will interpret as the binary relations $\sqsubset$, $\sqsubseteq$, $\circ$ and $\wr$, where the last three relations are defined by (df $\sqsubseteq$), df $\circ$ and (df $\circ$), respectively.

Let us build in the language $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$ the theory $\mathbf{M}$ (first-order with identity) which will be determined by the infinite set of non-logical axioms defined below. To this set belongs, inter alia, the following $\mathrm{L}_{\sqsubset}$-sentences ($\lambda 1$) and ($\lambda 2$) (given on p. 72):

$$\forall_x \forall_y (x \sqsubset y \rightarrow \neg\, y \sqsubset x) \tag{$\lambda 1$}$$

$$\forall_x \forall_y \forall_z (x \sqsubset y \wedge y \sqsubset z \to x \sqsubset z) \tag{$\lambda 2$}$$

and the definitions ($\delta \sqsubseteq$), ($\delta \circ$) and ($\delta \wr$) of the predicates "$\sqsubseteq$", "$\circ$" and "$\wr$", respectively.

In any $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-structure $\mathfrak{M} = \langle M, \sqsubset, \sqsubseteq, \circ, \wr \rangle$, axioms ($\lambda 1$) and ($\lambda 2$) say, respectively, that the relation $\sqsubset$ is asymmetric and transitive, i.e., that it satisfies conditions (L1) and (L2) (so also $\sqsubset$ is irreflexive). Axiom ($\delta \sqsubseteq$) says that the relation $\sqsubseteq$ satisfies condition (df $\sqsubseteq$). Therefore the relation $\sqsubseteq$ is reflexive, antisymmetric and transitive. From this and from Gödel's completeness theorem, it follow that from ($\lambda 1$), ($\lambda 2$) and ($\delta \sqsubseteq$) we can derive the following theses:[1]

$$\forall_x \neg x \sqsubset x \tag{$\mathrm{irr}_{\sqsubset}$}$$

$$\forall_x x \sqsubseteq x \tag{$\mathrm{r}_{\sqsubseteq}$}$$

$$\forall_x \forall_y \forall_z (x \sqsubseteq y \wedge y \sqsubseteq x \to x = y) \tag{$\mathrm{antis}_{\sqsubseteq}$}$$

$$\forall_x \forall_y \forall_z (x \sqsubseteq y \wedge x \sqsubseteq z \to x \sqsubseteq z) \tag{$\mathrm{t}_{\sqsubseteq}$}$$

Moreover, axioms ($\delta \circ$) and ($\delta \wr$) say that the relations $\circ$ and $\wr$ satisfy conditions (df $\circ$) and (df $\wr$), respectively. Hence the relation $\circ$ is reflexive and symmetric, and the relation $\wr$ is irreflexive and symmetric. Also from ($\mathrm{r}_{\sqsubseteq}$), ($\delta \circ$) and ($\delta \wr$) we get:

$$\forall_x x \circ x \tag{$\mathrm{r}_{\circ}$}$$

$$\forall_x \forall_y (x \circ y \equiv y \circ x) \tag{$\mathrm{s}_{\circ}$}$$

$$\forall_x \neg x \wr x \tag{$\mathrm{irr}_{\wr}$}$$

$$\forall_x \forall_y (x \wr y \equiv y \wr x) \tag{$\mathrm{s}_{\wr}$}$$

Since axioms ($\delta \sqsubseteq$), ($\delta \circ$) and ($\delta \wr$) are definitions of the predicates "$\sqsubseteq$", "$\circ$" and "$\wr$" in the theory **M**, then they may be eliminated.

Let $\mathfrak{M}$ be any $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-structure and φ be any $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-formula such that $\mathrm{vf}(\varphi) = \{\mathtt{x}_1, \ldots, \mathtt{x}_{k+1}\}$, for some $k \geqslant 0$ (note that $\mathtt{x} := \mathtt{x}_1$). If $k = 0$ then we will connect with φ the set $M^{\mathtt{x}}_{\varphi}$ of elements of M satisfying φ in $\mathfrak{M}$

$$M^{\mathtt{x}}_{\varphi} := \{x \in M : \mathfrak{M} \models \varphi\,[x/\mathtt{x}]\}\,.$$

The set $M^{\mathtt{x}}_{\varphi}$ is elementarily definable in $\mathfrak{M}$ with the help of φ (see p. 291 in Appendix II). For example, if $\varphi :=$ "$\mathtt{x} = \mathtt{x}$" then we obtain:

$$M^{\mathtt{x}}_{\mathtt{x}=\mathtt{x}} := \{x \in M : \mathfrak{M} \models \mathtt{x} = \mathtt{x}\,[x/\mathtt{x}]\} = \{x \in M : x = x\} = M.$$

[1] This is easy to do with the help of the resources of the elementary theories themselves.

If $k > 0$, however, then for arbitrary $y_1, \ldots, y_k$ from M we put:

$$M^{\mathtt{x}}_{\varphi}(y_1, ..., y_k) := \{x \in M : \mathfrak{M} \models \varphi\,[x/\mathtt{x}, y_1/\mathtt{x}_2, ..., y_k/\mathtt{x}_{k+1}]\,\}.$$

The set $M^{\mathtt{x}}_{\varphi}(y_1, ..., y_k)$ is elementarily definable in $\mathfrak{A}$ with parameters y_1, ..., y_k and with the help of φ (see p. 292 in Appendix II). For example, for any $y \in M$ and $\varphi :=$ "$\mathtt{x} = \mathtt{y}$", $\varphi :=$ "$\mathtt{x} \sqsubset \mathtt{y}$", $\varphi :=$ "$\mathtt{x} \sqsubseteq \mathtt{y}$" we get, respectively (note that $\mathtt{y} := \mathtt{x}_2$):

$$M^{\mathtt{x}}_{\mathtt{x}=\mathtt{y}}(y) := \{x \in M : \mathfrak{M} \models \mathtt{x} = \mathtt{y}\,[x/\mathtt{x}, y/\mathtt{y}]\,\} = \{y\},$$
$$M^{\mathtt{x}}_{\mathtt{x}\sqsubset\mathtt{y}}(y) := \{x \in M : \mathfrak{M} \models \mathtt{x} \sqsubset \mathtt{y}\,[x/\mathtt{x}, y/\mathtt{y}]\,\} = \mathbb{P}(y),$$
$$M^{\mathtt{x}}_{\mathtt{x}\sqsubseteq\mathtt{y}}(y) := \{x \in M : \mathfrak{M} \models \mathtt{x} \sqsubseteq \mathtt{y}\,[x/\mathtt{x}, y/\mathtt{y}]\,\} = \mathbb{I}(y).$$

Moreover, to the formula φ we assign the following $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-formula:

$$\sigma^{\mathtt{x}}_{\varphi} := \ulcorner \forall_{\mathtt{x}}(\varphi \rightarrow \mathtt{x} \sqsubseteq \mathtt{x}_{k+2}) \wedge \forall_{\mathtt{x}_{k+3}}(\mathtt{x}_{k+3} \sqsubseteq \mathtt{x}_{k+2} \rightarrow \exists_{\mathtt{x}}(\varphi \wedge \mathtt{x} \circ \mathtt{x}_{k+3}))\urcorner$$

Notice that $\mathrm{vf}(\sigma^{\mathtt{x}}_{\varphi}) = \{\mathtt{x}_2, \ldots, \mathtt{x}_{k+2}\}$; so if $k = 0$ then $\mathrm{vf}(\sigma^{\mathtt{x}}_{\varphi}) = \{\mathtt{y}\}$.

If $k = 0$ then we get (notice that $\mathtt{x} := \mathtt{x}_1$, $\mathtt{y} := \mathtt{x}_2$ and $\mathtt{z} := \mathtt{x}_3$; $\mathrm{vf}(\varphi) = \{\mathtt{x}\}$ and $\mathrm{vf}(\sigma^{\mathtt{x}}_{\varphi}) = \{\mathtt{y}\}$):

$$\sigma^{\mathtt{x}}_{\varphi} := \ulcorner \forall_{\mathtt{x}}(\varphi \rightarrow \mathtt{x} \sqsubseteq \mathtt{y}) \wedge \forall_{\mathtt{z}}(\mathtt{z} \sqsubseteq \mathtt{y} \rightarrow \exists_{\mathtt{x}}(\varphi \wedge \mathtt{x} \circ \mathtt{z}))\urcorner$$

Thus, when $k = 0$, the formula $\sigma^{\mathtt{x}}_{\varphi}$ says that the object represented by the free variable "$\mathtt{y}$" ("$\mathtt{x}_2$") is the mereological sum of all $\mathtt{x}$'s satisfying φ, i.e., that this object is the mereological sum of the set $M^{\mathtt{x}}_{\varphi}$. Formally:

LEMMA 1.1. *If $k = 0$ then for any $y \in M$:*

$$\mathfrak{M} \models \sigma^{\mathtt{x}}_{\varphi}\,[y/\mathtt{y}] \iff y \;\mathsf{Sum}\; M^{\mathtt{x}}_{\varphi}.$$

PROOF. If $k > 0$ and $y \in M$, then: $y \;\mathsf{Sum}\; M^{\mathtt{x}}_{\varphi}$ iff $\forall_x(x \in M^{\mathtt{x}}_{\varphi} \Rightarrow x \sqsubseteq y)$ and $\forall_z(z \sqsubseteq y \Rightarrow \exists_x(x \in M^{\mathtt{x}}_{\varphi} \wedge x \circ z))$ iff the valuation $[y/\mathtt{y}]$ satisfies in $\mathfrak{M}$ the formula $\sigma^{\mathtt{x}}_{\varphi}$. □

For example, if $\varphi :=$ "$\mathtt{x} = \mathtt{x}$" then we obtain:

$$\sigma^{\mathtt{x}}_{\mathtt{x}=\mathtt{x}} := \text{“}\forall_{\mathtt{x}}(\mathtt{x} = \mathtt{x} \rightarrow \mathtt{x} \sqsubseteq \mathtt{y}) \wedge \forall_{\mathtt{z}}(\mathtt{z} \sqsubseteq \mathtt{y} \rightarrow \exists_{\mathtt{x}}(\mathtt{x} = \mathtt{x} \wedge \mathtt{x} \circ \mathtt{z}))\text{”}$$

Thus, the formula $\sigma^{\mathtt{x}}_{\mathtt{x}=\mathtt{x}}$ is logically equivalent to "$\forall_{\mathtt{x}}\, \mathtt{x} \sqsubseteq \mathtt{y} \wedge \forall_{\mathtt{z}}(\mathtt{z} \sqsubseteq \mathtt{y} \rightarrow \exists_{\mathtt{x}}\, \mathtt{x} \circ \mathtt{z})$". Hence for any $y \in M$: $\mathfrak{M} \models \sigma^{\mathtt{x}}_{\mathtt{x}=\mathtt{x}}\,[y/\mathtt{y}] \iff y \;\mathsf{Sum}\; M$.

If, however, $k > 0$, then the formula $\sigma^{\mathtt{x}}_{\varphi}$ says that for arbitrary members $y_1, \ldots, y_k$ of M represented by the free variables "$\mathtt{x}_2$", ..., "$\mathtt{x}_{k+1}$", the object represented by the free variable "$\mathtt{x}_{k+2}$" is the mereological sum of all $\mathtt{x}$'s satisfying $\varphi(\mathtt{x}, y_1/\mathtt{x}_2, \ldots, y_k/\mathtt{x}_{k+1})$, i.e., that this object is the mereological sum of the set $M^{\mathtt{x}}_{\varphi}(y_1, ..., y_k)$. Formally:

LEMMA 1.2. *If $k > 0$ then for all $y_1, \ldots y_k, z \in M$:*

$$\mathfrak{M} \models \sigma^{\mathtt{x}}_{\varphi}[y_1/\mathtt{x}_2, ..., y_k/\mathtt{x}_{k+1}, z/\mathtt{x}_{k+2}] \iff z \ \mathsf{Sum}\ M^{\mathtt{x}}_{\varphi}(y_1, ..., y_k)$$
$$\iff z \ \mathsf{Sum}\ \{x \in M : \mathfrak{M} \models \varphi\,[x/\mathtt{x}_1, y_1/\mathtt{x}_2, ..., y_k/\mathtt{x}_{k+1}]\,\}.$$

PROOF. Let $k > 0$ and $y_1, \ldots, y_k, z \in M$. Then: $z \ \mathsf{Sum}\ M^{\mathtt{x}}_{\varphi}(y_1, ..., y_k)$ iff $\forall_x(x \in M^{\mathtt{x}}_{\varphi}(y_1, ..., y_k) \Rightarrow x \sqsubseteq z)$ and $\forall_u(u \sqsubseteq z \Rightarrow \exists_x(x \in M^{\mathtt{x}}_{\varphi}(y_1, ..., y_k) \wedge x \bigcirc u))$ iff the valuation $[y_1/\mathtt{x}_2, ..., y_k/\mathtt{x}_{k+1}, z/\mathtt{x}_{k+2}]$ satisfies in $\mathfrak{M}$ the formula $\sigma^{\mathtt{x}}_{\varphi}$. □

For example, if $\varphi :=$ "$\mathtt{x} = \mathtt{y}$" then we obtain:

$$\sigma^{\mathtt{x}}_{\mathtt{x}=\mathtt{y}} := \text{“}\forall_{\mathtt{x}}(\mathtt{x} = \mathtt{y} \rightarrow \mathtt{x} \sqsubseteq \mathtt{z}) \wedge \forall_{\mathtt{u}}(\mathtt{u} \sqsubseteq \mathtt{z} \rightarrow \exists_{\mathtt{x}}(\mathtt{x} = \mathtt{y} \wedge \mathtt{x} \bigcirc \mathtt{u}))\text{”}$$

The formula $\sigma^{\mathtt{x}}_{\mathtt{x}=\mathtt{y}}$ logically equivalent to "$\mathtt{y} \sqsubseteq \mathtt{z} \wedge \forall_{\mathtt{u}}(\mathtt{u} \sqsubseteq \mathtt{z} \rightarrow \mathtt{y} \bigcirc \mathtt{z})$". Hence for all $y, z \in M$: $\mathfrak{M} \models \sigma^{\mathtt{x}}_{\mathtt{x}=\mathtt{y}}\,[y/\mathtt{y}, z/\mathtt{z}] \iff z \ \mathsf{Sum}\ M^{\mathtt{x}}_{\mathtt{x}=\mathtt{y}}(y) \iff z \ \mathsf{Sum}\ \{y\}$. Since, by ($\mathtt{r}_{\sqsubseteq}$), the formula "$\mathtt{y} \sqsubseteq \mathtt{y} \wedge \forall_{\mathtt{u}}(\mathtt{u} \sqsubseteq \mathtt{y} \rightarrow \mathtt{y} \bigcirc \mathtt{y})$" is a thesis, then for any $y \in M$ we have: $\mathfrak{M} \models \sigma^{\mathtt{x}}_{\mathtt{x}=\mathtt{y}}\,[y/\mathtt{y}, y/\mathtt{z}]$, which means that $y \ \mathsf{Sum}\ \{y\}$.

For example, if $\varphi :=$ "$\mathtt{x} \sqsubseteq \mathtt{y}$" then we obtain:

$$\sigma^{\mathtt{x}}_{\mathtt{x}\sqsubseteq\mathtt{y}} := \text{“}\forall_{\mathtt{x}}(\mathtt{x} \sqsubseteq \mathtt{y} \rightarrow \mathtt{x} \sqsubseteq \mathtt{z}) \wedge \forall_{\mathtt{u}}(\mathtt{u} \sqsubseteq \mathtt{z} \rightarrow \exists_{\mathtt{x}}(\mathtt{x} \sqsubseteq \mathtt{y} \wedge \mathtt{x} \bigcirc \mathtt{u}))\text{”}$$

Hence for all $y, z \in M$: $\mathfrak{M} \models \sigma^{\mathtt{x}}_{\mathtt{x}\sqsubseteq\mathtt{y}}\,[y/\mathtt{y}, z/\mathtt{z}] \iff z \ \mathsf{Sum}\ M^{\mathtt{x}}_{\mathtt{x}=\mathtt{y}}(y) \iff z \ \mathsf{Sum}\ \mathbb{I}(y)$. Since, by ($\mathtt{r}_{\sqsubseteq}$), the formula "$\forall_{\mathtt{x}}(\mathtt{x} \sqsubseteq \mathtt{y} \rightarrow \mathtt{x} \sqsubseteq \mathtt{y}) \wedge \forall_{\mathtt{u}}(\mathtt{u} \sqsubseteq \mathtt{y} \rightarrow \exists_{\mathtt{x}}(\mathtt{x} \sqsubseteq \mathtt{y} \wedge \mathtt{x} \bigcirc \mathtt{u}))$" is a thesis, then for any $y \in M$ we have: $\mathfrak{M} \models \sigma^{\mathtt{x}}_{\mathtt{x}\sqsubseteq\mathtt{y}}\,[y/\mathtt{y}, y/\mathtt{z}]$, which means that $y \ \mathsf{Sum}\ \mathbb{I}(y)$.

For any $\mathrm{L}^{\mathrm{d}}_{\sqsubseteq}$-formula φ such that $\mathrm{vf}(\varphi) = \{\mathtt{x}_1, \ldots, \mathtt{x}_{k+1}\}$, from the formula $\sigma^{\mathtt{x}}_{\varphi}$ we obtain the $\mathrm{L}^{\mathrm{d}}_{\sqsubseteq}$-formula $\sigma^{\mathtt{x}*}_{\varphi}$ by the substitution $\mathtt{x}_{k+4}/\mathtt{x}_{k+2}$. Clearly, this substitution is allowed and we have $\mathrm{vf}(\sigma^{\mathtt{x}*}_{\varphi}) = \{\mathtt{x}_2, \ldots, \mathtt{x}_{k+1}, \mathtt{x}_{k+4}\}$ (for $k = 0$ we have $\mathrm{vf}(\sigma^{\mathtt{x}*}_{\varphi}) = \{\mathtt{x}_4\}$).

As the next axioms of theory **M** we adopt the following $\mathrm{L}^{\mathrm{d}}_{\sqsubseteq}$-sentences:

$$\forall_{\mathtt{x}_2} \ldots \forall_{\mathtt{x}_{k+1}} \forall_{\mathtt{x}_{k+2}} \forall_{\mathtt{x}_{k+4}} (\sigma^{\mathtt{x}}_{\varphi} \wedge \sigma^{\mathtt{x}*}_{\varphi} \rightarrow \mathtt{x}_{k+2} = \mathtt{x}_{k+4}) \qquad (\lambda 3^k_{\varphi})$$

$$\forall_{\mathtt{x}_2} \ldots \forall_{\mathtt{x}_{k+1}} (\exists_{\mathtt{x}}\, \varphi \rightarrow \exists_{\mathtt{x}_{k+2}}\, \sigma^{\mathtt{x}}_{\varphi}) \qquad (\lambda 4^k_{\varphi})$$

Therefore — taking into account the interpretation of the formulae σ^x_φ and σ^{x*}_φ in the structure $\mathfrak{M}$ — axiom ($\lambda 3^k_\varphi$) says that at most one mereological sum of the set $M^x_\varphi(y_1, \dots, y_k)$, i.e., we have:

$$\forall_{z,v,y_1,\dots,y_k \in M}\big(z \ \mathbb{Sum}\ M^x_\varphi(y_1,\dots,y_k) \wedge v \ \mathbb{Sum}\ M^x_\varphi(y_1,\dots,y_k) \Longrightarrow z = v\big),$$

and axiom ($\lambda 4^k_\varphi$) says that if the set $M^x_\varphi(y_1, \dots, y_k)$ is not empty, then there is at least one mereological sum of it, i.e., we have:

$$\forall_{y_1,\dots,y_k \in M}\big(M^x_\varphi(y_1,\dots,y_k) \neq \emptyset \implies \exists_{z \in M}\ z \ \mathbb{Sum}\ M^x_\varphi(y_1,\dots,y_k)\big).$$

Clearly, for $k = 0$ we obtain (note that $y := x_2$ and $u := x_4$):

$$\forall_y \forall_u (\sigma^x_\varphi \wedge \sigma^{x*}_\varphi \rightarrow y = u) \qquad (\lambda 3^0_\varphi)$$

$$\exists_x \varphi \rightarrow \exists_y \sigma^x_\varphi \qquad (\lambda 4^0_\varphi)$$

In Section 10 we will show that in formulating axioms ($\lambda 3^k_\varphi$) and ($\lambda 4^k_\varphi$) we can not accept that the formula φ should have only one free variable, that is, be limited to $k = 0$.[2]

2. Quasi-mereological structures – models of theory M

We will define and examine the class **qMS** composed of $\mathrm{L}_\sqsubset$-structures, which we mentioned on p. 72. We will prove that it is a class of the models of theory **M**.[3]

As in Chapter II, for an arbitrary $\mathrm{L}_\sqsubset$-structure we will define auxiliary relations $\sqsubseteq$, $\bigcirc$, $\wr$ and $\mathbb{Sum}$ by applying definitions (df $\sqsubseteq$), (df $\bigcirc$), (df $\wr$) and (df $\mathbb{Sum}$), respectively. From these definitions it follows that relations $\sqsubseteq$ and $\bigcirc$ are reflexive (so also for any $x \in M$ we have $x \ \mathbb{Sum}\ \{x\}$ and $x \ \mathbb{Sum}\ \mathbb{I}(x)$), the relation $\wr$ is irreflexive, $\bigcirc$ and $\wr$ are symmetric, $\sqsubseteq$ is included in $\bigcirc$, and the identity $\wr = -\bigcirc$ holds.

An $\mathrm{L}_\sqsubset$-structure $\mathfrak{M} = \langle M, \sqsubset\rangle$ we will call *quasi-mereological* iff in $\mathfrak{M}$ conditions (L1) and (L2) hold, along with counterparts of conditions (L3) and (L4) with one universal quantifier restricted to parametrically

[2] Such an insufficient solution was adopted, for example, by Smith [1993].

[3] In Section 9 will we show that it is possible to base theory **M** on a finite number of specific axioms; in other words, that the class is finitely elementarily axiomatisable.

e-definable (in short: pe-definable) sets in $\mathfrak{M}$.[4] In order to present these definitions formally, we must accept that $\mathfrak{pe}\mathcal{P}(\mathfrak{M})$ be the family of sets which are parametrically elementarily definable (or pe-definable) in $\mathfrak{M}$ and that $\mathfrak{pe}\mathcal{P}_+(\mathfrak{M}) := \mathfrak{pe}\mathcal{P}(\mathfrak{M}) \setminus \{\emptyset\}$. Therefore an $\mathrm{L}_{\sqsubset}$-structure $\mathfrak{M} = \langle M, \sqsubset\rangle$ is quasi-mereological (in short: *q-mereological*) iff the relation $\sqsubset$ satisfies conditions (L1) and (L2), and the conditions below hold in $\mathfrak{M}$:

$$\forall_{S\in\mathfrak{pe}\mathcal{P}(\mathfrak{M})}\forall_{x,y\in M}(x \mathrel{\mathsf{Sum}} S \wedge y \mathrel{\mathsf{Sum}} S \implies x = y), \tag{peL3}$$

$$\forall_{S\in\mathfrak{pe}\mathcal{P}_+(\mathfrak{M})}\exists_{x\in M}\ x \mathrel{\mathsf{Sum}} S\,. \tag{peL4}$$

Let **qMS** be a class of all q-mereological structures. It follows from the definition, that each mereological structure is q-mereological.[5]

LEMMA 2.1. $\mathbf{MS} \subseteq \mathbf{qMS}$.

However, we have:

PROPOSITION 2.2. *Each q-mereological structure with a finite universe belongs to the class* **MS**.

PROOF. If $\mathfrak{M}$ has an finite universe, then $\mathfrak{pe}\mathcal{P}(\mathfrak{M}) = \mathcal{P}(M)$, because each finite set is e-definable with parameters. □

For theory **M** the following lemmas hold.

LEMMA 2.3. *If an* $\mathrm{L}_{\sqsubset}$*-structure* $\langle M, \sqsubset\rangle$ *belongs to the class* **qMS**, *then the* $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$*-structure* $\langle M, \sqsubset, \sqsubseteq, \circ, \wr\rangle$ *is a model of the theory* **M**.

PROOF. Given the assumptions made so far, it is obvious that in $\mathfrak{M}^{\mathrm{d}} := \langle M, \sqsubset, \sqsubseteq, \circ, \wr\rangle$ the axioms (λ1), (λ2), ($\delta\sqsubseteq$), ($\delta\circ$) and ($\delta\wr$) hold. It remains to show that in $\mathfrak{M}^{\mathrm{d}}$, all axioms of the form ($\lambda 3^k_\varphi$) and ($\lambda 4^k_\varphi$) hold. Let us take, therefore, an arbitrary $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-formula for which $\mathrm{vf}(\varphi) = \{\mathtt{x}_1, \ldots, \mathtt{x}_{k+1}\}$, for some $k \geqslant 0$.

(a) For $k > 0$. *Ad* ($\lambda 3^k_\varphi$): Take an arbitrary valuation $[y_1/\mathtt{x}_2, \ldots, y_k/\mathtt{x}_{k+1}, z/\mathtt{x}_{k+2}, u/\mathtt{x}_{k+4}]$ in M. If the valuation $[y_1/\mathtt{x}_2, \ldots, y_k/\mathtt{x}_{k+1},$

---

[4] Recall that elementarily definable sets with empty sets of parameters are also parametrically elementarily definable sets (cf. p. 291 in Appendix II).

In the case of (L3), it will turn out that restricting the range of the universal quantifier will not be essential (see Proposition 2.7). In Section 10 we will show how weak we get the conditions if the range of this quantifier will be restricted to definable sets without parameters.

[5] On p. 200, we will show that $\mathbf{MS} \subsetneq \mathbf{qMS}$, i.e., that restricting the range of the universal quantifier in (peL4) is essential.

$z/\mathtt{x}_{k+2}, u/\mathtt{x}_{k+4}]$ satisfies in $\mathfrak{M}^{\mathrm{d}}$ the conjunction $\ulcorner\sigma^{\mathtt{x}}_{\varphi} \wedge \sigma^{\mathtt{x}*}_{\varphi}\urcorner$ in $\mathfrak{M}^{\mathrm{d}}$, then — on the basis of Lemma 1.2 — we have x Sum $M^{\mathtt{x}}_{\varphi}(y_1, \dots, y_k)$ and y Sum $M^{\mathtt{x}}_{\varphi}(y_1, \dots, y_k)$. Hence $z = u$, in virtue of (peL3). Therefore the arbitrarily chosen valuation satisfies the implication ($\lambda 3^k_{\varphi}$).

Ad ($\lambda 4^k_{\varphi}$): Take an arbitrary valuation $[y_1/\mathtt{x}_2, \dots, y_k/\mathtt{x}_{k+1}]$ in M. If it satisfies in the antecedent of the implication ($\lambda 4^k_{\varphi}$), then for a certain $x_0 \in M$, the valuation $[x_0/\mathtt{x}, y_1/\mathtt{x}_2, \dots, y_k/\mathtt{x}_{k+1}]$ satisfies φ. Hence $M_{\varphi}(y_1, \dots, y_k)^{\mathtt{x}} \neq \emptyset$. Therefore, in virtue of (peL4), for some $z_0 \in M$ we have z_0 Sum $M^{\mathtt{x}}_{\varphi}(y_1, \dots, y_k)$. Hence, in virtue of Lemma 1.2, $\mathfrak{M}^{\mathrm{d}} \models \sigma^{\mathtt{x}}_{\varphi}[y_1/\mathtt{x}_2, \dots, y_k/\mathtt{x}_{k+1}, x_0/\mathtt{x}_{k+2}]$, i.e., $\mathfrak{M}^{\mathrm{d}} \models \exists_{\mathtt{x}_{k+2}} \sigma^{\mathtt{x}}_{\varphi}[y_1/\mathtt{x}_2, \dots, y_k/\mathtt{x}_{k+1}]$. Therefore, the initial valuation satisfies the implication ($\lambda 4^k_{\varphi}$).

(b) For $k = 0$. The proof is as in case (a) but without parameters (so we use Lemma 1.1). □

Lemma 2.4. *If an* $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$*-structure* $\mathfrak{M}^{\mathrm{d}} = \langle M, \sqsubset, \sqsubseteq, \circ, \wr \rangle$ *is a model of theory* **M** *then the relations* $\sqsubseteq$, $\circ$ *and* $\wr$ *satisfy conditions* (df $\sqsubseteq$), (df $\circ$) *and* (df $\wr$), *respectively, and* $\langle M, \sqsubset \rangle$ *belongs to class* **qMS**.

Proof. Given the assumption made, it is clear that the relations $\sqsubseteq$, $\circ$ and $\wr$ satisfy conditions (df $\sqsubseteq$), (df $\circ$) and (df $\wr$), respectively, and that in $\mathfrak{M} = \langle M, \sqsubset \rangle$ conditions (L1) and (L2) hold. It remains to show that in $\mathfrak{M}$ conditions (peL3) and (peL4) hold.

Ad (peL3): Let us take an arbitrary set $S \in \mathrm{pe}\mathcal{P}(\mathfrak{M})$. Then there is an $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-formula such that for some $k \geqslant 0$ we obtain: $\mathrm{vf}(\varphi) = \{\mathtt{x}_1, \dots, \mathtt{x}_{k+1}\}$ and for some $y_1, \ldots, y_k$ from M, we have $S = M^{\mathtt{x}}_{\varphi}(y_1, \dots, y_k)$. As in the proof of Theorem 2.3 we can assume that $k > 0$.

Now assume that z Sum S and u Sum S. Then, by virtue of Lemma 1.2, for $\mathfrak{M}^{\mathrm{d}}$ we have $\mathfrak{M}^{\mathrm{d}} \models \ulcorner\sigma^{\mathtt{x}}_{\varphi} \wedge \sigma^{\mathtt{x}}_{\varphi}(\mathtt{x}_{k+4}/\mathtt{x}_{k+2})\urcorner[y_1/\mathtt{x}_2, \dots, y_k/\mathtt{x}_{k+1}, z/\mathtt{x}_{k+2}, u/\mathtt{x}_{k+4}]$. Hence, since the implication ($\lambda 3^k_{\varphi}$) is true $\mathfrak{M}^{\mathrm{d}}$, the valuation $[z/\mathtt{x}_{k+2}, u/\mathtt{x}_{k+4}]$ satisfies the formula "$\ulcorner \mathtt{x}_{k+2} = \mathtt{x}_{k+4}\urcorner$", i.e., $z = u$.

Ad (peL4): Let us take an arbitrary set $S \in \mathrm{pe}\mathcal{P}_{+}(\mathfrak{M})$, and an $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-formula φ and parameters $y_1, \ldots y_k \in M$, as in the proof for (peL3). Since $S \neq \emptyset$, then for some $x_0 \in M$ we have $\mathfrak{M}^{\mathrm{d}} \models \varphi[x_0/\mathtt{x}, y_1/\mathtt{x}_2, \dots, y_k/\mathtt{x}_{k+1}]$ and $\mathfrak{M}^{\mathrm{d}} \models \exists_{\mathtt{x}} \varphi[y_1/\mathtt{x}_2, \dots, y_k/\mathtt{x}_{k+1}]$. Since the implication ($\lambda 4^k_{\varphi}$) is true in $\mathfrak{M}^{\mathrm{d}}$, the valuation $[y_1/\mathtt{x}_2, \dots, y_k/\mathtt{x}_{k+1}]$ satisfies in $\mathfrak{M}^{\mathrm{d}}$ the $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-formula $\ulcorner\exists_{\mathtt{x}_{k+1}} \sigma^{\mathtt{x}}_{\varphi}\urcorner$. Therefore, for some $z_0 \in M$ the valuation $[y_1/\mathtt{x}_2, \dots, y_k/\mathtt{x}_{k+1}, z_0/\mathtt{x}_{k+2}]$ satisfies the formula $\sigma^{\mathtt{x}}_{\varphi}$, i.e., in virtue of Lemma 1.2, z_0 Sum S. □

Remark 2.1. For convenience, we will identify any structure $\mathfrak{M} = \langle M, \sqsubset \rangle$ with its definitional extension $\mathfrak{M}^{\mathrm{d}} = \langle \sqsubset, \sqsubseteq, \circ, \wr \rangle$, where relations $\sqsubseteq$, $\circ$ and $\wr$ are defined by (df $\sqsubseteq$), df $\circ$ and (df $\circ$), respectively. Thus, we will further recognize that the class **qMS** consists of structures of the form $\langle \sqsubset, \sqsubseteq, \circ, \wr \rangle$. □

From lemmas 2.3 and 2.4, and Remark 2.1 we obtain:

Theorem 2.5. *The class* **qMS** *is elementarily axiomatisable as the class of models of the theory* $\mathbf{M}$, *i.e.,*

$$\mathbf{qMS} = \mathrm{Mod}(\mathrm{Ax}^{\mathbf{M}}) = \mathrm{Mod}(\mathbf{M})\,.$$

Proof. Assume that $\mathfrak{M} = \langle M, \sqsubset \rangle$ belongs to **qMS**. Then, by virtue of Lemma 2.3, $\mathfrak{M}^{\mathrm{d}} = \langle M, \sqsubset, \sqsubseteq, \circ, \wr \rangle$ is a model of theory $\mathbf{M}$. Conversely, if $\mathfrak{M}^{\mathrm{d}} = \langle M, \sqsubset, \sqsubseteq, \circ, \wr \rangle$ is a model of the set $\mathrm{Ax}^{\mathbf{M}}$ then — by using Lemma 2.4 — $\mathfrak{M}$ belongs to **qMS**. □

Thus, we obtain:

$$\mathrm{Th}(\mathbf{qMS}) = \overline{\mathbf{M}}\,, \tag{2.1}$$

where Th(**qMS**) is the set of all true $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-sentences in all structures from **qMS** and $\overline{\mathbf{M}}$ is the set of all theses of $\mathbf{M}$ which are sentences. In fact, from Gödel's completeness theorem we have: $\mathrm{Th}(\mathbf{qMS}) := \bigcap\{\mathrm{Th}(\mathfrak{M}) : \mathfrak{M} \in \mathbf{qMS}\} = \bigcap\{\mathrm{Th}(\mathfrak{M}) : \mathfrak{M} \in \mathrm{Mod}(\mathbf{M})\} = \overline{\mathbf{M}}$.

Moreover, from Lemma 2.1 and theorems III.2.4 and 2.5 we obtain:

Theorem 2.6. $\mathbf{MS} \subsetneq \mathbf{qMS}$.[6]

Proof. Theorem III.2.4 says that **MS** is not elementarily axiomatisable, but **qMS** is. Hence $\mathbf{MS} \neq \mathbf{qMS}$. Therefore, the inclusion $\mathbf{MS} \subseteq \mathbf{qMS}$ is strict. □

The following fact will come in handy later:

Proposition 2.7. *In all structures from* **qMS**, *the following conditions hold*: (L3), (SSP) *and* (WSP). *Therefore, in quasi-mereological structures, all those conditions are in force that we demonstrated in Chapter II using just* (L1)–(L3), (WSP) *and* (SSP).

[6] Besides this fact we will show belong that Th(**qMS**) = Th(**MS**), i.e., in both classes **qMS** and **MS** the same sentences are true.

PROOF. In the proof of Theorem II.6.1, which said that sentence (SSP) holds in the class **MS**, axioms (L3) and (L4) apply only to pe-definable sets. Therefore, we may repeat that proof, applying axioms (peL3) and (peL4). By Lemma II.6.2, condition (L3) follows from (L1), (L2) and (SSP). Hence, by Lemma II.4.1, condition (WSP) also holds. □

Moreover, since $M \in \mathrm{pe}\mathcal{P}_+(\mathfrak{M})$, there exists therefore in the structure $\mathfrak{M}$ the unity $\mathbb{1}$ defined by condition (df $\mathbb{1}$) and satisfying condition (II.5.1), i.e., we have:

$$\mathbb{1} := (\iota\, x)\; x \;\mathbb{S}\mathrm{um}\; M\,,$$
$$\forall_{z\in M}\; z \sqsubseteq \mathbb{1}\,.$$

3. The relation $\mathbb{S}\mathrm{um}$ versus the relation of supremum in qMS

From Proposition 2.7, conditions (L1), (L2) and (SSP) hold in **qMS**. As in Section 8 of Chapter II we showed that these conditions entail the inclusion:

$$\mathbb{S}\mathrm{um} \subseteq \mathsf{sup}_{\sqsubseteq}\,. \tag{3.1}$$

Furthermore, we obtain:

THEOREM 3.1. *In an arbitrary structure* $\mathfrak{M}$ *from* **qMS**, *the relation* is a mereological sum *of all elements of a given non-empty set that is pe-definable in* $\mathfrak{M}$ *coincides with the relation* is a least upper bound *of all elements of the set. That is, we have*:

$$\forall_{S\in \mathrm{pe}\mathcal{P}_+(\mathfrak{M})}\forall_{x\in M}\big(x \;\mathbb{S}\mathrm{um}\; S \iff S \neq \emptyset \wedge x \;\mathsf{sup}_{\sqsubseteq}\; S\big).$$

PROOF. '⇒' By using (II.3.2) and (3.1).

'⇐' Assume that $S \in \mathrm{pe}\mathcal{P}_+(\mathfrak{M})$ and $x \;\mathsf{sup}_{\sqsubseteq}\; S$. Then, in virtue of (peL4), for a certain y_0 we have $y_0 \;\mathbb{S}\mathrm{um}\; S$. Hence $y_0 \;\mathsf{sup}_{\sqsubseteq}\; S$, by (3.1). Therefore $x = y_0$, because $\mathsf{sup}_{\sqsubseteq}$ satisfies condition ($\mathrm{U}_{\mathsf{sup}}$). Therefore, we have $x \;\mathbb{S}\mathrm{um}\; S$. □

Thus, by using a singular description, we may write Theorem 3.1 as:

$$\forall_{S\in \mathrm{pe}\mathcal{P}_+(\mathfrak{M})}\; (\iota\, x)\; x \;\mathbb{S}\mathrm{um}\; S = (\iota\, x)\; x \;\mathsf{sup}_{\sqsubseteq}\; S\,. \tag{3.2}$$

In fact, thanks to (peL3) and (peL4), for an arbitrary $S \in \mathrm{pe}\mathcal{P}_+(\mathfrak{M})$ there is exactly one x such that $x \;\mathbb{S}\mathrm{um}\; S$. In virtue of (3.1) and ($\mathrm{U}_{\mathsf{sup}}$), in an equivalent way, x is the only element in M such that $x \;\mathsf{sup}_{\sqsubseteq}\; S$.

4. The operations of sum, produce and supplementation in the class qMS

Let $\mathfrak{M} \in \mathbf{qMS}$. Thanks to (peL3) and (peL4), in $\mathfrak{M}$ we may define ON the set $\mathfrak{pe}\mathcal{P}_+(\mathfrak{M})$ the unary operation $\bigsqcup\colon \mathfrak{pe}\mathcal{P}_+(\mathfrak{M}) \to M$ the *sum of all elements of a given set* with the help of the condition from p. 94:

$$\bigsqcup S := (\iota\, x)\; x \;\mathsf{Sum}\; S\,. \tag{df $\bigsqcup$}$$

On the strength of (3.2), we have:

$$\forall_{S\in\mathfrak{pe}\mathcal{P}_+(\mathfrak{M})}\; \bigsqcup S = \sup_{\sqsubseteq}(S). \tag{4.1}$$

Since for arbitrary $x, y \in M$ we have $\{x, y\} \in \mathfrak{pe}\mathcal{P}_+(\mathfrak{M})$, then we may create a binary operation $\sqcup\colon M \times M \to M$ the *sum of two elements* with the help of condition (df $\sqcup$) from p. 95:

$$x \sqcup y := \bigsqcup\{x, y\}.$$

It satisfies conditions (II.7.3)–(II.7.13) (condition (II.7.3) holds because for arbitrary $x, y \in M$ the set $\{z \in M : z \sqsubseteq x \vee z \sqsubseteq y\}$ belongs to $\mathfrak{pe}\mathcal{P}_+(\mathfrak{M})$, as e-definable with parameters x and y with the help of the $\mathrm{L}_{\sqsubset}$-formula "$\mathtt{z \sqsubset x \vee z = x \vee z \sqsubset y \vee z = y}$").

In order to introduce further operations in quasi-mereological structures, the lemma below will come in handy:

LEMMA 4.1. *In each* $\mathrm{L}_{\sqsubset}$*-structure* $\mathfrak{M} = \langle M, \sqsubset\rangle$ *for arbitrary* $S \in \mathfrak{pe}\mathcal{P}(\mathfrak{M})$ *and* $x, y \in M$ *we have:*

(i) $\bigcap \mathbb{I}(S) \in \mathfrak{pe}\mathcal{P}(\mathfrak{M})$,
(ii) $\{z \in M : z \sqsubseteq x \wedge z \sqsubseteq y\} \in \mathfrak{pe}\mathcal{P}(\mathfrak{M})$,
(iii) $\{y \in M : y \wr x\} \in \mathfrak{pe}\mathcal{P}(\mathfrak{M})$.

PROOF. *Ad* (i): We have $\bigcap \mathbb{I}(S) = \{y \in X : \forall_{z\in S}\; y \sqsubseteq z\}$; cf. Remark II.9.1. The proof runs just like the proof of Lemma 3.2(i) in Appendix II. □

Applying Lemma 4.1, and conditions (peL3) and (peL4), we may define IN the set $\mathfrak{pe}\mathcal{P}(\mathfrak{M})$ a PARTIAL unary operation $\bigsqcap\colon \mathfrak{pe}\mathcal{P}(\mathfrak{M}) \to M$ by using condition (def $\bigsqcap$) given on p. 99:

$$\bigcap \mathbb{I}[S] \neq \emptyset \implies \bigsqcap S := \bigsqcup \bigcap \mathbb{I}[S].$$

This operation has those properties given on pp. 99–100. By making use of the mutual definition of the relations of supremum and infimum, for arbitrary $S \in \mathcal{P}(M)$ and $x \in M$ we get: $x \operatorname{\mathsf{sup}}_{\sqsubseteq} \{y \in M : \forall_{z \in S}\, y \sqsubseteq z\}$ iff $x \operatorname{\mathsf{inf}}_{\sqsubseteq} S$. Therefore from (4.1) we have:

$$\forall_{S \in \mathrm{pe}\mathcal{P}_+(\mathfrak{M})}\big(\bigcap \mathbb{I}(S) \neq \emptyset \implies \bigsqcap S = \inf_{\sqsubseteq}(S)\big). \tag{4.2}$$

In an analogous manner with respect to the operation $\sqcup$ we can create a PARTIAL binary operation $\sqcap\colon M \times M \to M$ the *product of two elements* with the domain i$\{\langle x, y\rangle \in M \times M : x \bigcirc y\}$, by condition (df $\sqcap$) from p. 101:

$$x \bigcirc y \implies x \sqcap y := \bigsqcap\{x, y\} = \inf_{\sqsubseteq}\{x, y\}.$$

From Lemma 4.1, it follows that the operation $\sqcap$ satisfies condition (II.9.9). It is also easy to show that the operation $\sqcap$ satisfies conditions (II.9.6)–(II.9.15). As was shown on p. 148, condition (II.9.10) entails:

$$\forall_{x,y \in M}\big(x \bigcirc y \implies \exists_{u \in M}\forall_{z \in M}(z \sqsubseteq u \iff z \sqsubseteq x \wedge z \sqsubseteq y)\big). \tag{$\exists\sqcap$}$$

We can repeat in its entirety the analysis from Section 10 in Chapter II. Therefore, in all quasi-mereological structures the following two distributivity conditions hold for all $x, y, z \in M$:

$$(x \sqcup y) \sqcap (x \sqcup z) = \begin{cases} x \sqcup (y \sqcap z) & \text{if } y \bigcirc z, \\ x & \text{if } y \wr z. \end{cases} \tag{Δ_1}$$

$$x \bigcirc y \vee x \bigcirc z \implies$$
$$x \sqcap (y \sqcup z) = \begin{cases} (x \sqcap y) \sqcup (x \sqcap z) & \text{if } x \bigcirc y \text{ and } x \bigcirc z, \\ x \sqcap y & \text{if } x \bigcirc y \text{ and } x \wr z, \\ x \sqcap z & \text{if } x \wr y \text{ and } x \bigcirc z. \end{cases} \tag{Δ_2}$$

Similarly, as on p.104, we can show that for any $x \in M$: $x \neq \mathbb{1}$ iff $\{y \in M : y \wr x\} \neq \emptyset$. From this, Lemma 4.1 and the facts proved on p. 104, it follows that if $M \neq \{\mathbb{1}\}$, then ON the set $M \setminus \{\mathbb{1}\}$ we can define the mereological complement operation ${}^{\mathsf{c}}\colon M \setminus \{\mathbb{1}\} \to M \setminus \{\mathbb{1}\}$:

$$x^{\mathsf{c}} := \bigsqcup\{y \in M : y \wr x\} = \sup_{\sqsubseteq}\{y \in M : y \wr x\}. \tag{df ${}^{\mathsf{c}}$}$$

It is easy to show that the mereological complement operation in quasi-mereological structures also has the properties presented by conditions (II.11.3)–(II.11.13) (DM_1) and (DM_1).

On the strength of (II.11.8) and (II.11.3) in all quasi-mereological structures the following condition holds:

$$\forall_{x \in M}\big(x \neq \mathbb{1} \implies \exists_{y \in M}(x \sqcup y = \mathbb{1} \wedge x \wr y)\big). \tag{Υ}$$

5. The class qMS on diagram 3

Let us note that in quasi-mereological structures — by making use of (II.9.8), (II.9.10), (II.11.3), (II.11.4) and (II.11.6) — we can derive the Super Strong Supplementation Principle from p. 106:

$$\forall_{x,y\in M}\big(x \not\sqsubseteq y \implies \exists_{z\in M}(z \sqsubseteq x \wedge z \wr y \wedge \forall_{u\in M}(u \sqsubseteq x \wedge u \wr y \Rightarrow u \sqsubseteq z))\big), \tag{SSP+}$$

Therefore every axiom of the class **GMS** holds in all structures from **qMS**. Thus, **qMS** $\subseteq$ **GMS**. We prove, however, that:

THEOREM 5.1. **qMS** $\subsetneq$ **GMS**.

PROOF. For any set X, the model $\langle \mathrm{F}_+(X), \subsetneq\rangle$ from the proof of Theorem 7.1(ii) belong to the class **GMS**. If X is infinite then the universe $\mathrm{F}_+(X)$ has no mereological sum in $\langle \mathrm{F}_+(X), \subsetneq\rangle$. Of course, the universe is e-definable by the formula "`x = x`". So, condition (peL4) does not hold in $\langle \mathrm{F}_+(X), \subsetneq\rangle$. Thus, this condition does not follow from the axioms of the class **GMS**. □

From theorems 2.6 and 5.1 we can supplement diagram 3 to diagram 4.

6. Atoms and atomic and atomless elements in quasi-mereological structures

Let $\mathfrak{M} \in$ **qMS** and $\mathfrak{at}$ be a set of mereological atoms in M as defined on p. 75 with the following condition:

$$x \in \mathfrak{at} :\iff \neg\exists_{z\in M}\, z \sqsubset x \iff \mathbb{P}(x) = \emptyset\,. \tag{df $\mathfrak{at}$}$$

In all quasi-mereological structures conditions (2.4) and (2.5) from Chapter II hold:[7]

$$x \in \mathfrak{at} \iff \mathbb{I}(x) = \{x\} \iff \forall_{z\in M}(z \sqsubseteq x \Leftrightarrow z = x),$$
$$x \in \mathfrak{at} \iff \forall_{y\in M}(x \circ y \Rightarrow x \sqsubseteq y).$$

[7] Since **MS** $\subsetneq$ **qMS**, then all the considerations of this section concern mereological structures too.

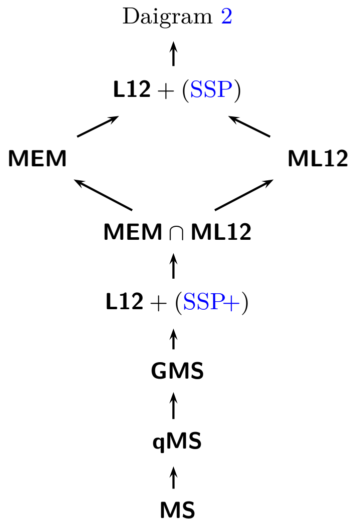


Diagram 4. The lattice of certain superclasses of the classes **qMS** and **MS**

Below, we will make use of the following property of the set $\mathfrak{at}$:

$$a \in \mathfrak{at} \implies \forall_{x,y \in M}(a \sqsubseteq x \sqcup y \;\Leftrightarrow\; a \sqsubseteq x \vee a \sqsubseteq y). \tag{6.1}$$

In fact, applying ($\sqsubseteq\subseteq\bigcirc$), (II.2.5) and (II.7.7) we have: $a \sqsubseteq x \sqcup y$ iff $a \bigcirc x \sqcup y$ iff either $a \bigcirc x$ or $a \bigcirc y$ iff either $a \sqsubseteq x$ or $a \sqsubseteq y$.

Notice that we have:

$$\operatorname{Card} M = 1 \iff \mathbb{1} \in \mathfrak{at}\,. \tag{6.2}$$

In fact, if $\operatorname{Card} M = 1$, then $M = \{\mathbb{1}\}$ and $\mathbb{1} \not\sqsubset \mathbb{1}$, in virtue of ($\mathrm{irr}_{\sqsubset}$). Conversely, if $\operatorname{Card} M > 1$, then there exists a $y \in M$ such that $y \sqsubset \mathbb{1}$, i.e., $\mathbb{1} \notin \mathfrak{at}$. Furthermore, in virtue of ($\nexists 0$), besides the trivial case where $\operatorname{Card} M = 1$, no element in M is a ‘zero’, and therefore no mereological atom is either.

We say that an element x from M is *atomic* in $\mathfrak{M}$ iff each ingrediens x has some ingrediens that is an atom. Let $\mathfrak{atc}$ be the set of all atomic elements in $\mathfrak{M}$, i.e., for any $x \in M$:

$$x \in \mathfrak{atc} :\iff \forall_{y \in M}(y \sqsubseteq x \implies \exists_{a \in \mathfrak{at}}\, a \sqsubseteq y). \tag{df $\mathfrak{atc}$}$$

Obviously, $\mathfrak{at} \subseteq \mathfrak{atc}$, by (II.2.4) and ($\mathrm{r}_{\sqsubseteq}$).

Atomic elements have the following property:

$$\forall_{x,y\in M}\big(x \in \mathfrak{atc} \wedge y \in \mathfrak{atc} \iff x \sqcup y \in \mathfrak{atc}\big). \tag{6.3}$$

In fact, let $x \in \mathfrak{atc}$, $y \in \mathfrak{atc}$ and take a z such that $z \sqsubseteq x \sqcup y$. If $x \wr z$, then — in virtue of (II.7.12) — we have $z \sqsubseteq y$, and so we make use of our assumptions. Let therefore $x \circ z$. Since $x \sqcap z \sqsubseteq x$, there therefore exists an $a \in \mathfrak{at}$ such that $a \sqsubseteq x \sqcap z$. Hence $a \sqsubseteq z$. The converse implication we obtain from the fact that $x \sqsubseteq x \sqcup y$ and $y \sqsubseteq x \sqcup y$ and by condition ($\mathrm{t}_{\sqsubseteq}$).

We call the structure $\mathfrak{M}$ *atomic* iff all elements are atomic, i.e., we have $\mathfrak{atc} = M$.

Lemma 6.1. *A structure $\mathfrak{M}$ is atomic iff any one of the (equivalent) conditions holds:*

1° $\forall_{x\in M}\exists_{a\in\mathfrak{at}}\ a \sqsubseteq x$,
2° $\mathbb{1} \in \mathfrak{atc}$,
3° $\mathbb{1}\ \mathsf{Sum}\ \mathfrak{at}$.

Proof. *Ad* (1°): If $M \subseteq \mathfrak{atc}$ then for any $x \in M$, then there is an $a \in \mathfrak{at}$ such that $a \sqsubseteq x$, by ($\mathrm{r}_{\sqsubseteq}$). Of course, if $\forall_{x\in M}\exists_{a\in\mathfrak{at}}\ a \sqsubseteq x$ holds, then $M \subseteq \mathfrak{atc}$.

Ad (2°): Of course, if $M \subseteq \mathfrak{atc}$ then $\mathbb{1} \in \mathfrak{atc}$. If $\mathbb{1} \in \mathfrak{atc}$ then $M \subseteq \mathfrak{atc}$, by (1°), ($\mathrm{t}_{\sqsubseteq}$) and the fact that: $\forall_{z\in M}\ z \sqsubseteq \mathbb{1}$.

Ad (3°): If $M \subseteq \mathfrak{atc}$ then $\mathbb{1}\ \mathsf{Sum}\ \mathfrak{at}$, by (2°) and (II.2.5). If $\mathbb{1}\ \mathsf{Sum}\ \mathfrak{at}$ then $M \subseteq \mathfrak{atc}$, by ($\mathrm{t}_{\sqsubseteq}$) and (II.2.5). □

We say that an element $x \in M$ is *atomless* iff no atom in $\mathfrak{M}$ is an ingrediens of x. Let $\mathfrak{atl}$ be a set of all atomless elements in $\mathfrak{M}$, i.e.,

$$x \in \mathfrak{atl} \ :\Longleftrightarrow \neg\exists_{a\in\mathfrak{at}}\ a \sqsubseteq x\,. \tag{df $\mathfrak{atl}$}$$

We have $\mathfrak{atl} \cap \mathfrak{at} = \emptyset = \mathfrak{atl} \cap \mathfrak{atc}$. From (6.1) it follows that

$$\forall_{x,y\in M}\big(x \in \mathfrak{atl} \wedge y \in \mathfrak{atl} \iff x \sqcup y \in \mathfrak{atl}\big). \tag{6.4}$$

We call a structure $\mathfrak{M}$ *atomless* iff each of its elements is atomless, i.e., $M = \mathfrak{atl}$. From ($\mathrm{r}_{\sqsubseteq}$) it follows that:

Lemma 6.2. *A structure $\mathfrak{M}$ is atomless iff $\mathfrak{at} = \emptyset$.*

Notice that all atomic elements are exterior to all atomless element:

$$\forall_{x,y\in M}\big(x \in \mathfrak{atc} \wedge y \in \mathfrak{atl} \implies x \wr y\big). \tag{6.5}$$

Assume for a contradiction that (a) $x \in \mathfrak{atc}$, (b) $y \in \mathfrak{atl}$ and (c) $x \bigcirc y$. From (c), for some z_0 we have (d) $z_0 \sqsubseteq x$ and (e) $z_0 \sqsubseteq y$. From (a) and (d), for some $a_0 \in \mathfrak{at}$ we have $a_0 \sqsubseteq z_0$. Hence, by (e) end ($\mathrm{t}_{\sqsubseteq}$), also $a_0 \sqsubseteq y$. But this contradicts (b).

THEOREM 6.3. *For any quasi-mereological structure $\mathfrak{M} = \langle M, \sqsubset\rangle$ exactly one of the conditions below holds*:

(a) *$\mathfrak{M}$ is atomless,*
(b) *$\mathfrak{M}$ is atomic,*
(c) *there are $x \in \mathfrak{atc}$ and $y \in \mathfrak{atl}$ such that $\mathbb{1} = x \sqcup y$.*

In other words, in any $\mathfrak{M} \in \mathbf{qMS}$ the following condition hold:

$$\mathfrak{at} = \emptyset \ \vee\ \forall_{x\in M}\exists_{a\in\mathfrak{at}}\, a \sqsubseteq x \ \vee\ \exists_{x\in\mathfrak{atc}}\exists_{y\in\mathfrak{atl}}\, \mathbb{1} = x \sqcup y\,. \qquad (\Sigma_1)$$

Moreover, in all quasi-mereological structures condition (c) is equivalent to the following:

(d) *there is an $x \in M \setminus \{\mathbb{1}\}$ such that $x \in \mathfrak{atc}$ and $x^{\mathsf{c}} \in \mathfrak{atl}$.*

In other words, in any $\mathfrak{M} \in \mathbf{qMS}$ the following condition hold:

$$\mathfrak{at} = \emptyset \ \vee\ \forall_{x\in M}\exists_{a\in\mathfrak{at}}\, a \sqsubseteq x \ \vee\ \exists_{x\in M\setminus\{\mathbb{1}\}}(x \in \mathfrak{atc} \wedge x^{\mathsf{c}} \in \mathfrak{atl}). \qquad (\Sigma_2)$$

PROOF. Let $\mathfrak{M} \in \mathbf{qMS}$ and assume that $\mathfrak{M}$ is neither atomless nor atomic. Hence $\mathfrak{at} \neq \emptyset$. So $\mathbb{1} \notin \mathfrak{atl}$. Moreover, on p. 74 in footnote 8 we noted that the set $\mathfrak{at}$ belongs to $\mathrm{e}\mathcal{P}(\mathfrak{M})$. Hence, via (peL4), there exists an $x \in M$ such that $x \ \mathsf{Sum}\ \mathfrak{at}$. By Lemma 6.1, we have $x \neq \mathbb{1}$. It is obvious that $-x \in \mathfrak{atl}$. By (3.1) we have $x \ \mathsf{sup}_{\sqsubseteq}\ \mathfrak{at}$.

It is obvious that (a) $x \notin \mathfrak{atl}$. Assume for a contradiction that $x \notin \mathfrak{atc}$. Then there is a y such that (b) $y \sqsubseteq x$ and (c) $y \in \mathfrak{atl}$. From (a) and (c) we have $x \neq y \neq \mathbb{1}$, i.e., $y \sqsubset x$. Hence and from (b) we have (d) $x \not\sqsubseteq -y$. Moreover, on the strength of (WSP), there exists a z such that $z \sqsubset x$ and $z \wr y$, i.e., $z \sqsubseteq -y$. Therefore, $x \bigcirc -y$. From (d) it follows that $x \sqcap -y \sqsubset x$. Furthermore, on the strength of (b) and (Δ_1) we have $(x \sqcap -y) \sqcup y = (x \sqcup y) \sqcap (-y \sqcup y) = x \sqcap \mathbb{1} = x$. Hence, from (c) and (6.1) it follows that $x \sqcap -y$ is an upper bound of the set $\mathfrak{at}$. And that contradicts the fact that $x \ \mathsf{sup}_{\sqsubseteq}\ \mathfrak{at}$.

Of course, condition (d) entails condition (c). Moreover, by virtue of (II.11.10) and (6.5), condition (c) entails condition (d). □

7. Quasi-mereological structures versus e-complete Boolean lattices

In Section 1 of Chapter III we examined the connection between the class **MS** and the class **CBL** of complete Boolean lattices. An analogous connection holds between the class **qMS** and the class **ecBL** (= **pecBL**) of e-complete (also pe-complete) Boolean lattices.[8]

As in the case of Theorem III.1.1, we prove the theorem below:

THEOREM 7.1. *Let* $\mathfrak{B} = \langle B, \leq \rangle$ *be a non-trivial e-complete Boolean lattice with the zero* o *and the unity* $\mathrm{1}$ ($\mathrm{o} \neq \mathrm{1}$). *We put* $M := B \setminus \{\mathrm{o}\}$ *and* $\sqsubset\, := \,\leq|_M \setminus \mathrm{id}_M$. *Then* $\mathfrak{M} = \langle M, \sqsubset \rangle$ *is a quasi-mereological structure in which conditions from Theorem III.1.1 hold, where conditions* (iv) *and* (vii) *only for non-empty parametrically elementarily definable sets in* $\mathfrak{M}$.

PROOF. We demonstrate (i)–(iii) just as in the case of the proof of Theorem III.1.1.

Ad (iv): Let $S \in \mathrm{pe}\mathcal{P}_+(\mathfrak{M})$. We must show that $S \in \mathrm{pe}\mathcal{P}_+(\mathfrak{B})$ too. On the basis of the assumption for a certain $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-formula, for which $\mathrm{vf}(\varphi) = \{\mathtt{x}, \mathtt{x}_2, \ldots, \mathtt{x}_{1+k}\}$ for some $k \geqslant 0$, and for certain parameters $y_1, \ldots, y_k$ from M, we have $S = \{x \in M : \mathfrak{M} \models \varphi\,[x/\mathtt{x}, y_1/\mathtt{x}_2, \ldots, y_k/\mathtt{x}_{1+k}]\}$. It is clear that if we put $\mathfrak{A} := \mathfrak{M}$ and $\mathit{0} := \mathrm{o}$ in Lemma III.2.1, then $\mathfrak{A}^o = \mathfrak{B}$. With the help of that transformed lemma, let us assign to formula φ a certain $\mathrm{L}^{\mathrm{o}}_{\leq}$-formula φ^*. It is easy to prove that for each $x \in B$ we have:

$$\mathfrak{M} \models \varphi\,[x/\mathtt{x}, y_1/\mathtt{x}_2, \ldots, y_k/\mathtt{x}_{1+k}] \iff \mathfrak{B} \models \ulcorner \mathtt{x} \neq \mathtt{0} \ \wedge\ \varphi^{*}\urcorner\,[x/\mathtt{x}, y_1/\mathtt{x}_2, \ldots, y_k/\mathtt{x}_{1+k}]\,.$$

Therefore, S is pe-definable in $\mathfrak{B}$ with parameters $y_1, \ldots, y_k$ from M. Since the lattice is pe-complete, then S has the least upper bound in this lattice. Moreover, $\sup_{\leq} S \neq \mathrm{o}$, since $\emptyset \neq S \subseteq M$. As in the proof of Theorem III.1.1, we show that $\sup_{\leq} S \;\mathbb{S}\mathrm{um}\; S$. And this proves that sentence (peL4) is true in $\mathfrak{M}$.

We prove the further conditions just as we did in the case of Theorem III.1.1. □

The theorem below is the counterpart of Theorem III.1.2.

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[8] We discuss these lattices in Section 1 of Appendix II. We show there also that **CBL** $\subsetneq$ **ecBL**.

THEOREM 7.2. *Let* $\mathfrak{M} = \langle M, \sqsubset \rangle \in \mathbf{qMS}$, $0 \notin M$, $M^0 := M \cup \{0\}$ *and* $\leq := \sqsubseteq \cup (\{0\} \times M^0)$, *i.e., for arbitrary* $x, y \in M^0$

$$x \leq y \;:\Longleftrightarrow x \sqsubseteq y \vee x = 0\,.$$

Then $\mathfrak{M}^o := \langle M^0, \leq, 0, \mathbb{1} \rangle$ *is a non-trivial e-complete Boolean lattice in which* 0 *is the zero,* $\mathbb{1}$ *is the unity,* $\sqsubseteq\, = \,\leq|_M$ *and in which for each* $S \in \mathrm{pe}\mathcal{P}_+(\mathfrak{M})$ *conditions* (iii) *and* (iv) *hold from Theorem III.1.2.*

PROOF. The identity $\sqsubseteq\, = \,\leq|_M$ we prove as we did on p. 117.

If $\mathfrak{M} \in \mathbf{qMS}$ then the facts given in Section 2 on pp. 202–203 permit us to repeat the proof (without the axiom of choice) of Theorem III.1.2 to the point at which we stated that the structure $\mathfrak{M}^o := \langle M^0, \leq, 0, \mathbb{1} \rangle$ is a non-trivial Boolean lattice with the zero 0 and the unity $\mathbb{1}$. It therefore remains to show that it is an e-complete lattice. To this end, we make use Lemma 17.1(ii) from Appendix I and Theorem 6.6 from Appendix II, and we show that $\mathbb{1} \in \mathrm{E}(\mathfrak{M}^o)$.

Let us observe that in the lattice $\mathfrak{M}^o$ the operation $+$ coincides with the operation $\sqcup$ on $M \times M$ and the operation $\cdot$ coincides with the operation $\sqcap$ on $\{\langle x, y \rangle \in M \times M : x \bigcirc y\}$. Moreover, $\mathrm{At}(\mathfrak{M}^o) = \mathfrak{at}$, $\mathrm{Atc}(\mathfrak{M}^o) = \mathfrak{atc} \cup \{0\}$ and $\mathrm{Atl}(\mathfrak{M}^o) = \mathfrak{atl} \cup \{0\}$. We must therefore show that there exists an $x \in \mathrm{Atc}(\mathfrak{M}^o)$ and a $y \in \mathrm{Atl}(\mathfrak{M}^o)$ such that $\mathbb{1} = x + y$.

If the structure $\mathfrak{M}$ is atomic, then $\mathbb{1} \in \mathfrak{atc}$ and $\mathbb{1} = \mathbb{1} + 0$. If the structure $\mathfrak{M}$ is atomless, then $\mathbb{1} \in \mathfrak{atl}$ and $\mathbb{1} = 0 + \mathbb{1}$. In the remaining case — on the basis of Theorem 6.3 — there are an $x \in \mathfrak{atc}$ and a $y \in \mathfrak{atl}$ such that $\mathbb{1} = x \sqcup y$. □

By making use of the two theorems below, we may prove the identity $\mathrm{Th}(\mathbf{qMS}) = \mathrm{Th}(\mathbf{MS})$ that we mentioned earlier. To this end, the auxiliary fact below will be necessary:

LEMMA 7.3. *Assign to an arbitrary* $\mathrm{L}_{\sqsubset}$*-sentence an* $\mathrm{L}^{0}_{\leq}$*-sentence as in Lemma III.2.1. Then:*

(i) $\varphi \in \mathrm{Th}(\mathbf{MS})$ *iff* $\varphi^* \in \mathrm{Th}(\mathbf{CBL})$,
(ii) $\varphi \in \mathrm{Th}(\mathbf{qMS})$ *iff* $\varphi^* \in \mathrm{Th}(\mathbf{ecBL})$.

PROOF. *Ad* (i): Assume that $\varphi \in \mathrm{Th}(\mathbf{MS})$ and take an arbitrary $\mathfrak{B} \in \mathbf{CBL}$, in which o is the zero. To the lattice $\mathfrak{B}$ we assign the structure $\mathfrak{B}^+_{\sqsubset} \in \mathbf{MS}$, which arose from $\mathfrak{B}$ as a result of the operation carried out in Theorem III.1.1. On the strength of the assumption, $\varphi \in \mathrm{Th}(\mathfrak{B}^+_{\sqsubset})$. Putting $0 := \mathrm{o}$ we assign to the structure $\mathfrak{B}^+_{\sqsubset}$ a complete Boolean lattice

$\mathfrak{M}^o = \langle M^0, \leq \rangle$, as described in Theorem III.1.2. Obviously, $\mathfrak{M}^o = \mathfrak{B}$. From this and from Lemma III.2.1, we have $\varphi^* \in \mathrm{Th}(\mathfrak{B})$.

Conversely, let $\varphi^* \in \mathrm{Th}(\mathbf{CBL})$ and take an arbitrary $\mathfrak{M} \in \mathbf{MS}$. Then for an arbitrarily chosen $0 \notin M$, on the strength of Theorem III.1.2, we have $\mathfrak{M}^o \in \mathbf{CBL}$. Therefore $\varphi^* \in \mathrm{Th}(\mathfrak{M}^o)$. Hence, on the strength of Lemma III.2.1, we have $\varphi \in \mathrm{Th}(\mathfrak{M})$.

Ad (ii): As in (i) but changing **MS** to **qMS**, and **CBL** to **ecBL**, and theorems III.1.1 and III.1.2 to theorems 7.1 and 7.2. □

8. $\overline{\mathbf{M}} = \mathrm{Th}(\mathbf{qMS}) = \mathrm{Th}(\mathbf{MS})$

Although $\mathbf{MS} \neq \mathbf{qMS}$, then in these classes the same elementary sentences are true.

THEOREM 8.1. $\mathrm{Th}(\mathbf{qMS}) = \mathrm{Th}(\mathbf{MS})$.

PROOF. Since $\mathbf{MS} \subseteq \mathbf{qMS}$ then $\mathrm{Th}(\mathbf{qMS}) \subseteq \mathrm{Th}(\mathbf{MS})$.

To prove the converse inclusion, take arbitrary $\varphi \in \mathrm{Th}(\mathbf{MS})$ and $\mathfrak{M} \in \mathbf{qMS}$. We will show that $\varphi \in \mathrm{Th}(\mathfrak{M})$, which will prove the inclusion we want.

Let φ^* be an $\mathrm{L}^0_{\leq}$-sentence assigned to φ in Lemma III.2.1. On the strength of Lemma 7.3(i), we have $\varphi^* \in \mathrm{Th}(\mathbf{CBL})$. Hence, in virtue of Proposition II.6.8, we have $\varphi^* \in \mathrm{Th}(\mathbf{ecBL})$. Therefore $\varphi \in \mathrm{Th}(\mathbf{qMS})$, on the strength of Lemma 7.3(ii). □

Thus, by virtue of (2.1), we obtain:

$$\overline{\mathbf{M}} = \mathrm{Th}(\mathbf{qMS}) = \mathrm{Th}(\mathbf{MS})\,.$$

9. The class qMS is finitely elementarily axiomatisable – finite axiomatization of the theory M

To prove that the class **qMS** is finitely elementarily axiomatisable, or — in other words — that theory **M** has a finite axiomatization, the following theorem will be needed:

THEOREM 9.1. *Let $\mathfrak{M} = \langle M, \sqsubset, \sqsubseteq, \bigcirc, \wr \rangle$ be an arbitrary $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-structure, in which the sets $\mathfrak{at}$, $\mathfrak{atc}$ and $\mathfrak{atl}$ are defined by the conditions (df $\mathfrak{at}$), (df $\mathfrak{atc}$) and (df $\mathfrak{atl}$), respectively. Then, for structure $\mathfrak{M}$ to belong to* *the class* **qMS**, *it is both necessary and sufficient that all the conditions below be satisfied*:

(a) *The relation* $\sqsubset$ *is asymmetric and transitive.*
(b) *For for arbitrary* $x, y \in M$ *there exists a* $z \in M$ *such that* $z \operatorname{sup}_{\sqsubseteq} \{x, y\}$, *and if* $x \bigcirc y$ *then for some* $u \in M$ *we have* $u \operatorname{inf}_{\sqsubseteq} \{x, y\}$; *so, by* (a), *we can define two binary operators* $\sqcup \colon M \times M \to M$ *and* $\sqcap \colon \{\langle x, y\rangle \in M \times M : x \bigcirc y\} \to M$, *with the equalities* $x \sqcup y := (\iota\, z)\; z \operatorname{sup}_{\sqsubseteq} \{x, y\}$ *and* $x \sqcap y := (\iota\, z)\; z \operatorname{inf}_{\sqsubseteq} \{x, y\}$).
(c) *There exists an* $x \in M$ *such that for each* $y \in M$ *we have* $y \sqsubseteq x$; *so, by* (a), *we can distinguish in* $\mathfrak{M}$ *an element* $\mathbb{1}$ *which satisfies the condition*: $\forall_{y \in M}\; y \sqsubseteq \mathbb{1}$.
(d) *The relation* $\wr$ *and the operation* $\sqcup$ *satisfy the condition* (Υ).
(e) *The operations* $\sqcup$ *and* $\sqcap$ *satisfy condition* (Δ_1), *for all* $x, y, z \in M$.
(f) *For the operation* $\sqcup$, *for the element* $\mathbb{1}$ *and for the sets* $\mathbb{c}\mathbb{l}\mathbb{c}$ *and* $\mathbb{c}\mathbb{l}\mathbb{l}$, *condition* ($\Sigma_1$) *holds.*

PROOF. '⇒' Let $\mathfrak{M} \in$ **qMS**. Then the relation $\sqsubset$ is asymmetric and transitive. Moreover, on p. 202–203 we showed that $\mathfrak{M}$ satisfies conditions (b)–(e). The satisfaction of condition (f) holds in virtue of Theorem 6.3.

'⇐' If a structure $\mathfrak{M}$ satisfies conditions (a)–(e), then we may repeat the proof (without the axiom of choice) of Theorem III.1.2 to the point where we stated that the structure $\mathfrak{M}^o := \langle M^{\mathit{0}}, \leq, \mathit{0}, \mathbb{1}\rangle$ is a non-trivial Boolean lattice with the zero $\mathit{0}$ and the unity $\mathbb{1}$ (see pp. 117–119).[9]

Using condition (f) just as we did in the proof of Theorem 7.2, we can show that the lattice $\mathfrak{M}^o$ is pe-complete.[10]

We have shown, therefore, that $\mathfrak{M}^o \in$ **ecBL**. It is obvious that by applying the procedure of 'throwing out the zero' applied in Theorem 7.1, we have come back to the initial structure $\mathfrak{M}$. Therefore, by Theorem 7.1, the structure $\mathfrak{M}$ belongs to **qMS**. □

We may now give the finite set $\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}$ of $\mathrm{L}^{\mathrm{d}}_{\sqsubseteq}$-sentences as an axiomatization of theory **M**. The first five axioms in the set $\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}$ are: (λ1), (λ2), (δ⊑), (δ○) and (δ≀). The extension of the language $\mathrm{L}_{\sqsubseteq}$ by the additional

[9] This situation has occurred already in the proof of Theorem 7.2, where we assumed that $\mathfrak{M} \in$ **qMS** and the facts given in Section 2 on pp. 202–203 confirmed that conditions (a)–(d) hold for the theorem currently being proven.

[10] In the proof of Theorem 7.2 we have only made use of the fact that condition (Σ_1) holds in $\mathfrak{M}$.

predicates "$\sqsubseteq$", "$\circ$" and "$\wr$" allows for the straightforward formulation of the next axioms from the set $\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}$. Let us give below the first three of them:

$$\forall_x \forall_y \exists_z \forall_u (z \sqsubseteq u \equiv x \sqsubseteq u \wedge y \sqsubseteq u) \quad (\exists\sqcup)$$

$$\forall_x \forall_y (x \circ y \rightarrow \exists_z \forall_u (u \sqsubseteq z \equiv u \sqsubseteq x \wedge u \sqsubseteq y)) \quad (\exists\sqcap)$$

$$\exists_z \forall_u \; u \sqsubseteq z \quad (\exists 1)$$

From ($\mathtt{r}_{\sqsubseteq}$) and ($\mathtt{antis}_{\sqsubseteq}$) it follows that in axioms ($\exists\sqcup$), ($\exists\sqcap$) and ($\exists 1$) the quantifier "$\exists_z$" may be replaced with the singular quantifier "$\exists!_z$":

$$\forall_x \forall_y \exists!_z \forall_u (z \sqsubseteq u \equiv x \sqsubseteq u \wedge y \sqsubseteq u) \quad (\exists!\sqcup)$$

$$\forall_x \forall_y (x \circ y \rightarrow \exists!_z \forall_u (u \sqsubseteq z \equiv u \sqsubseteq x \wedge u \sqsubseteq y)) \quad (\exists!\sqcap)$$

$$\exists!_z \forall_u \; u \sqsubseteq z \quad (\exists! 1)$$

Therefore, one can define two new constants for our theory: the two-place functional symbol "$\sqcup$" and the name constant "1", the following being their definitions:

$$\forall_x \forall_y \forall_u (x \sqcup y \sqsubseteq u \equiv x \sqsubseteq u \wedge y \sqsubseteq u) \quad (\delta\sqcup)$$

$$\forall_u \; u \sqsubseteq 1 \quad (\delta 1)$$

The constant "$\sqcup$" is obviously the symbol of a binary sum (corresponding to the least upper bound) and the constant "1" is the symbol of the unity. Let us note that from ($\delta\sqcup$) and ($\mathtt{r}_{\sqsubseteq}$) we obtain the following theses:

$$x = x \sqcup x$$

$$x \sqcup y = y \sqcup x$$

$$x \sqsubseteq x \sqcup y$$

$$(x \sqcup y) \circ (x \sqcup z)$$

We can also conditionally[11] define a two-place constant function "$\sqcap$", which corresponds to the 'partial' product or the greatest lower bound:

$$\forall_x \forall_y (x \circ y \rightarrow \forall_u (u \sqsubseteq x \sqcap y \equiv u \sqsubseteq x \wedge u \sqsubseteq y) \quad (\delta\sqcap)$$

By applying these definitions we can more easily formulate the remaining axioms of the theory **M**. The first of them is the elementary

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[11] On the subject of conditional definitions, see, for example, [Grzegorczyk, 1974, pp. 212–213].

form of condition (Υ), and the two following axioms corresponding to condition (Δ_1):

$$\forall_x (x \neq 1 \to \exists_y (x \sqcup y = 1 \wedge x \wr y)) \qquad (\Upsilon)$$

$$y \wr z \to x = (x \sqcup y) \sqcap (x \sqcup z) \qquad (\Delta_1^1)$$

$$y \circ z \to x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z) \qquad (\Delta_1^2)$$

To formulate the final axiom, we will need the help of the three one-place predicates "at", "atc" and "atl" which are defined as follows:

$$\mathtt{at}\, x \equiv \neg \exists_y\, y \sqsubset x \qquad (\delta\,\mathtt{at})$$

$$\mathtt{atc}\, x \equiv \forall_y (y \sqsubseteq x \to \exists_z (\mathtt{at}\, z \wedge z \sqsubseteq y)) \qquad (\delta\,\mathtt{atc})$$

$$\mathtt{atl}\, x \equiv \neg \exists_z (\mathtt{at}\, z \wedge z \sqsubseteq x) \qquad (\delta\,\mathtt{atl})$$

Obviously, the predicate "at" may be read as "is an atom"; the predicate "atc" as "is atomic", and the predicate "atl" as "is atomless". Definitions (δ at), (δ atc) and (δ atl) correspond to (df $\mathfrak{at}$), (df $\mathfrak{atc}$) and (df $\mathfrak{atl}$).

The final axiom from the set $\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}$ is the elementary counterpart of condition (Σ_1):

$$\neg \exists_x\, \mathtt{at}\, x \vee \forall_x \exists_y (\mathtt{at}\, y \wedge y \sqsubseteq x) \vee \exists_x \exists_y (\mathtt{atc}\, x \wedge \mathtt{atl}\, y \wedge 1 = x \sqcup y) \qquad (\Sigma 1)$$

On the basis of Theorem 9.1, we can prove that

THEOREM 9.2. *The class* **qMS** *is finitely elementarily axiomatisable by the set of axioms* $\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}$, *i.e., we have* **qMS** $= \mathrm{Mod}(\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}})$.

Hence, by (2.1) and Gödel's completeness theorem, we have:

THEOREM 9.3. *The theory* **M** *is finitely axiomatisable by the set of axioms* $\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}$, *i.e., we have* $\mathbf{M} = \mathrm{Cn}\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}$.

PROOF. For any formula φ in $\mathrm{L}^{\mathrm{d}}_{\mathfrak{r}}$: $\varphi \in \mathrm{Cn}\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}$ iff $\overline{\varphi} \in \mathrm{Th}(\mathrm{Mod}(\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}))$ iff $\overline{\varphi} \in \mathrm{Th}(\mathbf{qMS})$ iff $\overline{\varphi} \in \overline{\mathbf{M}}$ iff $\varphi \in \mathbf{M}$. □

10. A particular weakening of conditions (peL3) and (peL4)

In this section, we will answer the question: *What sort of class of structures do we obtain when the universal quantifier in sentences* (L3) *and* (L4) *is restricted to e-definable sets (without parameters)?* This question is equivalent to the following one: *What theory do we obtain, when, in* *formulating axioms* ($\lambda 3^k_\varphi$) *and* ($\lambda 4^k_\varphi$), *we accept that the formula* φ *has only one free variable, i.e., we restrict ourselves to* ($\lambda 3^0_\varphi$) *and* ($\lambda 4^0_\varphi$), *the case where* $k = 0$?

Let $\mathrm{e}\mathcal{P}(\mathfrak{M})$ be the family of all e-definable sets (without parameters) in $\mathfrak{M}$ and $\mathrm{e}\mathcal{P}_+(\mathfrak{M}) := \mathrm{e}\mathcal{P}(\mathfrak{M}) \setminus \{\emptyset\}$. Let **E** be the class of all $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-structures in which the relations $\sqsubset$, $\sqsubseteq$, $\circ$ and $\wr$ satisfy conditions: (L1), (L2), (df $\sqsubseteq$), (df $\circ$) and (df $\wr$), and the following conditions holds in $\mathfrak{M}$:

$$\forall_{S \in \mathrm{e}\mathcal{P}(\mathfrak{M})} \forall_{x,y \in M} (x \ \mathrm{Sum}\ S \wedge y \ \mathrm{Sum}\ S \implies x = y), \tag{eL3}$$

$$\forall_{S \in \mathrm{e}\mathcal{P}_+(\mathfrak{M})} \exists_{x \in M}\ x \ \mathrm{Sum}\ S\,. \tag{eL4}$$

From the same definitions we have the inclusion **qMS** $\subseteq$ **E**, but we will prove that **qMS** $\subsetneq$ **E**.

Proposition 10.1. *Both conditions* (peL3) *and* (WSP) *do not follow from the set* $\{$(L1), (L2), (eL3), (L4)$\}$, *so also it does not follow either from* $\{$(L1), (L2), (eL3), (peL4)$\}$ *or* $\{$(L1), (L2), (eL3), (eL4)$\}$. *Thus, we obtain*:

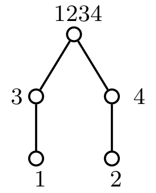
- **qMS** $\subsetneq$ **E**.
- **E** $\nsubseteq$ **L12** + (WSP).

Proof. In model 15, the following conditions hold: (L1), (L2), (L4) and (eL3). For (L4): $x \ \mathrm{Sum}\ \{x\}$. Moreover, if S includes at least one of the sets $\{1234\}$, $\{1,2\}$, $\{3,4\}$, $\{2,3\}$ and $\{1,4\}$, then 1234 is the only sum of S. To finish, 3 and 4 are, respectively, the only sums of the sets $\{1,3\}$ and $\{2,4\}$. For (eL3): the only subsets of the universe, for which (L3) does not hold, are the singletons $\{1\}$ and $\{2\}$ ($3 \ \mathrm{Sum}\ \{1\}$ and $4 \ \mathrm{Sum}\ \{2\}$). But these singletons are not e-definable in model 15. In fact, with respect to the symmetry that holds, it is not possible in the language $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$ to distinguish elements 1 and 2. Therefore, in model 15, condition (eL3) holds, but (peL3) and (WSP) do not hold. □

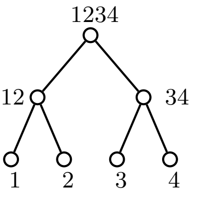
Moreover, we can obtain:

Proposition 10.2. *Condition* (peL4) *does not follow from the set* $\{$(L1), (L2), (L3), (eL4)$\}$.

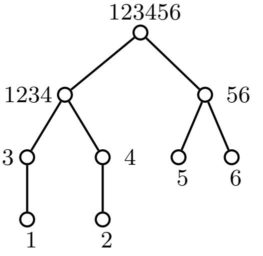
Proof. In model 16, for example, the set $\{2,3\}$ does not have a mereological sum, but all subsets of the universe which do not have one are not e-definable. Therefore, in model 16, condition (eL4) holds, but (peL4) does not hold. □



Model 15. Conditions (L1), (L2), (eL3) and (L4) hold, but conditions (peL3) and (WSP) do not hold



Model 16. Conditions (L1), (L2), (L3) and (eL4) hold, but (peL4) does not hold



Model 17. Conditions (L1), (L2), (eL3) and (eL4) hold, but both conditions (peL3) and (peL4) do not hold

PROPOSITION 10.3. *Condition* ⌜(peL3) $\vee$ (peL4)⌝ *does not follow from the set* {(L1), (L2), (eL3), (eL4)}.

PROOF. In model 17, conditions (L1), (L2), (eL3) and (eL4) hold, but both conditions (peL3) and (peL4) do not hold. (Models 15 and 16 are submodels of model 17.) □

PROPOSITION 10.4. *Condition* ($\exists\wr$) *does not follow from the set* {(L1), (L2), (eL3), (eL4)}. *Thus, we obtain*:

- $\mathbf{E} \not\subseteq \mathbf{L12} + (\exists\wr)$.

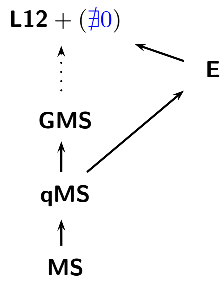


Diagram 5. The class **E** in the lattice of certain superclasses of **MS**

PROOF. Take a model whose universe is composed of rational non-positive numbers, and where $\sqsubset$ is the relation $<$. In this model, (L1) and (L2) hold, but ($\exists\wr$) does not hold. Furthermore, the only e-definable sets are: $\emptyset$, the whole universe, $\{0\}$ (by the formula "$\neg\exists_y\ x \sqsubset y$"), and the set of negative rational numbers. The only mereological sum of the last three sets is 0. Therefore, in this model, (eL3) and (eL4) hold. $\square$

PROPOSITION 10.5. *Condition* ($\nexists 0$) *follows from the set* $\{$(L1), (L2), (eL3)$\}$. *Thus, we obtain*:

- $\mathbf{E} \subsetneq \mathbf{L12} + (\nexists 0)$.

PROOF. if $\operatorname{Card} M = 1$ them the only element of the universe is its own ingrediens. Conversely, assume for a contradiction that (a) $\operatorname{Card} M > 1$ and (b) for a certain $x_0 \in M$ we have $\forall_{y \in M}\ x_0 \sqsubseteq y$. In virtue of (antis$_{\sqsubseteq}$), the singleton $\{x_0\}$ is e-definable in $\mathfrak{M}$ with the help of the formula "$\forall_y\ x \sqsubseteq y$". We have $x_0 \operatorname{Sum} \{x_0\}$. In virtue of (a) there exists a y_0 such that $y_0 \neq x_0$. In virtue of (b) we have $y_0 \operatorname{Sum} \{x_0\}$. Therefore, we have obtained a contradiction from (eL3). Thus, $\mathbf{E} \subseteq \mathbf{L12} + (\nexists 0)$.

Finally, since $\mathbf{E} \nsubseteq \mathbf{L12} + (\exists\wr) \subsetneq \mathbf{L12} + (\nexists 0)$, then $\mathbf{E} \subsetneq \mathbf{L12} + (\nexists 0)$. $\square$

Repeating the proof of Theorem 5.1, we obtain:

PROPOSITION 10.6. $\mathbf{GMS} \nsubseteq \mathbf{E}$.

We can therefore supplement diagrams 2, 3 and 4 to diagram 5.

In the language $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$ we create the theory **E** using the following axioms: (λ1), (λ2), ($\delta \sqsubseteq$), ($\delta \circ$), ($\delta \wr$), ($\lambda 3^0_\varphi$) and ($\lambda 4^0_\varphi$). That is, we omit from the theory **M** all axioms (λ3) and (λ4) for $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-formulae that have more than one free variable (the case where $k > 0$). From the facts proved earlier, it follows that the class E is elementarily axiomatisable, because it is a class of models of theory $\mathbf{E}$:

$$\mathsf{E} = \mathrm{Mod}(\mathbf{E})\,.$$

Assume that $\mathfrak{M}$ belongs to E. Then it is easy to show that $\mathfrak{M}$ is a model of theory $\mathbf{E}$, if we make use of Lemma 1.1 and rework the proof of Theorem 2.3 for the case where $k = 0$. Conversely, if $\mathfrak{M}$ is a model of the set of axioms of theory $\mathbf{E}$, then — using Lemma 1.1 — we can rework the proof of Theorem 2.4 for the case where $k = 0$.

Chapter VII

Mereological sets of distributive classes

We will present a first-order theory **MDC** (see Section 4) in which we can speak of mereological sets (sums) composed of distributive classes. Besides the concepts of *being a distributive class* and *being a member of a distributive set*, it will involve the concepts of *being a mereological set* (*sum*) *of* and *being a mereological part of*. We will interpret in **MDC** Morse's first-order class theory **MT** [1965] (see Section 3). We will show that our theory **MDC** has a model only if Morse's theory has one (see Section 4).

This chapter derived from earlier work but has undergone significant modification [see Pietruszczak, 1995, 1996].

1. Motivations

In von Neumann-Gödel type set theories (NG for short) we distinguish between *sets* and *classes*. Every set is a class, yet not every class is a set. Sets are those and only those classes which are members of at least one class. Classes that are not sets are called *proper classes*; for instance: the class of all sets, the class of all singleton sets, the class of all groups, etc.

Within set theory we cannot, however, deal with 'objects' ('collections', 'complexes', 'multitudes', 'assemblies') whose elements are proper classes. Such 'objects', nevertheless, are quite handy in some cases. We will mention three examples: from mathematics, from meta-mathematics, and from the philosophy of science.

1. A definition of category frequently begins somewhat like: "We say that a category $\mathfrak{A}$ is defined if the following are specified: [...]". Then, three objects are mentioned, all of which can be proper classes: for instance, when we deal with the category of all sets, all groups, or all metric spaces [cf. Dold, 1972]. Sometimes it is straightforwardly said that a category is a triplet, consisting of these objects. In such cases, obviously, the category $\mathfrak{A}$ cannot be an object in an NG-type set theory. This barrier can be bypassed in several ways, which are presented, for instance, in [Cohn, 1969; Semadeni and Wiweger, 1978]. A 'partial' solution is to define categories in set theories tailored especially to suit this purpose, e.g., in Grothendieck's, or MacLane's systems. A drawback of this solution is that we are not able to consider 'the whole' category of all sets, 'the whole' category of all groups, etc. A 'full' resolution of the difficulties has been proposed by Lawvere, who, instead of construing category theory within a set theory, has built up an axiomatic category theory and construed his set theory within it.

2. To deal with models of Zermelo-Fraenkel set theory in Morse's class theory the following approach is often adopted: Take a class C and a relation $e \subseteq C \times C$, which is to be the interpretation of the predicate "ϵ" [cf. Jech, 1971, p. 25; Guzicki and Zbierski, 1992, p. 21].[1] Then, introduce, inductively, the notion of a formula being *true* in the class C, even though it is clear that this notion depends on the relation e as well. Consequently, the class C alone is called a *model of the theory*, despite the fact that, with different interpretations of the predicate "ϵ" ($e_1 \neq e_2$), the very notion of model becomes equivocal.[2] Within Morse's class theory we cannot, obviously enough, define the model as a pair $\langle C, e\rangle$ [such as in Jech, 1971].[3]

3. In [Nowaczyk, 1985] theories are associated with "resources" or "systems" of their concepts. The author uses the term "resources" because the concepts themselves are proper classes already. He writes:

> This, of course, brings about certain difficulties in formulating theorems and leads to employing, «unofficially», terms like 'resources', or 'systems' (but only in such a way that they could be eliminated altogether).
> [Nowaczyk, 1985, p. 105]

[1] The counterpart of "ϵ" in the metalanguage, i.e., the language of the paper, will be the symbol "$\in$".

[2] Equivocality disappears only when the standard interpretation of "ϵ" is adopted, under which e is the 'natural' membership relation in M, i.e., when $e = \{\langle x, y\rangle \in C^2 : x \in y\}$.

[3] In Morse's class theory, we could define a model of Zermelo-Fraenkel set theory as a function from the class $C \times C$ into the set $\{0, 1\}$ (such a function is a proper class whenever C is). Thus, to define a model we would need only one object, unequivocally determining both its universe (the class C) and the interpretation of the predicate "ϵ" in it (the above function is the characteristic function of a binary relation).

The above examples evidence the 'natural' need for a simple and consistent way of construing objects composed of proper classes. Any such construction must, of course, go beyond the framework of the theory of distributive classes.

To discover a solution we will look into mereology. We will construct mereological sums (or: collective sets, mereological sets, mereological fusions) composed of distributive classes.[4]

2. Outline of a set-theoretic ontology

The ontology proposed below is, obviously, of a metaphorical character. It is but the outline of a project to establish a set-theoretic ontology. Its full formulation must be left to a formal and relatively consistent theory.

1. (a) Among other ontological assumptions of set theory, we could adopt the principle that — apart from *distributive classes* — there exist some objects, which (together with the *empty class*) are so-called *urelements* of non-empty distributive classes. These have been named variously: "non-classes" [cf. Mendelson, 1964, p. 160], "individuals" [cf. Nowaczyk, 1985, p. 48] and "atoms" [cf. Jech, 1971, Sec. 26]. We will adopt this latter term[5] prefixed with the adjective "distributive".[6] No urelement has distributive elements.

Distributive atoms can, however, be mereological sums (i.e., collective sets).[7] Therefore, they can possess components, fragments, chunks, pieces, proper mereological parts (collective parts), or mereological elements (collective elements), which are not dealt with in set theory.

[4] Basic intuitions connected with the notion of *collective set* can be found, for instance in [Kotarbiński, 1966; Quine, 1953; Słupecki and Borkowski, 1984]. It is sometimes said that mereological sets (sums) — as opposed to distributive classes — are so-called concrete objects. This standpoint precludes collective sets with abstract elements. However, Leśniewski [1991c] allowed such objects. He considered a geometrical interval as a collective set composed of other intervals (there seems to be no reason to believe that geometrical intervals, which he analyzed in a paper on the 'foundations of mathematics', were for Leśniewski concrete objects). The present author is of the opinion that a mereological set is concrete if and only if all its elements are.

[5] The first has to broad a meaning, the second is used in philosophy in a technical sense; both are inappropriate for our purpose.

[6] Apart from distributive atoms, we will be considering so-called mereological atoms.

[7] These can, for example, be forests, herds, solar systems, or the like; hence our reluctance to call them "individuals".

We assume that the following principle is one of the ontological assumptions of set theory:

Principle. *No distributive class is a mereological part of any distributive atom.*[8]

Let us underline again that the principle above is never adopted explicitly in set theory. And quite naturally so: in set theory we never speak about mereological parts of anything. However, it seems that any other solution would contradict the intuitive meaning of the notion of *urelement*.

(b) The most frequently encountered version of set theory assumes that its universe[9] consists entirely of so-called *pure classes*. These comprise the empty class and the classes which have the former as their only urelement.

(c) It can be assumed that the universe of mereology consists of the above distributive atoms. If one's adopted version of mereology is atomic, then the objects under consideration are mereological sums uniquely defined by their *mereological atoms*, i.e., objects that have no proper mereological parts.

To expand our ontology *ad infinitum*:

2. The relation of *is a mereological part of* we extend onto distributive classes, assuming that each of these is a mereological atom.[10]

[8] This principle is also — in some version — in [Lewis, 1991] as the Priority Thesis: "No class is part of any individual" (p. 7) and the Second Thesis: "No class has any part that is not a class" (p. 6). Note, however, that Lewis understands the word "part" in a sense which allows for improper parts, i.e., ingredienses in our sense. Lewis' thesis therefore says, respectively: *no class is ingrediens of any individual* and *no class has any ingrediens that is not a class*. Thus, in our terminology, we also have: *no class is part of any individual*.

[9] The universe of a theory is the 'range' of objects considered in the theory. The universe itself cannot, obviously, be one of the objects investigated within the theory.

[10] As Lewis [1991], we mean that not only urelements have mereological parts. But the First Thesis of Lewis says (see footnote 8): "One class is [an ingrediens] of another iff the first is a subclass of the second" (p. 4). "The conjunction of the First and Second Thesis is our *Main Thesis*: The [ingredienses] of a class are all and only its subclasses" (p. 6–7). Lewis adds:

> To explain what the First Thesis means, I must hasten to tell you that my usage is a little idiosyncratic. By 'classes' I mean things that have members. By 'individuals' I mean things that are members, but do not themselves have members. Therefore there is no such class as the null class. I don't mind calling some memberless thing — some individual — the null *set*. But that

3. Then, we form mereological sums (sets) of all distributive atoms and distributive classes. Thus, we assume that there exists mereological sums that have at least two mereological parts, of which at least one is a distributive class. The objects we have construed this way are neither distributive classes (by 2), nor distributive atoms (by our Principle).

4. To the objects postulated in 3, we add distributive atoms and distributive classes.

5. The 'formation' of distributive classes from the urelements can be metaphorically described as a certain construction. To find a precise and metaphor-free formulation we turn to the formal set theory. Such a 'construction', starting, as it does, from urelements (described in 1), cannot lead outside the universe of distributive classes (also considered in 1), since this is precisely how the latter has been formed.

Applying the 'construction' to the universe of 4, we obtain new distributive classes. These will have at least one (distributive) element that is neither a distributive atom nor a 'standard' distributive class.

6. The universe of 4 we further extend by adding the non-standard distributive classes described in 5.

7. We extend the relation *is a mereological part* into the universe of 6, assuming that each distributive class (be it standard or not) is a mereological atom.

8. We form mereological sums of all objects mentioned in 6. In other words, we repeat step 3 for the objects of 6. We assume, thus, that there exist mereological objects that have at least two mereological parts, of which at least one is a 'non-standard' distributive class.

9. We extend the universe of 6 to include the objects postulated in 8.

And so on, *ad infinitum*.

The realization of the above project — embodied in a formal system — we will postpone to another occasion. Here, we will focus on a small fragment of it. Starting from pure classes (1b), we will extend the universe to include the objects postulated in point 3 and stop at this stage. The extended universe will comprise solely distributive classes and mereological sums of them. Other formal systems, which will be certain implementations of this project, will be presented in Chapter VIII.

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doesn't make it a memberless class. Rather, that makes it a 'set' that is not a class. Standardly, all sets are classes and none are individuals.

[Lewis, 1991, p. 4]

We will present a formal theory, suitable to deal with such a universe. We will show that the set-theoretic universe will remain the same. We will also show its relative consistency.

3. Morse's first-order class theory

Morse's first-order class theory **MT** will be expressed in a first-order language with identity $\mathrm{L}_{\mathbf{MT}}$. Its sole specific symbol is the two-place predicate "$\in$" to be read as "is an element of" (or "belongs to").

Axioms of the theory **MT**, as presented here, are modelled on [Guzicki and Zbierski, 1992, pp. 9–11]. First comes the extensionality axiom:

$$\forall_{x,y}(\forall_z(z \in x \equiv z \in y) \to x = y) \tag{MT1}$$

To give the next axioms it will be best to extend the language $\mathrm{L}_{\mathbf{MT}}$ with some predicates, which will be definable in the theory **MT** by the following definitions:

$$\mathsf{Null}\, x \equiv \neg \exists_y\, y \in x \tag{δ Null}$$

$$\mathsf{Set}\, x \equiv \exists_y\, x \in y \tag{δ Set}$$

$$z\ \mathsf{Pair}\ xy \equiv \forall_u(u \in z \equiv u = x \lor u = y) \tag{δ Pair}$$

$$y\ \mathsf{Un}\ x \equiv \forall_z(z \in y \equiv \exists_u(u \in x \land z \in u)) \tag{δ Un}$$

$$x \subset y \equiv \forall_z(z \in x \to z \in y) \tag{$\delta \subset$}$$

$$y\ \mathsf{Pow}\ x \equiv \forall_z(z \in y \equiv \mathsf{Set}\, z \land z \subset x) \tag{δ Pow}$$

$$z\ \mathsf{IS}\ xy \equiv \forall_u(u \in z \equiv u \in x \land u \in y) \tag{δ IS}$$

We will read these new defined subformulae as follows:

- $\mathsf{Null}\, x$ – x is a null class
- $\mathsf{Set}\, x$ – x is a set
- $z\ \mathsf{Pair}\ xy$ – z is the pair of x and y
- $y\ \mathsf{Un}\ x$ – y is the union (generalised sum) of x
- $x \subset y$ – x is a subset of y
- $y\ \mathsf{Pow}\ x$ – y is the power class of x
- $z\ \mathsf{IS}\ xy$ – z is the intersection of x and y

Employing the above definitions, we formulate the following axioms for sets:

$$\forall_x(\mathsf{Null}\, x \to \mathsf{Set}\, x) \tag{MT2}$$

$$\forall_{x,y,z}(\mathsf{Set}\, x \land \mathsf{Set}\, y \land z\ \mathsf{Pair}\ xy \to \mathsf{Set}\, z) \tag{MT3}$$

$$\forall_{x,y}(\mathsf{Set}\,x \wedge y\ \mathsf{Un}\,x \rightarrow \mathsf{Set}\,y) \tag{MT4}$$

$$\forall_{x,y}(\mathsf{Set}\,x \wedge y\ \mathsf{Pow}\,x \rightarrow \mathsf{Set}\,y) \tag{MT5}$$

That is, any null class is a set; any pair of sets is a set; any union of a set is a set; any power class of a set is a set.

We can also formulate the following axiom of foundation (regularity):

$$\forall_x(\neg\ \mathsf{Null}\,x \rightarrow \exists_y(y \in x \wedge \forall_z(z\ \mathsf{IS}\ xy \rightarrow \mathsf{Null}\,z))) \tag{MT6}$$

That is, for any non-empty class, some element of it has an empty intersection with it.

The next axiom says that there is an infinity class:

$$\exists_x(\forall_u(\mathsf{Null}\,u \rightarrow u \in x) \wedge \forall_{y,z}(y \in x \wedge \forall_u(u \in z \equiv u \in y \vee u = y)) \rightarrow z \in x) \tag{MT7}$$

In order to more 'legibly' to formulate the axiom of replacement, we will need some more definitions. The first one introduces the predicate "oPair" that can be read as "is an ordered pair of":

$$z\ \mathsf{oPair}\ xy \equiv \forall_u(u \in z \equiv u\ \mathsf{Pair}\ xy \vee u\ \mathsf{Pair}\ xx) \tag{δ oPair}$$

The second one introduces the predicate "Im":

$$z\ \mathsf{Im}\ xy \equiv \forall_u(u \in z \equiv \mathsf{Set}\,u \wedge \exists_v(v \in y \wedge \forall_w(w\ \mathsf{oPair}\ vu \rightarrow w \in x))) \tag{δ Im}$$

The new defined subformula "z Im xy" can be read as "z is an image of x, restricted to y". The last definition employed will be:

$$\mathsf{Fun}\,x \equiv \forall_y(y \in x \rightarrow \exists_{z,u}(\mathsf{Set}\,z \wedge \mathsf{Set}\,u \wedge y\ \mathsf{oPair}\ zu)) \wedge \forall_{y,z,u,v,w}(y\ \mathsf{oPair}\ uv \wedge y \in x \wedge z\ \mathsf{oPair}\ uw \wedge z \in x \rightarrow v = w) \tag{δ Fun}$$

i.e., "Fun x" can be read as "x is a function". Employing the above definitions we may now state the axiom of foundation:

$$\forall_x(\mathsf{Fun}\,x \rightarrow \forall_{y,z}(\mathsf{Set}\,y \wedge z\ \mathsf{Im}\ xy \rightarrow \mathsf{Set}\,z)) \tag{MT8}$$

That is, any image of a function, restricted to a set, is a set.

We now extend the language $\mathrm{L}_{\mathbf{MT}}$ to the first-order language$\mathrm{L}^{\mathrm{d}}_{\mathbf{MT}}$ by the predicates we have defined.

Next we assume an infinite set of axioms of class existence, in the form of the following schema for an arbitrary $\mathrm{L}^{\mathrm{d}}_{\mathbf{MT}}$-formula $\varphi(u)$ in which the variable "u" is free but the variable "x" is not free:

$$\exists_x \forall_u (u \in x \equiv \mathsf{Set}\, u \wedge \varphi(u)) \tag{MT9}$$

From (MT1) and (MT9), for any $\mathrm{L}_{\mathbf{MT}}$-formula φ, in which "u" is free, but "x" is not free, we therefore obtain:

$$\exists!_x \forall_u (u \in x \equiv \mathsf{Set}\, u \wedge \varphi(u)) \tag{3.1}$$

Using (3.1) for the formula $\varphi(u) :=$ "$u \neq u$" we have the following thesis of **MT**:

$$\exists!_x \forall_u (u \in x \equiv \mathsf{Set}\, u \wedge u \neq u)$$

So we obtain:

$$\exists!_x \ \mathrm{Null}\, x$$

We can therefore introduce the following definition of the individual constant "$\emptyset$" ("empty class"):

$$x = \emptyset \equiv \mathrm{Null}\, x$$

From (MT2) we obtain the thesis "$\mathsf{Set}\, \emptyset$", i.e., the null class $\emptyset$ is a set.

Using (3.1) for the formula $\varphi(u) :=$ "$u = y \vee u = z$" we have the following theses of **MT**:

$$\forall_{y,z} \exists!_x \forall_u (u \in x \equiv \mathsf{Set}\, u \wedge (u = y \vee u = z))$$
$$\forall_{y,z} (\mathsf{Set}\, y \wedge \mathsf{Set}\, z \rightarrow \exists!_x \ x \ \mathsf{Pair}\, yz)$$

The class x postulated in the above thesis we will call the *pair* of sets y and z, and signify it by: $\{y, z\}$. Instead of "$\{y, y\}$" we write "$\{y\}$". From (MT3) we obtain: if y and z are sets then $\{y, z\}$ is a set.

Moreover, for the predicate "oPair", using (3.1) and (MT3), we have the following theses of **MT**:

$$\forall_{y,z} \exists!_x \forall_u (u \in x \equiv \mathsf{Set}\, u \wedge (u \ \mathsf{Pair}\, yz \vee u \ \mathsf{Pair}\, yy))$$
$$\forall_{y,z} (\mathsf{Set}\, y \wedge \mathsf{Set}\, z \rightarrow \exists!_x \forall_u (u \in x \equiv u \ \mathsf{Pair}\, yz \vee u \ \mathsf{Pair}\, yy))$$
$$\forall_{y,z} (\mathsf{Set}\, y \wedge \mathsf{Set}\, z \rightarrow \exists!_x \forall_u (u \in x \equiv u = \{y, z\} \vee u = \{y\}))$$
$$\forall_{y,z} (\mathsf{Set}\, y \wedge \mathsf{Set}\, z \rightarrow \exists!_x \ x \ \mathsf{oPair}\, yz)$$

The class x postulated in the above thesis we will call the *ordered pair* of sets y and z, and signify it by: $<y, z>$. We see that for arbitrary sets

y and z we have $<y, z> = \{\{y\}, \{y, z\}\}$. From (MT3) we obtain: if y and z are sets then $<y, z>$ is a set.

By (3.1), for arbitrary sets y and z there is exactly one class which is a union (generalised sum) of the set $\{y, z\}$ and which we denote by "$y \cup z$". So we have: $\forall_u (u \in y \cup z \equiv u \in y \vee u \in z)$. Hence, by (MT4), we obtain: if y and z are sets then $y \cup z$ is a set.

Using (3.1) and the observations above on the formula $\varphi(u) :=$ "$u \in y \wedge u \in z$" we have the following theses of **MT**:

$$\forall_{y,z} \exists!_x \forall_u (u \in x \equiv \mathsf{Set}\, u \wedge u \in y \wedge u \in z)$$

$$\forall_y (\mathsf{Set}\, y \rightarrow \exists!_x\ x \ \mathrm{IS}\ yz)$$

For any set y and any class z, the class x postulated in the above thesis we will call the *intersection* of y and z, and signify it by: $y \cap z$. Of course, we have the theses: "$y \cap z = z \cap z$", "$y \cap z \subset y$" and "$y \cap z \subset z$".

Using (3.1) for the formula $\varphi(u) :=$ "$\exists_v (v \in y \wedge u = <v, v>)$" we have the following theses of **MT**:

$$\forall_z \exists!_x \forall_u (u \in x \equiv \mathsf{Set}\, u \wedge \exists_v (v \in z \wedge u = <v, v>))$$

$$\forall_z \exists!_x \forall_u (u \in x \equiv \exists_v (v \in z \wedge u = <v, v>))$$

For any set y, the class x postulated in the above thesis is a function. Notice that for any set y and any class z we have: $y \cap z$ is an image of x, restricted to y. Hence $y \cap z$ is a set, by (MT8).

By the above, if x is a set then $x \cap y$ is a set. Moreover, if $y \subset x$ then $x \cap y = y$. So we obtain the following thesis:

$$\forall_{x,y} (\mathsf{Set}\, x \wedge y \subset x \rightarrow \mathsf{Set}\, y)$$

Finally, notice that by (MT6) we obtain that no class is its own element, i.e.:

$$\forall_x \neg\, x \in x$$

Assume for a contradiction that for some x we have $x \in x$. Then x is a set and so we can consider the set $\{x\}$. Since $\{x\} \neq \emptyset$, then — by (MT6) — for some z we have: $z \in \{x\}$ and $\{x\} \cap z = \emptyset$. But $z = x$ and $x \in \{x\}$. Thus, we obtain a contradiction: $\{x\} \cap z = \{x\} \cap x \neq \emptyset$.

Apart from the above axioms, we can also assume the axiom of choice.

4. Elementary mereology of distributive classes

4.1. The language and definitions

The first-order theory with identity **MDC** (elementary *Mereology of Distributive Classes*) will be expressed in the first-order language $L_{\mathbf{MDC}}$ with the identity predicate "=". The primitive notions of $L_{\mathbf{MDC}}$ are the (set-membership) predicate "$\in$", the two-place predicate "$\sqsubset$" (to be read as "is a part of") and the one-place predicate "Cl" (to be read as "is a distributive class").[11]

In order to more easily formulate the theory **MDC**, we will extend the language $L_{\mathbf{MDC}}$ by the following symbols:

- All symbols defined in the theory **MT** in Section 3.
- All symbols defined in the theory **M** in sections 1 and 9 of Chapter VI.

The predicates "$\sqsubseteq$", "at", "atc" and "atl" may be read as "is an ingrediens of", "is an mereological atom", "is mereologically atomic" and "is mereologically atomless", respectively. The two-place functional symbols "$\sqcup$" and "$\sqcap$" are the symbols for binary mereological sum and binary mereological product, respectively. Finally, the constant "1" is the symbol for unity.

4.2. The first axioms from elementary mereology

We want to formulate an extension of **M** in the language $L_{\mathbf{MDC}}$. We can therefore take any group of axioms of theory **M** as the first axioms of the new theory. For example, it we might take the finite set $\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}$ composed of the following $L^{d}_{\sqsubset}$-formulae:

- ($\lambda 1$), ($\lambda 2$), ($\delta \sqsubseteq$), ($\delta \circ$), ($\delta \wr$), ($\exists \sqcup$), ($\exists \sqcap$), ($\exists 1$), ($\delta \sqcup$), ($\delta\, 1$), ($\delta \sqcap$), (Υ), (Δ^1_1), (Δ^2_1), (δ at) (δ atc), (δ atl) and ($\Sigma 1$).

We could also take the infinite group of axioms composed of the following $L_{\mathbf{MDC}}$-sentences:

- ($\lambda 1$), ($\lambda 2$), ($\lambda 3^k_\varphi$) and ($\lambda 4^k_\varphi$), for arbitrary $k \geqslant 0$ and $L_{\mathbf{MDC}}$-formula φ such that $\mathrm{vf}(\varphi) = \{x_1, \ldots, x_{k+1}\}$.

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[11] The predicate "Cl" was redundant in the theory **MT**, as all objects from its universe were distributive classes (i.e., we would have had an axiom "$\forall_x \mathrm{Cl}\, x$").

4.3. The syntactic interpretation the theory MT in the theory MDC. Further axioms of MDC

Let $\Im$ be a syntactic interpretation [cf. e.g. Shoenfield, 1977, Section 4.7] of the language $L_{\mathbf{MT}}$ in the language $L_{\mathbf{MDC}}$ such that:

- $\Im(\text{“}\epsilon\text{”}) := \text{“}\epsilon\text{”}$,
- the *universe* of $\Im$ is the predicate “$\mathtt{Cl}$”.

For any $L_{\mathbf{MT}}$-formula φ we define an $L_{\mathbf{MDC}}$-formula $\varphi^{\Im}$, which is to be the $\Im$-*interpretation* of φ. First, let $\varphi_{\mathtt{Cl}}$ be an $L_{\mathbf{MDC}}$-formula produced by the relativisation of φ to “$\mathtt{Cl}$”, i.e., by replacing any sub-formula of the form $\ulcorner\forall_{x_i}\,\psi(x_i)\urcorner$ by $\ulcorner\forall_{x_i}\,(\mathtt{Cl}\,x_i \rightarrow \psi(x_i))\urcorner$ and any sub-formula of the form $\ulcorner\exists_{x_i}\,\psi(x_i)$ by $\ulcorner\exists_{x_i}\,(\mathtt{Cl}\,x_i \wedge \psi(x_i))\urcorner$. Second, if $x_{i_1}, \ldots, x_{i_n}$ are all the variables that occur free in φ (hence, in $\varphi_{\mathtt{Cl}}$), enumerated in alphabetic order, then $\varphi^{\Im}$ is $\ulcorner\mathtt{Cl}\,x_{i_1} \wedge \cdots \wedge \mathtt{Cl}\,x_{i_n} \rightarrow \varphi_{\mathtt{Cl}}\urcorner$.

The first axioms of theory **MDC** will be the interpretations, by $\Im$, of the definitions and axioms of the theory **MT**. That is, we accept:

- if δ is one of the definitions ($\delta\,\mathtt{Null}$), ($\delta\,\mathtt{Set}$), ($\delta\,\mathtt{Pair}$), ($\delta\,\mathtt{Un}$), ($\delta\,\subset$), ($\delta\,\mathtt{Pow}$) and ($\delta\,\mathtt{IS}$) in **MT**, then $\delta^{\Im}$ is an axiom of **MDC**;
- if α is one of the axioms (MT1)–(MT8) of **MT** then $\alpha^{\Im}$ is an axiom of **MDC**;
- for any $L^{d}_{\mathbf{MT}}$-formula $\varphi(u)$ in which the variable “u” is free but the variable “x” is not free, the formula $(\text{MT9})^{\Im}$ is an axiom of **MDC**.

Hence, the following $L^{d}_{\mathbf{MT}}$-sentences are theses of **MDC**:

$$\exists_x\,(\mathtt{Cl}\,x \wedge \forall_u\,(\mathtt{Cl}\,u \rightarrow (u \in x \equiv \mathtt{Set}\,u \wedge u = u)))$$
$$\exists_x\,(\mathtt{Cl}\,x \wedge \forall_u\,(\mathtt{Cl}\,u \rightarrow (u \in x \equiv \mathtt{Set}\,u)))$$
$$\exists_x\ \mathtt{Cl}\,x \tag{4.1}$$

From (4.1) we therefore obtain [cf. Shoenfield, 1977, Section 4.7]:

- $\Im$ is an interpretation of the language $L_{\mathbf{MT}}$ in the theory **MDC**;
- $\Im$ is an interpretation of the theory **MT** in **MDC**.

Therefore — by virtue of the Interpretation Theorem [cf. Shoenfield, 1977, Section 4.7] — we obtain:

- for any thesis φ of **MT**, the $L_{\mathbf{MDC}}$-formula $\varphi^{\Im}$ is a thesis of **MDC**.

We proceed by construing **MDC** in such a way that will ensure the validity of the converse as well (see Theorem 4.3).

Remark 4.1. The sentence "$\exists_x\ x = x$" is a thesis of **MT**, as it is logically valid. We have $(\exists_x\ x = x)^{\Im} :=$ "$\exists_x(\mathrm{Cl}\,x \wedge x = x)$". Of course, the last sentence is logically equivalent to (4.1).

However, in order to apply the Interpretation Theorem, we first had to prove thesis (4.1) in **MDC** which then allowed us to interpret the language $L_{\mathbf{MT}}$ in **MDC**. Otherwise, we would not have known whether $(\exists_x\ x = x)^{\Im}$ was a thesis of **MDC**. □

4.4. The last two axioms of MDC. Relative consistency of MDC

The first of the additional axioms says that:

- any object that any element has is a distributive class,
- only distributive classes (sets) are elements.

Formally:[12]

$$\forall_{x,y}(x \in y \rightarrow \mathrm{Cl}\,x \wedge \mathrm{Cl}\,y) \tag{MDC1}$$

The last axiom says that all distributive classes are mereological atoms:

$$\forall_x(\mathrm{Cl}\,x \rightarrow \mathrm{at}\,x) \tag{MDC2}$$

From ($\mathtt{r}_{\sqsubseteq}$), (MDC2), ($\delta\,\mathtt{at}$) and ($\delta\sqsubseteq$) we obtain:

$$\forall_x(\mathrm{Cl}\,x \rightarrow \neg\exists_y\ y \sqsubset x) \tag{4.2}$$

$$\forall_x(\mathrm{Cl}\,x \rightarrow \forall_y(y \sqsubset x \equiv y \neq y)) \tag{4.3}$$

$$\forall_{x,y}(\mathrm{Cl}\,x \wedge \mathrm{Cl}\,y \rightarrow (y \sqsubset x \equiv y \neq y)) \tag{4.4}$$

$$\forall_x(\mathrm{Cl}\,x \rightarrow \forall_y(y \sqsubseteq x \equiv x = y)) \tag{4.5}$$

$$\forall_{x,y}(\mathrm{Cl}\,x \wedge \mathrm{Cl}\,y \rightarrow (y \sqsubseteq x \equiv x = y)) \tag{4.6}$$

$$\forall_{x,y}(\mathrm{Cl}\,x \wedge \mathrm{Cl}\,y \rightarrow (y \circ x \equiv x = y)) \tag{4.7}$$

$$\forall_{x,y}(\mathrm{Cl}\,x \wedge \mathrm{Cl}\,y \rightarrow (y \wr x \equiv x \neq y)) \tag{4.8}$$

$$\forall_x(\mathrm{Cl}\,x \rightarrow (\mathrm{at}\,x \equiv x = x)) \tag{4.9}$$

$$\forall_x(\mathrm{Cl}\,x \rightarrow (\mathrm{atc}\,x \equiv x = x)) \tag{4.10}$$

$$\forall_x(\mathrm{Cl}\,x \rightarrow (\mathrm{atl}\,x \equiv x \neq x)) \tag{4.11}$$

Due to the presence of (MDC1), we could in certain cases dispense with restrictions on some of the quantifiers. For example, instead of $(\mathrm{MT1})^{\Im}$, $(\delta\,\mathtt{Null})^{\Im}$, $(\delta\,\mathtt{Set})^{\Im}$ and $(\mathrm{MT2})^{\Im}$ we could take, respectively:

$$\forall_{x,y}(\mathrm{Cl}\,x \wedge \mathrm{Cl}\,y \wedge \forall_z(z \in x \equiv z \in y) \rightarrow x = y) \tag{4.12}$$

$$\forall_x(\mathrm{Cl}\,x \rightarrow (\mathrm{Null}\,x \equiv \neg\exists_y\ y \in x)) \tag{4.13}$$

[12] We assume that the universe of class theory consists entirely of pure classes.

$$\forall_{x,y}(\mathrm{Cl}\,x \rightarrow (\mathrm{Set}\,x \equiv \exists_y\, x \in y)) \tag{4.14}$$

$$\forall_x(\mathrm{Cl}\,x \rightarrow (\neg\exists_y\, y \in x \rightarrow \exists_y\, x \in y)) \tag{4.15}$$

Moreover, notice that from (MT9)$^{\mathfrak{I}}$ and (4.3)–(4.11), for an arbitrary $\mathrm{L}_{\mathbf{MDC}}$-formula $\varphi(u)$ with $k+1$ ($k \geqslant 0$) free variables "u", $x_{i_1}, \ldots, x_{i_k}$ such that the variable "x" is not free in $\varphi(u)$ and all quantifiers in $\varphi(u)$ are restricted by "Cl"[13], we obtain the following thesis:

$$\mathrm{Cl}\,x_{i_1} \wedge \cdots \wedge \mathrm{Cl}\,x_{i_k} \rightarrow \exists_x(\mathrm{Cl}\,x \wedge \forall_u(u \in x \equiv \mathrm{Set}\,u \wedge \varphi(u))) \tag{4.16}$$

The following relative consistency theorem holds for **MDC**:

THEOREM 4.1. *Let $\mathfrak{m} = \langle C_{\mathfrak{m}}, \epsilon_{\mathfrak{m}}\rangle$, where $C_{\mathfrak{m}}$ is a non-empty set and $\epsilon_{\mathfrak{m}} \subseteq C_{\mathfrak{m}}^2$, be a normal*[14] *model of* **MT**. *Let $\mathfrak{M} = \langle \mathcal{M}, \sqsubset_{\mathfrak{M}}, \epsilon_{\mathfrak{M}}, \mathrm{Cl}_{\mathfrak{M}}\rangle$ be an $\mathrm{L}_{\mathbf{MDC}}$-structure in which $\mathcal{M} := \mathcal{P}_+(C_{\mathfrak{m}})$ and:*

$$\sqsubset_{\mathfrak{M}} := \{\langle X, Y\rangle \in \mathcal{M}^2 : X \subsetneq Y\},$$
$$\epsilon_{\mathfrak{M}} := \{\langle X, Y\rangle \in \mathcal{M}^2 : \exists_{x,y\in C_{\mathfrak{m}}}(X = \{x\} \wedge Y = \{y\} \wedge x \,\epsilon_{\mathfrak{m}}\, y)\},$$
$$\mathrm{Cl}_{\mathfrak{M}} := \{X \in \mathcal{M} : \exists_{x\in C_{\mathfrak{m}}}\, X = \{x\}\}.$$

Then $\mathfrak{M}$ is a normal atomic model of **MDC** *such that*

$$\sqsubseteq_{\mathfrak{M}} = \{\langle X, Y\rangle \in \mathcal{M}^2 : X \subseteq Y\},$$
$$\mathsf{O}_{\mathfrak{M}} = \{\langle X, Y\rangle \in \mathcal{M}^2 : X \cap Y \neq \emptyset\}, \qquad \wr_{\mathfrak{M}} = \mathcal{M}^2 \setminus \mathsf{O}_{\mathfrak{M}},$$
$$\mathrm{at}_{\mathfrak{M}} = \mathrm{Cl}_{\mathfrak{M}}, \qquad \mathrm{atc}_{\mathfrak{M}} = \mathcal{M}, \qquad \mathrm{atl}_{\mathfrak{M}} = \emptyset\,,$$
$$\forall_{S\in\mathcal{M}}\exists_{X\in \mathrm{at}_{\mathfrak{M}}}\, X \sqsubseteq_{\mathfrak{M}} S\,.$$

PROOF. For any set X, the structure $\mathfrak{P}_X := \langle \mathcal{P}(X), \subseteq, \emptyset, X\rangle$ is an atomic Boolean lattice. Thus, $\langle \mathcal{P}_+(X), \subsetneq\rangle$ is an atomic mereological structure in which atoms are singletons and for any non-empty family $\mathcal{F}$ the sum $\bigcup \mathcal{F}$ is the mereological sum of $\mathcal{F}$. Hence, it is clear that (λ1) and (λ2) hold in $\mathfrak{M}$. We will show that (λ3^k_φ) and (λ4^k_φ) also hold in $\mathfrak{M}$.

Suppose that for arbitrary $k \geqslant 0$ and $\mathrm{L}_{\mathbf{MDC}}$-formula φ with $k+1$ free variables "x", $x_{i_1}, \ldots, x_{i_k}$ we have $\mathfrak{M} \models \exists_x\, \varphi(x)\,[S_1/x_{i_1}, \ldots S_k/x_{i_k}]$, i.e., the valuation $[S_1/x_{i_1}, \ldots S_k/x_{i_k}]$ in the family $\mathcal{M}$ satisfies the antecedent of (λ4^k_φ). Hence the non-empty family $M^x_\varphi(S_1, \ldots, S_k) := \{X \in \mathcal{M} :$

[13] That is, quantifiers occur only in its subformulae that have the forms $\ulcorner\forall_{x_i}(\mathrm{Cl}\,x_i \rightarrow \psi)\urcorner$ and $\ulcorner\exists_{x_i}(\mathrm{Cl}\,x_i \wedge \psi)\urcorner$.

[14] In the sense that the interpretation of the predicate "=" is 'true' identity.

$\mathfrak{M} \models \varphi\,[X/\mathtt{x}, S_1/\mathtt{x}_{i_1}, \ldots, S_k/\mathtt{x}_{i_k}]\,\}$ is pe-definable in $\mathfrak{M}$ with parameters $S_1, \ldots, S_k$ (see p. 195). Because $\langle \mathcal{M}, \subsetneq \rangle$ is a mereological structure, $\bigcup M^{\mathtt{x}}_{\varphi}(S_1, \ldots, S_k)$ is the mereological sum of $M^{\mathtt{x}}_{\varphi}(S_1, \ldots, S_k)$. So the valuation $[S_1/x_{i_1}, \ldots, S_k/\mathtt{x}_{i_k}]$ satisfies the consequent of ($\lambda 4^k_{\varphi}$) (see p. 196). For similar reasons as above ($\lambda 3^k_{\varphi}$) holds in $\mathfrak{M}$.

It is also easy to show that structure $\mathfrak{M}$ meets the finite set $\mathrm{Ax}^{\mathbf{M}}_{\mathrm{fin}}$ which is composed of the following $\mathrm{L}^{\mathrm{d}}_{\sqsubset}$-formulae: ($\lambda$1), ($\lambda$2), ($\delta \sqsubseteq$), ($\delta \circ$), ($\delta \wr$), ($\exists \sqcup$), ($\exists \sqcap$), ($\exists 1$), ($\delta \sqcup$), ($\delta 1$), ($\delta \sqcap$), ($\Upsilon$), ($\Delta^1_1$), ($\Delta^2_1$), ($\delta\,\mathtt{at}$) ($\delta\,\mathtt{atc}$), ($\delta\,\mathtt{atl}$) and ($\Sigma 1$) (because the formula "$\forall_{\mathtt{x}} \exists_{\mathtt{y}} (\mathtt{at}\, y \wedge y \sqsubseteq x)$" holds).

The axiom (MDC1) holds in $\mathfrak{M}$, as a consequence of the fact that, if $X \in_{\mathfrak{M}} Y$, then by the interpretation of "$\in$'; in $\mathfrak{M}$', the sets X and Y are singleton sets, hence they belong to $\mathsf{Cl}_{\mathfrak{M}}$. Of course, axiom (MDC2) holds in $\mathfrak{M}$.

That the axioms 'coming from' **MT** are true as well follows from the fact that in $\mathfrak{M}$ they act only on singletons created from elements of the set $C_{\mathfrak{m}}$. □

Directly from Theorem 4.1 we obtain:

THEOREM 4.2. *If* **MT** *is consistent, then* **MDC** *is too.*

We have also the 'conservative interpretation theorem' for **MDC**:[15]

THEOREM 4.3. *For any* $\mathrm{L}_{\mathbf{MT}}$*-formula* φ: $\varphi^{\mathfrak{I}}$ *is a thesis of* **MDC** *if and only if* φ *is a thesis of* **MT**.

PROOF. '$\Rightarrow$' Let φ be an arbitrary $\mathrm{L}_{\mathbf{MT}}$-formula. In the proof we will make use of a well-known fact [cf. e.g. van Dalen, 1994, p. 79], implying that, for any structure $\mathfrak{A}$ of $\mathrm{L}_{\mathbf{MDC}}$: if $\mathfrak{A}^{\mathsf{Cl}}$ is a structure for $LtMDC$ with the universe $\mathsf{Cl}_{\mathfrak{M}}$ ($\mathsf{Cl}_{\mathfrak{M}} \neq \emptyset$) and with relations from $\mathfrak{A}$ restricted to $\mathsf{Cl}_{\mathfrak{M}}$, then $\mathfrak{A} \models \varphi^{\mathfrak{I}}$ iff $\mathfrak{A}^{\mathsf{Cl}} \models \varphi$.

Let us assume that $\varphi^{\mathfrak{I}}$ is a thesis of **MDC**. Take any model $\mathfrak{m}$ of **MT**. With no loss of generality we can assume that $\mathfrak{m}$ is a normal model. We will show that $\mathfrak{m} \models \varphi$. Thus, by Gödel's completeness theorem (the model was chosen arbitrarily), it will follow that φ is a thesis of **MT**.

In Theorem 4.1, starting from a model $\mathfrak{m}$, we have built a structure $\mathfrak{M}$, such that the former is a model of **MDC**. Thus, by the assumption

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[15] The interpretation $\mathfrak{I}$ of **MT** in **MDC** can be termed "conservative". Notice that **MDC** is not a conservative extension of **MT**, as it is not even an extension of it (for instance formula (MT1) is not a thesis of **MDC**).

and the completeness theorem, we obtain $\mathfrak{M} \vDash \varphi^{\mathfrak{I}}$. Hence, by the fact mentioned above, we have $\mathfrak{M}^{\mathrm{Cl}} \vDash \varphi$.

By the construction of $\mathfrak{M}$, it follows that $\epsilon_{\mathfrak{M}}|_{\mathrm{Cl}_{\mathfrak{M}}} := \epsilon_{\mathfrak{M}} \cap (\mathrm{Cl}_{\mathfrak{M}} \times \mathrm{Cl}_{\mathfrak{M}}) = \epsilon_{\mathfrak{M}}$. Since only the predicate "ϵ" from $\mathrm{L}_{\mathbf{MDC}}$ occurs in φ, we have that $\langle \mathrm{Cl}_{\mathfrak{M}}, \epsilon_{\mathfrak{M}} \rangle \vDash \varphi$ as well. This, by the definitions of $\mathrm{Cl}_{\mathfrak{M}}$ and $\epsilon_{\mathfrak{M}}$ in Theorem 4.1, implies that $\mathfrak{m} \vDash \varphi$.

'$\Leftarrow$' It follows from the Interpretation Theorem [cf. Shoenfield, 1977, Section 4.7]. □

Remark 4.2. It is easy to see that to meet the goals set out in the opening "Motivations" section, we can adopt a weaker theory in which the axiom (λ4) takes the special case only for the formula $\varphi(x) :=$ "$x = y \vee x = z$", i.e., we take:

$$\forall_{y,z} \exists_u \, \sigma^x_{x=y \vee x=z}$$

where $\mathrm{vf}(\sigma^x_{x=y \vee x=z}) = \{y, z, u\}$ and

$$\sigma^x_{x=y \vee x=z} := \text{“} y \sqsubseteq u \wedge z \sqsubseteq u \wedge \forall_v (v \sqsubseteq u \rightarrow y \circ v \vee z \circ v) \text{”}$$

In other words, for arbitrary objects we assume that their mereological sum exists. □

Remark 4.3. Assume that instead of (MDC2) we adopt as axiom the formula:

$$\forall_x (\mathrm{Cl}\, x \rightarrow \forall_y (y \sqsubset x \equiv (\mathrm{Cl}\, y \wedge y \neq \emptyset \wedge y \subset x \wedge y \neq x)))$$

i.e., the parts of a class are all and only its non-empty proper subclasses.[16] Then for the formula $\varphi(x) :=$ "$(x = y \vee x = z)$" we get the thesis:

$$\forall_{y,z} (\mathrm{Cl}\, y \wedge \mathrm{Cl}\, z \rightarrow (\sigma^x_{x=y \vee x=z} \equiv \mathrm{Cl}\, u \wedge \forall_v (v \in u \leftrightarrow v \in y \vee v \in z)$$

which says that the mereological sum of two distributive classes is the distributive sum of them.

Informally: for two distributive classes c_1 and c_2, the mereological sum $[\![c_1, c_2]\!]$ is equal to the class $c_1 \cup c_2$. Thus, we have no pair. Moreover, if $c_1 \subseteq c_2$ then $[\![c_1, c_2]\!] = c_2$. □

[16] Cf. [Lewis, 1991, p. 6–7] and my footnote 10.

Chapter VIII

Unitary theories of individuals and sets

1. Introduction

In this chapter, we will be concerned with a certain first-order theory in which we may talk of collective sets composed both of individuals and distributive sets. Besides the concepts of *being a distributive set* and the relation of *belonging to* (to a distributive set) it will contain the concepts: of *being a collective set*, *the relation of being a collective part of* and of *being an individual* .

A set theory built not for the aims of 'pure mathematics' but for application elsewhere (e.g., in physics or philosophy) permits the existence of objects other than distributive sets (classes). These objects have no element because only distributive classes possess them. Objects distinct from distributive sets (classes) are often called "individuals". One may indeed adopt the position that their whole is a (distributive) set and that the class of all sets is built out of such individuals. This means — to put it vividly — that the 'ultimate elements' from which all sets are built are individuals or empty sets.[1] If one does not apply the axiom of foundation (regularity), a better name for these objects is "non-classes", as it carries no implications; this name is adopted by Mendelson [1964]. Nothing is assumed about such 'non-sets' in set theory.

On the other hand, mereology — that is, the theory of collective sets and their parts — is often entitled "the calculus of individuals". In mereology, various relations between objects are studied, determined by the relation *is a part of* (in a collective sense). We believe that one need not assume that only individuals have (collective) parts. David Lewis in [1991] does not assume they do (he, however, adopts a different approach to the one developed below in Section 3).

[1] In so-called 'pure' set theory, which suffices for mathematics, it is taken that the 'ultimate' element of each set is just the empty set.

The conceptual apparatus which are employing in this work allows us to define the concept of *being an individual*. This will be an object which has no distributive elements but which also does not possess any distributive sets as its parts (in a collective sense).

In Chapter I and in [Pietruszczak, 1997], the distinctions between a collective and distributive set, and between the relation of belonging to a distributive set and being a part of a collective set, were drawn in an intuitive way. In this chapter we will formally realise the project of the set-theoretic ontology we set out in Chapter VII (as well in [Pietruszczak, 1995, 1996]), where we are considering a theory in which it is possible to speak of collective sets composed of distributive proper classes without contradiction.

2. First-order set theory

2.1. The theory ZF

In this subsection we will introduce a certain version of Zermelo-Fraenkel first-order set theory (**ZF**). The language $L_{\mathbf{ZF}}$ of **ZF** is a first-order language with identity "=". Its specific constants (primitive in **ZF**) are the two-place predicate "∈" and the one-place predicate "Set" to be read as "is a distributive set". Other constants will be definable with the help of "∈" and "Set", which will expand the language.

We model the set of axioms of **ZF** on [Grzegorczyk, 1974, pp. 172–176] (with some small changes). Besides the logical axioms relating to the aforementioned constants, we also have the following specific axioms.

The first is the axiom of extensionality:

$$\forall_{x,y}(\mathrm{Set}\,x \wedge \mathrm{Set}\,y \wedge \forall_z(z \in x \equiv z \in y) \rightarrow x = y) \tag{ZF1}$$

We also adopt the principle (not mentioned in [Grzegorczyk, 1974]), that if an object has even one element, it is a set:

$$\forall_{x,y}(x \in y \rightarrow \mathrm{Set}\,y) \tag{ZF2}$$

The given set we call a *family of sets* when all its elements are sets. We therefore extend the language by a one-place predicate "F" ("is a family of sets") by adopting the definition:

$$\mathrm{F}\,x \equiv \mathrm{Set}\,x \wedge \forall_y(y \in x \rightarrow \mathrm{Set}\,y)$$

The next axiom says that, for an arbitrary family of sets, there exists a set such that to it belong all and only those objects which are elements of some element of the family:

$$\mathrm{F}\,x \rightarrow \exists_z (\mathsf{Set}\, z \wedge \forall_u (u \in z \equiv \exists_y (z \in y \wedge y \in x))) \tag{ZF3}$$

To give the next axiom it will be best to extend the language with a two-place predicate "$\subset$", read as "is a subset":

$$x \subset y \equiv \mathsf{Set}\, x \wedge \mathsf{Set}\, y \wedge \forall_z (z \in x \rightarrow z \in y)$$

The power-set axiom states that, for an arbitrary set there exists a set set to which belong all the subsets of the first set:

$$\mathsf{Set}\, x \rightarrow \exists_y (\mathsf{Set}\, y \wedge \forall_z (z \in y \equiv z \subset x)) \tag{ZF4}$$

The axiom of infinity says firstly that a set exists and secondly that that set is a family of sets having at least one element each and such that each of its elements is a subset of another of its elements:

$$\exists_x (\mathrm{F}\,x \wedge \exists_y y \in x \wedge \forall_y (y \in x \rightarrow \exists_z (z \in x \wedge z \neq y \wedge y \subset z))) \tag{ZF5}$$

Let $\varphi(x_i, x_j)$ be a formula with at least two free variables x_i and x_j $(i \neq j)$. We say that the formula $\varphi(x_i, x_j)$ is univocal if for an arbitrary x_k, which does not occur in $\varphi(x_i, x_j)$, we have the following thesis:

$$\forall_{x_i, x_j, x_k} (\varphi(x_i, x_j) \wedge \varphi(x_i, x_k/x_j) \rightarrow x_j = x_k)$$

If in the arbitrarily-chosen univocal formula $\varphi(x, y)$ with at least two free variables "x" and "y" the variable "u" is not freer, then we adopt the axiom of substitution according to the following schema:

$$\mathsf{Set}\, z \rightarrow \exists_u (\mathsf{Set}\, u \wedge \forall_y (y \in u \equiv \exists_x (x \in z \wedge \varphi(x, y)))) \tag{ZF6}$$

This schema says that, with the help of the univocal formula $\varphi(x, y)$, we can turn an arbitrary set into a set (i.e., the 'image of a set is a set, if the transformation is carried out with the help of the univocal formula'). From axioms of the form (ZF6) follow schemas distinguishing subsets. For an arbitrary formula ψ with at least one free variable "y", if the variable "u" does not appear in the formula, we obtain:

$$\mathsf{Set}\, z \rightarrow \exists_u (\mathsf{Set}\, u \wedge \forall_y (y \in u \equiv (y \in z \wedge \psi(y)))) \tag{2.1}$$

In fact, it suffices to apply the axiom of substitution to the univocal formula $\varphi(x, y) = \ulcorner x = y \wedge \psi(y) \urcorner$.

Thesis (2.1) states that for an arbitrary set and arbitrary property formulable in the language of **ZF** there exists a subset of that set to which belong all and only those objects which also have that property. That is, for an arbitrary set z there is a set such that to it belong all and only those elements of z which satisfy the formula $\psi(y)$. By virtue of the axiom of extensionality (ZF1) there is exactly one such set. We will signify it by: $\{y \in z \mid \psi(y)\}$.[2]

By virtue of the axiom of infinity (ZF5) there exists at most one set; that is, we have the thesis "$\exists_z \mathrm{Set}\, z$". Therefore, by (2.1), we have a thesis that says that there exists a set which we may signify by the metalinguistic abbreviation "$\{y \in z \mid y \neq y\}$". From the axiom of extensionality it follows that this set is independent of the set represented by the variable "z". Therefore we will signify it by "$\varnothing$" and call it the *empty set*. From this definition and axioms (ZF1) and (ZF2) we obtain the following thesis:

$$\forall_x (\mathrm{Set}\, x \equiv x = \varnothing \lor \exists_y\, y \in x) \tag{2.2}$$

That is, something is a set iff either it is the empty set or it has elements.

We will also adopt the axiom of regularity in a form taken from [Jech, 1971, sections 9 and 26]:

$$\exists_u\, u \in z \to \exists_x (x \in z \land x \cap z = \varnothing) \tag{ZF7}$$

In this axiom the abbreviation "$x \cap z$" signifies the set $\{y \in z \mid y \in x \land y \in z\}$ (from the antecedent of (ZF7) and from (ZF2) it follows that the object signified by "z" is a set, so this is meaningful). If x is not a set (z is not a family of sets), then $x \cap z$ is equivalent to $\varnothing$.[3] Axiom (ZF7) has significant meaning in the case where z is a family of sets. For then

[2] This is a metalinguistic abbreviation. It would be somewhat tiresome to have to present the theory with everything written as formulae in the object language (i.e., without any metalinguistic abbreviations, any commentaries, etc.) The metalanguage itself will, however, include the language of the theory of classes. The theory of classes will be used in analysing models of our theory. We do not want to mix abbreviations for **ZF** with the corresponding fragment of the metalanguage (for example, in the metalanguage, we will write the class of objects satisfying condition $\Phi(a)$ with the help of the standard symbolism, i.e., as $\{x : \Phi(x)\}$). In the metalanguage, we will use the symbol "$\in$" for the predicate of membership (but note that in both the metalanguage and object-language we will use the same symbol for the identity predicate).

[3] Then axiom (ZF7) states that for an arbitrary non-empty set z which is not a family of sets, it has an element x which has no common element with the set z. This follows from the definition of a family of sets: since the set z is non-empty and is not a family, it therefore has an element x which is not a set, i.e., x has no elements.

its element x is a set and the set $x \cap z$ is the intersection of x and z. In this case, the axiom says that to the family z belongs a set disjoint from it.

As is well-known, it follows from (ZF7) that there does not exist an infinite sequence of sets x_0, x_1, x_2, ... such that: ... $x_2 \in x_1 \in x_0$. There does not also exist a set x such that $x \in x$ [see Jech, 1971, sec. 26]. Hence from this and axiom (ZF2) it follows that

$$\forall_x \neg x \in x$$

As is well-known, a thesis of **ZF** is a sentence asserting the existence of an unordered pair:

$$\forall_{x,y} \exists_z (\mathsf{Set}\, z \wedge \forall_u (u \in z \equiv u = x \vee u = y))$$

The set z postulated in the above thesis we will call a *pair* of x and y, and signify it by: $\{x, y\}$. Instead of "$\{x, x\}$" we write "$\{x\}$". Furthermore, we introduce a metalinguistic abbreviation for ordered pairs: $<x, y>$. Standardly, this abbreviates "$\{\{x\}, \{x, y\}\}$".

2.2. The theory ZFA

In [Jech, 1971, sec. 26] the theory **ZFA**' — "the set theory with atoms" — was introduced and examined. This is, **ZF** with set-theoretic *atoms* allowed, i.e., object different from sets and not having elements.[4] The theory **ZFA** is built in a first-order language with identity $L_{\mathbf{ZFA}}$ which has three (primitive) specific constants: a two-place predicate "$\in$" and two name constants: "$\emptyset$" (a constant signifying the empty set, as in Section 2.1) and "a" (a constant signifying a *set of atoms*).

Remark 2.1. These considerations already bring out the essential differences between **ZFA** and **ZF**, which also allows for "atoms". In **ZF**, we cannot prove the existence of a set to which belong all non-sets and only them (non-sets are "atoms" — object satisfying the formula "$\neg\, \mathsf{Set}\, x$"). Only with the addition of the axiom "$\forall_x \mathsf{Set}\, x$", in **ZF** can we prove that such a set exists and is empty. □

The first two axioms of **ZFA** may be presented as follows:

$$\neg \exists_x\; x \in \emptyset \tag{ZFA1}$$

$$\forall_x (x \in \mathsf{a} \equiv x \neq \emptyset \wedge \neg \exists_y\, y \in x) \tag{ZFA2}$$

[4] In [Jech, 1971], **ZF** is a 'pure' theory of sets, i.e., it does not feature the predicate "Set".

Therefore, the empty set has no elements and an atom is an object which has no elements but which is different from the empty set.

Below is the definition of the concept of *being a set*:

$$\mathsf{Set}\, x \equiv \neg\, x \in \mathsf{a}$$

That is, sets are objects that do not have atoms. With this definition we adopt axioms (ZF1), (ZF3)–(ZF7) as formulated in Section 2.1. Axiom (ZF2) results from (ZFA2) because we have the following thesis:

$$\forall_{x,y}\, (x \in y \to \neg\, y \in \mathsf{a})$$

Therefore theory **ZFA** is stronger than **ZF**.

If we add the following formula as an axiom to **ZF**:

$$\exists_x \forall_y\, (y \in x \equiv \neg\, \mathsf{Set}\, y)$$

then both theories are definitionally equivalent. In fact, in **ZF** we can define a name constant "$\varnothing$" such that (ZFA1) becomes a thesis. Furthermore, the axiom allows us to define a name constant "a" for which we have the thesis "$\forall_x\, (x \in \mathsf{a} \equiv \neg\, \mathsf{Set}\, x)$". Hence and from (2.2) we obtain (ZFA2).

3. The set theory with classical mereology

A unitary theory of individuals and sets will be the 'joining together' of theory **ZF** from Section 2.1 with first-order classical mereology. We build the first-order theory **MZF** in a first-order language $\mathrm{L}_{\mathbf{MZF}}$ with identity, it being an extension of the languages $\mathrm{L}_{\mathbf{ZF}}$ and $\mathrm{L}_{\sqsubset}$. The language $\mathrm{L}_{\mathbf{MZF}}$ therefore has three primitive constants. Two of them are the constants introduced in Section 2.1: "Set" and '$\in$". The third is a two-place predicate "$\sqsubset$" (which we will read as "is a mereological part of").

All the axioms of **ZF** are also axioms of **MZF**. We apply the axiom-schema (ZF6) also to arbitrary univocal formulae of the language $\mathrm{L}_{\mathbf{MZF}}$. We may therefore apply theses of the form (2.1) with arbitrary formulae of this language.

A second group of axioms is comprised of Leśniewski's axioms for mereology written in the first-order forms (λ1) and (λ2) (see p. 72), from which we obtain the following thesis:

$$\forall_x\ \neg\, x \sqsubset x \qquad (\mathrm{irr}_{\sqsubset})$$

which is the counterpart of condition ($\mathrm{irr}_{\sqsubset}$).

We extend the language $L_{\mathbf{MZF}}$ by three auxiliary two-place predicates "$\sqsubseteq$" ("is an ingrediens of"), "$\circ$" ("overlaps with"), and '$\wr$' ("is exterior to"). These predicates we define, respectively (as on p. 86):

$$\forall_x \forall_y (x \sqsubseteq y \equiv (x \sqsubset y \lor x = y)) \tag{$\delta\sqsubseteq$}$$

$$\forall_x \forall_y (x \circ y \equiv \exists_z (z \sqsubseteq z \land z \sqsubseteq y)) \tag{$\delta\circ$}$$

$$\forall_x \forall_y (x \wr y \equiv \neg\exists_z (z \sqsubseteq z \land z \sqsubseteq y)) \tag{$\delta\wr$}$$

Using "$\sqsubseteq$" and "$\circ$", we can introduce another two-argument predicate "Sum" ("is a mereological sum of"), which we define with the help of the formula below:

$$x \,\mathrm{Sum}\, z \equiv \mathrm{Set}\, z \land \forall_y (y \in z \to y \sqsubseteq x) \land \forall_y (y \sqsubseteq x \to \exists_u (u \in z \land u \circ y))$$

With the help of the so-defined predicate "Sum", we adopt the two following first-order axioms of Leśniewski's mereology:

$$\forall_{x,y,z} (x \,\mathrm{Sum}\, z \land y \,\mathrm{Sum}\, z \to x = y) \tag{$\lambda 3$}$$

$$\forall_z (\exists_y \, y \in z \to \exists_x \, x \,\mathrm{Sum}\, z) \tag{$\lambda 4$}$$

The axioms above are obviously counterparts of conditions (L3) and (L4) from the definition of mereological structures. In other words, we are assuming that the predicate "Sum" is univocal with respect to its first argument and that mereological sum exists for each non-empty subset. The condition "$\exists_y y \in z$" in (λ4) — by virtue of (ZF2) — is equivalent to the condition "$\mathrm{Set}\, z \land z \neq \emptyset$" ("$z$ is a non-empty set").

In **MZF** we assume that no distributive set has any mereological part:

$$\forall_z (\mathrm{Set}\, z \to \neg\exists_x \, x \sqsubset z) \tag{α}$$

In other words, distributive sets should be mereological atoms.

From axiom (α) and definition ($\delta\sqsubseteq$) we have the following thesis:

$$\forall_z (\mathrm{Set}\, z \to \forall_x (x \sqsubseteq z \equiv x = z))$$

This differs from what David Lewis calls (see footnote 8 on p. 221):

> *First Thesis*: One class is [an ingrediens] of another iff the first is a subclass of the second. [Lewis, 1991, p. 4]

Moreover, Lewis writes:

> By 'classes' I mean things that have members. By 'individuals' I mean things that are members, but do not themselves have members. There-

> fore there is no such class as the null class. I don't mind calling some memberless thing — some individual — the null *set*.
>
> [Lewis, 1991, p. 4]

Therefore — as a result of the considerations above — the "empty set" (or the "null set") counts as an individual in Lewis' theory and is supposed to be a mereological sum (fusion) of all individuals:

> *Redefinition*: The *null set* is the fusion of all individuals.
>
> [Lewis, 1991, p. 14].

Our approach as expressed in axiom (α) is 'minimalist' and it is in accordance with Lewis' second thesis (see footnote 8 on p. 221):

> *Second Thesis*: No class has any [ingrediens] that is not a class.
>
> [Lewis, 1991, p. 6]

In other words: *every ingrediens of a class is a class.* In fact, in line with the definition of the predicate "$\sqsubseteq$", each class (as with other objects) is its own ingrediens and — in virtue of (α) — has no other ingredienses.

The predicate "Ind" ("is an individual") we define in the following way:

$$\mathrm{Ind}\, x \equiv \forall_y (y \sqsubseteq x \to \neg \mathrm{Set}\, y)$$

This says that a given object is an individual when no ingrediens of it is a (distributive) set. Therefore, it is just what Lewis wants with his (see footnote 8 on p. 221):

> *Priority Thesis*: No class is [ingrediens] of any individual.
>
> [Lewis, 1991, p. 7]

Thus, in our terminology, we also have: *no class is part of any individual.*

The following thesis follows directly from the definitions:

$$\mathrm{Ind}\, x \equiv (\neg \mathrm{Set}\, x \wedge \forall_y (y \sqsubset x \to \neg \mathrm{Set}\, y))$$

Therefore no individual is a set:

$$\forall_x (\mathrm{Ind}\, x \to \neg \mathrm{Set}\, x)$$

Furthermore, individuals do not have elements:

$$\forall_x (\mathrm{Ind}\, x \to \neg \exists_y\, y \in x)$$

An individual is not a set and by (ZF2) only sets have elements.

We can therefore in our theory prove the following Lewis' thesis:

> *Fusion Thesis*: Any fusion of individuals is itself an individual.
>
> [Lewis, 1991, p. 7]

This may be written in the first-order language thus:

$$\forall_{x,z}(x \ \mathtt{Sum}\ z \wedge \forall_y (y \in z \rightarrow \mathtt{Ind}\, y) \rightarrow \mathtt{Ind}\, x) \tag{FT}$$

In fact, let x Sum z and all elements of the set z be individuals. We assume indirectly that x is not an individual. From this assumption it follows that a certain set y_0 is an ingrediens of x, i.e., $\mathtt{Set}\, y_0 \wedge y_0 \sqsubseteq x$. Hence, in virtue of the definition of "Sum", for some u_0 we have: $u_0 \in z \wedge y_0 \bigcirc u_0$. Since, by (α), the set y_0 is a mereological atom, then $y_0 \sqsubseteq u_0$ (see condition (II.2.5)). Hence u_0 is not an individual, which contradicts the assumption, that all elements of the set z are individuals.

The axioms we have adopted do not, however, allow us to state that there exist any individuals. An axiom of existence of mereological sums would not help us here. It is true that it could 'generate' new objects that are not sets, but these objects would also not be individuals. For example, take the set $\{\emptyset, \{\emptyset\}\}$. In virtue of (λ3) and (λ4) there exists exactly one object which is a mereological sum of the elements of that set. Let us signify it by "$[\![\emptyset, \{\emptyset\}]\!]$". In virtue of the definition of "Sum", the sets $\emptyset$ and $\{\emptyset\}$ are ingredienses of $[\![\emptyset, \{\emptyset\}]\!]$, i.e., we have $\emptyset \sqsubseteq [\![\emptyset, \{\emptyset\}]\!]$ and $\{\emptyset\} \sqsubseteq [\![\emptyset, \{\emptyset\}]\!]$. Because $\emptyset \neq \{\emptyset\}$, then both $\neg\, \emptyset = [\![\emptyset, \{\emptyset\}]\!]$ and $\neg\, \{\emptyset\} = [\![\emptyset, \{\emptyset\}]\!]$. Hence $\emptyset$ and $\{\emptyset\}$ are parts of $[\![\emptyset, \{\emptyset\}]\!]$, i.e., $\emptyset \sqsubset [\![\emptyset, \{\emptyset\}]\!]$ and $\{\emptyset\} \sqsubset [\![\emptyset, \{\emptyset\}]\!]$.[5] Therefore, $[\![\emptyset, \{\emptyset\}]\!]$ is not a set, as it has parts. Furthermore, the object $[\![\emptyset, \{\emptyset\}]\!]$ is also not an individual, because individuals do not have parts that are sets.

We must assume the existence of at least one object which is an individual:

$$\exists_x \ \mathtt{Ind}\, x \tag{ι1}$$

It does not follow from the axioms we have accepted, however, that there exist at least two individuals. Indeed, axiom (ι_1) says that there exists a certain individual. Let us signify it by "i". Then the formula "$\exists_y\ y \in \{i\}$" is true. Therefore, on the basis of axioms (λ3) and (λ4), there exists exactly mereological sum of the set $\{i\}$. Let us signify this object by "$[\![i]\!]$". From the definitions of the predicates "$\sqsubseteq$", "$\bigcirc$" and "Sum" the thesis "$[\![i]\!] = i$" follows, however.

Only the assumption that there exist at least two individuals allows us to 'generate' a third individual. To this end, we note that, from axioms

[5] It can be shown that they are the only parts of the object $[\![\emptyset, \{\emptyset\}]\!]$.

(λ1)–(λ3) we can derive a thesis which is the counterpart of condition (WSP) from Chapter II:

$$x \sqsubset y \rightarrow \exists_z (z \sqsubset y \wedge z \wr x) \tag{WSP}$$

Let us assume, therefore, that there exist at least two individuals i_1 and i_2. If i_1 is a part of i_2 then a certain object o is a part of i_2, which has no common ingrediens with i_1. Hence object o is a third individual. A similar result obtains if i_2 is a part of i_1. In the third case, where neither i_1 is a part of i_2 nor i_2 a part of i_1, then — by virtue of (λ3) and (λ4) — for the two-element set $\{i_1, i_2\}$ there exists exactly one mereological sum of it. Let us signify this object by "$[\![i_1, i_2]\!]$". On the basis of (FT), the object $[\![i_1, i_2]\!]$ is an individual. It is different from individuals i_1 and i_2, by virtue of our assumption.

In section 2.2 we assumed that the 'whole' of all set-theoretic atoms ("non-classes"') creates a set. In the same way, in our theory let us assume that the 'whole' of all individuals is a set:

$$\exists_z \forall_x (x \in z \equiv \mathrm{Ind}\, x) \tag{ι_2}$$

Without this assumption, we can only guarantee the existence of sets of individuals included in some other sets (see schema (2.1)). Let us signify the postulated set by "i". Axiom (ι_1) guarantees us also the thesis "$\mathrm{i} \neq \emptyset$", i.e., that the set of individuals is not empty.

Thesis (FT) may also be written in the following form:

$$\forall_{x,z} (x \,\mathrm{Sum}\, z \wedge z \subset \mathrm{i} \rightarrow x \in \mathrm{i}) \tag{FT$'$}$$

Without going into the technical details, we can prove that there exists a set of all ordered pairs composed of individuals:

$$\mathrm{i}^2 = \{\langle x, y\rangle \mid x \in \mathrm{i} \wedge y \in \mathrm{i}\}$$

That is, there exists the Cartesian product of the set i, of all individuals.

Using the set of individuals and applying (2.1) we can distinguish subsets of ordered pairs with the predicates "$\sqsubset$", "$\sqsubseteq$" and "Sum", i.e., the relations $\mathrm{part}_{\mathrm{i}}$, $\mathrm{ingr}_{\mathrm{i}}$ and $\mathrm{sum}_{\mathrm{i}}$ can be defined by the equivalences below:

$$\langle x, y\rangle \in \mathrm{part}_{\mathrm{i}} \equiv x \in \mathrm{i} \wedge y \in \mathrm{i} \wedge x \sqsubset y$$

$$\langle x, y\rangle \in \mathrm{ingr}_{\mathrm{i}} \equiv x \in \mathrm{i} \wedge y \in \mathrm{i} \wedge x \sqsubseteq y$$

$$\langle x, z\rangle \in \mathrm{sum}_{\mathrm{i}} \equiv x \in \mathrm{i} \wedge z \subset \mathrm{i} \wedge x \,\mathrm{Sum}\, z$$

Therefore, $\mathtt{part_i}$ (resp. $\mathtt{ingr_i}$) is the set of all ordered pairs of individuals, in which the first member of the pair is a part (resp. individual) of the second member. Furthermore, $\mathtt{sum_i}$ is the set of all ordered pairs in which the first member is an individual which is the mereological sum of some set of individuals.

From the observations made in this section, we also obtain the following result:

the pair $<\mathtt{i}, \mathtt{part_i}>$ *is a mereological structure.*

In other words:

1. The relation $\mathtt{part_i}$ is irreflexive and transitive in the set $\mathtt{i}$ of all individuals.

This follows directly from axioms (λ1) and (λ2).

2. The relation $\mathtt{sum_i}$ is a function of the second argument.

This follows directly from definitions and axiom (λ3).

The final condition defining a mereological structure boils down to the thesis:

3. $z \subset \mathrm{i} \wedge z \neq \emptyset \rightarrow \exists_x <x, z> \in \mathrm{sum_i}$

This follows from the definitions along with (λ4) and (FT′). From the antecedent of the implication we have: $\mathtt{Set}\, z \wedge z \neq \emptyset$. Hence, in virtue of (λ4), we have $\exists_x\, x\ \mathtt{Sum}\ z$. Therefore, applying (FT′), we obtain the consequent.

4. Other unitary theories of individuals and sets

By creating other unitary theories of individuals and sets, we can use weaker axioms than (λ3) and (λ4). In other words, we can create weaker mereological theories.

For example, we can build a unitary theory of individuals and sets related to the theory of the class **L12** + (SSP), i.e., to *Neutral Existential Mereology* (see Section 2 of Chapter V). Thus — instead of axioms (λ3) and (λ4) — we use the following axiom

$$\neg\, x \sqsubseteq y \rightarrow \exists_z (z \sqsubseteq x \wedge z \wr y) \tag{SSP}$$

which is the counterpart of condition (SSP). Clearly, (λ3) is a thesis of this theory.

The third unitary theory of individuals and sets is related to the theory of the class **ML12**; that is, to the class of mereological strictly partially ordered sets (see Section 4 of Chapter V). To provide the first-order counterpart of condition ($\mathbb{S}$um-sup$_\sqsubseteq$) we introduce the following auxiliary binary predicate "sup":

$$x \,\mathtt{sup}\, z \;\equiv\; \mathtt{Set}\, z \wedge \forall_u (u \in z \rightarrow u \sqsubseteq x) \wedge \forall_y (\forall_u (u \in z \rightarrow u \sqsubseteq y) \rightarrow x \sqsubseteq y)$$

Of course, the formula "$x \,\mathtt{sup}\, z$" says that the object x is the supremum of the set z. We obtain this third theory by using the following mereological axiom:

$$\forall_z (\mathtt{Set}\, z \rightarrow \forall_x (x \;\mathtt{Sum}\; z \;\equiv\; z \neq \varnothing \wedge x \,\mathtt{sup}\, z)) \qquad (\mathtt{Sum\text{-}sup})$$

instead of axioms (λ3) and (λ4). Clearly, (λ3) and (SSP) are theses of this theory.

The fourth unitary theory of individuals and sets will be a combination of set theory with Grzegorczyk's mereological theory (see Section 7 in Chapter V). Thus — instead of axioms (λ3) and (λ4) — we use the following two axioms

$$\exists_u \; u \,\mathtt{sup}\, \{x, y\} \qquad (\exists_{\mathrm{pair}}\mathtt{sup})$$

$$\neg\, x \sqsubseteq y \rightarrow \exists_z (z \sqsubseteq x \wedge z \wr y \wedge \forall_u (u \sqsubseteq x \wedge u \wr y \rightarrow u \sqsubseteq z)) \qquad (\mathrm{SSP}+)$$

which are the counterparts of conditions ($\exists_{\mathrm{pair}}$sup) and (SSP+), respectively. Clearly, (λ3), (SSP) and (Sum-sup) are theses of this theory.

5. Relative consistency of unitary theories of individuals and sets

If the theory **ZF** is consistent then it has a model of the form $\langle M_\mathfrak{m}, \in_\mathfrak{m}, \mathtt{Set}_\mathfrak{m}\rangle$, where $M_\mathfrak{m}$ is a non-empty set, $\in_\mathfrak{m} \subseteq M_\mathfrak{m}^2$ and $\mathtt{Set}_\mathfrak{m} \subseteq M_\mathfrak{m}$. Then the theory **ZFA** should also be consistent and, in addition, **ZFA** should have some model of the form $\langle M_\mathfrak{m}, \in_\mathfrak{m}, \varnothing_\mathfrak{m}, \mathtt{a}_\mathfrak{m}\rangle$, where $M_\mathfrak{m}$ is a non-empty set, $\in_\mathfrak{m} \subseteq M_\mathfrak{m}^2$, $\varnothing_\mathfrak{m}, \mathtt{a}_\mathfrak{m} \in M_\mathfrak{m}$ and $\mathtt{a}_\mathfrak{m} \neq \varnothing_\mathfrak{m}$, i.e., $\{x \in M_\mathfrak{m} : x \in_\mathfrak{m} \mathtt{a}_\mathfrak{m}\} \neq \emptyset$.[6]

Otherwise, if **ZFA** had only models, in which $\mathtt{a}_\mathfrak{m} = \varnothing_\mathfrak{m}$, then **ZFA** would not differ from 'pure' set theory with the thesis "$\forall_x \; \mathtt{Set}\, x$", i.e., we would have $\mathtt{Set}_\mathfrak{m} := M_\mathfrak{m} \setminus \{x \in M_\mathfrak{m} : x \in_\mathfrak{m} \mathtt{a}_\mathfrak{m}\} = M_\mathfrak{m} \setminus \emptyset = M_\mathfrak{m}$.

[6] Notice that if **ZFA** has a model then it has a normal model, in the sense that the interpretation of the predicate "=" is 'true' identity.

It is easy to show that if **ZFA** has a model in which $\mathsf{a}_{\mathfrak{m}} \neq \varnothing_{\mathfrak{m}}$ then **ZFA** also has a model in which there is exactly one $a \in M_{\mathfrak{m}}$ such that $a \in_{\mathfrak{m}} \mathsf{a}_{\mathfrak{m}}$ (i.e., there is exactly one atom and the formulae "$\exists!_x\ \mathtt{x} \in \mathtt{a}$" and "$\exists_x\ \mathtt{a} = \{x\}$" hold in $\mathfrak{m}$).

The following relative consistency theorem holds for **MZF**:

THEOREM 5.1. *Let $\mathfrak{m} = \langle M_{\mathfrak{m}}, \in_{\mathfrak{m}}, \varnothing_{\mathfrak{m}}, \mathsf{a}_{\mathfrak{m}}\rangle$ be a normal model of* **ZFA** *in which there is exactly one $a \in M_{\mathfrak{m}}$ such that $a \in_{\mathfrak{m}} \mathsf{a}_{\mathfrak{m}}$. Let $\mathfrak{M} = \langle \mathcal{M}, \sqsubset_{\mathfrak{M}}, \in_{\mathfrak{M}}, \mathsf{Set}_{\mathfrak{M}}\rangle$ be an $\mathrm{L}_{\mathbf{MZF}}$-structure in which $\mathcal{M} := \mathcal{P}_{+}(M_{\mathfrak{m}})$ and:*

$$\sqsubset_{\mathfrak{M}} := \{\langle X, Y\rangle \in \mathcal{M}^2 : X \subsetneq Y\},$$
$$\in_{\mathfrak{M}} := \{\langle X, Y\rangle \in \mathcal{M}^2 : \exists_{x,y \in M_{\mathfrak{m}}}(X = \{x\} \wedge Y = \{y\} \wedge x \in_{\mathfrak{m}} y)\},$$
$$\mathsf{Set}_{\mathfrak{M}} := \{S \in \mathcal{M} : \exists_{s \in M_{\mathfrak{m}} \setminus \{a\}}\ S = \{s\}\}.$$

Then $\mathfrak{M}$ is a normal atomic model of **MZF** *such that*

$$\sqsubseteq_{\mathfrak{M}} = \{\langle X, Y\rangle \in \mathcal{M}^2 : X \subseteq Y\},$$
$$\mathsf{O}_{\mathfrak{M}} = \{\langle X, Y\rangle \in \mathcal{M}^2 : X \cap Y \neq \emptyset\}, \qquad \wr_{\mathfrak{M}} = \mathcal{M}^2 \setminus \mathsf{O}_{\mathfrak{M}},$$
$$\mathsf{at}_{\mathfrak{M}} = \mathsf{Set}_{\mathfrak{M}} \cup \{a\}, \qquad \mathsf{atc}_{\mathfrak{M}} = \mathcal{M}, \qquad \mathsf{atl}_{\mathfrak{M}} = \emptyset\,,$$
$$\mathsf{Ind}_{\mathfrak{M}} = \{\{a\}\}, \qquad \mathsf{i}_{\mathfrak{M}} = \{\mathsf{a}_{\mathfrak{m}}\},$$

and also for any $X, S \in \mathcal{M}$ we have

$$X\ \mathsf{Sum}_{\mathfrak{M}}\ S \iff S \in \mathsf{Set}_{\mathfrak{M}} \wedge X = \{y \in M_{\mathfrak{m}} : \{y\} \in_{\mathfrak{M}} S\}. \tag{\%}$$

PROOF. It is clear that $\mathsf{at}_{\mathfrak{M}} = \mathsf{Set}_{\mathfrak{M}} \cup \{a\}$, $\mathsf{atc}_{\mathfrak{M}} = \mathcal{M}$, $\mathsf{Ind}_{\mathfrak{M}} = \{\{a\}\}$, and axioms ($\lambda$1), ($\lambda$2), ($\alpha$) and ($\iota_1$) hold in $\mathfrak{M}$.

Ad (ι_2): For any $X \in \mathcal{M}$ we have: $X \in_{\mathfrak{M}} \{\mathsf{a}_{\mathfrak{m}}\}$ iff there is an $x \in M_{\mathfrak{m}}$ such that $X = \{x\}$ and $x \in_{\mathfrak{m}} \mathsf{a}_{\mathfrak{m}}$ iff $X = \{a\}$ iff $X \in \mathsf{Ind}_{\mathfrak{M}}$. Thus, $\mathsf{i}_{\mathfrak{M}} = \{\mathsf{a}_{\mathfrak{m}}\}$.

Ad (%): '$\Rightarrow$' Suppose that X $\mathsf{Sum}_{\mathfrak{M}}$ S, i.e., the following three conditions hold:

1. $S \in \mathsf{Set}_{\mathfrak{M}}$,
2. for any $Y \in \mathcal{M}$, if $Y \in_{\mathfrak{M}} S$ then $Y \sqsubseteq_{\mathfrak{M}} X$,
3. for any $Y \in \mathcal{M}$, if $Y \sqsubseteq_{\mathfrak{M}} X$ then for some $U \in \mathcal{M}$ we have $U \in_{\mathfrak{M}} S$ and $U \mathsf{O}_{\mathfrak{M}} Y$.

Thus, for some $s \in M_{\mathfrak{m}} \setminus \{i\}$ we have $S = \{s\}$ and:

2′. for any $y \in M_{\mathfrak{m}}$, if $\{y\} \in_{\mathfrak{M}} S$ then $y \in X$,

3′. for any $Y \in \mathcal{M}$, if $Y \subseteq X$ then for some $u \in M_{\mathfrak{m}}$ we have $\{u\} \in_{\mathfrak{M}} S$ and $\{u\} \subseteq Y$.

By $2'$ we have $\{y \in M_{\mathfrak{m}} : \{y\} \in_{\mathfrak{M}} S\} \subseteq X$. Moreover, by $3'$, for any $y \in M_{\mathfrak{m}}$, if $\{y\} \subseteq X$ then for some $u \in M_{\mathfrak{m}}$ we have $\{u\} \in_{\mathfrak{M}} S$ and $\{u\} \subseteq \{y\}$. Therefore for any $y \in M_{\mathfrak{m}}$, if $y \in X$ then for some $u \in M_{\mathfrak{m}}$ we have $\{u\} \in_{\mathfrak{M}} S$ and $u = y$. Thus, $X \subseteq \{y \in M_{\mathfrak{m}} : \{y\} \in_{\mathfrak{M}} S\}$.

'$\Leftarrow$' It is obvious.[7]

Ad (λ3) *and* (λ4): Directly from (%) we obtain that axiom (λ3) holds in $\mathfrak{M}$. Moreover, for any $S \in \mathcal{M}$, if $\mathfrak{M} \models \exists_y\, y \in z\,[S/z]$, then $S \in \mathsf{Set}_{\mathfrak{m}}$ and $\{y \in M_{\mathfrak{m}} : \{y\} \in_{\mathfrak{M}} S\} \neq \emptyset$; and so $\{y \in M_{\mathfrak{m}} : \{y\} \in_{\mathfrak{M}} S\}$ belongs to $\mathcal{M}$. Thus, in virtue of (%), axiom (λ4) holds in $\mathfrak{M}$.

That the axioms 'coming from' **ZF** are true as well follows from the fact that in $\mathfrak{M}$ they act only on singletons created from elements of the set $M_{\mathfrak{m}}$. □

Directly from Theorem 5.1 we obtain:

THEOREM 5.2. *If* **ZFA** *is consistent, then* **MZF** *is too.*

Because the second, third and fourth of the mereological theories which we propose are weaker than **MZF**, they are also consistent, if **ZFA** is consistent. However, the second and third of these theories can be given a model with a very simple construction.

THEOREM 5.3. *Let* $\mathfrak{m} = \langle M_{\mathfrak{m}}, \in_{\mathfrak{m}}, \varnothing_{\mathfrak{m}}, \mathsf{a}_{\mathfrak{m}}\rangle$ *be a model of* **ZFA** *such that* $\mathsf{a}_{\mathfrak{m}} \neq \varnothing_{\mathfrak{m}}$. *Let* $\mathfrak{M} = \langle M_{\mathfrak{m}}, \sqsubset_{\mathfrak{M}}, \in_{\mathfrak{M}}, \mathsf{Set}_{\mathfrak{M}}\rangle$ *be an* $\mathrm{L}_{\mathbf{MZF}}$*-structure in which* $\sqsubset_{\mathfrak{M}} = \emptyset$, $\in_{\mathfrak{M}} = \in_{\mathfrak{m}}$ *and* $\mathsf{Set}_{\mathfrak{M}} = M_{\mathfrak{m}} \setminus \{a : a \in_{\mathfrak{m}} \mathsf{a}_{\mathfrak{m}}\}$. *Then* $\mathfrak{M}$ *is a atomic model of the second and third theories such that:* $\sqsubseteq_{\mathfrak{M}} = \mathrm{id}_{M_{\mathfrak{m}}} = \mathsf{O}_{\mathfrak{M}}$, $\mathsf{at}_{\mathfrak{M}} = M_{\mathfrak{m}}$, $\mathsf{Ind}_{\mathfrak{M}} = \{a : a \in_{\mathfrak{m}} \mathsf{a}_{\mathfrak{m}}\}$, $\mathsf{i}_{\mathfrak{M}} = \{\mathsf{a}_{\mathfrak{m}}\}$, *and for any* $x, y \in M_{\mathfrak{m}}$:

$$y \;\mathsf{Sum}_{\mathfrak{M}}\; x \text{ iff } y \text{ is the only member of } M_{\mathfrak{m}} \text{ such that } y \in_{\mathfrak{m}} x.$$[8]

PROOF. It is clear that the formulae (λ1), (λ2), (λ3),(α), (ι_1), (ι_2), (SSP) and (Sum-sup) hold in $\mathfrak{M}$. □

[7] Notice that: $X \;\mathsf{Sum}_{\mathfrak{M}}\; \mathsf{i}_{\mathfrak{M}}$ iff $X \;\mathsf{Sum}_{\mathfrak{M}}\; \{\mathsf{a}_{\mathfrak{m}}\}$ iff $X = \{y \in M_{\mathfrak{m}} : y \in_{\mathfrak{m}} \mathsf{a}_{\mathfrak{m}}\}$ iff $X = \{a\}$. It may be informally expressed with the help of the formulae "$a \;\mathsf{Sum}\; \{a\}$" and "$[\![a]\!] = a$".

[8] Informally speaking, the mereological sum only exists for singletons of the form $\{x\}$ and it is equal to x, where $x \in M_{\mathfrak{m}}$.

APPENDICES

Both appendices are fundamentally algebraic. Only in sections 1–4 of Appendix II do we take a look at some concepts of first-order (elementary) theories and their models. The reader familiar with lattice theory may wish to take no more than a glance at Appendix I in order simply to familiarise themselves with the terminology used in this book.

In Appendix II we will be concerned with the 'elementary aspects' of Boolean lattices. We will be examining a first-order (elementary) theory with identity connected with Boolean lattices. To begin, we shall remind ourselves of some key concepts of elementary theories with identity.

Appendix I

Essential set theory and algebra

1. Sets, operation on sets, family of sets

Let X be any (distributive) set. If an object x is a member of X then we write: $x \in X$. In such case we also say that x *belongs to* X or that X *contains* x. If an object x is not a member of X then we write: $x \notin X$. In such case we also say that x *does not belong to* X or that X *does not contain* x. A set Y is a *subset* of X (we write: $Y \subseteq X$ iff every member of Y is a member of X. In such case we also say Y *is included in* X or X *includes* Y. Formally:

$$Y \subseteq X \;:\Longleftrightarrow \forall_x (x \in Y \;\Rightarrow\; x \in X).$$

If a set Y is not a *subset* of X then we write: $Y \nsubseteq X$. Of course:

$$Y \nsubseteq X \;\Longleftrightarrow\; \exists_x (x \in Y \wedge x \notin X).$$

Note that for a given sets X and Y we have:

$$\begin{aligned} X = Y \;&\Longleftrightarrow\; \forall_x (x \in X \;\Leftrightarrow\; x \in Y), \\ &\Longleftrightarrow\; X \subseteq Y \wedge Y \subseteq X. \end{aligned}$$

We have exactly one set that has no element. It is called the *empty set* and is denoted by "$\emptyset$". Of course, $\emptyset$ is a subset of any set.

A set Y is a *proper subset* of X (we write $Y \subsetneq X$) iff $Y \subseteq X$, but $X \neq Y$. Of course, the empty set $\emptyset$ does not have any proper subsets. So $\emptyset$ is a proper subset of all non-empty sets and only of non-empty sets.

A set all of whose members are sets is called a *family of sets*. The empty family of set is just the empty set $\emptyset$. Examples of families of sets are, for a set X, the family of all subsets of X, $\mathcal{P}(X)$, so-called the *power set* of X, and the family $\mathcal{P}_+(X)$ of all non-empty subsets of X.

The set $\{x, y\}$ is called the *unordered pair* of objects x and y. Its only elements are x and y, i.e.,

$$\{x, y\} := \{z : z = x \vee z = y\}.$$

So if $x = y$ then we get the so-called *singleton*

$$\{x\} := \{x, x\} = \{z : z = x\}.$$

So $\mathcal{P}(\emptyset) = \{\emptyset\}$ and $\mathcal{P}_{+}(\emptyset) = \emptyset$.

Furthermore, for any objects $x_1, \dots x_n$ $(n > 0)$ we have the following finite set:

$$\{x_1, \dots x_n\} := \{z : z = x_1 \vee \cdots \vee z = x_n\}.$$

The family $\langle x, y\rangle$ of sets $\{x\}$ and $\{x, y\}$ is called the *ordered pair* of objects x and y. An ordered pair can be defined in term of unordered pairs in the following way:

$$\langle x, y\rangle := \{\{x\}, \{x, y\}\}.$$

For any objects x, y, z and u we have: $\langle x, y\rangle = \langle z, u\rangle$ iff $x = z$ and $y = u$.

The *set-theoretic sum* of a given sets X and Y is the set $X \cup Y$ of all members X and Y, that is, all objects which belong either X or Y, i.e.:

$$X \cup Y := \{x : x \in X \vee x \in Y\}.$$

Both X and Y are subsets of $X \cup Y$, i.e., $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$. Of course, $X \cup X = X$. Moreover, $Y \subseteq X$ iff $X \cup Y = X$.

The *set-theoretic product* of a given sets X and Y is the set $X \cap Y$ of all common members X and Y, that is, all objects which belong both X and Y, i.e.:

$$X \cap Y := \{x : x \in X \wedge x \in Y\}.$$

The set $X \cap Y$ is a subset of both X and Y, i.e., $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$. Of course, $X \cap X = X$. Moreover, $Y \subseteq X$ iff $X \cap Y = Y$.

The *set-theoretic difference* of a given sets X and Y (or the *relative set-theoretic complement of* Y *with respect to* X) is the set $X \setminus Y$ of all members of X which are not members of X, i.e.,

$$X \setminus Y := \{x : x \in X \wedge x \notin Y\}.$$

We have: $X \setminus Y \subseteq X$, $(X \setminus Y) \cap Y = \emptyset$, $X \setminus X = \emptyset$, and: $Y \subseteq X$ iff $Y \setminus X = \emptyset$. Moreover: $Z \subseteq X \setminus Y$ iff $Z \subseteq X$ and $Z \cap Y = \emptyset$.

Let $\mathcal{F}$ be any family of sets. The *set-theoretic sum of family* $\mathcal{F}$ is the set $\bigcup \mathcal{F}$ of objects that belong to at least one set from $\mathcal{F}$, i.e.,

$$\bigcup \mathcal{F} := \{x : \exists_{S \in \mathcal{F}}\ x \in S\}.$$

In particular, for the family $\{X, Y\}$ of sets X and Y we have $\bigcup\{X, Y\} = X \cup Y$. Of course, $\bigcup \emptyset = \emptyset$.

The *set-theoretic product of non-empty family* $\mathcal{F}$ is the set $\bigcap \mathcal{F}$ of objects that belong to all sets from $\mathcal{F}$, i.e.,

$$\bigcap \mathcal{F} := \{x : \forall_{S \in \mathcal{F}}\ x \in S\}.$$

In particular, for the family $\{X, Y\}$ of sets X and Y we have $\bigcap\{X, Y\} = X \cap Y$.

If we consider a family $\mathcal{F}$ of sets included in a given set X, we have

$$\bigcap \mathcal{F} = \{x \in X : \forall_{S \in \mathcal{F}}\ x \in S\}.$$

In this case we have $\bigcap \mathcal{F} \subseteq X$. We can, moreover, also consider the product of the empty family of sets. It will be just the set X, since $\bigcap \emptyset = \{x \in X : \forall_S (S \in \emptyset \Rightarrow x \in S)\} = X$.

A non-empty family $\mathcal{F}$ is a *field of sets* iff $\mathcal{F}$ satisfies the following three conditions:

- $\bigcup \mathcal{F} \setminus X \in \mathcal{F}$, for any $X \in \mathcal{F}$;
- $X \cap Y \in \mathcal{F}$, for all $X, Y \in \mathcal{F}$;
- $X \cup Y \in \mathcal{F}$, for all $X, Y \in \mathcal{F}$.

Of course, then we also have

- $\emptyset \in \mathcal{F}$ and $\bigcup \mathcal{F} \in \mathcal{F}$;

Since for some set S_0 we have $S_0 \in \mathcal{F}$, then also $\bigcup \mathcal{F} \setminus S_0 \in \mathcal{F}$ and $\emptyset = S_0 \cap (\bigcup \mathcal{F} \setminus S_0) \in \mathcal{F}$. Hence $\bigcup \mathcal{F} = \bigcup \mathcal{F} \setminus \emptyset \in \mathcal{F}$.

If all members of a field $\mathcal{F}$ are subsets of a set X and $X \in \mathcal{F}$, then $\mathcal{F}$ is called a *field of sets over* X or an *algebra of sets over* X. In such case we have $X = \bigcup \mathcal{F}$, since $X \in \mathcal{F}$.

Example 1.1. The power set $\mathcal{P}(X)$ is an algebra of sets over a set X. □

Example 1.2. Let $\mathcal{FC}(X)$ be the family of all finite subsets of a set X and all co-finite subsets of X, i.e., those subsets of X whose complements are finite. That is, for any $S \in \mathcal{P}(X)$:

$$S \in \mathcal{FC}(X) \ :\Longleftrightarrow S \text{ is finite} \vee X \setminus S \text{ is finite}.$$

(i) $\mathcal{FC}(X)$ is an algebra of sets over X.
(ii) $\mathcal{FC}(X) = \mathcal{P}(X)$ iff X is finite. □

An algebra of sets $\mathcal{A}$ over X is called *complete* iff for each its subfamily $\mathcal{F}$ we have $\bigcup \mathcal{F} \in \mathcal{A}$ (then also $\bigcap \mathcal{F} \in \mathcal{A}$).

Example 1.3. $\mathcal{P}(X)$ is a complete algebra of sets over X. □

Example 1.4. Let be $\mathbb{N}$ be the set of all natural numbers. Then $\mathcal{FC}(\mathbb{N})$ is not a complete algebra of sets (see Example 1.2). For example, the sum of the family of all finite subsets made up of even numbers is equal to the set of all even numbers, but this set does not belong to $\mathcal{FC}(\mathbb{N})$. □

2. Binary relations

For arbitrary sets X and Y, the *Cartesian product* $X \times Y$ is the set of all ordered pairs $\langle x, y\rangle$, where $x \in X$ and $y \in Y$, i.e.,

$$X \times Y := \{\langle x, y\rangle : x \in X \wedge y \in Y\}.$$

Of course, for $X = Y$ we have:

$$X \times X = \{\langle x, y\rangle : x, y \in X\}.$$

Any subset of the Cartesian product $X \times Y$ is called a *binary relation* in $X \times Y$. In other words, R is a binary relation in $X \times Y$ iff R belongs to $\mathcal{P}(X \times Y)$. So $\emptyset$ is also a binary relation in $X \times Y$.

Next, we will look at binary relations in the Cartesian product $X \times X$. Such relations will simply be called binary relations in X. So a binary relation in X is any subset of the Cartesian product $X \times X$. Let $\mathcal{B}(X)$ be the family of all binary relations in X, i.e.,

$$\mathcal{B}(X) := \mathcal{P}(X \times X).$$

In $\mathcal{B}(X)$ we distinguish the set-theoretic relation id_X of *identity* on X, i.e.,

$$\mathrm{id}_X := \{\langle x, y\rangle : x, y \in X \wedge x = y\} = \{\langle x, x\rangle : x \in X\}.$$

Since the elements of the family $\mathcal{B}(X)$ are sets, the 'usual' two-place set-theoretic operations are defined in it: sum $\cup$, product $\cap$, and relative complement $\setminus$, i.e., for arbitrary $R_1, R_2 \in \mathcal{B}(X)$ we have:

$$R_1 \cup R_2 := \{\langle x, y\rangle : \langle x, y\rangle \in R_1 \vee \langle x, y\rangle \in R_2\},$$
$$R_1 \cup R_2 := \{\langle x, y\rangle : \langle x, y\rangle \in R_1 \wedge \langle x, y\rangle \in R_2\},$$
$$R_1 \setminus R_2 := \{\langle x, y\rangle : \langle x, y\rangle \in R_1 \wedge \langle x, y\rangle \notin R_2\}.$$

In addition, we have two operations on binary relations. The *converse relation* to a relation R is the following binary relation

$$\breve{R} := \{\langle x,y\rangle : \langle y,x\rangle \in R\}.$$

Obviously, $R = \breve{\breve{R}}$ and $(R \cup \mathrm{id}_X)\breve{} = \breve{R} \cup \mathrm{id}_X$. Moreover, for arbitrary relations $R_1, R_2 \in \mathcal{B}(X)$ their relative product is the following binary relation

$$R_1 \circ R_2 := \{\langle x,y\rangle : \exists_{z\in X}(\langle x,z\rangle \in R_1 \wedge \langle z,y\rangle \in R_2)\}.$$

Of course, for all $R, R_1, R_2 \in \mathcal{B}(X)$ we have: $R = R\circ\mathrm{id}_X$; $R\circ(R_1 \cup R_2) = (R \circ R_1) \cup (R \circ R_2)$; and $(R_1 \cup R_2) \circ R = (R_1 \circ R) \cup (R_2 \circ R)$.

A relation $R \in \mathcal{B}(X)$ is *reflexive* iff $\mathrm{id}_X \subseteq R$, i.e.:

$$\forall_{x\in X}\ \langle x,x\rangle \in R\,. \tag{r$_R$}$$

A relation $R \in \mathcal{B}(X)$ is *irreflexive* iff $\mathrm{id}_X \cap R = \emptyset$, i.e.:

$$\forall_{x\in X}\ \langle x,x\rangle \notin R\,. \tag{irr$_R$}$$

A relation $R \in \mathcal{B}(X)$ is *symmetric* iff $R \subseteq \breve{R}$ (iff $R = \breve{R}$), i.e.:

$$\forall_{x,y\in X}(\langle x,y\rangle \in R \iff \langle y,x\rangle \in R). \tag{s$_R$}$$

A relation $R \in \mathcal{B}(X)$ is *asymmetric* iff $R \cap \breve{R} = \emptyset$, i.e.:

$$\forall_{x,y\in X}\, \neg(\langle x,y\rangle \in R \wedge \langle y,x\rangle \in R). \tag{as$_R$}$$

A relation $R \in \mathcal{B}(X)$ is *antisymmetric* iff $R \cap \breve{R} \subseteq \mathrm{id}_X$, i.e.,

$$\forall_{x,y\in X}(\langle x,y\rangle \in R \wedge \langle y,x\rangle \in R \implies x = y)\,. \tag{antis$_R$}$$

A relation $R \in \mathcal{B}(X)$ is *transitive* iff $R \circ R \subseteq R$, i.e.,

$$\begin{aligned} &\forall_{x,y,z\in X}(\langle x,y\rangle \in R \wedge \langle y,z\rangle \in R \implies \langle x,z\rangle \in R), \qquad \text{or} \\ &\forall_{x,y\in X}\big(\exists_{z\in X}(\langle x,y\rangle \in R \wedge \langle y,z\rangle \in R) \implies \langle x,z\rangle \in R\big). \end{aligned} \tag{t$_R$}$$

Obviously, for any $R \in \mathcal{B}(X)$, the reflexivity (resp. irreflexivity, symmetry, asymmetry, antisymmetry, transitivity) of R is equivalent to the reflexivity (resp. irreflexivity, symmetry, asymmetry, antisymmetry, transitivity) of the converse relation $\breve{R}$.

For ease of expression, instead of "$\langle x, y\rangle \in R$" we will write for short "$x\,R\,y$" (we will do likewise for other combinations of variables "x", "y", "z" etc., and also for other variables referring to binary relations).[1]

We will give below several facts about binary relations which will come in handy in both parts of the book and in both appendices.

LEMMA 2.1. *For any $R \in \mathcal{B}(X)$:*

(i) *R is reflexive and antisymmetric iff $\mathrm{id}_X = R \cap \breve{R}$.*
(ii) *R is reflexive and transitive iff for arbitrary $x, y \in X$ we have*

$$x\,R\,y \iff \forall_{z\in X}(z\,R\,x \Rightarrow z\,R\,y).$$

PROOF. *Ad* (i): R is reflexive and antisymmetric iff $\mathrm{id}_X \subseteq R$, $\mathrm{id}_X \subseteq \breve{R}$ and $R \cap \breve{R} \subseteq \mathrm{id}_X$ iff $\mathrm{id}_X \subseteq R \cap \breve{R} \subseteq \mathrm{id}_X$.

Ad (ii): Suppose that R is reflexive and transitive and $x\,R\,y$. Then, by (t_R), for any $z \in X$: if $z\,R\,x$ then $z\,R\,y$. We now assume that $\forall_{z\in X}(z\,R\,x \Rightarrow z\,R\,y)$. Then, by ($\mathrm{r}_R$), we have $x\,R\,x$. Therefore, by our assumption, $x\,R\,y$. Conversely, assume that the given condition holds. Then condition (t_R) is the only other way of writing the simple implication in the assumed condition. Furthermore, by substituting "x" for "y" we obtain: $x R x \iff \forall_{z\in X}(z R x \Rightarrow z R x)$. Since the right-hand side is tautologous, we get $x\,R\,x$. □

LEMMA 2.2. *For any relation $R \in \mathcal{B}(X)$:*

(i) *If R is asymmetric then R is irreflexive.*
(ii) *If R is irreflexive and transitive, then R is asymmetric.*
(iii) *R is asymmetric iff R is irreflexive and antisymmetric.*

PROOF. *Ad* (i): If for some $x \in X$ we have $x\,R\,x$, then we get a contradiction with the assumption, by (as_R).

Ad (ii): By (t_R), for arbitrary $x, y \in X$: $x\,R\,y$ and $y\,R\,x$ entail $x\,R\,x$. Hence and from (irr_R) we have (as_R).

Ad (iii): If R is asymmetric, then R is irreflexive, by (i), and $R \cap \breve{R} = \emptyset \subseteq \mathrm{id}_X$, so R is antisymmetric. Conversely, if R irreflexive and antisymmetric, then $R \cap \breve{R} \subseteq R \cap \mathrm{id}_X = \emptyset$, so R is asymmetric. □

[1] From a formal point of view, the expression "$x\;R\;y$" is not correct. From a grammatical point of view, it has to be a sentential type of expression, but it is composed of three variables each of which is a name type of expression (for these variables occur either in place of the name of some object from X or in place of the name of some relation from $\mathcal{B}(X)$). We will, however, treat the form "$x\,R\,y$" as short for the sentential form "$\langle x, y\rangle \in R$". We may also take it to be an abbreviation of the sentential form "object x is in a relation R with object y".

LEMMA 2.3. *For any relation $R \in \mathcal{B}(X)$:*

(i) *The relation $R \cup \mathrm{id}_X$ is reflexive.*
(ii) *R is irreflexive iff $R = (R \cup \mathrm{id}_X) \setminus \mathrm{id}_X$.*
(iii) *R is asymmetric iff R is irreflexive and the relation $R \cup \mathrm{id}_X$ is antisymmetric. In both cases, we have : $R = (R \cup \mathrm{id}_X) \setminus (R \cup \mathrm{id}_X)\breve{}$.*
(iv) *If R is antisymmetric then $R \cup \mathrm{id}_X$ also is antisymmetric.*
(v) *If R is transitive, then $R \circ (R \cup \mathrm{id}_X) \subseteq R$, $(R \cup \mathrm{id}_X) \circ R \subseteq R$ and so the relations $R \cup \mathrm{id}_X$ also is transitive.*

PROOF. *Ad* (i): We have $\mathrm{id}_X \subseteq R \cup \mathrm{id}_X$; so $R \cup \mathrm{id}_X$ is reflexive.

Ad (ii): If R is irreflexive, then $R \cap \mathrm{id}_X = \emptyset$. Therefore $R = (R \cup \mathrm{id}_X) \setminus \mathrm{id}_X$. Conversely, if $R = (R \cup \mathrm{id}_X) \setminus \mathrm{id}_X$ then $R \cap \mathrm{id}_X = ((R \cup \mathrm{id}_X) \setminus \mathrm{id}_X) \cap \mathrm{id}_X = \emptyset$.

Ad (iii): If R is asymmetric, then both R and $\breve{R}$ are irreflexive, by Lemma 2.2(i). Therefore we have $R \cap \breve{R} = \emptyset$ and $R \cap \mathrm{id}_X = \emptyset = \breve{R} \cap \mathrm{id}_X$. Hence $(R \cup \mathrm{id}_X) \cap (R \cup \mathrm{id}_X)\breve{} = (R \cup \mathrm{id}_X) \cap (\breve{R} \cup \mathrm{id}_X) = (R \cap \breve{R}) \cup (R \cap \mathrm{id}_X) \cup (\breve{R} \cap \mathrm{id}_X) \cup \mathrm{id}_X = \emptyset \cup \emptyset \cup \emptyset \cup \mathrm{id}_X = \mathrm{id}_X$. So $R \cup \mathrm{id}_X$ is antisymmetric.

Conversely, if R is irreflexive and $R \cup \mathrm{id}_X$ is antisymmetric, then — in the light of (ii) — we have $R \cap \breve{R} = ((R \cup \mathrm{id}_X) \setminus \mathrm{id}_X) \cap ((R \cup \mathrm{id}_X) \setminus \mathrm{id}_X)\breve{} = ((R \cup \mathrm{id}_X) \cap (\breve{R} \cup \mathrm{id}_X)) \setminus \mathrm{id}_X = \mathrm{id}_X \setminus \mathrm{id}_X = \emptyset$, since $(R \cup \mathrm{id}_X) \cap (\breve{R} \cup \mathrm{id}_X) \subseteq \mathrm{id}_X$. So R is asymmetric.

Moreover, firstly, $(R \cup \mathrm{id}_X) \setminus (\breve{R} \cup \mathrm{id}_X) \subseteq (R \cup \mathrm{id}_X) \setminus \mathrm{id}_X = (R \setminus \mathrm{id}_X) \cup (\mathrm{id}_X \setminus \mathrm{id}_X) = R \setminus \mathrm{id}_X \subseteq R$. Secondly, if R is asymmetric, then R is irreflexive and $R \cap (\breve{R} \cup \mathrm{id}_X) = (R \cap \breve{R}) \cup (R \cap \mathrm{id}_X) = \emptyset$. Hence $R \subseteq (R \cup \mathrm{id}_X) \setminus (\breve{R} \cup \mathrm{id}_X)$.

Ad (iv): If R is antisymmetric then $(R \cup \mathrm{id}_X) \cap (\breve{R} \cup \mathrm{id}_X) = (R \cap \breve{R}) \cup (R \cap \mathrm{id}_X) \cup (\breve{R} \cap \mathrm{id}_X) \cup \mathrm{id}_X \subseteq \mathrm{id}_X$. So $R \cup \mathrm{id}_X$ is also antisymmetric.

Ad (v): Assume that R is transitive, i.e., $R \circ R \subseteq R$. Then $R \circ (R \cup \mathrm{id}_X) = (R \circ R) \cup (R \circ \mathrm{id}_X) = (R \circ R) \cup R \subseteq R$. Similarly we obtain: $(R \cup \mathrm{id}_X) \circ R \subseteq R$. Furthermore, $(R \cup \mathrm{id}_X) \circ (R \cup \mathrm{id}_X) = (R \circ R) \cup (R \circ \mathrm{id}_X) \cup (\mathrm{id}_X \circ \mathrm{id}_X) \subseteq R \cup \mathrm{id}_X$. Hence $R \cup \mathrm{id}_X$ is transitive. □

LEMMA 2.4. *For any relation $R \in \mathcal{B}(X)$:*

(i) *$R \setminus \mathrm{id}_X$ is irreflexive.*
(ii) *R is reflexive iff $R = (R \setminus \mathrm{id}_X) \cup \mathrm{id}_X$.*
(iii) *R is antisymmetric iff $R \setminus \mathrm{id}_X$ is asymmetric.*
(iv) *If R is antisymmetric and transitive, then $R \circ (R \setminus \mathrm{id}_X) \subseteq R \setminus \mathrm{id}_X$, $(R \setminus \mathrm{id}_X) \circ R \subseteq R \setminus \mathrm{id}_X$, so $R \setminus \mathrm{id}_X$ is transitive.*
(v) *If R is antisymmetric then $R \setminus \mathrm{id}_X = R \setminus \breve{R}$.*

PROOF. *Ad* (i): Obviously, $\mathrm{id}_X \cap (R \setminus \mathrm{id}_X) = \emptyset$. So $R \setminus \mathrm{id}_X$ is irreflexive.

Ad (ii): We have $(R \setminus \mathrm{id}_X) \cup \mathrm{id}_X = (R \cup \mathrm{id}_X) \cap (\mathrm{id}_X \setminus \mathrm{id}_X) = R \cup \mathrm{id}_X$. Therefore, R is reflexive $\mathrm{id}_X \subseteq R$ iff $R = R \cup \mathrm{id}_X = (R \setminus \mathrm{id}_X) \cup \mathrm{id}_X$.

Ad (iii): Observe that $(R \setminus \mathrm{id}_X) \cap (R \setminus \mathrm{id}_X)\breve{} = (R \setminus \mathrm{id}_X) \cap (\breve{R} \setminus \mathrm{id}_X) = (R \cap \breve{R}) \setminus \mathrm{id}_X$. Thus, $R \setminus \mathrm{id}_X$ is asymmetric iff $(R \setminus \mathrm{id}_X) \cap (R \setminus \mathrm{id}_X)\breve{} = \emptyset$ iff $(R \cap \breve{R}) \setminus \mathrm{id}_X = \emptyset$ iff $R \cap \breve{R} \subseteq \mathrm{id}_X$ iff R is antisymmetric.

Ad (iv): Suppose that R is antisymmetric and transitive, i.e., $R \cap \breve{R} \subseteq \mathrm{id}_X$ and $R \circ R \subseteq R$. Then if $\langle x, y\rangle \in (R \setminus \mathrm{id}_X) \circ R$ then for some $z \in X$ we have (A) $x\,R\,z$, (B) $x \neq z$ and (C) $z\,R\,y$. Therefore, from (A), (C) and (t_R) we get $x\,R\,y$. It is therefore necessary to show that $x \neq y$. Briefly, if $x = y$, then $z\,R\,x$, by (C). Now, since $x\,R\,z$, we get a contradiction with (B), by the antisymmetry of R. We prove the second condition in a similar way. From each of these conditions the transitivity of R follows.

Ad (v): First, note that $R \setminus \breve{R} \subseteq R \setminus \mathrm{id}_X$. Essentially, for all $x, y \in X$: if $x\,R\,y$ and $\neg\, y\,R\,x$, then $x \neq y$. Second, assume that R is antisymmetric, i.e., $R \cap \breve{R} \subseteq \mathrm{id}_X$; so $(R \cap \breve{R}) \setminus \mathrm{id}_X = (R \setminus \mathrm{id}_X) \cap \breve{R} = \emptyset$. Then $R \setminus \mathrm{id}_X \subseteq (X \times X) \setminus \breve{R}$; and so $R \setminus \mathrm{id}_X \subseteq R \setminus \breve{R}$. □

From the above lemmas the following well-known fact follows:

FACT 2.5. *For all $R_1, R_2 \in \mathcal{B}(X)$ the following conditions are equivalent*:

(a) *R_1 is irreflexive and transitive, and $R_2 = R_1 \cup \mathrm{id}_X$.*
(b) *R_1 is asymmetric and transitive, and $R_2 = R_1 \cup \mathrm{id}_X$.*
(c) *R_2 is reflexive, antisymmetric and transitive, and $R_1 = R_2 \setminus \mathrm{id}_X$.*
(d) *R_2 is reflexive, antisymmetric and transitive, and $R_1 = R_2 \setminus \breve{R}_2$.*

PROOF. "(a) ⇔ (b)" By Lemma 2.2(i,ii). "(b) ⇒ (c)" By Lemma 2.3. "(c) ⇒ (b)" By Lemma 2.4(i–iv). "(c) ⇔ (d)" By Lemma 2.4(v). □

We say that $R \in \mathcal{B}(X)$ is a *preorder* (or *quasiorder*) in X iff R is reflexive and transitive. Any ordered pair $\langle X, R\rangle$, where $R \in \mathcal{B}(X)$ and R is a preorder in X, is called a *preordered set* (or *proset*).

We say that $R \in \mathcal{B}(X)$ *partially orders* X iff R is reflexive, transitive, and antisymmetric. If R partially orders X then we also say that R is a *partial order* in X, and the pair $\langle X, R\rangle$ we call a *partially ordered set* (or *poset*). Of course, R is a partial order in X iff R is a antisymmetric preorder in X.

We say that $R \in \mathcal{B}(X)$ is an *equivalence relation* in X iff R is reflexive, symmetric, and transitive. For example, id_X is an equivalence relation in X. Of course, R is an equivalence relation in X iff R is a symmetric preorder in X.

We say that R *strictly partially orders* X iff R satisfies one of the following equivalent conditions (see Lemma 2.2(i,ii)):

(irr-t) R is irreflexive and transitive,
(as-t) R is asymmetric and transitive.

So if R strictly partially orders X, then R is both irreflexive, asymmetric, and transitive. In such case we also say that R is a *strict partial order* in X, and the pair $\langle X, R\rangle$ we call a *strictly partially ordered set*.

3. Strict partial orders

Let **SPOS** be the class of all strictly partially ordered sets.

Let X be any non empty set and $R \in \mathcal{B}(X)$. For ease of expression, we shall symbolise that R is a strict partial order in X with the symbol "$\prec$" and we shall subscript any relation with this symbol to signify that it is a relation which imposes a strict partial order on a set. The set-theoretic complement of the relation $\prec$, i.e., $(X \times X) \setminus \prec$, we shall symbolise by "$\not\prec$".[2]

Remark 3.1. On this way of expressing things, the symbol "$\prec$" is a variable ranging over strict partial orders. □

With this in mind, for arbitrary $x, y, z \in W$ we have :

$$x \not\prec x\,, \tag{irr$_\prec$}$$

$$x \prec y \implies y \not\prec x\,, \tag{as$_\prec$}$$

$$x \prec y \wedge y \prec z \implies x \prec z\,. \tag{t$_\prec$}$$

Let $\leqq$ be the sum of the relations $\prec$ and id_X:

$$\leqq\, := \,\prec \cup\, \mathrm{id}_X\,. \tag{df $\leqq$}$$

Therefore, for arbitrary $x, y \in X$ we have:

$$x \leqq y \iff x \prec y \vee x = y\,.$$

Applying Lemma 2.3 for arbitrary $x, y, z \in X$ we get:

$$x \leqq x\,,$$

$$x \prec y \iff x \leqq y \wedge x \neq y\,,$$

[2] The struck-through version of a symbol for a given relation signifies the set-theoretic complement of that relation.

$$x \leqq y \wedge y \leqq x \implies x = y\,,$$
$$x \prec y \iff x \leqq y \wedge y \nleqq x\,,$$
$$x \prec y \wedge y \leqq z \implies x \prec z\,,$$
$$x \leqq y \wedge y \prec z \implies x \prec z\,,$$
$$x \leqq y \wedge y \leqq z \implies x \leqq z\,.$$

4. Partial orders

Let **POS** be a class of partially ordered sets.

Let X be any non empty set and $R \in \mathcal{B}(X)$. For ease of expression, we shall symbolise that R is a partial order in X with the symbol "$\leq$" and we shall subscript any relation with this symbol to signify that it is a relation which imposes a partial order on a set.

Remark 4.1. (i) On this way of expressing things, the symbol "$\leq$" is a variable ranging over partial orders.

(ii) We have not used the symbol $\leqq$ although the relation $\leq \setminus \mathrm{id}_X$ is a strict partial order in X and $\leq = (\leq \setminus \mathrm{id}_X) \cup \mathrm{id}_X$ (cf. Lemma 2.4(i–iv). In X may exist two binary relations $\prec$ and $\leq$ such that $\leq \neq \leqq$.

(iii) We do not have to think that the symbol "$\leq$" 'says' that the relation $\leq$ is the sum of the relations id_X and some relation $<$. We may also well accept that the symbol "$\leq$" 'says' that the relation id_X is identical to the product of the relations $\leq$ and its converse, i.e., $\mathrm{id}_X = \leq \cap \geq$, where $\geq := \breve{\leq}$ (cf. Lemma 2.1(i)).

(iv) All these results concerning the relation $\leq$ apply also to the relation $\leqq$ defined with the help of (df $\leqq$), since $\leqq$ is reflexive, antisymmetric, and transitive (cf. Lemma 2.3(i,iii,v) along with the corresponding formulae in Section 3). □

With this in mind, for arbitrary $x, y, z \in X$ we have:

$$x \leq x\,, \tag{$\mathrm{r}_\leq$}$$
$$x \leq y \wedge y \leq x \implies x = y\,, \tag{$\mathrm{antis}_\leq$}$$
$$x \leq y \wedge y \leq z \implies x \leq z\,. \tag{$\mathrm{t}_\leq$}$$

By Lemma 2.1(ii), for arbitrary $x, y \in X$ we have:

$$x \leq y \iff \forall_{z \in X}(z \leq x \Rightarrow z \leq y). \tag{4.1}$$

From this and the antisymmetry of the relation follows:

$$x = y \iff \forall_{z \in X}(z \leq x \Leftrightarrow z \leq y).$$

Let $\lneq$ be the difference between relations $\leq$ and id_X:

$$\lneq \,:=\, \leq \setminus \mathrm{id}_X\,. \tag{df $\lneq$}$$

Therefore, for arbitrary $x, y \in X$ we have:

$$x \lneq y \iff x \leq y \wedge x \neq y\,.$$

Applying Lemma 2.4 for arbitrary $x, y, z \in X$ we get:

$$\neg\, x \lneq x\,, \tag{irr$_{\lneq}$}$$
$$x \leq y \iff x \lneq y \vee x = y\,,$$
$$x \lneq y \implies \neg\, y \lneq x\,, \tag{as$_{\lneq}$}$$
$$x \lneq y \wedge y \leq z \implies x \lneq z\,,$$
$$x \leq y \wedge y \lneq z \implies x \lneq z\,,$$
$$x \lneq y \wedge y \lneq z \implies x \lneq z\,, \tag{t$_{\lneq}$}$$
$$x \lneq y \iff x \leq y \wedge y \nleq x\,. \tag{def$'$ $\lneq$}$$

Obviously, formula (def$'$ $\lneq$) could be taken as a definition of $\lneq$ (cf. Lemma 2.4(v)).

It follows from the conditions above that the relation $\lneq$ is a strict partial order in X and that for the relations $\lneq$ and $\leq$ hold counterparts of those conditions given for the relations $\prec$ and $\leqq$ in the previous section.

Example 4.1. For any non-empty family $\mathcal{F}$ of sets the pair $\langle \mathcal{F}, \subseteq \rangle$ is a partially ordered set. We will use this writing instead of $\langle \mathcal{F}, \leq \rangle$, where $\leq\, := \{\langle X, Y\rangle \in \mathcal{F} \times \mathcal{F} : X \subseteq Y\}$.[3] □

LEMMA 4.1. *For any $x \in X$ we put*

$$X \restriction x := \{y \in X : y \leq x\}.$$

Then $\mathfrak{X} \restriction x := \langle X \restriction x, \leq|_{X\restriction x}\rangle$ is a partially ordered set, where $\leq|_{X\restriction x}$ is the restriction of $\leq$ to the set $X \restriction x$.

Let $\mathfrak{X}_1 = \langle X_1, \leq_1\rangle$ and $\mathfrak{X}_2 = \langle X_2, \leq_2\rangle$ be any partially ordered sets. We say that a function f from X_1 into X_2 is a *homomorphism* from $\mathfrak{X}_1$ to $\mathfrak{X}_2$ iff for all $x, y \in X_1$ we have:

$$x \leq_1 y \iff f(x) \leq_2 f(y).$$

[3] Formally, "$\subseteq$" is a set-theoretic predicate and not a relation in $\mathcal{F}$.

A *monomorphism* from $\mathfrak{X}_1$ to $\mathfrak{X}_2$ is any injective homomorphism. A *isomorphism* from $\mathfrak{X}_1$ onto $\mathfrak{X}_2$ is any injective and surjective (i.e., bijective) homomorphism. We say the $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are *isomorphic* iff there is an isomorphism from $\mathfrak{X}_1$ onto $\mathfrak{X}_2$.

Let $\mathfrak{X} = \langle X, \leq \rangle$ be a partially ordered set. Take arbitrary $x \in X$ and $S \in \mathcal{P}(X)$. We say that x is an *upper* (resp. *lower*) *bound* of S in $\mathfrak{X}$ iff for each $z \in S$ we have $z \leq x$ (resp. $x \leq z$). Let $\mathrm{UB}(S)$ (resp. $\mathrm{LB}(S)$) be the set of all upper (resp. lower) bounds of a set S in $\mathfrak{X}$. Moreover, we say that x is a *greatest* (resp. *least*) *element of* S in $\mathfrak{X}$ iff $x \in S \cap \mathrm{UB}(S)$ (resp. $x \in S \cap \mathrm{LB}(S)$). There can at most be one greatest (resp. least) element of S in $\mathfrak{X}$. Briefly, if x and y are greatest (resp. least) elements of S, then $x \leq y$ and $y \leq x$. Therefore, $x = y$, by ($\mathrm{antis}_{\leq}$). If $S = X$ then we will shortly say that x is a *greatest* (resp. *least*) *element of* $\mathfrak{X}$.

We say that x is a *supremum* of a set S in $\mathfrak{X}$ (we write: $x \,\mathsf{sup}_{\leq}\, S$) iff x is the least upper bound of S. In other words, $x \,\mathsf{sup}_{\leq}\, S$ iff x is the least element in the set $\mathrm{UB}(S)$. This may be put symbolically as follows for all $x \in X$ and $S \in \mathcal{P}(X)$:

$$x \,\mathsf{sup}_{\leq}\, S \iff \forall_{z \in S}\, z \leq x \wedge \forall_{y \in X}(\forall_{z \in S}\, z \leq y \implies x \leq y). \quad (\mathrm{df}\ \mathsf{sup}_{\leq})$$

It follows from the fact that in the set $\mathrm{UB}(S)$ there can be at most one least element that if S has a supremum then it is unique, i.e.:

$$\forall_{x,y \in X}(x \,\mathsf{sup}_{\leq}\, S \wedge y \,\mathsf{sup}_{\leq}\, S \implies x = y). \qquad (\mathrm{U}_{\mathsf{sup}})$$

Moreover, by definitions, we obtain:

$$\forall_{S \in \mathcal{P}(X)} \forall_{x \in X}(x \text{ is a greatest element of } S \implies x \,\mathsf{sup}_{\leq}\, S). \qquad (4.2)$$

Suppose that x is a greatest element in S, i.e., $x \in S$ and $\forall_{z \in S}\, z \leq x$. Take an arbitrary $y \in X$ such that $\forall_{z \in S}\, z \leq y$. Since $x \in S$, so $x \leq y$. Therefore, by ($\mathrm{df}\ \mathsf{sup}_{\leq}$), we have $x \,\mathsf{sup}_{\leq}\, S$.[4]

Of course, for $S = X$ from condition (4.2) we obtain:

$$\forall_{x \in X}(x \text{ is a greatest element in } \mathfrak{X} \iff x \,\mathsf{sup}_{\leq}\, X). \qquad (4.3)$$

Furthermore, note that only by ($\mathrm{df}\ \mathsf{sup}_{\leq}$) the relation $\mathsf{sup}_{\leq}$ is monotonic, i.e.:

$$\forall_{S,Z \in \mathcal{P}(X)} \forall_{x,y \in M}(x \,\mathsf{sup}_{\leq}\, S \wedge y \,\mathsf{sup}_{\leq}\, Z \wedge Y \subseteq Z) \implies x \sqsubseteq y. \qquad (\mathrm{M}_{\sup})$$

[4] Note that a given set may have a supremum that does not belong to the set, and then it will not be a greatest element of this set.

From ($r_\leq$) it follows that:

$$\forall_{x\in X}\ x \,\mathsf{sup}_\leq\, \{x\}\,, \qquad (4.4)$$
$$\forall_{x\in X}\ x \,\mathsf{sup}_\leq\, \{z \in X : z \leq x\}.$$

From ($r_\leq$) and ($\text{antis}_\leq$) it follows that:

$$\forall_{x,y\in X}(y \,\mathsf{sup}_\leq\, \{x\} \implies x = y). \qquad (\mathrm{S}_{\mathsf{sup}})$$

By ($r_\leq$) and ($t_\leq$) we have:

$$x \,\mathsf{sup}_\leq\, S \iff \forall_{y\in X}(x \leq y \Leftrightarrow \forall_{z\in S}\ z \leq y). \qquad (4.5)$$

'$\Rightarrow$' Suppose that $x \,\mathsf{sup}_\leq\, S$. Then if $x \leq y$ then $\forall_{z\in S}\ z \leq y$, by ($t_\leq$), since $\forall_{z\in S}\ z \leq x$. Conversely, if $\forall_{z\in S}\ z \leq y$ then $x \leq y$, by (df $\mathsf{sup}_\leq$). '$\Leftarrow$' If $\forall_{y\in P}(x \leq y \Leftrightarrow \forall_{z\in S}\ z \leq y)$ then $\forall_{z\in S}\ z \leq x$, by ($r_\leq$). Therefore, $x \,\mathsf{sup}_\leq\, S$.

For finite sets, the relation $\mathsf{sup}_\leq$ has an interesting property, namely that, for arbitrary $y_1, \ldots, y_n \in X$ $(n > 0)$ the following holds:

$$x \,\mathsf{sup}_\leq\, \{y_1, \ldots, y_n\} \iff x \,\mathsf{sup}_\leq\, \{z \in X : \exists_{i\in\{1,\ldots,n\}}\ z \leq y_i\}. \qquad (4.6)$$

Let $x \,\mathsf{sup}_\leq\, \{y_1, \ldots, y_n\}$, i.e., we have (a) $\forall_{i\in\{1,\ldots,n\}}\ y_i \leq x$ and (b) $\forall_{y\in X}(\forall_{i\in\{1,\ldots,n\}}\ y_i \leq y \Rightarrow x \leq y)$. Then, by (4.1), (a) is equivalent to (c): $\forall_{z\in X, i\in\{1,\ldots,n\}}(z \leq y_i \Rightarrow z \leq x)$; and (b) gives:

$$\forall_{y\in X}(\forall_{i\in\{1,\ldots,n\}, z\in X}(z \leq y_i \Rightarrow z \leq y)) \implies x \leq y). \qquad (d)$$

Furthermore, (c) is equivalent to (e): $\forall_{z\in X}(\exists_{i\in\{1,\ldots,n\}}\ z \leq y_i \Rightarrow z \leq x)$. Condition (d) is equivalent to (f): $\forall_{y\in X}(\forall_{z\in X}(\exists_{i\in\{1,\ldots,n\}}\ z \leq y_i \Rightarrow z \leq y) \implies x \leq y)$. The conjunction of (e) and (f), in virtue of (df $\mathsf{sup}_\leq$), is equivalent to $x \,\mathsf{sup}_\leq\, \{z \in X : \exists_{i\in\{1,\ldots,n\}}\ z \leq y_i\}$.

Example 4.2. For a non-empty family of sets $\mathcal{F}$ let $\mathsf{sup}_\subseteq$ be the supremum relation in the partially ordered set $\langle \mathcal{F}, \subseteq \rangle$ (see Example 4.1), which is included in $\mathcal{F} \times \mathcal{P}(\mathcal{F})$ and for any subset Y of $\mathcal{F}$ and any subfamily $\mathcal{S}$ of $\mathcal{F}$ we have:

$$Y \,\mathsf{sup}_\subseteq\, \mathcal{S} \iff \forall_{Z\in\mathcal{S}}\ Z \subseteq Y \wedge \forall_{S\in\mathcal{F}}(\forall_{Z\in\mathcal{S}}\ Z \subseteq S \implies Y \subseteq S).$$

Of course, a subfamily $\mathcal{S}$ of $\mathcal{F}$ may not have a supremum in $\langle \mathcal{F}, \subseteq \rangle$. But if $\mathcal{S}$ has a supremum in $\langle \mathcal{F}, \subseteq \rangle$, then we obtain:

- For any $Y \in \mathcal{F}$: if $Y \,\mathsf{sup}_\subseteq\, \mathcal{S}$ then $\bigcup \mathcal{S} \subseteq Y$.
  In fact, if $Y \,\mathsf{sup}_\subseteq\, \mathcal{S}$ then $\forall_{Z\in\mathcal{S}}\ Z \subseteq Y$; so $\bigcup \mathcal{S} \subseteq Y$.

- If $\bigcup\mathcal{S} \notin \mathcal{F}$ and $Y \ \mathsf{sup}_{\subseteq}\ \mathcal{S}$, then $\bigcup\mathcal{S} \subsetneq Y$.
- If $\bigcup\mathcal{S} \in \mathcal{F}$ then $\bigcup\mathcal{S} \ \mathsf{sup}_{\subseteq}\ \mathcal{S}$.

Indeed, firstly, $\forall_{Z\in\mathcal{S}}\ Z \subseteq \bigcup\mathcal{S}$. Secondly, for any $S \in \mathcal{S}$: if $\forall_{Z\in\mathcal{S}}\ Z \subseteq S$ then $\bigcup\mathcal{S} \subseteq S$. □

Example 4.3. For any subfamily $\mathcal{S}$ of the power set $\mathcal{P}(X)$ we have:

- $\bigcup\mathcal{S} \ \mathsf{sup}_{\subseteq}\ \mathcal{S}$ in $\langle\mathcal{P}(X), \subseteq\rangle$. □

We say that x is a *infimum* of S in $\mathfrak{X}$ (we write: $x \ \mathsf{inf}_{\leq}\ S$) iff x is the greatest lower bound of S. In other words, $x \ \mathsf{inf}_{\leq}\ S$ iff x is the greatest element in the set $\mathrm{LB}(S)$. This may be put symbolically as follows for all $x \in X$ and $S \in \mathcal{P}(X)$:

$$x \ \mathsf{inf}_{\leq}\ S \iff \forall_{z\in S}\ x \leq z \wedge \forall_{y\in X}(\forall_{z\in S}\ y \leq z \implies y \leq x). \qquad (\mathrm{df}\ \mathsf{inf}_{\leq})$$

It follows from the fact that in the set $\mathrm{LB}(S)$ there can be at most one greatest element that if S has a infimum then it is unique, i.e.:

$$\forall_{x,y\in X}(x \ \mathsf{inf}_{\leq}\ S \wedge y \ \mathsf{inf}_{\leq}\ S \implies x = y). \qquad (\mathrm{U}_{\mathsf{inf}})$$

Moreover, from our definitions, we obtain:

$$\forall_{S\in\mathcal{P}(X)}\forall_{x\in X}(x \text{ is a least element of } S \implies x \ \mathsf{inf}_{\leq}\ S). \qquad (4.7)$$

Suppose that x is a least element in S, i.e., $x \in S$ and $\forall_{z\in S}\ x \leq z$. Take an arbitrary $y \in X$ such that $\forall_{z\in S}\ z \leq y$. Since $x \in S$, so $y \leq x$. Therefore, by (df $\mathsf{inf}_{\leq}$), we have $x \ \mathsf{sup}_{\leq}\ S$.[5]

Of course, for $S = X$ from condition (4.7) we obtain:

$$\forall_{x\in X}(x \text{ is a least element in } \mathfrak{X} \iff x \ \mathsf{inf}_{\leq}\ X). \qquad (4.8)$$

Furthermore, only by (df $\mathsf{inf}_{\leq}$) the relation $\mathsf{inf}_{\leq}$ is monotonic, i.e.:

$$\forall_{S_1,S_2\in\mathcal{P}(M)}\forall_{x,y\in M}(x \ \mathsf{inf}_{\leq}\ S_1 \wedge y \ \mathsf{inf}_{\leq}\ S_2 \wedge S_1 \subseteq S_2) \implies y \sqsubseteq x. \qquad (\mathrm{M}_{\mathsf{inf}})$$

From ($\mathrm{t}_{\leq}$) it follows that:

$$\forall_{x\in M}\ x \ \mathsf{inf}_{\leq}\ \{x\}\,. \qquad (4.9)$$

Example 4.4. For a non-empty family of sets $\mathcal{F}$ let $\mathsf{inf}_{\subseteq}$ be the infimum relation in the partially ordered set $\langle\mathcal{F}, \subseteq\rangle$ (see Example 4.1), which is included in $\mathcal{F} \times \mathcal{P}(\mathcal{F})$ and for any subset Y of $\mathcal{F}$ and any subfamily $\mathcal{S}$ of $\mathcal{F}$ we have:

[5] Note that a given set may have a infimum that does not belong to the set, and then it will not be a least element of this set.

$$Y \,\mathsf{inf}_{\subseteq}\, \mathcal{S} \iff \forall_{Z\in\mathcal{S}}\, Y \subseteq Z \wedge \forall_{S\in\mathcal{F}}(\forall_{Z\in\mathcal{S}}\, S \subseteq Z \implies S \subseteq Y).$$

Of course, a subfamily $\mathcal{S}$ of $\mathcal{F}$ may not have a infimum in $\langle\mathcal{F}, \subseteq\rangle$. But if $\mathcal{S}$ has a infimum in $\langle\mathcal{F}, \subseteq\rangle$, then we obtain:

- For any $Y \in \mathcal{F}$: if $Y \,\mathsf{inf}_{\subseteq}\, \mathcal{S}$ then $Y \subseteq \bigcap\mathcal{S}$.
  Indeed, if $Y \,\mathsf{inf}_{\subseteq}\, \mathcal{S}$ then $\forall_{Z\in\mathcal{S}}\, Y \subseteq Z$; so $Y \subseteq \bigcap\mathcal{S}$.
- If $\bigcap\mathcal{S} \notin \mathcal{F}$ and $Y \,\mathsf{inf}_{\subseteq}\, \mathcal{S}$, then $Y \subsetneq \bigcap\mathcal{S}$.
- If $\bigcap\mathcal{S} \in \mathcal{F}$ then $\bigcap\mathcal{S} \,\mathsf{inf}_{\subseteq}\, \mathcal{S}$.
  Indeed, firstly, $\forall_{Z\in\mathcal{S}}\, \bigcap\mathcal{S} \subseteq Z$. Secondly, for any $S \in \mathcal{S}$: if $\forall_{Z\in\mathcal{S}}\, S \subseteq Z$ then $S \subseteq \bigcap\mathcal{S}$. □

Example 4.5. For any subfamily $\mathcal{S}$ of the power set $\mathcal{P}(X)$ we have:

- $\bigcap\mathcal{S} \,\mathsf{inf}_{\subseteq}\, \mathcal{S}$ in $\langle\mathcal{P}(X), \subseteq\rangle$. □

The relations $\mathsf{inf}_{\leq}$ and $\mathsf{sup}_{\leq}$ are interdefinable, i.e., for all $S \in \mathcal{P}(X)$ and $x \in X$ the following conditions hold:

$$x \,\mathsf{inf}_{\leq}\, S \iff x \,\mathsf{sup}_{\leq}\, \mathrm{LB}(S), \tag{4.10}$$

$$x \,\mathsf{sup}_{\leq}\, S \iff x \,\mathsf{inf}_{\leq}\, \mathrm{UB}(S). \tag{4.11}$$

For (4.10): Assume that $x \,\mathsf{inf}_{\leq}\, S$, i.e., that x is the greatest in the set $\mathrm{LB}(S)$. Then, $x \,\mathsf{sup}_{\leq}\, \mathrm{LB}(S)$, by (4.2). Conversely, let $x \,\mathsf{sup}_{\leq}\, \mathrm{LB}(S)$. Then, by (df $\mathsf{sup}_{\leq}$), we have $\forall_{y\in\mathrm{LB}(S)}\, y \leq x$, i.e., (a) $\forall_{y\in X}(\forall_{z\in S}\, y \leq z \Rightarrow y \leq x)$, and (b) $\forall_{y\in X}(\forall_{u\in\mathrm{LB}(S)}\, u \leq y \Rightarrow x \leq y)$. Moreover, for any $z \in S$ we have $\forall_{u\in\mathrm{LB}(S)}\, u \leq z$. Hence $x \leq z$, by (b). From this and from (a) and (df $\mathsf{inf}_{\leq}$) we obtain $x \,\mathsf{inf}_{\leq}\, S$. We prove (4.11) in a similar way (but we use (4.7) instead of (4.2)).

From (4.10) and (4.11) we obtain respectively:

$$x \,\mathsf{inf}_{\leq}\, \emptyset \iff x \,\mathsf{sup}_{\leq}\, X\,, \tag{4.12}$$

$$x \,\mathsf{inf}_{\leq}\, \{y_1, \ldots, y_n\} \iff x \,\mathsf{sup}_{\leq}\, \{z \in X : \forall_{i\in\{1,\ldots,n\}}\, z \leq y_i\}, \tag{4.13}$$

$$x \,\mathsf{sup}_{\leq}\, \emptyset \iff x \,\mathsf{inf}_{\leq}\, X\,. \tag{4.14}$$

Moreover, in a similar way as (4.5) we can show that:

$$x \,\mathsf{inf}_{\leq}\, S \iff \forall_{y\in X}(y \leq x \Leftrightarrow \forall_{z\in S}\, y \leq z). \tag{4.15}$$

For any $S \in \mathcal{P}(X)$ we put:

$$\mathsf{max}_{\leq}(S) := \{x \in S : \neg\exists_{z\in S}\, x \lneq z\} = \{x \in S : \forall_{z\in S}(x \leq z \Rightarrow z = x\}, \tag{df $\mathsf{max}_{\leq}$}$$

$$\mathsf{min}_{\leq}(S) := \{x \in S : \neg\exists_{z\in S}\, z \lneq x\} = \{x \in S : \forall_{z\in S}(z \leq x \Rightarrow z = x\}. \tag{df $\mathsf{min}_{\leq}$}$$

The members of the set $\mathsf{max}_{\leq}(S)$ (resp. $\mathsf{min}_{\leq}(S)$ we call *maximal* (resp. *minimal*) *elements* in the set S. It is easy to see that if x is the greatest element in a set S, then $\mathsf{max}_{\leq}(S) = \{x\}$. Similarly, if x is the least element than $\mathsf{min}_{\leq}(S) = \{x\}$.

5. Bounded partial orders

Let Let $\mathfrak{X} = \langle X, \leq \rangle$ be a partially ordered set. By (4.3) and (4.12), for an arbitrary $x \in X$ the following equivalence holds: x is a greatest element in $\mathfrak{X}$ iff $x \ \mathsf{sup}_{\leq} \ X$ iff $x \ \mathsf{inf}_{\leq} \ \emptyset$. The greatest element in $\mathfrak{X}$ (if exists) we call the *unity* of $\mathfrak{X}$ and we signify it by "1". Obviously, there can only exist at most one such greatest element in $\mathfrak{X}$. If in $\mathfrak{X}$ there exists a unity then we write $\langle X, \leq, 1 \rangle$. It follows from the previous section that if in $\mathfrak{X}$ there exists a unity, then $\mathsf{max}_{\leq}(X) = \{1\}$.

Note that in the light of Lemma 4.1 we obtain:

LEMMA 5.1. *For any $x \in X$ the partially ordered set $\mathfrak{X}\restriction x$ has the unity x.*

By (4.8) and (4.14), for an arbitrary $x \in X$ the following equivalence holds: x is a least element in $\mathfrak{X}$ iff $x \ \mathsf{inf}_{\leq} \ X$ iff $x \ \mathsf{sup}_{\leq} \ \emptyset$. The least element in $\mathfrak{X}$ (if exists) we call the *zero* of $\mathfrak{X}$ and we signify it by "0". Obviously, there can only exist at most one such least element in $\mathfrak{X}$. If in $\mathfrak{X}$ there exists a zero then we write $\langle X, \leq, 0 \rangle$. It follows from the previous section that if in $\mathfrak{X}$ there exists a zero, then $\mathsf{min}_{\leq}(X) = \{0\}$.

We say that a partially ordered set $\mathfrak{X}$ is *bounded* iff $\mathfrak{X}$ has a zero and a unity. We will then write $\langle X, \leq, 0, 1 \rangle$. Obviously, if the structure $\mathfrak{X}$ is trivial, i.e., X has one element, then $0 = 1$.

Let $\mathfrak{X}_1 = \langle X_1, \leq_1, 0_1, 1_1 \rangle$ and $\mathfrak{X}_2 = \langle X_2, \leq_2, 0_2, 1_2 \rangle$ be any bounded partially ordered sets. A *homomorphism* from $\mathfrak{X}_1$ to $\mathfrak{X}_2$ is any order-homomorphism (see p. 259) such that $f(0_1) = 0_2$ and $f(1_1) = 1_2$.

6. Lattices

Lattices may be considered to be a certain kind of algebra, i.e., like sets with certain (primitive) operations of sum and product. They may also be treated as a kind of order, in which we can define those operations [cf. Grätzer, 1971, pp. 4–7]. It will be more convenient for us, as regards their application to mereology, to take the second approach.

A partial order $\langle L, \leq\rangle$ we call a *lattice* iff for arbitrary $x, y \in L$ the set $\{x, y\}$ has a supremum and an infimum, i.e.:

$$\forall_{x,y\in L}\exists_{z,u\in L}(z \,\mathsf{sup}_{\leq}\, \{x,y\} \wedge u \,\mathsf{inf}_{\leq}\, \{x,y\}). \tag{L}$$

Remark 6.1. (i) With respect to (4.4) and (4.9), condition (L) suffices only for the case where $x \neq y$.

(ii) To define a lattice, it is not essential that the relations $\mathsf{sup}_{\leq}$ and $\mathsf{inf}_{\leq}$ are interdefinable (cf. (4.10)) and (4.11)). Indeed, for a set $\{x, y\}$ at least one of the sets $\{z \in L : z \leq x \wedge z \leq y\}$ and $\{z \in L : x \leq z \wedge y \leq z\}$ might not be a one- or two-element set, and only in such cases do suprema and infima definitely exist (these sets may even be infinite). □

With respect to $\mathrm{U}_{\mathsf{sup}}$) and ($\mathrm{U}_{\mathsf{inf}}$ there exists only one supremum and only one infimum for the set $\{x, y\}$. We may therefore define on the Cartesian product $L \times L$ the binary operations *sum* $+$ and *product* $\cdot$ which take values in L. For arbitrary $x, y \in L$ we put:

$$x + y := (\iota\, z)\; z \,\mathsf{sup}_{\leq}\, \{x, y\}, \tag{df +}$$

$$x \cdot y := (\iota\, z)\; z \,\mathsf{inf}_{\leq}\, \{x, y\}. \tag{df $\cdot$}$$

We obtain directly from (4.6) and (4.13) the following for all $x, y, z \in L$:

$$z = x + y \iff z \,\mathsf{sup}_{\leq}\, \{u \in P : u \leq x \vee u \leq y\},$$

$$z = x \cdot y \iff z \,\mathsf{sup}_{\leq}\, \{u \in P : u \leq x \wedge u \leq y\}.$$

For arbitrary $x, y, z \in L$ the operations $+$ and $\cdot$ satisfy the following well-known conditions [cf. Grätzer, 1971; Traczyk, 1970]:

$$x + y = y + x \qquad\qquad x \cdot y = y \cdot x \tag{6.1}$$

$$x + (y + z) = (x + y) + z \qquad\qquad x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{6.2}$$

$$x + x = x \qquad\qquad x \cdot x = x \tag{6.3}$$

$$x + (x \cdot y) = x \qquad\qquad x \cdot (x + y) = x \tag{6.4}$$

Therefore, the operations $+$ and $\cdot$ are commutative, associative and idempotent. Furthermore, we may prove that for arbitrary $x, y, z \in L$:

$$x + (y \cdot z) \leq (x + y) \cdot (x + z), \tag{6.5}$$

$$(x \cdot y) + (x \cdot z) \leq x \cdot (y + z)\,, \tag{6.6}$$

$$x + y = y \iff x \leq y \iff x \cdot y = x\,, \tag{6.7}$$

$$x \leq x + y, \qquad\qquad x \cdot y \leq x\,, \tag{6.8}$$

$$x \leq y \implies x + z \leq y + z\,, \tag{6.9}$$

$$x \leq y \implies x \cdot z \leq y \cdot z\,, \tag{6.10}$$

$$x \leq z \wedge y \leq z \implies x + y \leq z\,. \tag{6.11}$$

It may be proven by induction [cf. Grätzer, 1971, p. 4] that the partially ordered set $\langle L, \leq\rangle$ is a lattice iff each non-empty finite subset of the set L has a supremum and an infimum, i.e.,:

$$\forall_{S\in\mathcal{P}(L)}\big(0 < \operatorname{Card} S < \aleph_0 \implies \exists_{z,u\in L}(z \;\mathsf{sup}_{\leq}\; S \wedge u \;\mathsf{inf}_{\leq}\; S)\big). \tag{L'}$$

To prove this it suffices to show that for each $n > 0$ and arbitrary $z, x_1, \ldots, x_n \in L$ the following hold:

$$z \;\mathsf{sup}_{\leq}\; \{x_1, \ldots, x_n\} \iff z = x_1 + \cdots + x_n,$$
$$z \;\mathsf{inf}_{\leq}\; \{x_1, \ldots, x_n\} \iff z = x_1 \cdot \;\cdots\; \cdot x_n.$$

In the light of (6.8), (6.11) and lemmas 4.1 and 5.1, for any lattice $\mathfrak{L} = \langle L, \leq\rangle$ we obtain:

LEMMA 6.1. *For any $x \in L$ the partially ordered set $\mathfrak{L}\restriction x$ is a lattice with the unity x. Moreover, the operations in $\mathfrak{L}\restriction x$ are the restriction of operations in $\mathfrak{L}$.*

If there exists a unity for the lattice $\mathfrak{L} = \langle L, \leq\rangle$ then for each $x \in L$ the following hold:

$$x + 1 = 1\,, \qquad\qquad x \cdot 1 = x\,. \tag{6.12}$$

Furthermore, for any $x \in L$ we have:

$$\forall_{z\in L}\, z + x = x \iff x = 1 \iff \forall_{z\in L}\, z \cdot x = z\,.$$

If there exists a zero for $\mathfrak{L}$, then for each $x \in L$ the following hold:

$$x + 0 = x, \qquad\qquad x \cdot 0 = 0\,. \tag{6.13}$$

Furthermore, for any $x \in L$ we have:

$$\forall_{z\in L}\, z + x = z \iff x = 0 \iff \forall_{z\in L}\, z \cdot x = x\,.$$

In any lattice $\langle L, \leq, 0\rangle$ with zero, for arbitrary $x, y \in L$ we have:

$$x \cdot y = 0 \iff \forall_{z\in L}(z \leq x \wedge z \leq y \Rightarrow z = 0). \tag{6.14}$$

From the fact that $x \cdot y = \mathrm{o}$ and that for some $z \neq \mathrm{o}$ we have $z \leq x$ and $z \leq y$, a contradiction follows: $\mathrm{o} \neq z = z \cdot z = z \cdot x \cdot z \cdot y = z \cdot x \cdot y = z \cdot \mathrm{o} = \mathrm{o}$. Conversely, if $x \cdot y \neq \mathrm{o}$ then — by virtue of (6.1) and (6.8) — x and y do not satisfy the right-hand side of the equality.

Moreover, the following holds:

$$\forall_{x,y \in L}\big(x \leq y \implies \forall_{u \in L}(u \leq x \wedge u \cdot y = \mathrm{o} \Rightarrow u = \mathrm{o})\big). \tag{6.15}$$

Assume that $x \leq y$, $u \leq x$, and $u \cdot y = \mathrm{o}$. Then, by ($\mathrm{t}_{\leq}$), we have $u \leq y$. Hence $u = u \cdot y = \mathrm{o}$.

We say that a lattice $\mathfrak{L}$ is *bounded* iff $\mathfrak{L}$ has a zero and a unity. We will then write $\langle L, \leq, \mathrm{o}, \mathrm{1} \rangle$.

For any bounded lattices $\mathfrak{L}_1 = \langle L_1, \leq_1, \mathrm{o}_1, \mathrm{1} \rangle$ and $\mathfrak{L}_2 = \langle L_2, \leq_2, \mathrm{o}_2, \mathrm{1}_2 \rangle$ we put $\leq\, := \,\leq_1 \times \leq_2$, i.e.: for arbitrary $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$:

$$\langle x_1, x_2 \rangle \leq \langle y_1, y_2 \rangle \;:\Longleftrightarrow x_1 \leq_1 y_1 \wedge x_2 \leq_2 y_2.$$

Lemma 6.2. *The pair* $\mathfrak{L}_1 \times \mathfrak{L}_2 := \langle L_1 \times L_2, \leq, \langle \mathrm{o}_1, \mathrm{o}_2 \rangle, \langle \mathrm{1}_1, \mathrm{1}_2 \rangle \rangle$ *is a bounded lattice, where we carry out the operations over the coordinates, which we call the* product of lattices $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$.

Let $\mathfrak{L}_1 = \langle L_1, \leq_1, \mathrm{o}_1, \mathrm{1}_1 \rangle$ and $\mathfrak{L}_2 = \langle L_2, \leq_2, \mathrm{o}_2, \mathrm{1}_2 \rangle$ be any bounded lattices and $h \colon L_1 \to L_2$ be any any order-homomorphism for bounded partially ordered sets (see p. 264). Then for all $x, y \in B_1$ we have: $h(-_1 x) = -_2 h(x)$, $h(x +_1 y) = h(x) +_2 h(y)$, and $h(x \cdot_1 y) = h(x) \cdot_2 h(y)$.

At the end of this section we will deal with so-called separative lattices. For these lattices we will prove two facts that we will use in Section 11.

We say that a lattice $\mathfrak{L} = \langle L, \leq, \mathrm{o} \rangle$ is *separative* iff $\mathfrak{L}$ satisfies the converse implication to (6.15), i.e., the following condition:

$$\forall_{x,y \in L}\big(\forall_{u \in L}(u \leq x \wedge u \cdot y = \mathrm{o} \Rightarrow u = \mathrm{o}) \implies x \leq y\big). \tag{sep}$$

Thus, in virtue of (6.15), we obtain:

Lemma 6.3. *In any separative lattice* $\langle L, \leq, \mathrm{o} \rangle$ *the following two (logically equivalent) conditions hold*:

$$\forall_{x,y \in L}\big(x \leq y \iff \forall_{u \in L}(u \leq x \wedge u \cdot y = \mathrm{o} \Rightarrow u = \mathrm{o})\big), \tag{6.16}$$

$$\forall_{x,y \in L}\big(x \not\leq y \iff \exists_{u \in L}(u \neq \mathrm{o} \wedge u \leq x \wedge u \cdot y = \mathrm{o})\big). \tag{6.17}$$

The first condition states that separative lattices have an interesting property which we will make use of in Section 11.

LEMMA 6.4. *In any separative lattice* $\langle L, \leq, \mathrm{o}\rangle$, *for arbitrary* $x, y \in L$ *the following condition holds*:

$$\exists_{S\in\mathcal{P}(L)}\big(\forall_{z\in S}\ z \leq y \wedge \forall_{u\in L}((u \leq x \wedge \forall_{z\in S}\ z\cdot u = \mathrm{o}) \Rightarrow u = \mathrm{o})\big) \implies x \leq y\,.$$

PROOF. Let $\langle L, \leq, \mathrm{o}\rangle$ be a separation lattice and $x, y \in L$. Assume that a set S_0 satisfies two conditions: (a) $\forall_{z\in S_0}\ z \leq y$ and (b) $\forall_{u\in L}(u \leq x \wedge \forall_{z\in S_0}\ z\cdot u = \mathrm{o}) \Rightarrow u = \mathrm{o}$. To take advantage of condition (sep), let us take any $u \in L$ such that (c) $u \leq x$ and (d) $u \cdot y = \mathrm{o}$. Then, by (a), (d), (6.1), and (6.10) we obtain: $\forall_{z\in S_0}\ z\cdot u \leq y \cdot u = \mathrm{o}$. Therefore, $u = \mathrm{o}$, by (b) and (c). So we obtain: $\forall_{u\in L}(u \leq x \wedge u\cdot y = \mathrm{o} \Rightarrow u = \mathrm{o})$. Hence $x \leq y$, by (sep). □

The second condition gives a sufficient condition for being a separative structure.

LEMMA 6.5. *Let* $\mathfrak{L} = \langle L, \leq, \mathrm{o}\rangle$ *be any lattice with zero in which the relation* $\leq$ *satisfies the following condition*:

for any $S \in \mathcal{P}(L)$ *there is exaxtly one* $x \in L$ *such that*
(a) $\forall_{z\in S}\ z \leq x$ *and* ($\star$)
(b) $\forall_{u\in L}\big(u \leq x \wedge \forall_{z\in S}\ u\cdot z = \mathrm{o} \implies u = \mathrm{o}\big)$.

Then $\mathfrak{L}$ *is a separative lattice.*

PROOF. Assume for a contradiction that condition ($\star$) holds in $\mathfrak{L}$ and (sep) does not. Therefore, for some $x_0, y_0 \in L$ we have: (A) $x_0 \nleq y_0$ and (B) $\forall_{z\in L}(z \leq x_0 \wedge z\cdot y_0 = \mathrm{o} \Rightarrow z = \mathrm{o})$.

We put $S_0 := \{z \in L : z \leq y_0\}$. Since $x_0\cdot y_0 \leq y_0$, then (i) $x_0\cdot y_0 \in S_0$.

To start, observe that y_0 satisfies conditions (a) and (b) from ($\star$) for the set S_0. Condition (a) is a logical tautology. *Ad* (b): for any $u \in L$, if $u \leq y_0$ and $\forall_{z\in S_0}\ u\cdot z = \mathrm{o}$, then both $u \in S_0$ and so $u = u\cdot u = \mathrm{o}$.

We can find a y_1 such that $y_1 \neq y_0$ and y_1 satisfies conditions (a) and (b) from ($\star$) for the set S_0. And that contradicts ($\star$).

Now note that, by ($\star$), for the set $S_1 := S_0 \cup \{x_0\}$ there is exactly one $y_1 \in L$ such that (ii) $\forall_{z\in S_0}\ z \leq y_1$, (iii) $x_0 \leq y_1$, and (iv) for any $u \in L$, if $u \leq y_1$ and $\forall_z(z \in S_0 \vee z = x_0 \Rightarrow u\cdot z = \mathrm{o})$, then $u = \mathrm{o}$.

From (A) and (iii) it follows that $y_0 \neq y_1$.

Finally, we show that the set S_0 satisfies conditions (a) and (b) from ($\star$). In fact, condition (a) we obtain by (ii). *Ad* (b): We take an arbitrary $u \in L$ such $u \leq y_1$ and $\forall_{z\in S_0}\ u\cdot z = \mathrm{o}$. From this and (i) we have (v) $u \cdot x_0 \cdot y_0 = 0$. Since $u \cdot x_0 \leq x_0$, then by (B) and (v) we have $u \cdot x_0 = 0$. Therefore, we may apply (iv) and obtain: $u = 0$. □

7. Distributive lattices

We say that a lattice $\mathfrak{L} = \langle L, \leq \rangle$ is *distributive* iff we may reverse inequalities (6.5) and (6.6). Therefore, by virtue of the antisymmetry of the relation $\leq$, the lattice $\mathfrak{L}$ is distributive iff the following conditions (equivalent in $\mathfrak{L}$) are satisfied:

$$\forall_{x,y,z \in L}\big(x + (y \cdot z) = (x + y) \cdot (x + z)\big), \tag{7.1}$$

$$\forall_{x,y,z \in L}\big(x \cdot (y + z) = (x \cdot y) + (x \cdot z)\big). \tag{7.2}$$

Furthermore, in any distributive lattice $\mathfrak{L}$ the following are true:

$$\forall_{S \in \mathcal{P}(L)} \forall_{x,y \in L}\big(y \ \mathsf{inf}_{\leq}\ S \implies (x + y) \ \mathsf{inf}_{\leq}\ \{x + z : z \in S\}\big), \tag{7.3}$$

$$\forall_{S \in \mathcal{P}(L)} \forall_{x,y \in L}\big(y \ \mathsf{sup}_{\leq}\ S \implies (x \cdot y) \ \mathsf{sup}_{\leq}\ \{x \cdot z : z \in S\}\big). \tag{7.4}$$

In each lattice each of conditions (7.1)–(7.4) entails the other three.

Thus, in any distributive lattice $\langle L, \leq, 0 \rangle$ with zero we obtain:

$$\forall_{S \in \mathcal{P}(L)} \forall_{x,y \in L}\big(y \ \mathsf{sup}_{\leq}\ S \wedge x \leq y \wedge \forall_{z \in S}\, x \cdot z = 0 \implies x = 0\big). \tag{7.5}$$

Suppose that $x \leq y \ \mathsf{sup}_{\leq}\ S$ and for any $z \in S$ we have $z \cdot x = 0$. Then, by (6.7) and (7.4), we have: $x = x \cdot y$ and $x \ \mathsf{sup}_{\leq}\ \{x \cdot z : z \in S\}$. Therefore, $x \ \mathsf{sup}_{\leq}\ \{0\}$, and so $x = 0$.

8. Complements in bounded distributive lattices

Let $\langle L, \leq, 0, 1 \rangle$ be a bounded lattice and $x, y \in L$. Then we say that y is a *complement* of x iff both $x \cdot y = 0$ and $x + y = 1$.

A *complemented lattice* is any bounded lattice $\mathfrak{L} = \langle L, \leq, 0, 1 \rangle$ in which every member has a complement, i.e., $\mathfrak{L}$ satisfies the following condition:

$$\forall_{x \in L} \exists_{y \in L}\big(x \cdot y = 0 \wedge x + y = 1\big). \tag{c}$$

In general, there are bounded lattices in which some elements may have more than one complement. However, in any bounded distributive lattice every member has at most one complement. Indeed, we have:

Lemma 8.1. *In any bounded distributive lattice* $\mathfrak{L} = \langle L, \leq, 0, 1\rangle$.

(i) *For arbitrary* $x, y, z \in L$, *if* y *is a complement of* x *and* $x \cdot z = 0$, *then* $z \leq y$.
(ii) *Every member of* L *has at most one complement.*

Proof. *Ad* (i): Suppose that y is a complement of x and $x \cdot z = 0$. Then $z = z \cdot 1 = z \cdot (x + y) = (z \cdot x) + (z \cdot y) = 0 + (z \cdot y) = z \cdot y$. Hence $z \leq y$, by (6.7).

Ad (ii): Suppose that y and z are complements of x. Then $x \cdot y = 0 = x \cdot z$. Hence, by (i), we have $z \leq y$ and $y \leq z$. So $y = z$, by ($\mathrm{antis}_{\leq}$). □

A *uniquely complemented lattice* $\mathfrak{L}$ is any bounded lattice in which every member has exactly one complement, i.e., $\mathfrak{L}$ satisfies the following condition:

$$\forall_{x \in L} \exists!_{y \in L} (x \cdot y = 0 \wedge x + y = 1). \tag{c!}$$

Thus, in any uniquely complemented lattice we can define on the set L a unary *complement operation* $-\colon L \to L$. For any $x \in L$ we put:

$$-x := (\iota y)\,(x \cdot y = 0 \wedge x + y = 1). \tag{df $-$}$$

Notice that, in the light of Lemma 8.1, all distributive complemented lattices (i.e., *Boolean lattices*; see Section 9) are uniquely complemented lattices.

At the end of this section, we prove two lemmas that will be needed in sections 9 and 11.

Lemma 8.2. *If* $\mathfrak{L} = \langle L, \leq, 0, 1\rangle$ *a bounded distributive lattice and* y *is a complement of* x, *then the following equivalent conditions hold*:

(a) y *is the greatest element of* $\{z \in L : x \cdot z = 0\}$,
(b) $x \cdot y = 0$ *and* $y \ \mathsf{sup}_{\leq} \{z \in L : x \cdot z = 0\}$.

Proof. If y is a complement of x, then $x \cdot y = 0$ and—in the light of Lemma 8.1(i)—for arbitrary $z \in L$: if $x \cdot z = 0$ then $z \leq y$. The equivalence of (a) and (b) we will get using (4.2). □

Lemma 8.3. *Let* $\mathfrak{L} = \langle L, \leq, 0, 1\rangle$ *be any bounded distributive and separative lattice. Then for arbitrary* $x, y \in L$ *we have*:

$$y \text{ is a complement of } x \iff y \ \mathsf{sup}_{\leq} \{z \in L : x \cdot z = 0\}.$$

Proof. '⇒' By virtue of Lemma 8.2.

'⇐' Let y $\mathsf{sup}_{\leq}$ $\{z \in L : x \cdot z = 0\}$. Since $\mathfrak{L}$ is distributive, then — by (7.4) — we have: $(x \cdot y)$ $\mathsf{sup}_{\leq}$ $\{x \cdot z \in L : x \cdot z = 0\}$. Therefore, $(x \cdot y)$ $\mathsf{sup}_{\leq}$ $\{0\}$. That is, $x \cdot y = 0$. Now take an arbitrary z such that $z{\cdot}(x{+}y) = 0$. Since $\mathfrak{L}$ is distributive, then $z{\cdot}(x{+}y) = (z{\cdot}x){+}(z{\cdot}y) = 0$, by (7.2). Hence, by (6.8), we have (A) $z{\cdot}x = 0$ and (B) $z{\cdot}y = 0$. From (A) — since y $\mathsf{sup}_{\leq}$ $\{z \in L : x \cdot z = 0\}$ — we have $z \leq y$. Hence $z = z \cdot y = 0$, by virtue of (B). Applying condition (sep) to 1 and $x+y$, we get: $1 \leq x+y$. Therefore, $x + y = 1$. □

9. Boolean lattices (Boolean algebras)

As we mentioned earlier, any complemented distributive lattice is called a *Boolean lattice*. We also remember that all Boolean lattices are uniquely complemented lattices. Let **BL** be a class of all Boolean lattices.

Let $\langle B, \leq, 0, 1\rangle$ be a Boolean lattice and $x \in B$. Since there is a $y \in L$ such that y is a complement of x, then — in virtue of Lemma 8.3 — there is a unique least upper bound of the set $\{z \in L : x \cdot z = 0\}$ and

$$-x = \sup_{\leq}\{z \in B : x \cdot z = 0\}.$$

In any Boolean lattice $\langle B, \leq, 0, 1\rangle$ the following conditions are well-known. For arbitrary $x, y \in B$ we have:

$$-(x + y) = (-x) \cdot (-y) \qquad -(x \cdot y) = (-x) + (-y) \tag{9.1}$$

$$x + (-x) = 1 \qquad x \cdot (-x) = 0 \tag{9.2}$$

$$-1 = 0 \qquad x \cdot (-x) = 0 \tag{9.3}$$

$$--x = x \tag{9.4}$$

$$x \leq y \iff -y \leq -x \tag{9.5}$$

$$(-x) + y = 1 \iff x \leq y \iff x \cdot (-y) = 0\,, \tag{9.6}$$

Furthermore, we obtain:

Lemma 9.1. *All Boolean lattices are separative. So all Boolean lattices satisfy both* (6.16) *and* (6.17).

Proof. Assume the antecedent of (sep). For $u := x \cdot -y$ we have both $x \cdot -y \leq x$ and $x \cdot -y \cdot y = x \cdot 0 = 0$. Hence $x \cdot -y = 0$. Therefore $x \leq y$, by (9.6). (The quantifier "∃" in (6.17) can refer, for example, to $u := x \cdot -y$.) □

For any $x, y \in M$ we also put:

$$x - y := x \cdot -y\,.$$

Of course, if $x \leq y$ then $x - y = \mathrm{o}$. The object $x - y$ can be treated as the *difference* of x and y, or the *relative complement of* y *with respect to* x. For any elements $x, y \in B$, the symmetric difference of x and y is

$$x \triangle y = (x - y) + (y - x).$$

Remark 9.1. (i) Let A be a non-empty set, $\mathbf{+}$ and $\ast$ be binary operations in A, $\mathbf{-}$ be a unary operation in A, and $\mathsf{0}$ and $\mathsf{1}$ be elements of A. Then an algebraic structure $\langle A, \mathbf{+}, \ast, \mathbf{-}, \mathsf{0}, \mathsf{1}\rangle$ is a *Boolean algebra* iff its components satisfy the equalities (6.1), (6.2), (6.12), (6.13), (9.2) and (7.1) (resp. (7.2)), if the symbols "$+$", "$\cdot$", "$-$", "o", and "$\mathrm{ı}$" are replaced by the symbols "$\mathbf{+}$", "$\ast$", "$\mathbf{-}$", "$\mathsf{0}$", and "$\mathsf{1}$", respectively. It is enough to have all equalities listed in sections 6–9 fulfilled.

(ii) In this appendix we show that if $\langle B, \leq, \mathrm{o}, \mathrm{ı}\rangle$ is a Boolean lattice then the algebraic structure $\langle B, +, \cdot, -, \mathrm{o}, \mathrm{ı}\rangle$ is a Boolean algebra.

(iii) If $\langle A, \mathbf{+}, \ast, \mathbf{-}, \mathsf{0}, \mathsf{1}\rangle$ is a Boolean algebra and a binary relation $\leq$ is defined in A by: $x \leq y \iff x \mathbf{+} y = y$, then the structure $\langle A, \leq, \mathsf{0}, \mathsf{1}\rangle$ is a Boolean lattice. Indeed, the relation $\leq$ partially orders the set A and satisfies conditions $\sup_{\leq}\{x, y\} = x + y$ and $\inf_{\leq}\{x, y\} = x \cdot y$ [see, e.g., Frankiewicz and Zbierski, 1992, pp. 10–11]. □

Example 9.1. For any field of sets $\mathcal{F}$,

- the structure $\mathfrak{B}_{\mathcal{F}} := \langle \mathcal{F}, \subseteq, \emptyset, \bigcup \mathcal{F}\rangle$ is a Boolean lattice;
- for all $S, Z \in \mathcal{F}$ we have: $S + Z = S \cup Z$, $S \cdot Z = S \cap Z$, and $-S = \bigcup \mathcal{F} \setminus S$.
- the algebraic structure $\langle \mathcal{F}, \cup, \cap, -, \emptyset, \bigcup \mathcal{F}\rangle$ is a Boolean algebra, where for arbitrary $S, Z \in \mathcal{F}$: $S \subseteq Z$ iff $S \cup Z = Z$ iff $S \cap Z = S$.

Thus, for any algebra of sets $\mathcal{A}$ over a set X,

- the structure $\mathfrak{B}_{\mathcal{A}} := \langle \mathcal{A}, \subseteq, \emptyset, X\rangle$ is a Boolean lattice;
- the algebraic structure $\langle \mathcal{A}, \cup, \cap, -, \emptyset, X\rangle$ is a Boolean algebra, where for arbitrary $S, Z \in \mathcal{A}$: $S \subseteq Z$ iff $S \cup Z = Z$ iff $S \cap Z = S$.

So also for any set X,

- the structures $\mathfrak{P}_X := \langle \mathcal{P}(X), \subseteq, \emptyset, X\rangle$ and $\mathfrak{FC}_X := \langle \mathcal{FC}(X), \subseteq, \emptyset, X\rangle$ are Boolean lattices. □

Example 9.2 (Frankiewicz and Zbierski, 1992, p. 27). Let $\mathcal{A}$ be an algebra of sets over a set X such that all singletons of X belong to $\mathcal{A}$. Then for any subfamily $\mathcal{S}$ of $\mathcal{A}$ and any $Y \in \mathcal{A}$:

- if $Y \ \mathsf{sup}_{\subseteq} \ \mathcal{S}$ in the Boolean lattice $\mathfrak{B}_{\mathcal{A}}$, then $Y = \bigcup \mathcal{S}$. □

Remark 9.2. There is a family $\mathcal{F}$ of subsets of a set X such that $\langle \mathcal{F}, \subseteq, \emptyset, X \rangle$ is a Boolean lattice, but $\mathcal{F}$ is not an algebra of set over X.

(i) For example, let $\mathcal{F}$ be the family of sets composed of: $\emptyset$, $\{0,1\}$, $\{1,2\}$, $X := \{0,1,2,3\}$. Then $\langle \mathcal{F}, \subseteq, \emptyset, X \rangle$ is a Boolean lattice, but it is not an algebra of set over X. Clearly, we have $\{0,1\} + \{1,2\} = X$, $\{0,1\} \cdot \{1,2\} = \emptyset$, and $\{0,1\} = -\{1,2\}$.

(ii) We will also give another, more interesting example. Let $\mathscr{T} = \langle X, \mathscr{O} \rangle$ a topological space, where X is a non-empty set and $\mathscr{O}$ is a family of all open subsets of X in $\mathscr{T}$. A subset U of X is a *regular open set* of $\mathscr{T}$ iff $U = \operatorname{Int} \operatorname{Cl} U$, where Int and Cl are standard interior and closure operations of $\mathscr{T}$.

Let $\mathrm{r}\mathscr{O}$ be the family of regular open sets of $\mathscr{T}$. Of course, $\bigcup \mathrm{r}\mathscr{O} = X$, since $X \in \mathrm{r}\mathscr{O}$. It is known that the structure $\langle \mathrm{r}\mathscr{O}, \subseteq, \emptyset, X \rangle$ is a Boolean lattice such that for any $U, V \in \mathrm{r}\mathscr{O}$ we have: $U + V = \operatorname{Int} \operatorname{Cl}(U \cup V)$, $U \cdot V = U \cap V$, $-U = \operatorname{Int}(X \setminus U)$, and $U \subseteq V$ iff $\operatorname{Cl}(U) \subseteq \operatorname{Cl}(V)$ [see, e.g., Koppelberg, 1998a, p. 26, and Frankiewicz and Zbierski, 1992, pp. 13–14].

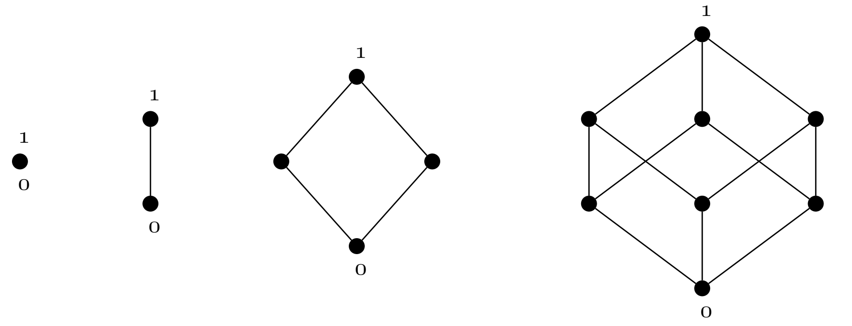
Notice that the family $\mathrm{r}\mathscr{O}$ may not be an algebra of sets over X, since there may be $U \in \mathrm{r}\mathscr{O}$ such that $X \setminus U \notin \mathrm{r}\mathscr{O}$. For example, let X be the set $\mathbb{R}$ of real numbers and $\mathscr{O}_{\mathbb{R}}$ be the 'natural' topology in $\mathbb{R}$ determined by the metric $\rho(t_1, t_2) := |t_1 - t_2|$, for all $t_1, t_2 \in \mathbb{R}$. Then the open interval $(0,1)$ belongs to the family $\mathrm{r}\mathscr{O}_{\mathbb{R}}$ of all regular open subset of $\mathscr{O}_{\mathbb{R}}$, but $\mathbb{R} \setminus (0,1)$ do not belong to $\mathscr{O}_{\mathbb{R}}$. □

The universe of a finite Boolean lattice has 2^n elements, for $n \in \mathbb{N}$ (see p. 282). For $n = 0, 1, 2, 3$ we have only four (up to isomorphism) lattices. These are represented by the diagrams in Model 18.[6]

In the light of Lemma 6.2, for arbitrary Boolean lattices $\mathfrak{B}_1 = \langle B_1, \leq_1, \mathrm{o}_1, \mathrm{1}_1 \rangle$ and $\mathfrak{B}_2 = \langle B_2, \leq_2, \mathrm{o}_2, \mathrm{1}_2 \rangle$ we have:

Lemma 9.2. *The product $\mathfrak{B}_1 \times \mathfrak{B}_2$ is Boolean lattice, in which for arbitrary $x \in L_1$ and $y \in L_2$, $-\langle x, y \rangle = \langle -x, -y \rangle$.*

[6] In these diagrams "•" says that a given element is in the relation $\leq$ with itself (the relation $\leq$ is reflexive). An upwards line leading from x to y says that $x \lneq y$ (the relation $\leq$ is transitive and antisymmetric). Otherwise, $x \nleq y$.



Model 18. Examples of finite Boolean lattices

In the light of Lemma 6.1, for any Boolean lattice $\langle B, \leq, \mathrm{o}, \mathrm{1}\rangle$ we obtain [cf. Koppelberg, 1998b, p. 39, or Frankiewicz and Zbierski, 1992, pp. 16 and 30]:

LEMMA 9.3. *For any $x \in B$:*

(i) *The lattice $\mathfrak{B}{\restriction}x$ is a Boolean lattice with the zero* o *and the unity x.*
(ii) *Each $y \in B$ in $\mathfrak{B} \restriction x$ has $x - y$ as a complement.*
(iii) *The "projection" map $\mathrm{p}_x \colon B \to B{\restriction}x$, defined by condition $\mathrm{p}_x(y) := y \cdot x$, is a homomorphism from $\mathfrak{B}$ onto the lattice $\mathfrak{B} \restriction x$.*
(iv) *The function $y \mapsto \langle y \cdot x, y \cdot -x\rangle$ is an isomorphism from $\mathfrak{B}$ onto the product $\mathfrak{B} \restriction x \times \mathfrak{B} \restriction -x$.*

The Boolean lattice $\mathfrak{B}{\restriction}x$ is called the *relative lattice* or *factor lattice* of $\mathfrak{B}$ with respect to x.

10. Complete lattices

We say that a partially ordered set $\mathfrak{X} = \langle X, \leq\rangle$ is *complete* iff each subset of the set X has a supremum, i.e., $\mathfrak{X}$ satisfies the following condition:

$$\forall_{S \in \mathcal{P}(X)} \exists_{x \in X}\ x \ \mathsf{sup}_{\leq}\ S\,. \tag{cL}$$

By (4.10) and (4.11), condition (cL) is equivalent to the following:

$$\forall_{S \in \mathcal{P}(X)} \exists_{x \in X}\ x \ \mathsf{inf}_{\leq}\ S\,. \tag{cL$'$}$$

Each of conditions (cL) and (cL′) entails both conditions (L) and (L′). Therefore, every complete partially ordered set is a lattice. So all complete partially ordered sets we will call *complete lattices*. Of course, a lattice $\mathfrak{L}$ is complete iff $\mathfrak{L}$ is complete as a partially ordered set. Note that, in the light of (L'), each finite lattice is complete.

Every complete lattice $\mathfrak{L} = \langle L, \leq \rangle$ is bounded; we put:

$$\mathrm{o} := (\iota\, x)\, x \;\mathsf{inf}_{\leq}\; L\,,$$
$$\mathrm{1} := (\iota\, x)\, x \;\mathsf{sup}_{\leq}\; L\,.$$

With respect to (cL), (cL'), ($\mathrm{U}_{\mathsf{sup}}$) and ($\mathrm{U}_{\mathsf{inf}}$) in any complete lattice $\langle L, \leq \rangle$ we may define on the set $\mathcal{P}(L)$ the *supremum function* and the *infimum function* which take values in the set L:

$$\sup S := (\iota\, x)\, x \;\mathsf{sup}_{\leq}\; S\,, \tag{df sup}$$
$$\inf S := (\iota\, x)\, x \;\mathsf{inf}_{\leq}\; S\,. \tag{df inf}$$

In any complete lattice $\mathfrak{L}$ conditions (6.5) and (6.6) may be generalised. For arbitrary $x \in L$ and $S \in \mathcal{P}(L)$ we have:

$$x + \inf S \leq \inf\{x + z : z \in S\}, \tag{10.1}$$
$$\sup\{x \cdot z : z \in S\} \leq x \cdot \sup S\,. \tag{10.2}$$

Let us prove (10.2). From the definition of a supremum we have $\forall_{z \in S}\, z \leq \sup S$. Hence, by virtue of (6.10), we have $\forall_{z \in S}(x \cdot z \leq x \cdot \sup S)$. Therefore, $x \cdot \sup S$ is an upper bound of the set $\{x \cdot z : z \in S\}$. From this and the definition of a supremum we obtain the desired inequality. By making use of the definition of an infimum and (6.9) we may similarly prove inequality (10.1).

In a complete lattice $\mathfrak{L}$ distributivity conditions (7.3) and (7.4) may be written as equalities having the following form:

$$\forall_{S \in \mathcal{P}(L)} \forall_{x \in L}\; x + \inf S = \inf\{x + z : z \in S\}, \tag{10.3}$$
$$\forall_{S \in \mathcal{P}(L)} \forall_{x \in L}\; x \cdot \sup S = \sup\{x \cdot z : z \in S\}. \tag{10.4}$$

It is well known that in any complete lattice, any of conditions (7.1), (7.2), (10.3) and (10.4) entails the other three.

Lemma 10.1. *Let $\mathfrak{L}_1 = \langle B_1, \leq_1 \rangle$ and $\mathfrak{L}_2 = \langle L_2, \leq_2, \mathrm{o}_2, \mathrm{1}_2 \rangle$ be lattices. Then the product $\mathfrak{L}_1 \times \mathfrak{L}_2$* (see Lemma 6.2) *is complete iff both lattices $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are complete.*

Proof. '$\Rightarrow$' For any $S_1 \in \mathcal{P}(L_1)$ we put $S^* := S_1 \times \{\mathrm{o}_2\}$, where $\mathrm{o}_2 := \inf_{\leq_2} L_2$. By virtue of our assumption, there are $x_1 \in L_1$ and $x_2 \in L_2$ such that $\langle x_1, x_2 \rangle = \sup_{\leq} S^*$. Obviously, $x_2 = \mathrm{o}_2$ and $x_1 = \sup_{\leq_1} S_1$. We proceed similarly for any $S_2 \in \mathcal{P}(L_2)$.

'$\Leftarrow$' For any $S \in \mathcal{P}(L_1 \times L_2)$ let $S_1^* := \{z_1 \in L_1 : \exists_{z_2 \in L_2} \langle z_1, z_2\rangle \in S\}$ and $S_2^* := \{z_2 \in L_2 : \exists_{z_1 \in L_1} \langle z_1, z_2\rangle \in S\}$. By virtue of our assumption, there are $x_1 \in L_1$ and $x_2 \in L_2$ such that $x_1 = \sup_{\leq_1} S_1$ and $x_2 = \sup_{\leq_2} S_2$. It is easy to see that $\langle x_1, x_2\rangle = \sup_{\leq} S$. □

11. Complete Boolean lattices

Let **CBL** be the class of all complete Boolean lattices. Since each finite lattice is complete, so $\emptyset \neq$ **CBL** $\subseteq$ **BL**.

Example 11.1. (i) For any complete field $\mathcal{F}$, the Boolean lattice $\mathfrak{B}_{\mathcal{F}}$ is complete (see Example 9.1) and for any subfamily $\mathcal{S}$ of $\mathcal{F}$ we have: $\sup_{\subseteq} \mathcal{S} = \bigcup \mathcal{S}$ and $\inf_{\subseteq} \mathcal{S} = \bigcap \mathcal{S}$ (see also examples 4.1, 4.2, and 4.4).

(ii) For any set X, the Boolean lattice $\mathfrak{P}_X$ is complete (see examples 1.3, 4.3, and 4.5). □

Example 11.2. For any topological space $\mathscr{T} = \langle X, \mathscr{O}\rangle$, the Boolean lattice $\langle \mathrm{r}\mathscr{O}, \subseteq, \emptyset, X\rangle$ is complete (see Remark 9.2) such that for any subfamily $\mathscr{U}$ of $\mathrm{r}\mathscr{O}$ we have $\sup_{\subseteq} \mathscr{U} = \mathrm{Int}\,\mathrm{Cl} \bigcup \mathscr{U}$ and $\inf_{\subseteq} \mathscr{U} = \mathrm{Int}\,\mathrm{Cl} \bigcap \mathscr{U}$ [see, e.g., Koppelberg, 1998a, p. 26, and Frankiewicz and Zbierski, 1992, pp. 13–14]. □

However, we have **CBL** $\subsetneq$ **BL**, because we have the following examples of Boolean lattices which are incomplete.

Example 11.3. Let $\mathcal{A}$ be any algebra of sets over X such that all singletons of X belong to $\mathcal{A}$. Then in the Boolean lattice $\mathfrak{B}_{\mathcal{A}}$ (cf. Example 9.1), for any subfamily $\mathcal{S}$ of $\mathcal{F}$, in the light of Example 9.2, we have:

- if $\bigcup \mathcal{S} \notin \mathcal{A}$, then $\mathcal{S}$ does not have a supremum in $\mathfrak{B}_{\mathcal{A}}$.

Indeed, in virtue of Example 9.2, if $\bigcup \mathcal{S} \notin \mathcal{A}$, i.e., there is no $Y \in \mathcal{A}$ such that $Y \neq \bigcup \mathcal{S}$, then there is no $Y \in \mathcal{A}$ such that $Y \sup_{\subseteq} \mathcal{S}$.

Thus, if $\mathcal{A}$ is incomplete (as an algebra of sets), then the Boolean latice $\mathfrak{B}_{\mathcal{A}}$ is incomplete. □

Example 11.4. (i) In Example 1.4 we show that the algebra of sets $\mathcal{FC}(\mathbb{N})$ over the set $\mathbb{N}$ is not complete (as an algebra of sets). Since $\mathcal{FC}(\mathbb{N})$ contains all singletons of $\mathbb{N}$, so — in the light of examples 9.1 and 11.3 — the Boolean lattice $\mathfrak{FC}_{\mathbb{N}}$ is not complete.

(ii) Also incomplete is the Boolean lattice which we obtain from the field of Borel subsets of the topological space of real numbers (see Remark 9.2) [cf., e.g., Traczyk, 1970, p. 55]. □

From lemmas 9.2 and 10.1 we obtain:

LEMMA 11.1. *Let $\mathfrak{B}_1 = \langle B_1, \leq_1, 0_1, 1 \rangle$ and $\mathfrak{B}_2 = \langle B_2, \leq_2, 0_2, 1_2 \rangle$ be Boolean lattices. Then the Boolean lattice $\mathfrak{B}_1 \times \mathfrak{B}_2$ is complete iff both lattices $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are complete.*

In [1935], Tarski studied the equivalence of various axiomatisations of complete Boolean algebras (complete Boolean lattices; cf. Remark 9.1). We will present Tarski's conclusion in the next section (as Theorem 12.1). Theorem 11.2 is also associated with Tarski, which we will present and prove in a moment. It gives a necessary and sufficient condition for lattices with zero to be complete Boolean lattices. To prove Theorem 11.2 we will not need to use Theorem 12.1. We need only draw upon the results we have established already.

THEOREM 11.2. *Let $\mathfrak{L} = \langle L, \leq, 0 \rangle$ be a lattice with zero. Then for $\mathfrak{L}$ to be a complete Boolean lattice, it is both necessary and sufficient that the relation $\leq$ satisfies condition $(\star)$ from Lemma 6.5.*

PROOF. '$\Rightarrow$' Let $\mathfrak{L}$ be a complete Boolean lattice. Take an arbitrary $S \in \mathcal{P}(L)$ and put $x := \sup_{\leq} S$. From the definition of supremum and by (7.5), x satisfies conditions (a) and (b) from $(\star)$ for the set S.

Finally, we will show that x is the only element from L satisfying conditions (a) and (b) from $(\star)$ for the set S. Assume that $x' \in L$ also satisfies conditions (a) and (b) from $(\star)$ for the set S. Since $\mathfrak{L}$ is a Boolean lattice, the sentence (sep) holds in it (cf. (6.16)). Therefore, from Lemma 6.4 we get $x \leq x'$ and $x' \leq x$. Hence $x = x'$.

'$\Leftarrow$' Let $\mathfrak{L}$ be a lattice with zero, in which $(\star)$ holds. By Lemma 6.5, $\mathfrak{L}$ is separative, i.e., the condition (sep) holds in $\mathfrak{L}$.

First, we prove that $\mathfrak{L}$ is a complete lattice. Take any $S \in \mathcal{P}(L)$. Since $(\star)$ holds in $\mathfrak{L}$, then there exists exactly one x_0 that satisfies conditions (a) and (b) from $(\star)$ for the set S. So, by (a), x_0 is an upper bound of S. Moreover, if $y \in L$ is an upper bound of S, then — by (b) for x_0 and S and Lemma 6.4 — we have $x_0 \leq y$. This shows that x_0 is the least upper bound of S, i.e., that $x_0 = \sup_{\leq} S$. Moreover, it follows from this that $\sup S$ is the only element in L which satisfies condition (b) from $(\star)$:

$$\forall_{y \in L} \big(y \leq \sup\nolimits_{\leq} S \wedge \forall_{z \in S}\ y \cdot z = 0 \implies y = 0\big). \tag{b$'$}$$

It follows from the arbitrary choice of S that lattice $\mathfrak{L}$ is complete. Therefore $\mathfrak{L}$ is also bounded: $1 = \sup_{\leq} L$.

We prove now the distributivity condition (10.4). Let $x \in L$ and $S \in \mathcal{P}(L)$. By (10.2) we have $\sup_{\leq}\{x \cdot z : z \in S\} \leq x \cdot \sup_{\leq} S$. It therefore suffices to show the inverse inequality.

We put $x_S := \sup_{\leq}\{x \cdot z : z \in S\}$. It follows directly from the definition of a supremum that (A) $\forall_{u \in S}\, x \cdot u \leq x_S$.

Assume for a contradiction that $x \cdot \sup S \not\leq x_S$. Hence, by virtue of (sep), there exists a v such that (B) $v \neq \mathrm{o}$, (C) $v \leq x \cdot \sup_{\leq} S$, and (D) $v \cdot x_S = \mathrm{o}$.

From (A), by virtue of (6.10), we get (i) $\forall_{u \in S}\, v \cdot x \cdot u \leq v \cdot x_S$. From this and (D) we get (ii) $\forall_{u \in S}\, v \cdot x \cdot u = \mathrm{o}$. From (C) it follows that $v = v \cdot x \cdot \sup S$. Hence, $x \cdot v = x \cdot v \cdot x \cdot \sup S = x \cdot v \cdot \sup S$. Hence we have (iii) $x \cdot v \leq \sup_{\leq} S$. By applying (b′), (ii), (iii) and (C) we get $\mathrm{o} = x \cdot v = v$. And this contradicts (B). □

12. Tarski's theorem

The result Tarski obtained in [1935] concerns the equivalence of different systems of axioms of complete Boolean algebras and may be presented — in the terminology we have been using in this appendix — in the following form:

THEOREM 12.1 (Tarski, 1935, theorems 1 and 2). *Let X be any non-empty set and $<$ be any transitive binary relation in X. Then for the relation $<$ to be reflexive and antisymmetric, and the partially ordered set $\langle X, < \rangle$ to be a complete Boolean lattice, it is both necessary and sufficient that the relation $<$ satisfies the following condition*:

for any $S \in \mathcal{P}(X)$ there exists exactly one $x \in X$ such that

$$\begin{array}{ll}\text{(a)} & \forall_{z \in S}\; z < x \text{ and} \\ \text{(b)} & \forall_{y \in X}\big(y < x \wedge \forall_{z \in S}\forall_{u \in X}\big(u < y \wedge u < z \Rightarrow \forall_{v \in X}\; u < v\big) \\ & \quad \Longrightarrow\; \forall_{v \in X}\; y < v\big).\end{array} \tag{$\bigstar$}$$

Remark 12.1. Condition ($\bigstar$) has a rather complicated form. This is caused by the fact that it has been formulated just with the relation $<$ (whose transitivity we established earlier). Were we to put the lattice $\langle L, \leq, \mathrm{o} \rangle$ with zero in place of the pair $\langle X, < \rangle$, then condition ($\bigstar$) would reduce to condition ($\star$) from Theorem 11.2.[7] To start, observe that, in

[7] Theorem 12.1 says indeed that the pair $\langle X, < \rangle$, in which the relation $<$ is transitive and in which condition ($\bigstar$) holds, is a bounded lattice.

the modified version(★), point (a) is simply point (a) from (⋆). Observe next that the subformulae "$\forall_{u\in L}\ u \leq v$" and "$\forall_{v\in L}\ y \leq v$" are the only definitions of a zero in the lattice $\langle L, \leq, \mathrm{o}\rangle$ and so therefore say respectively no more than the formulae "$u = \mathrm{o}$" and "$y = \mathrm{o}$". Thanks to this, the subformula "$\forall_{u\in L}(u \leq y \wedge u \leq z \Rightarrow \forall_{v\in L}\ u \leq v)$" takes the form "$\forall_{u\in L}(u \leq y \wedge u \leq z \ \Rightarrow\ u = \mathrm{o})$". This may be simplified to "$y \cdot z = \mathrm{o}$" thanks to condition (6.14) which holds in every lattice with zero [cf. Tarski, 1956c, p. 324]. Therefore point (b) from (★) for lattices with zero reduces to point (b) from (⋆). □

From the above remark and Theorem 12.1 follows Theorem 11.2 in an obvious way.

13. Atoms and atomic elements in lattices with zero

Let $\mathfrak{L} = \langle L, \leq, \mathrm{o}\rangle$ be any lattice with zero. For convenience, if $y \leq x$, then either we say that y is contained in x or we say that x includes y. If $y \lneq x$ then we say that y is less than x.

All minimal elements of the set $L \setminus \{\mathrm{o}\}$ we call *atoms* in $\mathfrak{B}$. That is, x is an atom in $\mathfrak{L}$ iff $x \in \mathsf{min}_{\leq}(L \setminus \{\mathrm{o}\})$ By, (df $\mathsf{min}_{\leq}$), for any $x \in L$:

$$\begin{aligned} x \text{ is an atom in } \mathfrak{L} &:\Longleftrightarrow x \neq \mathrm{o} \wedge \neg\exists_{z\in L\setminus\{\mathrm{o}\}}\ z \lneq x \\ &\Longleftrightarrow x \neq \mathrm{o} \wedge \forall_{z\in L\setminus\{\mathrm{o}\}}(z \leq x \Rightarrow z = x) \qquad (\mathrm{df\,At}(\mathfrak{L})) \\ &\Longleftrightarrow x \neq \mathrm{o} \wedge \forall_{z\in L}(z \lneq x \Rightarrow z = \mathrm{o}). \end{aligned}$$

In other words, x is an atom in $\mathfrak{L}$ iff x is not the zero and o is the only element in L less than x. Let $\mathrm{At}(\mathfrak{L})$ be the set of all atoms in $\mathfrak{L}$.

Atoms in $\mathfrak{L}$ have the following properties:

$$x \in \mathrm{At}(\mathfrak{L}) \Longleftrightarrow x \neq \mathrm{o} \wedge \forall_{y\in L}(x \leq y \vee x \cdot y = \mathrm{o}), \tag{13.1}$$

$$x \in \mathrm{At}(\mathfrak{L}) \Longrightarrow \forall_{y,z\in L}(x \leq y + z \ \Leftrightarrow\ x \leq y \vee x \leq z). \tag{13.2}$$

For (13.1): By (df $\mathrm{At}(\mathfrak{L})$), if $x \in \mathrm{At}(\mathfrak{L})$, then both $x \neq \mathrm{o}$ and, since $x \cdot y \leq x$, so if $x \cdot y \neq \mathrm{o}$ then $x \cdot y = x$. Conversely, let x satisfy the right-hand side and take an arbitrary $z \in L$ such that $z \lneq x$. Then $x \nleq z$. Hence $x \cdot z = \mathrm{o}$. Therefore, $z = x \cdot z = \mathrm{o}$. For (13.2): Let $x \in \mathrm{At}(\mathfrak{L})$. Then $x \neq \mathrm{o}$. Suppose that $x \leq y + z$. Then $\mathrm{o} \neq x = x \cdot (y + z)$. Assume for a contradiction that $x \nleq y$ and $x \nleq z$. Then $x \cdot y = \mathrm{o}$ and $x \cdot z = \mathrm{o}$, by (13.1). Hence $\mathrm{o} = (x \cdot y) + (x \cdot z) = x \cdot (y + z)$. Conversely, suppose that either $x \leq y$ or $x \leq z$. Then $x = x \cdot y = x \cdot z$. Assume for a contradiction that $x \nleq y+z$. Then, by (13.1), $\mathrm{o} = x\cdot(y+z) = (x\cdot y)+(x\cdot z) = x+x = x$.

For an arbitrary $x \in L$ we put:

$$\mathrm{At}_x(\mathfrak{L}) := \{a \in \mathrm{At}(\mathfrak{L}) : a \leq x\}.$$

LEMMA 13.1. *For any $x \in L$ the following conditions are equivalent:*

(a) $x \ \mathsf{sup}_{\leq} \ \mathrm{At}_x(\mathfrak{L})$,
(b) $\forall_{y \in L}(\mathrm{At}_x(\mathfrak{L}) \subseteq \mathrm{At}_y(\mathfrak{L}) \implies x \leq y)$,
(c) $\forall_{y \in L}(x \leq y \iff \mathrm{At}_x(\mathfrak{L}) \subseteq \mathrm{At}_y(\mathfrak{L}))$.

PROOF. '(a) ⇒ (b)' If $x = \mathrm{o}$, then $\mathrm{o} \leq y$, for any $y \in L$. Thus, suppose that $x \neq \mathrm{o}$ and $\mathrm{At}_x(\mathfrak{L}) \subseteq \mathrm{At}_y(\mathfrak{L})$. Then $\forall_{a \in \mathrm{At}_x(\mathfrak{L})} \, a \leq y$. Hence $x \leq y$, since $x \ \mathsf{sup}_{\leq} \ \mathrm{At}_x(\mathfrak{L})$.

'(b) ⇒ (a)' Clearly, we have $\forall_{a \in \mathrm{At}_x(\mathfrak{L})} \, a \leq x$. Moreover, assume that $\forall_{a \in \mathrm{At}_x(\mathfrak{L})} \, a \leq y$. Then $\mathrm{At}_x(\mathfrak{L}) \subseteq \mathrm{At}_y(\mathfrak{L})$. Hence $x \leq y$, by (b). Thus, $x \ \mathsf{sup}_{\leq} \ \mathrm{At}_x(\mathfrak{L})$.

'(b) ⇔ (c)' By ($\mathrm{t}_{\sqsubseteq}$). □

For any $\mathfrak{L} = \langle L, \leq, \mathrm{o}\rangle$ we say that a given element x from L is *atomic* in $\mathfrak{L}$ iff each non-zero element contained in x includes some atom, i.e.,

$$x \text{ is atomic in } \mathfrak{L} :\iff \forall_{y \in L\setminus\{\mathrm{o}\}}(y \leq x \implies \exists_{a \in \mathrm{At}(\mathfrak{L})} \, a \leq y). \qquad (\mathrm{df\,Atc}(\mathfrak{L}))$$

Let $\mathrm{Atc}(\mathfrak{L})$ be the set of all atomic elements in $\mathfrak{L}$. All atoms are atomic, but not vice versa, i.e., formally:

$$\mathrm{At}(\mathfrak{L}) \subsetneq \mathrm{Atc}(\mathfrak{L}).$$

Firstly, if $a \in \mathrm{At}(\mathfrak{L})$, $y \neq \mathrm{o}$, and $y \leq a$, then $y = a$, by ($\mathrm{df\,At}(\mathfrak{L})$). Secondly, $\mathrm{o} \in \mathrm{Atc}(\mathfrak{L})$ and $\mathrm{o} \notin \mathrm{At}(\mathfrak{L})$.

Atomic elements have the following property:

$$x \in \mathrm{Atc}(\mathfrak{L}) \wedge y \in \mathrm{Atc}(\mathfrak{L}) \iff x + y \in \mathrm{Atc}(\mathfrak{L})\,. \qquad (13.3)$$

Let $x, y \in \mathrm{Atc}(\mathfrak{L})$ and take an arbitrary $z \neq \mathrm{o}$ such that $z \leq x + y$. If $x \cdot z = \mathrm{o}$ then $z \leq y$, and therefore we make use of our assumption. So, let $x \cdot z \neq \mathrm{o}$. Since $x \cdot z \leq x$, then there is $a \in \mathrm{At}(\mathfrak{L})$ such that $a \leq x \cdot z$. Hence $a \leq z$. The converse implication follows from ($\mathrm{t}_{\leq}$) and the fact that $x \leq x + y$ and $y \leq x + y$.

LEMMA 13.2. *If $\mathfrak{L}$ is separative, then for any $x \in L$ such that $x \in \mathrm{Atc}(\mathfrak{L})$ we have $x \ \mathsf{sup}_{\leq} \ \mathrm{At}_x(\mathfrak{L})$.*

PROOF. Let $x \in \mathrm{Atc}(\mathfrak{L})$. Clearly, we have $\forall_{a \in \mathrm{At}_x(\mathfrak{L})}\, a \leq x$. Moreover, assume for a contradiction that $\forall_{a \in \mathrm{At}_x(\mathfrak{L})}\, a \leq y$ and $x \nleq y$. Then $x \neq 0$ and $\mathrm{At}_x(\mathfrak{L}) \neq \emptyset$, by (df Atc($\mathfrak{L}$)) and ($\mathrm{r}_{\sqsubseteq}$). Therefore $y \neq 0$. Moreover, by (sep), for some $u_0 \in L$ we have $0 \neq u_0 \leq x$ and $u_0 \cdot y = 0$. Hence, by (df Atc($\mathfrak{L}$)), for some $a_0 \in \mathrm{At}(\mathfrak{L})$ we have $a_0 \leq u_0$; and so $a_0 \cdot y = 0$. We obtain therefore a contradiction: $0 = a_0 \cdot y = y \neq 0$. □

Finally, we will consider the case where $\mathfrak{L}$ is a Boolean lattice.

LEMMA 13.3. *If $\mathfrak{L} = \langle L, \leq, 0, 1\rangle$ is a Boolean lattice, then for any $x \in L$ such that $x \operatorname{\mathsf{sup}}_{\leq} \mathrm{At}(\mathfrak{L})$ we have $x \in \mathrm{Atc}(\mathfrak{L})$.*

PROOF. Suppose that $x \operatorname{\mathsf{sup}}_{\leq} \mathrm{At}(\mathfrak{L})$. If $\mathrm{At}(\mathfrak{L}) = \emptyset$ then $x = 0$, and so $x \in \mathrm{Atc}(\mathfrak{L})$. Suppose therefore that $\mathrm{At}(\mathfrak{L}) \neq \emptyset$. Then $x \neq 0$. Assume for a contradiction that $x \notin \mathrm{Atc}(\mathfrak{L})$. Then for some $y_0 \in L$ we have: (a) $0 \neq y_0 \leq x$ and (b) there is no $a \in \mathrm{At}(\mathfrak{L})$ such that $a \leq y_0$. From (a) we have $x \nleq -y_0$. Hence $x \cdot -y_0 \lneq x$. Furthermore, by (a) we have $x = x + y_0 = (x + y_0) \cdot 1 = (x + y_0) \cdot (-y_0 + y_0) = (x \cdot -y_0) + y_0$. From this, (b) and (13.2), for any $a \in \mathrm{At}(\mathfrak{L})$: $a \leq x$ iff $a \leq x \cdot -y_0$. So $x \cdot -y_0$ is an upper bound of $\mathrm{At}(\mathfrak{L})$. And this contradicts the claim that x is the least upper bound of $\mathrm{At}(\mathfrak{L})$. □

14. Atomic and atomistic lattices

Let $\mathfrak{L} = \langle L, \leq, 0\rangle$ be any lattice with zero. We call $\mathfrak{L}$ *atomic* iff each its non-zero element includes some atom, i.e.,

$$\mathfrak{L} \text{ is atomic } :\Longleftrightarrow \forall_{x \in L \setminus \{0\}} \exists_{a \in \mathrm{At}(\mathfrak{L})}\, a \leq x\,.$$

From ($\mathrm{r}_{\leq}$) we have: $\mathfrak{L}$ is *atomic* iff each of its elements is atomic, i.e.:

$$\mathfrak{L} \text{ is atomic } \Longleftrightarrow \mathrm{Atc}(\mathfrak{L}) = L\,. \tag{14.1}$$

If $\mathfrak{L}$ has also the greatest element 1, then by ($\mathrm{t}_{\leq}$) we obtain:

$$\mathfrak{L} \text{ is atomic iff } 1 \in \mathrm{Atc}(\mathfrak{L}).$$

From the above, in the light of Lemma 6.1we obtain:

LEMMA 14.1. *For any $x \in L$: the lattice $\mathfrak{B} \restriction x$ is atomic iff $x \in \mathrm{Atc}(\mathfrak{B})$.*

A lattice $\mathfrak{L} = \langle L, \leq, 0\rangle$ with zero is called *atomistic* iff each of elements is the least upper bound of a set of atoms, i.e.,

$$\mathfrak{L} \text{ is atomistic } :\Longleftrightarrow \forall_{x \in L} \exists_{A \subseteq \mathrm{At}(\mathfrak{L})}\, x \operatorname{\mathsf{sup}}_{\leq} A\,.$$

We obtain:

$$\mathfrak{L} \text{ is atomistic } \iff \forall_{x \in L}\, x \,\mathsf{sup}_{\leq}\, \mathrm{At}_x(\mathfrak{L}). \tag{14.2}$$

'⇒' Let $\mathfrak{L}$ be atomistic and $x \in L$. Then for some $A_0 \subseteq \mathrm{At}(\mathfrak{L})$ we have $x \,\mathsf{sup}_{\leq}\, A_0$. Hence (a) $A_0 \subseteq \mathrm{At}_x(\mathfrak{L})$ and (b) for any $y \in L$, if $\forall_{a \in A_0}\, a \leq y$, then $x \leq y$. Clearly $\forall_{a \in \mathrm{At}_x(\mathfrak{L})}\, a \leq x$. Assume for a contradiction that for some $y_0 \in L$ we have (c) $\forall_{a \in \mathrm{At}_x(\mathfrak{L})}\, a \leq y$ and (d) $x \nleq y_0$. From (a)–(c) we obtain $x \leq y_0$. But this contradicts (d). '⇐' It is obvious.

Now note that every atomistic lattice is also atomic:

Lemma 14.2. *Suppose that $\mathfrak{L} = \langle L, \leq, \mathrm{o}\rangle$ is atomistic. Then $\mathfrak{L}$ is atomic.*

Proof. Suppose that $\mathfrak{L} = \langle L, \leq, \mathrm{o}\rangle$ is atomistic and $x \in L \setminus \{\mathrm{o}\}$. Then for some $A_0 \subseteq \mathrm{At}(\mathfrak{L})$ we have $x \,\mathsf{sup}_{\leq}\, A_0$. Hence $A_0 \neq \emptyset$; and so for some $a_0 \in A_0$ we have $a_0 \leq x$. □

Finally, by (14.1) and Lemma 13.2 we obtain:

Lemma 14.3. *Suppose that $\mathfrak{L} = \langle L, \leq, \mathrm{o}\rangle$ is atomic and satisfies* (sep). *Then $\mathfrak{L}$ is also atomistic.*

15. Atomic Boolean lattices

Note that Lemma 14.3 we can use for all all Boolean latices, because they satisfy the condition (sep) (see condition (6.16)). Thus, a given Boolean lattice is atomic iff it is atomistic (see also Lemma 14.2).

It is known that for any Boolean lattice $\mathfrak{B}$ [see, e.g., Frankiewicz and Zbierski, 1992, pp. 26–27 and Traczyk, 1970, pp. 53 and 55]:

- If $\mathfrak{B}$ is finite then it is atomic.
- If $\mathfrak{B}$ is atomic and has $n \in \mathbb{N}$ atoms, then $\mathfrak{B}$ has 2^n members.
- If $\mathfrak{B}$ is complete and atomic, then $\mathfrak{B}$ is isomorphic with $\langle \mathcal{P}(\mathrm{At}(\mathfrak{B})), \subseteq, \emptyset, \mathrm{At}(\mathfrak{B})\rangle$.

Example 15.1. (i) For any set X, the Boolean lattices $\mathfrak{P}_X$ and $\mathfrak{FC}_X$ are atomic, where all singletons of X are atoms (see Example 9.1).

(ii) The Boolean lattice which we obtain from the field of Borel subsets of the topological space of real numbers is also atomic, where all singletons of $\mathbb{R}$ are atoms [cf., eg., Traczyk, 1970, p. 55]. □

Remark 15.1. For some set X and some family $\mathcal{F}$ of subsets of X such that $\langle \mathcal{F}, \subseteq, \emptyset, X\rangle$ is an atomic complete Boolean lattice, but none of its atoms is a singleton of X. See, e.g., the first example in Remark 9.2. □

16. Atomless elements. Atomless lattices

Let $\mathfrak{L} = \langle L, \leq, \mathrm{o}\rangle$ be any lattice with zero. We say that a given element of L is *atomless* in $\mathfrak{L}$ iff it does not include any atoms, i.e.:

$$x \text{ is atomless in } \mathfrak{L} \;:\Longleftrightarrow \neg\exists_{a\in \mathrm{At}(\mathfrak{L})}\, a \leq x\,. \qquad (\mathrm{df\,Atl}(\mathfrak{L}))$$

Let $\mathrm{Atl}(\mathfrak{L})$ be the set of all atomless elements in $\mathfrak{L}$. We have:

$$\mathrm{Atl}(\mathfrak{L}) \cap \mathrm{At}(\mathfrak{L}) = \emptyset\,, \qquad (16.1)$$
$$\mathrm{Atl}(\mathfrak{L}) \cap \mathrm{Atc}(\mathfrak{L}) = \{\mathrm{o}\}.$$

For (16.1): We use ($\mathrm{r}_{\leq}$). For the second one: First, $\mathrm{o} \in \mathrm{Atc}(\mathfrak{L}) \cap \mathrm{Atl}(\mathfrak{L})$. Second, assume for a contradiction that $x \in \mathrm{Atc}(\mathfrak{L}) \cap \mathrm{Atl}(\mathfrak{L})$ and $x \neq \mathrm{o}$. Since $x \leq x$, we therefore have: $\exists_{a\in \mathrm{At}(\mathfrak{L})}\, a \leq x$ and $\neg\exists_{a\in \mathrm{At}(\mathfrak{L})}\, a \leq x$.

Analogously to (13.3), for arbitrary $x, y \in L$ we obtain:

$$x \in \mathrm{Atl}(\mathfrak{L}) \wedge y \in \mathrm{Atl}(\mathfrak{L}) \iff x + y \in \mathrm{Atl}(\mathfrak{L})\,. \qquad (16.2)$$

Using the definition of *being atomless* and (13.2) we have: $x+y \notin \mathrm{Atl}(\mathfrak{L})$ iff for some $a \in \mathrm{At}(\mathfrak{L})$ we have $a \leq x + y$ iff for some $a \in \mathrm{At}(\mathfrak{L})$ either $a \leq x$ or $a \leq y$ iff either for some $a \in \mathrm{At}(\mathfrak{L})$ we have $a \leq x$ or for some $a \in \mathrm{At}(\mathfrak{L})$ we have $a \leq y$ iff $x \notin \mathrm{Atl}(\mathfrak{L})$ or $y \notin \mathrm{Atl}(\mathfrak{L})$.

We call $\mathfrak{L}$ *atomless* iff each element of $\mathfrak{L}$ is atomless, i.e.:

$$\mathfrak{L} \text{ is atomless } :\Longleftrightarrow \mathrm{Atl}(\mathfrak{L}) = L\,.$$

If $\mathfrak{L}$ also has the greatest element $\mathrm{1}$, then by ($\mathrm{t}_{\leq}$) we obtain:

$$\mathfrak{L} \text{ is atomless iff } \mathrm{1} \in \mathrm{Atl}(\mathfrak{L}).$$

From the above, in the light of Lemma 6.1we obtain:

Lemma 16.1. *For any $x \in L$: the lattice $\mathfrak{L}\restriction x$ is atomless iff $x \in \mathrm{Atl}(\mathfrak{L})$.*

Moreover, from definitions and (16.1) we have: $\mathfrak{L}$ is atomless iff it has no atoms; formally:

$$\mathfrak{L} \text{ is atomless } \iff \mathrm{At}(\mathfrak{L}) = \emptyset\,.$$

Lemma 16.2. *$\mathfrak{L}$ is atomic and atomless iff $\mathfrak{L}$ is trivial* (*i.e.*, $L = \{\mathrm{o}\}$).

Proof. Assume that $\mathfrak{L}$ s non-trivial and atomic. Then $\mathrm{1} \neq \mathrm{o}$ and there is $a \in \mathrm{At}(\mathfrak{L})$ such that $a \leq \mathrm{1}$. Therefore, $\mathfrak{L}$ is not atomless. The converse implication is obvious. □

Since every finite Boolean lattice is atomic, every non-trivial atomless Boolean lattice has infinitely many elements.

Example 16.1. The complete Boolean lattice $\mathrm{RO}(\mathbb{R}) := \langle \mathrm{r}\mathscr{O}_{\mathbb{R}}, \subseteq, \emptyset, \mathbb{R}\rangle$ is atomless (see examples see examples 9.2 and 11.2) [cf. Frankiewicz and Zbierski, 1992, p. 26]. □

17. Ideals in Boolean lattices

Let $\mathfrak{B} = \langle B, \leq, 0, 1\rangle$ be any Boolean lattice. We say that a subset I of B is an *ideal* of $\mathfrak{B}$ iff I satisfies the following condition:

1. $0 \in I$,
2. for any $x, y \in B$: if $x, y \in I$ then $x + y \in I$,
3. for any $x, y \in B$: if $x \in I$ and $y \leq x$, then $y \in I$.

Obviously, both the singleton $\{0\}$ and the universe B are ideals of $\mathfrak{B}$.

If $\mathfrak{B}_1 = \langle B_1, \leq_1, 0_1, 1_1\rangle$ and $\mathfrak{B}_2 = \langle B_2, \leq_2, 0_2, 1_2\rangle$ are Boolean lattices, $h\colon B_1 \to B_2$ is a homomorphism, and I is an ideal of $\mathfrak{B}_2$, then the set $h^{-1}[I] := \{x \in B_1 : h(x) \in I\}$ is an ideal of $\mathfrak{B}_1$ (cf. p. 267).

Let $\mathrm{E}(\mathfrak{B})$ be the set of these and only those elements that are sums of atomic elements and atomless elements, i.e., we put:

$$\mathrm{E}(\mathfrak{B}) := \{x \in B : \exists_{y \in \mathrm{Atc}(\mathfrak{B})} \exists_{z \in \mathrm{Atl}(\mathfrak{L})}\ x = y + z\}.$$

Lemma 17.1 (Koppelberg, 1998b, p. 288).

(i) *The set* $\mathrm{E}(\mathfrak{B})$ *is an ideal of* $\mathfrak{B}$.
(ii) $1 \in \mathrm{E}(\mathfrak{B})$ *iff* $\mathrm{E}(\mathfrak{B}) = B$.

Proof. *Ad* (i): First, $0 = 0 + 0$ and $0 \in \mathrm{Atc}(\mathfrak{B}) \cap \mathrm{Atl}(\mathfrak{B})$.

Second, let $x_1, x_2 \in \mathrm{E}(\mathfrak{B})$, i.e., $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$, for some $y_1, y_2 \in \mathrm{Atc}(\mathfrak{B})$ and $z_1, z_2 \in \mathrm{Atl}(\mathfrak{B})$. Then $x_1 + x_2 = (y_1 + y_2) + (z_1 + z_2)$. By (13.3), $y_1 + y_2 \in \mathrm{Atc}(\mathfrak{B})$ and, by (16.2), $z_1 + z_2 \in \mathrm{Atl}(\mathfrak{B})$.

Let $x_1 \leq x_2$ and $x_2 \in \mathrm{E}(\mathfrak{B})$, i.e., $x_1 = x_1 \cdot x_2$ and $x_2 = y_2 + z_2$, for some $y_2 \in \mathrm{Atc}(\mathfrak{B})$ and $z_2 \in \mathrm{Atl}(\mathfrak{B})$. Then $x_1 = x_1 \cdot (y_2 + z_2) = (x_1 \cdot y_2) + (x_1 \cdot z_2)$, $x_1 \cdot y_2 \in \mathrm{Atc}(\mathfrak{B})$, and $x_1 \cdot z_2 \in \mathrm{Atl}(\mathfrak{B})$.

Ad (ii): If $1 \in \mathrm{E}(\mathfrak{B})$ then $\mathrm{E}(\mathfrak{B}) = B$, since $\mathrm{E}(\mathfrak{B})$ is an ideal and $y \leq 1$, for any $y \in B$. □

Lemma 17.2. $1 \in \mathrm{E}(\mathfrak{B})$ *iff there is* $x \in \mathrm{Atc}(\mathfrak{B})$ *such that* $-x \in \mathrm{Atl}(\mathfrak{B})$.

PROOF. '⇒' Let $1 \in \mathrm{E}(\mathfrak{B})$, i.e., $1 = x + y$, for some $x \in \mathrm{Atc}(\mathfrak{B})$ and $y \in \mathrm{Atl}(\mathfrak{B})$. Assume for a contradiction that $x \cdot y \neq 0$. Then there is $a \in \mathrm{At}(\mathfrak{B})$ such that $a \leq x \cdot y \leq x$. Hence also $a \leq y$. A contradiction. So $y = -x$. '⇐' Suppose that $x \in \mathrm{Atc}(\mathfrak{B})$ and $-x \in \mathrm{Atl}(\mathfrak{B})$. Since $1 = x + -x$, so $1 \in \mathrm{E}(\mathfrak{B})$. □

LEMMA 17.3. (i) $\mathrm{Atc}(\mathfrak{B}) \cup \mathrm{Atl}(\mathfrak{B}) \subseteq \mathrm{E}(\mathfrak{B})$.
(ii) *If* $\mathfrak{B}$ *is atomic or atomless, then* $\mathrm{E}(\mathfrak{B}) = B$.

PROOF. *Ad* (i): Since $0 \in \mathrm{Atc}(\mathfrak{B}) \cap \mathrm{Atl}(\mathfrak{B})$ and for any $x \in B$ we $x + 0 = x = 0 + x$, so have $\mathrm{Atc}(\mathfrak{B}) \cup \mathrm{Atl}(\mathfrak{B}) \subseteq \mathrm{E}(\mathfrak{B})$.

Ad (ii): If $\mathfrak{B}$ is atomic or atomless, then either $B = \mathrm{Atc}(\mathfrak{B}) \subseteq \mathrm{E}(\mathfrak{B}) \subseteq B$ or $B = \mathrm{Atl}(\mathfrak{B}) \subseteq \mathrm{E}(\mathfrak{B}) \subseteq B$. □

LEMMA 17.4. (i) *If* $\mathrm{At}(\mathfrak{B})$ *has a supremum, then* $\mathrm{E}(\mathfrak{B}) = B$.
(ii) *If* $\mathfrak{B}$ *is complete then* $\mathrm{E}(\mathfrak{B}) = B$.

PROOF. *Ad* (i): Suppose that $x \ \mathsf{sup}_{\leq} \ \mathrm{At}(\mathfrak{B})$. Then $-x \in \mathrm{Atl}(\mathfrak{B})$. If $\mathrm{At}(\mathfrak{B}) = \emptyset$, then $\mathfrak{B}$ is atomless and so $\mathrm{E}(\mathfrak{B}) = B$, by Lemma 17.3(ii). Assume therefore that $\mathrm{At}(\mathfrak{B}) \neq \emptyset$. Then $x \neq 0$, $x \in \mathrm{Atc}(\mathfrak{B})$, and $1 \in \mathrm{E}(\mathfrak{B})$, by lemmas 13.3 and 17.2, respectively. Hence $\mathrm{E}(\mathfrak{B}) = B$, by Lemma 17.1(ii).

Ad (ii): By (i), since if $\mathfrak{B} \in \mathbf{CBL}$ then $\mathrm{At}(\mathfrak{B})$ has a supremum. □

18. Quotient Boolean lattices

Let $\mathfrak{B} = \langle B, \leq, 0, 1 \rangle$ be a Boolean lattice and I be an ideal of $\mathfrak{B}$. Then the following binary relation in B

$$x \cong_I y \;:\Longleftrightarrow x \bigtriangleup y \in I$$

is *congruence relation* on B, i.e., it is an equivalence relation on B such that, for all x_1, x_2, y_1, y_2 in B, $x_1 \cong_I x_2$, $y_1 \cong_I y_2$, and $x_1 \leq y_1$ imply $x_2 \leq y_2$. Hence we obtain: $x_1 \cong_I x_2$, $y_1 \cong_I y_2$ imply $-x_1 \cong_I -x_2$, $x_1 + y_1 \cong_I x_2 + y_2$, and $x_1 \cdot y_1 \cong_I x_2 \cdot y_2$.

For any $x \in B$ let $[x]_I$ be the equivalence class of x with respect to $\cong_I$, i.e., $[x]_I := \{y \in B : x \cong_I y\}$, and let B/I be the set of equivalence classes of $\cong_I$, i.e.,

$$B/I := \{[x]_I : x \in B\}.$$

Note that

$$\cong_{\{0\}} = \mathrm{id}_B \quad \text{and} \quad \cong_B = B \times B\,.$$

Hence for any $x \in B$ we have:

$$[x]_{\{o\}} = \{x\} \quad \text{and} \quad [x]_B = B\,.$$

Therefore,

$$B/\{o\} = \{\{x\} : x \in B\}, \tag{18.1}$$
$$B/B = \{B\}. \tag{18.2}$$

Moreover, in the set B/I we can define the following partial order:

$$[x]_I \leq_I [y]_I \ :\Longleftrightarrow x \leq y\,.$$

Then we obtain the *quotient Boolean lattice* $\mathfrak{B}/I := \langle B/I, \leq_I, o_I, 1_I \rangle$ with respect to $\cong_I$, in which $o_I = [o]_I$ and $1_I = [1]_I$. In this lattice for any $x, y \in B$ we have: $[x]_I +_I [y]_I = [x+y]_I$, $[x]_I \cdot_I [y]_I = [x \cdot y]_I$, and $-_I[x]_I = [-x]_I$.

By definition, the mapping $\pi_I\colon x \mapsto [x]_I$ is a homomorphism from B onto B/I. The homomorphism π_I is called a *canonical map*.

Lemma 18.1. (i) *If $\mathfrak{B}/I$ is trivial, then $I = B$ and $B/I = \{B\}$.*
(ii) *If $I = B$ then $\mathfrak{B}/I$ is trivial.*

Proof. *Ad* (i): Assume that $\mathfrak{B}/I$ is trivial. Then $B/I = \{o_I\} = \{[o]_I\}$. So for any $x \in B$ we have $x \in I$, since $x \triangle o \in I$ and $x = x \triangle o$. Hence $I = B$ and $B/I = \{B\}$, by (18.2).

Ad (ii): Suppose that $I = B$. Then $B/_B = \{B\} = \{[o]_B\}$, by (18.2). Thus, $\mathfrak{B}/_{E_i}$ is trivial. □

As in [Koppelberg, 1998b, pp. 288–289], by induction, we define for any $n \in \mathbb{N}$ an ideal E_n of $\mathfrak{B}$ and then let

$$\mathfrak{B}^{(n)} := \mathfrak{B}/E_n,$$
$$E_0 := \{o\}, \quad E_{n+1} := \pi_{E_n}^{-1}[\mathrm{E}(\mathfrak{B}/E_n)].$$

Thus, E_0 is an ideal of $\mathfrak{B}$ and therefore: $\mathfrak{B}/E_0$ is a Boolean lattice; $\mathrm{E}(\mathfrak{B}/E_0)$ is an ideal of $\mathfrak{B}/E_0$; E_1 is an ideal of $\mathfrak{B}$; $\mathfrak{B}/E_1$ is a Boolean lattice; $\mathrm{E}(\mathfrak{B}/E_1)$ is an ideal of $\mathfrak{B}/E_1$; E_2 is an ideal of $\mathfrak{B}$; $\mathfrak{B}/E_2$ is a Boolean lattice; and so on for any $n \in \mathbb{N}$.

In the light of (18.1) we have:

Lemma 18.2. *The canonical map $\pi_{\{o\}}$ is an isomorphism from $\mathfrak{B}$ onto the lattice $\mathfrak{B}/\{o\}$.*

Thus, by the above lemma,

$$\mathrm{E}(\mathfrak{B}/\{0\}) = \{\, \{x\} : x \in \mathrm{E}(\mathfrak{B}) \,\},$$
$$E_1 = \mathrm{E}(\mathfrak{B}). \tag{18.3}$$

Moreover, we obtain:

LEMMA 18.3. *For any* $n \in \mathbb{N}$:

(i) *If* $E_n = B$ *then for any* $m > n$ *also* $E_m = B$.
(ii) *If* $\mathfrak{B}/E_n$ *is trivial then for any* $m > n$ *also* $\mathfrak{B}/_{E_m}$ *is trivial.*

PROOF. *Ad* (i): By Lemma 18.1, if $E_n = B$, then $\mathfrak{B}/E_n$ is trivial and $B/E_n = \{B\}$. Hence $E_{n+1} := \pi_{E_n}^{-1}[\mathfrak{B}/E_n] = B$. So also $E_m = B$, for any $m > n$.

Ad (ii): Directly from (i) and Lemma 18.1. □

Appendix II

Elementarily complete Boolean lattices

1. Basic concepts of elementary theories and their models

Throughout the book we have been using first-order (elementary) languages with the identity predicate "=" that are built from the following symbols:[1]

- a countable number of variables "x_1", "x_2", "x_3", ...; the set of variables we denote by "Var"; the first six initial variables will be replaced below by the six following letters: "x", "y", "z", "u", "v", and "w";
- logical constants, these being the truth-connectives "¬", "∨", "∧", "→", and "≡"; the quantifiers "∀" and "∃"; and the identity predicate "=";
- left and right brackets: "(" and ")".

A given first-order language with the identity predicate "=" will have a (finite and non-empty) set of specific (non-logical) constants, which may be predicate constants, function constants, and individual constants. With the help of the function and individual constants we construct in a standard way the set of terms of a given language (if we only have predicates then the set of terms is equal to Var). From these terms, the identity predicate, and specific predicates (if there are any), we build in a standard way the *atomic formulae* of the language.

For a given first-order language L with the identity predicate "=", the set of L-formulae is the smallest set including the set of atomic formulae of L and to which belong the formulae $\ulcorner \neg\varphi \urcorner$, $\ulcorner (\varphi \mathbin{\S} \psi) \urcorner$ and $\ulcorner Q x_i \varphi \urcorner$, where φ and ψ are L-formulae, $\S \in \{\vee, \wedge, \rightarrow, \equiv\}$, $Q \in \{\forall, \exists\}$, and $x_i \in \mathrm{Var}$;

[1] The introduction of elementary languages and, later, elementarily definable sets and relations with or without parameters in relational structures, reveals that various results we have discovered within the framework of set theory could also have been made within the structure of various first-order theories, and so without the need to use the concepts of *being a set* and of *being a relation* from set theory.

we will use $\ulcorner x_i \neq x_j \urcorner$ as an abbreviation for $\ulcorner \neg\, x_i = x_j \urcorner$. (We will omit brackets where there is no risk of ambiguity in the usual way.) Let $\mathrm{v}(\varphi)$ and $\mathrm{vf}(\varphi)$ be respectively the set of all variables and the set of all free variables of an L-formula φ. The sentences of L are the L-formulae without any free variables.

Let L be an arbitrary first-order language with the identity predicate "=". An L-*structure* is an ordered pair $\mathfrak{A} = \langle A, \Im \rangle$, where A is a non-empty set (*the universe of* $\mathfrak{A}$) and $\Im$ is a set-theoretic interpretation of non-logical constants of L. Interpretations of the predicates, function and individual constants of the language L (where only some of these may occur in L) are certain subsets of A, relations in A, operators in A, and distinguished elements in A, respectively. If L has a finite sequence $s_1, \ldots, s_k$ $(k > 0)$ of non-logical constants, then for L-structures we will write this in the form $\langle A, \Im(s_1), \ldots, \Im(s_k) \rangle$. We take the identity relation id_A to be the interpretation of the identity predicate "=".

For any L-structure $\mathfrak{A} = \langle A, \Im \rangle$, an arbitrary function $V \colon \mathrm{Var} \to A$ is a valuation of the variables. By induction we extend V to the set of terms of the language L and similarly define the satisfaction of L-formulae in the structure $\mathfrak{A}$ by the valuation V. Then we write: $\mathfrak{A} \models \varphi[V]$. For atomic L-formulae with an n-place predicate π and terms $\tau_1, \ldots, \tau_n$ we have: $\mathfrak{A} \models \pi(\tau_1, ..., \tau_n)\, [V]$ iff $\langle V(\tau_1), ..., V(\tau_n) \rangle \in \Im(\pi)$, where the relation $\Im(\pi)$ interprets the predicate π and $V(\tau_1), \ldots, V(\tau_n)$ are values of terms $\tau_1, \ldots, \tau_n$ in $\mathfrak{A}$. Moreover, we have always: $\mathfrak{A} \models \ulcorner \tau_1 = \tau_2 \urcorner [V]$ iff $V(\tau_1) = V(\tau_2)$. We interpret the connectives and quantifiers classically.

A given L-formula is true in an L-structure $\mathfrak{A}$ iff it is satisfied in $\mathfrak{A}$ by all valuations. Notice that for the L-sentences to be true — taking into account the interpretation of the quantifiers — it suffices that they satisfy at least one valuation (which is in any case not essential, because the sentences have no free variables).

We say that an L-structure $\mathfrak{A}$ is a model of a set of L-formulae Φ iff in $\mathfrak{A}$ all formulae in Φ are true. Let $\mathrm{Mod}(\Phi)$ be the class of all L-structures which are models of Φ. If $\Phi \subseteq \Psi$ then $\mathrm{Mod}(\Psi) \subseteq \mathrm{Mod}(\Phi)$.

The set of all true L-sentences in an L-structure $\mathfrak{A}$ is signified by $\mathrm{Th}(\mathfrak{A})$. We say that L-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* iff $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$.

For any class composed exclusively of L-structures $\boldsymbol{K}$, let $\mathrm{Th}(\boldsymbol{K})$ be the set of all L-sentences which are true in all structures of $\boldsymbol{K}$. Formally, $\mathrm{Th}(\boldsymbol{K}) := \bigcap \{\, \mathrm{Th}(\mathfrak{A}) : \mathfrak{A} \in \boldsymbol{K} \,\}$. Note that, if $\boldsymbol{K} \subseteq \boldsymbol{K}'$ then $\mathrm{Th}(\boldsymbol{K}') \subseteq \mathrm{Th}(\boldsymbol{K})$.

A class $\boldsymbol{K}$ composed of L-structures is *elementarily axiomatisable* (or *elementary in the wider sense*) iff there exists a set Σ of L-sentences such that $\boldsymbol{K} = \mathrm{Mod}(\Sigma)$. If additionally the set Σ is finite, then we say that $\boldsymbol{K}$ is *finitely elementarily axiomatisable* (or *elementary in the narrow sense*).

Directly from our definitions we obtain:

PROPOSITION 1.1. *Every elementarily (resp. finite elementary) axiomatisable class of L-structures is closed under elementary equivalence. In other words, for any elementarily (resp. finite elementary) axiomatizable class $\boldsymbol{K}$ of L-structures and arbitrary L-structures $\mathfrak{A}$ and $\mathfrak{B}$: if $\mathfrak{A} \in \boldsymbol{K}$ and $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$, then $\mathfrak{B} \in \boldsymbol{K}$.*

Moreover, we prove:

PROPOSITION 1.2. *Let $\boldsymbol{K}$ be a class of L-structures and let Σ be a set of L-sentences:*

(i) $\boldsymbol{K} \subseteq \mathrm{Mod}(\Sigma)$ *iff* $\Sigma \subseteq \mathrm{Th}(\boldsymbol{K})$.
(ii) $\boldsymbol{K} \subseteq \mathrm{Mod}(\mathrm{Th}(\boldsymbol{K}))$.
(iii) *The class $\boldsymbol{K}$ is elementarily axiomatisable iff* $\boldsymbol{K} = \mathrm{Mod}(\mathrm{Th}(\boldsymbol{K}))$.

PROOF. *Ad* (i): Directly from definitions we obtain: $\boldsymbol{K} \subseteq \mathrm{Mod}(\Sigma)$ iff $\forall_{\mathfrak{A}\in\boldsymbol{K}}\, \mathfrak{A} \in \mathrm{Mod}(\Sigma)$ iff $\forall_{\mathfrak{A}\in\boldsymbol{K}}\, \Sigma \subseteq \mathrm{Th}(\mathfrak{A})$ iff $\Sigma \subseteq \bigcap\{\mathrm{Th}(\mathfrak{A}) : \mathfrak{A} \in \boldsymbol{K}\}$.

Ad (ii): Directly from (i) for $\Sigma := \mathrm{Th}(\boldsymbol{K})$.

Ad (iii): Assume that $\boldsymbol{K}$ is elementarily axiomatisable, i.e., for some set Σ_0 of L-sentences we have $\boldsymbol{K} = \mathrm{Mod}(\Sigma_0)$. Then $\Sigma_0 \subseteq \mathrm{Th}(\boldsymbol{K})$, by (i). Hence $\mathrm{Mod}(\mathrm{Th}(\boldsymbol{K})) \subseteq \mathrm{Mod}(\Sigma_0) = \boldsymbol{K}$. Furthermore, we use (ii). Conversely, If $\boldsymbol{K} = \mathrm{Mod}(\mathrm{Th}(\boldsymbol{K}))$ then $\boldsymbol{K}$ is elementarily axiomatisable (by the set $\mathrm{Th}(\boldsymbol{K})$ of L-sentences). □

Let L be a first-order language with the identity predicate "$=$" and Cn_L be the standard operation of consequence of first-order logic with identity for L. An arbitrary set T of L-formulae such that $T = \mathrm{Cn}_L(T)$ we call a *first-order* (or *elementary*) *theory with identity* built in a language L. Let $\overline{T}$ be the set of L-sentences in T. If for an arbitrary L-formula φ we take it that $\overline{\varphi}$ is an L-sentence which is the closure of the formula φ,[2] then: $\varphi \in T$ iff $\overline{\varphi} \in \overline{T}$. For an arbitrary set Φ of L-formulae then set $\mathrm{Cn}_L(\Phi)$ is a theory, because $\mathrm{Cn}_L(\Phi) = \mathrm{Cn}_L(\mathrm{Cn}_L(\Phi))$. An arbi-

[2] If the formula φ is a sentence, then $\overline{\varphi} = \varphi$. Otherwise, $\overline{\varphi}$ arises by placing in front of the formula φ universal quantifiers binding all free variables in φ in numerical order.

trary set Φ of L-formulae such that $T = \mathrm{Cn}_L(\Phi)$ we call an *axiomsatisation of theory T*. Note that, if Φ is an axiomsatisation of theory T then $\mathrm{Mod}(T) = \mathrm{Mod}(\Phi)$, because $\Phi \subseteq T$ and, moreover, every formula from T is true in all models of Φ.

Gödel's Completeness Theorem states that the last statement can be reversed, i.e., also every formula that is true in all models of Φ belongs to $\mathrm{Cn}_L(\Phi)$.

Gödel's Completeness Theorem. *For any set Φ of L-formulae and any L-formula φ: $\varphi \in \mathrm{Cn}_L(\Phi)$ iff $\overline{\varphi}$ is true in all models of Φ. Formally,*

$$\varphi \in \mathrm{Cn}_L(\Phi) \iff \overline{\varphi} \in \mathrm{Th}(\mathrm{Mod}(\Phi)).$$

Let T be an elementary theory in a language L. We say T is *complete* iff for any L-sentence σ either $\sigma \in T$ or $\ulcorner\neg\sigma\urcorner \in T$.

Proposition 1.3. *For any complete elementary theory T in L:*

(i) *If $\mathfrak{A}$ is a model of T then $\overline{T} = \mathrm{Th}(\mathfrak{A})$.*
(ii) *All models of T are elementarily equivalent.*

Proof. *Ad* (i): First, by definitions, $\overline{T} \subseteq \mathrm{Th}(\mathfrak{A})$. Second, for any L-sentence: if $\sigma \notin \overline{T}$ then $\ulcorner\neg\sigma\urcorner \in \overline{T}$. Hence $\ulcorner\neg\sigma\urcorner \in \mathrm{Th}(\mathfrak{A})$; and so $\sigma \notin \mathrm{Th}(\mathfrak{A})$. Thus, also $\mathrm{Th}(\mathfrak{A}) \subseteq \overline{T}$.

Ad (ii): By (i), if $\mathfrak{A}$ and $\mathfrak{B}$ are models of T then $\mathrm{Th}(\mathfrak{A}) = \overline{T} = \mathrm{Th}(\mathfrak{B})$. □

2. Elementary definability with or without parameters

Let L be any first-order language with identity predicate "=" and $\mathfrak{A} = \langle A, \mathfrak{I}\rangle$ be any L-structure. We say that a subset S of A is *elementarily definable* (or shortly: *e-definable*) *in* $\mathfrak{A}$ iff there is an L-formula φ such that $\mathrm{vf}(\varphi) = \{\mathtt{x}_1\}$ and for any $x \in A$: $x \in S$ iff $\mathfrak{A} \models \varphi\,[x/\mathtt{x}_1]$. Then we also say that S is e-definable in $\mathfrak{A}$ with the help of $\varphi(\mathtt{x}_1)$. Let $\mathrm{e}\mathcal{P}(\mathfrak{A})$ be the family of sets which are e-definable in $\mathfrak{A}$ and let $\mathrm{e}\mathcal{P}_+(\mathfrak{A}) := \mathrm{e}\mathcal{P}(\mathfrak{A}) \setminus \{\emptyset\}$.

Obviously, all sets occurring as components of $\mathfrak{A}$ are elementarily definable with the help of appropriate atomic L-formulae. Furthermore, the sets $\emptyset$ and A are elementarily definable in $\mathfrak{A}$ with the help of the L-formulae "$\mathtt{x}_1 \neq \mathtt{x}_1$" and "$\mathtt{x}_1 = \mathtt{x}_1$", respectively.

Let $k > 0$ and $y_1, \ldots, y_k \in A$. We say that a subset S of A is *elementarily definable in* $\mathfrak{A}$ *with parameters* y_1, …, y_k iff there is an L-formula φ such that $\mathrm{vf}(\varphi) = \{\mathtt{x}_1, \ldots, \mathtt{x}_{k+1}\}$ and for any $x \in A$: $x \in S$

iff $\mathfrak{A} \vDash \varphi\,[x/\mathrm{x}_1, y_1/\mathrm{x}_2, ..., y_k/\mathrm{x}_{k+1}]$. Moreover, we say that a subset of A is elementarily definable in $\mathfrak{A}$ with parameters iff it is elementarily definable in $\mathfrak{A}$ with the help of some non-empty finite set of parameters from A.

All non-empty finite subsets of A are elementarily definable with parameters. In fact, for any $k > 0$ and $y_1, \ldots, y_k \in A$, the set $\{y_1, \ldots y_k\}$ is elementarily definable with parameters $y_1, \ldots, y_k$. We use the formula "$\mathrm{x}_1 = \mathrm{x}_2 \vee \cdots \vee \mathrm{x}_1 = \mathrm{x}_k$", since $\{y_1, \ldots, y_k\} = \{x \in A : x = y_1 \vee \cdots \vee x = y_k\}$.

Notice that all e-definable in $\mathfrak{A}$ sets are also elementary definable with parameters. In fact, if a set is e-definable by a formula $\varphi(\mathrm{x})$, then it is elementarily definable with parameters by the formula $\ulcorner\varphi(\mathrm{x}) \wedge (\mathrm{x} = y \vee \neg\, \mathrm{x} = y)\urcorner$. Thus, all sets which are elementarily definable in $\mathfrak{A}$ with or without parameters will be called *parametrically elementarily definable* (or shortly: *pe-definable*). Let $\mathfrak{pe}\mathcal{P}(\mathfrak{A})$ be the family of sets which are parametrically elementarily definable (pe-definable) in $\mathfrak{A}$ and let $\mathfrak{pe}\mathcal{P}_+(\mathfrak{A}) := \mathfrak{pe}\mathcal{P}(\mathfrak{A}) \setminus \{\emptyset\}$. We have $\mathrm{e}\mathcal{P}(\mathfrak{A}) \subseteq \mathfrak{pe}\mathcal{P}(\mathfrak{A})$ and $\mathrm{e}\mathcal{P}_+(\mathfrak{A}) \subseteq \mathfrak{pe}\mathcal{P}_+(\mathfrak{A})$.

For any $a \in A$ we say that a is *e-definable in* $\mathfrak{A}$ (without parameters) iff the singleton $\{a\}$ is e-definable in $\mathfrak{A}$, i.e., there is an L-formula φ such that $\mathrm{vf}(\varphi) = \{\mathrm{x}_1\}$ and for any $x \in A$: $x = a$ iff $\mathfrak{A} \vDash \varphi\,[x/\mathrm{x}_1]$.

Now let R be any relation on A, i.e., $R \subseteq A^n$, for some $n > 1$. Then we say that R is *e-definable in* $\mathfrak{A}$ (without parameters) iff there is an L-formula φ such that $\mathrm{vf}(\varphi) = \{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$ and for all $x_1, \ldots, x_n \in A$: $\langle x_1, \ldots, x_n\rangle \in R$ iff $\mathfrak{A} \vDash \varphi\,[x_1/\mathrm{x}_1, \ldots, x_n/\mathrm{x}_n]$. Note that the relation id_A is e-definable in $\mathfrak{A}$ with the help of the L-formula "$\mathrm{x}_1 = \mathrm{x}_2$".

We say that $f\colon A^n \to A$ $(n > 0)$ is *e-definable in* $\mathfrak{A}$ iff for some L-formula φ we have $\mathrm{vf}(\varphi) = \{\mathrm{x}_1, \ldots, \mathrm{x}_{n+1}\}$ and for all $x_1, \ldots, x_n, y \in A$: $y = f(x_1, ..., x_n)$ iff $\mathfrak{A} \vDash \varphi\,[x_1/\mathrm{x}_1, ..., x_n/\mathrm{x}_n, y/\mathrm{x}_{n+1}]$.

Clearly, for all specific constants of L their interpretations in any L-structure $\mathfrak{A}$ are e-definable with the help of appropriate atomic L-formulae. For any individual constant ν, the element $\Im(\nu)$ is e-definable in $\mathfrak{A}$ with the help of the L-formula $\ulcorner\mathrm{x}_1 = \nu\urcorner$. For any n-ary predicate constant π $(n > 0)$, the relation $\Im(\pi) \subseteq A^n$ is e-definable in $\mathfrak{A}$ with the help of the L-formula $\ulcorner\pi(\mathrm{x}_1, ..., \mathrm{x}_n)\urcorner$. For any n-ary function constant θ, the function $\Im(\theta)\colon A^n \to A$ is e-definable in $\mathfrak{A}$ with the help of the L-formula $\ulcorner\mathrm{x}_{n+1} = \theta(\mathrm{x}_1, ..., \mathrm{x}_n)\urcorner$.

Finally, a relation $R \subseteq A^n$ is *elementarily definable in* $\mathfrak{A}$ *with parameters* $y_1, \ldots, y_k$ from A $(k > 0)$ iff there is an L-formula φ such that

$\mathrm{vf}(\varphi) = \{\mathtt{x}_1, \ldots, \mathtt{x}_{n+k}\}$ and for any $x_1, \ldots, x_n \in A$: $\langle x_1, \ldots, x_n \rangle \in R$ iff $\mathfrak{A} \models \varphi\,[x_1/\mathtt{x}_1, \ldots, x_n/\mathtt{x}_n, y_1/\mathtt{x}_{n+1}, \ldots, y_k/\mathtt{x}_{n+k}]$. Moreover, we say that a relation in A is *elementarily definable in* $\mathfrak{A}$ *with parameters* iff it is elementarily definable in $\mathfrak{A}$ with the help of some non-empty finite set of parameters from A.

Notice that all e-definable in $\mathfrak{A}$ relations are also elementarily definable with parameters. In fact, if a relation is e-definable by a formula $\varphi(\mathtt{x}_1, \ldots, \mathtt{x}_n)$, then it is elementarily definable with parameters by the formula by $\ulcorner \varphi(\mathtt{x}_1, \ldots, \mathtt{x}_n) \wedge (\mathtt{x}_1 = \mathtt{x}_{n+1} \vee \neg\, \mathtt{x}_1 = \mathtt{x}_{n+1}) \urcorner$. Thus, all relations which are elementarily definable in $\mathfrak{A}$ with or without parameters will be called *parametrically elementarily definable* (or shortly: *pe-definable*).

3. Elementary theory of Boolean lattices

To the class **POS** of partially ordered sets of the form $\langle X, \leq \rangle$ we may join the first-order language $\mathrm{L}_{\leq}$ with the identity predicate "$=$" and a single specific constant, the two-place predicate "$\leq$". Thus, the class **POS** is included in the class of all $\mathrm{L}_{\leq}$-structures.

We may also join the language $\mathrm{L}_{\leq}$ with the class **BL** of all Boolean lattices. In this language we can formulate the elementary theory **B** of Boolean lattices by adopting, inter alia, the following specific axioms:

$$\forall_x\; x \leq x \tag{β1}$$

$$\forall_x \forall_y \forall_z (x \leq y \wedge y \leq x \rightarrow x = y) \tag{β2}$$

$$\forall_x \forall_y \forall_z (x \leq y \wedge x \leq z \rightarrow x \leq z) \tag{β3}$$

$$\forall_x \forall_y \exists_z \forall_u (z \leq u \equiv x \leq u \wedge y \leq u) \tag{β4}$$

$$\forall_x \forall_y \exists_z \forall_u (u \leq z \equiv u \leq x \wedge u \leq y) \tag{β5}$$

$$\exists_z \forall_u\; u \leq z \tag{β6}$$

$$\exists_z \forall_u\; z \leq u \tag{β7}$$

If the axioms above are true in a $\mathrm{L}_{\leq}$-structure $\mathfrak{X} = \langle X, \leq \rangle$, then the first three say that $\mathfrak{X}$ is a partially ordered set.[3] The following two axioms say that an arbitrarily chosen pair of elements from X have a supremum and a infimum with respect to the relation $\leq$.[4] The final two axioms say

[3] Cf. conditions ($\mathrm{r}_{\leq}$), ($\mathrm{antis}_{\leq}$), and ($\mathrm{t}_{\leq}$).

[4] Cf. conditions (df $\mathsf{sup}_{\leq}$), (4.5) and (L) from Appendix I.

that $\mathfrak{X}$ is bounded, i.e., that it has a unity and a zero. Therefore, the structure $\mathfrak{X}$ is a bounded lattice.

Conversely, if $\langle X, \leq, 0, 1\rangle$ is a bounded lattice and we interpret the predicate "≤" as the relation $\leq$ then this lattice is an $L_{\leq}$-structure which is a model of the first seven axioms of theory **B**.

Applying axioms (β1)–(β3), conditions (U_{sup}) and (U_{inf}), and Gödel's Completeness Theorem, it is easy to see that in axioms (β4)–(β7) we may replace the existential quantifier "∃" with the uniqueness existential quantifier "∃!". Therefore, we may extend the language $L_{\leq}$ by adding two binary function constants "+" and "·", and two individual constants "0" and "1". In this extended language we definitionally extend the theory **B** by adding the following definitions (axioms):

$$\forall_x \forall_y \forall_u (x + y \leq u \equiv x \leq u \wedge y \leq u) \qquad (\delta_+)$$

$$\forall_x \forall_y \forall_u (u \leq x \cdot y \equiv u \leq x \wedge u \leq y) \qquad (\delta_\cdot)$$

$$\forall_u \; u \leq 1 \qquad (\delta_1)$$

$$\forall_u \; 0 \leq u \qquad (\delta_0)$$

Since the presence of function constants and individual constants considerably streamlines the presentation of certain results, we will freely make use of the above definitional extension of the theory **B**, which we will also call **B** in a harmless violation of terminological convention.

With the above definitions in hand, it is now easier for us to formulate the last two axioms of theory **B**:

$$\forall_x \forall_y \forall_z (x + (y \cdot z) = (x + y) \cdot (x + z)) \qquad (\beta 8)$$

$$\forall_x \exists_y (x + y = 1 \wedge x \cdot y = 0) \qquad (\beta 9)$$

We stated above that if the axioms of theory **B** are true in the $L_{\leq}$-structure $\mathfrak{X} = \langle X, \leq\rangle$ then $\mathfrak{X}$ is a bounded lattice. If the function constants "+" and "·", and the individual constants "0" and "1" are interpreted by the functions $+, \cdot \colon X \times X \to X$ and by the elements 0 and 1 of X, respectively, and the last two axioms of theory **B** are true in $\mathfrak{X}$, then axiom (β8) says that $\mathfrak{X}$ is a distributive lattice and axiom (β9) says that $\mathfrak{X}$ is a complemented lattice (so also a uniquely complemented lattices). Therefore, $\mathfrak{X}$ is a Boolean lattice.

Conversely, if $\mathfrak{B} = \langle B, \leq, 0, 1\rangle$ is a Boolean lattice, then — as we saw above — it is a model of axioms (β1)–(β7) of theory **B**. If the function constants "+" and "·", and the individual constants "0" and "1" are interpreted as the functions $+$ and $\cdot$, and the elements 0 and 1 of B, respectively, then the definitions of theory **B** are true in $\mathfrak{B}$. Therefore, axioms (β8) and (β9) of the theory **B** are also true in $\mathfrak{B}$, i.e., $\mathfrak{B}$ is a model of theory **B**.

By making use of Lemma 8.1 from Appendix I and Gödel's Completeness Theorem, we may replace the existential quantifier "$\exists_y$" in the last axiom with the uniqueness existential quantifier "$\exists!_y$". Therefore, we may definitionally expand theory **B** by adding to it a one-place function constant "$-$" defined thus:

$$\forall_x (x + - x = 1 \wedge x \cdot - x = 0) \tag{δ₋}$$

Let $L^d_{\leq}$ be the extension of language $L_{\leq}$ by defined constants: "0", "1", "+", "·", and "−". Henceforth in this appendix all Boolean lattices will be treated as $L^d_{\leq}$-structures.

Now let $\mathrm{Ax}^{\mathbf{B}}$ be the set of specific axioms (including definitions) of the theory **B**, i.e., $\mathbf{B} := \mathrm{Cn}_{L^d_{\leq}}(\mathrm{Ax}^{\mathbf{B}})$. To recapitulate, a given $L^d_{\leq}$-structure is a model of the set $\mathrm{Ax}^{\mathbf{B}}$ (of the theory **B**) iff it is a Boolean lattice (with respect to which the defined constants "+", "·", "−", "0" and "1" are interpreted in the lattice by the operations $+$, $\cdot$, $-$, and elements o and $\mathrm{1}$, respectively). We therefore have:

$$\mathsf{BL} = \mathrm{Mod}(\mathrm{Ax}^{\mathbf{B}})\,, \tag{3.1}$$

and from Gödel's Completeness Theorem we get:

$$\mathrm{Th}(\mathsf{BL}) = \overline{\mathbf{B}}\,.$$

Thus, we see that the class **BL** of all Boolean lattices is finitely elementarily axiomatisable (which is no surprise at all). Below we prove that the class **CBL** of all complete Boolean lattices is not elementarily axiomatisable (see Theorem 5.2). For this purpose we use the elementary invariants (see the next section).

At the end of this section we prove two lemmas.

LEMMA 3.1. *For any Boolean lattice* $\mathfrak{B} = \langle B, \leq, \mathrm{o}, \mathrm{1}\rangle$ *the sets* $\emptyset$, B, $\mathrm{At}(\mathfrak{B})$, $\mathrm{Atc}(\mathfrak{B})$, *and* $\mathrm{Atl}(\mathfrak{B})$ *belong to* $\mathrm{e}\mathcal{P}(\mathfrak{B})$.

PROOF. *Ad* (3.1): For the sets $\emptyset$ and B we use the $L^d_{\leq}$-formulae "$\neg\, x = x$" and "$x = x$", respectively. Moreover, for $\mathrm{At}(\mathfrak{B})$, $\mathrm{Atc}(\mathfrak{B})$, and $\mathrm{Atl}(\mathfrak{B})$ we write the definitions (df At($\mathfrak{L}$)), (df Atc($\mathfrak{L}$)), and (df Atl($\mathfrak{L}$)) in the language $L^d_{\leq}$. We obtain the following $L^d_{\leq}$-formulae, respectively:

$$\mathtt{at}\, x := \quad \neg\, x = 0 \wedge \neg \exists_u (\neg\, u = 0 \wedge \neg\, u = x \wedge u \leq x)$$

$$\mathtt{atc}\,x := \quad \forall_v ((\neg\, v = 0 \wedge v \leq x) \to \exists_u (\mathtt{at}\, u \wedge u \leq v))$$
$$\mathtt{atl}\,x := \quad \neg \exists_u (\mathtt{at}\, u \wedge u \leq x)$$ □

LEMMA 3.2. (i) *If S belongs to $\mathrm{pe}\mathcal{P}(\mathfrak{X})$ (resp. $\mathrm{e}\mathcal{P}(\mathfrak{X})$) for a partially ordered set $\mathfrak{X} = \langle X, \leq \rangle$, then both sets $\mathrm{UB}(S)$ and $\mathrm{LB}(S)$ belong to $\mathrm{pe}\mathcal{P}(\mathfrak{X})$ (resp. $\mathrm{e}\mathcal{P}(\mathfrak{X})$).*

(ii) *If S belongs to $\mathrm{pe}\mathcal{P}(\mathfrak{L})$ (resp. $\mathrm{e}\mathcal{P}(\mathfrak{L})$) for a lattice $\mathfrak{L}$, then also both sets $\mathrm{UB}(S)$ and $\mathrm{LB}(S)$ belong to $\mathrm{pe}\mathcal{P}(\mathfrak{L})$ (resp. $\mathrm{e}\mathcal{P}(\mathfrak{L})$).*

(iii) *If S belongs to $\mathrm{pe}\mathcal{P}(\mathfrak{B})$ (resp. $\mathrm{e}\mathcal{P}(\mathfrak{B})$) for a Boolean lattice $\mathfrak{B}$, then also both sets $\mathrm{UB}(S)$ and $\mathrm{LB}(S)$ belong to $\mathrm{pe}\mathcal{P}(\mathfrak{B})$ (resp. $\mathrm{e}\mathcal{P}(\mathfrak{B})$).*

PROOF. *Ad* (i): Suppose that S is elementarily definable in $\mathfrak{X}$ with parameters $y_1, \ldots, y_k$ from X, where either $k = 0$ or $k > 0$, i.e., there is formula φ of $\mathrm{L}_{\leq}$ such that $\mathrm{vf}(\varphi) = \{\mathtt{x}_1, \ldots, \mathtt{x}_{k+1}\}$ and for any $x \in X$: $x \in S$ iff $\mathfrak{B} \models \varphi\,[x/\mathtt{x}_1, y_1/\mathtt{x}_2, \ldots, y_k/\mathtt{x}_{k+1}]$. Then: $y \in \mathrm{UB}(S)$ iff $\forall_{x \in X} (x \in S \Rightarrow x \leq y)$ iff $\mathfrak{B} \models \forall_{\mathtt{x}_1} (\varphi \to \mathtt{x}_1 \leq \mathtt{x}_2)\,[y_1/\mathtt{x}_2, \ldots, y_k/\mathtt{x}_{k+1}, y/\mathtt{x}_{k+2}]$. So $\mathrm{UB}(S) \in \mathrm{pe}\mathcal{P}(\mathfrak{X})$. The result for $\mathrm{LB}(S)$ is derived in a similar way. The result for $k = 0$ we obtain also in a similar way

Ad (ii) *and* (iii): As for (i), but with using a suitable first-order language. □

4. The elementary invariants

We will present the theory of elementary invariants based on [Ershov, 1980] and [Koppelberg, 1998b, Section 18.1]. For Boolean lattices we define the set of elementary invariants consisting of certain ordered triples $\langle k, l, m \rangle$ such that $k, m \in \mathbb{N} \cup \{\omega\}$ and $l \in \{0, 1\}$, where ω is the cardinality of $\mathbb{N}$.

For any Boolean lattice $\mathfrak{B} = \langle B, \leq, 0, 1 \rangle$ we define the triple $\mathrm{inv}(\mathfrak{B}) = \langle \mathrm{inv}_1(\mathfrak{B}), \mathrm{inv}_2(\mathfrak{B}), \mathrm{inv}_3(\mathfrak{B}) \rangle$ of *elementary invariants* of $\mathfrak{B}$. In Section 18 of Appendix I it was defined as a sequence $\mathfrak{B}^{(0)}$, $\mathfrak{B}^{(1)}$, ... of Boolean lattices composed of quotient Boolean lattices: $\mathfrak{B}/E_0$, $\mathfrak{B}/E_1$, ...

If $\mathfrak{B}$ is trivial then we put $\mathrm{inv}(\mathfrak{B}) := \langle 0, 0, 0 \rangle$. By Lemma 18.3 from Appendix I, if $\mathfrak{B}$ is trivial then all lattices in the sequence $\mathfrak{B}^{(0)}$, $\mathfrak{B}^{(1)}$, ... are trivial, too.

Remark 4.1. Koppelberg [1998b, pp. 289–290] adopted the invariants $\langle -1, 0, 0 \rangle$ for a trivial lattice $\mathfrak{B}$. In [Koppelberg, 1998b], however, no lattice has the invariants $\langle 0, 0, 0 \rangle$. Therefore, our definition is equivalent to the one given there. □

Now we consider those cases when $\mathfrak{B}$ is non-trivial. In such cases, also $\mathfrak{B}^{(0)}$ is non-trivial, by Lemma 18.2 from Appendix I.

In the first of such cases, when for any $n \in \mathbb{N}$ the lattice $\mathfrak{B}^{(n)}$ is non-trivial, we put $\mathrm{inv}(\mathfrak{B}) := \langle \omega, 0, 0 \rangle$.

In the second of such cases, when for some $n \in \mathbb{N} \setminus \{0\}$ the lattice $\mathfrak{B}^{(n)}$ is trivial, let:

1. $\mathrm{inv}_1(\mathfrak{B})$:= the only number $i \in \mathbb{N}$ such that $\mathfrak{B}^{(0)}, \ldots, \mathfrak{B}^{(i)}$ are not-trivial and for any $j > i$: $\mathfrak{B}^{(j)}$ is trivial (cf. Lemma 18.3 from Appendix I);
2. $\mathrm{inv}_2(\mathfrak{B}) = 0$ if $\mathfrak{B}^{(\mathrm{inv}_1(\mathfrak{B}))}$ is atomic; and $\mathrm{inv}_2(\mathfrak{B}) = 1$, otherwise;
3. $\mathrm{inv}_3(\mathfrak{B}) := \min(\omega, \operatorname{Card} \operatorname{At}(\mathfrak{B}^{(\mathrm{inv}_1(\mathfrak{B}))}))$.

Note that if $\mathrm{inv}_1(\mathfrak{B}) = 0$ then both $\mathrm{inv}_2(\mathfrak{B})$ and $\mathrm{inv}_3(\mathfrak{B})$ relate to $\mathfrak{B}$, since $\mathfrak{B}$ and $\mathfrak{B}^{(0)}$ are isomorphic (cf. Lemma 18.2 from Appendix I). Moreover, $\mathrm{inv}_3(\mathfrak{B})$ indicates the number of atoms in a quotient lattice distinguished by $\mathrm{inv}_1(\mathfrak{B})$. If the lattice $\mathfrak{B}$ has infinitely many atoms, then we let $\mathrm{inv}_3(\mathfrak{B}) = \omega$.

All trivial lattices are atomic and atomless. Therefore, the acceptance for them $\mathrm{inv}(\mathfrak{B}) = \langle 0, 0, 0 \rangle$ is also consistent with the definition of the invariants $\mathrm{inv}_2(\mathfrak{B})$ and $\mathrm{inv}_3(\mathfrak{B})$.

PROPOSITION 4.1. (i) *If $\mathfrak{B}$ is non-trivial and $\mathrm{inv}_1(\mathfrak{B}) \neq \omega$, then either $\mathrm{inv}_2(\mathfrak{B}) \neq 0$ or $\mathrm{inv}_3(\mathfrak{B}) \neq 0$.*
(ii) *$\mathrm{inv}(\mathfrak{B}) = \langle 0, 0, 0 \rangle$ iff $\mathfrak{B}$ is trivial.*
(iii) *If $0 \neq \mathrm{inv}_1(\mathfrak{B}) \neq \omega$ then $\mathrm{inv}_2(\mathfrak{B}) + \mathrm{inv}_3(\mathfrak{B}) > 0$.*
(iv) *If $\mathrm{inv}_2(\mathfrak{B}) = 0 = \mathrm{inv}_3(\mathfrak{B})$ then either $\mathrm{inv}_1(\mathfrak{B}) = \omega$ or $\mathrm{inv}_1(\mathfrak{B}) = 0$.*
(v) *$\mathrm{inv}(\mathfrak{B}) = \langle \omega, 0, 0 \rangle$ iff all lattices $\mathfrak{B}, \mathfrak{B}^{(0)}, \mathfrak{B}^{(1)}, \ldots$ are non-trivial.*

PROOF. *Ad* (i): Let $\mathfrak{B}$ be non-trivial and $\mathrm{inv}_1(\mathfrak{B}) \in \mathbb{N}$. Then lattices $\mathfrak{B}, \ldots, \mathfrak{B}^{(\mathrm{inv}_1(\mathfrak{B}))}$ are non-trivial. So if $\mathrm{inv}_2(\mathfrak{B}) = 0$ and $\mathrm{inv}_3(\mathfrak{B}) = 0$, then we obtain a contradiction: $\mathfrak{B}^{(\mathrm{inv}_1(\mathfrak{B}))}$ is non-trivial and atomic, but it has no atoms.

Ad (ii)–(iv): Directly from (i) and definitions.

Ad (v): By definition. Note that if there is $n \in \mathbb{N}$ such that $\mathfrak{B}$, $\mathfrak{B}^{(n)}$ is trivial trivial, then $\mathrm{inv}_1(\mathfrak{B}) \in \mathbb{N}$. □

LEMMA 4.2. (i) $\mathrm{inv}_1(\mathfrak{B}) = 0$ *iff* $\mathrm{E}(\mathfrak{B}) = B$.
(ii) *If $\mathfrak{B}$ is atomic or atomless, then* $\mathrm{inv}_1(\mathfrak{B}) = 0$.
(iii) *If $\mathfrak{B}$ is complete them* $\mathrm{inv}_1(\mathfrak{B}) = 0$.

Proof. *Ad* (i): '⇒' If $\mathrm{inv}_1(\mathfrak{B}) = 0$ then either $\mathfrak{B}$ is trivial or only two lattices $\mathfrak{B}$ and $\mathfrak{B}^{(0)}$ are not trivial. In both cases $\mathfrak{B}^{(1)}$ is trivial. Hence $E_1 = \mathrm{E}(\mathfrak{B}) = B$, by (18.3) and Lemma 18.1 from Appendix I.

'⇐' Suppose that $\mathrm{E}(\mathfrak{B}) = B$. Then $E_1 = B$, by (18.3) from Appendix I. Hence $\mathfrak{B}^{(1)}$ is trivial, by Lemma 18.1 from Appendix I. Therefore either $\mathfrak{B}$ is trivial or only two lattices $\mathfrak{B}$ and $\mathfrak{B}^{(0)}$ are not trivial, by lemmas 18.2 and 18.3 from Appendix I. Hence $\mathrm{inv}_1(\mathfrak{B}) = 0$.

Ad (ii): In the light of Lemma 17.3(ii) from Appendix I, if $\mathfrak{B}$ is atomic or atomless, then $\mathrm{E}(\mathfrak{B}) = B$. So we use (i).

Ad (iii): By (i) and Lemma 17.4(ii) from Appendix I. □

Lemma 4.3. *$\mathfrak{B}$ is atomic iff $\mathrm{inv}_1(\mathfrak{B}) = 0 = \mathrm{inv}_2(\mathfrak{B})$.*

Proof. '⇒' If $\mathfrak{B}$ is atomic then $\mathrm{inv}_1(\mathfrak{B}) = 0$, by Lemma 4.2. Moreover, $\mathrm{inv}_2(\mathfrak{B}) = 0$, since only $\mathfrak{B}$ and $\mathfrak{B}^{(0)}$ are non-trivial and $\mathfrak{B}^{(0)}$ is atomic.

'⇐' Let $\mathrm{inv}_1(\mathfrak{B}) = 0 = \mathrm{inv}_2(\mathfrak{B})$. Then only $\mathfrak{B}$ and $\mathfrak{B}^{(0)}$ are non-trivial and $\mathfrak{B}^{(0)}$ is atomic. Hence also $\mathfrak{B}$ is atomic. □

Lemma 4.4. *$\mathrm{inv}(\mathfrak{B}) = \langle 0, 1, 0\rangle$ iff $\mathfrak{B}$ is non-trivial and atomless.*

Proof. '⇒' If $\mathrm{inv}(\mathfrak{B}) = \langle 0, 1, 0\rangle$, then $\mathfrak{B}$ is non-trivial and only $\mathfrak{B}$ and $\mathfrak{B}^{(0)}$ are non-trivial. Therefore $\mathfrak{B}^{(0)}$ and $\mathfrak{B}$ are not atomic. So $\mathfrak{B}^{(0)}$ and $\mathfrak{B}$ are atomless. '⇐' If $\mathfrak{B}$ is atomless, then $\mathrm{inv}_1(\mathfrak{B}) = 0$, by Lemma 4.2(ii). Hence, if $\mathfrak{B}$ is also non-trivial then only $\mathfrak{B}$ and $\mathfrak{B}^{(0)}$ are non-trivial and $\mathfrak{B}^{(0)}$ is atomless. Hence $\mathrm{inv}(\mathfrak{B}) = \langle 0, 1, 0\rangle$. □

Proposition 4.5. *If $\mathfrak{B}$ is atomic with infinitely many atoms then $\mathrm{inv}(\mathfrak{B}) = \langle 0, 0, \omega\rangle$.*

Proof. If $\mathfrak{B}$ is atomic then $\mathrm{inv}_1(\mathfrak{B}) = 0 = \mathrm{inv}_2(\mathfrak{B})$, by Lemma 4.3. Hence only $\mathfrak{B}$ and $\mathfrak{B}^{(0)}$ are non-trivial and also $\mathfrak{B}^{(0)}$ has infinitely many atoms. So $\mathrm{inv}_3(\mathfrak{B}) = \omega$. □

In [Koppelberg, 1998b, sections 18.1 and 18.2] the following is proven:

Theorem 4.6. *For all $\mathfrak{A}, \mathfrak{B} \in \mathbf{BL}$: $\mathrm{inv}(\mathfrak{A}) = \mathrm{inv}(\mathfrak{B})$ iff $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$. That is, any two Boolean lattices are elementarily equivalent iff they have the same elementary invariants.*

5. The class **CBL** is not elementarily axiomatisable

For all Boolean lattices we will use the first-order language $\mathrm{L}^{\mathrm{d}}_{\leq}$ with the identity predicate "=". Let us begin with the following lemma:

Lemma 5.1. *Boolean lattices* $\mathfrak{P}_{\mathbb{N}}$ *and* $\mathfrak{FC}_{\mathbb{N}}$ *have the same elementary invariants* $\langle 0, 0, \omega\rangle$. *So they are elementarily equivalent.*

Proof. The lattices $\mathfrak{P}_{\mathbb{N}}$ and $\mathfrak{FC}_{\mathbb{N}}$ are non-trivial and atomic. So, by Lemma 4.2(ii) and the definition of $\mathrm{inv}_2(\cdot)$, we have $\mathrm{inv}_1(\mathfrak{P}_{\mathbb{N}}) = \mathrm{inv}_1(\mathfrak{FC}_{\mathbb{N}}) = \mathrm{inv}_2(\mathfrak{P}_{\mathbb{N}}) = \mathrm{inv}_2(\mathfrak{FC}_{\mathbb{N}}) = 0$. Moreover, $\mathrm{inv}_3(\mathfrak{P}_{\mathbb{N}}) = \omega = \mathrm{inv}_3(\mathfrak{FC}_{\mathbb{N}})$, since both lattices have infinitely many atoms. So, by Theorem 4.6, they are elementarily equivalent. □

By Proposition 1.1 and the above lemma we obtain:

Theorem 5.2. *The class* **CBL** *is not elementarily axiomatisable.*

Proof. The Boolean lattices $\mathfrak{P}_{\mathbb{N}}$ and $\mathfrak{FC}_{\mathbb{N}}$ are elementarily equivalent, but the first one is complete and the second is not complete (see examples 11.1 and 11.4 from Appendix I). Thus, by Proposition 1.1, the class of all complete Boolean lattices is not elementarily axiomatisable, since it is not closed under elementary equivalence. □

6. ‘Elementary completeness’ in the class of Boolean lattices

It is known that not all Boolean lattices are complete, i.e., we have **CBL** $\subsetneq$ **BL** (see Example 11.4 in Appendix I). Now we want to examine the class of such Boolean lattices that are not complete, but satisfy condition (cL) of completeness, if we limit it to e-definable sets in a given Boolean lattice. Below we will prove that this limit can be increased to all pe-definable sets (see Theorem 6.6). In the next section we prove that this class is finitely elementarily axiomatisable (see Theorem 7.1).

To start, we will prove two lemmas.

Lemma 6.1. *Let* $\mathfrak{X} = \langle X, \leq\rangle$ *be a partially ordered set.*

(i) $\mathfrak{X}$ *satisfies*

$$\forall_{S\in \mathrm{pe}\mathcal{P}(\mathfrak{X})}\exists_{x\in X}\ x \ \mathsf{sup}_{\leq}\ S\,. \tag{pecX}$$

iff $\mathfrak{X}$ *satisfies*

$$\forall_{S\in \mathrm{pe}\mathcal{P}(\mathfrak{X})}\exists_{x\in X}\ x \ \mathsf{inf}_{\leq}\ S\,. \tag{epcX$'$}$$

If $\mathfrak{X}$ *satisfies the above conditions then it is a bounded lattice.*

(ii) $\mathfrak{X}$ *satisfies*

$$\forall_{S\in \mathrm{e}\mathcal{P}(\mathfrak{X})}\exists_{x\in X}\ x \ \mathsf{sup}_{\leq}\ S\,. \tag{ecX}$$

iff $\mathfrak{X}$ *satisfies*

$$\forall_{S\in \mathrm{e}\mathcal{P}(\mathfrak{X})}\exists_{x\in X}\ x \ \mathsf{inf}_{\leq}\ S\,. \tag{ecX$'$}$$

If $\mathfrak{X}$ *satisfies the above conditions then it bounded.*

PROOF. *Ad* (i): Firstly, we use Lemma 3.2(i) and conditions (4.10) and (4.11) from Appendix I. Secondly, the given conditions imply that each finite subset of X has a supremum and a infimum, since such sets belong to $\mathrm{pe}\mathcal{P}(\mathfrak{X})$. So if $\mathfrak{X}$ satisfies the above conditions then it is a lattice. It is bounded. We have $0 \ \mathsf{sup}_{\leq} \ \emptyset$ and $1 \ \mathsf{sup}_{\leq} \ X$, since $\emptyset, X \in \mathrm{e}\mathcal{P}(\mathfrak{X})$.

Ad (ii): As for (i). □

LEMMA 6.2. *A partially ordered set* $\mathfrak{X} = \langle X, \leq \rangle$ *satisfies* (pecX) *iff for any* $\mathrm{L}_{\leqslant}$*-formula* φ *such that for some* $k \geqslant 0$ *we have* $\mathrm{vf}(\varphi) = \{\mathtt{x}_1, \ldots, \mathtt{x}_{k+1}\}$, *the closure of the following formula*[5] *is true in* $\mathfrak{X}$:

$$\exists_{\mathtt{x}_{k+2}} \forall_{\mathtt{x}_{k+3}} (\mathtt{x}_{k+2} \leq \mathtt{x}_{k+3} \equiv \forall_{\mathtt{x}_1} (\varphi \rightarrow \mathtt{x}_1 \leq \mathtt{x}_{k+3})) \qquad (\mathsf{sup}^k_{\varphi})$$

PROOF. '⇒' Let $\mathfrak{X}$ satisfy (pecX) and φ be any $\mathrm{L}_{\leqslant}$-formula such that $\mathrm{vf}(\varphi) = \{\mathtt{x}_1, \ldots, \mathtt{x}_{k+1}\}$, for some $k \geqslant 0$. If $k > 0$ then $\mathrm{vf}(\mathsf{sup}^k_{\varphi}) = \{\mathtt{x}_2, \ldots, \mathtt{x}_{k+1}\}$; otherwise, $\mathrm{vf}(\mathsf{sup}^0_{\varphi}) = \emptyset$. If $k > 0$ then we take arbitrary $y_1, \ldots, y_k$ from X; otherwise, we do not select any object. Notice that the following set

$$S^k_{\varphi} := \{z \in X : \mathfrak{B} \models \varphi[z/\mathtt{x}_1, y_1/\mathtt{x}_2, ..., y_k/\mathtt{x}_{k+1}]\}$$

belongs to $\mathrm{pe}\mathcal{P}(\mathfrak{X})$. If $k = 0$ then the set $S^0_{\varphi} := \{z \in X : \mathfrak{B} \models \varphi[z/\mathtt{x}_1]\}$ belongs to $\mathrm{e}\mathcal{P}(\mathfrak{X})$.

By virtue of our assumption, for some $x_0 \in X$ we have $x_0 \ \mathsf{sup}_{\leq} \ S^k_{\varphi}$. Hence, by (4.5) from Appendix I, for any $u \in X$ we have: $x_0 \leq u$ iff $\forall_{z \in S_{\varphi}} z \leq u$. So the formula "$\forall_{\mathtt{x}_{k+3}} (\mathtt{x}_{k+2} \leq \mathtt{x}_{k+3} \equiv \forall_{\mathtt{x}_1} (\varphi \rightarrow \mathtt{x}_1 \leq \mathtt{x}_{k+3}))$" is satisfied in $\mathfrak{X}$ by the valuation $[y_1/\mathtt{x}_2, ..., y_k/\mathtt{x}_{k+1}, x_0/\mathtt{x}_{k+2}]$. Hence the formula "$\exists_{\mathtt{x}_{k+2}} \forall_{\mathtt{x}_{k+3}} (\mathtt{x}_{k+2} \leq \mathtt{x}_{k+3} \equiv \forall_{\mathtt{x}_1} (\varphi \rightarrow \mathtt{x}_1 \leq \mathtt{x}_{k+3}))$" is satisfied in $\mathfrak{X}$ by the valuation $[y_1/\mathtt{x}_2, ..., y_k/\mathtt{x}_{k+1}]$. Since $y_1, \ldots, y_{k+1}$ were arbitrarily chosen, then the closure of (sup^k_{φ}) is true in $\mathfrak{X}$.

'⇐' Suppose that $\mathfrak{X}$ fulfills the condition on the right-hand side. We take an arbitrary $S \in \mathrm{pe}\mathcal{P}(\mathfrak{X})$, i.e., for some $k \geqslant 0$, $y_1, \ldots, y_k \in X$, and $\mathrm{L}_{\leqslant}$-formula φ_0 for which $\mathrm{vf}(\varphi_0) = \{\mathtt{x}_1, \ldots, \mathtt{x}_{k+1}\}$, for any $x \in X$ we have: $x \in S$ iff $\mathfrak{B} \models \varphi_0[x/\mathtt{x}_1, y_1/\mathtt{x}_2, ..., y_k/\mathtt{x}_{k+1}]$. By virtue of our assumption, the closure of the formula ($\mathsf{sup}^k_{\varphi_0}$) is true in $\mathfrak{X}$. Hence the formula "$\exists_{\mathtt{x}_{k+2}} \forall_{\mathtt{x}_{k+3}} (\mathtt{x}_{k+2} \leq \mathtt{x}_{k+3} \equiv \forall_{\mathtt{x}_1} (\varphi_0 \rightarrow \mathtt{x}_1 \leq \mathtt{x}_{k+3}))$" is satisfied in $\mathfrak{X}$ by the valuation $[y_1/\mathtt{x}_2, ..., y_k/\mathtt{x}_{k+1}]$. Hence, by (4.5) from Appendix I, there is an $x \in X$ such that $x \ \mathsf{sup}_{\leq} \ S$. □

[5] For $k = 0$ the following formula is a sentence which is its own closure.

Of course, the above two lemmas also hold for all lattices, and so for all Boolean lattices. All you need to do is take a suitable first-order language. Then we will use Lemma 3.2(ii) or Lemma 3.2(iii) instead of Lemma 3.2(i).

Let $\mathfrak{B} = \langle B, \leq, 0, 1\rangle$ be a Boolean lattice. We call $\mathfrak{B}$ *elementarily complete* (or shortly: *e-complete*) iff each e-definable set in $\mathfrak{B}$ has a supremum, i.e., the following holds:

$$\forall_{S\in \mathrm{e}\mathcal{P}(\mathfrak{B})}\exists_{x\in B}\, x \,\mathsf{sup}_{\leq}\, S\,. \tag{ecB}$$

Moreover, we call $\mathfrak{B}$ *parametrically elementarily complete* (or shortly: *pe-complete*) iff each pe-definable set in $\mathfrak{B}$ has a supremum, i.e., the following holds:

$$\forall_{S\in \mathrm{pe}\mathcal{P}(\mathfrak{B})}\exists_{x\in B}\, x \,\mathsf{sup}_{\leq}\, S\,. \tag{epcB}$$

Let **ecBL** (resp. **pecBL**) be the class of e-complete (resp. pe-complete) Boolean lattices.. Directly from our definitions we have **CBL** $\subseteq$ **pecBL** $\subseteq$ **ecBL**, but we will prove that **CBL** $\subsetneq$ **pecBL** $=$ **ecBL** (see Theorem 6.7).

Directly from Lemma 6.2 we obtain:

Lemma 6.3. (i) *A Boolean lattice $\mathfrak{B}$ is pe-complete iff for any $\mathrm{L}^{\mathrm{d}}_{\leqslant}$-formula φ such that for some $k \geqslant 0$ we have $\mathrm{vf}(\varphi) = \{x_1, \ldots, x_{k+1}\}$, the closure of the formula (sup^k_φ) is true in $\mathfrak{B}$.*

(ii) *A Boolean lattice $\mathfrak{B}$ is e-complete iff for any $\mathrm{L}^{\mathrm{d}}_{\leqslant}$-formula φ such that $\mathrm{vf}(\varphi) = \{x_1\}$, the sentence ($\mathsf{sup}^0_\varphi$) is true in $\mathfrak{B}$.*

Proof. *Ad* (i): For a given Boolean lattice $\mathfrak{B}$ as for Lemma 6.2, but with using the language $\mathrm{L}^{\mathrm{d}}_{\leqslant}$.

Ad (ii): From (i) for $k = 0$. □

Directly from the above lemma we obtain:

Proposition 6.4. *The classes* **ecBL** *and* **pecBL** *are closed under elementary equivalence.*

By using the above proposition we obtain:

Proposition 6.5. *All atomic Boolean lattices and all atomless Boolean lattices belong to the class* **pecBL**; *so also to* **ecBL**.

Proof. All trivial Boolean lattices belong to **CBL**. So we suppose that $\mathfrak{B}$ is non-trivial Boolean lattice.

By Lemma 4.3, if $\mathfrak{B}$ is atomic then $\mathrm{inv}_1(\mathfrak{B}) = 0 = \mathrm{inv}_2(\mathfrak{B})$. Thus, we will consider two cases. First, assume that $\mathrm{inv}_3(\mathfrak{B}) = n$, for some

$n > 0$. Since $\mathrm{inv}_1(\mathfrak{B}) = 0$, so $\mathrm{inv}_3(\mathfrak{B})$ relates to the same lattice $\mathfrak{B}$, i.e., it has n atoms. Therefore $\mathfrak{B}$ is finite, and so it is complete. Hence $\mathfrak{B} \in$ **CBL** $\subseteq$ **pecBL**. Second, assume that $\mathrm{inv}_3(\mathfrak{B}) = \omega$, i.e., $\mathfrak{B}$ has infinitely many atoms. By Lemma 5.1 the Boolean lattice $\mathfrak{P}_{\mathbb{N}}$ has the elementary invariants $\langle 0, 0, \omega\rangle$. By virtue of Theorem 4.6, the lattices $\mathfrak{B}$ and $\mathfrak{P}_{\mathbb{N}}$ are elementarily equivalent. Since $\mathfrak{P}_{\mathbb{N}} \in$ **CBL**, so also $\mathfrak{B} \in$ **pecBL**, by virtue of Proposition 6.4.

In the light of Lemma 4.4, if $\mathfrak{B}$ is non-trivial and atomless then $\mathrm{inv}(\mathfrak{B}) = \langle 0, 1, 0\rangle$. In addition, the atomless complete Boolean lattice $\mathrm{RO}(\mathbb{R})$, considered in Example 16.1 from Appendix I, also has the elementary invariants $\langle 0, 1, 0\rangle$. By virtue of Theorem 4.6, these lattices are elementarily equivalent. Since $\mathrm{RO}(\mathbb{R}) \in$ **CBL**, so also $\mathfrak{B} \in$ **pecBL**, by Proposition 6.4. □

THEOREM 6.6. *For any Boolean lattice* $\mathfrak{B}$:

$$\mathfrak{B} \in \textbf{ecBL} \iff \mathrm{E}(\mathfrak{B}) = B \iff \mathfrak{B} \in \textbf{pecBL}\,.$$

Thus, **ecBL** = **pecBL**.

PROOF. '$\mathfrak{B} \in$ **ecBL** $\Rightarrow \mathrm{E}(\mathfrak{B}) = B$' Assume that $\mathfrak{B} \in$ **ecBL**. Because $\mathrm{At}(\mathfrak{B}) \in \mathrm{e}\mathcal{P}(\mathfrak{B})$, so there is an $x \in B$ such that $x \,\mathsf{sup}_{\leq}\, \mathrm{At}(\mathfrak{B})$. Therefore $\mathrm{E}(\mathfrak{B}) = B$, by Lemma 17.4(i) from Appendix I.

'$\mathrm{E}(\mathfrak{B}) = B \Rightarrow \mathfrak{B} \in$ **pecBL**' Let $\mathrm{E}(\mathfrak{B}) = B$. Then $\mathrm{inv}_1(\mathfrak{B}) = 0$, by Lemma 4.2(i). Therefore, $\mathfrak{B}$ has one of the following elementary invariants:

1. $\langle 0, 0, n\rangle$, for some $n \in \mathbb{N}$;
2. $\langle 0, 0, \omega\rangle$;
3. $\langle 0, 1, 0\rangle$;
4. $\langle 0, 1, n\rangle$, for some $n \in \mathbb{N} \setminus \{0\}$;
5. $\langle 0, 1, \omega\rangle$.

In cases 1–3, as in the proof of Proposition 6.5, we can show that $\mathfrak{B} \in$ **pecBL**.

In case 4, let $\mathfrak{B}_1$ be an arbitrary finite Boolean lattice with n atoms and let $\mathfrak{B}_2 := \mathrm{RO}(\mathbb{R})$ (see Example 16.1 from Appendix I). Since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are complete then the lattice $\mathfrak{B}_1 \times \mathfrak{B}_2$ is complete, by Lemma 11.1 from Appendix I. Moreover, $\mathrm{E}(\mathfrak{B}_1 \times \mathfrak{B}_2) = \mathbb{N} \times \mathbb{R}$, by Lemma 17.4(ii) from Appendix I. Hence $\mathrm{inv}_1(\mathfrak{B}_1 \times \mathfrak{B}_2) = 0$, by Lemma 4.2(i). Since $\mathfrak{B}_2$ is atomless, so $\mathrm{At}(\mathfrak{B}_1 \times \mathfrak{B}_2) = \{\langle i, \emptyset\rangle : i \in \mathrm{At}(\mathfrak{B}_1)\}$ and $\langle \mathrm{o}, \mathbb{R}\rangle$ belongs to $\mathrm{Atl}(\mathfrak{B}_1 \times \mathfrak{B}_2)$. Thus, $\mathfrak{B}_1 \times \mathfrak{B}_2$ is not atomic and $\mathrm{inv}(\mathfrak{B}_1 \times \mathfrak{B}_2) = \langle 0, 1, n\rangle$.

In case 5, we put $\mathfrak{B}_1 := \mathfrak{P}_{\mathbb{N}}$ and $\mathfrak{B}_2 := \mathrm{RO}(\mathbb{R})$. The lattice $\mathfrak{B}_1 \times \mathfrak{B}_2$ is complete, $\mathrm{E}(\mathfrak{B}_1 \times \mathfrak{B}_2) = \mathbb{N} \times \mathbb{R}$, and $\mathrm{inv}_1(\mathfrak{B}_1 \times \mathfrak{B}_2) = 0$, as above. Since $\mathrm{At}(\mathfrak{B}_1 \times \mathfrak{B}_2) = \{\langle i, \emptyset\rangle : i \in \mathbb{N}\}$ and $\langle \text{o}, \mathbb{R}\rangle$ belongs to $\mathrm{Atl}(\mathfrak{B}_1 \times \mathfrak{B}_2)$, so $\mathrm{inv}(\mathfrak{B}_1 \times \mathfrak{B}_2) = \langle 0, 1, \omega\rangle$.

Thus, also in cases 4 and 5, as in the proof of Proposition 6.5, we can show that $\mathfrak{B} \in$ **pecBL**.

'$\mathfrak{B} \in$ **pecBL** $\Rightarrow \mathfrak{B} \in$ **ecBL**' By definitions. □

From Proposition 6.5 and Theorem 6.6 we obtain:

Theorem 6.7. **CBL** $\subsetneq$ **pecBL** $=$ **ecBL** $\subsetneq$ **BL**.

Proof. First, in examples 11.4 and 15.1 from Appendix I show two atomic Boolean lattice which are not complete. Thus, **CBL** $\subsetneq$ **pecBL**, by Proposition 6.5. Second, by Proposition 18.5 from [Koppelberg, 1998b], there is a Boolean lattice $\mathfrak{B}$ such that $\mathrm{inv}_1(\mathfrak{B}) \neq 0$. Hence $\mathrm{E}(\mathfrak{B}) \neq B$, by Lemma 17.1(ii) from Appendix I. And therefore $\mathfrak{B} \notin$ **ecBL**, by Theorem 6.6. Thus, **ecBL** $\subsetneq$ **BL**. □

Although **CBL** $\neq$ **ecBL**, in these classes the same elementary sentences are true.

Proposition 6.8. $\mathrm{Th}($**CBL**$) = \mathrm{Th}($**ecBL**$) = \mathrm{Th}($**pecBL**$)$.

Proof. Since **CBL** $\subseteq$ **ecBL**, then $\mathrm{Th}($**ecBL**$) \subseteq \mathrm{Th}($**CBL**$)$. To show the converse, we take arbitrary $\varphi \in \mathrm{Th}($**CBL**$)$ and $\mathfrak{B} \in$ **ecBL**. We will show that $\varphi \in \mathrm{Th}(\mathfrak{B})$ which will prove the the inclusion in question.

Because $\mathfrak{B} \in$ **ecBL**, then $\mathrm{inv}_1(\mathfrak{B}) = 0$, by Lemma 4.2(i). Therefore, as in the proof of Theorem 6.6, we consider the following possibilities for elementary invariants of $\mathfrak{B}$:

1. $\langle 0, 0, n\rangle$, for some $n \in \mathbb{N}$;
2. $\langle 0, 0, \omega\rangle$;
3. $\langle 0, 1, 0\rangle$;
4. $\langle 0, 1, n\rangle$, for some $n \in \mathbb{N} \setminus \{0\}$;
5. $\langle 0, 1, \omega\rangle$.

In case 1 in the proof of Proposition 6.5 we show that $\mathfrak{B} \in$ **CBL**. Thus, $\varphi \in \mathrm{Th}(\mathfrak{B})$.

In cases 2–5 in the proof of Theorem 6.6 we showed that there exists a lattice $\mathfrak{B}'$ from **CBL** with the same characteristic. Hence, by Theorem 4.6, we have $\mathrm{Th}(\mathfrak{B}) = \mathrm{Th}(\mathfrak{B}')$. Therefore, $\varphi \in \mathrm{Th}(\mathfrak{B})$, since $\varphi \in \mathrm{Th}($**CBL**$)$ and $\mathfrak{B}' \in$ **CBL**. □

Remark 6.1. Consider the following three sentences of the language $\mathrm{L}^{\mathrm{d}}_{\leq}$:

$$\exists_x \exists_y (\mathtt{atc}\, x \wedge \mathtt{atl}\, y \wedge 1 = x + y) \tag{εβ1}$$

$$\exists_x (\mathtt{atc}\, x \wedge \mathtt{atl} - x) \tag{εβ2}$$

$$\forall_z \exists_x \exists_y (\mathtt{atc}\, x \wedge \mathtt{atl}\, y \wedge z = x + y) \tag{εβ3}$$

where the $\mathrm{L}^{\mathrm{d}}_{\leq}$-formulae "$\mathtt{atc}\, x$" and "$\mathtt{atc}\, y$" are given in the proof of Lemma 3.1. They say: "x is atomic" and "y is atomless", respectively.

Sentence (εβ3) says that the condition "$\mathrm{E}(\mathfrak{B}) = B$" holds. Sentence (εβ1) says that "$1 \in \mathrm{E}(\mathfrak{B})$" holds. In the light of Lemma 17.1(ii) from Appendix I, these sentences are equivalent in the elementary theory **B** of Boolean lattices. Moreover, by virtue of Lemma 17.2 from Appendix I, sentences (εβ1) and (εβ2) are equivalent, too. Our analysis shows that (εβ2)–(εβ2) belong to Th(**ecBL**) (= Th(**CBL**)). But there is a Boolean lattice $\mathfrak{B}$ such that $\mathrm{E}(\mathfrak{B}) \neq B$ (it follows, for example, from Proposition 18.5 in [Koppelberg, 1998b]). Hence none of the sentences (εβ1)–(εβ3) belongs to Th(**BL**). Thus, Th(**BL**) $\subsetneq$ Th(**CBL**). □

At the end of this section we will investigate a counterpart of Theorem 11.2 from Appendix I in which we swap just the family $\mathcal{P}(L)$ for the family $\mathfrak{pe}\mathcal{P}(\mathfrak{L})$ in condition ($\star$). This new theorem yields a necessary and sufficient condition for lattices with zero for them to be e-complete Boolean lattices.

Theorem 6.9. *Let $\mathfrak{L} = \langle L, \leq, \mathrm{o} \rangle$ be a lattice with zero. Then for $\mathfrak{L}$ to be an e-complete Boolean lattice it is both necessary and sufficient that the relation $\leq$ satisfies the following condition:*

for any $S \in \mathfrak{pe}\mathcal{P}(\mathfrak{L})$ there is exactly one $x \in L$ such that
(a) $\forall_{z \in S}\ z \leq x$ *and* ($\star\star$)
(b) $\forall_{u \in L} \big(u \leq x \wedge \forall_{z \in S}\ u \cdot z = \mathrm{o} \implies u = \mathrm{o}\big)$.

Proof. '⇒' Let $\mathfrak{L}$ be an e-complete Boolean lattice. Take an arbitrary $S \in \mathfrak{pe}\mathcal{P}(\mathfrak{L})$. Since $\mathfrak{L} \in$ **pecBL**, we can put $x := \sup S$. From the definition of supremum and by virtue of (7.5) from Appendix I, x satisfies conditions (a) and (b) from ($\star\star$) for the set S. In a manner analogous to that employed in the proof of Theorem 11.2 from Appendix I, we will show that x is the only element in $\mathfrak{L}$ satisfying conditions (a) and (b) from ($\star\star$) for the set S.

'⇐' Let $\mathfrak{L}$ be a lattice with zero in which ($\star\star$) holds. Observe that by carrying out a proof analogous to the proof of Lemma 6.5 from Appendix I, it is possible to show that that sentence (sep) (p. 267) is true in $\mathfrak{L}$. In essence, in the proof of this lemma we made use of condition ($\star$) only for the sets $S_0, S_1 \in \mathcal{P}(L)$ which were pe-definable in $\mathfrak{L}$. Therefore, we repeat the proof with the weaker condition ($\star\star$).

We show, analogously to the proof of Theorem 11.2 from Appendix I, that the lattice $\mathfrak{L}$ satisfies condition (epcB). Moreover, since $L \in \mathrm{e}\mathcal{P}(\mathfrak{L})$, then the lattice $\mathfrak{L}$ is also bound: $1 \ \mathsf{sup}_{\leq}\ L$.

We prove the distributivity of $\mathfrak{L}$ by repeating the appropriate fragment of the proof of the above-mentioned theorem, taking for arbitrary $u, v \in L$ the set $S := \{u, v\}$, which is pe-definable with parameters u and v. Obviously, $x_S := \sup\{x \cdot z : z \in S\} = (x \cdot u) + (x \cdot v)$.

Finally, since the lattice $\mathfrak{L}$ satisfies condition (epcB) and for any $x \in L$ the set $\{z \in L : x \cdot z = 0\}$ is pe-definable with the parameter x, so this set has a least upper bound, which is — by virtue of Lemma 8.3 from Appendix I — the complement of x. □

7. The class ecBL is finitely elementarily axiomatisable

By applying Lemma 6.2, we see that the class of partially ordered sets satisfying (pecX) is finitely elementarily axiomatisable by the set composed of sentences (β1)–(β3) and an infinite number of $\mathrm{L}_{\leq}$-formulae of the form (sup^k_φ) defined in the lemma. As we showed in Lemma 6.1, each partially ordered set satisfying (pecX) is a bounded lattice. Therefore, from Lemma 6.2 and the considerations of Section 3, it follows that we can derive $\mathrm{L}_{\leq}$-sentences (β4)–(β7) from the infinite set of axioms: (β1)–(β3) and (sup^k_φ) defined in Lemma 6.2.[6]

As in Section 3, we can extend the language $\mathrm{L}_{\leq}$ to the language $\mathrm{L}^{\mathrm{d}}_{\leq}$ and the elementary theory above through definitions (δ_+), ($\delta_\cdot$), (δ_1), and (δ_0). With them in hand, we can add the two axioms (β8) and (β9), and definition (δ_-). Let $\mathrm{Ax}^{\mathbf{pecB}}$ be the infinite set of specific axioms (including definitions) of the theory discussed here: (β1)–(β3), (β8), (β9), (δ_+), ($\delta_\cdot$), (δ_1), (δ_0), (δ_-), and (sup^k_φ), for any $k \geqslant 0$ and any formula φ of $\mathrm{L}^{\mathrm{d}}_{\leq}$ with $k+1$ free variables.

We may also consider the following infinite set $\mathrm{Ax}^{\mathbf{ecB}}$ of specific axioms (including definitions): (β1)–(β5), (β8), (β9), (δ_+), ($\delta_\cdot$), (δ_1), (δ_0), (δ_-), and sup^0_φ, for any formula φ of $\mathrm{L}^{\mathrm{d}}_{\leq}$ with one free variable. As shown

[6] This can be easily checked directly by applying axioms (β1)–(β3) and (sup^k_φ) in turn to the $\mathrm{L}_{\leq}$-formulae "$u = x \wedge u = y$", "$u \leq x \wedge u \leq y$", "$u = u$", and "$\neg\, u = u$" (with appropriate changes to the variables).

in Lemma 6.1(ii), we can derive $\mathrm{L}_{\leqslant}$-sentences (β6) and (β7) from our axioms.

We put **pecB** $:= \mathrm{Cn}_{\mathrm{L}^{\mathrm{d}}_{\leqslant}}(\mathrm{Ax}^{\mathbf{pecB}})$ and **ecB** $:= \mathrm{Cn}_{\mathrm{L}^{\mathrm{d}}_{\leqslant}}(\mathrm{Ax}^{\mathbf{ecB}})$, i.e., **pecB** (resp. **ecB**) is the elementary theory defined by the set of specific axioms $\mathrm{Ax}^{\mathbf{pecB}}$ (resp. $\mathrm{Ax}^{\mathbf{pecB}}$). From the facts presented above and in Section 3, it follows that:

$$\mathrm{Mod}(\mathrm{Ax}^{\mathbf{ecB}}) = \mathsf{ecBL} = \mathsf{pecBL} = \mathrm{Mod}(\mathrm{Ax}^{\mathbf{pecB}}),$$

and from Gödel's theorem, we get

$$\overline{\mathbf{ecB}} = \mathrm{Th}(\mathsf{ecBL}) = \mathrm{Th}(\mathsf{pecBL}) = \overline{\mathbf{pecB}}. \tag{7.1}$$

We may therefore call the theory **pecB** (resp. **ecB**) an *elementary theory of pe-complete* (resp. e-complete) *Boolean lattices*. We have also:

$$\mathbf{ecB} = \mathbf{pecB}.$$

Proposition 6.8 gives: $\mathrm{Th}(\mathsf{CBL}) = \mathrm{Th}(\mathsf{ecBL}) = \mathrm{Th}(\mathsf{pecBL})$. Therefore to the theory **ecB** (= **pecB**) belong those and only those $\mathrm{L}^{\mathrm{d}}_{\leqslant}$-formulae which are true in all complete Boolean lattices, i.e., $\mathrm{Th}(\mathsf{CBL}) = \overline{\mathbf{ecB}}$. From this perspective, we may simply call the theory **ecB** an *elementary theory of the class* **CBL** and use "**CB**" to signify it.[7]

From Theorem 6.6 it follows that there exists a finite axiomatisation for the theory **CB**, because because we have:

THEOREM 7.1. *The class* **ecBL** *is finitely elementarily axiomatisable.*

PROOF. We will give the finite axiomatisation for the class **ecBL** in the extension of the language $\mathrm{L}^{\mathrm{d}}_{\leqslant}$. It is composed of the sentences of the set $\mathrm{Ax}^{\mathbf{B}}$ and and one of the three sentences ($\varepsilon\beta$3)–($\varepsilon\beta$2).

Let us put $\mathrm{Ax}^{\mathbf{CB}} := \mathrm{Ax}^{\mathbf{B}} \cup \{(\varepsilon\beta1)\}$. From (3.1) and Theorem 6.6 it follows that the given structure is a model of the set $\mathrm{Ax}^{\mathbf{CB}}$ iff it belongs to **ecBL**.

Moreover, in Remark 6.1 we show the $\mathrm{L}^{\mathrm{d}}_{\leqslant}$-sentence ($\varepsilon\beta$3), ($\varepsilon\beta$1), and ($\varepsilon\beta$2) are equivalent in the elementary theory **B** of Boolean lattices. □

[7] Obviously, $\mathsf{CBL} \subsetneq \mathrm{Mod}(\mathbf{CB}) = \mathsf{ecBL}$ (cf. Theorem 6.7 and (7.1)).

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List of symbols

$\neg$	truth-connective of negation, page 17
$\wedge$	truth-connective of conjunction, page 17
$\vee$	truth-connective of inclusive disjunction, page 17
$\Rightarrow$	truth-connective of material implication, page 17
$\Leftrightarrow$	truth-connective of material biconditional, page 17
$\forall$	universal quantifier, page 17
$\exists$	existential quantifier, page 17
$\neq$	symbol of non-identity, page 17
$=$	symbol of identity, page 18
$\sqsubset$	relation *is a part of*, page 19
$\not\sqsubset$	negation of $\sqsubset$, page 19
$\in$	predicate “is an element of”, page 19
$\sqsubseteq$	binary relation *is an ingrediens of*, page 20
Cl	predicate “is a class”, page 32
Set	predicate “is a set”, page 32
pCl	predicate “is a proper class”, page 32
(L1)	first axiom of Leśniewski’s mereology, page 71
(L2)	second axiom of Leśniewski’s mereology, page 71
L12	class of structures satisfying conditions (L1) and (L2) (= the class of strictly partially ordered sets), page 71
SPOS	class of strictly partially ordered sets, page 71
MS	class of mereological structures, page 71
L_{c}	elementary language for the class **MS**, page 71
c	predicate “is a part of” in the language L_{c}, page 71
$\mathbb{P}$	parts-function, page 73
$\mathbb{I}$	ingredienses-function, page 73
$\mathbb{at}$	set of mereological atoms, page 75
O	relation *overlaps*, page 75
$\mathbb{O}$	overlapping-function, page 76
$\wr$	relation *is exterior to*, page 76
$\mathbb{Sum}$	relation *is a mereological sum of* a given subset of the universe of a given mereological structure, page 78

$\mathbb{O}[S]$	family of set which is the image of a set S determined by the function $\mathbb{O}$, page 78
$\bigcup \mathbb{O}[S]$	set-theoretical sum of the image of a set S determined by the function $\mathbb{O}$, page 78
(L3)	third axiom of Leśniewski's mereology, page 82
Card	Cardinality of sets, page 84
L123	class of structures satisfying conditions (L1), (L2), and (L3), page 86
$\leq$	predicate "is an ingrediens of" in elementary languages, page 86
O	predicate "overlaps with" in elementary languages, page 86
$\rceil$	predicate "is exterior to" in elementary languages, page 86
(L4)	fourth axiom of Leśniewski's mereology, page 87
(ιx)	description operator, page 88
$\exists!$	uniqueness existential quantifier, page 89
$\mathbb{1}$	unity in a given mereological structure = the mereological sum of all members of the universe of a given mereological structure, page 89
$\bigsqcup$	unary operation of mereological sum of a given set in a given mereological structure, page 94
$\bigsqcup S$	mereological sum of members of a set S in a given mereological structure, page 94
$\sqcup$	binary operation of mereological sum of elements in a given mereological structure, page 95
$\mathsf{sup}_{\sqsubseteq}$	relation *is a supremum of* with respect to $\sqsubseteq$, page 97
$\mathbb{I}[S]$	family of set which is the image of a set S determined by the function $\mathbb{I}$, page 99
$\bigcap \mathbb{I}[S]$	set-theoretical product of the image of a set S determined by the function $\mathbb{I}$, page 99
$\bigsqcap$	unary operation of mereological product of a given set in a given mereological structure, page 99
$\bigsqcap S$	mereological product of members of a set S in a given mereological structure, page 99
$\mathsf{inf}_{\sqsubseteq}$	relation *is an infimum of* with respect to $\sqsubseteq$, page 100
$\sqcap$	binary operation of mereological product of elements in a given mereological structure, page 101
c	mereological complement operation, page 104
$\smallsetminus$	relative mereological complement (difference) in a given mereological structure, page 106
$\mathrm{F}_{\mathfrak{M}}$	family of all filters in a mereological structure $\mathfrak{M}$, page 107
$\mathrm{Ult}_{\mathfrak{M}}$	family of ultrafilters (maximal filters) in a mereological structure $\mathfrak{M}$, page 108
$\mathrm{PF}_{\mathfrak{M}}$	family of primary filters in $\mathfrak{M}$, page 109
s	Stone map in a given mereological structure, page 110

$(\mathfrak{B}_4)$ one of the Tarski's postulates of the extended system of Boolean algebra, page 113
$(\mathfrak{B}_4^*)$ one of the Tarski's postulates of the extended system of Boolean algebra, page 113
$\mathrm{L}_{\leq}^{0}$ elementary language with identity and two specific constants, page 122
$\leq$ two-place predicate constant, page 122
0 individual constant, page 122
TS class of mereological structures in Tarski's sense, page 125
$\blacktriangleleft$ binary relation in mereological structures in Tarski's sense, page 126
$\mathrm{L}_{\leq}$ elementary language for the class **TS**, page 129
Δ bijection from **MS** onto **TS**, page 129
MS* class of generalised mereological structures, page 131
$\lhd$ transitive relation of the class **MS***, page 131
$\unlhd$ reflexive and transitive relation of the class **MS***, page 131
$\mathbb{Fu}$ relation *is a fusion of* a given subset of the universe of a given mereological structure, page 136
$\mathbb{Nu}$ relation of *being a product* (*nucleus*) in Leonard and Goodman's system, page 141
$\mathbb{Agr}$ relation *is an aggregate of* a given subset of the universe, page 151
Σ partial operator of the sum (fusion), page 168
σ relation of the sum (fusion), page 169
Π partial operator of the product, page 171
ML12 class of mereological strictly partially ordered sets, page 181
MEM class of models of Simons' Minimal Extensional Mereology, page 182
GMS class of Grzegorczyk's mereological structures, page 186
M elementary mereology, page 193
M axioms of elementary mereology, page 193
$\mathrm{L}_{\mathsf{c}}^{\mathrm{d}}$ extension of language L_{c} by defined constants, page 193
M_{φ}^{x} set of elements of M satisfying φ with one variable, page 194
σ_{φ}^{x} formula that represents the relation of mereological sum, page 195
$\mathfrak{pe}\mathcal{P}(\mathfrak{M})$ family of sets which are elementarily definable in a structure $\mathfrak{M}$ with or without parameters, page 198
$\mathfrak{pe}\mathcal{P}_{+}(\mathfrak{M})$ family of non-empty sets which are elementarily definable in a structure $\mathfrak{M}$ with or without parameters, page 198
qMS class of quasi-mereological structures, page 198
$\mathrm{Mod}(\mathrm{Ax}^{\mathbf{M}})$ class of all models of axioms of the theory **M**, page 200
$\mathrm{Mod}(\mathbf{M})$ class of all models of the theory **M**, page 200
$\underline{\mathrm{Mod}(\mathbf{M})}$ set of all true sentences in all structures of **qMS**, page 200
$\overline{\mathbf{M}}$ set of all theses of **M** which are sentences, page 200
$\mathbb{atc}$ set of atomic elements, page 205
$\mathbb{atl}$ set of atomless elements, page 206

$e\mathcal{P}(\mathfrak{M})$ family of sets which are elementarily definable (without parameters) in a structure $\mathfrak{M}$, page 214
$e\mathcal{P}_{+}(\mathfrak{M})$ family of non-empty sets which are elementarily definable (without parameters) in a structure $\mathfrak{M}$, page 214
E class of all L^{d}_{c}-structures in which the relations $\sqsubset$, $\sqsubseteq$, $\circ$ and $\wr$ satisfy given conditions, page 214
ϵ predicate "is an element of", page 223
Null predicate "is a null class", page 223
Set predicate "is a distributive set", page 223
Pair predicate "is the pair of", page 223
Un predicate "is the union (generalised sum) of", page 223
$\subset$ predicate "is a subset of", page 223
Pow predicate "is the power class of", page 223
IS predicate "is the intersection of", page 223
$L^{d}_{\mathbf{MT}}$ extension of language $L_{\mathbf{MT}}$, page 224
$\varnothing$ empty class in **MT**, page 225
$\{y, z\}$ unordered pair in **MT**, page 225
$<x, y>$ ordered pair in **MT**, page 225
$y \cap z$ intersection of y and z in **MT**, page 226
ZF Zermelo-Fraenkel first-order set theory, page 234
$L_{\mathbf{ZF}}$ first-order language of **ZF**, page 234
Set predicate "is a distributive set", page 234
F predicate "is a family of sets", page 234
$\varnothing$ empty set in **ZF**, page 236
$\cap$ product of sets in **ZF**, page 236
$\{x, y\}$ unordered pair in **ZF**, page 237
$<x, y>$ ordered pair in **ZF**, page 237
ZFA **ZF** with set-theoretic atoms, page 237
$L_{\mathbf{ZFA}}$ first-order language of **ZFA**, page 237
a set of atoms in **ZFA**, page 237
MZF set theory with classical mereology, page 238
$L_{\mathbf{MZF}}$ first-order language of **MZF**, page 238
Sum predicate "is a mereological sum of" in **MZF**, page 239
Ind predicate "is an individual" in **MZF**, page 240
i set of individuals in **MZF**, page 242
$\in$ set-theoretic predicate "is a member of" ("belongs to"), page 249
$\notin$ set-theoretic predicate "is not a member of" ("does not belong to"), page 249
$\subseteq$ set-theoretic predicate "is a subset of", page 249
$\nsubseteq$ set-theoretic predicate "is not a subset of", page 249
$\emptyset$ empty set, page 249
$\subsetneq$ predicate 'is a proper subset of', page 249

$\mathcal{P}(X)$	power set of a set X (the set of all subsets of X), page 249
$\mathcal{P}_+(X)$	set of all non-empty subsets of a set X, page 249
$\cup$	set-theoretic operation of sum of two sets, page 250
$\cap$	set-theoretic operation of product of two sets, page 250
$\setminus$	set-theoretic operation of difference of two sets, page 250
$\bigcup$	set-theoretic operation of sum of families of sets, page 251
$\bigcup \mathcal{F}$	set-theoretic sum of a family $\mathcal{F}$ of sets, page 251
$\bigcap$	set-theoretic operation of product of families of sets, page 251
$\bigcap \mathcal{F}$	set-theoretic product of a family $\mathcal{F}$ of sets, page 251
$\mathcal{FC}(X)$	family of all finite subsets of a set X and all those subsets of X whose complements are finite, page 251
$\mathbb{N}$	set of all natural numbers, page 252
$\times$	operation of Cartesian product of sets, page 252
$X \times Y$	Cartesian product of sets X and Y, page 252
$X \times X$	Cartesian product of a given set X, page 252
$\mathcal{B}(X)$	family of all binary relation in a given set X, page 252
id_X	set-theoretic binary relation *identity* on a given set X, page 252
$\breve{R}$	converse relation to a relation R, page 253
$\circ$	operation of relative product of relations, page 253
$R_1 \circ R_2$	relative product of relations R_1 and R_2, page 253
SPOS	class of strictly partially ordered sets, page 257
$\prec$	strict partial order in given set, page 257
$\not\prec$	complement of the relation $\prec$, page 257
$\leqq$	set-theoretic sum of the relations $\prec$ and *identity*, page 257
POS	class of partially ordered sets, page 258
$\leq$	partial order in given set, page 258
$\lneq$	difference between relations $\leq$ and id_X, page 259
UB	set of all upper bounds of a given set in a given partially ordered set $\mathfrak{X}$, page 260
LB	set of all lower bounds of a given set in a given partially ordered set $\mathfrak{X}$, page 260
$\mathsf{sup}_{\leq}$	supremum relation with respect to $\leq$, page 260
$\mathsf{inf}_{\leq}$	infimum relation with respect to $\leq$, page 262
$\mathsf{max}_{\leq}$	set of maximal elements in a given set with respect to $\leq$, page 264
$\mathsf{min}_{\leq}$	set of minimal elements in a given set with respect to $\leq$, page 264
1	unity of a given partially ordered set, page 264
0	zero of a given partially ordered set, page 264
$+$	operation of sum of two elements in a given lattice, page 265
$\cdot$	operation of product of two elements in a given lattice, page 265
$-$	complement operation, page 270
BL	class of all Boolean lattices, page 271
$\triangle$	symmetric difference, page 272

$L_{\leq}$	first-order language with the single specific constant "≤", page 293
B	elementary theory of Boolean lattices, page 293
$L^{d}_{\leq}$	extension of language $L_{\leq}$ by defined constants, page 295
$Ax^{\mathbf{B}}$	set of specific axioms of theory **B**, page 295
ω	cardinality of $\mathbb{N}$, page 296
$inv(\mathfrak{B})$	elementary invariants of a given Boolean lattice, page 296
ecBL	class of e-complete Boolean lattices, page 301
pecBL	class of pe-complete Boolean lattices, page 301
$Ax^{\mathbf{pecB}}$	set of specific axioms of the theory **pecB**, page 305
$Ax^{\mathbf{ecB}}$	set of specific axioms of the theory **ecB**, page 305
pecB	elementary theory of pe-complete Boolean lattices, page 306
ecB	elementary theory of e-complete Boolean lattices, page 306
CB	elementary theory of complete Boolean lattices, page 306

List of featured formulas

Index of names and terms